Non-stationary non-Gaussian random vibration analysis of Duffing systems based on explicit time-domain method

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Abstract: Non-stationary non-Gaussian random vibration problems of structures are challenging and drawing increasing attention. In the present study, firstly, an explicit time-domain method (ETDM) is proposed to determine the higher-order response statistics of linear systems subjected to non-stationary non-Gaussian random excitations, in which the first four orders of cumulants of dynamic responses are directly formulated through the cumulant operation rule based on the explicit expressions of responses. Secondly, an equivalent linearization – explicit time-domain method (EL-ETDM) is further developed to solve the non-stationary non-Gaussian random vibration problems of Duffing systems, in which the equivalent linear system is derived discarding the assumption of Gaussian response, and the corresponding higher-order cumulant analyses of the linearized system are accomplished by the efficient ETDM. The present approach can account for non-Gaussian random excitations with arbitrary forms, and two specific applications to the Poisson white noise and the square form of Gaussian random process are investigated. Four numerical examples are presented to demonstrate the effectiveness of the proposed methods.

Keywords: non-Gaussian; non-stationary; random vibration; Duffing system; equivalent linearization method; explicit time-domain method

1 Introduction

The external loads exerting on engineering structures may exhibit significant non-Gaussian random characteristics, such as the earthquake load [1], wind load [2] and wave load [3], among others. In most cases, the above external loads are assumed as Gaussian random processes for the convenience of statistical description and random vibration analysis. However, such approximation may lead to an underestimation of structural peak response and an overestimation of structural fatigue life [4-5], which will pose a potential threat to the structural safety. Therefore, random vibration analysis of structures should be conducted considering the non-Gaussian nature of random excitations and it is of great necessity to develop an effective method for non-Gaussian random vibration analysis.

For random vibration analysis of linear systems under Gaussian excitations, extensive research has been done on this aspect and several analysis methods have been well developed [6-8]. By contrast, the research considering non-Gaussian random excitations relatively lags behind but also receives certain attention [9-15]. In particular, Grigoriu and Ariaratnam [10] investigated the higher-order moments and mean crossing rates of responses of linear oscillators under polynomials of stationary Gaussian processes by use of Ito's calculus. Hu [11] derived the analytical solutions to the higher-order moments and cumulants of responses of a linear oscillator excited by stationary Poisson white noise also via Ito's calculus. Settineri and Falsone [14] employed the probability transformation method to obtain the evolutionary probability density functions of responses of linear systems subjected to the square form of a non-stationary Gaussian random process. The above methodologies were only developed for non-Gaussian random excitations with specific forms. Sheng et al. [15] extended the power spectrum method (PSM) to solve the random vibration problems of linear systems considering general non-Gaussian excitations, in which the higher-order spectra of responses can be determined once the higher-order spectra of the non-Gaussian random excitations are provided. However, to evaluate the time-varying higher-order spectra of responses under non-stationary non-Gaussian random excitations, a large number of linear time-history analyses need to be conducted at different frequency intervals, which will be very time-consuming for large-scale systems.

Over the past few decades, significant research effort has been devoted to the random vibration analysis of nonlinear systems subjected to Gaussian excitations, and various nonlinear random vibration analysis methods have been developed [16-22]. In comparison, the research on non-Gaussian random vibration analysis of nonlinear systems has been limited and deserves more attention. Zeng and Zhu [23] and Zeng and Li [24] investigated the stationary responses of different kinds of nonlinear oscillators driven by Poisson white noise using the stochastic averaging method. Guo et al. [25] developed an exponential polynomial closure approximate method to analyze the non-stationary responses of Duffing oscillators excited by filtered Poisson white noise. Grigoriu [26], Sobiechowski and Socha [27] and Cai and Suzuki [28] addressed the stationary non-Gaussian random vibration problems of nonlinear oscillators via the statistical linearization technique, in which the non-Gaussian excitations are modelled by a Poisson white noise, a polynomial of a Gaussian process and an approach of nonlinear filter, respectively. It can be seen from the above literatures that the research on nonlinear random vibration under non-Gaussian excitations is mainly restricted to single-degree-of-freedom problems with specific non-Gaussian forms.

In view of the above limitations, the current study is dedicated to developing an effective method for non-stationary random vibration analysis of linear and nonlinear systems subjected to general non-Gaussian excitations. In recent years, an efficient explicit time-domain method (ETDM) [8] and a fast equivalent linearization – explicit time-domain method (EL-ETDM) [29-31] have been proposed for solving the non-stationary Gaussian random vibration problems of linear and nonlinear systems, respectively. In the present study, the ETDM is further extended for non-Gaussian random vibration analysis of linear systems, in which the first four orders of cumulants of dynamic responses are directly formulated by the cumulant operation rule based on the explicit expressions of responses. Thereafter, the EL-ETDM is further developed for non-Gaussian random vibration analysis of Duffing systems, in which the equivalent linear system is derived without introducing the traditional assumption of Gaussian response, and the numerous higher-order cumulant analyses of the linearized system involved in non-stationary problems are accomplished efficiently by ETDM. The present ETDM and EL-ETDM can be implemented given the first four orders of cumulant functions of the non-Gaussian random excitations, and the two methods are therefore applicable to arbitrary forms of non-Gaussian excitations. Four numerical examples including a linear oscillator with the stationary Poisson white noise, a 20-degree-of-freedom linear system with the square form of a non-stationary Gaussian random process, a Duffing oscillator with the square form of a non-stationary Gaussian random process and a 5-degree-of-freedom Duffing system with the non-stationary Poisson white noise are presented to validate the feasibility of the proposed methods.

2 Moment and cumulant functions of non-Gaussian random processes

Through introducing the characteristic and log-characteristic function of a non-Gaussian process, the relationships between the moment and cumulant functions of the non-Gaussian random process are established in this section. On this basis, the analytical cumulant functions of the Poisson white noise and the square form of a Gaussian random process are further presented.

2.1 Moment and cumulant functions

Suppose $X(t)$ is a non-Gaussian random process, and the *k*th-order joint probability density function of $X(t)$ can be denoted as $p_X(x_1,t_1;x_2,t_2;\cdots;x_k,t_k)$. Define the *k*th-order characteristic function of $X(t)$ as the Fourier transform of the *k*th-order probability density function, i.e.

$$\begin{aligned} M_X(\theta_1,t_1;\theta_2,t_2;\cdots;\theta_k,t_k) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} p_X(x_1,t_1;x_2,t_2;\cdots;x_k,t_k)\exp(\mathrm{i}\sum\nolimits_{j=1}^{k}\theta_j x_j)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_k \\ &= \mathrm{E}[\exp(\mathrm{i}\sum\nolimits_{j=1}^{k}\theta_j X_j)] \end{aligned} \tag{1}$$

where $X_j = X(t_j)\ (j=1,2,\cdots,k)$; $\mathrm{E}[\bullet]$ denotes the mathematical expectation; and i denotes the imaginary unit.

From Equation (1), one can derive the *k*th-order moment function of $X(t)$ as follows:

$$m_{X,k}(t_1,t_2,\cdots,t_k) = \mathrm{E}[X_1X_2\cdots X_k] = \frac{1}{\mathrm{i}^k}\left.\frac{\partial^k M_X(\theta_1,t_1;\theta_2,t_2;\cdots;\theta_k,t_k)}{\partial\theta_1\partial\theta_2\cdots\partial\theta_k}\right|_{\theta_1=\theta_2=\cdots=\theta_k=0} \tag{2}$$

Subsequently, define the *k*th-order cumulant function of $X(t)$ as

$$\chi_{X,k}(t_1,t_2,\cdots,t_k) = \mathrm{cum}[X_1,X_2,\cdots,X_k] = \frac{1}{\mathrm{i}^k}\left.\frac{\partial^k \ln M_X(\theta_1,t_1;\theta_2,t_2;\cdots;\theta_k,t_k)}{\partial\theta_1\partial\theta_2\cdots\partial\theta_k}\right|_{\theta_1=\theta_2=\cdots=\theta_k=0} \tag{3}$$

where $\mathrm{cum}[\bullet]$ is the cumulant operator; and $\ln M_X(\theta_1,t_1;\theta_2,t_2;\cdots;\theta_k,t_k)$ is termed as the *k*th-order log-characteristic function of $X(t)$.

Based on Equations (2) and (3), the relationships between the moment and cumulant functions of the non-Gaussian random process $X(t)$ can be determined [32]. For instance, the first four orders of cumulant functions can be expressed in terms of the first four orders of moment functions as follows:

$$\begin{cases} \chi_{X,1}(t_1) = m_{X,1}(t_1) \\ \chi_{X,2}(t_1,t_2) = m_{X,2}(t_1,t_2) - m_{X,1}(t_1)m_{X,1}(t_2) \\ \chi_{X,3}(t_1,t_2,t_3) = m_{X,3}(t_1,t_2,t_3) - m_{X,1}(t_1)m_{X,2}(t_2,t_3) - m_{X,1}(t_2)m_{X,2}(t_1,t_3) - m_{X,1}(t_3)m_{X,2}(t_1,t_2) \\ \qquad +2m_{X,1}(t_1)m_{X,1}(t_2)m_{X,1}(t_3) \\ \chi_{X,4}(t_1,t_2,t_3,t_4) = m_{X,4}(t_1,t_2,t_3,t_4) - m_{X,2}(t_1,t_2)m_{X,2}(t_3,t_4) - m_{X,2}(t_1,t_3)m_{X,2}(t_2,t_4) - m_{X,2}(t_1,t_4)m_{X,2}(t_2,t_3) \\ \qquad -m_{X,1}(t_1)m_{X,3}(t_2,t_3,t_4) - m_{X,1}(t_2)m_{X,3}(t_1,t_3,t_4) - m_{X,1}(t_3)m_{X,3}(t_1,t_2,t_4) - m_{X,1}(t_4)m_{X,3}(t_1,t_2,t_3) \\ \qquad +2\begin{bmatrix} m_{X,1}(t_1)m_{X,1}(t_2)m_{X,2}(t_3,t_4) + m_{X,1}(t_1)m_{X,1}(t_3)m_{X,2}(t_2,t_4) + m_{X,1}(t_1)m_{X,1}(t_4)m_{X,2}(t_2,t_3) \\ +m_{X,1}(t_2)m_{X,1}(t_3)m_{X,2}(t_1,t_4) + m_{X,1}(t_2)m_{X,1}(t_4)m_{X,2}(t_1,t_3) + m_{X,1}(t_3)m_{X,1}(t_4)m_{X,2}(t_1,t_2) \end{bmatrix} \\ \qquad -6m_{X,1}(t_1)m_{X,1}(t_2)m_{X,1}(t_3)m_{X,1}(t_4) \end{cases} \tag{4}$$

