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Doubly Critical Semilinear Schrödinger Equations.

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Doctor of Philosophy in Mathematics

by

Alexander Adam Azzam

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ABSTRACT OF THE DISSERTATION

Doubly Critical Semilinear Schrödinger Equations.

by

Alexander Adam Azzam
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2017
Professor Monica Vişan, Co-Chair
Professor Rowan Brett Killip, Co-Chair

We present several new results regarding the two and three dimensional energy-critical nonlinear Schrödinger equation in the presence of a second critical nonlinearity.

The dissertation of Alexander Adam Azzam is approved.

Per Kraus

Rowan Brett Killip, Committee Co-Chair

Monica Vişan, Committee Co-Chair

University of California, Los Angeles

2017

To my parents: my first advisors.

You made our home a classroom…

...where every hour was an office hour…

…and the only tuition was loading the dishwasher.

Your love and support made this possible…

…I am proud to call you colleagues.

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CHAPTER 1

Introduction

In this thesis we consider the existence and behavior of solutions to semilinear Schrödinger equations, and discuss harmonic analysis questions related to their associated operators. In general, a semilinear Schrödinger equation takes the form

$$i\partial_t u = -\tfrac{1}{2}\Delta u + Vu + \mu F(u), \qquad u(0,x) = u_0(x). \tag{1.0.1}$$

Here $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ is a complex-valued function of time and space and u_0 is the given initial data.

The semilinear Schrödinger model (1.0.1) strikes an important balance between being robust enough to describe a wide range of physical phenomena, and being lean enough to make it vulnerable to fruitful mathematical analysis. For example, when $V = 0$ and $F \equiv 0$, we recover the traditional linear Schrödinger equation, which models the evolution of a quantum system with a free particle. When $V = 0$ and $F(u) = |u|^p u$ ($p > 0$), we recover the nonlinear Schrödinger equation, which models a wealth of physical phenomena like light in nonlinear media, Bose–Einstein condensates, and water waves. To model Bose–Einstein condensates in a trap, for example, one may study the model with $V = \frac{|x|^2}{2}$ and $F(u) = |u|^p u$ ($p > 0$).

1.1 Background

In the last thirty years, considerable effort has been spent trying to understand the existence and behavior of solutions to semilinear Schrödinger equations of the form

$$\begin{cases} iu_t = -\Delta u + \mu|u|^p u, \qquad p > 0 \\ u(0,x) = u_0(x) \in \dot{H}^s_x(\mathbb{R}^d). \end{cases} \tag{1.1.1}$$

In fact, this model is so ubiquitous we will simply refer to it as *the* nonlinear Schrödinger equation (NLS). Since the linearization of the nonlinearity in (1.1.1) is diagnolized by the Fourier Transform, harmonic analysis techniques are well-suited to analyze this model. We say that (1.1.1) is *defocusing* if $\mu = 1$ and *focusing* if $\mu = -1$. Solutions to (1.1.1) conserve both their mass

$$M(u(t)) := \int_{\mathbb{R}^d} |u(t,x)|^2 \, dx, \tag{1.1.2}$$

and their energy

$$H(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+2}|u(t,x)|^{p+2} \, dx.$$

1.1.1 Energy-Criticality with Scaling Symmetry.

The class of solutions to (1.1.1) enjoy a number of useful symmetries. Of particular importance is the invariance of the class of solutions under the scaling

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x). \tag{1.1.3}$$

The effect of the scaling on the initial data is given by

$$||u_\lambda(0)||_{\dot{H}^s_x(\mathbb{R}^d)} = \lambda^{s-(\frac{d}{2}-\frac{2}{p})}||u(0)||_{\dot{H}^s_x(\mathbb{R}^d)}.$$

Thus, when $s = \frac{d}{2} - \frac{2}{p}$, the scaling (1.1.3) leaves invariant both the class of solutions and the size of the initial data.

Definition 1.1.1. Consider the initial value problem (1.1.1). Let

$$s_c = \tfrac{d}{2} - \tfrac{2}{p}.$$

We say (1.1.1) is *critical* when $s = s_c$, *subcritical* when $s > s_c$, and *supercritical* when $s < s_c$.

When $s = s_c = 1$, the energy $H(u(t))$ is left invariant by the scaling (1.1.3), giving rise to the *energy-critical* NLS

$$\begin{cases} iu_t + \Delta u = |u|^{\frac{4}{d-2}} u, \qquad d \geq 3 \\ u(0,x) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^d). \end{cases} \tag{1.1.4}$$

When $s = 1$ and $p < \frac{4}{d-2}$ (i.e. $s_c < 1$), we say that (1.1.1) is *energy-subcritical.*

In the last two decades the energy-critical NLS (1.1.4) has been the subject of a relentless, inspiring, and successful campaign to understand the local and global behavior of its solutions. Local well-posedness was first proved by Cazenave and Weissler in [CW90], who showed that the length of a solution's lifespan depends on the profile of u_0 rather than the norm of u_0. The first victory on the front of large-data global well-posedness was made by J. Bourgain in [Bou99], who proved global well-posedness and exhibited global spacetime bounds for spherically symmetric initial data in $d = 3, 4$ by introducing what is now known as the 'induction on energy' paradigm. Using this paradigm, and introducing a wealth of new ideas and techniques, the authors of [CKS08] managed to remove Bourgain's assumption of spherically symmetric data in $d = 3$. Adapting these techniques to handle newfound difficulties in high dimensions, the problem was completely resolved by the authors of [RV07] ($d = 4$) and [Vis07] ($d \geq 5$).

1.1.2 Energy-Criticality without Scaling Symmetry

Though the notion of energy-criticality in $d \geq 3$ is defined through an available scaling symmetry, it is important to understand the characteristic features of the energy-subcritical and energy-critical nonlinearities independent of this symmetry.

As (1.1.3) reveals, energy-criticality is determined by the response of the $\dot{H}^1_x$ norm at fine scales or, equivalently, high-frequencies. In the energy-critical case, the kinetic and potential energy norms are equally strong at all scales. In the energy-subcritical case, the kinetic energy norm dominates the potential energy norm at fine scales. This phenomenon is expressed concisely in the Sobolev inequality, which in $d \geq 3$ reads

$$||f||_{L_x^{\frac{2d}{d-2}}} \leq C_d ||\nabla f||_{L_x^2}, \tag{1.1.5}$$

$$\sup_{||u||_{H_x^1} \leq 1} ||u||_{L_x^p} \leq C(p,d) \qquad\qquad 2 \leq p \leq \frac{2d}{d-2}. \tag{1.1.6}$$

Heuristically, Sobolev embedding informs us of how strong our nonlinearity may be, i.e. how large p may be, before the potential energy norm overpowers the kinetic energy norm.

These features present themselves in the well-posedness theory in a few ways. In the energy-subcritical setting the time of existence guaranteed in the local theory depends only on the size of the initial data; in the energy-critical setting, the time of existence depends on its profile. The local theory may be iterated to extend the lifespan of a local solution provided there is no energy concentration. A computation shows that

$$H(u_\lambda) = \lambda^{2(1-s_c)} H(u) \to \infty,$$

which converges to ∞ as $\lambda \to \infty$ in the energy-subcritical case $s_c < 1$. Thus, energy conservation discourages concentration in this case. In the energy-critical setting $s_c = 1$, however, the conservation of energy does not immediately rule out the possibility of concentration.

1.2 Doubly Critical Semilinear Schrödinger Equations

In this thesis, we consider two "doubly critical" semilinear Schrödinger equations. Heuristically, we say a nonlinearity is "doubly critical" if its nonlinearity is energy-critical and critical in another sense. We clarify this point below.

In Chapter 2, we follow [Azz15] and consider the Cauchy problem for a defocusing nonlinear Schrödinger (NLS) equation on $\mathbb{R}^2$:

$$\begin{cases} iu_t + \Delta u = F_1(u) := u(e^{4\pi|u|^2} - 1) \\ u(0,x) = u_0(x) \in H_x^1(\mathbb{R}^2) \end{cases} \tag{1.2.1}$$

We address the question of whether global solutions to (1.2.1) obey global spacetime bounds and scatter in H_x^1 in the energy-subcritical case. This equation enjoys no scaling symmetry, and so some explanation is needed to justify in what sense we regard (1.2.1) as energy-critical. In $\mathbb{R}^2$, Sobolev embedding guarantees that every polynomial nonlinearity is energy-subcritical. Indeed, if $u \in H^1_x$ is localized to frequency $\sim N$ and $||u||_{H^1_x(\mathbb{R}^2)} \leq 1$ (say), then Bernstein's inequalities show that

$$||u||^p_{L^p_x} \lesssim_p N^{-2}||\nabla u||^2_{L^2_x}.$$

At fine scales ($N \gg 1$), we see that the kinetic energy dominates the potential energy. Thus, if we are to identify an energy-critical nonlinearity, it is natural to consider an exponential nonlinearity. For more insight on our use of the honorific "energy-critical" for (1.2.1), see Chapter 2.

We regard (1.2.1) as doubly critical in light of the main obstacle to scattering. The chief difficulty in establishing scattering results for (1.2.1) stem from the poor decay of the cubic term in the Taylor expansion of $F_1(u)$:

$$F_1(u) = 4\pi u|u|^2 + 8\pi^2 u|u|^4 + \cdots .$$

Indeed, $F_1(u)$ can only decay at least as slow as the cubic nonlinearity $4\pi u|u|^2$ does. Thus, we may only expect global solutions to (1.2.1) to scatter in H^1_x if we expect scattering in H^1_x for global solutions to the the associated Cauchy problem

$$\begin{cases} iu_t + \Delta u = 4\pi|u|^2 u \\ u(0,x) = u_0(x) \in H^1_x(\mathbb{R}^2). \end{cases} \tag{1.2.2}$$

In $\mathbb{R}^2$, this corresponds to the *mass-critical* NLS. Indeed, when $s = s_c = 0$ the mass $M(u(t))$ is left invariant by the scaling (1.1.3), giving rise to the *mass-critical* NLS. In two space dimensions, this takes the form:

$$\begin{cases} iu_t + \Delta u = g(u) := |u|^2 u \\ u(0,x) = u_0(x) \in L^2_x(\mathbb{R}^2). \end{cases} \tag{1.2.3}$$

The local theory for (1.2.3) was established by Cazenave and Weissler in [CW89]. Analogously to the local theory for the energy-critical case in $d \geq 3$, the lifespan of the local solutions they constructed depended on the profile of the initial data and not just the L^2_x-norm. Two decades later, in [KTV09], Killip–Tao–Vişan showed that for radial initial data, (1.2.3) is globally well-posed and that solutions obey global spacetime bounds; in particular, scattering holds. Soon after, in [Dod16], Dodson removed the radiality assumption and established the theorem in its full generality.

To better understand (1.2.1), we decompose the nonlinearity

$$F_1(u) = 4\pi|u|^2u + F_2(u), \qquad F_2(u) := u(e^{4\pi|u|^2} - 4\pi|u|^2 - 1),$$

and thus view (1.2.1) as a perturbation of the mass-critical NLS (1.2.2) with error $F_2(u)$. We exploit the insights of Tao-Vişan-Zhang from [TVZ07] and the estimates from [IMM12] to show that the error term $F_2(u)$ is mild enough to derive global spacetime bounds from those enjoyed by global solutions of the mass-critical problem. These results are summarized in the following theorem.

Theorem 2.1.3 For $u_0 \in H^1_x$ with $H_1(u_0) \le 1$, there exists a unique strong solution $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ to (1.2.1). If $H_1(u_0) < 1$, then u satisfies

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^4 + |\nabla u(t,x)|^4 \; dx \; dt < C(||u_0||_{H^1_x})$$

and scatters both backwards and forwards in time.

In Chapter 3 we follow [Azz17] and turn our attention to another doubly critical semilinear Schrödinger equation. We consider, for $a \le \frac{1}{4}$, the energy-critical NLS with inverse-square potential

$$\begin{cases} i\partial_t u = (-\Delta - \frac{a}{|x|^2})u + |u|^4 u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^3) \end{cases} . \tag{1.2.4}$$

The restriction $a \le \frac{1}{4}$ will be motivated below. Note that when $a = 0$, the model (1.2.4) reduces to the traditional 3D energy-critical NLS.

The family of operators

$$H_a := -\Delta - \frac{a}{|x|^2} \qquad a \le \tfrac{1}{4} \tag{1.2.5}$$

appear often in the study of quantum mechanics, combustion theory, and electrostatics (see [Cas50], [KSW75]). Since the potential term is homogenous of order -2, the potential and Laplacian scale exactly the same. For this reason, this makes the study (1.2.4) of particular mathematical interest. Indeed, since the potential and Laplacian are of equal strength at every scale, analysis of (1.2.4) is immune to standard perturbative methods. Thus, Schrödinger equations with inverse square potentials have necessitated a more delicate approach. This challenge has been met with a long campaign to understand those properties enjoyed by H_a and the local and global behavior of solutions to (1.2.4).

Substantial progress has been made in understanding the operator H_a and the model (1.2.4). In Chapter 3 we give a detailed history of this progress. Of note are the advances made in [KMV15b], where Killip-Miao-Vişan-Zhang-Zheng prove a variety of harmonic analysis tools for studying H_a when $a \leq \frac{1}{4}$. In particular, they develop a Littlewood-Paley theory for H_a, which allows them to prove in [KMV15a] that (1.2.4) is globally well-posed when $a < \frac{1}{4} - \frac{1}{25}$. This represents the first progress made in treating the energy critical nonlinearity $|u|^4 u$ in the presence of an inverse square potential.

In Chapter 3 we consider harmonic analysis questions related to the operator

$$H := H_{\frac{1}{4}} = -\Delta - \frac{1}{4|x|^2}$$

The choice of the constant $a = \frac{1}{4}$ in this chapter makes the nonlinearity in (1.2.4) critical in another sense. This constant corresponds to the sharp constant in the Hardy inequality.

Lemma 1.2.1 (Hardy Inequality). *If $u \in C_0^\infty(\mathbb{R}^3)$, then*

$$\frac{1}{4} \int \frac{|u(x)|^2}{|x|^2} \, dx \leq \int |\nabla u(x)|^2 \, dx. \tag{1.2.6}$$

Moreover, (1.2.6) *fails if the constant $\frac{1}{4}$ is replaced by any constant $a > \frac{1}{4}$.*

When $a \leq \frac{1}{4}$ and $u \in C_0^\infty(\mathbb{R}^3)$ we see that

$$\langle H_a u, u \rangle_{L_x^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u(x)| \, dx - a \int \tfrac{|u(x)|^2}{|x|^2} \, dx \geq 0, \tag{1.2.7}$$

and so H_a is positive semi-definite. When $a > \frac{1}{4}$, all self-adjoint extensions of H_a are unbounded below. In this case the spectrum of H_a has infinitely many negative eigenvalues that diverge to $-\infty$ ([PST03]), making it impossible to obtain global dispersive estimates.

In this chapter, we advocate for the idea that any problem involving the critical inverse square potential should be decomposed into a system of equations: one for the radial part of the solution (which is more susceptible to classical ODE techniques), and another for the remainder. Indeed, what is clear from previous advances in [BPS03], [Miz16], and [BM16] is that data in $L^2 \ominus L^2_{\mathrm{rad}}$ enjoy a wealth of useful estimates for H_a which in turn yield Strichartz estimates that are not immediately available in the general case. When restricted to higher angular momentum, the operator H_a enjoys estimates that mirror those of the Laplacian.

To make this discussion more precise, let $P : L^2(\mathbb{R}^3) \to L^2_{\mathrm{rad}}(\mathbb{R}^3)$ be the projection defined by

$$Pf(x) = \frac{1}{4\pi^2} \int_{\mathbb{S}^2} f(|x|\omega) d\sigma(\omega).$$

We define $P^\perp = I - P$ to be the orthogonal projection onto $L^2_\perp := L^2 \ominus L^2_{\mathrm{rad}}$, the space of angular functions. As P and $P^\perp$ commute with Δ and $-\frac{1}{4|x|^2}$, they commute with H. We will often write

$$H^\perp := P^\perp H = HP^\perp$$

when convenient.

The advantage of restricting to functions in $L^2_\perp$ is due, in part, to an improvement to Hardy's inequality in this setting. For a clear exposition, see [EF06].

Lemma 1.2.2 (Improved Hardy Inequality). *If $f : \mathbb{R}^3 \to \mathbb{C}$ is a Schwartz function, then*

$$\frac{9}{4} \int_{\mathbb{R}^3} \frac{|P^\perp f|^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^3} |\nabla P^\perp f|^2\, dx. \tag{1.2.8}$$

Moreover, (1.2.8) *fails if the constant $\frac{9}{4}$ is replaced by any constant $a > \frac{9}{4}$.*

For data in $L^2_\perp$ this improvement allows one to apply techniques available only in the subcritical regime $a < \frac{1}{4}$ at the critical constant $a = \frac{1}{4}$. In this paper, we exploit this improvement to expand upon the harmonic analysis tools developed in [KMV15b]. We seek out which embeddings enjoyed by Δ are also enjoyed by H in order to develop a Littlewood-Paley theory for H. Towards this end, we study the resolvent of H and, in particular, how it differs from the resolvent of Δ when projected to angular functions.

In Chapter 3 we improve upon the best known resolvent estimates for H and prove the following:

Theorem 3.2.2. Let $(q,s) \in [1,\infty) \times [1,\infty)$ satisfy

$$\tfrac{4}{3} \le \tfrac{1}{q} + \tfrac{1}{s'} \le \tfrac{5}{3}. \tag{1.2.9}$$

If, additionally,

$$\sqrt{2} > \max\{2 - \tfrac{3}{s'}, 2 - \tfrac{3}{q}, \tfrac{3}{q} - \tfrac{5}{2}, \tfrac{3}{s'} - \tfrac{5}{2}\}, \tag{1.2.10}$$

then

$$\|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})\|_{L^q_x(\mathbb{R}^3) \to L^s_x(\mathbb{R}^3)} < \infty. \tag{1.2.11}$$

This estimate allows us to improve the range of Bernstein estimates enjoyed by $H^{\perp}$ that are currently known. Indeed, we obtain that $H^{\perp}$ obeys many of the useful Bernstein estimates enjoyed by the Laplacian. More precisely, we prove the following:

Corollary 3.2.3 (Sharp Sobolev Embedding). If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\|P^{\perp}(H+1)^{-1} f\|_{L^6_x(\mathbb{R}^3)} \lesssim ||f||_{L^2_x(\mathbb{R}^3)}. \tag{1.2.12}$$

Corollary 3.2.4. If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\|P^{\perp}(H+1)^{-1} f\|_{L^\infty_x(\mathbb{R}^3)} \lesssim ||f||_{L^2_x(\mathbb{R}^3)} \tag{1.2.13}$$

$$\|P^{\perp}(H+1)^{-1} f\|_{L^2_x(\mathbb{R}^3)} \lesssim ||f||_{L^1_x(\mathbb{R}^3)}. \tag{1.2.14}$$

Theorem 3.2.5 (Bernstein Estimates). The operator $P^{\perp} e^{-H} : L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ is bounded.

CHAPTER 2

The 2D Nonlinear Schrödinger Equation with exponential nonlinearity.

2.1 Introduction

In this chapter, we consider the Cauchy problem for a pair of defocusing nonlinear Schrödinger (NLS) equations on $\mathbb{R}^2$:

$$\begin{cases} iu_t + \Delta u = F_1(u) := u(e^{4\pi|u|^2} - 1) \\ u(0,x) = u_0(x) \in H^1_x(\mathbb{R}^2) \end{cases} \tag{2.1.1}$$

$$\begin{cases} iu_t + \Delta u = F_2(u) := u(e^{4\pi|u|^2} - 4\pi|u|^2 - 1) \\ u(0,x) = u_0(x) \in H^1_x(\mathbb{R}^2). \end{cases} \tag{2.1.2}$$

Here $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is a complex-valued function of time and space. In this paper, our chief interest will be to understand the long-time behavior of solutions to (2.1.1) and (2.1.2). Of course, before we do this, we must first clarify what we mean by a solution.

Definition 2.1.1. Let $I \subseteq \mathbb{R}$ be an interval containing the origin. We say $u : I \times \mathbb{R}^2 \to \mathbb{C}$ is a (strong) *solution* to (2.1.1) (resp. (2.1.2)) if it belongs to $C_t(K; H^1_x)$ for every compact interval $K \subseteq I$ and satisfies the Duhamel formula

$$u(t) = e^{it\Delta}u(0) - i\int_0^t e^{i(t-s)\Delta}(iu_t + \Delta u)(s)\, ds, \tag{2.1.3}$$

for all $t \in I$. We refer to the interval I as the *lifespan* of u. We say u is a global solution if $I = \mathbb{R}$.

Solutions to (2.1.1) and (2.1.2) conserve, respectively, the following energies:

$$H_1(u(t)) := \int_{\mathbb{R}^2} |\nabla u(t,x)|^2 + \tfrac{1}{4\pi}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2\right)(t,x)\ dx, \tag{2.1.4}$$

$$H_2(u(t)) := \int_{\mathbb{R}^2} |\nabla u(t,x)|^2 + \tfrac{1}{4\pi}\left(e^{4\pi|u|^2} - 1 - 4\pi|u|^2 - 8\pi^2|u|^4\right)(t,x)\ dx. \tag{2.1.5}$$

Moreover, solutions to (2.1.1) and (2.1.2) both enjoy the conservation of mass

$$M(u(t)) := \int_{\mathbb{R}^2} |u(t,x)|^2\ dx. \tag{2.1.6}$$

When there is no chance of confusion, we will write H_1 for $H_1(u(t))$, H_2 for $H_2(u(t))$ and M for $M(u(t))$.

The study of (2.1.1) (resp. (2.1.2)) began in [CIM09], where it was shown that global solutions exist provided $H_1(u_0) \leq 1$ (resp. $H_2(u_0) \leq 1$). The different techniques, estimates, and difficulties involved in the study of (2.1.1) and (2.1.2) in the cases $H_i(u_0) < 1$, $H_i(u_0) = 1$, and $H_i(u_0) > 1$ prompted the authors to adopt the following trichotomy.

Definition 2.1.2. We say that (2.1.1) (resp. (2.1.2)) is *energy-subcritical* if $H_1(u_0) < 1$ (resp. $H_2(u_0) < 1$), *energy-critical* if $H_1(u_0) = 1$ (resp. $H_2(u_0) = 1$), and *energy-supercritical* if $H_1(u_0) > 1$ (resp. $H_2(u_0) > 1$).

Traditionally, the honorific "energy-critical" has been given to a family of semilinear NLS in $d \geq 3$, where an available scaling symmetry leaves invariant both the energy and class of solutions. In our case neither equation enjoys a scaling symmetry, and so some explanation is needed to justify in what sense we regard (2.1.1) and (2.1.2) as energy-critical. To properly explain this, we will begin with a brief review of the familiar energy-critical NLS in $d \geq 3$. Our goal in doing so is to understand the defining features of energy-criticality independent of scaling, and how these features manifest themselves in the theory of well-posedness. Once this is accomplished, we will draw analogies between the theory of (2.1.1) and (2.1.2) and the theory of the energy-critical NLS in dimension $d \geq 3$.

