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A Simple Way to Characterize Linear Coupling in a Storage Ring

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Abstract

The techniques of normal form analysis, well known in the literature [1], can be used to provide a straightforward characterization of linear betatron dynamics in a coupled lattice. Here, we consider both the beam distribution and the betatron oscillations in a storage ring, assuming that the beam emittances and betatron actions respectively are provided as parameters. We find that the beta functions for uncoupled motion generalize in a simple way to the coupled case. Defined in the way that we propose, the beta functions remain well behaved (positive and finite) under all circumstances, and have essentially the same physical significance for the beam size and betatron oscillations as in the uncoupled case. We discuss a technique for making direct measurements of the ratio of the coupled lattice functions at different points in the lattice.

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1 Introduction

Optimal performance of electron storage rings in synchrotron light sources and circular colliders often depends on good control of the betatron coupling. Characterizing the coupling in a straightforward fashion becomes particularly important when the lattice includes regions where the beam is significantly coupled by design, as in the solenoid field of the interaction region of a collider. In this note, we revisit the basic principles of betatron motion in coupled systems, and propose a simple way to characterize the coupling, relating to the equilibrium beam sizes and tilts and to the trajectories of free betatron oscillations. We also discuss a possible method for direct detailed measurements of the quantities we use for describing the coupled dynamics, analogous to the phase advance measurements already used in some machines, for example PEP-II [2]. This work is motivated by possible application to PEP-II, and we illustrate some of the ideas we present by simulations using the PEP-II LER lattice.

Characterization of the dynamics in an uncoupled lattice is achieved using the familiar Twiss parameters. These parameters give the beam size locally throughout the lattice, if the beam emittance in each plane is known. The same parameters give the local amplitude of oscillation of a single particle trajectory, if the betatron action of the trajectory is known. In a coupled lattice, there are various techniques to choose from in characterizing the lattice. In one commonly used approach [3], one finds a transformation that puts the single-turn 4×4 matrix into block diagonal form, and then performs the standard Twiss analysis on each of the decoupled 2×2 submatrices. In some circumstances, the beta functions found from this method can be infinite or negative, and it can be difficult to get a clear idea of the dynamics simply by looking at a plot of the beta functions. There are also sometimes subtleties involved in identifying the different modes at different places in the lattice [4].

An alternative and (we believe) simpler approach is provided by the standard normal form analysis. Here, one again reduces the single-turn matrix to block diagonal form, but each submatrix is now simply a rotation. The dynamics are characterized by elements of the matrix required to carry out this transformation. The essential results are:

- The required normalizing transformation is easily constructed from the eigenvectors of the single-turn matrix.
- In the uncoupled case, elements of the normalizing transformation are identified with the beta functions, and this identification generalizes naturally to the coupled case.
- The beta functions defined in this way are always positive and finite, and are still associated with the beam size and amplitude of betatron motion in the same way as the uncoupled case.
- One can calculate complex functions (here called zeta functions) associated with the beta functions, that characterize the coupling. For given emittances, the beam sizes and tilt are then given simply by:

$$\begin{aligned}
\langle x^2 \rangle &= \beta_x \varepsilon_{\mathrm{I}} + |\zeta_y|^2 \varepsilon_{\mathrm{II}} \\
\langle y^2 \rangle &= \beta_y \varepsilon_{\mathrm{II}} + |\zeta_x|^2 \varepsilon_{\mathrm{I}} \\
\langle xy \rangle &= \sqrt{\beta_x}\,\mathrm{Re}(\zeta_x)\varepsilon_{\mathrm{I}} + \sqrt{\beta_y}\,\mathrm{Re}(\zeta_y)\varepsilon_{\mathrm{II}}
\end{aligned} \tag{1}$$

where the brackets $\langle\ \rangle$ indicate an average over all particles in the beam.

- One avoids geometric transformations of the axes, so there is never any confusion between horizontal and vertical motion.
- It is possible to make direct measurements (i.e. without reference to a lattice model) of the phase advance and the ratios of the beta and coupling functions between different points in the lattice.

The normal form analysis is well known in the literature [1]. None of the results presented here relating to the theory are new, but we stress their use for characterizing the dynamics in a coupled lattice in a simple and straightforward way. We find it convenient to treat the problem of the beam distribution separately from the problem of single particle (or coherent) betatron oscillations. In each case, we begin our analysis by reviewing the useful results from uncoupled dynamics, and then see how the theory may be generalized to the coupled case. Further generalization is possible to include longitudinal dynamics: Forest [[5]] has proposed a treatment of the lattice functions essentially similar to that used here, but explicitly including the longitudinal dynamics.

We consider only linear transverse dynamics. We also assume for our treatment of the beam distribution, that the horizontal and vertical emittances are known by some other means, or are put in as parameters. These quantities cannot, in fact, be calculated simply from the single-turn map, since they are global quantities dependent, for example, on the dispersion arising from steering errors. Methods are available (for example [6]) for calculating the equilibrium emittances in a general coupled electron storage ring, but to obtain agreement with reality it is necessary to know the detailed dispersion and alignment errors. For practical purposes, it is often sufficient (and highly convenient) to have a method for calculating the beam distribution *given* the emittances as parameters – one can assume, for example, that the horizontal emittance is close to the natural emittance of the lattice, and the vertical emittance is a few percent of this.

In considering the beam sizes, we assume that the beam distribution is Gaussian. This is generally the case for electron storage rings, and it is sufficient to know just the second order moments of the phase space variables. Furman [7] has shown how to calculate higher order moments for general beam distributions of the form $\rho(\mathbf{x}) \propto g(s(\mathbf{x}))$, where g is a non-negative function of a single variable, and s is a quadratic function of the phase-space variables.