Conversely, the first four orders of moment functions can also be written in terms of the first four orders of cumulant functions as follows:

$$\begin{cases} m_{X,1}(t_1) = \chi_{X,1}(t_1) \\ m_{X,2}(t_1,t_2) = \chi_{X,2}(t_1,t_2) + \chi_{X,1}(t_1)\chi_{X,1}(t_2) \\ m_{X,3}(t_1,t_2,t_3) = \chi_{X,3}(t_1,t_2,t_3) + \chi_{X,1}(t_1)\chi_{X,2}(t_2,t_3) + \chi_{X,1}(t_2)\chi_{X,2}(t_1,t_3) + \chi_{X,1}(t_3)\chi_{X,2}(t_1,t_2) \\ \qquad +\chi_{X,1}(t_1)\chi_{X,1}(t_2)\chi_{X,1}(t_3) \\ m_{X,4}(t_1,t_2,t_3,t_4) = \chi_{X,4}(t_1,t_2,t_3,t_4) + \chi_{X,2}(t_1,t_2)\chi_{X,2}(t_3,t_4) + \chi_{X,2}(t_1,t_3)\chi_{X,2}(t_2,t_4) + \chi_{X,2}(t_1,t_4)\chi_{X,2}(t_2,t_3) \\ \qquad +\chi_{X,1}(t_1)\chi_{X,3}(t_2,t_3,t_4) + \chi_{X,1}(t_2)\chi_{X,3}(t_1,t_3,t_4) + \chi_{X,1}(t_3)\chi_{X,3}(t_1,t_2,t_4) + \chi_{X,1}(t_4)\chi_{X,3}(t_1,t_2,t_3) \\ \qquad +\begin{bmatrix} \chi_{X,1}(t_1)\chi_{X,1}(t_2)\chi_{X,2}(t_3,t_4) + \chi_{X,1}(t_1)\chi_{X,1}(t_3)\chi_{X,2}(t_2,t_4) + \chi_{X,1}(t_1)\chi_{X,1}(t_4)\chi_{X,2}(t_2,t_3) \\ +\chi_{X,1}(t_2)\chi_{X,1}(t_3)\chi_{X,2}(t_1,t_4) + \chi_{X,1}(t_2)\chi_{X,1}(t_4)\chi_{X,2}(t_1,t_3) + \chi_{X,1}(t_3)\chi_{X,1}(t_4)\chi_{X,2}(t_1,t_2) \end{bmatrix} \\ \qquad +\chi_{X,1}(t_1)\chi_{X,1}(t_2)\chi_{X,1}(t_3)\chi_{X,1}(t_4) \end{cases} \tag{5}$$

Setting $t_1 = t_2 = t_3 = t_4 = t$, Equations (4) and (5) become

$$\begin{cases} \chi_{X,1}(t) = m_{X,1}(t), \quad \chi_{X,2}(t) = m_{X,2}(t) - m_{X,1}^2(t), \quad \chi_{X,3}(t) = m_{X,3}(t) - 3m_{X,1}(t)m_{X,2}(t) + 2m_{X,1}^3(t) \\ \chi_{X,4}(t) = m_{X,4}(t) - 3m_{X,2}^2(t) - 4m_{X,1}(t)m_{X,3}(t) + 12m_{X,1}^2(t)m_{X,2}(t) - 6m_{X,1}^4(t) \end{cases} \tag{6}$$

and

$$\begin{cases} m_{X,1}(t) = \chi_{X,1}(t), \quad m_{X,2}(t) = \chi_{X,2}(t) + \chi_{X,1}^2(t), \quad m_{X,3}(t) = \chi_{X,3}(t) + 3\chi_{X,1}(t)\chi_{X,2}(t) + \chi_{X,1}^3(t) \\ m_{X,4}(t) = \chi_{X,4}(t) + 3\chi_{X,2}^2(t) + 4\chi_{X,1}(t)\chi_{X,3}(t) + 6\chi_{X,1}^2(t)\chi_{X,2}(t) + \chi_{X,1}^4(t) \end{cases} \tag{7}$$

respectively.

It can be seen from Equations (4)-(7) that the moment and cumulant functions of a non-Gaussian random process are interconvertible, and both of them can be used for time-domain statistical description of the random process. The cumulant functions are typically preferred for a non-Gaussian white noise since the second-order cumulant function and higher-order cumulant functions of the noise are in the form of impulse function, and the corresponding power spectrum and higher-order spectra via Fourier transform are flat [32]. Therefore, in the context of non-Gaussian random vibration, it is generally recommended that the cumulant functions be employed for description of the time-domain statistics of the non-Gaussian random excitation.

It is noteworthy that the research on analytical models of cumulant functions of non-Gaussian random processes is limited, and to the best knowledge of the authors, only the cumulant functions of certain types of non-Gaussian random processes can be analytically derived, e.g., the non-Gaussian white noise and the polynomial form of a Gaussian random process. In what follows, the cumulant functions of the Poisson white noise, a special type of non-Gaussian white noise, and the square form of a Gaussian random process will be given.

2.2 Poisson white noise

Suppose $X(t)$ is a non-stationary Poisson white noise and can be expressed as $X(t) = g(t)\hat{X}(t)$, in which $g(t)$ is the modulation function and $\hat{X}(t)$ is a stationary Poisson white noise defined as [33]

$$\hat{X}(t) = \begin{cases} 0 & N(t) = 0 \\ \sum_{j=1}^{N(t)} Z_j \delta(t - t_j) & N(t) > 0 \end{cases} \tag{8}$$

where $\delta(\cdot)$ is the Dirac function; $N(t)$ is a time homogeneous Poisson counting process with a mean arrival rate of λ; t_j $(j = 1, 2, \cdots, N(t))$ are the arrival time instants of the random impulses; and Z_j $(j = 1, 2, \cdots, N(t))$ are the amplitudes of the random impulses, which are independent and identically distributed random variables.

The *k*th-order cumulant function of the stationary Poisson white noise $\hat{X}(t)$ can be expressed as [33]

$$\chi_{\hat{X},k}(t_1, t_2, \cdots, t_k) = \lambda \mathrm{E}[Z^k]\delta(t_2 - t_1)\delta(t_3 - t_1)\cdots\delta(t_k - t_1) \tag{9}$$

where $\mathrm{E}[Z^k]$ is the kth-order moment of an arbitrary Z_j. Correspondingly, the kth-order cumulant function of the non-stationary Poisson white noise $X(t)$ can be expressed as

$$\chi_{X,k}(t_1,t_2,\cdots,t_k)=g(t_1)g(t_2)\cdots g(t_k)\chi_{\hat{X},k}(t_1,t_2,\cdots,t_k) \tag{10}$$

It can be seen from Equations (9) and (10) that the cumulant functions of a Poisson white noise are equal to zeros as long as $t_1=t_2=\cdots=t_k$ is not satisfied. Such delta-correlated property can significantly simplify the analysis of non-Gaussian random vibration problems, which will be demonstrated in Sections 3 and 4.

2.3 Square form of Gaussian random process

For fluid-structure interaction problems, the fluid-induced forces can be expressed in terms of the square form of the fluid velocities. Therefore, even though the fluid velocities, e.g., the wind velocity and the wave-particle velocity, can be modeled as Gaussian processes, the fluid-induced forces, e.g., the aerodynamic force and the hydrodynamic force, should be considered as non-Gaussian processes. To investigate this aspect, now suppose $X(t)$ is of the square form of a non-stationary Gaussian random process $Y(t)$, i.e., $X(t)=Y^2(t)$. Then, based on the Gaussian closure technique [16], the moment functions of $X(t)$ can be formulated only in terms of the second-order moment function of $Y(t)$. For instance, the first four orders of moment functions of $X(t)$ can be expressed as

$$\begin{cases}
m_{X,1}(t_1)=m_{Y,2}(t_1)\\
m_{X,2}(t_1,t_2)=m_{Y,2}(t_1)m_{Y,2}(t_2)+2m_{Y,2}^2(t_1,t_2)\\
m_{X,3}(t_1,t_2,t_3)=m_{Y,2}(t_1)m_{Y,2}(t_2)m_{Y,2}(t_3)+2m_{Y,2}(t_1)m_{Y,2}^2(t_2,t_3)+2m_{Y,2}(t_2)m_{Y,2}^2(t_1,t_3)+2m_{Y,2}(t_3)m_{Y,2}^2(t_1,t_2)\\
\qquad +8m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_2,t_3)\\
m_{X,4}(t_1,t_2,t_3,t_4)=m_{Y,2}(t_1)m_{Y,2}(t_2)m_{Y,2}(t_3)m_{Y,2}(t_4)\\
\qquad +2m_{Y,2}(t_1)m_{Y,2}(t_2)m_{Y,2}^2(t_3,t_4)+2m_{Y,2}(t_1)m_{Y,2}(t_3)m_{Y,2}^2(t_2,t_4)+2m_{Y,2}(t_1)m_{Y,2}(t_4)m_{Y,2}^2(t_2,t_3)\\
\qquad +2m_{Y,2}(t_2)m_{Y,2}(t_3)m_{Y,2}^2(t_1,t_4)+2m_{Y,2}(t_2)m_{Y,2}(t_4)m_{Y,2}^2(t_1,t_3)+2m_{Y,2}(t_3)m_{Y,2}(t_4)m_{Y,2}^2(t_1,t_2)\\
\qquad +8m_{Y,2}(t_1)m_{Y,2}(t_2,t_3)m_{Y,2}(t_2,t_4)m_{Y,2}(t_3,t_4)+8m_{Y,2}(t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_1,t_4)m_{Y,2}(t_3,t_4)\\
\qquad +8m_{Y,2}(t_3)m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_4)m_{Y,2}(t_2,t_4)+8m_{Y,2}(t_4)m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_2,t_3)\\
\qquad +4m_{Y,2}^2(t_1,t_2)m_{Y,2}^2(t_3,t_4)+4m_{Y,2}^2(t_1,t_3)m_{Y,2}^2(t_2,t_4)+4m_{Y,2}^2(t_1,t_4)m_{Y,2}^2(t_2,t_3)\\
\qquad +16m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_2,t_4)m_{Y,2}(t_3,t_4)+16m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_4)m_{Y,2}(t_2,t_3)m_{Y,2}(t_3,t_4)\\
\qquad +16m_{Y,2}(t_1,t_3)m_{Y,2}(t_1,t_4)m_{Y,2}(t_2,t_3)m_{Y,2}(t_2,t_4)
\end{cases} \tag{11}$$

Substitution of Equation (11) into Equation (4) yields the first four orders of cumulant functions of $X(t)$ as follows:

$$\begin{cases} \chi_{X,1}(t_1) = m_{Y,2}(t_1) \\ \chi_{X,2}(t_1,t_2) = 2m_{Y,2}^2(t_1,t_2) \\ \chi_{X,3}(t_1,t_2,t_3) = 8m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_2,t_3) \\ \chi_{X,4}(t_1,t_2,t_3,t_4) = 16m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_3)m_{Y,2}(t_2,t_4)m_{Y,2}(t_3,t_4) + 16m_{Y,2}(t_1,t_2)m_{Y,2}(t_1,t_4)m_{Y,2}(t_2,t_3)m_{Y,2}(t_3,t_4) \\ \qquad +16m_{Y,2}(t_1,t_3)m_{Y,2}(t_1,t_4)m_{Y,2}(t_2,t_3)m_{Y,2}(t_2,t_4) \end{cases} \tag{12}$$

It is implied from Equation (12) that, once the analytical model of the second-order moment function of the Gaussian process $Y(t)$ is known, the first four orders of cumulant functions of the non-Gaussian process $X(t)$ can then be analytically derived, which will greatly reduce the storage space required for the first four orders of cumulant functions of $X(t)$.

It should be noted that, given the covariance function of the non-Gaussian random process and the corresponding non-Gaussian marginal distribution, the covariance function of the underlying Gaussian random process needs to be determined based on the translation process theory. In this case, certain compatibility conditions of the non-Gaussian random process need to be satisfied to ensure that the covariance function of the underlying Gaussian random process is obtainable [34-35], from which the non-Gaussian random process can be generated in a more general sense.

3 Non-stationary non-Gaussian random vibration analysis of linear systems by ETDM

In this section, the ETDM [8] originally proposed for non-stationary random vibration analysis of linear systems under Gaussian excitations is extended to the case considering non-Gaussian excitations. The explicit expressions of dynamic responses are first derived for the linear system based on the Newmark kinematic assumptions, and the first four orders of cumulants of an arbitrary critical response are then explicitly formulated through the cumulant operation rule with non-Gaussian random excitations.