2.1.1 Energy-Critical NLS in $\mathbb{R}^d$, $d \geq 3$.

In dimension $d \geq 3$, consider the defocusing nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = |u|^p u, \qquad p > 0 \\ u(0,x) = u_0(x) \in \dot{H}^s_x(\mathbb{R}^d). \end{cases} \tag{2.1.7}$$

Solutions to (2.1.7) also enjoy the conservation of mass (2.1.6) and that of energy

$$H(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{p+2}|u(t,x)|^{p+2}\, dx.$$

The class of solutions to (2.1.7) is invariant under the scaling

$$u(t,x) \mapsto u_\lambda(t,x) := \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x). \tag{2.1.8}$$

The effect of the scaling on the initial data is given by

$$||u_\lambda(0)||_{\dot{H}^s_x(\mathbb{R}^d)} = \lambda^{s-(\frac{d}{2}-\frac{2}{p})}||u(0)||_{\dot{H}^s_x(\mathbb{R}^d)}.$$

Thus, when $s = \frac{d}{2} - \frac{2}{p}$, the scaling (2.1.8) leaves invariant both the class of solutions and the size of the initial data.

Definition 2.1.3. Consider the initial value problem (2.1.7). Let

$$s_c = \tfrac{d}{2} - \tfrac{2}{p}.$$

We say the problem is *critical* when $s = s_c$, *subcritical* when $s > s_c$, and *supercritical* when $s < s_c$.

When $s = s_c = 1$, the energy $H(u(t))$ is left invariant by the scaling (2.1.8), giving rise to the *energy-critical* NLS

$$\begin{cases} iu_t + \Delta u = |u|^{\frac{4}{d-2}} u, \qquad d \geq 3 \\ u(0,x) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^d). \end{cases} \tag{2.1.9}$$

When $s = 1$ and $p < \frac{4}{d-2}$ (i.e. $s_c < 1$), we say that (2.1.7) is *energy-subcritical.*

In the last two decades the energy-critical NLS (2.1.9) has been the subject of a relentless, inspiring, and successful campaign to understand the local and global behavior of its solutions. Local well-posedness was first proved by Cazenave and Weissler in [CW90], who showed that the length of a solution's lifespan depends on the profile of u_0 rather than the norm of u_0. The first victory on the front of large-data global well-posedness was made by J. Bourgain in [Bou99], who proved global well-posedness and exhibited global spacetime bounds for spherically symmetric initial data in $d = 3, 4$ by introducing what is now known as the 'induction on energy' paradigm. Using this paradigm, and introducing a wealth of new ideas and techniques, the authors of [CKS08] managed to remove Bourgain's assumption of spherically symmetric data in $d = 3$. Adapting these techniques to handle newfound difficulties in high dimensions, the problem was completely resolved by the authors of [RV07] ($d = 4$) and [Vis07] ($d \geq 5$).

Though the notion of energy-criticality in $d \geq 3$ is defined through an available scaling symmetry, it is important to understand the characteristic features of the energy-subcritical and energy-critical nonlinearities independent of this symmetry.

As (2.1.8) reveals, energy-criticality is determined by the response of the $\dot{H}^1_x$ norm at fine scales or, equivalently, high-frequencies. In the energy-critical case, the kinetic and potential energy norms are equally strong at all scales. In the energy-subcritical case, the kinetic energy norm dominates the potential energy norm at fine scales. This phenomenon is expressed concisely in the Sobolev inequality, which in $d \geq 3$ reads

$$||f||_{L_x^{\frac{2d}{d-2}}} \leq C_d ||\nabla f||_{L^2_x}, \tag{2.1.10}$$

$$\sup_{||u||_{H^1_x} \leq 1} ||u||_{L^p_x} \leq C(p, d) \qquad\qquad 2 \leq p \leq \frac{2d}{d-2}. \tag{2.1.11}$$

Heuristically, Sobolev embedding informs us of how strong our nonlinearity may be, i.e. how large p may be, before the potential energy norm overpowers the kinetic energy norm.

These features present themselves in the well-posedness theory in a few ways. In the energy-subcritical setting the time of existence guaranteed in the local theory depends only on the size of the initial data; in the energy-critical setting, the time of existence depends on its profile. The local theory may be iterated to extend the lifespan of a local solution provided there is no energy concentration. A computation shows that

$$H(u_\lambda) = \lambda^{2(1-s_c)} H(u) \to \infty,$$

which converges to ∞ as $\lambda \to \infty$ in the energy-subcritical case $s_c < 1$. Thus, energy conservation discourages concentration in this case. In the energy-critical setting $s_c = 1$, however, the conservation of energy does not immediately rule out the possibility of concentration.

2.1.2 Energy-Critical NLS in $\mathbb{R}^2$

In $\mathbb{R}^2$, Sobolev embedding guarantees that every polynomial nonlinearity is energy-subcritical. Indeed, if $u \in H_x^1$ is localized to frequency $\sim N$ and $||u||_{H_x^1(\mathbb{R}^2)} \leq 1$ (say), then Bernstein's inequalities show that

$$||u||_{L_x^p}^p \lesssim_p N^{-2} ||\nabla u||_{L_x^2}^2.$$

At fine scales ($N \gg 1$), we see that the kinetic energy dominates the potential energy. Thus, if we are to identify an energy-critical nonlinearity, it is natural to consider an exponential nonlinearity.

In [CIM09] and [IMM12], the authors identified the nonlinearities $F_1(u)$ and $F_2(u)$ in (2.1.1) and (2.1.2) as energy-critical when $H_1(u_0) = 1$ and $H_2(u_0) = 1$, respectively, using a substitute for the end-point Sobolev inequality (2.1.10) known as the Moser–Trudinger inequality. We will discuss the Moser–Trudinger inequality and its many variants in detail below. In analogy to the end-point Sobolev embedding (2.1.11) in $d \geq 3$, the Moser–Trudinger inequality informs us of the exact speed at which a nonlinearity may grow before the potential energy term overpowers the kinetic energy norm. Unlike the end-point Sobolev embedding however, the Moser–Trudinger inequality holds only for functions with sufficiently small $\dot{H}_x^1$ norm. Thus, in the case $d = 2$, we fix one nonlinearity $F_1(u)$ or $F_2(u)$ and classify energy-criticality depending on the size of initial energy.

In [CIM09], the authors established a complete trichotomy analogous to the energy-critical case in $d \geq 3$, corresponding to the size of the initial data's energy. In the terminology of Definition 2.1.2, the Cauchy problems (2.1.1) and (2.1.2) are globally well-posed in the energy-subcritical and energy-critical cases. In analogy to the $d \geq 3$ case, the lifespan of a local solution in the energy-subcritical cases of (2.1.1) and (2.1.2) depend on the size of the initial data, whereas the lifespan in the energy-critical cases depend fully on the profile of the initial data. Moreover, they demonstrate that (2.1.1) and (2.1.2) are ill-posed for a subset of initial data in the energy-supercritical case (though no ill-posedness results are known for slightly energy-supercritical data).

With global well-posedness established in the energy-critical and energy-subcritical cases, the next natural question to investigate was whether global solutions scatter in H_x^1. In [IMM12], the authors proved the existence of global spacetime bounds which imply scattering for global solutions to (2.1.2) in the energy-subcritical case. The key insight was to use the *a-priori* Morawetz estimate established independently by Colliander–Grillakis–Tzirakis in [CGT09] and Planchon–Vega in [PV09]. In [BIP14], the authors expanded on [IMM12], by proving the existence of global spacetime bounds which imply scattering for global solutions in the energy-critical case of (2.1.2) under the additional assumption of radial initial data.

The chief difficulty in establishing similar results for the corresponding cases of (2.1.1) stem from the poor decay of the cubic term in the Taylor expansion of $F_1(u)$:

$$F_1(u) = 4\pi u|u|^2 + 8\pi^2 u|u|^4 + \cdots .$$

Indeed, $F_1(u)$ can only decay at least as slow as the cubic nonlinearity $4\pi u|u|^2$ does. Thus, we may only expect global solutions to (2.1.1) to scatter in H_x^1 if we expect scattering in H_x^1 for global solutions to the the associated Cauchy problem

$$\begin{cases} iu_t + \Delta u = 4\pi |u|^2 u \\ u(0,x) = u_0(x) \in H_x^1(\mathbb{R}^2). \end{cases} \tag{2.1.12}$$

In $\mathbb{R}^2$, this corresponds to the mass-critical NLS, whose theory we review briefly below.

2.1.3 Mass-Critical NLS in $\mathbb{R}^2$

When $s = s_c = 0$ the mass $M(u(t))$ is left invariant by the scaling (2.1.8), giving rise to the *mass-critical* NLS. In two space dimensions, this takes the form:

$$\begin{cases} iu_t + \Delta u = g(u) := |u|^2 u \\ u(0,x) = u_0(x) \in L_x^2(\mathbb{R}^2). \end{cases} \tag{2.1.13}$$

The local theory for (2.1.13) was established by Cazenave and Weissler in [CW89]. Analogously to the local theory for the energy-critical case in $d \geq 3$, the lifespan of the local solutions they constructed depended on the profile of the initial data and not just the L_x^2-norm. Two decades later, in [KTV09], Killip–Tao–Visan showed that for radial initial data, (2.1.13) is globally well-posed and that solutions obey global spacetime bounds; in particular, scattering holds. Soon after, in [Dod16], Dodson removed the radiality assumption and established the theorem in its full generality.

Theorem 2.1.4 (Radial [KTV09], Non-Radial [Dod16]). *For $u_0 \in L_x^2$, there exists a unique strong solution $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ to* (2.1.13)*. Moreover, u satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^4 \, dx \, dt < C(||u_0||_{L_x^2})$$

and scatters both backwards and forwards in time.

Thus, when $u_0 \in L_x^2$ the global solution $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ scatters in L_x^2. Returning to (2.1.12), we would like to know, for H_x^1 data, whether the global solution guaranteed by Theorem 2.1.4 scatters in H_x^1. Luckily, this follows from a standard lemma (See, for example, Lemma 3.10 in [TVZ07]).

Lemma 2.1.5 (Persistence of Regularity). *Let $k = 0, 1$ and I be a compact time interval. Let v be the unique solution to* (2.1.13) *on $I \times \mathbb{R}^2$ with*

$$||v||_{L_t^4 L_x^4(I \times \mathbb{R}^2)} \leq L. \tag{2.1.14}$$

Then, if $t_0 \in I$ and $v(t_0) \in H_x^k$, we have

$$||v||_{S^k(I \times \mathbb{R}^2)} \leq C(L) ||v(t_0)||_{H_x^k}. \tag{2.1.15}$$

2.1.4 Main Results

In this paper we address the question of whether global solutions to (2.1.1) obey global spacetime bounds and scatter in H_x^1 in the energy-subcritical case. To do so, we exploit the insights of Tao-Visan-Zhang from [TVZ07].

In [TVZ07], the authors embark on a systematic study of Cauchy problems of the form

$$\begin{cases} iu_t + \Delta u = |u|^{p_1}u + |u|^{p_2}u \\ u(0,x) = u_0(x) \in H_x^1, \end{cases} \tag{2.1.16}$$

with $u : \mathbb{R}_t \times \mathbb{R}_x^d \to \mathbb{C}$ and $0 < p_1 < p_2 \leq \frac{4}{d-2}$. They show that for various values of p_1 and p_2, if (2.1.16) is globally well-posed then a solution to (2.1.16) can be viewed as a perturbation of a solution to

$$\begin{cases} iu_t + \Delta u = |u|^{p_1}u \\ u(0,x) = u_0(x) \in H_x^1, \end{cases} \tag{2.1.17}$$

with error $|u|^{p_2}u$. In these cases, they show, the solution to (2.1.16) may inherit global spacetime bounds, if they exist, for (2.1.17).

Our approach is similar. We write

$$F_1(u) = 4\pi|u|^2u + F_2(u),$$

and thus view (2.1.1) as a perturbation of the mass-critical NLS (2.1.12) with error $F_2(u)$. We exploit the estimates from [IMM12] in the energy-subcritical case of (2.1.2) to show that the error term $F_2(u)$ is mild enough to derive global spacetime bounds from those enjoyed by global solutions of the mass-critical problem. These results are summarized in the following theorem.

Theorem 2.1.6. *For $u_0 \in H_x^1$ with $H_1(u_0) \leq 1$, there exists a unique strong solution $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ to (2.1.1). If $H_1(u_0) < 1$, then u satisfies*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^2} |u(t,x)|^4 + |\nabla u(t,x)|^4 \; dx \; dt < C(||u_0||_{H_x^1})$$

and scatters both backwards and forwards in time.

The merit of our approach is that it provides a framework to treat a doubly critical nonlinearity. Our techniques provide a method to show that estimates proven for solutions to (2.1.2), whose nonlinearity enjoys better spacetime estimates, may be inherited for solutions to (2.1.1).

To establish spacetime bounds on global solutions to (2.1.1) or (2.1.2), the Strichartz inequality (see Section 2.2) informs us that we need only investigate which spacetime bounds are available to estimate the nonlinearity. This requires an inquiry into which $1 \leq p, q \leq \infty$ do we have control over

$$||F_i(u)||_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^2)}, \tag{2.1.18}$$

for $i \in \{1, 2\}$. Indeed, the analysis in [BIP14],[CIM09], and [IMM12] relied crucially on estimating terms like (2.1.18). Our second main result describes a wide range of exponents for which one may control (2.1.18).

Theorem 2.1.7 (Moser–Trudinger–Strichartz)**.** *Let $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ satisfy*

$$||u(t)||_{\dot{H}_x^1(\mathbb{R}^2)} \leq 1 \qquad \textit{and} \qquad ||u(t)||^2_{L_x^2(\mathbb{R}^2)} = M, \tag{2.1.19}$$

for every $t \in \mathbb{R}$. If $s \geq 4$ and $1 \leq p, q \leq \infty$ are such that

$$\frac{q'}{p} > 1, \tag{2.1.20}$$

then

$$\left|\left||u|^s e^{4\pi|u|^2}\right|\right|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^2)} \lesssim_M ||u||^{\frac{4}{p}}_{L_t^4 H_x^{1,4}(\mathbb{R}\times\mathbb{R}^2)}. \tag{2.1.21}$$

Conversely, if (2.1.21) *holds for all u satisfying* (2.1.19)*, then q and p must satisfy* (2.1.20).

It is important to understand that Theorem 2.1.7 holds for arbitrary spacetime functions satisfying (2.1.19). Though every solution to (2.1.1) and (2.1.2) satisfies (2.1.19) in the energy-subcritical and energy-critical cases, it is obvious that not every spacetime function satisfying (2.1.19) solves (2.1.1) or (2.1.2). Thus, improvements into the range of exponents may be made if we restrict ourselves to only consider solutions to (2.1.1) and (2.1.2). For example, in [BIP14], the authors demonstrate that one may improve (2.1.20) if $u(t)$ was sufficiently small in an Orlicz space for all times $t \in \mathbb{R}$. While many specific instances of Theorem 2.1.7 have been important in the study of 2.1.2 to date, to the author's knowledge, there has yet to be a systematic treatment of such inequalities. The importance of this approach is to see that several of the *ad-hoc* uses of space-time Moser-Trudinger-type inequalities are special instances of Theorem 2.1.7. It is our hope that this treatment simplifies these arguments moving forward.

The endpoint in (2.1.2) represents a frustrating obstacle in extending Theorem 2.1.6 to the energy-critical case of (2.1.1). Indeed, even assuming conditionally the existence of global spacetime bounds in (2.1.2), we are unable to extend the results to similar ones for (2.1.1). This obstacle arises in the perturbation theory, when one has to estimate a dual Strichartz norm of the form

$$||\nabla F_2(u)||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}},$$

for $0 < \delta \leq \frac{1}{2}$. At times when $||u(t)||_{L_x^\infty(\mathbb{R}^2)}$ is large, this term behaves like

$$||\nabla u|u|^4 e^{4\pi|u|^2}||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}}.$$

To close our perturbation argument, we may only estimate ∇u in $L_t^\infty L_x^2$ or $L_t^4 L_x^4$. By interpolation and Hölder this requires one to estimate the term $|u|^4 e^{4\pi|u|^2}$ in a space $L_t^p L_x^q$ with $\frac{q'}{p} = 1$, the unavailable end-point case of Theorem 2.1.7.

The investigation into which exponents in (2.1.20) are permissible for solutions to (2.1.1) or (2.1.2) is an important line of future inquiry.

2.2 Preliminaries

We begin by fixing some notation. We will write $X \lesssim Y$ if there exists a constant C so that $X \leq CY$. When we wish to stress the dependence of this implicit constant on a parameter ε (say), so that $C = C(\varepsilon)$, we write $X \lesssim_\varepsilon Y$. We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. If there exists a *small* constant c for which $X \leq cY$ we will write $X \ll Y$.

For $1 \leq r < \infty$ we recall the Lebesgue space $L_x^r(\mathbb{R}^2)$, which is the completion of smooth compactly supported functions $f:\mathbb{R}^2\to\mathbb{C}$ under the norm

$$||f||_r = ||f||_{L^r_x(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^2} |f(x)|^r\ dx\right)^{\frac{1}{r}}.$$

When $r=\infty$, we employ the essential supremum norm. For $f:\mathbb{R}\times\mathbb{R}^2\to\mathbb{C}$ we use $L^p_tL^q_x$ to denote the spacetime norm

$$||f||_{p,q} = ||f||_{L^p_tL^q_x(\mathbb{R}\times\mathbb{R}^2)} = \left(\int_{\mathbb{R}}\left(\int_{\mathbb{R}^2} |f(t,x)|^q\ dx\right)^{\frac{p}{q}}\ dt\right)^{\frac{1}{p}}$$

with the natural modifications when either p or q is infinity, or when $\mathbb{R}\times\mathbb{R}^2$ is replaced by some other spacetime region. In particular, on the spacetime slab $[-T,T]\times\mathbb{R}^2$ we write

$$||f||_{L^p_tL^q_x([-T,T]\times\mathbb{R}^2)} = ||f||_{L^p_TL^q_x}.$$

When $q=p$ we write $L^p_{t,x}=L^p_tL^p_x$.

Our convention for the Fourier transform on $\mathbb{R}^2$ is

$$\hat{f}(\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\ dx.$$

The Fourier transform allows us to define the fractional differentiation operators

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s\hat{f}(\xi) \qquad \text{and} \qquad \widehat{\langle\nabla\rangle^s f}(\xi) = (1+|\xi|^2)^{\frac{s}{2}}\hat{f}(\xi).$$

The fractional differentiation operators give rise to the (in-)homogenous Sobolev spaces. We define $\dot{H}^{1,r}(\mathbb{R}^2)$ and $H^{1,r}(\mathbb{R}^2)$ to be the completion of smooth compactly supported functions $f:\mathbb{R}^2\to\mathbb{C}$ under the norms

$$||f||_{\dot{H}^{1,r}(\mathbb{R}^2)} = |||\nabla|f||_{L^r_x} \qquad \text{and} \qquad ||f||_{H^{1,r}(\mathbb{R}^2)} = |||\langle\nabla\rangle f||_{L^r_x}.$$

When $r=2$, we simply write $H^{1,r}=H^1$ and $\dot{H}^{1,r}=\dot{H}^1$.

Throughout this paper, we will often need to dampen the mass term in the Sobolev norm and so for $0<\mu\le 1$ we define

$$||u||^2_{H^1_\mu} = \mu||u||^2_{L^2_x} + ||\nabla u||^2_{L^2_x}.$$

In two dimensions, we say that a pair of exponents (q, r) is Schrödinger-*admissible* if $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$ and $2 \leq q, r \leq \infty$ but $(q, r) \neq (2, \infty)$. We say that a pair of exponents (q, r) is *dual* Schrödinger-admissible if their Hölder conjugates (q', r') are Schrödinger-admissible. It is straightforward to show that (q, r) is dual Schrödinger-admissible if and only if $\frac{1}{q} + \frac{1}{r} = \frac{3}{2}$ and $(q, r) \neq (2, 1)$. If $I \times \mathbb{R}^2$ is a spacetime slab, we define the $S^0(I \times \mathbb{R}^2)$ *Strichartz norm* by

$$||u||_{S^0(I\times\mathbb{R}^2)} := \sup_{(q,r) \text{ admissible}} ||u||_{L_t^q L_x^r(I\times\mathbb{R}^2)}.$$

The attentive reader will notice that since we are in two dimensions, we need to restrict the supremum to a closed subset of admissible pairs (as to avoid the inadmissible endpoint). Every argument in this paper requires finitely many admissible pairs, so this caveat matters little to us. Similarly, we define the $S^1(I \times \mathbb{R}^2)$ and $S(I \times \mathbb{R}^2)$ norms to be

$$||u||_{S^1(I\times\mathbb{R}^2)} := ||\nabla u||_{S^0(I\times\mathbb{R}^2)} \qquad \text{and} \qquad ||u||_{S(I\times\mathbb{R}^2)} := ||\langle\nabla\rangle u||_{S^0(I\times\mathbb{R}^2)}.$$

We also use $N^0(I \times \mathbb{R}^2)$ to denote the dual space of $S^0(I \times \mathbb{R}^2)$ and

$$N^1(I \times \mathbb{R}^2) := \{u;\, \nabla u \in N^0(I \times \mathbb{R}^2)\}.$$

As before, we define the $N(I \times \mathbb{R}^2)$ norm to be

$$||u||_{N(I\times\mathbb{R}^2)} = ||\langle\nabla\rangle u||_{N^0(I\times\mathbb{R}^2)}.$$

Lemma 2.2.1 (Strichartz Estimates, [KT98])**.** *Let I be a compact time interval, $k \in \{0, 1\}$, and let $u : I \times \mathbb{R}^2 \to \mathbb{C}$ be a solution to the Schrödinger equation*

$$iu_t + \Delta u = F,$$

for a function F. Then

$$|||\nabla|^k u||_{S^0(I\times\mathbb{R}^2)} \lesssim ||u(t_0)||_{\dot{H}^k(\mathbb{R}^2)} + |||\nabla|^k F||_{N^0(I\times\mathbb{R}^2)}.$$

2.2.1 Pointwise Estimates

In this subsection, we record pointwise estimates needed for the well-posedness and perturbation theory in Sections 4 and 5, respectively. We assume that $F : \mathbb{C} \to \mathbb{C}$ is continuously differentiable, and use ∂_z and $\partial_{\bar{z}}$ to denote the usual Wirtinger derivatives

$$\partial_z F = F_z := \tfrac{1}{2}\left(\tfrac{\partial F}{\partial x} - i\tfrac{\partial F}{\partial y}\right) \qquad \text{and} \qquad \partial_{\bar{z}} F = F_{\bar{z}} := \tfrac{1}{2}\left(\tfrac{\partial F}{\partial x} + i\tfrac{\partial F}{\partial y}\right).$$

The Fundamental Theorem of Calculus permits us to write

$$F(z) - F(w) = \int_0^1 (z-w)F_z(w + \theta(z-w)) + \overline{(z-w)}F_{\bar{z}}(w+\theta(z-w))\, d\theta. \tag{2.2.1}$$

We may use (2.2.1) to bound

$$|F(z) - F(w)| \lesssim |z-w|(|F_z(z)| + |F_z(w)| + |F_{\bar{z}}(z)| + |F_{\bar{z}}(w)|). \tag{2.2.2}$$

Of course, if F is continuously twice differentiable we may obtain (2.2.2) for F_z and $F_{\bar{z}}$ in place of F.

