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CANCELLATION OF DIVERGENCES
IN REGGEIZED ELECTROMAGNETIC MASS DIFFERENCES*

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ABSTRACT

By use of Reggeology and the Cottingham formula, the divergences of electromagnetic mass differences are isolated in Minkowski space. Sum rules are derived which are necessary and sufficient to cancel them; they also turn out to be the necessary and sufficient conditions for the Wick rotation to Euclidean space via $q_0 \to iq_0$. Bjorken's argument for logarithmic divergences due to current algebra is extended to the case where subtractions are needed in the dispersion relations. The connection between the divergence, the commutator $\left[[j_\mu(0,x),H],\ j^\mu(0)\right]$, and the fixed pole at $J=0$ is clarified. The asymptotic value of the fixed-pole residue is found in terms of this commutator.

I. INTRODUCTION

In Cottingham's formulation electromagnetic mass differences are calculated in Euclidean space, i.e. via the substitution $q_0 \to iq_0$ in the integral over the photon 4-momentum q. This trick, the so-called Wick rotation, has the advantage of putting the entire integrand into the physical region. However, since several authors[1,2] have obtained divergent results by this method, the validity of this shift of the integration contour ought to be investigated. We shall therefore leave everything in Minkowski space, use Reggeology to isolate the divergences, and establish sum rules which are necessary and sufficient for their cancellation; the latter will also turn out to be equivalent to Wick rotatability.

Next we shall show that Bjorken's argument[1] for logarithmic divergences due to the nonvanishing of the commutator $\left[[j_\mu(0,x),H],\ j^\mu(0)\right]$ holds even in the $\Delta I=1$ case where a subtraction is needed in the dispersion relation. The connection between the logarithmic divergence, the quantity $C = \int d^3x\ \langle p|\left[[j_\mu(0,x),H],\ j^\mu(0)\right]|p\rangle$, and the fixed[3] pole at $J = 0$ is clarified. In particular we shall show that the fixed-pole residue $B_0(q^2) \to C/q^2$ at infinity. It is this property of the fixed pole which causes the logarithmic divergence, both in Minkowski and in Euclidean space. While it is probable that the Regge divergences will be cancelled by our sum rules, there does not seem to be any way out of Bjorken's logarithmic divergence except by demanding that $C = 0$.

Finally we shall point out why the divergence claimed by Cottingham and Gibb[2] is spurious (they make the wrong subtraction in their dispersion relation).

II. COMPTON SCATTERING AND ELECTROMAGNETIC MASS SHIFTS

In perturbation theory the electromagnetic self mass of a hadron to first order in $\alpha = e^2/4\pi$ is given by[4]

$$\Delta m = \frac{i}{(2\pi)^2} \, \frac{1}{2} \int \frac{d^4q}{q^2+i\epsilon} \, T(\underset{\sim}{q}, q_0), \tag{2.1}$$

where $T = T_\mu{}^\mu$, and $\epsilon^{*\mu} \, T_{\mu\lambda}(\underset{\sim}{q}, q_0) \, \epsilon^\lambda$ is the forward Compton scattering amplitude for virtual photons of mass $q^2 = q_0{}^2 - \underset{\sim}{q}^2$ and polarizations ϵ_μ and ϵ_λ (Fig. 1). The usual assumptions of Lorentz invariance and current conservation allow us to write

$$\begin{aligned} T_{\mu\lambda} = {} & [q^2 g_{\mu\lambda} - q_\mu \, q_\lambda] t_1 \\ & + [\nu^2 g_{\mu\lambda} + \frac{q^2}{m^2} p_\mu \, p_\lambda - \frac{\nu}{m}(p_\mu q_\lambda + p_\lambda q_\mu)] t_2 \; . \end{aligned} \tag{2.2}$$

Since an average over the hadron spin is understood, there are only two invariant amplitudes; they are functions of the Lorentz scalars q^2 and $\nu = p \cdot q/m$. We shall work in the rest frame of the hadron, where $\underset{\sim}{p} = 0$, $p_0 = m$, and $\nu = q_0$.

The t_i and hence T are crossing-symmetric in $\nu = (s-u)/4m$:

$$t_i(q^2, \nu) = t_i(q^2 \, , -\nu).$$

With spherical and crossing symmetry, Eq. (2.1) integrates to

$$\Delta m = \frac{i}{2\pi} \int_0^\infty d\nu \int_{-\infty}^{\nu^2} \frac{dq^2}{q^2 + i\epsilon} \, (\nu^2 - q^2)^{\frac{1}{2}} \; T(q^2, \nu). \tag{2.3}$$

The invariant amplitudes t_i, their relation to the helicity amplitudes, their analyticity properties, and their high energy $(\nu \to \infty)$ limits have been discussed in detail by, for example, Pagels and Gross.[5] Here we shall just summarize the results that we are going to use. The t_i are real analytic functions in the cut ν plane. The cut starts at the inelastic threshold $\nu_t = m_\pi + (m_\pi^2 - q^2)/2m$ and runs along the real axis to $+\infty$. The t_i contain a Born term

$$t_{iB} = \frac{4mq^2 f_i(q^2)}{q^4 - 4m^2\nu^2 + i\epsilon q^2} \tag{2.4}$$

where the f_i are bilinear combinations of form factors.