3.1 Explicit expressions of dynamic responses

The equation of motion for a linear system can be expressed as

$$\mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{C}\dot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t)=\mathbf{L}X(t) \tag{13}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrix of the system, respectively; $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$ are the displacement, velocity and acceleration vector of the system, respectively; $X(t)$ is the external excitation assumed to be a non-stationary non-Gaussian random process; and $\mathbf{L}$ is the orientation vector of the external excitation.

Equation (13) can be recast in the form of state equation as follows:

$$\dot{\mathbf{V}}(t)=\mathbf{H}\mathbf{V}(t)+\mathbf{W}X(t) \tag{14}$$

where $\mathbf{V}(t)=[\mathbf{U}^{\mathrm{T}}(t)\ \dot{\mathbf{U}}^{\mathrm{T}}(t)]^{\mathrm{T}}$ is the state vector of the system; and $\mathbf{H}$ and $\mathbf{W}$ are expressed as

$$\mathbf{H}=\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C}\end{bmatrix},\quad \mathbf{W}=\begin{bmatrix}\mathbf{0}\\ \mathbf{M}^{-1}\mathbf{L}\end{bmatrix} \tag{15}$$

in which $\mathbf{0}$ and $\mathbf{I}$ are the zero matrix and the unit matrix, respectively.

Suppose the system is initially at rest. Then, solving Equation (14) by the use of Newmark kinematic assumptions [36], one can derive the explicit expression of the state vector as

$$\mathbf{V}_i=\mathbf{A}_{i,1}X_1+\mathbf{A}_{i,2}X_2+\cdots+\mathbf{A}_{i,i-1}X_{i-1}+\mathbf{A}_{i,i}X_i \quad (i=1,2,\cdots,n) \tag{16}$$

where n is the number of time steps for time-history analysis; $\mathbf{V}_i=\mathbf{V}(t_i)$ with $t_i=i\Delta t$ and Δt being the time step; $X_j=X(t_j)$ with $t_j=j\Delta t\ (j=1,2,\cdots,i)$; and $\mathbf{A}_{i,j}\ (j=1,2,\cdots,i)$ are the coefficient vectors with regard to the state vector $\mathbf{V}_i$, which can be expressed in closed form as

$$\begin{cases} \mathbf{A}_{1,1} = \mathbf{Q}_2, \ \mathbf{A}_{2,1} = \mathbf{T}\mathbf{Q}_2 + \mathbf{Q}_1, \ \mathbf{A}_{i,1} = \mathbf{T}\mathbf{A}_{i-1,1} \ (3 \le i \le n) \\ \mathbf{A}_{i,j} = \mathbf{A}_{i-1,j-1} \ (2 \le j \le i \le n) \end{cases} \tag{17}$$

where $\mathbf{T}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are expressed as [37]

$$\begin{cases} \mathbf{T} = -(\mathbf{H}-\mathbf{R}_1)^{-1}(\mathbf{R}_1+\mathbf{R}_2\mathbf{H}), \ \mathbf{Q}_1 = -(\mathbf{H}-\mathbf{R}_1)^{-1}\mathbf{R}_2\mathbf{W}, \ \mathbf{Q}_2 = -(\mathbf{H}-\mathbf{R}_1)^{-1}\mathbf{W} \\ \mathbf{R}_1 = \begin{bmatrix} a_3\mathbf{I} & \mathbf{0} \\ a_0\mathbf{I} & \mathbf{0} \end{bmatrix}, \ \mathbf{R}_2 = \begin{bmatrix} a_4\mathbf{I} & a_5\mathbf{I} \\ a_1\mathbf{I} & a_2\mathbf{I} \end{bmatrix} \\ a_0 = \dfrac{1}{\beta\Delta t^2}, \ a_1 = \dfrac{1}{\beta\Delta t}, \ a_2 = \dfrac{1}{2\beta}-1, \ a_3 = \dfrac{\gamma}{\beta\Delta t}, \ a_4 = \dfrac{\gamma}{\beta}-1, \ a_5 = \dfrac{\Delta t}{2}(\dfrac{\gamma}{\beta}-2) \end{cases} \tag{18}$$

where $\gamma = 0.5$ and $\beta = 0.25$ are adopted for unconditionally stable integration scheme.

It can be observed from Equation (17) that only the coefficient vectors $\mathbf{A}_{i,1}$ $(i = 1,2,\cdots,n)$ shown in the first row need to be calculated and stored, and based on the recursive relation shown in the second row, the other coefficient vectors can be directly obtained from $\mathbf{A}_{i,1}$ $(i = 1,2,\cdots,n)$. From the physical point of view, when the external excitation $X(t)$ takes the form of an impulse excitation applied at time t_1, as shown in Figure 1, one can easily obtain $\mathbf{V}_i = \mathbf{A}_{i,1}$ from Equation (16). This indicates that the coefficient vector $\mathbf{A}_{i,1}$ actually represents the state vector at time t_i induced by the aforementioned impulse excitation. Therefore, the computational cost for the coefficient vectors $\mathbf{A}_{i,1}$ $(i = 1,2,\cdots,n)$ is equal to that required by only one time-history analysis of the linear system.

Figure 1 The impulse excitation $X(t)$

3.2 Higher-order cumulant analysis

As the explicit expression of the state vector has been established in Equation (16), one can focus on any structural responses of interest for higher-order cumulant analysis, which implies that dimension-reduced statistical analysis can now be easily conducted. Suppose

$r = r(t)$ is an arbitrary critical response of the system. Then, from Equation (16), the explicit expression of the critical response $r_i = r(t_i)$ can be obtained as

$$r_i = \mathbf{q}^r \mathbf{V}_i = a_{i,1}^r X_1 + a_{i,2}^r X_2 + \cdots + a_{i,i-1}^r X_{i-1} + a_{i,i}^r X_i \quad (i = 1, 2, \cdots, n) \tag{19}$$

where $\mathbf{q}^r$ is the response transfer row vector for the critical response r; and $a_{i,j}^r = \mathbf{q}^r \mathbf{A}_{i,j}$ $(j = 1, 2, \cdots, i)$ are the coefficients with regard to r_i.

Suppose the first four orders of cumulants are of interest for description of the statistical characteristics of a non-Gaussian response. Based on the cumulant operation rule [32], the first four orders of cumulants of the critical response r_i can be directly formulated from Equation (19) as follows:

$$\begin{cases} \chi_{r,1}(t_i) = \mathrm{cum}(r_i) = \sum_{j=1}^{i} a_{i,j}^r \mathrm{cum}(X_j) \\ \chi_{r,2}(t_i) = \mathrm{cum}(r_i, r_i) = \sum_{j=1}^{i}\sum_{m=1}^{i} a_{i,j}^r a_{i,m}^r \mathrm{cum}(X_j, X_m) \\ \chi_{r,3}(t_i) = \mathrm{cum}(r_i, r_i, r_i) = \sum_{j=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i} a_{i,j}^r a_{i,m}^r a_{i,p}^r \mathrm{cum}(X_j, X_m, X_p) \\ \chi_{r,4}(t_i) = \mathrm{cum}(r_i, r_i, r_i, r_i) = \sum_{j=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i}\sum_{q=1}^{i} a_{i,j}^r a_{i,m}^r a_{i,p}^r a_{i,q}^r \mathrm{cum}(X_j, X_m, X_p, X_q) \end{cases} \quad (i = 1, 2, \cdots, n) \tag{20}$$

where $\mathrm{cum}(X_j)$, $\mathrm{cum}(X_j, X_m)$, $\mathrm{cum}(X_j, X_m, X_p)$ and $\mathrm{cum}(X_j, X_m, X_p, X_q)$ $(j, m, p, q = 1, 2, \cdots, i)$ can be determined from the first four orders of cumulant functions of the non-Gaussian random excitation $X(t)$, i.e., $\chi_{X,1}(t_1)$, $\chi_{X,2}(t_1, t_2)$, $\chi_{X,3}(t_1, t_2, t_3)$ and $\chi_{X,4}(t_1, t_2, t_3, t_4)$, respectively.

Thus far, the explicit formulation of the first four orders of cumulants of the response r_i has been achieved. Given the first four orders of cumulant functions of the non-Gaussian random excitation $X(t)$, the first four orders of cumulants of the response r_i can be directly calculated using Equation (20). In this sense, the present approach is applicable to arbitrary forms of non-Gaussian random excitations. Moreover, if the first four orders of response moments are desired, they can be easily determined by Equation (7) based on the first four orders of response cumulants obtained by Equation (20). Note that, as the explicit expression of response shown in Equation (19) holds for different time instants, one can also compute the cross cumulants/moments of responses with respect to different time instants using the cumulant/moment operation rule, which will certainly require more computational cost where necessary.

In particular, suppose $X(t)$ is a non-stationary Poisson white noise. Then, based on the delta-correlated property shown in Equations (9) and (10), Equation (20) can be simplified as

$$\begin{cases} \chi_{r,1}(t_i) = \mathrm{cum}(r_i) = \sum_{j=1}^{i} a_{i,j}^r \mathrm{cum}(X_j) \\ \chi_{r,2}(t_i) = \mathrm{cum}(r_i, r_i) = \sum_{j=1}^{i} (a_{i,j}^r)^2 \mathrm{cum}(X_j, X_j) \\ \chi_{r,3}(t_i) = \mathrm{cum}(r_i, r_i, r_i) = \sum_{j=1}^{i} (a_{i,j}^r)^3 \mathrm{cum}(X_j, X_j, X_j) \\ \chi_{r,4}(t_i) = \mathrm{cum}(r_i, r_i, r_i, r_i) = \sum_{j=1}^{i} (a_{i,j}^r)^4 \mathrm{cum}(X_j, X_j, X_j, X_j) \end{cases} \quad (i = 1, 2, \cdots, n) \qquad (21)$$

where $\mathrm{cum}(X_j)$, $\mathrm{cum}(X_j, X_j)$, $\mathrm{cum}(X_j, X_j, X_j)$ and $\mathrm{cum}(X_j, X_j, X_j, X_j)$ $(j = 1, 2, \cdots, i)$ can be determined from $\chi_{X,1}(t)$, $\chi_{X,2}(t)$, $\chi_{X,3}(t)$ and $\chi_{X,4}(t)$, respectively, which are presented in Equation (10).

Suppose $X(t)$ is of the square form of a non-stationary Gaussian random process $Y(t)$, i.e., $X(t) = Y^2(t)$. Then, based on Equation (12), Equation (20) can be further derived as

$$\begin{cases} \chi_{r,1}(t_i) = \mathrm{cum}(r_i) = \sum_{j=1}^{i} a_{i,j}^r \mathrm{E}(Y_j^2) \\ \chi_{r,2}(t_i) = \mathrm{cum}(r_i, r_i) = 2\sum_{j=1}^{i}\sum_{m=1}^{i} a_{i,j}^r a_{i,m}^r \mathrm{E}(Y_j Y_m)^2 \\ \chi_{r,3}(t_i) = \mathrm{cum}(r_i, r_i, r_i) = 8\sum_{j=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i} a_{i,j}^r a_{i,m}^r a_{i,p}^r \mathrm{E}(Y_j Y_m)\mathrm{E}(Y_j Y_p)\mathrm{E}(Y_m Y_p) \\ \chi_{r,4}(t_i) = \mathrm{cum}(r_i, r_i, r_i, r_i) = 48\sum_{j=1}^{i}\sum_{m=1}^{i}\sum_{p=1}^{i}\sum_{q=1}^{i} a_{i,j}^r a_{i,m}^r a_{i,p}^r a_{i,q}^r \mathrm{E}(Y_j Y_m)\mathrm{E}(Y_j Y_p)\mathrm{E}(Y_m Y_q)\mathrm{E}(Y_p Y_q) \end{cases} \quad (i = 1, 2, \cdots, n) \quad (22)$$

where the second-order moments involved can be completely determined from the second-order moment function of the non-stationary Gaussian random process $Y(t)$.