Recall our notation:

$$F_1(z) = z(e^{4\pi|z|^2} - 1), \qquad F_2(z) = z(e^{4\pi|z|^2} - 4\pi|z|^2 - 1), \qquad g(z) = z|z|^2.$$

Lemma 2.2.2. *If $z_1, z_2 \in \mathbb{C}$, then*

$$|g(z_1) - g(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} |z_j|^2 \tag{2.2.3}$$

$$|\partial_z g(z_1) - \partial_z g(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} |z_j| \tag{2.2.4}$$

$$|F_1(z_1) - F_1(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} (|z_j|^2 e^{4\pi|z_j|^2}) \tag{2.2.5}$$

$$|F_2(z_1) - F_2(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} |z_j|^2(e^{4\pi|z_j|^2} - 1) \tag{2.2.6}$$

$$|\partial_z F_1(z_1) - \partial_z F_1(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} (|z_j| + |z_j|^3)e^{4\pi|z_j|^2} \tag{2.2.7}$$

$$|\partial_z F_2(z_1) - \partial_z F_2(z_2)| \lesssim |z_1 - z_2| \sum_{j=1,2} (|z_j| + |z_j|^3)e^{4\pi|z_j|^2}. \tag{2.2.8}$$

Finally, for $i \in \{1,2\}$ and any $\varepsilon > 0$ we have

$$|F_i(z_1) - F_i(z_2)| \lesssim_\varepsilon |z_1 - z_2| \sum_{j=1,2} \left(e^{4\pi(1+\varepsilon)|z_j|^2} - 1\right) \tag{2.2.9}$$

$$|\partial_z F_i(z_1) - \partial_z F_i(z_2)| \lesssim_\varepsilon |z_1 - z_2| \sum_{j=1,2} \left(|z_j| + e^{4\pi(1+\varepsilon)|z_j|^2} - 1\right). \tag{2.2.10}$$

Moreover, (2.2.7), (2.2.8), *and* (2.2.10) *hold with $\partial_{\bar{z}}$ in place of ∂_z.*

2.2.2 Endpoint Sobolev Inequalities

In two space dimensions, Sobolev embedding guarantees that for each $2 \leq p < \infty$ or $2 < r \leq \infty$ there exists some $c_r, c_p > 0$ so that

$$||u||_{L^p_x(\mathbb{R}^2)} \leq c_p ||u||_{H^1_x(\mathbb{R}^2)}. \tag{2.2.11}$$

$$||u||_{L^\infty_x(\mathbb{R}^2)} \leq c_r ||u||_{H^{1,r}_x(\mathbb{R}^2)}. \tag{2.2.12}$$

The failure of (2.2.11) to hold at the endpoint $p = \infty$ and of (2.2.12) to hold with $r = 2$ has motivated mathematicians over the last half century to find appropriate substitutes. This has lead to two lines of inquiry, essentially dual to one another, and with essentially dual answers. The first line of inquiry attempts to discover for which functions $f : \mathbb{R} \to \mathbb{R}$ we have

$$\sup_{||u||_{H^1_x} \leq 1} \int_{\mathbb{R}^2} f(|u|)\ dx < \infty. \tag{2.2.13}$$

The second line of inquiry attempts to discover for which functions $g : \mathbb{R} \to \mathbb{R}$ we have

$$||u||_{L^\infty_x(\mathbb{R}^2)} \leq g(||u||_{H^{1,r}_x(\mathbb{R}^2)}), \tag{2.2.14}$$

for $r > 2$ and $u \in H^1_x$.

The first line of investigation lead to the family of Moser–Trudinger inequalities which show that (2.2.13) holds for a variety of functions of square exponential growth. The second line of investigation lead to the family of Brézis–Wainger–Gallouët inequalities, which show that (2.2.14) holds for a variety of functions of $\sqrt{\log}$ growth.

2.2.2.1 Moser–Trudinger Inequalities

In [Tru67], Trudinger observed that one could take $c_p = c \cdot \sqrt{p}$ in (2.2.11) for some constant c independent of p. By expanding into power series, he found that

$$||e^{\alpha|u|^2} - 1||_{L^1_x} = \sum_{k=1}^{\infty} \frac{\alpha^k ||u||_{2k}^{2k}}{k!} \leq \sum_{k=1}^{\infty} \frac{(2kc^2\alpha)^k}{k!} < \infty$$

for $0 < \alpha \ll 1$ sufficiently small. A few years later, in [Mos71], Moser applied symmetrization techniques and found that the constant $\alpha = 4\pi$ was optimal. Since then, mathematicians have discovered a wide variety of Moser–Trudinger(-type) inequalities, which are indispensable in our analysis. In this paper we rely on a few, which are listed below.

In the energy-subcritical setting, we see that the exponential nonlinearity behaves like the first nonzero term in its Taylor approximation.

Proposition ([AT00]). *For each $\alpha \in [0, 4\pi)$ there exists $c = c(\alpha)$ so that*

$$||e^{\alpha|u|^2} - 1||_{L^1_x(\mathbb{R}^2)} \leq c||u||^2_{L^2_x(\mathbb{R}^2)}, \tag{2.2.15}$$

uniformly for $u \in H^1_x(\mathbb{R}^2)$ with $||\nabla u||_{L^2_x(\mathbb{R}^2)} \leq 1$.

Proposition ([BIP14]). *For each $\alpha \in [0, 4\pi)$ and $s \geq 1$, there exists a constant $c = c(\alpha, s)$ so that*

$$|||u|^s e^{\alpha|u|^2}||_{L^1_x(\mathbb{R}^2)} \leq c||u||^s_{L^s_x}, \tag{2.2.16}$$

uniformly for $u \in H^1_x(\mathbb{R}^2)$ with $||\nabla u||_{L^2_x(\mathbb{R}^2)} \leq 1$.

In the energy-critical setting, the previous proposition fails if we only require control over the $\dot{H}^1$ norm. However, if we require the full H^1_x norm to be sufficiently small, we recover a useful substitute.

Proposition ([Ruf05]). *There exists a constant c so that*

$$||e^{4\pi|u|^2} - 1||_{L^1_x(\mathbb{R}^2)} \leq c, \tag{2.2.17}$$

uniformly for $u \in H^1_x(\mathbb{R}^2)$ with $||u||_{H^1_x(\mathbb{R}^2)} \leq 1$.

A natural question arises from the previous concession, namely: what is the best bound we may obtain by requiring only that the $\dot{H}^1$ norm be sufficiently small. This is answered in the following proposition.

Proposition ([IMN15]). *There exists a constant c so that*

$$\int_{\mathbb{R}^2} \frac{e^{4\pi|u|^2} - 1}{(1 + |u|)^2}\, dx \leq c||u||^2_{L^2_x}, \tag{2.2.18}$$

uniformly for $u \in H^1_x(\mathbb{R}^2)$ with $||\nabla u||_{L^2_x(\mathbb{R}^2)} \leq 1$.

2.2.2.2 Brézis–Wainger–Gallouët Inequalities

In our analysis we will see that the most difficult part in controlling the nonlinearities in (2.1.1) and (2.1.2) are those times when $u(t)$ is large in L_x^∞. We have two estimates in our arsenal to control the L_x^∞ norm, the latter of which is more powerful. The first is the Morrey Embedding (2.2.12), and the second is the sharp Brézis–Wainger–Gallouët inequality stated in Proposition 2.2.2.2. Though both will prove useful, the strength of Proposition 2.2.2.2 over (2.2.12) comes from our knowledge of the explicit constant.

Proposition (Sharp Brézis–Wainger–Gallouët, [Bir10, IMM07])**.** *Let* $2 < r < \infty$ *and* $0 < \mu \leq 1$. *Then for any* $\alpha > 1$ *there exists a constant* C_α *depending on* α *so that*

$$||u||_{L_x^\infty}^2 \leq \frac{\alpha}{4\pi}\frac{2r}{r-2}||u||_{H_\mu^1}^2 \log\left[C_\alpha + \frac{c_r(8/\mu)^{1-\frac{2}{r}}||u||_{H_x^{1,r}}}{||u||_{H_\mu^1}}\right], \tag{2.2.19}$$

where c_r *is the constant appearing in* (2.2.12)*.*

2.2.3 Morawetz Estimates

The Morawetz estimate, proved independently by Colliander–Grillakis–Tzirakis and Planchon–Vega, is essential to our analysis by providing an *a-priori* estimate to start from and build upon.

Lemma 2.2.3. *[CGT09, PV09] If* $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ *is a global solution of* (2.1.1)*,* (2.1.2)*, or* (2.1.13) *then*

$$||u||_{L_t^4 L_x^8(\mathbb{R}\times\mathbb{R}^2)} \lesssim ||u||_{L_t^\infty L_x^2(\mathbb{R}\times\mathbb{R}^2)}^{\frac{3}{4}}||\nabla u||_{L_x^\infty L_x^2(\mathbb{R}\times\mathbb{R}^2)}^{\frac{1}{4}} \lesssim ||u||_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^2)}.$$

2.3 Moser–Trudinger–Strichartz Inequalities

The Moser–Trudinger and Brézis–Wainger–Gallouët inequalities are invaluable when proving L_x^q-estimates involving the exponential nonlinearities appearing in (2.1.1) and (2.1.2). Indeed, one can (and should) regard the Moser–Trudinger inequality as an L_x^1 estimate on the nonlinearity and the Brézis–Wainger–Gallouët inequality as an L_x^∞ estimate, permitting one to interpolate in between. This technique has been successfully employed in [CIM09] and [IMM12].

In this section, we systematically study which *spacetime* bounds are available in this setting. Specifically, we prove a general family of spacetime bounds including, to the author's knowledge, the first that hold generally in the energy-critical case. Several instances of these estimates are implicit in the literature (see [BIP14],[CIM09],[IMM12]). We contend that this formulation will streamline the standard proofs of well-posedness and scattering in the energy-subcritical cases; see Sections 4 and 5.

2.3.1 Global Moser–Trudinger–Strichartz

The lack of an *a priori* L_x^∞ bound on solutions to (2.1.1) and (2.1.2) present a frustrating but manageable challenge in proving spacetime estimates. We can overcome this challenge by splitting the set of times into two parts. If $t \in \mathbb{R}$ satisfies $||u(t)||_{L_x^\infty} \leq K$ (say), then we use the trivial bound

$$|u(t,x)(e^{4\pi|u(t,x)|^2} - 4\pi|u(t,x)|^2 - 1)| \lesssim_K |u(t,x)|^5.$$

As we will see, this contribution is easily estimated in two dimensions. The enemy, then, in establishing good spacetime bounds are those times when the $||u(t)||_{L_x^\infty}$ norm is large. We begin with a trivial L_x^1 estimate, which, when combined with a hard-fought $L_t^p L_x^\infty$ estimate, will yield the full Moser–Trudinger–Strichartz estimate.

Lemma 2.3.1 (An L_x^1 estimate)**.** *If* $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ *satisfies* $||u||_{L_t^\infty \dot{H}_x^1} \leq 1$*, then*

$$||e^{4\pi|u(t)|^2} - 1||_{L_x^1(\mathbb{R}^2)} \lesssim ||u(t)||^2_{L_x^2(\mathbb{R}^2)}(1 + ||u(t)||^2_{L_x^\infty}).$$

Proof. As u satisfies the hypotheses of Proposition 2.2.2.1, we know that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi|u(t,x)|^2} - 1}{(1 + |u(t,x)|)^2}\, dx \lesssim ||u(t)||^2_{L_x^2}.$$

Thus, we see that

$$\int_{\mathbb{R}^2} e^{4\pi|u(t,x)|^2} - 1\, dx \lesssim ||u(t)||^2_{L_x^2(\mathbb{R}^2)}(1 + ||u(t)||_{L_x^\infty})^2 \lesssim ||u(t)||^2_{L_x^2(\mathbb{R}^2)}(1 + ||u(t)||^2_{L_x^\infty}),$$

as desired. $\square$

Although the proof of Lemma 2.3.1 is elementary, it further illustrates our point that the nonlinearity is well controlled at times when $||u(t)||_{L_x^\infty}$ is small. The next lemma will demonstrate how to handle those times when $||u(t)||_{L_x^\infty}$ is large.

Lemma 2.3.2 (An $L_t^p L_x^\infty$ estimate)**.** *Let $\alpha > 0$ and suppose $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is such that*

$$||u||_{L_t^\infty \dot{H}_x^1} \leq 1 \qquad \textit{and} \qquad ||u(t)||^2_{L_x^2(\mathbb{R}^2)} = M$$

for every $t \in \mathbb{R}$. Then for all $1 \leq p < \infty$ with

$$||u||^2_{L_t^\infty \dot{H}_x^1} < \frac{1}{p\alpha}$$

there exists some constant $K = K(M, \alpha, p) > 0$ so that

$$||e^{4\pi\alpha|u|^2}||_{L_t^p L_x^\infty(I \times \mathbb{R}^2)} \lesssim ||u||^{\frac{4}{p}}_{L_t^4 H_x^{1,4}}, \tag{2.3.1}$$

where $I = \{t \in \mathbb{R} : ||u(t)||_{L_x^\infty(\mathbb{R}^2)} > K\}$.

Proof. We begin by defining a few relevant parameters, which will help us in interpolating later in the proof and avoiding an unnecessary discussion of the dependence of some constants on others.

We first choose $0 < \mu < 1$ sufficiently small so that

$$\frac{1}{p\alpha} > ||u||^2_{L_t^\infty H_\mu^1}. \tag{2.3.2}$$

We then choose an increasing continuous function $f : (2, \infty) \to (1, \infty)$ so that

$$\lim_{r \to 2^+} f(r) = 1, \qquad \lim_{r \to \infty} f(r) = \infty, \qquad \text{and} \tag{2.3.3}$$

$$\sup_{r \in (2,\infty)} \frac{r-2}{2r} \cdot \frac{4}{p\alpha} \cdot \frac{1}{f(r)} \cdot \frac{1}{||u||^2_{L_t^\infty H_\mu^1}} < 1. \tag{2.3.4}$$

Now, for $2 < r < \infty$, define

$$\lambda(r) := \tfrac{f(r)}{4\pi} \cdot \tfrac{2r}{r-2} \qquad \text{and} \tag{2.3.5}$$

$$\theta(r) := \tfrac{1}{p\alpha} \cdot \tfrac{4}{4\pi\lambda(r)||u||^2_{L_t^\infty H_\mu^1}} = \tfrac{r-2}{2r} \cdot \tfrac{4}{p\alpha} \cdot \tfrac{1}{f(r)} \cdot \tfrac{1}{||u||^2_{L_t^\infty H_\mu^1}}. \tag{2.3.6}$$

By (2.3.4), we know $\lambda(r) > \frac{1}{\pi}$ and $0 < \theta(r) < 1$ uniformly in $(2, \infty)$.

Consider the continuous function $\phi : (2, \infty) \to \mathbb{R}$ given by

$$\phi(r) = \tfrac{1}{r} - \left[\tfrac{\theta(r)}{4} + \tfrac{1-\theta(r)}{2}\right] = \tfrac{r-2}{2r}\left[\tfrac{1}{p\alpha} \cdot \tfrac{1}{f(r)} \cdot \tfrac{1}{||u||^2_{L_t^\infty H_\mu^1}} - 1\right].$$

By (2.3.2) and (2.3.3),

$$\begin{aligned}
&\lim_{r\to 2^+} \frac{1}{p\alpha} \cdot \frac{1}{f(r)} \cdot \frac{1}{||u||^2_{L_t^\infty H_\mu^1}} - 1 = \frac{1}{p\alpha ||u||^2_{L_t^\infty H_\mu^1}} - 1 > 0 \quad \text{and} \\
&\lim_{r\to \infty} \frac{1}{p\alpha} \cdot \frac{1}{f(r)} \cdot \frac{1}{||u||^2_{L_t^\infty H_\mu^1}} - 1 = -1 < 0.
\end{aligned}$$

Since $\frac{r-2}{2r} > 0$ for $r > 2$, it follows by the intermediate value theorem that there exists some $r_0 \in (2, \infty)$ so that $\phi(r_0) = 0$. For this value of r_0, we have that

$$\frac{\theta(r_0)}{4} = \frac{\theta(r_0)}{4} + \frac{1-\theta(r_0)}{\infty} \qquad \text{and} \qquad \frac{1}{r_0} = \frac{\theta(r_0)}{4} + \frac{1-\theta(r_0)}{2}.$$

By interpolation we deduce that

$$||u||_{L_t^{\frac{4}{\theta(r_0)}} H_x^{1,r_0}} \le ||u||^{\theta(r_0)}_{L_t^4 H_x^{1,4}} ||u||^{1-\theta(r_0)}_{L_t^\infty H_x^1}. \tag{2.3.7}$$

Now, let

$$K = 2(1+M)C_{\lambda(r_0)} \qquad \text{and} \qquad I = \{t \in \mathbb{R} : ||u(t)||_{L_x^\infty(\mathbb{R}^2)} > K\},$$

where $C_{\lambda(r_0)}$ is the constant appearing in (2.2.19). Notice that if $t \in I$ we have

$$C_{\lambda(r_0)} ||u||^2_{L_t^\infty H_\mu^1} \le C_{\lambda(r_0)}(1+M) < \tfrac{1}{2}||u(t)||_{L_x^\infty} \le \tfrac{c_{r_0}}{2}||u(t)||_{H_x^{1,r_0}},$$

from which it follows that

$$C_{\lambda(r_0)} < \frac{c_{r_0}||u(t)||_{H_x^{1,r_0}}}{2||u||^2_{L_t^\infty H_\mu^1}}.$$

Thus

$$C_{\lambda(r_0)} + \frac{(8/\mu)^{1-\frac{2}{r_0}} c_{r_0} ||u(t)||_{H_x^{1,r_0}}}{||u||_{L_t^\infty H_\mu^1}} \lesssim_{M,\mu} ||u(t)||_{H_x^{1,r_0}}. \tag{2.3.8}$$

With our parameters now well understood, the proof is rather straightforward. By Proposition 2.2.2.2, (2.3.5), and (2.3.8),

$$\begin{aligned}
\exp[4\pi\alpha||u(t)||^2_{L^\infty_x(\mathbb{R}^2)}] &\leq \exp\left[4\pi\lambda(r_0)\alpha||u(t)||^2_{H^1_\mu}\log\left[C_{\lambda(r_0)} + \frac{(8/\mu)^{1-\frac{2}{r_0}}c_{r_0}||u(t)||_{H^{1,r_0}_x}}{||u(t)||_{H^1_\mu}}\right]\right] \\
&\leq \exp\left[4\pi\lambda(r_0)\alpha||u||^2_{L^\infty_t H^1_\mu}\log\left[C_{\lambda(r_0)} + \frac{(8/\mu)^{1-\frac{2}{r_0}}c_{r_0}||u(t)||_{H^{1,r_0}_x}}{||u||_{L^\infty_t H^1_\mu}}\right]\right] \\
&\lesssim_{M,\mu} ||u(t)||_{H^{1,r_0}_x}^{4\pi\alpha\lambda(r_0)||u||^2_{L^\infty_t H^1_\mu}} \\
&= ||u(t)||_{H^{1,r_0}_x}^{\frac{4}{p\theta(r_0)}}.
\end{aligned}$$

Note that the second inequality follows from the fact that if $a > 1$ and $b > 0$ then the function

$$x \mapsto x^2 \log(a + \tfrac{b}{x})$$

is an increasing function when $x > 0$. Integrating in time we obtain

$$|| \exp(4\pi\alpha|u|^2)||_{L^p_t L^\infty_x(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{M,\mu} ||u||^{\frac{4}{p\theta(r_0)}}_{L_t^{\frac{4}{\theta(r_0)}} H^{1,r_0}_x} \lesssim_{M,\mu} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x}$$

as desired. □

We divide the proof of Theorem 2.1.7 into two propositions.

Proposition. *Let $u : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{C}$ satisfy*

$$||u(t)||_{\dot{H}^1_x(\mathbb{R}^2)} \leq 1 \qquad \textit{and} \qquad ||u(t)||^2_{L^2_x(\mathbb{R}^2)} = M, \tag{2.3.9}$$

for every $t \in \mathbb{R}$. If $s \geq 4$ and $1 \leq p, q \leq \infty$ are such that

$$\frac{q'}{p} > 1, \tag{2.3.10}$$

then

$$|||u|^s e^{4\pi|u|^2}||_{L^p_t L^q_x(\mathbb{R}\times\mathbb{R}^2)} \lesssim_{M,q,p} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x(\mathbb{R}\times\mathbb{R}^2)}. \tag{2.3.11}$$

If $||u||_{L^\infty_t \dot{H}^1_x} < 1$, then we may replace condition (2.3.10) *with the condition*

$$\frac{q'}{p} \geq 1. \tag{2.3.12}$$

Proof. Let $K > 1$ be a large parameter whose value will be chosen later, and define

$$I_K := \{t \in \mathbb{R} : ||u(t)||_{L^\infty_x(\mathbb{R}^2)} > K\}.$$

If $t \notin I_K$, then since $s \geq 4$ we have that

$$|u(t,x)|^s e^{4\pi|u(t,x)|^2} \lesssim_K |u(t,x)|^4. \tag{2.3.13}$$

By Hölder and Sobolev Embedding it follows that

$$||u^4(t)||_{L^q_x} \leq ||u(t)||^{\frac{4}{q}}_{L^4_x} ||u(t)||^{\frac{4}{q'}}_{L^\infty_x} \lesssim ||u(t)||^{\frac{4}{q}}_{L^4_x} ||u(t)||^{\frac{4}{q'}}_{H^{1,4}_x}.$$

As $||u(t)||_{L^4_x} \leq ||u(t)||_{H^{1,4}_x}$ and $q' \geq p$ we see that

$$||u(t)||^{\frac{4}{q}}_{L^4_x} ||u(t)||^{\frac{4}{q'}}_{H^{1,4}_x} \leq ||u(t)||^{\frac{4}{p'}}_{L^4_x} ||u(t)||^{\frac{4}{q}-\frac{4}{p'}}_{H^{1,4}_x} ||u(t)||^{\frac{4}{q'}}_{H^{1,4}_x} = ||u(t)||^{\frac{4}{p'}}_{L^4_x} ||u(t)||^{\frac{4}{p}}_{H^{1,4}_x}.$$

Integrating in time, we obtain that

$$||u^4(t)||_{L^p_t L^q_x((\mathbb{R}\setminus I_K)\times\mathbb{R}^2)} \lesssim_K ||u||^{\frac{4}{p'}}_{L^\infty_t L^4_x} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x} \lesssim_{M,K} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x}. \tag{2.3.14}$$

In light of (2.3.13), we deduce that

$$|||u|^s e^{4\pi|u|^2}||_{L^p_t L^q_x((\mathbb{R}\setminus I_K)\times\mathbb{R}^2)} \lesssim_{M,K} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x}. \tag{2.3.15}$$

provided that $q' \geq p$.

