GEOMETRIC INFINITENESS IN NEGATIVELY PINCHED HADAMARD MANIFOLDS

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Abstract. We generalize Bonahon's characterization of geometrically infinite torsion-free discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ to geometrically infinite discrete isometry subgroups in the case of rank 1 symmetric spaces, and, under the assumption of bounded torsion, to the case of negatively pinched Hadamard manifolds. Every such geometrically infinite isometry subgroup Γ has a set of nonconical limit points with cardinality of continuum.

1. Introduction

In [4] Francis Bonahon proved the following characterization of geometrically infinite torsion-free discrete subgroups of the isometry group of the 3-dimensional hyperbolic space $\mathbb{H}^3$:

Theorem 1.1. *A discrete torsion-free subgroup $\Gamma < \mathrm{Isom}(\mathbb{H}^3)$ is geometrically infinite if and only if there exists a sequence of closed geodesics λ_i in the manifold $M = \mathbb{H}^3/\Gamma$ which "escapes every compact subset of M," i.e., for every compact subset $K \subset M$,*

$$\mathrm{card}\left(\{i : \lambda_i \cap K \neq \emptyset\}\right) < \infty.$$

In [12, Theorem 1.4] we generalized Bonahon's theorem to discrete torsion-free groups of isometries of negatively pinched Hadamard manifolds. Torsion-free discrete isometry subgroups only contain parabolic, loxodromic and identity isometries. In this paper, we also consider discrete isometry subgroups containing elliptic isometries. For rank 1 symmetric spaces X, we prove that Bonahon's theorem holds for any discrete isometry subgroup with torsion. For general negatively pinched Hadamard manifolds X, we generalize Bonahon's theorem to discrete isometry subgroups where the order of elliptic elements is bounded, i.e. discrete isometry subgroups with bounded torsion. Furthermore, we use an argument similar to the one in [12] to generalize a theorem of Chris Bishop and prove that the set of nonconical limit points of such geometrically infinite subgroups has cardinality of continuum. Our main result is:

Theorem 1.2. *Suppose that either X is a negatively curved symmetric space and $\Gamma < \mathrm{Isom}(X)$ is a discrete subgroup or X is a negatively pinched Hadamard manifold and $\Gamma < \mathrm{Isom}(X)$ is a discrete group with bounded torsion. Then the following are equivalent:*

1. *Γ is geometrically infinite.*
2. *There exists a sequence of closed geodesics $\lambda_i \subset M = X/\Gamma$ which escapes every compact subset of M.*
3. *The set of nonconical limit points of Γ has cardinality of continuum.*

As in [12], we then obtain the following form of the Beardon–Maskit criterion for geometric finiteness of discrete isometry groups of negatively pinched Hadamard manifolds:

Corollary 1.3. *If $\Gamma < \mathrm{Isom}(X)$ is a discrete isometry subgroup with bounded torsion (resp. with torsion) of a negatively pinched Hadamard manifold X (resp. a rank 1 symmetric*

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space), then Γ is geometrically finite if and only if every limit point of Γ is either a conical limit point or a parabolic fixed point.

For the 3-dimensional hyperbolic space $\mathbb{H}^3$, Bonahon proved that the product of two parabolic isometries which generate a nonelementary discrete isometry subgroup is loxodromic, and use this result to prove Theorem 1.1. We generalized this result to n-dimensional Hadamard manifolds with sectional curvatures pinched between $-\kappa^2$ and -1 , showing that there exists a number $l(n,\kappa)$ such that for any pair of parabolic isometries generating a nonelementary discrete subgroup, a certain word $w = w(g,h)$ of length $\leq l(n,\kappa)$ is loxodromic [12, Theorem 8.5]. In this paper, we prove a similar result for nonelementary discrete isometry subgroups generated by elliptic isometries. For rank 1 symmetric spaces, we use the following theorem proved by Emmanuel Breuillard in [9].

Theorem 1.4. *Let X be an n-dimensional symmetric space. Then there exists a constant $\mathfrak{C}(n)$ such that for every discrete non-amenable subgroup $\Gamma < \mathrm{Isom}(X)$ generated by two elements g, h, there exists a word $w = w(g,h)$ of length at most $\mathfrak{C}(n)$, which represents an infinite order element of Γ.*

Remark 1.5. While E. Breuillard proved Theorem 1.4 for arbitrary symmetric spaces (and, more generally, for subgroups of $GL(n,K)$, where K is a field), we apply his theorem only to rank 1 symmetric spaces.

For general negatively pinched n-dimensional Hadamard manifolds X, our argument is different, and we consider discrete isometry subgroups where the order of elliptic elements is no greater than some $u < \infty$. We prove that there exists a number $l = l(n,\kappa,u)$ such that for any pair of elliptic isometries $\gamma_1, \gamma_2 \in \mathrm{Isom}(X)$ generating a nonelementary discrete subgroup where the order of elliptic elements is no greater than u, a certain word $w = w(g,h)$ of length $\leq l$ has infinite order (i.e. is parabolic or loxodromic).

Organization of the paper. In Section 3 we discuss properties of elliptic isometries of negatively pinched Hadamard manifolds. In Section 4 we prove the existence of loxodromic elements of uniformly bounded word length in infinite elementary discrete isometry subgroups generated by elliptic elements. Furthermore, in Subsections 4.1 and 4.2, we prove the existence of loxodromic elements of uniformly bounded word length in discrete nonelementary isometry subgroups generated by two elliptic isometries in the case of rank 1 symmetric spaces and negatively pinched Hadamard manifolds, respectively. In Section 5, we use the results in Section 4 to generalize Bonahon's theorem and prove Theorem 1.2.

This paper follows our earlier work on torsion-free discrete isometry subgroups in [12]. Some arguments here are similar to the ones of our earlier paper. For detailed properties of negatively pinched Hadamard manifolds and discrete isometry subgroups see [12].

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2. Notation

In this paper, we use X to denote a negatively pinched Hadamard manifold of dimension n, unless otherwise stated. We assume that the sectional curvatures of X lie between $-\kappa^2$ and -1. Let $\partial_\infty X$ denote the ideal boundary sphere of X, and we use the notation $\bar{X} = X \cup \partial_\infty X$ for the visual compactification. Let $\mathrm{Isom}(X)$ denote the isometry group of X, and $\Gamma < \mathrm{Isom}(X)$ denote discrete isometry subgroups. Let δ denote the *hyperbolicity constant* of X; hence, $\delta \leq \cosh^{-1}(\sqrt{2})$. We use the notation $B(x,r)$ to denote the open r-ball centered at $x \in X$ and $N_r(A)$ for the open r-neighborhood of a subset $A \subset X$. Let $Vol(A)$ denote the volume of a measurable subset $A \subset X$.