For the $\nu \to \infty$ limit we follow Harari's suggestion[6] and use Regge theory. The cosine of the t-channel scattering angle at $t = 0$ is

$$z_t(t = 0) = \frac{s - u}{4\ m\ q} = \frac{\nu}{q} , \qquad q = \sqrt{q^2} \tag{2.5}$$

The Regge limit is extracted from a Sommerfeld-Watson transform when $z_t \to \infty$, that is in our case $\nu^2 \gg |q^2|$. Pure Regge behavior would imply

$$t_1(q^2,\nu) \longrightarrow B_1(q^2)\ \nu^\alpha \quad \text{and} \quad t_2(q^2,\nu) \longrightarrow B_2(q^2)\ \nu^{\alpha-2}$$

where $\alpha = \alpha^{(I)}(t = 0)$ is the $t = 0$ intercept of the leading Regge trajectory with the quantum numbers of the t-channel, in particular $\alpha^{(I = 1)} = \alpha_{A2} \approx 0.4 > 0$ for the $\Delta I = 1$ mass differences $m(n) - m(p)$, $m(\Sigma^-) - m(\Sigma^+)$, $m(\Xi^-) - m(\Xi^o)$, and $m(K^o) - m(K^+)$. For the $\Delta I = 2$ mass differences $m(\pi^+) - m(\pi^o)$ and $m(\Sigma^+) + m(\Sigma^-) - 2m(\Sigma^o)$, one assumes $\alpha^{(I = 2)} < 0$ in the absence of any known low-lying $I = 2$ mesons. This is Harari's familiar argument about the necessity of subtractions in the dispersion relations for the t_i.

Since we are dealing with a nonstrong amplitude, we have to admit the possibility of fixed poles.[3] Our amplitudes are crossing even and can have fixed poles at $J = 0, -2, -4 \cdots$. Therefore we must write the $\nu \to \infty$ limit in the form

$$t_1(q^2,\nu) \longrightarrow \sum_i B_{1i}(q^2)\,\nu^{\alpha_i} \quad \text{and} \quad t_2(q^2,\nu) \longrightarrow \sum_i B_{2i}(q^2)\,\nu^{\alpha_i - 2}\,, \tag{2.6}$$

where the sum goes over all Regge and fixed poles that contribute to the limit. The residue functions $B(q^2)$ for the Regge pole or poles contain the signature factor

$$B(q^2) = \left(\frac{e^{-i\pi\alpha} + 1}{\sin \pi\alpha}\right)\beta(q^2). \tag{2.7}$$

We shall omit the I-spin label; our results will apply to both cases.

In the following we shall work only with the complete amplitude

$$T = 3q^2 t_1 + (q^2 + 2\nu^2)t_2 \tag{2.8}$$

Its high energy limit is

$$T(q^2,\nu) \longrightarrow \sum_i B_i(q^2)\,\nu^{\alpha_i}\,, \tag{2.9}$$

with $B_i(q^2) = 3q^2 B_{1i}(q^2) + B_{2i}(q^2)$. Because $T(q^2,\nu)$ is analytic at $q^2 = 0$, Eq.(2.9) should remain valid when q^2 approaches zero, even though $z_t = \nu/q$ becomes singular. If a pole term does not contribute at some value of q^2, e.g. at $q^2 = 0$, then this would simply correspond to the vanishing of its residue.

III. DIVERGENCES IN MINKOWSKI SPACE AND CONDITIONS FOR CANCELLATIONS

A. Isolation of the Divergences

To compute Δm in Eq.(2.3), Cottingham[4] performs a Wick rotation from Minkowski to Euclidean space, i.e., he rotates the integration contour from q_0 to iq_0 in order to obtain an integral over spacelike photons only for which $T(q^2,\nu)$ can be measured in electron scattering experiments. In his formula,

$$\Delta m_{\text{Euclid}} = -\frac{1}{2\pi}\int_{-\infty}^{0}\frac{dq^2}{q^2}\int_0^{\sqrt{-q^2}} d\nu\,(-\nu^2-q^2)^{\frac{1}{2}}\;T(q^2,\,i\nu), \qquad (3.1)$$

the values of T at imaginary energies $i\nu$ are obtained from $T(q^2,\nu)$ via a dispersion relation. Cottingham justifies the Wick rotation by showing that no singularities in the q_0 plane are crossed. However, he did not investigate whether the quarter circle at $|q_0| = \infty$ contributes anything. As we shall see, this is not satisfied unless certain sum rules provide cancellations.

In Minkowski space Δm, Eq.(2.3), has contributions from the region $\nu^2 \to \infty$ at finite q^2 (see Fig. 2), where we can substitute the Regge expansion, Eq.(2.9). Obviously these terms diverge like $\Lambda^{2+\alpha}$ for $\alpha > -2$ or like $\log\Lambda$ for $\alpha = -2$:

$$\int^{\Lambda} d\nu \int\frac{dq^2}{q^2+i\epsilon}\,(\nu^2-q^2)^{\frac{1}{2}}\,B(q^2)\nu^{\alpha} = \frac{\Lambda^{2+\alpha}}{2+\alpha}\int\frac{dq^2}{q^2+i\epsilon}\,B(q^2), \qquad (3.2)$$

where Λ is a high-energy cutoff. If Δm is to be finite these divergences must cancel when summed over all q^2, which leads us to expect the sum rule

$$\int_{-\infty}^{\infty} \frac{dq^2}{q^2+i\epsilon} B(q^2) = 0 \tag{3.3a}$$

for each Regge or fixed-pole residue.

Such a sum rule is quite reasonable, because $T(q^2,\nu)$ and with it each $B(q^2)$ should be analytic in the cut q^2 plane. The cut extends from $q^2 = (2m_\pi)^2$ to $+\infty$ along the real axis, $(2m_\pi)^2$ being the lowest threshold for pair production by the photon. If $B(q^2)$ falls off fast enough at $|q^2| \to \infty$ then the contour in Eq. (3.3a) can be closed in the upper half plane, and the integral vanishes because there are no singularities inside the contour. In judging the behavior of $B(q^2)$ at $|q^2| \to \infty$, the Phragmen-Lindeloff theorem[7] is of great help. It tells us that the function $B(z)$ will reach the same limit Cz^δ, where C and δ are some constants, for all $|z| \to \infty$, provided it is analytic in the cut z plane, is $O\left\{\exp[|z|(\frac{1}{2} - \epsilon)]\right\}$ for $\epsilon > 0$ at infinity, and approaches Cz^δ when z approaches ∞ along the real axis.