We now turn to estimating u on I_K in the case that $q < \infty$. Let $\alpha \in (0,1)$ be close to 1. In light of Proposition 2.2.2.1 we have that for each $t \in I_K$ that

$$\begin{aligned}
|||u(t)|^s e^{4\pi|u(t)|^2}||^q_{L^q_x} &= \int_{\mathbb{R}^2} |u(t,x)|^{sq} e^{4\pi q|u(t,x)|^2}\, dx \\
&= e^{4\pi(q-\alpha)||u(t)||^2_{L^\infty_x(\mathbb{R}^2)}} \int_{\mathbb{R}^2} |u(t,x)|^{sq} e^{4\pi\alpha|u(t,x)|^2}\, dx \\
&\lesssim e^{4\pi(q-\alpha)||u(t)||^2_{L^\infty_x(\mathbb{R}^2)}} ||u(t)||^{sq}_{L^{sq}_x(\mathbb{R}^2)}.
\end{aligned}$$

Taking q^{th} roots, we arrive at

$$|||u|^s e^{4\pi|u|^2}||_{L^q_x(I_K\times\mathbb{R}^2)} \lesssim e^{4\pi\frac{q-\alpha}{q}||u(t)||^2_{L^\infty_x(\mathbb{R}^2)}} ||u||^s_{L^{sq}_x(\mathbb{R}^2)}.$$

Integrating in time and interpolating we deduce that

$$\begin{aligned}||\,|u|^s e^{4\pi|u|^2}||_{L^p_t L^q_x(I_K\times\mathbb{R}^2)} &\lesssim ||u||^s_{L^\infty_t L^{sq}_x(I_K\times\mathbb{R}^2)}||e^{4\pi\frac{q-\alpha}{q}|u|^2}||_{L^p_t L^\infty_x(I_K\times\mathbb{R}^2)}\\ &\lesssim_M ||e^{4\pi\frac{q-\alpha}{q}|u|^2}||_{L^p_t L^\infty_x(I_K\times\mathbb{R}^2)}.\end{aligned}$$

If $\frac{q'}{p} > ||u||^2_{L^\infty_t \dot{H}^1_x}$, then

$$\lim_{\alpha\to 1^-}\frac{1}{p\frac{q-\alpha}{q}} = \frac{q'}{p} > ||u||^2_{L^\infty_t\dot{H}^1_x(\mathbb{R}\times\mathbb{R}^2)},$$

and so we may apply Lemma 2.3.2 to obtain that for K sufficiently large,

$$||\,|u|^s e^{4\pi|u|^2}||_{L^p_t L^q_x(I_K\times\mathbb{R}^2)} \lesssim_{M,K} ||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x(\mathbb{R}\times\mathbb{R}^2)}. \tag{2.3.16}$$

Combining (2.3.15) and (2.3.16) we deduce that

$$||\,|u|^s e^{4\pi|u|^2}||_{L^p_t L^q_x}\lesssim_{M,K}||u||^{\frac{4}{p}}_{L^4_t H^{1,4}_x(\mathbb{R}\times\mathbb{R}^2)}$$

when $\frac{q'}{p}\geq 1$ and $\frac{q'}{p} > ||u(t)||^2_{L^\infty_t\dot{H}^1_x}$, as desired.

We now need to estimate u on I_K in the remaining case to consider when $q' = p = 1$ and $\|u\|_{L^\infty_t H^1_x(\mathbb{R}^2)} < 1$. In this case we see that for any $t\in I_K$ that

$$\begin{aligned}||\,|u(t)|^s e^{4\pi|u(t)|^2}||_{L^\infty_x(\mathbb{R}^2)} &\lesssim ||u(t)||^s_{L^\infty_x} e^{4\pi\|u(t)\|^2_{L^\infty_x}}\\ &\lesssim e^{4\pi(1+\varepsilon)||u(t)||^2_{L^\infty_x}},\end{aligned}$$

for every $\varepsilon > 0$. As $\|u\|_{L^\infty_t H^1_x} < 1$ we may choose ε sufficiently small so that $\|u\|_{L^\infty_t H^1_x} < \frac{1}{1+\varepsilon}$. Applying Lemma 2.3.2 with $\alpha = 1+\varepsilon$ we may find a $K>0$ so that

$$||e^{4\pi(1+\varepsilon)|u|^2}||_{L^1_t L^\infty_x(I_K\times\mathbb{R}^2} \lesssim ||u||^4_{L^4_t H^{1,4}_x},$$

as desired.

$\square$

A few remarks on the previous theorem are in order.

Remark 2.3.1. The condition that

$$\frac{q'}{p} > 1,$$

is equivalent to condition that

$$\tfrac{1}{p}+\tfrac{1}{q} > 1.$$

Remark 2.3.2. Let us pause to illustrate a slightly different technique in estimating the purely exponential nonlinearity

$$||e^{4\pi|u|^2} - 1||_{L_t^p L_x^q(I_K \times \mathbb{R}^2)}.$$

Rather than employ Proposition 2.2.2.1, we instead estimate using Lemma 2.3.1. Indeed, by interpolation and Lemma 2.3.1 we see that for any $\eta > 0$,

$$\begin{aligned} ||e^{4\pi|u|^2} - 1||_{L_x^q} &\le ||e^{4\pi|u|^2} - 1||_{L_x^1}^{\frac{1}{q}} ||e^{4\pi|u|^2} - 1||_{L_x^\infty}^{\frac{1}{q'}} \\ &\le C(M)(1 + ||u||_{L_x^\infty}^2)^{\frac{1}{q}} ||e^{4\pi|u|^2} - 1||_{L_x^\infty}^{\frac{1}{q'}} \\ &\le C(M,\eta)||e^{4\pi\frac{(1+\eta)}{q'}|u|^2}||_{L_x^\infty}. \end{aligned}$$

If $\frac{q'}{p} > ||u||_{L_t^\infty \dot{H}_x^1}^2$, we may choose η sufficiently small as to guarantee that

$$\frac{1}{\frac{(1+\eta)}{q'} \cdot p} > ||u||_{L_t^\infty \dot{H}_x^1}^2.$$

Applying Lemma 2.3.2 with $\alpha = \frac{1+\eta}{q'}$, there exists some $K > 0$ so that

$$||e^{4\pi|u|^2} - 1||_{L_t^p L_x^q(I_K \times \mathbb{R}^2)} \le C(M,\eta)||u||_{L_t^4 H_x^{1,4}}^{\frac{4}{p}}. \tag{2.3.17}$$

We observe that Theorem 2.1.7 fails at the end-point.

Proposition. *If $1 \le p, q \le \infty$ and $s > 0$ are such that*

$$|||u|^s e^{4\pi|u|^2}||_{L_t^p L_x^q(\mathbb{R} \times \mathbb{R}^2)} \lesssim ||u||_{L_t^4 H_x^{1,4}}^{\frac{4}{p}},$$

for every spacetime function $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ satisfying

$$||u(t)||_{\dot{H}_x^1(\mathbb{R}^2)} \le 1 \qquad \textit{and} \qquad ||u(t)||_{L_x^2(\mathbb{R}^2)}^2 = M, \tag{2.3.18}$$

for every $t \in \mathbb{R}$, then

$$\frac{q'}{p} > 1.$$

Proof. As in [IMM12], for $N \ge 3$ we consider the sequence of functions

$$v_N(t,x) = \sqrt{\frac{2\pi}{\log N}} \frac{1}{(2\pi)^2} \int_{1<|\xi|<e^{-1}N} |\xi|^{-2} e^{-it|\xi|^2 + i\xi \cdot x} \, d\xi.$$

Note that $v_N(t,x) = e^{it\Delta} v_N(0,x)$, where

$$v_N(0,x) = \sqrt{\frac{2\pi}{\log(N)}} \frac{1}{(2\pi)^2} \int_{1<|\xi|<e^{-1}N} |\xi|^{-2} e^{i\xi\cdot x}\, d\xi.$$

By conservation of mass and energy it follows that

$$||v_N||^2_{L^\infty_t L^2_x} = ||v_N(0)||^2_{L^2_x} < \frac{1}{2\log N}, \qquad \text{and} \tag{2.3.19}$$

$$||v_N||^2_{L^\infty_t \dot{H}^1_x} = ||\nabla v_N(0)||^2_{L^2_x} = \frac{\log(N)-1}{\log(N)}. \tag{2.3.20}$$

On the regions $T_N := \{(t,x) : t \sim \varepsilon N^{-2} \text{ and } |x| \sim \varepsilon N^{-1}\}$, we may estimate

$$\mathrm{Re}(v_N(t,x)) \geq \sqrt{\tfrac{\log(N)}{2\pi}} - \tfrac{1+\varepsilon^2}{\sqrt{2\pi \log N}} \text{ and } |v_N|^s e^{4\pi|v_N|^2} \gtrsim N^2 (\log(N))^{\frac{s}{2}},$$

when $\varepsilon > 0$ is sufficiently small and N is sufficiently large. But then we see that

$$|||v_N|^s e^{4\pi|v_N|^2}||_{L^p_t L^q_x} \geq |||v_N|^s e^{4\pi|v_N|^2}||_{L^p_t L^q_x(T_N)} \gtrsim N^{2(1-\frac{1}{p}-\frac{1}{q})} (\log N)^{\frac{s}{2}}.$$

By the Strichartz inequality, (2.3.19), and (2.3.20) we obtain that

$$(\log N)^{\frac{s}{2}} N^{2(1-\frac{1}{p}-\frac{1}{q})} \lesssim_M ||v_N||^{\frac{4}{p}}_{L^4_t H^{1,4}_x} \lesssim_M ||v_N(0)||^{\frac{4}{p}}_{H^1_x} \lesssim 1,$$

or, equivalently, that $\frac{1}{p} + \frac{1}{q} > 1$. □

Throughout the perturbative theory, we will need the following estimate. The presence of the Morawetz norm $L^4_t L^8_x$ in the statement means that we cannot solely rely on the Moser–Trudinger–Strichartz inequality, since this will only produce an estimate involving the $L^4_t H^{1,4}_x$ norm. The difference will be a superficial one as we merely repeat the proof of Moser–Trudinger–Strichartz with a timely Hölder's inequality.

Corollary 2.3.3 ([IMM12]). *For each $H \in (0,1)$ there exists $\delta = \delta(H) \in (0, 1/3)$ so that for any strong solution u of* (2.1.2) *on $I \times \mathbb{R}^2$ with $H_2(u) \leq H$ we have*

$$||F_2(u)||_{N(I)} \lesssim C(H,M) ||u||^{4\delta}_{L^4_t L^8_x(I\times\mathbb{R}^2)} ||u||^2_{L^4_t H^{1,4}_x(I\times\mathbb{R}^2)},$$

where $M = ||u(0)||^2_{L^2_x}$.

Proof. Let $0 < \delta < \frac{1}{3}$, $K \gg 1$, and, as before, $I_K = \{t : ||u(t)||_{L_x^\infty} > K\}$.

We note that

$$||\nabla|^k F_2(u)| \lesssim ||\nabla|^k u| \cdot |u|^2 \cdot |u|^2 e^{4\pi|u|^2},$$

for $k \in \{0, 1\}$.

We first estimate the nonlinearity when $t \notin I_K$. In this case, we have that

$$||\nabla|^k F_2(u(t,x))| \lesssim_K ||\nabla|^k u(t,x)||u(t,x)|^4.$$

So on the set $(\mathbb{R} \setminus I_K) \times \mathbb{R}^2$ we have

$$\begin{aligned} |||\nabla|^k F_2(u)||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}} &\lesssim_K |||\nabla|^k u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{2}{1-\delta}}} ||u^2||_{L_t^{\frac{1}{\delta}} L_x^{\frac{2}{\delta}}} ||u^2||_{L_t^{\frac{2}{1-\delta}} L_x^{\frac{2}{1-2\delta}}} \\ &\lesssim_K |||\nabla|^k u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{2}{1-\delta}}} ||u||^2_{L_t^{\frac{2}{\delta}} L_x^{\frac{4}{\delta}}} ||u^4||^{\frac{1}{2}}_{L_t^{\frac{1}{1-\delta}} L_x^{\frac{1}{1-2\delta}}}. \end{aligned}$$

By interpolation,

$$|||\nabla|^k u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{2}{1-\delta}}} \lesssim |||\nabla|^k u||^{1-2\delta}_{L_t^\infty L_x^2} |||\nabla|^k u||^{2\delta}_{L_t^4 L_x^4} \lesssim ||u||^{2\delta}_{L_t^4 H_x^{1,4}}. \tag{2.3.21}$$

We have seen from (2.3.14) in Proposition 2.3.1 that, since $(1-\delta) + (1-2\delta) > 1$,

$$||u^4||^{\frac{1}{2}}_{L_t^{\frac{1}{1-\delta}} L_x^{\frac{1}{1-2\delta}}} \lesssim (||u||^{4(1-\delta)}_{L_t^4 H_x^{1,4}})^{\frac{1}{2}} = ||u||^{2-2\delta}_{L_t^4 H_x^{1,4}}.$$

By the Gagliardo-Nirenberg interpolation inequality, we know that

$$||u(t)||_{L_x^{\frac{4}{\delta}}} \lesssim ||\nabla u(t)||^{1-2\delta}_{L_x^2} ||u(t)||^{2\delta}_{L_x^8}.$$

So, integrating in time, we see that

$$||u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{4}{\delta}}} \lesssim ||\nabla u||^{1-2\delta}_{L_t^\infty L_x^2} ||u||^{2\delta}_{L_t^4 L_x^8}. \tag{2.3.22}$$

Putting this together, we see that on $(\mathbb{R} \setminus I_K) \times \mathbb{R}^2$ that

$$|||\nabla|^k F_2(u)||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}} \lesssim ||u||^{4\delta}_{L_t^4 L_x^8} ||u||^2_{L_t^4 H_x^{1,4}},$$

as desired.

We now turn to estimate the nonlinearity when $t \in I_K$. By (2.3.17) in Remark 2.3.2, for each $1 \le p, q \le \infty$ with $\frac{q'}{p} > ||u||_{L_t^\infty \dot{H}_x^1}$ we may find a $K > 0$ so that

$$||e^{4\pi|u|^2} - 1||_{L_t^p L_x^q(I_K \times \mathbb{R}^2)} \lesssim ||u||_{L_t^4 H_x^{1,4}}^{\frac{4}{p}}.$$

So by Hölder we find that on the spacetime region $I_K \times \mathbb{R}^2$

$$|||\nabla|^k F_2(u)||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}} \lesssim |||\nabla|^k u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{2}{1-\delta}}} ||u||_{L_t^{\frac{2}{\delta}} L_x^{\frac{4}{\delta}}}^2 ||e^{4\pi|u|^2} - 1||_{L_t^{\frac{2}{1-\delta}} L_x^{\frac{2}{1-2\delta}}}.$$

As

$$\lim_{\delta \to 0} \frac{\left(\frac{2}{1-2\delta}\right)'}{\frac{2}{1-\delta}} = 1 > ||u||_{L_t^\infty \dot{H}_x^1},$$

for sufficiently small $\delta > 0$ we know that

$$||e^{4\pi|u|^2} - 1||_{L_t^{\frac{2}{1-\delta}} L_x^{\frac{2}{1-2\delta}}(I_K \times \mathbb{R}^2)} \lesssim ||u||_{L_t^4 H_x^{1,4}}^{2(1-\delta)}.$$

Estimating the other terms similarly, as in (2.3.22) and (2.3.21), and putting everything together, we see that

$$|||\nabla|^k F_2(u)||_{L_t^{\frac{2}{1+2\delta}} L_x^{\frac{1}{1-\delta}}} \lesssim C(H, M) ||u||_{L_t^4 L_x^8}^{4\delta} ||u||_{L_t^4 H_x^{1,4}}^2,$$

as desired. □

2.3.2 Local Moser–Trudinger–Strichartz

Employing similar techniques and using Hölder's inequality in time, we may derive a local-in-time Moser–Trudinger–Strichartz inequality. As in the global case, an estimate on the $L_t^\infty L_x^1$ norm will follow from the Moser–Trudinger inequality, an estimate on the $L_t^p L_x^\infty$ norm will follow from the Brézis–Wainger–Gallouët inequality, and a general spacetime estimate will follow from interpolating in-between. We give the statement in the energy-subcritical setting where it is of most use. As a corollary, we will derive dual Strichartz estimates on the nonlinearity which will streamline the well-posedness theory in the next section.

For the sake of exposition, we introduce the following function space: on a slab $I \times \mathbb{R}^2$, we define $X(I)$ to be the closure of test functions under the norm

$$||u||_{X(I)} := ||u||_{L_t^4 H_x^{1,4}(I \times \mathbb{R}^2)} + ||u||_{L_t^\infty H_x^1(I \times \mathbb{R}^2)}.$$

In the special case $I = [-T, T]$ we simply write $X_T = X([-T, T])$.

Lemma 2.3.4. *Suppose $u \in L_t^\infty H_x^1([-T,T] \times \mathbb{R}^2)$ satisfies*

$$||u||_{L_t^\infty \dot{H}_x^1} \leq A < 1 \qquad \textit{and} \qquad ||u||^2_{L_T^\infty L_x^2(\mathbb{R}^2)} = M.$$

If $\beta \in (0,1]$ and $1 \leq p < \infty$ satisfy

$$A^2 < \frac{1}{p\beta},$$

then there exists

$$0 < \gamma = \gamma(A, M, p, \beta) < \min\{4\beta, \tfrac{4}{p}\}$$

and $\varepsilon_0 = \varepsilon_0(A, M, \beta, p) > 0$ so that

$$||e^{4\pi(1+\varepsilon)\beta|u|^2} - 1||_{L_t^p L_x^\infty([-T,T]\times\mathbb{R}^2)} \lesssim_{A,M,p,\beta} T^{\frac{4-p\gamma}{4p}} (T^{\frac{1}{4}} + ||u||_{L_T^4 H_x^{1,4}})^\gamma. \tag{2.3.23}$$

for all $0 < \varepsilon < \varepsilon_0$. Moreover, if $p\beta \leq 1$, then $0 < \gamma = \gamma(A, M, \beta) < 4\beta$ and $\varepsilon_0 = \varepsilon_0(A, M)$.

Proof. We begin our proof, as we did in Lemma 2.3.2, by choosing some parameters first. Choose $1 \geq \mu = \mu(A, M, \beta, p) > 0$, $\varepsilon_0 = \varepsilon_0(A, M, \beta, p) > 0$, and $\alpha = \alpha(A, M, \beta, p) > 1$ so that

$$\alpha(1+\varepsilon_0)^2(A^2 + \mu M) < \min\{1, \tfrac{1}{p\beta}\}. \tag{2.3.24}$$

Let $0 < \varepsilon < \varepsilon_0$. By Proposition 2.2.2.2 we know that for some $C_\alpha > 1$ that

$$\exp(4\pi(1+\varepsilon)\beta||u(t)||^2_{L_x^\infty}) \leq \exp\left(4\alpha\beta(1+\varepsilon)||u(t)||^2_{H_\mu^1} \log\left(C_\alpha + \frac{c_4(8/\mu)^{\frac{1}{2}}||u(t)||_{H_x^{1,4}}}{||u(t)||_{H_\mu^1}}\right)\right)$$

We recall that for $a > 1$ and $b > 0$ the function

$$x \mapsto x^2 \log(a + \tfrac{b}{x})$$

is increasing for $x > 0$. Since $||u(t)||_{H_\mu^1} \leq \sqrt{A^2 + \mu M}$, it follows that

$$\exp(4\pi(1+\varepsilon)\beta||u(t)||^2_{L_x^\infty}) \lesssim_{A,\alpha,\mu} \left(1 + \frac{||u(t)||_{H_x^{1,4}}}{\sqrt{A^2+\mu M}}\right)^{4\alpha\beta(1+\varepsilon)(A^2+\mu M)} \tag{2.3.25}$$

Let $\gamma := 4\alpha\beta(1+\varepsilon)(A^2 + \mu M)$. By (2.3.24) we know that $0 < \gamma < \min\{4\beta, 4/p\}$ and, in particular, that $p\gamma < 4$.

Integrating in time, by Hölder's inequality we see that

$$\begin{aligned}
||e^{4\pi(1+\varepsilon)\beta|u|^2}||_{L_t^p L_x^\infty} &\lesssim_{A,M,\alpha,\mu} ||1+\frac{||u(t)||_{H_x^{1,4}}}{A}||_{L_t^{p\gamma}}^{\gamma} \\
&\lesssim_{A,M,\alpha,\mu} T^{\frac{4-p\gamma}{4p}}||1+\frac{||u(t)||_{H_x^{1,4}}}{A}||_{L_T^4}^{\gamma} \\
&\lesssim_{A,M,\alpha,\mu} T^{\frac{4-p\gamma}{4p}}(T^{\frac{1}{4}}+||u||_{L_T^4 H_x^{1,4}})^{\gamma},
\end{aligned}$$

as desired. □

Combining Lemma 2.3.4 with Proposition 2.2.2.1 we obtain the following corollary by interpolation.

Theorem 2.3.5 (Local Moser–Trudinger–Strichartz). *Suppose* $u \in L_t^\infty H_x^1([-T,T]\times\mathbb{R}^2)$ *satisfies*

$$||u||_{L_t^\infty \dot{H}_x^1} \leq A < 1 \qquad \textit{and} \qquad ||u||_{L_T^\infty L_x^2(\mathbb{R}^2)}^2 = M.$$

If $1 \leq p \leq \infty$ *and* $1 < q \leq \infty$ *satisfy*

$$\frac{q'}{p} > A^2$$

then there exists $0 < \gamma = \gamma(A,M,p,q) < \min\{\frac{4}{q'},\frac{4}{p}\}$ *and* $\varepsilon_0 = \varepsilon_0(A,M,p,q) > 0$ *so that*

$$||e^{4\pi(1+\varepsilon)|u|^2}-1||_{L_T^p L_x^q} \lesssim ||u||_{L_t^\infty L_x^2}^{\frac{2}{q}} T^{\frac{4-p\gamma}{4p}}(T^{\frac{1}{4}}+||u||_{L_T^4 H_x^{1,4}})^{\gamma}. \tag{2.3.26}$$

for every $0<\varepsilon<\varepsilon_0$*. Moreover, if* $\frac{q'}{p} \geq 1$*, then* $0<\gamma=\gamma(A,M,q)<\frac{4}{q'}$ *and* $\varepsilon_0=\varepsilon_0(A,M)$*.*

Proof. Taking $\beta = \frac{1}{q'}$ in Lemma 2.3.4 we obtain an $\varepsilon_0 = \varepsilon_0(A,M) > 0$ and a $0<\gamma<\frac{4}{q'}$ so that (2.3.23) holds for all $0<\varepsilon<\varepsilon_0$ and $p\in[1,q']$. By interpolation and Proposition 2.2.2.1 we see that

$$\begin{aligned}
||e^{4\pi|u(t)|^2}-1||_{L_x^q} &\leq ||e^{4\pi|u(t)|^2}-1||_{L_x^1}^{\frac{1}{q}}||e^{4\pi|u(t)|^2}-1||_{L_x^\infty}^{\frac{1}{q'}} \\
&\lesssim ||u(t)||_{L_x^2}^{\frac{2}{q}}||e^{\frac{4\pi}{q'}|u(t)|^2}-1||_{L_x^\infty}.
\end{aligned}$$

Integrating in time we see by Lemma 2.3.4 that

$$\begin{aligned}
||e^{4\pi|u|^2}-1||_{L_T^p L_x^q} &\lesssim ||u||_{L_t^\infty L_x^2}^{\frac{2}{q}}||e^{\frac{4\pi}{q'}|u|^2}-1||_{L_x^p L_x^\infty} \\
&\lesssim_{A,M} ||u||_{L_t^\infty L_x^2}^{\frac{2}{q}} T^{\frac{4-p\gamma}{4p}}(T^{\frac{1}{4}}+||u||_{L_T^4 H_x^{1,4}})^{\gamma},
\end{aligned}$$

as desired.