Every isometry g of X extends to a homeomorphism (still denoted by g) of $\bar{X}$. Let $F(g)$ denote the fixed point set of g in $\bar{X}$. For a discrete isometry subgroup Γ, let $F(\Gamma)$ denote the fixed point set of the group, i.e

$$F(\Gamma) := \bigcap_{g \in \Gamma} F(g).$$

Isometries of X are classifies as parabolic, elliptic and loxodromic based on their fixed point sets, see [1, 8, 12]. Every elliptic element g in a discrete isometry subgroup has finite order. We let $o(g)$ to denote the order of g.

For an isometry g of X, we define the *rotation* of g at $x \in X$ as:

$$r_g(x) = \max_{v \in T_x X} \angle(v, P_{g(x),x} \circ g_{*_x} v).$$

Here $g_{*_x} : T_x X \to T_{g(x)} X$ is the differential and $P_{g(x),x} : T_{g(x)} X \to T_x X$ is the parallel transport along the unique geodesic from $g(x)$ to x. Following [1], given $a \geq 8$ we define the *norm of* g at x as $n_g(x) = \max(r_g(x), a \cdot d_g(x))$ where $d_g(x) = d(x, g(x))$.

Define the *Margulis region* $Mar(g, \varepsilon)$ of $g \in \text{Isom}(X)$ as:

$$Mar(g, \varepsilon) = \{x \in X \mid d(x, g(x)) \leq \varepsilon\}.$$

By the convexity of the distance function, $Mar(g, \varepsilon)$ is convex.

Given $x \in X$ and a discrete subgroup $\Gamma < \text{Isom}(X)$, let $\mathcal{F}_\varepsilon(x) = \{\gamma \in \Gamma \mid d(x, \gamma x) \leq \varepsilon\}$ denote the set of isometries in Γ which move x a distance at most ε. Let $\Gamma_\varepsilon(x)$ denote the subgroup generated by $\mathcal{F}_\varepsilon(x)$. We use $\varepsilon(n, \kappa)$ to denote the Margulis constant of X. Then, the Margulis Lemma, $\Gamma_\varepsilon(x)$ is virtually nilpotent whenever $0 < \varepsilon \leq \varepsilon(n, \kappa)$. More precisely,

Proposition 2.1. [1, Theorem 9.5] *Given $0 < \varepsilon \leq \varepsilon(n, \kappa)$ and $x \in X$, the group N generated by the set $\{\gamma \in \Gamma_\varepsilon(x) \mid n_\gamma(x) \leq 0.49\}$ is a nilpotent subgroup of $\Gamma_\varepsilon(x)$ of a uniformly bounded index (where the bound depends only on κ and n). Moreover, each coset $\gamma N \subset \Gamma_\varepsilon(x)$ can be represented by an element γ of word length $\leq m(n, \kappa)$ in the generating set $\mathcal{F}_\varepsilon(x)$ of $\Gamma_\varepsilon(x)$. Here $m(n, \kappa)$ is a constant depending only on κ and n.*

Remark 2.2. $\Gamma_\varepsilon(x)$ is always finitely generated.

We will use the following important property of nilpotent groups in Section 4:

Theorem 2.3. [10, 13] *Let G be a nilpotent group. The set of all finite order elements of G forms a characteristic subgroup of G. This subgroup is called the* torsion subgroup *of G and denoted by $Tor(G)$.*

Given a set $T = \{g_1, \cdots, g_k\} \subset \text{Isom}(X)$, we let $\langle T \rangle$ denote the group generated by T. Following [12], we let $\Lambda = \Lambda(\Gamma) \subset \partial_\infty X$ denote the *limit set* of Γ, i.e. the accumulation set in $\partial_\infty X$ of one (equivalently, any) Γ-orbit in X. Let $\text{Hull}(A)$ denote the closed convex hull of a subset $A \subset \bar{X}$, and let $\text{QHull}(A)$ denote the quasiconvex hull of A.

The group Γ acts properly discontinuously on $\bar{X} \setminus \Lambda$, [8, Proposition 3.2.6]. We obtain an orbifold with boundary

$$\bar{M} = M_c(\Gamma) = \left(\bar{X} \setminus \Lambda\right) / \Gamma.$$

Let $Core(M)$ denote the *convex core* of M which is defined as the Γ-quotient of the closed convex hull of $\Lambda(\Gamma)$.

Given $0 < \varepsilon \leq \varepsilon(n, \kappa)$ and a discrete subgroup $\Gamma < \text{Isom}(X)$, define the set $T_\varepsilon(\Gamma) = \{p \in X \mid \Gamma_\varepsilon(p) \text{ is infinite}\}$. It is a disjoint union of subsets of the form $T_\varepsilon(G)$, where G ranges over all maximal infinite elementary subgroups of Γ, [8, Proposition 3.5.5]. If $G < \Gamma$ is a maximal parabolic subgroup, $T_\varepsilon(G)$ is precisely invariant and $\text{Stab}_\Gamma(T_\varepsilon(G)) = G$, [8, Corollary 3.5.6]. In this case, by abuse of notation, we regard $T_\varepsilon(G)/G$ as a subset of M, and call it a *Margulis cusp*. Let $\mathrm{cusp}_\varepsilon(M)$ denote the union of all Margulis cusps of M; it is called the *cuspidal* part of M, [8].

3. Elliptic isometries

An isometry g of X is elliptic if it fixes a point in X. In particular, the identity map is elliptic. The fixed point set $F(g) \cap X$ is a totally geodesic subspace of X invariant under g. Note that the fixed point set $F(g)$ of any elliptic isometry $g \neq Id_X$ has dimension at most $\dim X - 1$, [1, Lemma 6.3].

Any elliptic element in a discrete isometry subgroup $\Gamma < \mathrm{Isom}(X)$ has finite order.

Lemma 3.1. [1, 12.2] *Let g be an elliptic isometry of a Hadamard manifold X with $o(g) = m$. If $x \in X$ and $d(x, F(g)) = a$, then $d(x, gx) \geq 2a \sin(\pi/m)$.*

Proposition 3.2. *Suppose that $T = \{g_1, \cdots, g_b\} \subset \mathrm{Isom}(X)$ is a subset consisting of b nontrivial elliptic elements. Then for any $x \in X$ and $\varepsilon > 0$, there exists $y \in B(x, \varepsilon/4)$ such that for each i,*

$$d(y, F_i) \geq \varepsilon/2^{b+2},$$

where F_i is the fixed point set of g_i.

Proof. Since F_1 is a totally-geodesic lower dimensional submanifold, there is a point $y_1 \in B(x, \varepsilon/4)$ such that

$$d(y_1, F_1) \geq \varepsilon/8.$$

Next, consider the ball $B(y_1, \varepsilon/8)$; there is a point $y_2 \in B(y_1, \varepsilon/8) \subset B(x, \varepsilon/4)$ such that

$$d(y_2, F_2) \geq \varepsilon/16$$

by the same reason as above. Inductively, there exists a point $y \in B(x, \varepsilon/4)$ such that

$$d(y, F_i) \geq \varepsilon/2^{b+2}$$

for all i.