Actually we have been a bit cavalier about taking the limit $\nu \to \infty$ outside the dq^2 integral; this is all right if

$$\int_{-\infty}^{\nu^2} \frac{dq^2}{q^2 + i\epsilon} (\nu^2 - q^2)^{\frac{1}{2}} T(q^2,\nu) \tag{3.4}$$

converges uniformly for all ν. On the spacelike side, this is easily established for the real part of Δm:

$$\text{Re}\,\Delta m = -\frac{1}{2\pi} \int_0^\infty d\nu \; P\!\!\int_{-\infty}^{\nu^2} \frac{dq^2}{q^2} (\nu^2 - q^2)^{\frac{1}{2}} \;\text{Im}\,T(q^2,\nu)$$

$$+ \frac{1}{2} \int_0^\infty d\nu \; \nu \;\text{Re}\,T(q^2 = 0,\nu) \tag{3.5}$$

(we shall assume $\mathrm{Im}\,\Delta m = 0$). The absorptive part of $T(q^2,\nu)$ equals $\mathrm{Im}\,T(q^2,\nu)$ for $q^2 < (2m_\pi)^2$, and the threshold

$$\nu_t = m_\pi + \frac{m_\pi^2 - q^2}{2m} \xrightarrow[q^2\to -\infty]{} \frac{|q^2|}{2m} = \frac{|q|}{2m}\,|q| \gg |q| \,. \tag{3.6}$$

Hence at $q^2 \to -\infty$ the integrand vanishes outside the Regge region, and we have no problem taking ν to infinity.

For the timelike side we quote Bjorken's[1] limit [his Eqs.(2.5) through (2.7)]:

$$T(q^2=\nu^2,\nu) \xrightarrow[|q^2|\to\infty]{} \frac{C}{q^2} + O\left(\frac{1}{q^4}\right) \,, \tag{3.7}$$

with

$$C = \int d^3x \;\langle p|\Big[\,[j_\mu(0,\underset{\sim}{x}),\, H],\, j^\mu(0)\Big]\,|p\rangle,$$

which follows from a fixed q dispersion relation in the absence of Schwinger terms in $[j_\mu(0,\underset{\sim}{x}),\, j^\mu(0)]$. In general, an integral converges uniformly if the integrand has a convergent upper bound. In our case, if we have

$$|T(q^2,\nu)| \;<\; \mathrm{const.}(q^2)^\gamma \;\nu^\alpha \tag{3.8}$$

with $\gamma < 0$ at large q^2 and ν, then $\int^\infty \left(\frac{dq^2}{q^2}\right) q^{2\gamma}$ exists, and we have uniform convergence. Since the bound Eq.(3.8) is quite generous in view of Bjorken's limit, we shall henceforth assume uniform convergence.

A closer examination of the integral (3.4) reveals that the sum rule, Eq.(3.3a), is not strong enough if $\alpha \geqslant 0$. Expanding the square root in powers of ν^2/q^2, we obtain

$$\lim_{\nu\to\infty} \int_{-\infty}^{\infty} \frac{dq^2}{q^2+i\epsilon} \, \nu\left[1-\frac{q^2}{\nu^2}\right]^{\frac{1}{2}} B(q^2)\,\nu^{\alpha}$$

$$= \lim_{\nu\to\infty} \left[\nu^{1+\alpha} \int_{-\infty}^{\infty} \frac{dq^2}{q^2+i\epsilon} B(q^2) - \frac{1}{2}\nu^{-1+\alpha} \int_{-\infty}^{\infty} dq^2\, B(q^2) + \cdots\right].$$

(3.9)

Of course, this expansion has to be broken off at the <u>n</u>th term if $\int_{-\infty}^{\infty} dq^2\, B(q^2)\, q^{2(n+1)}$ diverges. The first term produces a $\Lambda^{2+\alpha}$ or a $\log\Lambda$ divergence in Δm for all $\alpha \geq -2$, while the second term will diverge like Λ^{α} or $\log\Lambda$ if $\alpha \geq 0$. Since all α are ≤ 1, the rest will converge. We see that the pole terms with $\alpha \geq 0$ require the additional sum rule

$$\int_{-\infty}^{\infty} dq^2\, B(q^2) = 0, \tag{3.3b}$$

which is satisfied by our analyticity assumptions provided $q^2\, B(q^2)$ approaches zero when $|q^2|$ approaches infinity.

Now let us examine the possibility of other divergences. There can be none as q^2 approaches zero because $T(q^2,\nu)$ is well-behaved here. Divergences in the region $0 \leq \nu^2 \leq {\nu_t}^2 \approx q^4/4m^2$ and $q^2 \to -\infty$ have been ruled out by the vanishing of $\mathrm{Im}\, T(q^2,\nu)$. Finally, divergences from $\nu^2 \sim q^2 \to +\infty$ are not allowed by Eq.(3.7). Therefore we conclude that the sum rule(s) Eq.(3.3) are necessary and sufficient to make Δm finite.

If for some reason the bound in Eq.(3.8) should fail, and this does happen with the Born term for point nucleons, then we can still prove that the sum rules

$$\int_{-\infty}^{\infty} \frac{dq^2}{q^2+i\epsilon} \; T(q^2,\nu) = 0 \tag{3.10a}$$

$$\int_{-\infty}^{\infty} dq^2 \; T(q^2,\nu) = 0 \tag{3.10b}$$

for all finite ν are sufficient to eliminate the divergences at $\nu \to \infty$. This can be seen by subtracting $\frac{i}{2\pi}\int_0^{\Lambda} d\nu\; \nu \int_{-\infty}^{\infty} [dq^2/(q^2+i\epsilon)]$ $\times\, T(q^2,\nu) = 0$ from Eq.(2.3) with cutoff Λ,

$$\Delta m = \frac{i}{2\pi}\int_0^{\Lambda} d\nu\; \nu \int_{-\infty}^{\infty} \frac{dq^2}{q^2+i\epsilon} \; T(q^2,\nu) \left\{\theta(\nu^2 - q^2)\left[1 - (q^2/\nu^2)\right]^{\frac{1}{2}} - 1\right\},$$

and expanding the square root as before. Eqs.(3.3) are simply the high-energy limit of Eqs. (3.10), if it exists.