It is worth noting that, in general, the response cumulants are slowly varying functions, and one can calculate these response statistics at a larger time interval, i.e., $\Delta\tau = N\Delta t$ with N being the times of Δt, by focusing on the specific time instants using the explicit formulations shown in Equations (20)-(22), which can further enhance the computational efficiency of ETDM for higher-order cumulant analysis.

From the above formulation of ETDM, it can be seen that there exist two advantages of the present approach over PSM. First, only the higher-order cumulant functions of excitations are required in ETDM, which are generally more easily obtained from the time-domain records of excitations, and in contrast, it is still a tough task to establish the evolutionary higher-order spectra of non-stationary non-Gaussian random excitations required by PSM. Second, as can be seen from the discussion on the physical meanings of the coefficient vectors shown in Equation (16), only one single impulse response time-history analysis of the system is involved in ETDM for constructing the explicit expressions of dynamic responses, while for PSM, a large number of time-history analyses need to be conducted at different frequency intervals for obtaining the evolutionary power spectra, bi-spectra and tri-spectra of non-stationary non-Gaussian responses, leading to much more computational time than ETDM.

4 Non-stationary non-Gaussian random vibration analysis of Duffing systems by EL-ETDM

In this section, the EL-ETDM [29-31] originally developed for non-stationary random vibration analysis of nonlinear systems subjected to Gaussian excitations is extended to the case considering non-Gaussian excitations for Duffing systems by combining the equivalent linearization (EL) method and the ETDM. Owing to the influence of non-Gaussian excitations, the traditional assumption of Gaussian response can no longer be adopted in the EL method, and a series of higher-order cumulant analyses of the linearized system need to be conducted, which can be accomplished by the efficient ETDM presented in Section 3.

4.1 Equivalent linear system

For a Duffing system with hardening springs, the nonlinear equation of motion can be expressed as

$$\mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{C}\dot{\mathbf{U}}(t)+\mathbf{K}\mathbf{U}(t)+\mathbf{F}_{\mathrm{NE}}(t)=\mathbf{L}X(t) \tag{23}$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and linear elastic stiffness matrix of the Duffing system, respectively; $\mathbf{U}(t)$, $\dot{\mathbf{U}}(t)$ and $\ddot{\mathbf{U}}(t)$ are the displacement, velocity and acceleration vector of the Duffing system, respectively; $X(t)$ is the non-stationary non-Gaussian random excitation and $\mathbf{L}$ is the corresponding orientation vector; and $\mathbf{F}_{\mathrm{NE}}(t)$ is the nonlinear elastic force vector of the Duffing system, which can be expressed as

$$\mathbf{F}_{\mathrm{NE}}(t)=\mathbf{E}_1 f_{\mathrm{NE},1}(t)+\mathbf{E}_2 f_{\mathrm{NE},2}(t)+\cdots+\mathbf{E}_{n_\mathrm{s}} f_{\mathrm{NE},n_\mathrm{s}}(t) \tag{24}$$

$$f_{\mathrm{NE},i}(t)=\eta_i k_i d_i^3(t) \quad (i=1,2,\cdots,n_\mathrm{s}) \tag{25}$$

where n_s is the number of hardening springs; $f_{\mathrm{NE},i}(t)$ $(i=1,2,\cdots,n_\mathrm{s})$ is the nonlinear elastic force of the *i*th hardening spring and $\mathbf{E}_i$ is the corresponding orientation vector; k_i and η_i are the linear elastic stiffness and the coefficient reflecting the nonlinearity of the *i*th hardening spring, respectively; and $d_i(t)$ is the nodal relative displacement of the *i*th hardening spring.

For a specific time instant τ, Equation (25) can be replaced by the following equivalent linear equation as

$$f_{\mathrm{NE},i}(t)=k_{\mathrm{e},i}(\tau)d_i(t) \quad (0\le t\le\tau;\ i=1,2,\cdots,n_\mathrm{s}) \tag{26}$$

where $k_{\mathrm{e},i}(\tau)$ is the equivalent stiffness of the *i*th hardening spring, which can be determined by minimizing the mean square of the difference between Equations (25) and (26) at time instant τ and can be expressed as

$$k_{\mathrm{e},i}(\tau)=\eta_i k_i \frac{\mathrm{E}[d_i^4(\tau)]}{\mathrm{E}[d_i^2(\tau)]} \quad (i=1,2,\cdots,n_{\mathrm{s}}) \tag{27}$$

The nodal relative displacement of the ith hardening spring can be written in terms of the nodal displacement vector of the Duffing system as follows:

$$d_i(t)=\mathbf{E}_i^{\mathrm{T}}\mathbf{U}(t) \quad (i=1,2,\cdots,n_{\mathrm{s}}) \tag{28}$$

where $\mathbf{E}_i$ is the orientation vector for $f_{\mathrm{NE},i}(t)$ $(i=1,2,\cdots,n_{\mathrm{s}})$ shown in Equation (24).

Substitution of Equations (24), (26) and (28) into Equation (23) yields the equation of motion for the equivalent linear system of the Duffing system corresponding to the time instant τ as follows:

$$\mathbf{M}\ddot{\mathbf{U}}(t)+\mathbf{C}\dot{\mathbf{U}}(t)+[\mathbf{K}+\mathbf{K}_{\mathrm{e}}(\tau)]\mathbf{U}(t)=\mathbf{L}X(t) \quad (0\le t\le \tau) \tag{29}$$

where $\mathbf{K}_{\mathrm{e}}(\tau)$ is the equivalent stiffness matrix expressed as

$$\mathbf{K}_{\mathrm{e}}(\tau)=\mathbf{E}_1 k_{\mathrm{e},1}(\tau)\mathbf{E}_1^{\mathrm{T}}+\mathbf{E}_2 k_{\mathrm{e},2}(\tau)\mathbf{E}_2^{\mathrm{T}}+\cdots+\mathbf{E}_{n_{\mathrm{s}}} k_{\mathrm{e},n_{\mathrm{s}}}(\tau)\mathbf{E}_{n_{\mathrm{s}}}^{\mathrm{T}} \tag{30}$$

It can be seen from Equations (27) and (30) that, for the specific time instant τ, the equivalent stiffness matrix $\mathbf{K}_{\mathrm{e}}(\tau)$ depends on the second-order and fourth-order moments of responses, i.e., $\mathrm{E}[d_i^2(\tau)]$ and $\mathrm{E}[d_i^4(\tau)]$ $(i=1,2,\cdots,n_{\mathrm{s}})$, which, in turn, need to be determined via the higher-order cumulant analysis of the linearized system shown in Equation (29). Therefore, an iterative process involving a series of non-stationary non-Gaussian linear random vibration analyses is required for obtaining the equivalent linear system and the corresponding response statistics, which will be elaborated in Section 4.2.

It should be noted that, for the case of Gaussian excitations, the Gaussian assumption of $d_i(\tau)$ is acceptable provided that the Duffing system is not heavily nonlinear, and thus Equation (27) can be reduced to $k_{\mathrm{e},i}(\tau)=3\eta_i k_i \mathrm{E}[d_i^2(\tau)]$ $(i=1,2,\cdots,n_{\mathrm{s}})$ [29], in which only the second-order moment of $d_i(\tau)$ is required. However, under non-Gaussian excitations, the response of the Duffing system is no doubt non-Gaussian regardless of the degree of system nonlinearity. Therefore, for non-Gaussian random vibration analysis of nonlinear systems within the framework of the EL method, the assumption of Gaussian response is by no means feasible and should be abandoned, leading to the requirement of higher-order moment analysis for determination of the equivalent stiffness.

4.2 Higher-order cumulant analysis of the linearized system

For the linearized system shown in Equation (29), the explicit expression of the state vector at the specific time instant τ can be obtained from Equation (16) as

$$\mathbf{V}(\tau)=\mathbf{V}_n=\mathbf{A}_{n,1}X_1+\mathbf{A}_{n,2}X_2+\cdots+\mathbf{A}_{n,n-1}X_{n-1}+\mathbf{A}_{n,n}X_n \tag{31}$$

where $n=\tau/\Delta t$ with Δt being the time step; $X_j=X(t_j)$ with $t_j=j\Delta t\ (j=1,2,\cdots,n)$; and $\mathbf{A}_{n,j}\ (j=1,2,\cdots,n)$ are the coefficient vectors with regard to the state vector $\mathbf{V}_n$, which, from Equation (17), can be expressed in closed form as

$$\mathbf{A}_{n,j}=\mathbf{T}\mathbf{A}_{n,j+1}\ (j=1,2,\cdots,n-2),\ \ \mathbf{A}_{n,n-1}=\mathbf{T}\mathbf{Q}_2+\mathbf{Q}_1,\ \ \mathbf{A}_{n,n}=\mathbf{Q}_2 \tag{32}$$

where $\mathbf{T}$, $\mathbf{Q}_1$ and $\mathbf{Q}_2$ can be determined using Equations (15) and (18) with $\mathbf{K}$ being replaced by $\mathbf{K}+\mathbf{K}_{\mathrm{e}}(\tau)$. Note that, to determine $\mathbf{K}_{\mathrm{e}}(\tau)$ using Equations (27) and (30), the initial values of $\mathrm{E}[d_i^2(\tau)]$ and $\mathrm{E}[d_i^4(\tau)]\ (i=1,2,\cdots,n_{\mathrm{s}})$ can be taken as the convergent results at the previous time instant.

From Equation (31), the explicit expression of the nodal relative displacement of the *i*th hardening spring, $d_i(\tau)$, can be readily obtained as

$$d_i(\tau)=\mathbf{q}^{d_i}\mathbf{V}(\tau)=a_{n,1}^{d_i}X_1+a_{n,2}^{d_i}X_2+\cdots+a_{n,n-1}^{d_i}X_{n-1}+a_{n,n}^{d_i}X_n\quad (i=1,2,\cdots,n_{\mathrm{s}}) \tag{33}$$

where $a_{n,j}^{d_i}=\mathbf{q}^{d_i}\mathbf{A}_{n,j}\ (j=1,2,\cdots,n)$ are the coefficients with regard to $d_i(\tau)$; and $\mathbf{q}^{d_i}$ is the corresponding response transfer row vector.