□

Given an elliptic isometry g, let $x \in X$ denote a fixed point of g. Consider the differential $g_{*_x} : T_xX \to T_xX$. It is linear and preserves the inner product in $\mathbb{R}^n = T_xX$. Hence $g_{*_x} \in O(n)$.

Lemma 3.3. [11, Proposition 7] *Assume that $A \in O(n)$ has finite order which is bounded (from above) by u; then*

$$\| A - Id \| \geq 2\sin(\pi/u),$$

where $\| \cdot \|$ denotes the operator norm.

Next, we discuss finite discrete elementary groups of X.

A group $G < \mathrm{Isom}(X)$ is elementary either if the fixed point set $F(G)$ in nonempty, or if G preserves setwise some bi-infinite geodesic in $\bar{X}$.

Lemma 3.4. *Assume that two elements $g_1, g_2 \in Isom(X)$ generate a nonelementary discrete subgroup. Then $Mar(g_1, \varepsilon)$ and $Mar(g_2, \varepsilon)$ are disjoint for every $\varepsilon \leq \varepsilon(n, \kappa)$.*

Proof. Suppose there exists a point $x \in Mar(g_1, \varepsilon) \cap Mar(g_2, \varepsilon)$. By the Margulis Lemma, $\langle g_1, g_2 \rangle$ is elementary which is a contradiction. Thus, $Mar(g_1, \varepsilon)$ and $Mar(g_2, \varepsilon)$ are disjoint. □

Based on the fixed point set, elementary groups are divided into the following three classes [8]:

(1) $F(G)$ is a nonempty subspace of $\bar{X}$.
(2) $F(G)$ consists of a single point of $\partial_\infty X$.
(3) G has no fixed point in X, and G preserves setwise a unique bi-infinite geodesic in X.

Remark 3.5. If $G < \mathrm{Isom}(X)$ is discrete and in the first class, then G is finite by discreteness and consists of elliptic isometries. If G is discrete and in the second class, it is called parabolic, and it contains a parabolic isometry [8, Proposition]. Discrete groups G in the third class will be called elementary loxodromic groups.

Lemma 3.6. *Suppose that $G < \mathrm{Isom}(X)$ is a discrete elementary subgroup, and every element in G is elliptic. Then G is finite.*

Proof. By Remark 3.5, G is either finite or loxodromic. Suppose that G is loxodromic and preserves a geodesic $l \subset X$ setwise. Let $\rho : G \to \mathrm{Isom}(l)$ denote the restriction homomorphism. Since G is loxodromic, the subgroup $\rho(G)$ has no fixed point in l. Hence, there exist two elements $g, h \in G$ such that $\rho(g), \rho(h)$ are distinct involutions. Their product $\rho(g)\rho(h)$ is a nontrivial translation of l. Hence, gh is a loxodromic isometry of X, contradicting our assumption. Hence, G is finite. □

Corollary 3.7. *Every discrete elementary loxodromic group contains a loxodromic isometry.*

Proposition 3.8. *There exists a constant $A(n, u, \varepsilon)$ with the following property. Suppose that $G < \mathrm{Isom}(X)$ is a finite elementary subgroup where the order of all elements is $\leq u$, and $\varepsilon > 0$. Then for any $x \in X$, there exists $y \in B(x, \varepsilon/4)$ such that $d(y, F(g)) \geq A(n, u, \varepsilon)$ for all $g \in \langle T \rangle - \{Id\}$.*

Proof. We will prove that the order of G is bounded by some $b = b(n, u)$. Let $z \in X$ be a fixed point of G. We will identify the group of linear isometries of T_zX with $O(n)$. Then for each $g \in G$, the differential $g_{*_z} : T_zX \to T_zX$ is in $O(n)$. Consider two elliptic elements $g, h \in \langle T \rangle$. By the assumption on the orders of elements of G, $o(g^{-1}h) \leq u$. Then the order of $(g^{-1}h)_{*_z}$ is also bounded by u. By Lemma 3.3,

$$\|(g)_{*_z} - (h)_{*_z}\| = \|(g^{-1}h)_{*_z} - Id\| \geq 2\sin(\pi/u).$$

Since $O(n)$ is compact, the above inequality implies that the order of G is at most $b(n, u)$. Now, the statement follows from Proposition 3.2 applied to the subset $G \subset \mathrm{Isom}(X)$. □

4. Loxodromic products

In [12, Proposition 8.4], we proved that given two parabolic isometries g_1, g_2 of X, if the distance of their Margulis regions is sufficiently large, then g_2g_1 is loxodromic. Below, we extend this result to elliptic isometries g_1, g_2.

Proposition 4.1. *There exists a constant $L = L(\varepsilon)$ such that given two elliptic isometries g_1, g_2 in a subgroup $\Gamma < Isom(X)$, if $d(Mar(g_1, \varepsilon), Mar(g_2, \varepsilon)) > L$ then $h = g_2g_1$ is loxodromic.*

Proof. The argument is exactly the same as the proof in [12, Proposition 8.4]. The infinite piecewise geodesic path γ in Figure 1 is a uniform quasigeodesic if L is sufficiently large [12], and $B_i = Mar(g_i, \varepsilon)$ for $i = 1, 2$. By the Morse Lemma, g_2g_1 preserves the unique geodesic γ^* in X which is Hausdorff close to γ, and acts on γ^* as a nontrivial translation. Thus g_2g_1 is loxodromic. For details, see [12, Proposition 8.4]. □

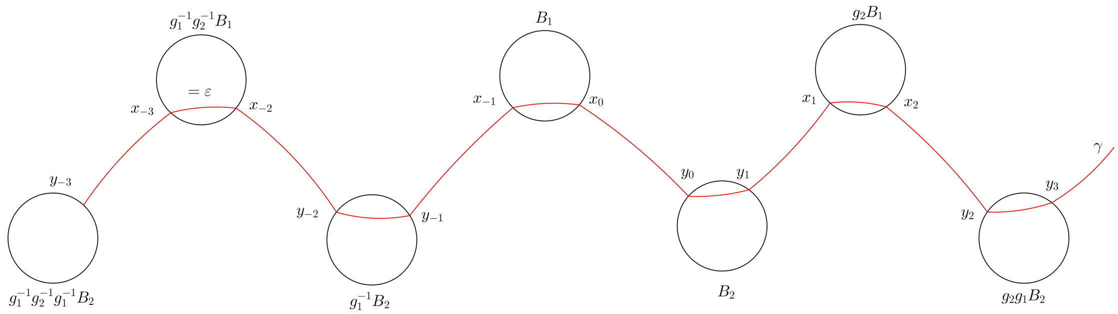

FIGURE 1.