B. The Born Term

The Born term

$$T_B(q^2,\nu) = 3q^2 \frac{4mq^2 f_1(q^2)}{q^4 - 4m^2\nu^2 + i\epsilon q^2} + (q^2+2\nu^2)\frac{4mq^2 f_2(q^2)}{q^4-4m^2\nu^2+i\epsilon q^2} \tag{3.11}$$

with its $\nu \to \infty$ limit

$$T_B(q^2,\nu) \longrightarrow \left[-\frac{2q^2}{m} f_2(q^2)\right]\nu^0 + \left[\frac{-3q^4}{m} f_1(q^2) - \frac{q^4}{m}\left(1+\frac{q^2}{2m^2}\right) f_2(q^2)\right]\nu^{-2} + \cdots \tag{3.12}$$

contributes to the fixed poles at $J = 0, -2, \cdots$. For spin one-half particles the f_i are

$$f_1 = \left(\frac{e}{2\pi}\right)^2 \frac{G_M^2 - G_E^2}{q^2 + 4m^2} \quad \text{and} \quad f_2 = \left(\frac{e}{2\pi}\right)^2 \frac{q^2 G_M^2 + 4m^2 G_E^2}{q^2(q^2+4m^2)} . \tag{3.13}$$

Since the electric and magnetic form factors G_E and G_M seem to decrease[8] at least like $1/q^4$ when $q^2 \to -\infty$ we have

$$f_i = O(1/q^{10}),$$

$$q^2 B_{0(\mathrm{Born})} = q^2 \left[- \frac{2q^2 f_2}{m} \right] = O\left(\frac{1}{q^6}\right),$$

and

$$B_{-2(\mathrm{Born})} = \left[- \frac{3q^4}{m} f_1 - \frac{q^4}{m}\left(1 + \frac{q^2}{m^2}\right) f_2 \right] = O\left(\frac{1}{q^4}\right) ,$$

which is several powers better than we need for the sum rules Eq. (3.3). Therefore the Born contribution to Δm converges.

For spinless hadrons there is only one form factor, the Born term is proportional to

$$\frac{-3q^4 + 4m^2 q^2 + 8m^2 \nu^2}{q^4 - 4m^2 \nu^2 + i\epsilon q^2} \; |F(q^2)|^2 ,$$

and the sum rules require that

$$(q^2)^{\frac{1}{2}} F(q^2) \to 0 \qquad (3.14)$$

at infinity.

For an illustration of our sum rules, consider point particles for which Δm is calculated according to Feynman's rules. For the point nucleon,

$$T_{B(\text{point nucleon})} = \left(\frac{e}{2\pi}\right)^2 4m \frac{q^2 + 2\nu^2}{q^4 - 4m^2\nu^2 + i\epsilon q^2}$$

has poles at $q^2 = \pm 2m\nu - i\epsilon$, is analytic in the upper half q^2 plane, decreases like $1/q^2$ as $|q^2|$ goes to infinity. Therefore we have

$$\int_{-\infty}^{\infty} \frac{dq^2}{q^2 + i\epsilon} T_{B(\text{point nucleon})}(q^2,\nu) = 0, \qquad (3.15)$$

cancelling the quadratic divergences in the spacelike, lightlike, and timelike contributions to Δm. However, $\int_{-\infty}^{\infty} dq^2\, T_{B(\text{point nucleon})}(q^2,\nu) = -i\pi 4m\left(\frac{e}{2\pi}\right)^2$ is not equal to zero and the well-known logarithmic divergence survives. Note that Eq.(3.15) does not converge uniformly, so we cannot take $\nu \to \infty$ and write it in the form of Eq.(3.3). For point pions the Born term is proportional to

$$\frac{-3q^4 + 4m^2q^2 + 8m^2\nu^2}{q^4 - 4m^2\nu^2 + i\epsilon q^2} .$$

Here the quadratic divergence will not be cancelled because $T_{B(\text{point pion})}$ approaches a constant as $|q^2|$ goes to ∞ and

$$\int_{-\infty}^{\infty} \frac{dq^2}{q^2 + i\epsilon} T_{B(\text{point pion})} \neq 0 ;$$

C. Wick Rotation

For the Wick rotation we have to continue T from real ν to $\nu = |\nu| e^{i\theta}$, $0 \leqslant \theta \leqslant \pi/2$. Assuming the high-energy limit Eq.(2.9) at all complex $\nu \to \infty$, we obtain for the quarter circle at infinity

$$\frac{i}{2\pi}\int_{\infty}^{i\infty} d\nu \int_{-\infty}^{\nu^2} \frac{dq^2}{q^2 + i\epsilon} (\nu^2 - q^2)^{\frac{1}{2}} T(q^2,\nu)$$

$$= \frac{i}{2\pi}\int_{\infty}^{i\infty} d\nu\, \nu^{1+\alpha} \int_{-\infty}^{|\nu^2| e^{2i\theta}} \frac{dq^2}{q^2 + i\epsilon} B(q^2) \left\{ 1 + O\left(\frac{q^2}{\nu^2}\right)\right\} \qquad (3.16)$$

This vanishes if and only if, for $\alpha < 0$

$$\int_{-\infty}^{\infty e^{2i\theta}} \frac{dq^2}{q^2 + i\epsilon} B(q^2) = 0 \ , \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2} \ , \quad (3.17a)$$

and for $\alpha > 0$

$$\int_{-\infty}^{\infty e^{2i\theta}} dq^2 \, B(q^2) = 0 \ , \qquad 0 \leqslant \theta \leqslant \frac{\pi}{2} \ . \quad (3.17b)$$

These conditions are satisfied automatically if they hold at $\theta = 0$ (see Fig. 3). At $\theta = \pi/2$ they become trivial, which explains why none of these divergences appeared in the usual calculations of Δm: they were carried out in Euclidean space.