Similar to Equation (20), the first four orders of cumulants of $d_i(\tau)$ can be directly formulated from Equation (33) as follows:

$$\begin{cases} \text{cum}[d_i(\tau)] = \sum_{j=1}^{n} a_{n,j}^{d_i} \text{cum}(X_j) \\ \text{cum}[d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} \sum_{m=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} \text{cum}(X_j, X_m) \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} a_{n,p}^{d_i} \text{cum}(X_j, X_m, X_p) \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} a_{n,p}^{d_i} a_{n,q}^{d_i} \text{cum}(X_j, X_m, X_p, X_q) \end{cases} \quad (i = 1, 2, \cdots, n_\text{s}) \quad (34)$$

In particular, when $X(t)$ is a non-stationary Poisson white noise, similar to Equation (21), Equation (34) can be reduced to

$$\begin{cases} \text{cum}[d_i(\tau)] = \sum_{j=1}^{n} a_{n,j}^{d_i} \text{cum}(X_j) \\ \text{cum}[d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} (a_{n,j}^{d_i})^2 \text{cum}(X_j, X_j) \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} (a_{n,j}^{d_i})^3 \text{cum}(X_j, X_j, X_j) \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau), d_i(\tau)] = \sum_{j=1}^{n} (a_{n,j}^{d_i})^4 \text{cum}(X_j, X_j, X_j, X_j) \end{cases} \quad (i = 1, 2, \cdots, n_\text{s}) \quad (35)$$

When $X(t)$ is of the square form of a non-stationary Gaussian random process $Y(t)$, i.e., $X(t) = Y^2(t)$, similar to Equation (22), Equation (34) can be further derived as

$$\begin{cases} \text{cum}[d_i(\tau)] = \sum_{j=1}^{n} a_{n,j}^{d_i} \text{E}(Y_j^2) \\ \text{cum}[d_i(\tau), d_i(\tau)] = 2 \sum_{j=1}^{n} \sum_{m=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} \text{E}(Y_j Y_m)^2 \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau)] = 8 \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} a_{n,p}^{d_i} \text{E}(Y_j Y_m) \text{E}(Y_j Y_p) \text{E}(Y_m Y_p) \\ \text{cum}[d_i(\tau), d_i(\tau), d_i(\tau), d_i(\tau)] = 48 \sum_{j=1}^{n} \sum_{m=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} a_{n,j}^{d_i} a_{n,m}^{d_i} a_{n,p}^{d_i} a_{n,q}^{d_i} \text{E}(Y_j Y_m) \text{E}(Y_j Y_p) \text{E}(Y_m Y_q) \text{E}(Y_p Y_q) \end{cases} \quad (i = 1, 2, \cdots, n_\text{s}) \quad (36)$$

Once the first four orders of cumulants $\text{cum}[d_i(\tau)]$, $\text{cum}[d_i(\tau), d_i(\tau)]$, $\text{cum}[d_i(\tau), d_i(\tau), d_i(\tau)]$ and $\text{cum}[d_i(\tau), d_i(\tau), d_i(\tau), d_i(\tau)]$ $(i = 1, 2, \cdots, n_\text{s})$ are obtained, they can be directly converted to the second-order and fourth-order moments of responses, $\text{E}[d_i^2(\tau)]$ and $\text{E}[d_i^4(\tau)]$ $(i = 1, 2, \cdots, n_\text{s})$, as shown in Equation (7), and the equivalent stiffness matrix $\mathbf{K}_\text{e}(\tau)$ can then be updated through Equations (27) and (30). Repeat the calculation process until the above response statistics are convergent. By now, the equivalent linear system of the Duffing system corresponding to the time instant τ has been obtained considering non-Gaussian random excitation, and the first four orders of cumulants of the other concerned responses can be calculated in the same way as those shown in Equation (34), which, if required, can be further utilized to determine the first four orders of moments of these concerned responses using Equation (7). Thereafter, one can move on to the next specified time instant $\tau+\Delta\tau$ and repeat the above calculation process until all the concerned time instants have been considered. It should be noted that the value of $\Delta\tau$ should be set to meet the requirement of describing the evolutionary higher-order statistics adequately, and in general, $\Delta\tau$ can be taken as certain times of Δt provided that the higher-order statistics are slowly varying functions.

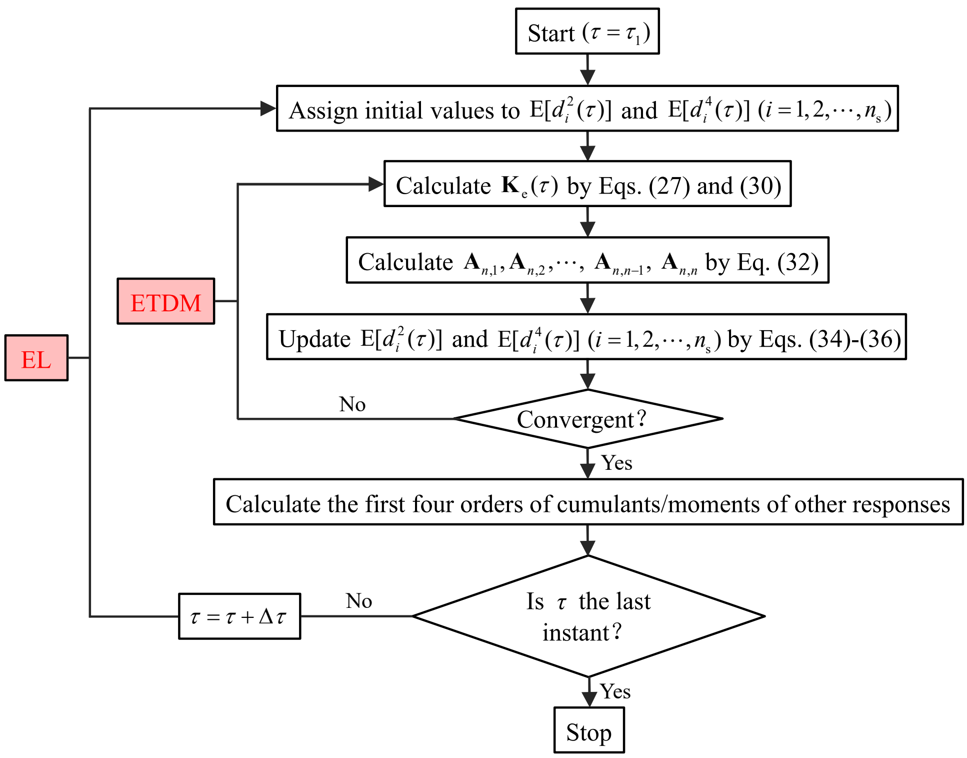


Figure 2 Solution procedure of EL-ETDM

For clarity, the procedure for the present EL-ETDM is illustrated in Figure 2, from which it can be seen that, although a series of higher-order cumulant/moment analyses of the linearized system need to be conducted, they can be accomplished by the efficient ETDM, making the EL method feasible for non-Gaussian problems. This will be validated in Section 5.

5 Numerical examples

In this section, two numerical examples including a linear oscillator subjected to stationary Poisson white noise and a 20-degree-of-freedom linear system under the square form of a non-stationary Gaussian random process are analyzed to validate the efficacy of the present ETDM for solving non-Gaussian random vibration problems of linear systems. Furthermore, the other two numerical examples involving a Duffing oscillator under the square form of a non-stationary Gaussian random process and a 5-degree-of-freedom Duffing system subjected to non-stationary Poisson white noise are investigated to demonstrate the feasibility of the present EL-ETDM for non-Gaussian random vibration analysis of nonlinear systems.

5.1 A linear oscillator

The equation of motion for a linear oscillator can be expressed as

$$\ddot{u}(t) + 2\zeta\omega\dot{u}(t) + \omega^2 u(t) = \hat{X}(t) \tag{37}$$

where $\omega = 10\text{rad/s}$ and $\zeta = 0.05$ are the natural frequency and damping ratio of the linear oscillator, respectively; and $\hat{X}(t)$ is the stationary Poisson white noise shown in Equation (8), in which the mean arrival rate is taken as $\lambda = 2\text{s}^{-1}$, and the amplitudes of random impulses are set to be mutually independent standard Gaussian random variables. The first four orders of cumulant functions of $\hat{X}(t)$ can then be readily determined using Equation (9).

The ETDM presented in Section 3 is utilized to conduct the non-Gaussian random vibration analysis of the linear oscillator, and the Monte Carlo simulation (MCS) with 5×10^4 samples is employed for obtaining the reference solutions to the response statistics. In the above analysis, the time duration and time step are taken as $T = 10\text{s}$ and $\Delta t = 0.002\text{s}$, respectively. Note that, for a linear oscillator under stationary Poisson white noise, the analytical solutions to the response cumulants and moments can be derived [27], which will also be used as the reference solutions.

The second-order and fourth-order cumulants of the displacement are shown in Figure 3 and Figure 4, respectively, and the second-order and fourth-order moments of the displacement are presented in Figure 5 and Figure 6, respectively. It can be observed from the above figures that the results obtained by ETDM agree well with those obtained by MCS, and the ETDM results are identical to the analytical solutions after they enter the stationary state, indicating the good accuracy of the present approach. Note that, for this example, as the amplitudes of random impulses involving in the Poisson white noise are mutually independent standard Gaussian random variables, the first-order and third-order cumulants and moments of the response are zeros, and the second-order cumulant is equal to the second-order moment of the response, as shown in Figure 3 and Figure 5. Furthermore, for the results of MCS, the higher-order cumulants of response generally require much more sample analyses to achieve the convergent results than the higher-order moments of response. In view of this, in the following examples, only the results of the response moments are presented for comparison.

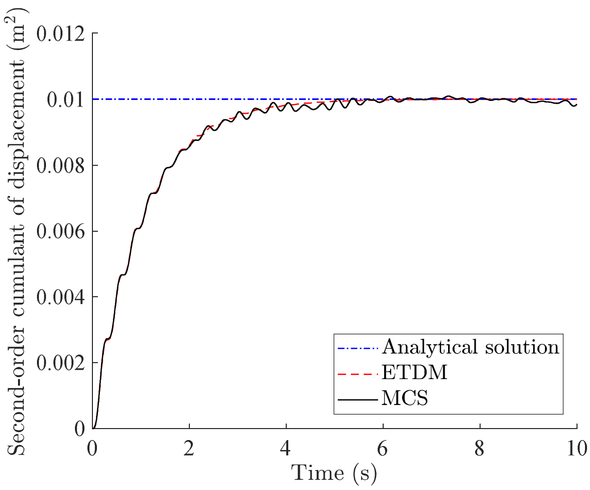


Figure 3 Second-order cumulant of displacement

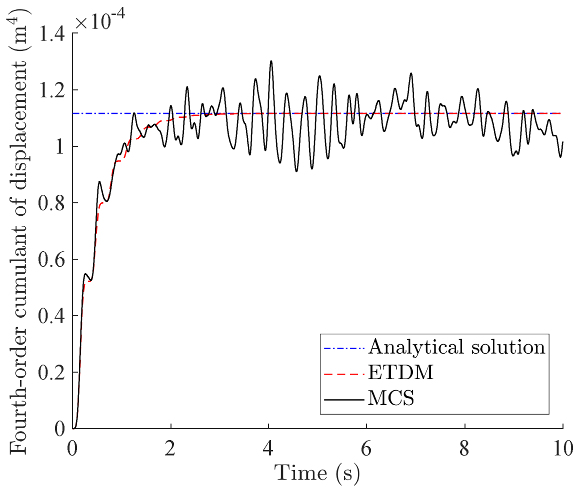


Figure 4 Fourth-order cumulant of displacement

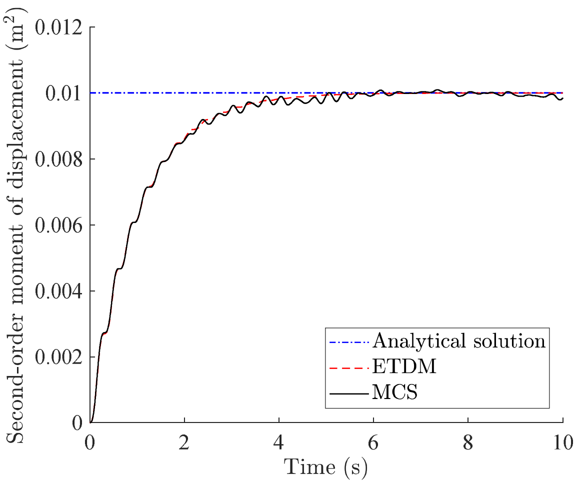


Figure 5 Second-order moment of displacement

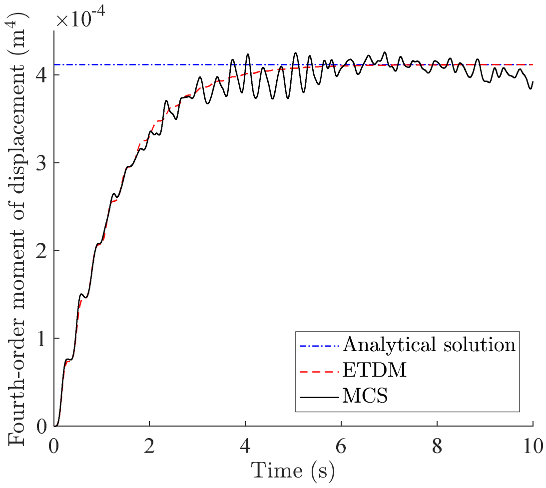


Figure 6 Fourth-order moment of displacement

5.2 A 20-degree-of-freedom linear system

For a 20-degree-of-freedom shear-type linear system, as shown in Figure 7, the mass and lateral stiffness of each storey are taken as $m_i = 1.8\times10^4\,\text{kg}$ and $k_i = 8.9\times10^5\,\text{kN/m}$ $(i = 1, 2, \cdots, 20)$, respectively, and the Rayleigh damping model is adopted with the damping ratio $\zeta = 0.05$. The system is subjected to a non-Gaussian random excitation $X(t) = Y^2(t)$, and $Y(t) = g(t)\hat{Y}(t)$ is a uniformly modulated non-stationary Gaussian random process, in which $g(t)$ is the modulation function expressed as