Consider an infinite elementary discrete subgroup of $\mathrm{Isom}(X)$ generated by elliptic isometries. It is either a parabolic elementary subgroup or a loxodromic elementary subgroup. Lemma 4.2 and Proposition 4.3 below prove the existence of infinite order elements of uniformly bounded word length in such groups.

Lemma 4.2. *Suppose that the set $T = \{g_1, g_2, \cdots, g_m\} \subset \mathrm{Isom}(X)$ consists of elliptic elements, and the group $\langle T\rangle$ is an elementary loxodromic group. Then there is a pair of indices $1 \le i, j \le m$ such that g_ig_j is loxodromic.*

Proof. Let l denote the geodesic preserved setwise by $\langle T\rangle$. We claim that there exists g_i such that it swaps the endpoints of l. Otherwise, l is fixed pointwise by $\langle T\rangle$, and $\langle T\rangle$ is a finite elementary subgroup of $\mathrm{Isom}(X)$ which is a contradiction. Since $g_i(l) = l$, there exists $x \in l$ such that $g_i(x) = x$. By the same argument as in Lemma 3.6, there exists g_j such that $g_j(x) \ne x$, and g_ig_j is loxodromic. □

For discrete parabolic elementary subgroups generated by elliptic isometries, we have the following result.

Proposition 4.3. *Given $x \in X, 0 < \varepsilon \le \varepsilon(n, \kappa)$ and a discrete subgroup $\Gamma < \mathrm{Isom}(X)$, suppose that the set $\mathcal{F}_\varepsilon(x) \subset \Gamma$ consists of elliptic elements and the group $\Gamma_\varepsilon(x) < \Gamma$ generated by this set is a parabolic elementary subgroup. Then there is a parabolic element $g \in \Gamma_\varepsilon(x)$ of word length in $\mathcal{F}_\varepsilon(x)$ uniformly bounded by a constant $C(n, \kappa)$.*

Proof. Let N be the subgroup of $\Gamma_\varepsilon(x)$ generated by the set $\{\gamma \in \Gamma_\varepsilon(x) \mid n_\gamma(x) \le 0.49\}$. By Proposition 2.1, N is a nilpotent subgroup of $\Gamma_\varepsilon(x) = s_1N \cup s_2N \cdots \cup s_IN$ where the index I is uniformly bounded and each s_i has uniformly bounded word length $\le m(n, \kappa)$ with respect to the generating set $\mathcal{F}_\varepsilon(x)$ of $\Gamma_\varepsilon(x)$.

Let $F = F_S$ denote the free group on $S = \mathcal{F}_\varepsilon(x)$. Consider the projection map $\pi : F \to \Gamma_\varepsilon(x)$, and the preimage $\pi^{-1}(N) < F$. Let T denote a left Schreier transversal for $\pi^{-1}(N)$ in F (i.e a transverse for $\pi^{-1}(N)$ in F so that every initial segment of an element of T itself belongs to T). By the construction, every element $t \in T$ in the Schreier transversal has the minimal word length among all the elements in $t\pi^{-1}(N)$. Then the word length of t is also bounded by $m(n, \kappa)$ since $t\pi^{-1}(N) = s_i\pi^{-1}(N)$ for some i. By the Reidemeister-Schreier Theorem, $\pi^{-1}(N)$ is generated by the set

$$\mathrm{Y} = \{t\gamma_is \mid t, s \in T, \gamma_i \in \mathcal{F}_\varepsilon(x), \text{ and } s\pi^{-1}(N) = t\gamma_i\pi^{-1}(N)\}.$$

Since the word length of elements in a Schreier transversal is not greater than $m(n, \kappa)$, then the word length of elements in the generating set Y is not greater than $2m(n, \kappa) + 1$.

Next, we claim that there exists a parabolic element in $\pi(\mathrm{Y})$. If not, then all the elements in $\pi(\mathrm{Y})$ are elliptic. By Theorem 2.3, all the torsion elements in N form a subgroup of N. Hence all elements in $N = \langle\pi(\mathrm{Y})\rangle$ are elliptic. By Lemma 3.6, N is finite, which contradicts our assumption that $\Gamma_\varepsilon(x)$ is infinite. Therefore, there exists a parabolic element in $\pi(Y)$ whose word length is $\leq 2m(n,\kappa)+1$. We let $C(n,\kappa) = 2m(n,\kappa)+1$.

□

Remark 4.4. The virtually nilpotent group $\Gamma_\varepsilon(x)$ is uniformly finitely generated by at most $S(n,\kappa)$ isometries α satisfying $d(x,\alpha(x)) \leq \varepsilon$, [1, Lemma 9.4]. Let F be the free group on the set A consisting of such elements α. Since the number of subgroups of F with a given finite index is uniformly bounded, and each subgroup has a finite free generating set it follows that $\pi^{-1}(N)$ has a generating set where each element has word length (with respect to A) uniformly bounded by some constant $C(n,\kappa)$. Hence there is a generating set of N where the word length of each element is uniformly bounded by $C(n,\kappa)$. Similarly, there exists a parabolic element g in this generating set of word length bounded by $C(n,\kappa)$ in elements α. This argument provides an alternative proof of the existence of a parabolic isometry of uniformly bounded word length in $\Gamma_\varepsilon(x)$.

In the rest of the section, we will prove the existence of a loxodromic isometry of uniformly bounded word length in a nonelementary discrete subgroup generated by two elliptic isometries. The arguments for rank 1 symmetric spaces and general negatively pinched Hadamard manifolds are different. For the latter case, we assume that the elliptic elements in the nonelementary isometry subgroup have order no greater than some $u < \infty$.

4.1. **Rank 1 symmetric spaces.** In this subsection, X denotes a rank 1 symmetric space. We use Theorem 1.4 proved by E. Breuillard in [9] to construct a loxodromic isometry of uniformly bounded word length.

Theorem 4.5. *Let $\Gamma < \mathrm{Isom}(X)$ be a discrete subgroup. Suppose that $g, h \in \Gamma$ generate a nonelementary subgroup of Γ. Then there exists a loxodromic element $w \in \langle g, h\rangle$ of uniformly bounded word length. The bound only depends on the symmetric space.*

Proof. By Theorem 1.4, there exists an infinite order element $w' \in \langle g, h\rangle$ of uniformly bounded word length. If w' is loxodromic, we let $w = w'$. Now we assume that w' is parabolic with the fixed point p. Then $hw'h^{-1}$ and $gw'g^{-1}$ are both parabolic elements with fixed points $h(p)$ and $g(p)$ respectively. We claim that at least one of $h(p)$ and $g(p)$ is different from p. Otherwise, $\langle g, h\rangle$ is an elementary subgroup (fixing p) which contradicts to our assumption. Without loss of generality, assume that $h(p) \neq p$. Then w' and $hw'h^{-1}$ are parabolic elements with different fixed points. Hence, there exists a loxodromic element $w \in \langle w', hw'h^{-1}\rangle$ of uniformly bounded word length [12, Theorem 8.5]. Thus in both cases, $w \in \langle g, h\rangle$ is a loxodromic element of uniformly bounded word length and the bound only depends on the symmetric space X.