To sum up, we have used Reggeology, as well as reasonable assumptions for the $q^2 \to \infty$ behavior, to derive necessary and sufficient conditions for a finite Δm. Furthermore, finiteness of Δm has turned out to be equivalent to Wick rotatability. In the case of a logarithmic divergence, for example the Born term with point nucleons, the Wick rotation only adds an imaginary constant to Δm, while the divergent real part remains the same.

IV. BJORKEN'S METHOD AND THE FIXED POLE

A. Inclusion of Subtractions

Bjorken calculated Δm in Euclidean space,[1] assumed unsubtracted dispersion relations, and discovered a logarithmic divergence due to the nonvanishing of $\left[[j_\mu(0,x),H],\ j^\mu(0)\right]$. Since a logarithmically divergent integral does not change on Wick rotation, apart from a finite imaginary constant, we should expect to find the same divergence in Minkowski space. This means that one of the sum rules for the $J = 0$ or $J = -2$ fixed-pole residues must fail. We shall indeed show that for the $J = 0$ pole

$$B_0(q^2) \longrightarrow \frac{C}{q^2} \tag{4.1}$$

with

$$C = \int d^3x \langle p| \left[[j_\mu(0,\underset{\sim}{x}),H],\ j^\mu(0)\right] |p\rangle$$

and therefore

$$\int_{-\infty}^{\infty} dq^2\ B_0(q^2) = -i\pi C,$$

which just reproduces Bjorken's result

$$\Delta m_{div} = -\frac{C}{4\pi} \log\Lambda \quad . \tag{4.2}$$

First let us review Bjorken's argument. He assumed an unsubtracted dispersion relation (USDR)[9]

$$T(q^2,\nu) = B_0(q^2) + \frac{1}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2\ \mathrm{Im}\ T(q^2,\nu')}{\nu'^2 - \nu^2} \tag{4.3}$$

and noted that for $q^2 \to -\infty$ the threshold becomes $\nu_t^2 \to \left(\frac{-q^2}{2m}\right)^2 = \frac{q^2}{4m^2}q^2$ $\gg q^2$. In Euclidean space one needs $T(q^2,\nu)$ only at values

$0 \leq |\nu^2| \leq -q^2$. In this range $|\nu^2| \ll \nu'^2$, and the denominator can be expanded in powers of ν^2/ν'^2:

$$T(q^2,\nu) = B_0(q^2) + \frac{1}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \,\mathrm{Im}\, T(q^2,\nu')}{\nu'^2} + \frac{\nu^2}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \,\mathrm{Im}\, T(q^2,\nu')}{\nu'^4}$$

$$= T(q^2,0)\left[1 + O\left(\frac{\nu^2}{\nu_t^2}\right)\right] . \qquad (4.4)$$

On the other hand, we have

$$T\left(q^2, (q^2)^{\frac{1}{2}}\right) \rightarrow \frac{C}{q^2} + O\left(\frac{1}{q^4}\right). \qquad (3.7)$$

These conditions imply, for $|\nu^2| \leq -q^2$,

$$T(q^2,\nu) \xrightarrow[q^2 \to -\infty]{} \frac{C}{q^2} + O\left(\frac{1}{q^4}\right) \qquad (4.5)$$

and cause the logarithmic divergence.

Now we generalize this argument to the $\Delta I = 1$ mass differences where one subtraction is necessary (and sufficient, since all $\alpha \leq 1$). The subtracted dispersion relation (SDR),

$$T(q^2,\nu) = T(q^2,N) + \frac{\nu^2 - N^2}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \,\mathrm{Im}\, T(q^2,\nu')}{(\nu'^2-\nu^2)(\nu'^2-N^2)} , \qquad (4.6)$$

holds for all subtraction points N, even for $N \to \infty$. If we take $N^2 \gg q^2$, $q^2 \gg 4m^2$ and $0 \leq |\nu^2| \leq q^2$ we can expand the integral in powers of ν^2/ν'^2 and ν^2/N^2:

$$T(q^2,\nu) = T(q^2,N) - \frac{N^2}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \,\mathrm{Im}\, T(q^2,\nu')}{\nu'^2(\nu'^2 - N^2)}\left[1 + O\left(\frac{\nu^2}{\nu_t^2}\right) + O\left(\frac{\nu^2}{N^2}\right)\right]$$

$$= T(q^2,0)\left[1 + O\left(\frac{\nu^2}{\nu_t^2}\right)\right] . \qquad (4.7)$$

Since this is the same as in Eq. (4.4), we see that Eq. (4.5) and the logarithmic divergence emerge in this case as well.[10]

B. The Fixed-Pole Residue

For the $\Delta I = 2$ case we can obtain the asymptotic value of the residue directly from the USDR (4.3) by taking $\nu^2 = q^2 \to -\infty$ and neglecting $\nu^2 \ll \nu'^2$:

$$B_0(q^2) = \frac{C}{q^2} - \frac{1}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \, \mathrm{Im}\, T(q^2,\nu')}{\nu'^2} + O\left(\frac{1}{q^4}\right) \qquad (4.8)$$

The integral must be $O(1/q^2)$ in order for the neglected terms in Eqs. (4.4) and (4.5) to be $O(1/q^4)$ as required by Bjorken's framework. We might wonder if the integral could go like $1/q^2$ asymptotically and maybe cancel the leading term in B_0. But then Bjorken's divergence would not be the same in Minkowski space, thus violating Wick rotatability. Therefore we suspect that the latter requires

$$q^2 \int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \, \mathrm{Im}\, T(q^2, \nu')}{\nu'^2} \longrightarrow 0.$$

As a matter of fact, this is precisely what we had assumed in the derivation of our sum rule Eq. (3.3).