$$g(t) = 4.0(\mathrm{e}^{-0.1t} - \mathrm{e}^{-0.2t}) \tag{38}$$

and $\hat{Y}(t)$ is a zero-mean stationary band-limited white noise.

The second-order moment function of $Y(t)$ can be expressed as [38]

$$m_{Y,2}(t_1, t_2) = g(t_1)g(t_2)m_{\hat{Y},2}(t_1, t_2) \tag{39}$$

$$m_{\hat{Y},2}(t_1, t_2) = \frac{2S_0}{t_1 - t_2}\sin[\omega_\mathrm{b}(t_1 - t_2)] \tag{40}$$

where $S_0 = 5\times10^3\,\text{N}\cdot\text{s}$ is the spectral density; and $\omega_\mathrm{b} = 100\,\text{rad/s}$ is the half-width of the frequency range. Based on the second-order moment function of $Y(t)$ shown in Equations (39) and (40), the first four orders of cumulant functions of $X(t)$ can be easily obtained using Equation (12).

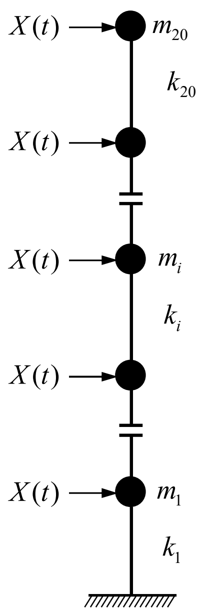


Figure 7 A 20-degree-of-freedom shear-type linear system

The ETDM and the MCS with 5×10^4 samples are utilized to solve the non-Gaussian random vibration problem of the linear system, in which the time duration and time step are taken as $T = 15\text{s}$ and $\Delta t = 0.02\text{s}$, respectively. The first four orders of moments of the top-storey lateral displacement of the system are depicted in Figures 8 to 11, from which it can be seen that the results obtained by ETDM and MCS are in good agreement, further demonstrating the good accuracy of the present approach.

It can be seen from Section 3.2 that the ETDM can achieve dimension-reduced analysis of higher-order statistics focusing on any arbitrary responses of interest. Furthermore, it can be observed from Figures 8 to 11 that the response moments are slowly varying with time, and thus it is not necessary to calculate the moments at so small a time interval as that used in establishing the explicit expression of the state vector shown in Equation (16). To validate the influence of the above considerations on the computational efficiency, the elapsed time of ETDM for different numbers of responses, i.e., 1, 10 and 20, and different time intervals, i.e., $10\Delta t = 0.2\text{s}$, $20\Delta t = 0.4\text{s}$ and $30\Delta t = 0.6\text{s}$, is presented in Table 1. It can be seen from Table 1 that, by taking advantage of the unique feature of dimension-reduced analysis just regarding the critical responses as well as the concerned time instants, the ETDM can achieve even higher efficiency in the process of statistical analysis.

Table 1 Elapsed time of ETDM

Number of responses	Time interval $\Delta\tau$		
	0.2s	0.4s	0.6s
1	214.4s	102.0s	75.4s
10	2135.6s	1011.3s	745.9s
20	4108.7s	2022.8s	1490.4s

Note: All the above computations were done on a laptop PC with an Intel Core i7-3632QM processor and 8 GB RAM.

To further demonstrate the efficiency of ETDM, the PSM is also employed to calculate the second- and third-order moments of the top-storey lateral displacement of the system, which are also depicted in Figures 9 and 10, respectively. It can be seen from Figures 9 and 10 that the results obtained by ETDM and PSM are both in good agreement with those obtained by MCS. However, for execution of PSM, the frequency domain of interest is discretized into 250 intervals, and a total of 250×2=500 time-history analyses of the system are required to obtain the evolutionary bi-spectra and third-order cumulants of responses, leading to much more computational cost than ETDM, in which only one single impulse response time-history analysis of the system is required. Furthermore, for obtaining the evolutionary tri-spectra and fourth-order cumulants of responses, the number of frequency intervals should be set much larger to ensure the accuracy due to the fast-varying property of the response tri-spectra, resulting in huge computational cost that has not been affordable in practice thus far, and therefore the fourth-order moment of the top-storey lateral displacement by PSM is not available in Figure 11.

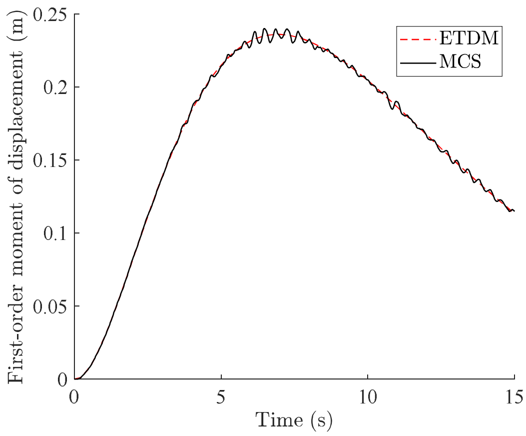


Figure 8 First-order moment of top-storey lateral displacement

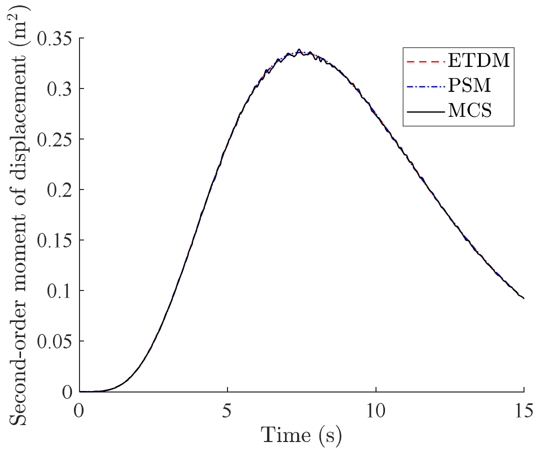


Figure 9 Second-order moment of top-storey lateral displacement

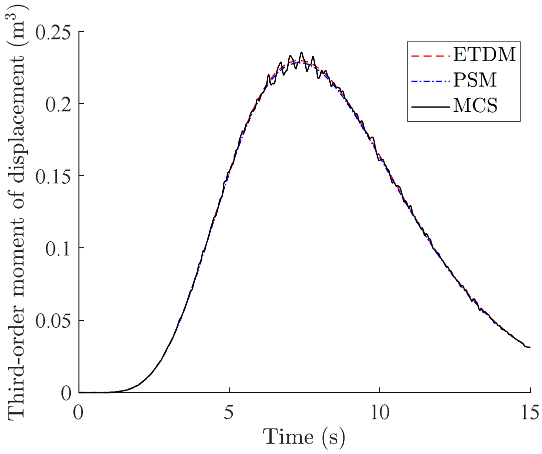


Figure 10 Third-order moment of top-storey lateral displacement

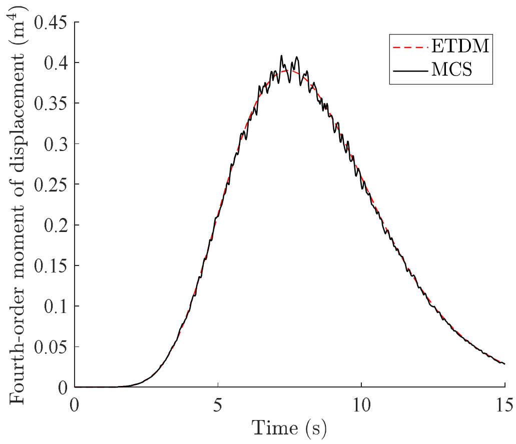


Figure 11 Fourth-order moment of top-storey lateral displacement

5.3 A Duffing oscillator

The equation of motion for a Duffing oscillator can be expressed as

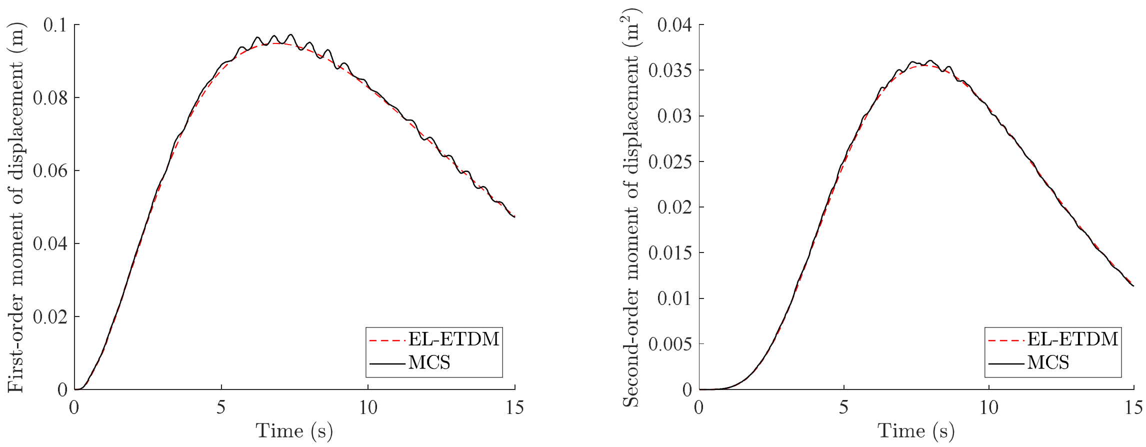
$$\ddot{u}(t)+2\zeta\omega\dot{u}(t)+\omega^2u(t)+\eta\omega^2u^3(t)=X(t) \tag{41}$$

where $\omega=10\text{rad/s}$ and $\zeta=0.05$ are respectively the natural frequency and damping ratio of the Duffing oscillator at the initial state; η is the coefficient reflecting the nonlinearity of the Duffing oscillator, which is taken as $\eta=0.5\text{m}^{-2}$ and $\eta=1.5\text{m}^{-2}$ for different levels of nonlinearity; and $X(t)=Y^2(t)$ is the non-Gaussian random acceleration excitation with $Y(t)$ being a zero-mean non-stationary Gaussian random process. The second-order moment function of $Y(t)$ is shown in Equations (39) and (40), in which $S_0=0.05\text{m/s}$ and