□

4.2. **Negatively pinched Hadamard manifolds.** In this subsection, X denotes a negatively pinched Hadamard manifold. Given a discrete isometry subgroup $\Gamma < \mathrm{Isom}(X)$ generated by a finite set S, the *entropy* of Γ with respect to S is defined as follows:

$$\mathrm{Ent}_S(\Gamma) = \lim_{m\to\infty} \frac{1}{m}(\log \mathrm{card}\,\{\gamma \in \Gamma \mid l_S(\gamma) \leq m\})$$

where $l_S(\gamma)$ denotes the word length of γ in terms of the generating set S. One also define the *entropy* of Γ by:

$$\mathrm{Ent}(\Gamma) = \inf_S\{\mathrm{Ent}_S(\Gamma) \mid S \text{ is a finite generating set of } \Gamma\}.$$

Theorem 4.6. [2, Theorem 1.1] *There exists a positive constant $D(n,\kappa)$ such that for any finitely generated discrete group Γ of isometries of X, either Γ is virtually nilpotent or $\mathrm{Ent}(\Gamma) \geq D(n,\kappa)$.*

Let $\Gamma < \mathrm{Isom}(X)$ be a discrete subgroup generated by a finite set S which is not virtually nilpotent. By Theorem 4.6,

$$\mathrm{Ent}_S(\Gamma) = \lim_{m\to\infty} \frac{1}{m}(\log \mathrm{card}\,\{\gamma \in \Gamma \mid l_S(\gamma) \le m\}) \ge D(n,\kappa).$$

Consider the sequence of non-negative real numbers $(a_m)_{m=1}^{\infty}$ where

$$a_m = \log \mathrm{card}\,\{\gamma \in \Gamma \mid l_S(\gamma) \le m\}.$$

The sequence is subadditive. Then, by Fekete's Lemma [6], the limit

$$\lim_{m\to\infty} \frac{a_m}{m}$$

exists and equals

$$\inf_{m\ge 1} \left(\frac{a_m}{m}\right).$$

Hence,

$$\mathrm{card}\,\{\gamma \in \Gamma \mid l_S(\gamma) \le m\} \ge e^{D(n,\kappa)m},$$

for all $m \ge 1$.

Proposition 4.7. *There exists a constant $N(n,\kappa,u)$ such that given elliptic isometries g_1, g_2 generating a discrete nonelementary subgroup $\langle g_1, g_2\rangle < \mathrm{Isom}(X)$, if all elliptic elements in this subgroup have order no greater than u, then there exists a loxodromic element in $\langle g_1, g_2\rangle$ of word length bounded by $N(n,\kappa,u)$.*

Proof. Let $\varepsilon = \varepsilon(n,\kappa)$ denote the Margulis constant, $V(r,n)$ be the volume of the r-ball in $\mathbb{H}^n$, $A(n,u,\varepsilon)$ be the constant in Proposition 3.8, $\eta = \max\{A(n,u,\varepsilon), \varepsilon/4\}$ and

$$R = R(n,\kappa,\varepsilon,u) = \frac{1}{D(n,\kappa)} \ln\left(\frac{V(\kappa(\mathcal{C}(L)),n)}{V(\rho/2,n)\kappa^n}\right) + 1$$

where $D(n,\kappa)$ is the constant in Theorem 4.6, $L = L(\varepsilon)$ is as in Proposition 4.1, $\rho = 2\eta\sin(\pi/u)$ and $\mathcal{C}(L) = L + 2n\delta + 2\varepsilon + \rho/2$.

Let $B(R)$ denote the set of all elements in $\langle g_1, g_2\rangle$ of word length at most R. If there exists a loxodromic element in $B(R)$, then its word length is bounded by $R(n,\kappa,\varepsilon,u)$. If there exists a parabolic element $g \in B(R)$ with the fixed point p, then at least one of the parabolic elements $g_1 g g_1^{-1}$ and $g_2 g g_2^{-1}$ has fixed point different from p. Otherwise, $\langle g_1, g_2\rangle$ is an elementary group (fixing p) which is a contradiction. Assume that $g_1 g g_1^{-1}(p) \ne p$. Then there is a loxodromic element of uniformly bounded word length in $\langle g, g_1 g g_1^{-1}\rangle$, [12, Theorem 8.5]. Now we assume that all the elements in $B(R)$ are elliptic. Let $B(R) = \{g_0 = e, g_1, g_2, \cdots g_N\}$ and $T = \{g_i^{-1}g_j \mid 0 \le i,j \le N, i \ne j\}$. Similarly, we assume that all the elements in T are elliptic.

If there exists a pair of indices $1 \le i,j \le N$ such that $d(Mar(g_i,\varepsilon), Mar(g_j,\varepsilon)) > L$, then by Proposition 4.1 $g_i g_j$ is a loxodromic element of word length $\le 2R$. Otherwise, consider the $L/2$-neighborhood of $Mar(g_i,\varepsilon)$ and denote it by B_i. Then $B_i \cap B_j \ne \varnothing$ for any $1 \le i,j \le N$. Similarly to the proof of [12, Proposition 8.2], there exists a point $x \in X$ such that $d(x,B_i) \le n\delta$ for all $0 \le i \le N$ where $\delta \le \cosh^{-1}\sqrt{2}$ is the hyperbolicity constant of X. Then

$$d(x, Mar(g_i,\varepsilon)) \le L/2 + n\delta.$$

For each i pick $y_i \in Mar(g_i,\varepsilon)$ such that $d(x,y_i) \le L/2 + n\delta$. Hence

$$d(x,g_i(x)) \le d(x,y_i) + d(y_i,g_i(y_i)) + d(g_i(y_i),g_i(x)) \le L + 2n\delta + \varepsilon.$$

Consider the ball $B(x,\varepsilon/2)$. Let

$$T' = \{g_i^{-1}g_j \mid 0 \le i,j \le N, i \ne j \text{ and } F(g_i^{-1}g_j) \cap B(x,\varepsilon/2) \ne \varnothing\}.$$

Then $d(x, g_i^{-1}g_j(x)) \leq \varepsilon$ for any $g_i^{-1}g_j \in T'$. By the Margulis Lemma, the subgroup $\langle T' \rangle$ is elementary.