To gain some more information we try the following approximation

$$\mathrm{Im}\, T(q^2,\nu') = \mathrm{Im} \sum_i B_i(q^2)\nu'^{\alpha_i} = -\sum_i \beta_i(q^2)\nu'^{\alpha_i} \qquad (4.9)$$

since we integrate only over the high-energy region $\nu'^2 \gtrsim \nu_t^2 \gg q^2$. Obviously this cannot be quite correct because $\mathrm{Im}\, T(q^2,\nu')$ vanishes for $\nu' < \nu_t$, whereas $\sum_i B_i(q^2)\nu'^{\alpha_i}$ has a cut all the way from

$\nu' = 0$ to ∞. Eventually, however, as $\nu_t^2 \ll \nu'^2 \to \infty$ Eq. (4.9) should be good, if Regge theory is to have any validity in Compton scattering. If the Regge limit (4.9) sets in at say $\nu'^2 \geqslant \eta\, \nu_t^2$, $\eta \gg 1$ fixed, and if the behavior of Im T near threshold is not too pathological, then this approximation should reproduce at least the correct power behavior of the integral in (4.8); it becomes

$$\frac{1}{\pi} \int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \operatorname{Im} T(q^2, \nu')}{\nu'^2} \propto \frac{2}{\pi} \sum_i \beta_i(q^2) \frac{1}{\alpha_i} \left(\frac{-q^2}{2m}\right)^{\alpha_i} \tag{4.10}$$

If B_0 is sufficiently regular at infinity, i.e., if $B_0 \to (C/q^2) + O(1/q^4)$, then $\beta(q^2)(-q^2)^{\alpha} = O(1/q^4)$ and the sum rule (3.3a) is satisfied for $-2 < \alpha < 0$. [If we only knew $\beta(q^2)(-q^2)^{\alpha} = O(1/q^2)$, then it could be broken for $\alpha < -1$.]

For the $\Delta I = 1$ case we take the real part (the imaginary part would just give $0 = 0$) of the SDR (4.6) with $\nu^2 = q^2 \gg 4m^2$ and $N \to \infty$. On the RHS all pole terms with $\alpha < 0$ disappear, only the $J = 0$ pole and the leading Regge pole with $\alpha = A = \alpha_{A2}$ survive. This yields

$$B_0(q^2) = \frac{C}{q^2} - \lim_{N\to\infty} \left[\cot\left(\frac{\pi A}{2}\right) \beta_A(q^2) N^A + \frac{N^2}{\pi} \int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \operatorname{Im} T(q^2, \nu')}{\nu'^2(\nu'^2 - N^2)} \right] \tag{4.11}$$

The leading Regge term cancels exactly the leading term from the integral, so we end up with just Eq. (4.8) but with the proviso that the $\beta_A(q^2)\nu'^A$ term be removed from Im T. We can see this explicitly by substituting approximation (4.9) into Eq. (4.11) and looking up the integral in a table of Hilbert transforms:

(a) for $\alpha = A > 0$

$$\frac{1}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 (\nu'^2)^{\frac{A}{2}-1}}{(\nu'^2 - N^2)} \xrightarrow[N\to\infty]{} -\cot\left(\frac{\pi A}{2}\right)\frac{N^A}{N^2}$$

(b) for $\alpha < 0$

$$\frac{1}{\pi}\int_{\nu_t^2}^{\infty} \frac{d\nu'^2 (\nu'^2)^{\frac{\alpha}{2}-1}}{(\nu'^2 - N^2)} \xrightarrow[N\to\infty]{} -\cot\left(\frac{\pi\alpha}{2}\right)\frac{N^\alpha}{N^2} + \frac{\nu_t^{\alpha}}{N^2\left(\frac{\pi\alpha}{2}\right)} .$$

The Regge terms will cancel exactly, while the $-N^2 \sum_{\alpha_i<0} \frac{\nu_t^{\alpha_i}}{N^2}\left(\frac{2}{\pi\alpha_i}\right)\beta_i$ part from (b) will contribute to the fixed-pole residue. This is the same as we had found in Eqs. (4.8) and (4.10) for $\Delta I = 2$; so we conclude that here, too,

$$B_0(q^2) \longrightarrow \frac{C}{q^2} .$$

V. CONCLUDING REMARKS

We have seen that the divergence of the $\Delta I = 1$ mass differences is no worse than for the $\Delta I = 2$ case (provided that the sum rule (3.3b) is satisfied for the α_{A2} Regge term). So what about the claim by Cottingham and Gibb[2] that Δm should diverge if computed via dispersion relations? Harari[6] makes a q^2-dependent subtraction at $\nu = 0$ and gets a convergent SDR of the form

$$T(q^2,\nu) = T(q^2,0) + \frac{\nu^2}{\pi} \int_{\nu_t^2}^{\infty} \frac{d\nu'^2 \; \mathrm{Im}\, T(q^2,\nu'^2)}{\nu'^2(\nu'^2 - \nu^2)} \tag{5.1}$$

This yields a finite Δm (apart from Bjorken's divergence). On the other hand, Cottingham and Gibb work with the Jost-Lehmann representation and make a q^2 <u>independent</u> subtraction at $q^2 = \epsilon^2$ and $\nu^2 = -\epsilon^2 \to 0$. Jost-Lehmann representation or not, such a subtraction leads to the dispersion relation

$$T(q^2,\nu) = T(\epsilon^2,i\epsilon) + \frac{1}{\pi} \int_{\nu_t^2}^{\infty} d\nu'^2 \left[\frac{\mathrm{Im}\, T(q^2,\nu')}{(\nu'^2 - \nu^2)} - \frac{\mathrm{Im}\, T(\epsilon^2,\nu')}{(\nu'^2 + \epsilon^2)}\right] . \tag{5.2}$$

Since Cottingham and Gibb do use dispersion relations to calculate Δm, and since they assume Regge behavior, they should realize that both $\mathrm{Im}\, T(q^2,\nu') \to \mathrm{Im}\, B(q^2)\nu'^{\alpha}$ and $\mathrm{Im}\, T(\epsilon^2,\nu') \to \mathrm{Im}\, B(\epsilon^2)\nu'^{\alpha}$ and therefore Eq. (5.2) still diverges for $\alpha > 0$. The subtraction at $q^2 = \epsilon^2 \to 0$ is useless, and consequently the expression for Δm diverges as if no subtraction had been made at all.