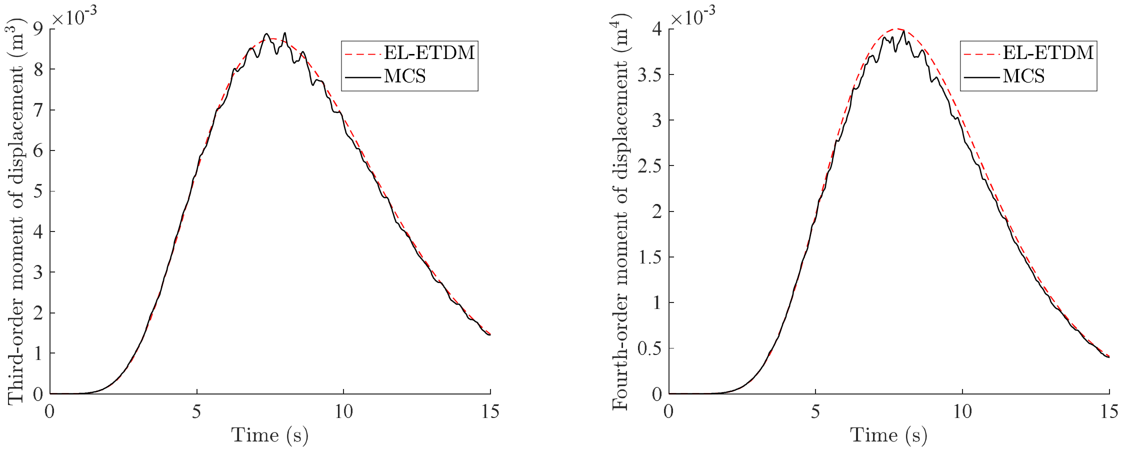
$\omega_b = 100\text{rad/s}$ are adopted, and on this basis, the first four orders of cumulant functions of $X(t)$ can be determined accordingly using Equation (12).

The EL-ETDM presented in Section 4 is employed for the non-Gaussian random vibration analysis of the Duffing oscillator, in which the time step is taken to be $\Delta t = 0.02\text{s}$ for explicit formulation of the response of the linearized systems, while the time step for EL analysis shown in Figure 2 is set to be $\Delta\tau = 0.2\text{s}$ with the duration $T = 15\text{s}$. For comparison, the MCS with 5×10^4 samples is also utilized to obtain the reference solutions to the response statistics, in which the duration and time step for time-history analysis are set to be $T = 15\text{s}$ and $\Delta t = 0.02\text{s}$, respectively.

The first four orders of moments of the displacement corresponding to $\eta = 0.5\text{m}^{-2}$ and $\eta = 1.5\text{m}^{-2}$ are presented in Figure 12 and Figure 13, respectively. It can be observed that the results obtained by EL-ETDM are in good agreement with those obtained by MCS, showing that the present approach is of good accuracy. Furthermore, by comparing the results shown in Figure 12 and Figure 13, it can be found that the accuracy of the statistical linearization technique decreases to a certain degree when the Duffing oscillator undergoes stronger nonlinearity, and the relative error of EL-ETDM may reach 6.1% for the fourth-order moment of displacement under the case of $\eta = 1.5\text{m}^{-2}$.

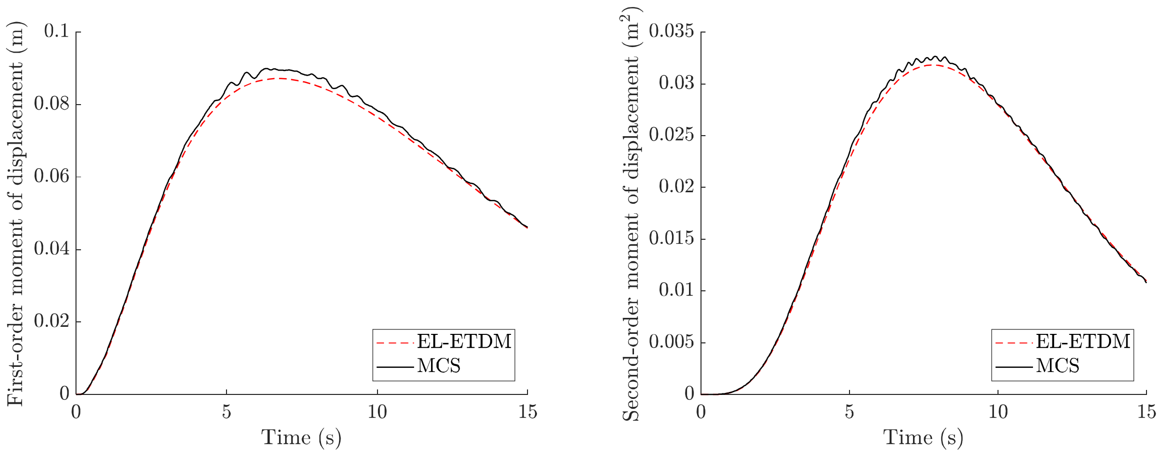


(a) First-order moment (b) Second-order moment

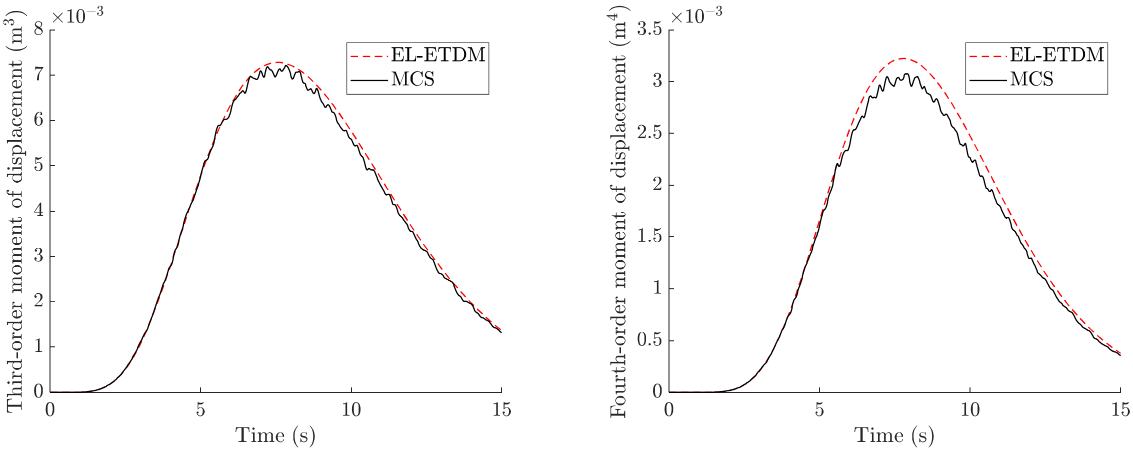


(c) Third-order moment (d) Fourth-order moment

Figure 12 First four orders of moments of displacement ($\eta = 0.5\text{m}^{-2}$)



(a) First-order moment (b) Second-order moment



(c) Third-order moment (d) Fourth-order moment

Figure 13 First four orders of moments of displacement ($\eta = 1.5\text{m}^{-2}$)

5.4 A 5-degree-of-freedom Duffing system

For a 5-degree-of-freedom shear-type Duffing system, as shown in Figure 14, the mass and stiffness of each storey are taken as $m_i = 3\times10^3\,\mathrm{kg}$ and $k_i = 3\times10^4\,\mathrm{kN/m}\ (i=1,2,\cdots,5)$, respectively, and the Rayleigh damping model with the damping ratio $\zeta = 0.05$ is adopted to define the damping matrix $\mathbf{C}$. To reflect different levels of nonlinearity, three cases are considered for the nonlinear coefficients, i.e., $\eta_i = 10\mathrm{m}^{-2}$, $\eta_i = 30\mathrm{m}^{-2}$ and $\eta_i = 50\mathrm{m}^{-2}$ $(i=1,2,\cdots,5)$.

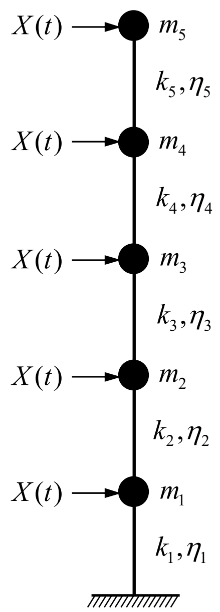


Figure 14 A 5-degree-of-freedom shear-type Duffing system

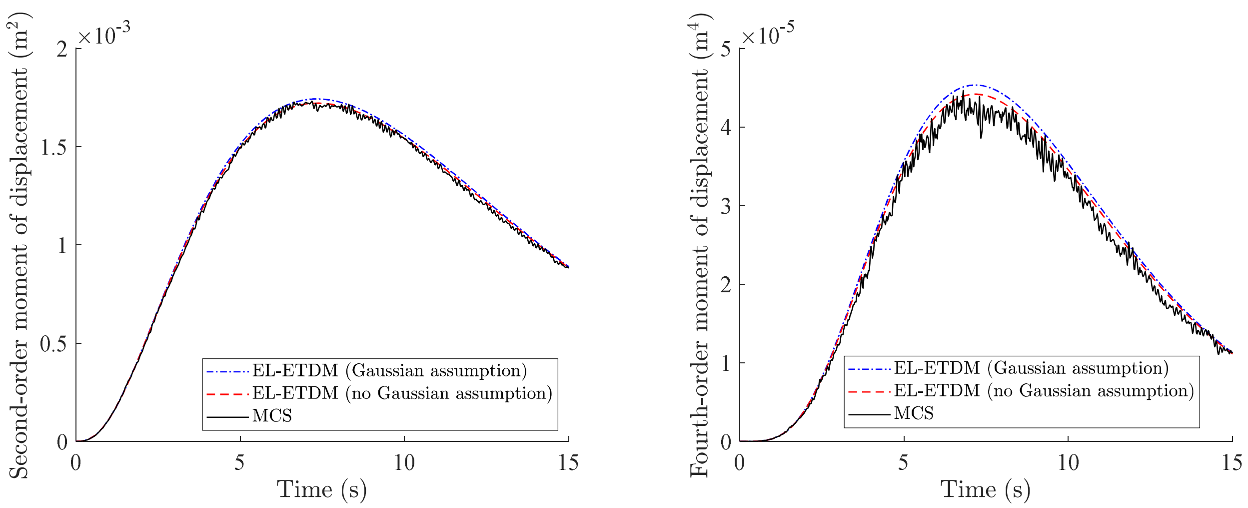
The Duffing system is subjected to a uniformly modulated non-stationary Poisson white noise $X(t) = g(t)\hat{X}(t)$, in which $g(t)$ is the modulation function shown in Equation (38), and $\hat{X}(t)$ is the stationary Poisson white noise shown in Equation (8). For the Poisson white noise, the mean arrival rate is set to be $\lambda = 0.5\mathrm{s}^{-1}$, and the amplitudes of random impulses are taken as mutually independent Gaussian random variables with the mean and standard deviation being 0 and $10\mathrm{kN\cdot s}$, respectively. The first four orders of cumulant functions of $X(t)$ can be easily determined using Equations (9) and (10).