$\square$

Corollary 2.3.6. *Suppose* $u_1, u_2 \in L_t^\infty H_x^1([-T,T]\times\mathbb{R}^2)$ *satisfy*

$$||u_i||_{L_t^\infty \dot{H}_x^1} \leq A < 1 \qquad \textit{and} \qquad ||u_i||^2_{L_T^\infty L_x^2(\mathbb{R}^2)} = M.$$

for each $i \in \{1,2\}$, *then there is some* $0 < \gamma = \gamma(A,M) < 3$ *so that*

$$||F_i(u_1) - F_i(u_2)||_{L_T^1 L_x^2} \lesssim_{A,M} ||u_1 - u_2||_{X_T} \sum_{j=1,2} ||u_j||_{X_T}^{\frac{1}{2}} T^{\frac{3-\gamma}{4}} (T^{\frac{1}{4}} + ||u_j||_{X_T})^\gamma.$$

Proof. Our pointwise estimates (2.2.5) and (2.2.6) imply that for any $\varepsilon > 0$ that

$$\begin{aligned} ||F_i(u_1) - F_i(u_2)||_{L_T^1 L_x^2} &\lesssim ||(u_1 - u_2) \sum_{j=1,2} e^{4\pi(1+\varepsilon)|u_j|^2} - 1||_{L_T^1 L_x^2} \\ &\lesssim ||u_1 - u_2||_{L_T^4 L_x^4} \sum_{j=1,2} ||e^{4\pi(1+\varepsilon)|u_j|^2} - 1||_{L_T^{\frac{4}{3}} L_x^4}. \end{aligned}$$

Applying Theorem 2.3.5 with $q' = p = \frac{4}{3}$ we obtain an $\varepsilon_0 = \varepsilon_0(A,M) > 0$ and a $0 < \gamma < 3$ so that (2.3.26) holds for all $0 < \varepsilon < \varepsilon_0$. This grants us that

$$||F_i(u_1) - F_i(u_2)||_{L_T^1 L_x^2} \lesssim ||u_1 - u_2||_{L_T^4 L_x^4} \sum_{j=1,2} ||u_j||^{\frac{1}{2}}_{L_t^\infty L_x^2} T^{\frac{3-\gamma}{4}} (T^{\frac{1}{4}} + ||u_j||_{L_T^4 H_x^{1,4}})^\gamma,$$

as desired. $\square$

Corollary 2.3.7. *Suppose* $u_1, u_2 \in L_t^\infty H_x^1([-T,T]\times\mathbb{R}^2)$ *satisfy*

$$||u_i||_{L_t^\infty \dot{H}_x^1} \leq A < 1 \qquad \textit{and} \qquad ||u_i||^2_{L_T^\infty L_x^2(\mathbb{R}^2)} = M$$

for each $i \in \{1,2\}$. *Then there is some* $0 < \gamma = \gamma(A,M) < 3$ *and* $\delta = \delta(A) > 0$ *so that*

$$\begin{aligned} ||F_i(u_1) - F_i(u_2)||_{L_T^1 \dot{H}_x^1} &\lesssim_A T^{\frac{3}{4}} ||u_1 - u_2||_{X_T} \sum_{j=1,2} (||u_j||_{X_T} + ||u_j||^2_{X_T}) \\ &\quad + T^{\frac{3-\gamma}{4}} ||u_1 - u_2||_{X_T} \sum_{j=1,2} (||u_j||_{X_T}^{\frac{1-4\delta}{2}} + ||u_j||_{X_T}^{\frac{3-4\delta}{2}})(T^{\frac{1}{4}} + ||u_j||_{X_T})^\gamma. \end{aligned}$$

Proof. Write

$$\begin{aligned} \nabla F_i(u_1) - \nabla F_i(u_2) &= [(\partial_z F_i)(u_2)](\nabla u_1 - \nabla u_2) + \nabla u_1 [(\partial_z F_i)(u_1) - (\partial_z F_i)(u_2)] \\ &\quad + [(\partial_{\bar{z}} F_i)(u_2)]\overline{(\nabla u_1 - \nabla u_2)} + \overline{\nabla u_1}[(\partial_{\bar{z}} F_i)(u_1) - (\partial_{\bar{z}} F_i)(u_2)) \\ &= (I) + (II) + (III) + (IV). \end{aligned}$$

We first bound $(I)+(III)$. Towards this end, note that by Hölder and our pointwise bound (2.2.10) we obtain that

$$\begin{aligned}||(I)+(III)||_{L^1_TL^2_x} &\lesssim ||\nabla u_1-\nabla u_2||_{L^4_TL^4_x}(||(\partial_z F_i)(u_2)||_{L^{\frac{4}{3}}_TL^4_x}+||(\partial_{\bar z} F_i)(u_2)||_{L^{\frac{4}{3}}_TL^4_x})\\ &\lesssim ||\nabla u_1-\nabla u_2||_{L^4_TL^4_x}|||u_2|(|u_2|+(e^{4\pi(1+\varepsilon)|u_2|^2}-1))||_{L^{\frac{4}{3}}_TL^4_x}\\ &\lesssim ||\nabla u_1-\nabla u_2||_{L^4_TL^4_x}(||u_2||^2_{L^{\frac{8}{3}}_TL^8_x}+||u_2(e^{4\pi(1+\varepsilon)|u_2|^2}-1)||_{L^{\frac{4}{3}}_TL^4_x})\\ &\lesssim ||\nabla u_1-\nabla u_2||_{L^4_TL^4_x}(T^{\frac{3}{4}}||u_2||^2_{L^\infty_TL^8_x}+||u_2(e^{4\pi(1+\varepsilon)|u_2|^2}-1)||_{L^{\frac{4}{3}}_TL^4_x}).\end{aligned}$$

To control the spacetime norm involving the exponential term, we argue as follows: for each $\delta>0$ we have by Hölder's inequality that

$$||u_2(e^{4\pi(1+\varepsilon)|u_2|^2}-1)||_{L^{\frac{4}{3}}_TL^4_x}\le ||u_2||_{L^\infty_TL^{\frac{1}{\delta}}_x}||e^{4\pi(1+\varepsilon)|u_2|^2}-1||_{L^{\frac{4}{3}}_TL^{\frac{4}{1-4\delta}}_x}.$$

Choose $\delta=\delta(A)$ small enough to guarantee that

$$A^2<\frac{(\frac{4}{1-4\delta})'}{\frac{4}{3}}=\frac{3}{3+4\delta}.$$

Applying Theorem 2.3.5 with $p=\frac{4}{3}$ and $q=\frac{4}{1-4\delta}$ we obtain an $\varepsilon_0=\varepsilon_0(A)>0$ and a $0<\gamma<3$ so that (2.3.26) holds for all $0<\varepsilon<\varepsilon_0$. This grants us that

$$||e^{4\pi(1+\varepsilon)|u_2|^2}-1||_{L^{\frac{4}{3}}_TL^{\frac{4}{1-4\delta}}_x}\lesssim_{A,M}||u_2||^{\frac{1-4\delta}{2}}_{L^\infty_tL^2_x}T^{\frac{3-\gamma}{4}}(T^{\frac{1}{4}}+||u_2||_{L^4_TH^{1,4}_x})^\gamma.$$

Putting these together and employing Sobolev Embedding (2.2.11) we see that

$$||(I)+(III)||_{L^1_TL^2_x}\lesssim_{A,M}||u_1-u_2||_{X_T}\left[T^{\frac{3}{4}}||u_2||^2_{X_T}+||u_2||^{\frac{3-4\delta}{2}}_{X_T}T^{\frac{3-\gamma}{4}}(T^{\frac{1}{4}}+||u_2||_{X_T}\right]^\gamma.$$

We now bound $(II)+(IV)$. Towards this end, we note again that by Hölder and applying our pointwise bound (2.2.10) with $z_2=0$ we obtain that

$$\begin{aligned}||(II)+(IV)||_{L^1_TL^2_x} &\lesssim ||\nabla u_1||_{L^4_TL^4_x}|||u_1-u_2|\sum_{j=1,2}(e^{4\pi(1+\varepsilon)|u_j|^2}-1+|u_j|)||_{L^{\frac{4}{3}}_TL^4_x}\\ &\lesssim ||\nabla u_1||_{L^4_TL^4_x}\sum_{j=1,2}|||u_1-u_2||u_j|||_{L^{\frac{4}{3}}_TL^4_x}+|||u_1-u_2|(e^{4\pi(1+\varepsilon)|u_j|^2}-1)||_{L^{\frac{4}{3}}_TL^4_x}.\end{aligned}$$

The first term is straightforward to bound, by Hölder and Sobolev Embedding we obtain that

$$|||u_1 - u_2||u_j|||_{L_T^{\frac{4}{3}} L_x^4} \leq T^{\frac{3}{4}} ||u_1 - u_2||_{L_T^\infty L_x^8} ||u_j||_{L_T^\infty L_x^8} \leq T^{\frac{3}{4}} ||u_1 - u_2||_{X_T} ||u_j||_{X_T}.$$

We deal with the exponential term exactly as we did before with the same choice of $\delta = \delta(A)$ to obtain that

$$\begin{aligned} |||u_1 - u_2|(e^{4\pi(1+\varepsilon)|u_j|^2} - 1)|||_{L_T^{\frac{4}{3}} L_x^4} &\lesssim ||u_1 - u_2||_{L_T^\infty L_x^{\frac{1}{\delta}}} ||e^{4\pi(1+\varepsilon)|u_j|^2} - 1||_{L_T^{\frac{4}{3}} L_x^{\frac{4}{1-4\delta}}} \\ &\lesssim_{A,M} ||u_1 - u_2||_{X_T} ||u_j||^{\frac{1-4\delta}{2}}_{L_t^\infty L_x^2} T^{\frac{3-\gamma}{4}} (T^{\frac{1}{4}} + ||u_j||_{L_T^4 H_x^{1,4}})^\gamma. \end{aligned}$$

Putting these together we see that

$$||(II) + (IV)||_{L_T^1 L_x^2} \lesssim_{A,M} ||u_1||_{X_T} ||u_1 - u_2||_{X_T} \sum_{j=1,2} T^{\frac{3}{4}} ||u_j||_{X_T} + T^{\frac{3-\gamma}{4}} ||u_j||^{\frac{1-4\delta}{2}}_{X_T} (T^{\frac{1}{4}} + ||u_j||_{X_T})^\gamma,$$

which yields the conclusion the theorem. □

2.4 Well-Posedness Revisited

In this section we revisit the well-posedness theory for (2.1.1) and (2.1.2) first proved in [CIM09] using the techniques developed in the previous section. We prove

Theorem 2.4.1 ([CIM09]). *Suppose $u_0 \in H_x^1(\mathbb{R}^2)$ and $||u_0||_{\dot{H}_x^1} < 1$. Then there exists $T > 0$ and a strong solution to* (2.1.1) *and* (2.1.2) *in $C([-T, T], H_x^1(\mathbb{R}^2))$.*

Proof. We give a proof by contraction mapping. For $T > 0$ let

$$X_T := C([-T, T], H_x^1(\mathbb{R}^2)) \cap L_t^4([-T, T], H_x^{1,4}(\mathbb{R}^2))$$

and note that $(X_T, ||\cdot||_{X_T})$ is a Banach space. Thus, for $T, \varepsilon > 0$

$$B(T, \varepsilon) = \{u \in X_T : ||u - e^{it\Delta} u_0||_{X_T} \leq \varepsilon\}$$

is a complete metric space. For what follows, we assume that $0 < \varepsilon < 1 - ||\nabla u_0||_{L_x^2}$. For $u \in B(T, \varepsilon)$ define

$$\Phi_j(u) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} F_j(u(s)) \, ds.$$

It suffices to prove that Φ_j is a self-map and has a fixed point.

We first prove that Φ_j is a self-map. To this end, we first note that since the $\dot{H}^1_x$ is a conserved quantity for the free Schrödinger equation it follows that

$$||u||_{L^\infty_t \dot{H}^1_x} \leq ||e^{it\Delta}u_0||_{L^\infty_t \dot{H}^1_x} + ||u - e^{it\Delta}u_0||_{L^\infty_t \dot{H}^1_x} \leq ||u_0||_{\dot{H}^1_x} + \varepsilon < 1$$

for any $u \in B(T, \varepsilon)$. Similarly, if $u \in B(T, \varepsilon)$ then

$$||u||_{X_T} \leq \varepsilon + C_{ST}||u_0||_{H^1_x},$$

where C_{ST} is the constant appearing in the Strichartz inequality. Thus, by virtue of the Strichartz Inequality and Corollaries 2.3.6 and 2.3.7 there exist $0 < \gamma < 3$ and $\delta > 0$ so that

$$\begin{aligned} ||\Phi_j(u) - e^{it\Delta}u_0||_{X_T} \lesssim ||F_j(u)||_{L^1_T H^1_x} &\lesssim T^{\frac{3-\gamma}{4}}(||u||^{\frac{3}{2}}_{X_T} + ||u||^{\frac{3-4\delta}{2}}_{X_T} + ||u||^{\frac{5-4\delta}{2}}_{X_T})(T^{\frac{1}{4}} + ||u||_{X_T})^\gamma \\ &\quad + T^{\frac{3}{4}}(||u||^2_{X_T} + ||u||^3_{X_T}). \\ &\lesssim C(||u_0||_{H^1_x})(T^{\frac{3}{4}} + T^{\frac{3-\gamma}{4}}). \end{aligned}$$

Thus, provided $T = T(\varepsilon) > 0$ is sufficiently small we obtain that

$$||\Phi_j(u) - e^{it\Delta}u_0||_{X_T} \leq \varepsilon,$$

and so we obtain that Φ_j is a self-map.

We now turn to proving that Φ_j is a contraction. Again, from Corollaries 2.3.6 and 2.3.7 we obtain that if $u_1, u_2 \in X(T)$ then

$$\begin{aligned} ||\Phi_j(u_1) - \Phi_j(u_2)||_{X_T} &\lesssim ||F_j(u_1) - F_j(u_2)||_{L^1_T H^1_x} \\ &\lesssim ||u_1 - u_2||_{X_T} C(||u_0||_{H^1_x})(T^{\frac{3}{4}} + T^{\frac{3-\gamma}{4}}). \end{aligned}$$

Thus, we may find $T = T(\varepsilon) > 0$ so that

$$||F_j(u_1) - F_j(u_2)||_{L^1_T H^1_x} < \frac{1}{2}||u_1 - u_2||_{X_T}$$

Thus, for $T = T(||u_0||_{H^1_x}) > 0$ sufficiently small we deduce that Φ_j is a contraction, as desired. $\square$

Remark 2.4.1. The observant reader will note that the above proof proves uniqueness only in the space X_T, and not in the larger space $C([-T,T], H_x^1(\mathbb{R}^2))$. Uniqueness in $C([-T,T], H_x^1(\mathbb{R}^2))$ is a consequence of the techniques of the previous section, but relies heavily on the convexity of the exponential. For details, see [CIM09].

For completeness and the sake of exposition, we include a proof of global well-posedness. As we will see, the local theory guarantees us that we may continue a strong solution on $[0,T)$ past time T provided the kinetic energy does not concentrate. Our strategy, then, is to quantify the speed at which the kinetic energy can concentrate. This is essentially a localization result due to Bourgain [Bou98] (see also Lemma 6.2 in [Nak99]).

Lemma 2.4.2. *Let u be a non-trivial solution of* (2.1.1) *or* (2.1.2) *on* $[0,T)$ *with* $0 < T \leq \infty$, *with* $\|u(t)\|_{L_x^2} = M$ *for all* $t \in [0,T)$. *If u solves* (2.1.1), *then there exists a positive constant* $C = C(M)$ *so that*

$$\frac{1}{\langle t\rangle^2} \leq C(M)(H_1(u_0) - ||\nabla u(t)||_{L_x^2}^2), \tag{2.4.1}$$

for every $0 < t < T$. *If u instead solves* (2.1.2), *then there exists a positive constant* $C = C(M)$ *so that*

$$\frac{1}{\langle t\rangle^4} \leq C(M)(H_2(u_0) - ||\nabla u(t)||_{L_x^2}^2), \tag{2.4.2}$$

for every $0 < t < T$.

Proof. Find $R > 1$ so that

$$\int_{\{|x|\leq R\}} |u_0(x)|^2 \, dx \geq \tfrac{1}{2}||u_0||_{L_x^2(\mathbb{R}^2)}^2. \tag{2.4.3}$$

Let $\phi : \mathbb{R} \to [0,1]$ be a smooth function satisfying $\phi(x) = 1$ for $x \leq 0$ and $\phi(x) = 0$ for $x \geq 1$. Define

$$\xi(x) := \phi\left(\tfrac{\mathrm{dist}(x,B(R))}{R'}\right),$$

where $\mathrm{dist}(x, B(R)) = \max\{|x| - R, 0\}$ is the distance from x to the ball $\{|x| \leq R\}$.

We note that $\xi(x) = 1$ when $|x| \leq R$, that $\xi(x) = 0$ when $|x| \geq R + R'$, and that $||\nabla \xi||_{L^\infty_x} \lesssim 1/R'$. This implies that

$$\int_{\{|x|\leq R+R'\}} |u(t,x)|^2 \, dx \geq \int_{\mathbb{R}^2} |\xi(x)|^2 |u(t,x)|^2 \, dx, \qquad \text{and}$$
$$\int_{\{|x|\leq R\}} |u_0(x)|^2 \, dx \leq \int_{\mathbb{R}^2} |\xi(x)|^2 |u_0(x)|^2 \, dx.$$

By taking their difference and employing the fundamental theorem of calculus and Fubini-Tonelli, we see that

$$\begin{aligned}\int_{\{|x|\leq R+R'\}} |u(t,x)|^2 \, dx - \int_{\{|x|\leq R\}} |u_0(x)|^2 dx &\geq \int_{\mathbb{R}^2} |\xi(x)|^2 (|u(t,x)|^2 - |u_0(x)|^2) \, dx \\ &= \int_{\mathbb{R}^2} \int_0^t |\xi(x)|^2 \partial_s |u(s,x)|^2 \, ds \, dx \\ &= \int_0^t \int_{\mathbb{R}^2} |\xi(x)|^2 \partial_s |u(s,x)|^2 \, dx ds.\end{aligned}$$

Since u satisfies (2.1.1) or (2.1.2), it follows that

$$|\int_{\mathbb{R}^2} |\xi(x)|^2 \partial_s |u(s,x)|^2| = |4\text{Im} \int_{\mathbb{R}^2} \xi(x) \nabla \xi(x) \cdot \nabla u(s,x) \bar{u}(s,x) \, dx| \leq \tfrac{C(M)}{R'}.$$

Here we used the fact that $||\nabla u(t)||^2_{L^2_x} \leq 1$. We deduce that

$$\int_{\{|x|\leq R+R'\}} |u(t,x)|^2 \, dx - \int_{\{|x|\leq R\}} |u_0(x)|^2 dx \geq -\tfrac{C(M)t}{R'}. \tag{2.4.4}$$

By expanding $e^{4\pi x^2}$ into its power series, we see that

$$8\pi^2 |u(t,x)|^4 \leq e^{4\pi |u(t,x)|^2} - 4\pi |u(t,x)|^2 - 1, \qquad \text{and} \tag{2.4.5}$$
$$\tfrac{32\pi^3}{3} |u(t,x)|^6 \leq e^{4\pi |u(t,x)|^2} - 4\pi |u(t,x)|^2 - 8\pi^2 |u(t,x)|^4 - 1. \tag{2.4.6}$$

This implies that

$$||u(t)||^4_{L^4_x} \leq \tfrac{1}{2\pi} (H_1(u(t)) - ||\nabla u(t)||^2_{L^2_x}), \qquad \text{and} \tag{2.4.7}$$
$$||u(t)||^6_{L^6_x} \leq \tfrac{3}{8\pi^2} (H_2(u(t)) - ||\nabla u(t)||^2_{L^2_x}). \tag{2.4.8}$$

If u satisfies (2.1.1), then by (2.4.3), (2.4.4) and (2.4.7) it follows from Cauchy-Schwarz that

$$\tfrac{M}{2} - \tfrac{C(M)t}{R'} \leq (\pi (R + R')^2)^{\frac{1}{2}} ||u(t)||^2_{L^4_x} \lesssim (R + R')(H_1(u(t)) - ||\nabla u(t)||^2_{L^2_x})^{\frac{1}{2}}.$$

Choosing $R' = \frac{4C(M)}{M}t$, we see from the conservation of energy that

$$\tfrac{1}{\langle t\rangle^2} \lesssim_M H_1(u_0) - ||\nabla u(t)||^2_{L^2_x}.$$

Similarly, if u satisfies (2.1.2), then by (2.4.3), (2.4.4) and (2.4.8) we have that

$$\tfrac{M}{2} - \tfrac{C(M)t}{R'} \le (\pi(R+R')^2)^{\frac{2}{3}}||u(t)||^2_{L^6_x} \lesssim (R+R')^{\frac{4}{3}}(H_2(u(t)) - ||\nabla u(t)||^2_{L^2_x})^{\frac{1}{3}}.$$

Again, choosing $R' = \frac{4C(M)}{M}t$ we see that

$$\tfrac{1}{\langle t\rangle^4} \lesssim_M H_2(u_0) - ||\nabla u(t)||^2_{L^2_x}.$$

$\square$

Theorem 2.4.3 ([CIM09])**.** *If $u_0 \in H^1_x$ then there exists a unique $C(\mathbb{R}, H^1_x(\mathbb{R}^2))$ solution to* (2.1.1) *and* (2.1.2) *provided $H_1(u_0) \le 1$ and $H_2(u_0) \le 1$, respectively.*

Proof. We only consider positive times, as the following argument can be repeated for negative times with no change. Suppose that u is a strong solution to (2.1.1) on some maximal interval $[0, T^*)$ with $T^* < \infty$. Suppose t_n is a sequence of times with $t_n \nearrow T^*$. Passing to a subsequence, we may assume that $||\nabla u(t_n)||_{L^2_x(\mathbb{R}^2)} \to L \le H_i(u_0)$. If u solves (2.1.1), then by Lemma 2.4.2 we know that

$$\tfrac{1}{\langle t_n\rangle^2} \le C(M)(H_1(u_0) - ||\nabla u(t_n)||^2_{L^2_x}).$$

We see then that

$$\lim_{n\to\infty} ||\nabla u(t_n)||^2_{L^2_x} = L < H_1(u_0) \le 1.$$

The local theory guarantees the existence of a local solution with initial data $u(t_n)$ with a lifetime at least $\tau = \tau(L) > 0$. But then for n satisfying $T^* - t_n < \tau$ we may continue u past T^*, producing a contradiction. A similar argument reaches the same conclusion if u instead solves (2.1.2). $\square$

2.5 Perturbation Theory

2.5.1 Stability Theory

In this section we derive the perturbation theory essential to the proof of Theorem 2.1.6. In the energy-subcritical setting, we will see that we may treat (2.1.2) as a perturbation of the mass-critical NLS. To do so, we must first understand how approximate solutions to the mass-critical NLS behave.