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SUMMARY

The high-energy limit of the spin-averaged Compton amplitude $T = T_\mu^{\ \mu}$ is $T(q^2,\nu) \longrightarrow \sum_i B_i(q^2)\nu^{\alpha_i}$ when $q^2 \ll \nu^2 \longrightarrow \infty$; the sum goes over all Regge and fixed poles. If

$$\Delta m = \frac{i}{2\pi}\int_0^\infty d\nu \int_{-\infty}^{\nu^2} [dq^2/(q^2+i\epsilon)](\nu^2 - q^2)^{\frac{1}{2}}\, T(q^2,\nu)$$

is evaluated in Minkowski space, the exponent α will produce a divergence $\Lambda^{2+\alpha}$. The sum rule $\int_{-\infty}^{\infty} [dq^2/(q^2+i\epsilon)]B_\alpha(q^2) = 0$ reduces this leading term to Λ^α; the sum rule $\int_{-\infty}^{\infty} dq^2\, B_\alpha(q^2) = 0$ reduces it to $\Lambda^{-2+\alpha}$. Table 1 summarizes the restrictions on the residue functions $B_i(q^2)$ that are necessary and sufficient to make Δm finite and to allow the Wick rotation.

Exponent of ν	Sum rule necess. + suffic. for finite Δm and for Wick rotation	Bounds at $\lvert q^2\rvert \longrightarrow \infty$ necessary and sufficient to satisfy the sum rules
$\alpha > 0$ $\Delta I = 1$ SDR	$\int_{-\infty}^{\infty} dq^2 B_\alpha(q^2) = 0$	$q^2 B_\alpha(q^2) \longrightarrow 0$
Fixed pole $\alpha = 0$ $\Delta I = 1,2$	$\int_{-\infty}^{\infty} dq^2 B_0(q^2) = 0$	$q^2 B_0(q^2) \longrightarrow 0$ but since $q^2 B_0(q^2) \longrightarrow C$ this requires $C = \int d^3x \langle p \vert \left[[j_\mu(0,x),H], j^\mu(0) \right] \vert p \rangle = 0$
$\alpha < 0$ $\Delta I = 1,2$ USDR	$\int_{-\infty}^{\infty} \frac{dq^2}{q^2 + i\epsilon} B_\alpha(q^2) = 0$	$B_\alpha(q^2) \longrightarrow 0$ probably satisfied

FOOTNOTES AND REFERENCES

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9. The USDR has to contain the fixed-pole term $B_0(q^2)$ separately. Although this was not included by Bjorken, it does not affect his conclusions.

10. We assume with Bjorken that Eq. (3.7) and the absence of Schwinger terms hold even if a subtraction is necessary.

FIGURE LEGENDS

Fig. 1. The forward Compton amplitude $T_{\mu\lambda}(q^2,\nu)$.

Fig. 2. The region of integration for Δm in (a) Minkowski space and (b) Euclidean space.

Fig. 3. Integration contour for the sum rules.