The EL-ETDM and the MCS with 5×10^4 samples are utilized to solve the non-Gaussian random vibration problem of the Duffing system. For EL-ETDM, the time step is taken to be

$\Delta t = 0.02\text{s}$ for explicit formulation of the responses of the linearized systems, and the time step for EL analysis shown in Figure 2 is set to be $\Delta\tau = 0.2\text{s}$ with the duration $T = 15\text{s}$. For MCS, the duration and time step for time-history analysis are set to be $T = 15\text{s}$ and $\Delta t = 0.02\text{s}$, respectively. To investigate the effects of the assumption of Gaussian response on the results of statistical linearization technique considering non-Gaussian random excitation, the EL-ETDM with Gaussian assumption is also adopted for the non-Gaussian random vibration analysis of the Duffing system, in which, instead of the formula shown in Equation (27), the equivalent stiffness of the ith hardening spring is expressed as $k_{\text{e},i}(\tau) = 3\eta_i k_i \text{E}[d_i^2(\tau)]\ (i = 1, 2, \cdots, n_\text{s})$ [29].

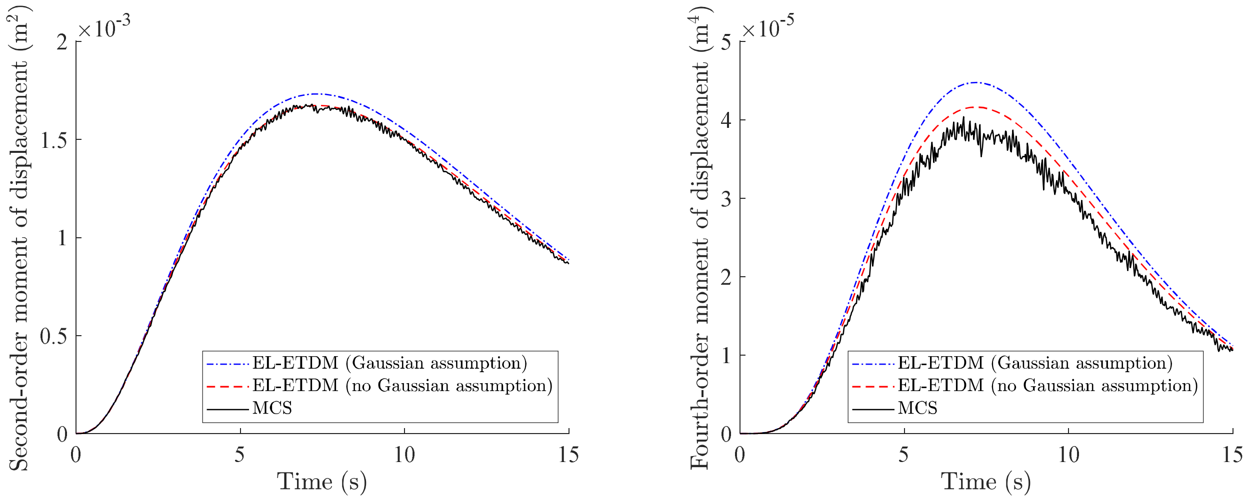
The second-order and fourth-order moments of the top-storey lateral displacement corresponding to $\eta_i = 10\text{m}^{-2}$, $\eta_i = 30\text{m}^{-2}$ and $\eta_i = 50\text{m}^{-2}$ are presented in Figures 15 to 17, respectively. It can be seen from the above figures that, although the accuracy of EL-ETDM (no Gaussian assumption) may decrease to a certain extent with the increase of the degree of system nonlinearity, the results obtained by EL-ETDM (no Gaussian assumption) are basically in good agreement with those obtained by MCS, validating the feasibility of the present approach. It can be further observed from Figures 15 to 17 that, if the assumption of Gaussian response is adopted, the accuracy of the statistical linearization technique deteriorates significantly and the results become unacceptable when stronger nonlinearity exists in the Duffing system. This is due to the fact that the response distribution of the Duffing system under Poisson white noise is never Gaussian regardless of the degree of system nonlinearity, and the error caused by the Gaussian assumption may increase as the degree of nonlinearity of the system increases.

Finally, in view of the fact that the Monte Carlo solutions may not be available for comparison when the problem becomes more complex, it is of great importance to have an estimate of the level of accuracy of the proposed EL-ETDM for nonlinear random vibration analysis. For this purpose, besides the comparisons shown in Figures 15 to 17, a series of nonlinear coefficients, i.e., $\eta_i = 10\text{m}^{-2}, 15\text{m}^{-2}, \cdots, 50\text{m}^{-2}$, have been investigated with EL-ETDM (Gaussian assumption), EL-ETDM (no Gaussian assumption) and MCS. The relative discrepancy between the results of EL-ETDM (Gaussian assumption) and EL-ETDM (no Gaussian assumption) as well as that between the results of EL-ETDM (no Gaussian assumption) and MCS are depicted in Figure 18. It can be observed from Figure 18 that the relative discrepancy between the results of EL-ETDM (Gaussian assumption) and EL-ETDM (no Gaussian assumption) is considerably larger than that between the results of EL-ETDM (no Gaussian assumption) and MCS, in particular for the second-order moments, indicating the error induced by the assumption of Gaussian response is the major error compared with that induced by the linearization criterion in the EL method. This implies that the relative discrepancy between the results of EL-ETDM (Gaussian assumption) and EL-ETDM (no Gaussian assumption) can be regarded as an upper bound for the error of the results of EL-ETDM (no Gaussian assumption). In this sense, one can have an estimate of the level of accuracy of the present EL-ETDM (no Gaussian assumption) without resort to the execution of MCS.



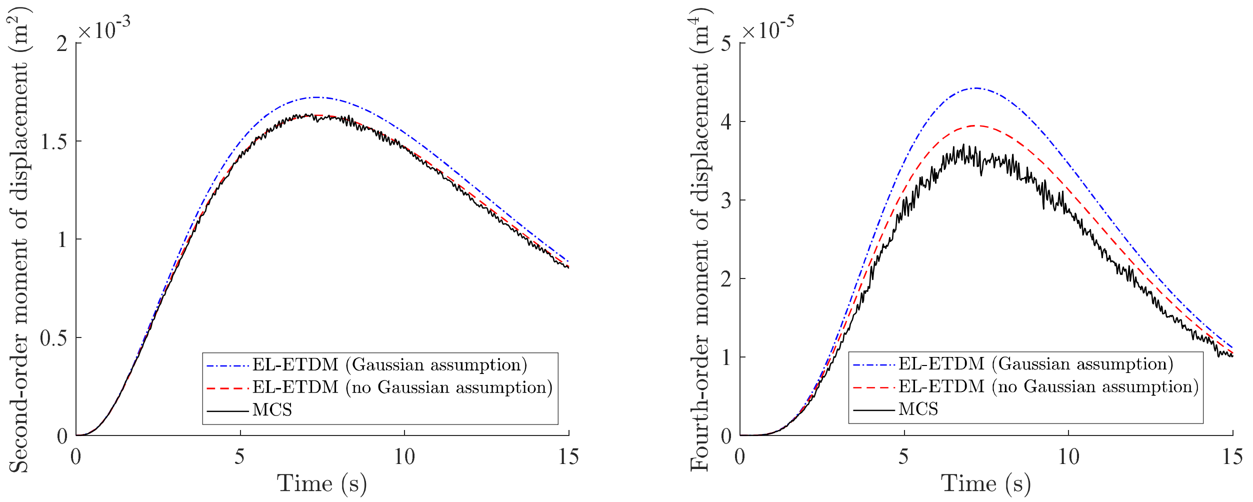
(a) Second-order moment (b) Fourth-order moment

Figure 15 Second-order and fourth-order moments of top-storey lateral displacement ($\eta_i = 10\text{m}^{-2}$)



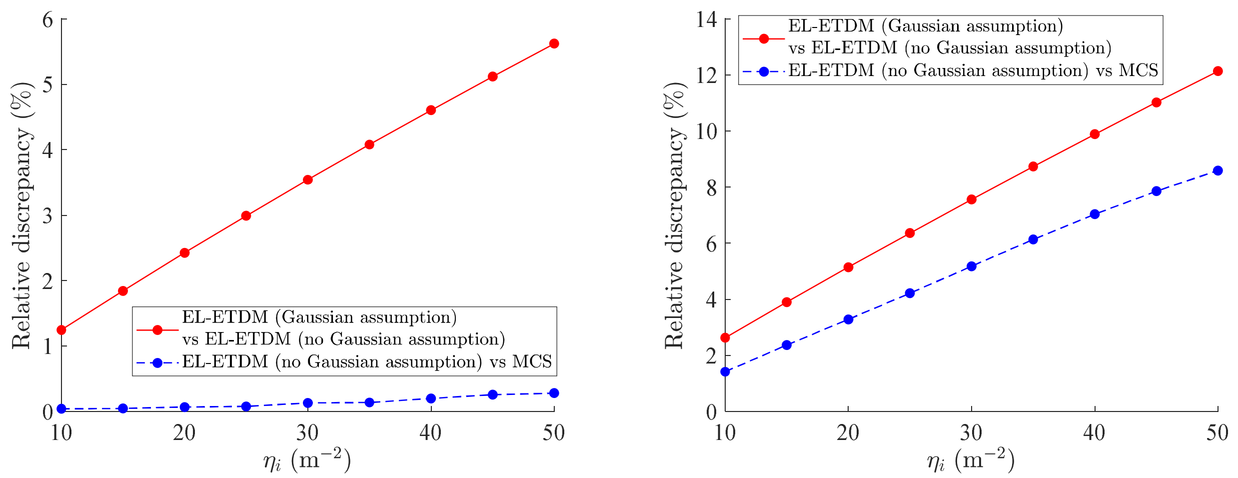
(a) Second-order moment (b) Fourth-order moment

Figure 16 Second-order and fourth-order moments of top-storey lateral displacement ($\eta_i = 30\text{m}^{-2}$)



(a) Second-order moment (b) Fourth-order moment

Figure 17 Second-order and fourth-order moments of top-storey lateral displacement ($\eta_i = 50\text{m}^{-2}$)



(a) Second-order moment (b) Fourth-order moment

Figure 18 Relative discrepancies among different methods for maximum values of second-order and fourth-order moments of top-storey lateral displacement

6 Conclusions

There exist two challenges involved in extending ETDM and EL-ETDM from Gaussian to non-Gaussian problems. The first challenge lies in the explicit formulation of the higher-order cumulants of non-Gaussian responses with much more concise forms compared with the traditional moment-based formulation adopted in ETDM for Gaussian problems, and such explicit formulation can significantly reduce the computational cost for evolutionary higher-order statistics of non-Gaussian responses compared with the existing PSM. The second challenge is to extend the EL method for nonlinear non-Gaussian problems without the use of the assumption of Gaussian responses, which can be readily accomplished by the present ETDM with high-efficient calculation of higher-order moments of non-Gaussian responses for the series of linearized systems involved in the linearization process.

The present approach is applicable to arbitrary forms of non-Gaussian random excitation since the only prerequisite for the approach is that the cumulant functions of the non-Gaussian random excitation are known. Four numerical examples considering two kinds of non-Gaussian random excitations, i.e., the Poisson white noise and the square form of Gaussian random process, have been investigated to demonstrate the effectiveness of the present approach.

It should be noted that, only uniform random excitations are considered in the present study, whereas the wind load and wave load that exert on a real structure are usually modelled as non-uniform random excitations. Therefore, the present approach needs to be further developed to account for the spatial correlation effects of non-uniform non-Gaussian random excitations in future study. Moreover, in this study, only Duffing systems are investigated in the nonlinear analysis, and the more general nonlinear systems, e.g., the nonlinear hysteretic systems and the nonlinear viscously damped systems, need to be further considered in the context of the present approach.

Acknowledgements

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