Lemma 2.5.1 (Short-Time Mass-Critical Perturbations)**.** *Let I be a compact interval and let $\tilde{v}$ be an approximate solution to* (2.1.13) *in the sense that*

$$(i\partial_t + \Delta)\tilde{v} = |\tilde{v}|^2\tilde{v} + e$$

for some spacetime function e. Assume that

$$||\tilde{v}||_{L_t^\infty H_x^1(I\times\mathbb{R}^2)} \leq M \tag{2.5.1}$$

for some positive constant M. Let $t_0 \in I$ and let $v(t_0)$ be close to $\tilde{v}(t_0)$ in the sense that

$$||v(t_0) - \tilde{v}(t_0)||_{H_x^1} \leq M', \tag{2.5.2}$$

for some $M' > 0$. Assume also the smallness conditions

$$||\tilde{v}||_{L_t^4 H_x^{1,4}} \leq \varepsilon_0, \tag{2.5.3}$$

$$||e^{i(t-t_0)\Delta}(v(t_0) - \tilde{v}(t_0))||_{L_t^4 H_x^{1,4}} \leq \varepsilon, \qquad \textit{and} \tag{2.5.4}$$

$$||e||_{N(I\times\mathbb{R}^2)} \leq \varepsilon \tag{2.5.5}$$

for some $0 < \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(M, M') > 0$ is a small constant. Then, there exists a solution v to (2.1.13) *on $I \times \mathbb{R}^2$ with initial data $v(t_0)$ at time $t = t_0$ satisfying*

$$||v - \tilde{v}||_{L_t^4 H_x^{1,4}} \lesssim \varepsilon, \tag{2.5.6}$$

$$||v - \tilde{v}||_{S(I\times\mathbb{R}^2)} \lesssim M', \tag{2.5.7}$$

$$||v||_{S(I\times\mathbb{R}^2)} \lesssim M + M', \qquad \textit{and} \tag{2.5.8}$$

$$||(i\partial_t + \Delta)(v - \tilde{v}) + e||_{N(I\times\mathbb{R}^2)} \lesssim \varepsilon. \tag{2.5.9}$$

Remark 2.5.1. Note that by Strichartz, hypothesis (2.5.4) is redundant if $M' = O(\varepsilon)$.

Proof. By time symmetry we may, and do, assume that $t_0 = \inf I$. Let $z := v - \tilde{v}$. Then z satisfies

$$\begin{cases} (i\partial_t + \Delta)z = |\tilde{v} + z|^2(\tilde{v} + z) - |\tilde{v}|^2\tilde{v} - e \\ z(t_0) = v(t_0) - \tilde{v}(t_0) \end{cases}$$

For $t \in I$ define

$$S(t) := ||(i\partial_t + \Delta)z + e||_{N([t_0,t]\times\mathbb{R}^2)} = |||\tilde{v} + z|^2(\tilde{v} + z) - |\tilde{v}|^2\tilde{v}||_{N([t_0,t]\times\mathbb{R}^2)}.$$

We will use the pointwise estimates (2.2.3) and (2.2.4) to estimate the N^0 and N^1 norms respectively. Indeed, with $g(z) = z|z|^2$, we have

$$|g(\tilde{v} + z) - g(\tilde{v})| \lesssim |z|(|z|^2 + |\tilde{v}|^2)$$
$$|\nabla g(\tilde{v} + z) - \nabla g(\tilde{v})| \lesssim |\tilde{v}|^2|\nabla z| + |z|^2|\nabla \tilde{v}| + |z|^2|\nabla z| + |\nabla \tilde{v}||z||\tilde{v}|.$$

Since $(\frac{4}{3}, \frac{4}{3})$ is a dual Strichartz pair, we know for any three spacetime functions $f_1(t,x)$, $f_2(t,x)$, and $f_3(t,x)$ that

$$||f_1 f_2 f_3||_{N^0} \lesssim ||f_1 f_2 f_3||_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \lesssim ||f_1||_{L_t^4 L_x^4} ||f_2||_{L_t^4 L_x^4} ||f_3||_{L_t^4 L_x^4}. \tag{2.5.10}$$

Repeated applications of (2.5.10) implies, by (2.5.3), that

$$S(t) \lesssim ||z||^3_{L_t^4 H_x^{1,4}} + ||z||^2_{L_t^4 H_x^{1,4}} ||v||_{L_t^4 H_x^{1,4}} + ||\tilde{v}||^2_{L_t^4 L_t^4} ||z||_{L_t^4 H_x^{1,4}} \tag{2.5.11}$$
$$\lesssim ||z||^3_{L_t^4 H_x^{1,4}} + \varepsilon_0^2 ||z||^2_{L_t^4 H_x^{1,4}} + \varepsilon_0 ||z||_{L_t^4 H_x^{1,4}} \tag{2.5.12}$$

On the other hand by Strichartz, (2.5.4), and (2.5.5), we have

$$||z||_{L_t^4 H_x^{1,4}} \lesssim ||e^{i(t-t_0)\Delta} z(t_0)||_{L_t^4 H_x^{1,4}} + S(t) + ||e||_{N([t_0,t]\times\mathbb{R}^2)} \lesssim S(t) + \varepsilon. \tag{2.5.13}$$

So we have that

$$S(t) \lesssim (S(t) + \varepsilon)^3 + \varepsilon_0^2 (S(t) + \varepsilon)^2 + \varepsilon_0 (S(t) + \varepsilon).$$

A continuity argument shows then that if ε_0 is taken suffiiently small, then

$$S(t) \leq \varepsilon \text{ for any } t \in I.$$

This implies (2.5.9). Using (2.5.9) and (2.5.13), one easily derives (2.5.6). Moreover, by Strichartz, (2.5.2), (2.5.5), and (2.5.9),

$$||z||_{S(I\times\mathbb{R}^2)} \lesssim ||z(t_0)||_{H^1_x} + ||(i\partial_t+\Delta)z+e||_{N(I\times\mathbb{R}^2)} + ||e||_{N([t_0,t]\times\mathbb{R}^2)} \lesssim M'+\varepsilon$$

which proves (2.5.7). To prove (2.5.8), we use Strichartz, (2.5.1), (2.5.2), (2.5.3), (2.5.5), and (2.5.9) we have

$$\begin{aligned}
||v||_S &\lesssim ||v(t_0)||_{H^1_x} + ||(i\partial_t+\Delta)v||_N \\
&\lesssim ||\tilde{v}(t_0)||_{H^1_x} + ||v(t_0)-\tilde{v}(t_0)||_{H^1_x} + ||(i\partial_t+\Delta)(v-\tilde{v})+e||_N + ||(i\partial_t+\Delta)\tilde{v}||_N + ||e||_N \\
&\lesssim M+M'+\varepsilon+||(i\partial_t+\Delta)\tilde{v}||_{L_t^{\frac{4}{3}}L_x^{\frac{4}{3}}} \\
&\lesssim M+M'+||\tilde{v}||^3_{L_t^4L_x^4} \\
&\lesssim M+M'+\varepsilon_0^3 \\
&\lesssim M+M'.
\end{aligned}$$

□

The following proposition is an expected consequence of the previous lemma. See, for example, Lemma 3.6 in [TVZ07] or Theorem 3.7 in [KV13].

Proposition. *Let I be a compact interval and let $\tilde{v}$ be an approximate solution in the sense that*

$$(i\partial_t+\Delta)\tilde{v} = |\tilde{v}|^2\tilde{v}+e$$

for some function e. Let ε_0 be as in the previous lemma. Assume that

$$||\tilde{v}||_{L_t^\infty H^1_x(I\times\mathbb{R}^2)} \leq M \tag{2.5.14}$$

$$||\tilde{v}||_{L_t^4 H_x^{1,4}} \leq L, \tag{2.5.15}$$

for some positive constant M and L. Let $t_0\in I$ and let $v(t_0)$ be close to $\tilde{v}(t_0)$ in the sense that

$$||v(t_0)-\tilde{v}(t_0)||_{H^1_x} \leq \varepsilon, \tag{2.5.16}$$

for some $\varepsilon_0 > \varepsilon > 0$. Assume also the smallness condition

$$||e||_{N(I\times\mathbb{R}^2)} \leq \varepsilon, \tag{2.5.17}$$

for some $0 < \varepsilon \leq \varepsilon_1$ where $\varepsilon_1 = \varepsilon_1(M, L) > 0$ is a small constant. Then, there exists a solution v to (2.1.13) *on $I \times \mathbb{R}^2$ with initial data $v(t_0)$ at time $t = t_0$ satisfiying*

$$||v - \tilde{v}||_{S(I\times\mathbb{R}^2)} \leq C(M, L)\varepsilon. \tag{2.5.18}$$

2.5.2 Proof of Theorem 2.1.6

In this section, we upgrade the *a priori* $L_t^4 L_x^8$ bound granted by Morawetz to derive global spacetime bounds that imply scattering for (2.1.1) in the energy-subcritical case. As in [TVZ07], our approach is perturbative. We split the nonlinear term in (2.1.1)

$$\begin{cases} iu_t + \Delta u = 4\pi|u|^2 u + F_2(u) \\ u(0, x) = u_0 \in H_x^1(\mathbb{R}^2) \end{cases}$$

and view (2.1.1) as a perturbation to the mass-critical NLS with F_2 as an error term. The dual Strichartz estimates on F_2 derived from Corollary 2.3.3 will grant us sufficiently good bounds to use the stability theory.

Proof of Theorem 2.1.6. Let u be the global solution to (2.1.1) given by Theorem 2.4.3. By Lemma 2.2.3 and the conservation of energy and mass, we see that

$$||u||_{L_t^4 L_x^8} \lesssim ||u||_{L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R}^2)} \leq C(H, M).$$

Let ε be a small constant to be chosen later. Split $\mathbb{R}$ into $J = J(H, M, \varepsilon)$ subintervals I_j, $0 \leq j \leq J - 1$ so that

$$||u||_{L_t^4 L_x^8(I_j)} \sim \varepsilon.$$

We'll show that u obeys good Strichartz bounds on each slab $I_j \times \mathbb{R}^2$. It follows from Corollary 2.3.3 that

$$||e||_{N(I_j\times\mathbb{R}^2)} \leq C(H, M)\varepsilon^{4\delta}||u||^2_{L_t^4 H_x^{1,4}(I_j)}, \tag{2.5.19}$$

for all $0 \leq j \leq J-1$. In what follows, we fix an interval $I_{j_0} = [a,b]$ and prove that u obeys good Strichartz estimates on the slab $I_{j_0} \times \mathbb{R}^2$. In order to do so, we view the solution u as a perturbation to a solution to (2.1.13)

$$\begin{cases} (i\partial_t + \Delta)v = 4\pi|v|^2 v \\ v(a) = u(a). \end{cases}.$$

This initial value problem is globally well-posed in H^1_x by Theorem 2.1.4, and the unique solution enjoys the global spacetime bound

$$||v||_S \leq C(H,M).$$

Thus, we can subdivide $\mathbb{R}$ into $K = K(M,H,\eta)$ subintervals J_k so that

$$||v||_{L^4_t H^{1,4}_x(J_k)} \sim \eta, \tag{2.5.20}$$

on each J_k, where $\eta > 0$ is a small constant to be chosen later. Without loss of generality, we assume that

$$[a,b] = \cup_{k=0}^{K'-1} J_k, t_0 = a, t_{K'} = b$$

The nonlinear evolution v being small on $J_k \times \mathbb{R}^2$ implies that the linear evolution $e^{i(t-t_k)\Delta}v(t_k)$ is small as well. By Strichartz, we have that

$$\begin{aligned} ||e^{i(t-t_k)\Delta}v(t_k)||_{L^4_t H^{1,4}_x(J_k)} &\leq ||v||_{L^4_t H^{1,4}_x} + C_{ST}|||v|^2 v||_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} + C_{ST}||\nabla|v|^2 v||_{L_t^{\frac{4}{3}} L_x^{\frac{4}{3}}} \\ &\leq ||v||_{L^4_t H^{1,4}_x} + C_{ST}||v||^2_{L^4_t L^4_x}||v||_{L^4_t L^4_x} + C_{ST}||v||^3_{L^4_t L^4_x} \\ &\leq \eta + 2C_{ST}\eta^3, \end{aligned}$$

where C_{ST} is the constant appearing in the Strichartz inequality. If $\eta < (4C_{ST})^{-1/2}$, then

$$||e^{i(t-t_k)\Delta}v(t_k)||_{L^4_t H^{1,4}_x(J_k)} \leq 2\eta. \tag{2.5.21}$$

Our goal is to apply the our mass-critical stability theory to compare u and v on the slab $[t_0,t_1] \times \mathbb{R}^2$. To do so, we must verify that u and v satisfy the hypotheses of Lemma 2.5.1. Once we do, this will guarantee that u and v remain close at time t_1. We will use this closeness to verify the hypotheses of Proposition 2.5.1 for u and v on the slab $[t_1,t_2] \times \mathbb{R}^2$, to show that they continue to remain close on the entirety of the slab. We continue in this fashion iteratively, recycling the closeness on one slab to verify the hypotheses of Proposition 2.5.1 for u and v on the next, to show that u and v remain close on all of $I_{j_0} \times \mathbb{R}^2$.

To establish the base case, recall that $u(t_0) = v(t_0)$. So, by Strichartz, (2.5.21), and (2.5.19) we know that

$$\begin{aligned} ||u||_{L_t^4 H_x^{1,4}([t_0,t_1])} &\leq ||e^{i(t-t_0)\Delta} u(t_0)||_{L_t^4 H_x^{1,4}([t_0,t_1])} + C_{ST} |||u|^2 u||_{N([t_0,t_1])} + C_{ST} ||F_2(u)||_{N([t_0,t_1])} \\ &\leq 2\eta + C_{ST} ||u||^3_{L_t^4 H_x^{1,4}([t_0,t_1])} + C(H,M)\varepsilon^{4\delta} ||u||^2_{L_t^4 H_x^{1,4}([t_0,t_1])} \end{aligned}$$

which, by a continuity argument, yields that

$$||u||_{L_t^4 H_x^{1,4}} \leq 4\eta \tag{2.5.22}$$

provided that η and $\varepsilon = \varepsilon(H, M)$ are sufficiently small. To apply our stability lemma, we need to check that the error term $F_2(u)$ is small in $N(J_0 \times \mathbb{R}^2)$. But this is straightforward since, by Corollary 2.3.3,

$$||F_2(u)||_{N([t_0,t_1])} \leq C(H,M)\varepsilon^{4\delta} ||u||^2_{L_t^4 H_x^{1,4}(J_0 \times \mathbb{R}^2)} \leq C(H,M)\eta^2 \varepsilon^{4\delta}. \tag{2.5.23}$$

Choosing ε sufficiently small depending only on H and M, we have that

$$||u - v||_{S(J_0 \times \mathbb{R}^2)} \leq \varepsilon^{2\delta}.$$

By Strichartz, this implies that

$$||u(t_1) - v(t_1)||_{H_x^1} \leq \varepsilon^{2\delta}, \tag{2.5.24}$$

$$||e^{i(t-t_1)\Delta}(u(t_1) - v(t_1))||_{L_t^4 H_x^{1,4}} \lesssim \varepsilon^{2\delta} \tag{2.5.25}$$

Now we'll use (2.5.24) and (2.5.25) to estimate u on the slab $J_1 \times \mathbb{R}^2$. Splitting the linear evolution, we see by Strichartz, (2.5.19), (2.5.21), and (2.5.25) that

$$\begin{aligned} ||u||_{L_t^4 H_x^{1,4}(J_1)} &\leq ||e^{i(t-t_k)\Delta} v(t_k)||_{L_t^4 H_x^{1,4}(J_1)} + ||e^{i(t-t_k)\Delta}(u(t_k) - v(t_k))||_{L_t^4 H_x^{1,4}(J_1)} \\ &\quad + C_{ST} ||u||^3_{L_t^4 H_x^{1,4}(J_1)} + C(H,M)\varepsilon^{4\delta} ||u||^2_{L_t^4 H_x^{1,4}(J_1)} \\ &\leq 2\eta + C_{ST}\varepsilon^{2\delta} + C_{ST} ||u||^3_{L_t^4 H_x^{1,4}(J_1)} + C(H,M)\varepsilon^{4\delta} ||u||^2_{L_t^4 H_x^{1,4}(J_1)}. \end{aligned}$$

A standard continuity argument then yields that $||u||_{L_t^4 H_x^{1,4}(J_1)} \leq 4\eta$ provided η and $\varepsilon = \varepsilon(H, M)$ are chosen sufficiently small.

This implies that

$$||F_2(u)||_{N(J_1 \times \mathbb{R}^2)} \leq C(H, M)\eta^2 \varepsilon^{4\delta}.$$

Choosing ε sufficiently small depending only on H and M, we can apply the stability lemma to derive that

$$||u - v||_{S(J_1 \times \mathbb{R}^2)} \leq \varepsilon^{\delta}.$$

By induction, if we take $\varepsilon = \varepsilon(H, M)$ smaller at each step, we obtain that

$$||u - v||_{S(J_k \times \mathbb{R}^2)} \leq \varepsilon^{\delta/2^{k-1}}. \tag{2.5.26}$$

Adding these bounds over all intervals J_k which meet I_{j_0} we deduce that

$$\sum_{j=0}^{K'-1} ||u||_{S([t_j, t_{j+1}] \times \mathbb{R}^n)} \lesssim ||v||_{S(I_{j_0} \times \mathbb{R}^2)} + \sum_{k=0}^{K'-1} ||u - v||_{S(J_k \times \mathbb{R}^n)} \leq C(H, M). \tag{2.5.27}$$

Thus, we find that (2.5.27) holds with K' in place of k. Since I_{j_0} was arbitrarily chosen, we may some over the finitely many J intervals to see that $||u||_{S(\mathbb{R} \times \mathbb{R}^2)} \leq JC(H, M) < \infty$, which completes the proof. □

2.5.3 Finite global Strichartz norms imply scattering

We show that finite global Strichartz norms imply scattering. We will only present the construction of the scattering states, and demonstrate that their linear flow asymptotically approximates our solution. In fact, we will only construct scattering states in the positive time direction, since identical arguments can be used in the negative time direction. Standard techniques can be used to construct wave operators, see [Caz03].

Suppose $u_0 \in H_x^1$ with $H(u_0) = H < 1$ and that $u(t)$ is a global solution to (2.1.1). Then, by Theorem 2.1.6, we know that

$$||u||_{L_t^4 H_x^{1,4}} \leq C(H, M).$$

For $0 < t < \infty$ define

$$u_+(t) = u_0 - i\int_0^t e^{-is\Delta} F_1(u(s))\ ds.$$

By the Strichartz inequality, we see that

$$||u_+||_{L_t^\infty H_x^1} \lesssim ||u_0||_{H_x^1} + ||F_1(u)||_{N(\mathbb{R})} \lesssim ||u_0||_{H_x^1} + ||4\pi|u|^2u||_{L_t^{\frac{4}{3}} H_x^{1,\frac{4}{3}}} + ||F_2(u)||_{L_t^{\frac{4}{3}} H_x^{1,\frac{4}{3}}}.$$

By Corollary 2.3.3, there exists $0 < \delta = \delta(H) < \frac{1}{3}$ so that

$$||4\pi|u|^2u||_{L_t^{\frac{4}{3}} H_x^{1,\frac{4}{3}}} \lesssim ||u||^3_{L_t^4 H_x^{1,4}} < \infty \qquad \text{and} \qquad ||F_2(u)||_N \lesssim ||u||^{4\delta}_{L_t^4 L_x^8} ||u||^2_{L_t^4 H_x^{1,4}} < \infty.$$

Thus $u_+(t) \in H_x^1$ for all $t \in \mathbb{R}$. A similar argument shows that $u_+(t)$ converges in H_x^1 as $t \to \infty$. Define

$$u_+ = u_0 - i\int_0^\infty e^{-is\Delta} F_1(u(s))\ ds.$$

With u_+ so defined, we see that

$$\begin{aligned} ||e^{it\Delta}u_+ - u(t)||_{H_x^1} &\lesssim ||\int_t^\infty e^{-is\Delta} F_1(u(s))\ ds||_{H_x^1} \\ &\lesssim ||u||^2_{L_t^4 H_x^{1,4}([t,\infty]\times\mathbb{R}^2)} + ||u||^{4\delta}_{L_t^4 L_x^8([t,\infty]\times\mathbb{R}^2)} ||u||^2_{L_t^4 H_x^{1,4}([t,\infty]\times\mathbb{R}^2)} \end{aligned}$$

which tends to 0 as $t \to \infty$.

CHAPTER 3

The 3D Schrödinger Operator with inverse-square potential.

3.1 Introduction

In this chapter, we will consider harmonic analysis questions concerning the unitary group e^{-itH} for the Schrödinger operator with inverse-square potential

$$H := -\Delta - \frac{1}{4|x|^2} \tag{3.1.1}$$

where Δ is the Laplacian in $\mathbb{R}^3$.

The operator H in (3.1.1) belongs to a family of operators

$$H_a := -\Delta - \frac{a}{|x|^2} \qquad a \leq \tfrac{1}{4} \tag{3.1.2}$$

of interest in quantum mechanics (see [Cas50], [KSW75]). Since the potential term is homogenous of order -2, the potential and Laplacian scale exactly the same. For this reason, this makes the study of the associated PDEs

$$\begin{cases} (i\partial_t - H_a)u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^3) \end{cases} \tag{3.1.3}$$

$$\begin{cases} (i\partial_t - H_a)u = |u|^p u, & (t,x) \in \mathbb{R} \times \mathbb{R}^3, \\ u(0,x) = u_0(x) \in \dot{H}^1_x(\mathbb{R}^3) \end{cases} \tag{3.1.4}$$

of particular mathematical interest. Indeed, since the potential and Laplacian are of equal strength at every scale, analysis of (3.1.3) is immune to standard perturbative methods. Thus, Schrödinger equations with inverse square potentials have necessitated a more delicate approach. This challenge has been met with a long campaign to understand those properties enjoyed by H_a and the local and global behavior of solutions to (3.1.4).

The choice of the constant $a = \frac{1}{4}$ in this paper is critical in another sense. This constant corresponds to the sharp constant in the Hardy inequality.

Lemma 3.1.1 (Hardy Inequality). *If $u \in C_0^\infty(\mathbb{R}^3)$, then*

$$\frac{1}{4}\int \frac{|u(x)|^2}{|x|^2}\, dx \leq \int |\nabla u(x)|^2\, dx. \tag{3.1.5}$$

Moreover, (3.1.5) *fails if the constant $\frac{1}{4}$ is replaced by any constant $a > \frac{1}{4}$.*

When $a \leq \frac{1}{4}$ and $u \in C_0^\infty(\mathbb{R}^3)$ we see that

$$\langle H_a u, u\rangle_{L_x^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |\nabla u(x)|\, dx - a\int \tfrac{|u(x)|^2}{|x|^2}\, dx \geq 0, \tag{3.1.6}$$

and so H_a is positive semi-definite. When $a > \frac{1}{4}$, all self-adjoint extensions of H_a are unbounded below. In this case the spectrum of H_a has infinitely many negative eigenvalues that diverge to $-\infty$ ([PST03]), making it impossible to obtain global dispersive estimates.