Assume that the subgroup $\langle T' \rangle$ is finite. By Proposition 3.8, there exists $y \in B(x, \varepsilon/4)$ such that $d(y, F(g_i^{-1}g_j)) \geq A(n, u, \varepsilon)$ for all $g_i^{-1}g_j \in T'$. For other elements $g_i^{-1}g_j$ in $T \setminus T'$, $F(g_i^{-1}g_j) \cap B(x, \varepsilon/2) = \varnothing$. Then $d(y, F(g_i^{-1}g_j)) \geq \varepsilon/4$. Thus, for any element $g_i^{-1}g_j \in T$, $d(y, F(g_i^{-1}g_j)) \geq \eta := \max\{A(n, u, \varepsilon), \varepsilon/4\}$. By Lemma 3.1, we have $d(y, g_i^{-1}g_j(y)) \geq \rho := 2\eta \sin(\pi/u)$. Thus,

$$d(y, g_i(y)) \geq \rho \text{ and } d(g_i(y), g_j(y)) \geq \rho, \quad 1 \leq i \neq j \leq N.$$

Observe that

$$d(y, g_i(y)) \leq d(y, x) + d(x, g_i(x)) + d(g_i(x), g_i(y)) \leq L + 2n\delta + 2\varepsilon.$$

Let $\mathcal{C}(L) = L + 2n\delta + 2\varepsilon + \rho/2$. Consider the balls $B(y, \mathcal{C}(L))$ and $B(g_i(y), \rho/2)$ for $0 \leq i \leq N$. The volume of $B(y, \mathcal{C}(L))$ is at most $V(\kappa\mathcal{C}(L), n)/\kappa^n$ and the volume of $B(y, \rho/2)$ is at least $V(\rho/2, n)$, [8, Proposition 1.1.12]. The balls $B(g_i(y), \rho/2), i = 0, \cdots, N$ are pairwise disjoint and contained in $B(y, \mathcal{C}(L))$. By Theorem 4.6,

$$V(\kappa\mathcal{C}(L), n)/\kappa^n \geq V(\rho/2, n)(N+1) \geq V(\rho/2, n)e^{D(n,\kappa)R}$$

which contradicts the definition of R.

Thus, $\langle T' \rangle$ is either parabolic or loxodromic. If $\langle T' \rangle$ is loxodromic, by Lemma 4.2, there exist elements γ_1 and γ_2 in T' such that $\gamma_1\gamma_2$ is loxodromic. If $\langle T' \rangle$ is parabolic, by Proposition 4.3, there exists a parabolic element in $\langle T' \rangle$ of uniformly bounded word length. By an argument similar to the proof of Theorem 4.5, there is a loxodromic element of uniformly bounded word length in $\langle g_1, g_2 \rangle$. Thus in all cases, there exists a loxodromic element in $\langle g_1, g_2 \rangle$ of word length bounded by a constant $N(n, \kappa, u)$.

□

Remark 4.8. The Cayley graph of the finitely generated group $\langle g_1, g_2 \rangle$ is an infinite connected locally finite graph, and contains a ray [5]. Hence,

$$\text{card}\,\{\gamma \in \Gamma \mid l_S(\gamma) \leq m\} \geq m,$$

for all $m \geq 1$. Instead of using Theorem 4.6, we use the inequality

$$V(\kappa\mathcal{C}(L), n)/\kappa^n \geq V(\rho/2, n)(N+1) \geq V(\rho/2, n)R.$$

Hence, alternatively, we can use

$$R = \frac{V(\kappa(\mathcal{C}(L)), n)}{V(\rho/2, n)\kappa^n} + 1$$

in the proof of the proposition.

5. Generalized Bonahon's theorem

In [12, Theorem 1.5], we generalized Bonahon's theorem to discrete geometrically infinite torsion-free subgroups $\Gamma < \text{Isom}(X)$. In this section, we use Theorem 4.5 and Proposition 4.7 to generalize Bonahon's theorem to discrete geometrically infinite isometry subgroups with torsion. Let X denote a rank 1 symmetric space or a negatively pinched Hadamard manifold. Correspondingly, let $\Gamma < \text{Isom}(X)$ denote a discrete geometrically infinite subgroup or a geometrically infinite subgroup with bounded torsion. The proofs in these two cases are the same, and the main idea is similar to the one in [12].

Proof of the implication $(1) \Rightarrow (2)$ **in Theorem 1.2:** If there exists a sequence of closed geodesics $\beta_i \subseteq M$ whose lengths go to 0 as $i \to \infty$, then the sequence (β_i) escapes every compact subset of M. From now on, we assume that there exists a constant $\epsilon > 0$ such that the length $l(\beta) \geq \epsilon$ for any closed geodesic β in M.

Set $\varepsilon = \varepsilon(n, \kappa)$. Recall that Margulis cusps in M are isometric to $T_\varepsilon(G)/G$ where $G < \Gamma$ is a maximal parabolic subgroup [8, 12]. There exists a universal constant $r \in [0, \infty)$ such that $\mathrm{Hull}(T_\varepsilon(G)) \subseteq N_r(T_\varepsilon(G))$ for any maximal parabolic subgroup G, [12, Corollary 6.5]. We let $B(G) := N_{2+4\delta}(\mathrm{Hull}(T_\varepsilon(G)))$ and let M^o be the union of subsets $B(G)/\Gamma$ where G ranges over all maximal parabolic subgroups of Γ. Let M^c denote the closure of $\mathrm{Core}(M) \setminus M^o$. Since Γ is geometrically infinite, the noncuspidal part of the convex core $\mathrm{Core}(M) \setminus \mathrm{cusp}_\varepsilon(M)$ is unbounded [8]. Then M^c is also unbounded since $M^o \subseteq N_{r+2+4\delta}(\mathrm{cusp}_\varepsilon(M))$.

Fix a point $x \in M^c$. Let $C_k = \{y \in M^c \mid d(x, y) \leq n\mathfrak{R}\}$ where $\mathfrak{R} = r + 2 + 4\delta + m\varepsilon$ and $m = C(n, \kappa)$ is the constant in Proposition 4.3. Let $\tilde{x}$ be a lift of x in X. For every C_k, there exists a sequence of geodesic loops (γ_i) connecting x to itself in $\mathrm{Core}(M)$ such that the Hausdorff distance $\mathrm{hd}(\gamma_i \cap M^c, C_k) \to \infty$ as $i \to \infty$, [12, Lemma 9.1]. Let $y_i \in \gamma_i \cap M^c$ be such that $d(y_i, C_k)$ is maximal on $\gamma_i \cap M^c$. We pick a component α_i of $\gamma_i \cap M^c$ containing the point y_i. Let $\boldsymbol{\delta} C_k$ denote the relative boundary $\partial C_k \setminus \partial M^c_{cusp}$ of C_k where $M^c_{cusp} = M^o \cap \mathrm{Core}(M)$. Consider the sequence of geodesic arcs (α_i).