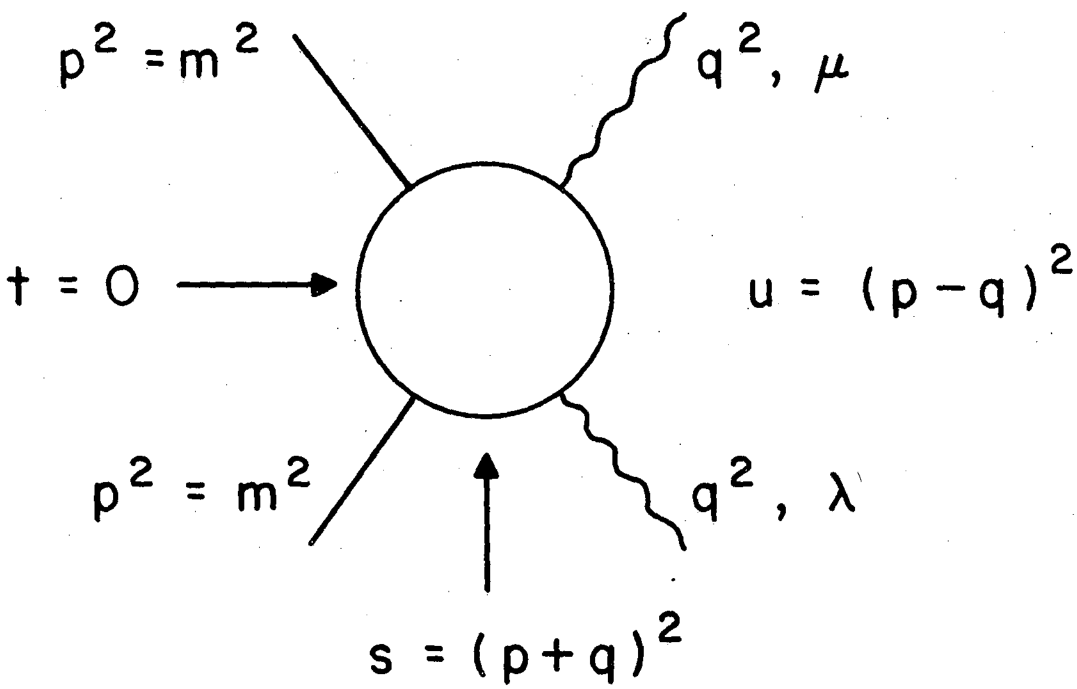

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Fig. 1

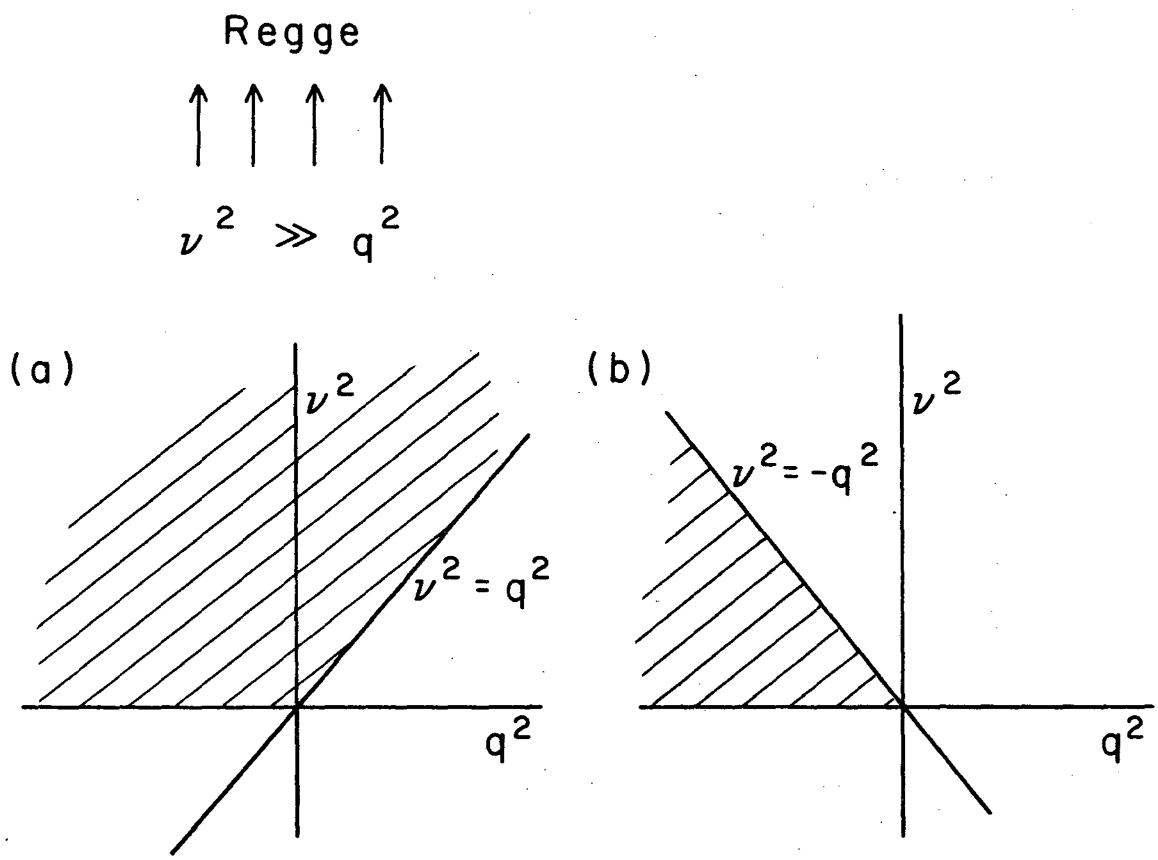

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Fig. 2

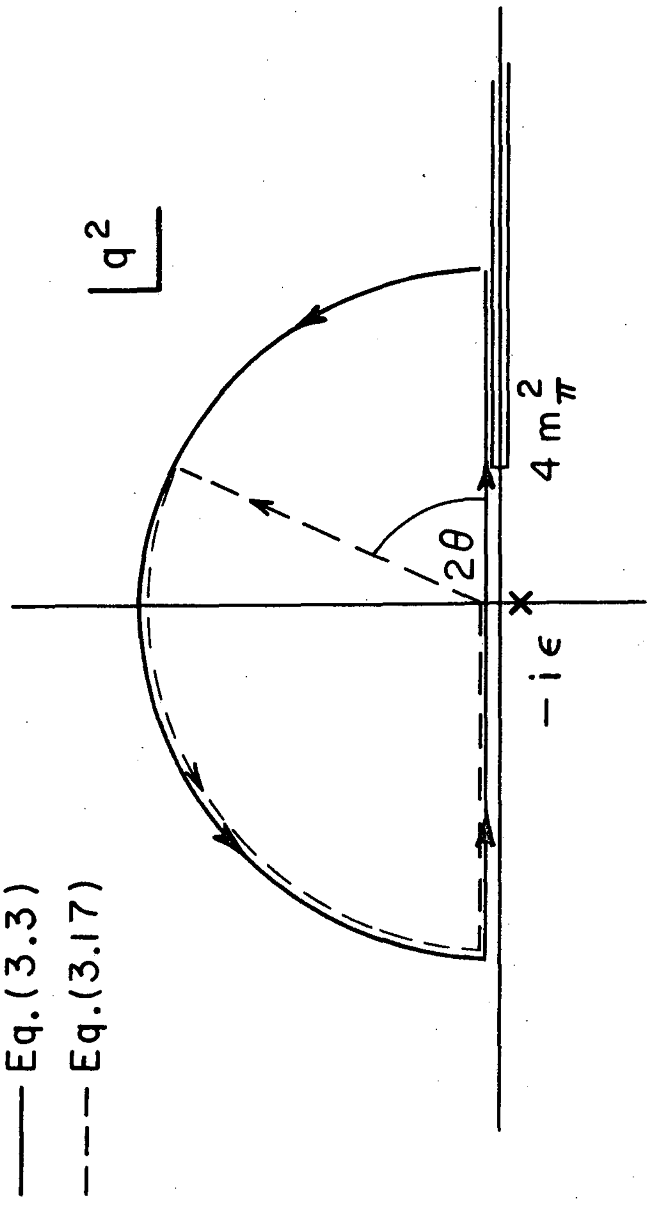

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Fig. 3