Substantial progress has been made in understanding H_a and (3.1.4) in both the critical case $a = 1/4$ and the subcritical case $a < \frac{1}{4}$. The first wave of progress in the linear problem came in the form of studying the full range of dispersive and Strichartz estimates for solutions to (3.1.3). When $a \leq \frac{1}{4}$, [BPS03], Burq–Planchon–Stalker–Tahvildar-Zadeh demonstrated local smoothing for solutions to (3.1.3). The nonlinear problem in this critical regime was advanced in [Suz16], when Suzuki demonstrated global wellposedness for (3.1.4) in $\mathbb{R}^3$ when $a \leq \frac{1}{4}$ and $p < 5$ for a suitable class of initial data by proving Strichartz estimates for (3.1.3) avoiding the sharp endpoint $(2, 6)$. Recently, in [Miz16], Mizutani recovers the sharp endpoint in a Lorentz space framework.

In [KMV15b], Killip-Miao-Visan-Zhang-Zheng prove a variety of harmonic analysis tools for studying H_a when $a \leq \frac{1}{4}$. In particular, they develop a Littlewood-Paley theory for H_a, which allows them to prove in [KMV15a] that (3.1.4) with $p = 5$ is globally well-posed when $a < \frac{1}{4} - \frac{1}{25}$. This represents the first progress made in treating the energy critical nonlinearity $|u|^4 u$ in the presence of an inverse square potential.

3.2 Main Results

In this paper, we advocate for the idea that any problem involving the critical inverse square potential should be decomposed into a system of equations: one for the radial part of the solution (which is more susceptible to classical ODE techniques), and another for the remainder. Indeed, what is clear from previous advances in [BPS03], [Miz16], and [BM16] is that data in $L^2 \ominus L^2_{\text{rad}}$ enjoy a wealth of useful estimates for H_a which in turn yield Strichartz estimates that are not immediately available in the general case. When restricted to higher angular momentum, the operator H_a enjoys estimates that mirror those of the Laplacian.

To make this discussion more precise, let $P : L^2(\mathbb{R}^3) \to L^2_{\text{rad}}(\mathbb{R}^3)$ be the projection defined by

$$Pf(x) = \frac{1}{4\pi^2} \int_{\mathbb{S}^2} f(|x|\omega) d\sigma(\omega).$$

We define $P^\perp = I - P$ to be the orthogonal projection onto $L^2_\perp := L^2 \ominus L^2_{\text{rad}}$, the space of angular functions. As P and $P^\perp$ commute with Δ and $-\frac{1}{4|x|^2}$, they commute with H. We will often write

$$H^\perp := P^\perp H = HP^\perp$$

when convenient.

The advantage of restricting to functions in $L^2_\perp$ is due, in part, to an improvement to Lemma 3.1.1 in this setting. For a clear exposition, see [EF06].

Lemma 3.2.1 (Improved Hardy Inequality). *If $f : \mathbb{R}^3 \to \mathbb{C}$ is a Schwartz function, then*

$$\frac{9}{4} \int_{\mathbb{R}^3} \frac{|P^\perp f|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^3} |\nabla P^\perp f|^2 \, dx. \tag{3.2.1}$$

Moreover, (3.2.1) *fails if the constant $\frac{9}{4}$ is replaced by any constant $a > \frac{9}{4}$.*

For data in $L^2_\perp$ this improvement allows one to apply techniques available only in the subcritical regime $a < \frac{1}{4}$ at the critical constant $a = \frac{1}{4}$. In this paper, we exploit this improvement to expand upon the harmonic analysis tools developed in [KMV15b]. We seek out which embeddings enjoyed by Δ are also enjoyed by H in order to develop a Littlewood-Paley theory for H. Towards this end, we study the resolvent of H and, in particular, how it differs from the resolvent of Δ when projected to angular functions.

In this paper we improve the resolvent estimate from [Miz16] to prove the following:

Theorem 3.2.2. *Let* $(q, s) \in [1, \infty) \times [1, \infty)$ *satisfy*

$$\tfrac{4}{3} \leq \tfrac{1}{q} + \tfrac{1}{s'} \leq \tfrac{5}{3}. \tag{3.2.2}$$

If, additionally,

$$\sqrt{2} > \max\{2 - \tfrac{3}{s'}, 2 - \tfrac{3}{q}, \tfrac{3}{q} - \tfrac{5}{2}, \tfrac{3}{s'} - \tfrac{5}{2}\}, \tag{3.2.3}$$

then

$$\|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})\|_{L_x^q(\mathbb{R}^3) \to L_x^s(\mathbb{R}^3)} < \infty. \tag{3.2.4}$$

This estimate allows us to improve the range of Bernstein estimates enjoyed by $H^{\perp}$ that are found in [Miz16]. Indeed, we obtain that $H^{\perp}$ obeys many of the useful Bernstein estimates enjoyed by the Laplacian. More precisely, we prove the following:

Corollary 3.2.3 (Sharp Sobolev Embedding)**.** *If* $f : \mathbb{R}^3 \to \mathbb{C}$ *is Schwartz, then*

$$\|P^{\perp}(H+1)^{-1} f\|_{L_x^6(\mathbb{R}^3)} \lesssim ||f||_{L_x^2(\mathbb{R}^3)}. \tag{3.2.5}$$

Corollary 3.2.4. *If* $f : \mathbb{R}^3 \to \mathbb{C}$ *is Schwartz, then*

$$\|P^{\perp}(H+1)^{-1} f\|_{L_x^\infty(\mathbb{R}^3)} \lesssim ||f||_{L_x^2(\mathbb{R}^3)} \tag{3.2.6}$$

$$\|P^{\perp}(H+1)^{-1} f\|_{L_x^2(\mathbb{R}^3)} \lesssim ||f||_{L_x^1(\mathbb{R}^3)}. \tag{3.2.7}$$

Theorem 3.2.5 (Bernstein Estimates)**.** *The operator* $P^{\perp} e^{-H} : L^1(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ *is bounded.*

Remark 3.2.1. The estimates in Theorem 3.2.2, Corollary 3.2.4, and Theorem 3.2.5 fail in the absence of the projection $P^{\perp}$.

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3.4 Preliminaries

We begin by fixing some notation. We will write $X \lesssim Y$ if there exists a constant C so that $X \leq CY$. When we wish to stress the dependence of this implicit constant on a parameter ε (say), so that $C = C(\varepsilon)$, we write $X \lesssim_\varepsilon Y$. We write $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. If there exists a *small* constant c for which $X \leq cY$ we will write $X \ll Y$.

For $1 \leq r < \infty$ we recall the Lebesgue space $L^r_x(\mathbb{R}^3)$, which is the completion of smooth compactly supported functions $f : \mathbb{R}^3 \to \mathbb{C}$ under the norm

$$||f||_r = ||f||_{L^r_x(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3} |f(x)|^r \, dx \right)^{\frac{1}{r}}.$$

When $r = \infty$, we employ the essential supremum norm. Our convention for the Fourier transform on $\mathbb{R}^3$ is

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x) \, dx.$$

The Fourier transform allows us to define the fractional differentiation operators

$$\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi) \qquad \text{and} \qquad \widehat{\langle\nabla\rangle^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi).$$

The fractional differentiation operators give rise to the (in-)homogenous Sobolev spaces. We define $\dot{H}^{1,r}(\mathbb{R}^3)$ and $H^{1,r}(\mathbb{R}^3)$ to be the completion of smooth compactly supported functions $f : \mathbb{R}^3 \to \mathbb{C}$ under the norms

$$||f||_{\dot{H}^{1,r}(\mathbb{R}^3)} = |||\nabla|f||_{L^r_x} \qquad \text{and} \qquad ||f||_{H^{1,r}(\mathbb{R}^3)} = |||\langle\nabla\rangle f||_{L^r_x}.$$

When $r = 2$, we simply write $H^{1,r} = H^1$ and $\dot{H}^{1,r} = \dot{H}^1$.

Let us define the operator $H = -\Delta - \frac{1}{4|x|^2}$ as the Friedrichs extension of the symmetric and non-negative sesquilinear form

$$Q(u,v) = \int \nabla u \cdot \overline{\nabla v} - \frac{1}{4|x|^2} u\bar{v} \, dx \qquad u, v \in C_0^\infty(\mathbb{R}^3). \tag{3.4.1}$$

More explicitly, let $\bar{Q}$ be the closure of Q with domain $D(\bar{Q})$ given by the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $(\|u\|^2_{L^2_x(\mathbb{R}^3)} + Q(u,u))^{\frac{1}{2}}$. For each element u in

$$D(H) = \{u \in D(\bar{Q}) : |\bar{Q}(u,v)| \leq C_u \|v\|_{L^2_x(\mathbb{R}^3)} \text{ for all } v \in D(\bar{Q})\},$$

we define Hu to be the unique element in $L^2_x(\mathbb{R}^3)$ satisfying $\bar{Q}(u,v) = (Hu, v)$ for all $v \in D(\bar{Q})$.

3.5 Combes-Thomas Estimates

In this section we prove a resolvent estimate for H, of which a special case is an essential stratagem in [Miz16]. To do this, we adopt a technique due to Combes and Thomas (see [L 73], [His00]) for proving decay estimates for the resolvent.

Lemma 3.5.1 (Combes-Thomas Estimate, [His00])**.** *Suppose A is a self-adjoint operator on a Hilbert space $\mathcal{H}$ and its spectrum, $\sigma(A)$, is positive and satisfies $d := dist(\sigma(A), 0) > 0$. If B is self-adjoint on $\mathcal{H}$, then $A + i\beta B$ is boundedly invertible for all $\beta \in \mathbb{R}$ and*

$$\|(A + i\beta B)^{-1}\| \leq d^{-1}.$$

Proof. If $u \in \mathcal{H}$, then

$$\|u\|\|(A + i\beta B)u\| \geq \mathrm{Re}\langle u, (A + i\beta B)u\rangle = \mathrm{Re}(\langle u, Au\rangle - i\beta\langle u, Bu\rangle) \geq d\|u\|^2.$$

□

To use Lemma 3.5.1 we will need a useful reformulation of Lemma 3.2.1.

Corollary 3.5.2. *If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz and $P^\perp f = f$, then*

$$\langle (H + \tfrac{8}{9}\Delta)f, f\rangle \geq 0 \tag{3.5.1}$$

and, likewise,

$$\langle (H - \tfrac{2}{|x|^2})f, f\rangle \geq 0. \tag{3.5.2}$$

Proof. Suppose f is Schwartz. As $P^\perp H = HP^\perp$, by (3.2.1) we see that

$$\begin{aligned}\langle (H + \tfrac{8}{9}\Delta)f, f\rangle_{L^2_x(\mathbb{R}^3)} &= \langle -\Delta f, f\rangle_{L^2_x(\mathbb{R}^3)} - \langle \tfrac{1}{4|x|^2} f, f\rangle_{L^2_x(\mathbb{R}^3)} + \langle \tfrac{8}{9}\Delta f, f\rangle_{L^2_x(\mathbb{R}^3)} \\ &= \tfrac{1}{9}(\langle \nabla f, \nabla f\rangle_{L^2_x(\mathbb{R}^3)} - \langle \tfrac{9}{4|x|^2} f, f\rangle_{L^2_x(\mathbb{R}^3)}) \\ &\geq 0.\end{aligned}$$

Similarly, we see that

$$\langle (H - \tfrac{2}{|x|^2})f, f\rangle_{L^2_x(\mathbb{R}^3)} = \langle -\Delta f, f\rangle_{L^2_x(\mathbb{R}^3)} - \langle \tfrac{9}{4|x|^2} f, f\rangle_{L^2_x(\mathbb{R}^3)} \geq 0.$$

□

Theorem 3.5.3 (Combes-Thomas Estimates)**.** *For $p \in \mathbb{R}$, define*

$$A := 1 - p^2 \left\{ (H^\perp + 1)^{-\frac{1}{2}} \tfrac{1}{|x|^2} (H^\perp + 1)^{-\frac{1}{2}} \right\}, \qquad \textit{and} \tag{3.5.3}$$

$$B := -ip(H^\perp + 1)^{-\frac{1}{2}} \left\{ \tfrac{x}{|x|^2} \cdot \nabla + \nabla \cdot \tfrac{x}{|x|^2} \right\} (H^\perp + 1)^{-\frac{1}{2}}. \tag{3.5.4}$$

If $|p| < \sqrt{2}$, then $A + iB$ is invertible.

Proof. First, note that A and B are self-adjoint and $\sigma(A) = \sigma(A) \cap \mathbb{R}^+$. By Lemma 3.5.1, $A + iB$ is invertible provided that $d = \operatorname{dist}(\sigma(A), 0) > 0$. To determine the range of p for which $d > 0$, it behooves us to compute

$$\|(H^\perp + 1)^{-\frac{1}{2}} \tfrac{1}{|x|^2} (H^\perp + 1)^{-\frac{1}{2}}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)}.$$

Towards this end we note that, by (3.5.2),

$$|x|(1 + H^\perp)^{\frac{1}{2}}(1 + H^\perp)^{\frac{1}{2}}|x| = |x|(1 + H^\perp)|x| \geq |x|(1 + \tfrac{2}{|x|^2})|x| = 2 + |x|^2.$$

It follows, then, that

$$\|(H^\perp + 1)^{-\frac{1}{2}} |x|^{-2} (H^\perp + 1)^{-\frac{1}{2}}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)} \leq \tfrac{1}{2}. \tag{3.5.5}$$

By a TT^* argument, it follows that

$$\||x|^{-1}(H^\perp + 1)^{-\frac{1}{2}}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)} = \|(H^\perp + 1)^{-\frac{1}{2}}|x|^{-1}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)} \leq \tfrac{1}{\sqrt{2}}. \tag{3.5.6}$$

Consequently,

$$\|(H^\perp + 1)^{-\frac{1}{2}} \tfrac{x}{|x|^2}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)} \leq \tfrac{1}{\sqrt{2}}. \tag{3.5.7}$$

Thus, $d \geq 1 - \frac{p^2}{2}$ so $d > 0$ provided $|p| < \sqrt{2}$. It follows that $A + iB$ is invertible provided that $|p| < \sqrt{2}$, as desired. □

Corollary 3.5.4 (Weighted Resolvent Estimate)**.** *The operator*

$$|x|^{1-p}(H^\perp + 1)|x|^{1+p} : L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)$$

is bounded for all $p \in (-\sqrt{2}, \sqrt{2})$.

Proof. In the notation of Lemma 3.5.3, with A as defined in (3.5.3) and B as defined in (3.5.4), we see that

$$|x|^{1-p}(H^{\perp}+1)|x|^{1+p} = |x|(H^{\perp}+1)^{\frac{1}{2}}\left\{A+iB\right\}(H^{\perp}+1)^{\frac{1}{2}}|x|. \tag{3.5.8}$$

By (3.5.6), the operators $(H^{\perp}+1)^{\frac{1}{2}}|x|$ and $|x|(H^{\perp}+1)^{\frac{1}{2}}$ are boundedly invertible. By Lemma 3.5.3, $A+iB$ is boundedly invertible provided that $p \in (-\sqrt{2}, \sqrt{2})$. Thus, $|x|^{1-p}(H^{\perp}+1)|x|^{1+p}$ is bounded provided that $p \in (-\sqrt{2}, \sqrt{2})$, as claimed. $\square$

3.6 Kernel Estimates

In this section we prove kernel estimates for the operator $P^{\perp}(-\Delta+1)^{-1}$.

Lemma 3.6.1. *Fix $y \in \mathbb{R}^3$ and suppose $x \in \mathbb{R}^3$. If $2|x| < |y|$, then*

$$|P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)| \lesssim |x|^{\pm p}|y|^{-2}e^{-|y|/4}. \tag{3.6.1}$$

If $2|y| < |x|$, then

$$|P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)| \lesssim |y||x|^{-3\pm p}e^{-|x|/4}. \tag{3.6.2}$$

Proof. As $|x|^{-1\pm p}$ is radial it commutes with $P^{\perp}$ and so

$$|P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)| = |x|^{-1\pm p}\cdot|P^{\perp}((-\Delta+1)^{-1}\delta_y)|. \tag{3.6.3}$$

It suffices, then, to bound

$$P^{\perp}((-\Delta+1)^{-1}\delta_y)(x) = \int_{\mathbb{S}^2}\frac{e^{-|x-y|}}{|x-y|} - \frac{e^{-|\omega|x|-y|}}{|\omega|x|-y|}d\sigma(\omega), \tag{3.6.4}$$

where $d\sigma$ denotes the uniform probability measure on the unit sphere $\mathbb{S}^2$. By the mean value theorem, for each $\omega \in \mathbb{S}^2$ there exists some $c(\omega)$ between $|x-y|$ and $|\omega|x|-y|$ so that

$$\left|\tfrac{e^{-|x-y|}}{|x-y|} - \tfrac{e^{-|\omega|x|-y|}}{|\omega|x|-y|}\right| = \left|-e^{-c(\omega)}\left(\tfrac{c(\omega)+1}{c(\omega)^2}\right)(|x-y|-|\omega|x|-y|)\right| \tag{3.6.5}$$

$$\leq e^{-c(\omega)}\left(\tfrac{c(\omega)+1}{c(\omega)^2}\right)\tfrac{2|x-\omega|x||\cdot|y|}{|x-y|+|\omega|x|-y|}. \tag{3.6.6}$$

Suppose for now that $2|x| < |y|$. It follows, then, that

$$c(\omega) \geq \min\{|x-y|, |\omega|x|-y|\} \geq ||x|-|y|| > \tfrac{|y|}{2}.$$

Since the function $t \mapsto e^{-t}\left(\frac{t+1}{t^2}\right)$ is decreasing, we see that

$$e^{-c(\omega)}\left(\frac{c(\omega)+1}{c(\omega)^2}\right) \leq e^{-|y|/2}\left(\tfrac{2|y|+4}{|y|^2}\right) \lesssim \tfrac{e^{-|y|/4}}{|y|^2}. \tag{3.6.7}$$

Moreover, we see that

$$\tfrac{2|x-\omega|x||\cdot|y|}{|x-y|+|\omega|x|-y|} \leq \tfrac{4|x||y|}{|y|} \leq 4|x|. \tag{3.6.8}$$

Combining (3.6.6), (3.6.7), and (3.6.8) we get

$$\left|\tfrac{e^{-|x-y|}}{|x-y|} - \tfrac{e^{-|\omega|x|-y|}}{|\omega|x|-y|}\right| \lesssim |x||y|^{-2}e^{-|y|/4}. \tag{3.6.9}$$

Integrating over the sphere, we combine (3.6.9) with (3.6.3) to deduce (3.6.1).

If instead $2|y| < |x|$, then the left hand side of (3.6.9) is bounded by the right hand side of (3.6.9) with $|y|$ replaced with $|x|$ and *vice versa*. This gives (3.6.2). $\square$

Lemma 3.6.2. *If $|p| < \frac{3}{2}$ and $y \in \mathbb{R}^3$, then*

$$\|P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)\|_{L^2_x(\mathbb{R}^2)} \lesssim |y|^{\pm p}\min\{|y|^{-\frac{1}{2}}, |y|^{-1}\}.$$

Proof. Fix some $y \in \mathbb{R}^3$. We split the integral into three regions

$$\left(\int_A + \int_B + \int_C\right)|P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)|^2\,dx = I_1 + I_2 + I_3, \tag{3.6.10}$$

where $A = \{2|x| < |y|\}$, $B = \{|x| > 2|y|\}$, and $C = \{|y|/2 \leq |x| \leq 2|y|\}$. On the region A, we employ (3.6.1) to see that

$$\begin{aligned}\int_A |P^{\perp}(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)|^2\,dx &\lesssim |y|^{-4}e^{-|y|/2}\int_A |x|^{\pm 2p}\,dx \\ &\lesssim e^{-|y|/2}|y|^{-1\pm 2p} \\ &\lesssim |y|^{\pm 2p}(|y|^{-2} \wedge |y|^{-1}),\end{aligned}$$

provided that $|p| < \frac{3}{2}$.

On the region B, we employ (3.6.2) to see that

$$\begin{aligned}\int_B |P^\perp(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)|^2\ dx &\lesssim |y|^2 \int_B |x|^{-6\pm 2p} e^{-|x|/2}\ dx \\ &\lesssim |y|^2 e^{-|y|} \int_{2|y|}^\infty r^{-4\pm 2p}\ dr \\ &\lesssim |y|^{-1\pm 2p} e^{-|y|} \\ &\lesssim |y|^{\pm 2p}(|y|^{-2} \wedge |y|^{-1}),\end{aligned}$$

provided that $|p| < \frac{3}{2}$.

Since $P^\perp : L^2_x(C) \to L^2_x(C)$ is bounded, we see that

$$\begin{aligned}\int_C |P^\perp(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)|^2\ dx &\lesssim \int_C |x|^{-2\pm 2p}|(-\Delta+1)^{-1}\delta_y)|^2\ dx && (3.6.11)\\ &\lesssim \int_C |x|^{-2\pm 2p} \tfrac{e^{-2|x-y|}}{|x-y|^2} dx && (3.6.12)\\ &\lesssim |y|^{-2\pm 2p} \int_C \tfrac{e^{-2|x-y|}}{|x-y|^2}\ dx. && (3.6.13)\end{aligned}$$

We may further split $C = C_1 \cup C_2$ where $C_1 = \{|x-y| < \frac{|y|}{2}\}$ and $C_2 = C \setminus C_1$. On C_1 we see by a change of coordinates that

$$\int_{C_1} \frac{e^{-2|x-y|}}{|x-y|^2}\ dx \lesssim \int_0^{\frac{|y|}{2}} e^{-2r}\ dr = 1 - e^{-|y|} \lesssim (1 \wedge |y|). \tag{3.6.14}$$

On C_2 we know that $|x-y| \geq \frac{|y|}{2}$ and so

$$\int_{C_2} \frac{e^{-2|x-y|}}{|x-y|^2}\ dx \lesssim \frac{e^{-|y|}}{|y|^2} \int_{C_2} 1\ dx \lesssim |y| e^{-|y|} \lesssim (1 \wedge |y|). \tag{3.6.15}$$

It follows, then, that

$$\int_C |P^\perp(|x|^{-1\pm p}(-\Delta+1)^{-1}\delta_y)|^2\ dx \lesssim |y|^{\pm 2p}(|y|^{-2} \wedge |y|^{-1}).$$

Summing over the regions, we have our desired bound. □

Corollary 3.6.3. *The integral kernel of $P^\perp((H+1)^{-1} - (-\Delta+1)^{-1})$, namely,*

$$G(z,y) = \langle \delta_z, P^\perp((H+1)^{-1} - (-\Delta+1)^{-1})\delta_y \rangle, \tag{3.6.16}$$

obeys the bound

$$|G(z,y)| \lesssim \left(\tfrac{|z|\wedge|y|}{|z|+|y|}\right)^p (|z|^{-1} \wedge |z|^{-\frac{1}{2}})(|y|^{-1} \wedge |y|^{-\frac{1}{2}}) \tag{3.6.17}$$

for all $p \in (-\sqrt{2}, \sqrt{2})$.