After passing to a subsequence in (α_i), one of the following three cases occurs:

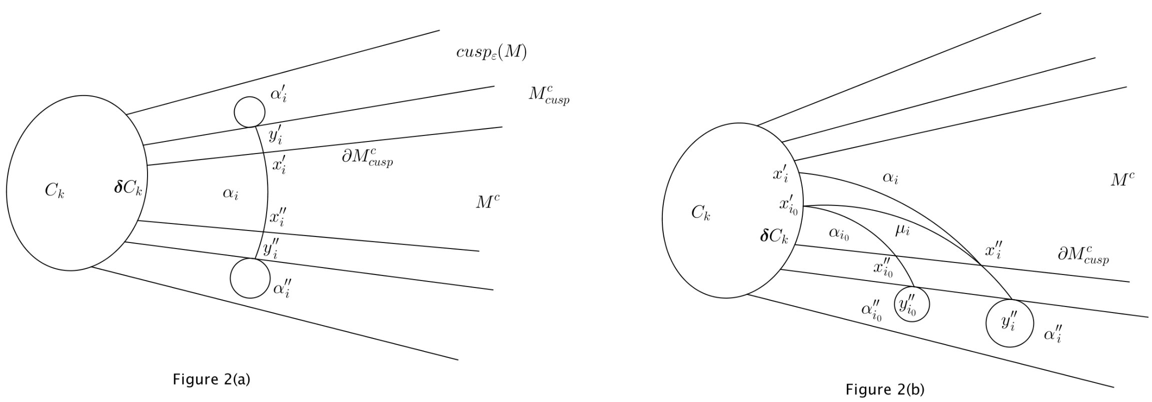

FIGURE 2.

Case (a): Each α_i has both endpoints x_i' and x_i'' on ∂M^c_{cusp} as in Figure 2(a). By construction, there exist y_i' and y_i'' in the cuspidal part $\mathrm{cusp}_\varepsilon(M)$ such that $d(x_i', y_i') \leq r_1, d(y_i', y_i'') \leq r_1$ where $r_1 = 2 + 4\delta + r$. Let $\tilde{y}_i'$ be a lift of y_i' such that $\tilde{y}_i' \in T_\varepsilon(G')$ for some maximal parabolic subgroup $G' < \Gamma$. By the definition, the subgroup $G_\varepsilon'(\tilde{y}_i')$ generated by the set

$$\mathcal{F}_\varepsilon(\tilde{y}_i') = \{\gamma \in G' \mid d(\tilde{y}_i', \gamma(\tilde{y}_i')) \leq \varepsilon\}$$

is infinite. We claim that there exists a parabolic element $g' \in G_\varepsilon'(\tilde{y}_i')$ such that $d(\tilde{y}_i', g'(\tilde{y}_i')) \leq m\varepsilon$. Assume that $\mathcal{F}_\varepsilon(\tilde{y}_i') = \{\gamma_1, \cdots, \gamma_b\}$. If γ_j is parabolic for some $1 \leq j \leq b$, we have $d(\tilde{y}_i', \gamma_j(\tilde{y}_i')) \leq \varepsilon$. Now assume that γ_j are elliptic for all $1 \leq j \leq b$. By Proposition 4.3, there is a parabolic element $g' \in \Gamma_\varepsilon(\tilde{y}_i')$ of word length in the generating set $\mathcal{F}_\varepsilon(\tilde{y}_i')$ bounded by m. By the triangle inequality, $d(\tilde{y}_i', g'(\tilde{y}_i')) \leq m\varepsilon$.

Then we find a nontrivial geodesic loop α_i' contained in the cuspidal part $\mathrm{cusp}_\varepsilon(M)$ such that α_i' connects y_i' to itself and has length $l(\alpha_i') \leq m\varepsilon$. Similarly, there exists a nontrivial geodesic loop α_i'' which connects y_i'' to itself and has length $l(\alpha_i'') \leq m\varepsilon$. Let

$$w' = x_i'y_i' * \alpha_i' * y_i'x_i' \in \Omega(M, x_i')$$

and

$$w'' = \alpha_i * x_i''y_i'' * \alpha_i'' * y_i''x_i'' * \alpha_i^{-1} \in \Omega(M, x_i')$$

where $\Omega(M, x_i')$ denotes the loop space of M. Observe that $w' \cap C_{n-1} = \emptyset, w'' \cap C_{n-1} = \emptyset$, and w', w'' represent parabolic elements $g', g'' \in \Gamma$ respectively.

We claim that g' and g'' have different fixed points. Otherwise, $g', g'' \in G'$ where $G' < \Gamma$ is some maximal parabolic subgroup. Then $y_i', y_i'' \in T_\varepsilon(G')/\Gamma$ and $x_i', x_i'' \in B(G')/\Gamma$. Since $\mathrm{Hull}(T_\varepsilon(G'))$ is convex, $B(G') = N_{2+4\delta}(\mathrm{Hull}(T_\varepsilon(G')))$ is also convex by the convexity of the distance function. Hence $x_i'x_i'' \subseteq B(G')/\Gamma$. However, $x_i'x_i''$ lies outside of $B(G')/\Gamma$ by the construction, which is a contradiction.

Hence, there exists a loxodromic element $\omega_k \in \langle g', g'' \rangle < \Gamma = \pi_1(M, x_i')$ of word length uniformly bounded by a constant $\mathfrak{L}$ depending only on X, [12, Theorem 8.5]. Let w_k be a concatenation of w', w'' and their reverses which represents ω_k. Then the number of geodesic arcs in w_k is uniformly bounded by $5\mathfrak{L}$. The piecewise geodesic loop w_k is freely homotopic to a closed geodesic w_k^* in M; hence, w_k^* is contained in the D-neighborhood of the loop w_k where $D = \cosh^{-1}(\sqrt{2})\lceil \log_2 5\mathfrak{L} \rceil + \sinh^{-1}(2/\epsilon) + 2\delta$, [12, Proposition 5.1]. Recall that $w' \cap C_{k-1} = \varnothing$ and $w'' \cap C_{k-1} = \varnothing$, so $w_k \cap C_{k-1} = \varnothing$. Thus $d(x, w_k^*) \geq (k-1)\mathfrak{R} - D$.

Case (b): For each i, the geodesic arc α_i connects $x_i' \in \boldsymbol{\delta} C_k$ to $x_i'' \in \partial M^c_{cusp}$, as in Figure 2(b). For each x_i'', there exists a point $y_i'' \in \mathrm{cusp}_\varepsilon(M)$ such that $d(x_i'', y_i'') \leq r_1$. By an argument similar to the one in Case (a), there exists a nontrivial geodesic loop α_i'' contained in the cuspidal part which connects y_i'' to itself and has length $l(\alpha_i'') \leq m\varepsilon$. The rest of the argument is exactly the same as the argument in the torsion-free case [12, Section 9].