Proof. Given two invertible operators A, B, we have the resolvent identity

$$A^{-1} - B^{-1} = B^{-1}(B - A)B^{-1} + B^{-1}(B - A)A^{-1}(B - A)B^{-1}. \tag{3.6.18}$$

As $(-\Delta + 1) - (H + 1) = \frac{1}{4}|x|^{-2}$, by (3.6.18) we see that

$$P^{\perp}(H + 1)^{-1} - P^{\perp}(-\Delta + 1)^{-1} = (\mathrm{I}) + (\mathrm{II}), \tag{3.6.19}$$

where

$$(\mathrm{I}) := \tfrac{1}{4} \cdot P^{\perp}(-\Delta + 1)^{-1}|x|^{-2}P^{\perp}(-\Delta + 1)^{-1} \qquad \text{and} \tag{3.6.20}$$

$$(\mathrm{II}) := \tfrac{1}{16} \cdot P^{\perp}(-\Delta + 1)^{-1}|x|^{-2}P^{\perp}(H + 1)^{-1}|x|^{-2}P^{\perp}(-\Delta + 1)^{-1}. \tag{3.6.21}$$

Let $G_{(\mathrm{I})}(z, y)$ and $G_{(\mathrm{II})}(z, y)$ be the integral kernels of (I) and (II), respectively, defined as above. It follows, then, from Cauchy-Schwarz and (3.6.2) that

$$\begin{aligned} |G_{(\mathrm{I})}(z, y)| &\lesssim \langle |x|^{-1-p}P^{\perp}(-\Delta + 1)^{-1}\delta_z, |x|^{-1+p}P^{\perp}(-\Delta + 1)^{-1}\delta_y\rangle_{L^2_x(\mathbb{R}^2)} \\ &\lesssim \||x|^{-1-p}P^{\perp}(-\Delta + 1)^{-1}\delta_z\|_{L^2_x(\mathbb{R}^3)}\||x|^{-1+p}P^{\perp}(-\Delta + 1)^{-1}\delta_y\|_{L^2_x(\mathbb{R}^3)} \\ &\lesssim |z|^{-p}|y|^{p}(|y|^{-\frac{1}{2}} \wedge |y|^{-1})(|z|^{-\frac{1}{2}} \wedge |z|^{-1}). \end{aligned} \tag{3.6.22}$$

Since (I) is self-adjoint, $G_{(\mathrm{I})}$ is conjugate symmetric and hence obeys the same bounds with the roles of $|z|$ and $|y|$ reversed. So

$$|G_{(\mathrm{I})}(z, y)|| \lesssim \left(\tfrac{|z| \wedge |y|}{|z| + |y|}\right)^{p} (|z|^{-1} \wedge |z|^{-\frac{1}{2}})(|y|^{-1} \wedge |y|^{-\frac{1}{2}}), \tag{3.6.23}$$

provided that $|p| < \frac{3}{2}$. For $|p| < \sqrt{2}$, we know by Corollary 3.5.4 that

$$C_p := \||x|^{-1+p}P^{\perp}(H + 1)^{-1}|x|^{-1-p}\|_{L^2_x(\mathbb{R}^3) \to L^2_x(\mathbb{R}^3)} \lesssim_p 1.$$

It follows then by (3.6.2), as before, that

$$\begin{aligned} |G_{(\mathrm{II})}(z, y)| &\lesssim C_p\||x|^{-1-p}P^{\perp}(-\Delta + 1)^{-1}\delta_z\|_{L^2_x(\mathbb{R}^3)}\||x|^{-1+p}P^{\perp}(-\Delta + 1)^{-1}\delta_y\|_{L^2_x(\mathbb{R}^3)} \\ &\lesssim C_p|z|^{-p}|y|^{p}(|y|^{-\frac{1}{2}} \wedge |y|^{-1})(|z|^{-\frac{1}{2}} \wedge |z|^{-1}). \end{aligned} \tag{3.6.24}$$

Again by conjugate symmetry of $G_{(\mathrm{II})}$, we see that $G_{(\mathrm{II})}$ obeys (3.6.23), as claimed. $\square$

3.7 Proofs of Main Results

In this section we begin by proving Theorem 3.2.2, and conclude by deducing a number of important Bernstein estimates.

Theorem 3.2.2. Let $(q, s) \in [1, \infty) \times [1, \infty)$ satisfy

$$\tfrac{4}{3} \leq \tfrac{1}{q} + \tfrac{1}{s'} \leq \tfrac{5}{3}. \tag{3.7.1}$$

If, additionally,

$$\sqrt{2} > \max\{2 - \tfrac{3}{s'}, 2 - \tfrac{3}{q}, \tfrac{3}{q} - \tfrac{5}{2}, \tfrac{3}{s'} - \tfrac{5}{2}\}, \tag{3.7.2}$$

then

$$\|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})\|_{L^q_x(\mathbb{R}^3) \to L^s_x(\mathbb{R}^3)} < \infty. \tag{3.7.3}$$

Proof. We will only treat the case $q, s \in [1,2] \times [1,2]$. For if $(q, s) \in [2, \infty) \times [2, \infty)$ satisfy (3.7.1) and (3.7.2), then (s', q') belongs to $[1, 2] \times [1, 2]$ and satisfy (3.7.1) and (3.7.2). If the theorem is known to be true in this case, then

$$\|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})\|_{L^{s'}_x(\mathbb{R}^3) \to L^{q'}_x(\mathbb{R}^3)} < \infty,$$

but then by duality it must also follow that

$$\|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})\|_{L^q_x(\mathbb{R}^3) \to L^s_x(\mathbb{R}^3)} < \infty.$$

Now, by duality, it suffices to show that

$$\sup_{||g||_{L^{s'}_y(\mathbb{R}^3)} \leq 1} \sup_{||f||_{L^q_x(\mathbb{R}^3)} \leq 1} \left| \iint_{\mathbb{R}^3 \times \mathbb{R}^3} G(x,y) f(x) g(y)\ dxdy \right| < \infty.$$

Suppose, then, that $||f||_{L^q_x(\mathbb{R}^3)} \leq 1$ and $||g||_{L^{s'}_y(\mathbb{R}^3)} \leq 1$. As G depends only on $|x|$ and $|y|$ we may, and do, assume without loss of generality that $g = g(|y|)$ and $f = f(|x|)$ are radial. There are precisely six regions to consider, each case corresponding to an ordering of $|x|$, $|y|$, and 1 (e.g. $|x| \geq |y| \geq 1$, $|x| \geq 1 \geq |y|$, etc.). By symmetry, it suffices to treat the three cases in which we always have $|x| \geq |y|$.

(1) With $A = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \geq |y| \geq 1\}$, we have by Corollary 3.6.3, for any $p \in (-\sqrt{2}, \sqrt{2})$, that

$$\begin{aligned}
&\int_A |G(x,y)f(x)g(y)|\ dxdy \\
&\lesssim \int_1^\infty \int_1^r r^{1-p}\rho^{1+p}|f(r)||g(\rho)|d\rho dr \\
&\lesssim \int_1^\infty \int_1^r r^{1-\frac{2}{q}-p}\rho^{1-\frac{2}{s'}+p}|f(r)r^{\frac{2}{q}}||g(\rho)\rho^{\frac{2}{s'}}|d\rho dr \\
&\lesssim \sum_{k=0}^\infty \sum_{j=0}^k 2^{k(1-\frac{2}{q}+\frac{1}{q'}-p)}2^{j(1-\frac{2}{s'}+\frac{1}{s}+p)}||f||_{L_x^q(|x|\sim 2^k)}||g||_{L_y^{s'}(|y|\sim 2^j)} \\
&= \sum_{0\leq j\leq k} \left(\tfrac{2^j}{2^k}\right)^{p+\frac{3}{q}-2} 2^{j(4-\frac{3}{q}-\frac{3}{s'})}||f||_{L_x^q(|x|\sim 2^k)}||g||_{L_y^{s'}(|y|\sim 2^j)}
\end{aligned} \tag{3.7.4}$$

We may choose p so that $2 - \frac{3}{q} - p < 0$. Since $\frac{1}{q} + \frac{1}{s'} \geq \frac{4}{3}$ we see that by Schur's Test

$$\begin{aligned}
\int_A |G(x,y)f(x)g(y)|\ dxdy &\leq \left(\sum_{k=0}^\infty ||f||^2_{L_x^q(|x|\sim 2^k)}\right)^{\frac{1}{2}} \left(\sum_{k=0}^\infty ||g||^2_{L_y^{s'}(|x|\sim 2^j)}\right)^{\frac{1}{2}} \\
&\lesssim ||f||_{L_x^q(\mathbb{R}^3)}||g||_{L_y^{s'}(\mathbb{R}^3)} \\
&\lesssim 1,
\end{aligned} \tag{3.7.5}$$

since $q, s' \leq 2$.

(2) With $B = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : |x| \geq 1 \geq |y|\}$, we similarly have by Corollary 3.6.3, for any $p \in (-\sqrt{2}, \sqrt{2})$, that

$$\begin{aligned}
&\int_B |G(x,y)f(x)g(y)|\ dxdy \\
&\lesssim \int_1^\infty \int_0^1 r^{1-\frac{2}{q}-p}\rho^{\frac{3}{2}-\frac{2}{s'}+p}|f(r)r^{\frac{2}{q}}||g(\rho)\rho^{\frac{2}{s'}}|d\rho dr \\
&\lesssim \sum_{k=0}^\infty \sum_{j=0}^\infty 2^{k(1-\frac{2}{q}+\frac{1}{q'}-p)}2^{-j(\frac{3}{2}-\frac{2}{s'}+\frac{1}{s}+p)}||f||_{L_x^q(|x|\sim 2^k)}||g||_{L_y^{s'}(|y|\sim 2^j)}.
\end{aligned} \tag{3.7.6}$$

Choosing p so that both $\frac{5}{2} - \frac{3}{s'} + p > 0$ and, simultaneously, $2 - \frac{3}{q} - p < 0$, by Cauchy-Schwarz, we see that

$$\int_B |G(x,y)f(x)g(y)|\ dxdy \lesssim ||f||_{L_x^q(\mathbb{R}^3)}||g||_{L_y^{s'}(\mathbb{R}^3)} \lesssim 1. \tag{3.7.7}$$

(3) With $C = \{(x,y) \in \mathbb{R}^3 \times \mathbb{R}^3 : 1 \geq |x| \geq |y|\}$, we similarly compute that

$$\begin{aligned}
&\int_C |G(x,y)f(x)g(y)| \ dxdy \\
&\lesssim \int_0^1 \int_0^r r^{\frac{3}{2}-p} \rho^{\frac{3}{2}+p} |f(r)||g(\rho)| d\rho dr \\
&\lesssim \int_0^1 \int_0^r r^{\frac{3}{2}-\frac{2}{q}-p} \rho^{\frac{3}{2}-\frac{2}{s'}+p} |f(r) r^{\frac{2}{q}}||g(\rho)\rho^{\frac{2}{s'}}| d\rho dr \\
&\lesssim \sum_{0 \leq k \leq j} \left(\frac{2^{-j}}{2^{-k}}\right)^{\frac{5}{2}-\frac{3}{s'}+p} 2^{-k(5-\frac{3}{s'}-\frac{3}{q})} ||f||_{L^q_x(|x|\sim 2^k)} ||g||_{L^{s'}_y(|y|\sim 2^j)} \qquad (3.7.8)
\end{aligned}$$

We may choose a suitable p satisfying $\frac{5}{2} - \frac{3}{s'} + p > 0$. As $\frac{3}{s'} + \frac{3}{q} \leq 5$, by Schur's Test we get

$$\int_C |G(x,y)f(x)g(y)| \ dxdy \lesssim 1, \qquad (3.7.9)$$

as desired.

□

We now pause to enjoy a few specific dividends of Theorem 3.2.2.

Corollary 3.2.3 (Sharp Sobolev Embedding). If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\|P^{\perp}(H+1)^{-1} f\|_{L^6_x(\mathbb{R}^3)} \lesssim ||f||_{L^2_x(\mathbb{R}^3)}. \qquad (3.7.10)$$

Proof. If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\begin{aligned}
&\|P^{\perp}(H+1)^{-1} f\|_{L^6_x(\mathbb{R}^3)} && (3.7.11)\\
&\leq \|P^{\perp}(-\Delta+1)^{-1} f\|_{L^6_x(\mathbb{R}^3)} + \|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1}) f\|_{L^6_x(\mathbb{R}^3)} && (3.7.12)\\
&\lesssim ||f||_{L^2_x(\mathbb{R}^3)} && (3.7.13)
\end{aligned}$$

by Theorem 3.2.2 since the pair $(2,6)$ satisfies

$$\tfrac{4}{3} = \tfrac{1}{2} + \tfrac{5}{6} \leq \tfrac{5}{3} \qquad \text{and} \qquad \sqrt{2} > \max\{-\tfrac{1}{2}, \tfrac{1}{2}, -1, 0\}.$$

□

Corollary 3.2.4. If $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\|P^{\perp}(H+1)^{-1}f\|_{L_x^\infty(\mathbb{R}^3)} \lesssim ||f||_{L_x^2(\mathbb{R}^3)} \tag{3.7.14}$$

$$\|P^{\perp}(H+1)^{-1}f\|_{L_x^2(\mathbb{R}^3)} \lesssim ||f||_{L_x^1(\mathbb{R}^3)}. \tag{3.7.15}$$

Proof. To see the first inequality, note that if $f : \mathbb{R}^3 \to \mathbb{C}$ is Schwartz, then

$$\|P^{\perp}(H+1)^{-1}f\|_{L_x^\infty(\mathbb{R}^3)} \tag{3.7.16}$$

$$\leq \|P^{\perp}(-\Delta+1)^{-1}f\|_{L_x^\infty(\mathbb{R}^3)} + \|P^{\perp}((H+1)^{-1} - (-\Delta+1)^{-1})f\|_{L_x^\infty(\mathbb{R}^3)} \tag{3.7.17}$$

$$\lesssim ||f||_{L_x^2(\mathbb{R}^3)}. \tag{3.7.18}$$

by Theorem 3.2.2 since the pair $(2, \infty)$ satisfies

$$\tfrac{4}{3} \leq \tfrac{1}{2} + 1 \leq \tfrac{5}{3} \qquad \text{and} \qquad \sqrt{2} > \max\{-1, \tfrac{1}{2}, -1, \tfrac{1}{2}\}.$$

The second inequality follows from duality. $\square$

Theorem 3.2.5 (Bernstein Estimates.) The operator $P^{\perp}e^{-H} : L^1 \to L^\infty$ is bounded.

Proof. First, write

$$P^{\perp}e^{-H} = (H+1)^{-1}P^{\perp}e^{-H}(H+1)^2(H+1)^{-1}P^{\perp}.$$

It follows that

$$\begin{aligned}\|P^{\perp}e^{-H}\|_{L_x^1 \to L_x^\infty} &= \|(H+1)^{-1}P^{\perp}e^{-H}(H+1)^2(H+1)^{-1}P^{\perp}\|_{L_x^1 \to L_x^\infty} \\ &\leq \|(H+1)^{-1}P^{\perp}\|_{L_x^2 \to L_x^\infty}\|e^{-H}(H+1)^2\|_{L_x^2 \to L_x^2}\|(H+1)^{-1}P^{\perp}\|_{L_x^1 \to L_x^2}.\end{aligned}$$

By (3.7.14), $\|(H+1)^{-1}P^{\perp}\|_{L_x^2 \to L_x^1} < \infty$. By (3.7.15), $\|(H+1)^{-1}P^{\perp}\|_{L_x^1 \to L_x^2} < \infty$. Lastly, $\|e^{-H}(H+1)^2\|_{L_x^2 \to L_x^2} < \infty$ by the Spectral Theorem. $\square$

REFERENCES

[AT00] S. Adachi and K. Tanaka. "Trudinger type inequalities in $\mathbf{R}^N$ and their best exponents." *Proc. Amer. Math. Soc.*, **128**(7):2051–2057, 2000.

[Azz15] Alexander Adam Azzam. "Scattering for the two-dimensional NLS with (full) exponential nonlinearity." *Preprint* `arXiv:1511.03386`, 2015.

[Azz17] Alexander Adam Azzam. "Resolvent Estimates for the 3D Schrödinger Operator with Inverse-Square Potential." *Preprint* `arXiv:1712.01366`, 2017.

[BIP14] H. Bahouri, S. Ibrahim, and G. Perelman. "Scattering for the critical 2-D NLS with exponential growth." *Differential Integral Equations*, **27**(3-4):233–268, 2014.

[Bir10] A. Biryuk. "An optimal limiting 2D Sobolev inequality." *Proc. Amer. Math. Soc.*, **138**(4):1461–1470, 2010.

[BM16] Jean-Marc Bouclet and Haruya Mizutani. "Uniform resolvent and Strichartz estimates for Schrödinger equations with critical singularities." *Preprint* `arXiv:1607.01187`, 2016.

[Bou98] J. Bourgain. "Scattering in the energy space and below for 3D NLS." *J. Anal. Math.*, **75**:267–297, 1998.

[Bou99] J. Bourgain. "Global wellposedness of defocusing critical nonlinear Schrödinger equation in the radial case." *J. Amer. Math. Soc.*, **12**(1):145–171, 1999.

[BPS03] Nicolas Burq, Fabrice Planchon, John G. Stalker, and A.Shadi Tahvildar-Zadeh. "Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential." *Journal of Functional Analysis*, **203**(2):519 – 549, 2003.

[Cas50] K. M. Case. "Singular Potentials." *Phys. Rev.*, **80**:797–806, Dec 1950.

[Caz03] T. Cazenave. *Semilinear Schrödinger equations*, volume 10 of *Courant Lecture Notes in Mathematics.* New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[CGT09] J. Colliander, M. Grillakis, and N. Tzirakis. "Tensor products and correlation estimates with applications to nonlinear Schrödinger equations." *Comm. Pure Appl. Math.*, **62**(7):920–968, 2009.

[CIM09] J. Colliander, S. Ibrahim, M. Majdoub, and N. Masmoudi. "Energy critical NLS in two space dimensions." *J. Hyperbolic Differ. Equ.*, **6**(3):549–575, 2009.

[CKS08] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. "Global well-posedness and scattering for the energy-critical nonlinear Schrödinger equation in $\mathbb{R}^3$." *Ann. of Math. (2)*, **167**(3):767–865, 2008.

[CW89] T. Cazenave and F. B. Weissler. "Some remarks on the nonlinear Schrödinger equation in the subcritical case." In *New methods and results in nonlinear field equations (Bielefeld, 1987)*, volume 347 of *Lecture Notes in Phys.*, pp. 59–69. Springer, Berlin, 1989.

[CW90] T. Cazenave and F. B. Weissler. "The Cauchy problem for the critical nonlinear Schrödinger equation in H^s." *Nonlinear Anal.*, **14**(10):807–836, 1990.

[Dod16] B. Dodson. "Global well-posedness and scattering for the defocusing, L^2-critical, nonlinear Schrödinger equation when $d = 2$." *Duke Math. J.*, **165**(18):3435–3516, 2016.

[EF06] T. Ekholm and R. L. Frank. "On Lieb-Thirring Inequalities for Schrödinger Operators with Virtual Level." *Communications in Mathematical Physics*, **264**(3):725–740, 2006.

[His00] P.D. Hislop. "Exponential decay of two-body eigenfunctions: A review." *Electronic Journal of Differential Equations*, (04):265–288, 2000.

[IMM07] S. Ibrahim, M. Majdoub, and N. Masmoudi. "Double logarithmic inequality with a sharp constant." *Proc. Amer. Math. Soc.*, **135**(1):87–97 (electronic), 2007.

[IMM12] S. Ibrahim, M. Majdoub, N. Masmoudi, and K. Nakanishi. "Scattering for the two-dimensional NLS with exponential nonlinearity." *Nonlinearity*, **25**(6):1843–1849, 2012.

[IMN15] S. Ibrahim, N. Masmoudi, and K. Nakanishi. "Trudinger - Moser inequality on the whole plane with the exact growth condition." *Journal of the European Mathematical Society*, pp. 819–835, 2015.

[KMV15a] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng. "The energy-critical NLS with inverse-square potential." *Preprint* *arXiv:1509.05822*, 2015.

[KMV15b] R. Killip, C. Miao, M. Visan, J. Zhang, and J. Zheng. "Sobolev spaces adapted to the Schrödinger operator with inverse-square potential." *Preprint* *arXiv:1503.02716*, 2015.

[KSW75] H. Kalf, U.-W. Schmincke, J. Walter, and R. Wüst. *On the spectral theory of Schrödinger and Dirac operators with strongly singular potentials*, pp. 182–226. Springer Berlin Heidelberg, Berlin, Heidelberg, 1975.

[KT98] M. Keel and T. Tao. "Endpoint Strichartz estimates." *Amer. J. Math.*, **120**(5):955–980, 1998.

[KTV09] R. Killip, T. Tao, and M. Visan. "The cubic nonlinear Schrödinger equation in two dimensions with radial data." *J. Eur. Math. Soc. (JEMS)*, **11**(6):1203–1258, 2009.

[KV13] R. Killip and M. Visan. *Evolution equations*, volume 17 of *Clay Mathematics Proceedings*. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2013. Lecture notes from the Clay Mathematics Institute Summer School held at the Eidgenössische Technische Hochschule (ETH), Zürich, June 23–July 18, 2008.

[L 73] J.M. Combes L. Thomas. "Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators." *Commun. Math. Phys*, **34**:251–270, 1973.

[Miz16] Haruya Mizutani. "Remarks on endpoint Strichartz estimates for Schrödinger equations with the critical inverse-square potential." *Preprint arXiv:1607.02848*, 2016.

[Mos71] J. Moser. "A sharp form of an inequality by N. Trudinger." *Indiana Univ. Math. J.*, **20**:1077–1092, 1970/71.

[Nak99] K. Nakanishi. "Energy scattering for nonlinear Klein-Gordon and Schrödinger equations in spatial dimensions 1 and 2." *J. Funct. Anal.*, **169**(1):201–225, 1999.

[PST03] Fabrice Planchon, John G. Stalker, and A. Shadi Tahvildar-Zadeh. "L^p Estimates for the wave equation with the inverse-square potential." *Discrete and Continuous Dynamical Systems*, **9**(2):427–442, 2003.

[PV09] F. Planchon and L. Vega. "Bilinear virial identities and applications." *Ann. Sci. Éc. Norm. Supér. (4)*, **42**(2):261–290, 2009.

[Ruf05] B. Ruf. "A sharp Trudinger-Moser type inequality for unbounded domains in $\mathbb{R}^2$." *J. Funct. Anal.*, **219**(2):340–367, 2005.

[RV07] E. Ryckman and M. Visan. "Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equation in $\mathbb{R}^{1+4}$." *Amer. J. Math.*, **129**(1):1–60, 2007.

[Suz16] Toshiyuki Suzuki. "Solvability of Nonlinear Schrödinger Equations with Some Critical Singular Potential via Generalized Hardy-Rellich Inequalities." *Funkcialaj Ekvacioj*, **59**(1):1–34, 2016.

[Tru67] N.S. Trudinger. "On imbeddings into Orlicz spaces and some applications." *J. Math. Mech.*, **17**:473–483, 1967.

[TVZ07] T. Tao, M. Visan, and X. Zhang. "The nonlinear Schrödinger equation with combined power-type nonlinearities." *Comm. Partial Differential Equations*, **32**(7-9):1281–1343, 2007.

[Vis07] M. Visan. "The defocusing energy-critical nonlinear Schrödinger equation in higher dimensions." *Duke Math. J.*, **138**(2):281–374, 2007.