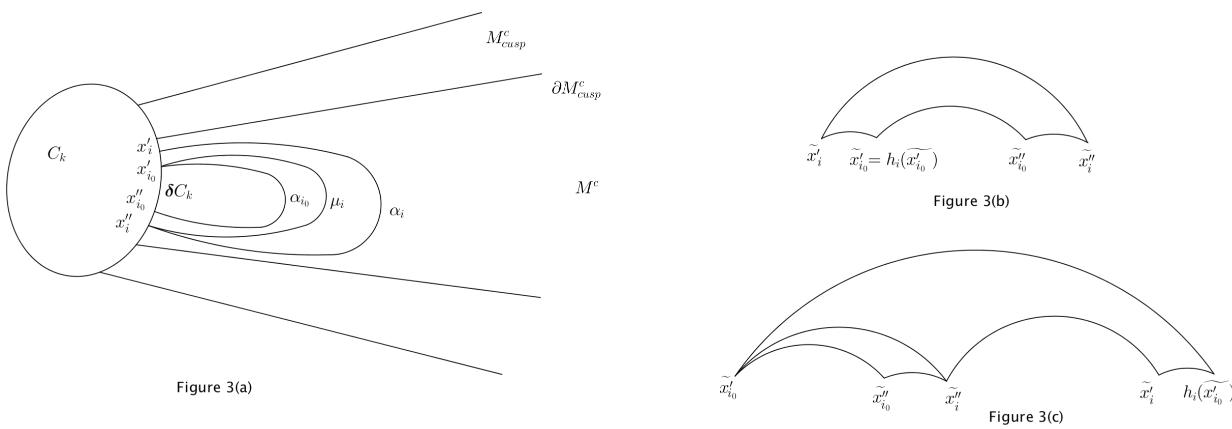

FIGURE 3.

Case (c): We assume that for each i, the geodesic arc α_i connects $x_i' \in \boldsymbol{\delta} C_k$ to $x_i'' \in \boldsymbol{\delta} C_k$. Since $\boldsymbol{\delta} C_k$ is compact, after passing to a further subsequence in (α_i), there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, $d(x_i', x_{i_0}') \leq 1$, $d(x_i'', x_{i_0}'') \leq 1$ and there are unique shortest geodesics $x_{i_0}'x_i'$ and $x_{i_0}''x_i''$. For each $i > i_0$ we define a geodesic $\mu_i = x_{i_0}'x_i''$, see Figure 3(a). Then, by δ-hyperbolicity of X, each μ_i is in the $(\delta+1)$-neighborhood of α_i. Let $v_i = \alpha_{i_0} * x_{i_0}''x_i'' * (\mu_i)^{-1} \in \Omega(M, x_{i_0}')$ for $i > i_0$. By the construction $v_i \cap C_{k-1} = \emptyset$.

Let h_i denote the element in $\Gamma = \pi_1(M, x_{i_0}')$ represented by v_i. If h_i is loxodromic for some $i > i_0$, there exists a closed geodesic w_k^* contained in the D-neighborhood of v_i, cf. Case (a). In this situation, $d(x, w_k^*) \geq (k-1)\mathfrak{R} - D$.

By an argument similar to the one in the torsion-free case, h_i cannot be the identity element of Γ for large i, see Figure 3(b) and [12, Section 9]. Assume, therefore, that h_i are not loxodromic and not the identity for all $i > i_0$. Then h_i could be either parabolic or elliptic for $i > i_0$.

Claim. *There exists a loxodromic element in* $\langle h_i, h_j\rangle$ *of uniformly bounded word length for some* $i, j > i_0$.

Proof. If there is a subsequence in $(h_i)_{i>i_0}$ consisting of parabolic elements, we can use the argument in the torsion-free case to find a loxodromic element in $\langle h_i, h_j\rangle$ of uniformly bounded word length for some $i, j > i_0$, [12]. Now assume that h_i are elliptic for all $i > i_0$.

If there exist $i, j > i_0$ such that $\langle h_i, h_j\rangle$ is nonelementary, by Theorem 4.5 (for rank 1 symmetric spaces) and Proposition 4.7 (for negatively pinched Hadamard manifolds), there exists a loxodromic element $\omega_k \in \langle h_i, h_j\rangle$ of word length uniformly bounded by a constant $\mathfrak{L}$. Now suppose that $\langle h_i, h_j\rangle$ is elementary for any pair of indices $i, j > i_0$. If one of the elementary subgroups is infinite and preserves a geodesic, by Lemma 4.2, $h_i h_j$ is loxodromic.

Assume that all the elementary subgroups $\langle h_i, h_j\rangle$ are either finite or parabolic for all $i, j > i_0$. Let B_i denote the closure of $Mar(h_i, \varepsilon)$ in $\bar{X}$. If there exist i, j such that B_i and B_j are disjoint, then $\langle h_i, h_j\rangle$ is nonelementary which contradicts our assumption. Thus for any pair of indices $i, j > i_0$, $B_i \cap B_j \neq \varnothing$. There exists a uniform constant r' such that $N_{r'}(B_i) \cap N_{r'}(B_j) \neq \varnothing$ in X. Hence there exists $\tilde{z} \in X$ such that for all $i > i_0$ we have $d(\tilde{z}, N_{r'}(B_i)) \leq n\delta$, [12, Proposition 8.2]. For any $q \in N_{r'}(B_i)$, $d(q, h_i(q)) \leq 2r' + \varepsilon$ by the triangle inequality. Thus,

$$d(\tilde{z}, h_i(\tilde{z})) \leq 2n\delta + 2r' + \varepsilon$$

for all $i > i_0$. Let $\widetilde{x}'_{i_0}$ denote a lift of x'_{i_0} in X, and $l = d(\tilde{z}, \widetilde{x}'_{i_0})$. Then

$$d(\widetilde{x}'_{i_0}, h_i(\widetilde{x}'_{i_0})) \leq 2l + 2n\delta + 2r' + \varepsilon$$

for all $i > i_0$. On the other hand, as in the argument of the torsion-free case (see [12, Section 9]), $d(\widetilde{x}'_{i_0}, h_i(\widetilde{x}'_{i_0})) \to \infty$ as $i \to \infty$, which is a contradiction.

□

Thus, for some pair of indices $i, j > i_0$, there exists a loxodromic element $\omega_k \in \langle h_i, h_j\rangle$ represented by a word length uniformly bounded by some constant $\mathfrak{L}$. By an argument similar to the one in Case (a), there exists a closed geodesic w_k^* such that $d(x, w_k^*) \geq (k-1)\mathfrak{R} - D$. The sequence of closed geodesics $\{w_k^*\}$, therefore, escapes every compact set of M. □

Corollary 5.1. *If* $\Gamma < \mathrm{Isom}(X)$ *is a discrete geometrically infinite subgroup with bounded torsion (resp. with torsion) of a negatively pinched Hadamard manifold* X *(resp. a rank 1 symmetric space* X*), then the set of nonconical limit points of* Γ *has cardinality of continuum.*

Proof. In view of the generalized Bonahon's theorem (Theorem 1.2 (1) ⇒ (2)), the proof is exactly the same as the one in the torsion-free case, see [12, Theorem 10.1].

□

The proofs of the implication (3) ⇒ (1) in Theorem 1.2 and Corollary 1.3 follow the proofs in the torsion-free case, see [12, Section 10].

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