The structure of this note is as follows. In section 2, we show how the matched beam distribution at any point in the lattice may be calculated from the single-turn matrix at that point and the beam emittances. Our final results are given in equations (11), (12) and (13). In section 3, we consider how to define the coupled lattice functions from consideration of the trajectory of a single particle through the lattice. It is here that we make use of the normal form analysis. The useful results are contained in equations (1), (22), (23), and (24). We finish section 3 by giving the simple relationships between the beam sizes and tilt, and the (well-behaved) lattice functions we have defined for coupled dynamics. Finally, in section 0, we consider how direct measurements may be made of the phase advances and the lattice functions that we have defined in the previous section. In principle, these measurements should be straightforward, similar to those already used in some places (for example PEP-II [2]) for uncoupled lattices.

2 Finding the Matched Beam Distribution

The matched beam distribution at any point in a storage ring is defined by the condition that, at any point in the lattice, after a full turn the distribution remains the same. Mathematically, if we define the covariance matrix (for motion in one dimension) in terms of averages over the co-ordinates of all particles in the beam:

$$\mathbf{\Sigma} = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix}$$

then the matched distribution condition implies that:

$$\mathbf{\Sigma} \to \mathbf{\Sigma}$$

under the transformation applied to all particles:

$$\mathbf{x} \to \mathbf{M} \cdot \mathbf{x} \tag{2}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ p_x \end{pmatrix}$$

and **M** is a matrix representing the single-turn map.

2.1 The Covariance Matrix in Uncoupled Motion

In the uncoupled case, the Twiss parameters and the phase advance specify the single-turn matrix **M**:

$$\mathbf{M} = \begin{pmatrix} \cos(\mu_x) + \alpha_x \sin(\mu_x) & \beta_x \sin(\mu_x) \\ -\gamma_x \sin(\mu_x) & \cos(\mu_x) - \alpha_x \sin(\mu_x) \end{pmatrix} \tag{3}$$

The condition that **M** is symplectic reduces the number of independent parameters in this matrix from four to three. We note three significant results:

1. The eigenvalues of **M** are $\exp(\pm i\mu_x)$.
2. The quantity J_x (the betatron action) defined by:

   $$2J_x = \gamma_x x^2 + 2\alpha_x x p_x + \beta_x p_x^2 \tag{4}$$

   is conserved under the transformation (2).
3. If we define the emittance of the beam as the average action over all particles in the beam:

   $$\varepsilon_x = \langle J_x \rangle = \tfrac{1}{2}\gamma_x \langle x^2 \rangle + \alpha_x \langle x p_x \rangle + \tfrac{1}{2}\beta_x \langle p_x^2 \rangle$$

   then the matched beam distribution is given by:

$$
\begin{aligned}
\langle x^2 \rangle &= \beta_x \varepsilon_x \\
\langle x p_x \rangle &= -\alpha_x \varepsilon_x \\
\langle p_x^2 \rangle &= \gamma_x \varepsilon_x
\end{aligned}
\tag{5}
$$

Before we generalize these results to coupled motion we write some of the above equations in a slightly different form. Using notation suggested by Furman [7], the action defined by (4), may be written:

$$2J_x = \langle \mathbf{x} | \mathbf{A}_x | \mathbf{x} \rangle \tag{6}$$

for an appropriate symmetric matrix $\mathbf{A}_x$:

$$\mathbf{A}_x = \begin{pmatrix} \gamma_x & \alpha_x \\ \alpha_x & \beta_x \end{pmatrix}$$

where:

$$\langle \mathbf{x} | = \begin{pmatrix} x & p_x \end{pmatrix} \qquad\qquad | \mathbf{x} \rangle = \begin{pmatrix} x \\ p_x \end{pmatrix}$$

(There is some risk of confusion of this notation for a vector with the brackets $\langle \; \rangle$ used to indicate an average over a number of particles; if a bracket is paired with a bar | then the quantity indicated is always a vector.) Since the equilibrium matched distribution must by definition be invariant under the transformation (2), it must be a function of the invariant action. Thus, for a Gaussian distribution:

$$\rho(x, p_x) = N \exp\left(-\frac{\langle \mathbf{x} | \mathbf{A}_x | \mathbf{x} \rangle}{2\varepsilon_x} \right)$$

where ρ is the density of particles in phase space and N is a normalization factor. The covariance matrix is then:

$$\mathbf{\Sigma} = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle x p_x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \varepsilon_x \mathbf{A}_x^{-1} \tag{7}$$

which is the same as equations (5).

In the uncoupled case there are of course two invariant actions, corresponding to horizontal and vertical motion. For coupled motion, there are still only two invariants: there are just more terms in each of the invariants in the coupled case, than the three terms on the right hand side of equation (4). If we can find these invariants in the general case (equivalent to finding the matrix **A**) then we can immediately write down the covariance matrix by a straightforward generalization of (7).

2.2 Generalization to Coupled Lattices

We now work in two transverse dimensions, so that:

$$\langle \mathbf{x}| = \begin{pmatrix} x & p_x & y & p_y \end{pmatrix} \qquad |\mathbf{x}\rangle = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix}$$

We wish to find a four-by-four symmetric matrix **A** such that:

$$2J_x = \langle \mathbf{x}|\mathbf{A}|\mathbf{x}\rangle$$

is invariant under

$$|\mathbf{x}\rangle \to \mathbf{M}|\mathbf{x}\rangle$$

(Note that we have for now dropped the subscripts identifying the action J and the symmetric matrix **A** with a particular plane; we shall find that there are in fact two solutions for **A**, corresponding to two emittances). The invariance of the action means that we require:

$$\mathbf{M}^{\mathrm{T}}\mathbf{A}\mathbf{M} = \mathbf{A} \tag{8}$$

To construct **A**, we use the eigenvectors of **M**:

$$\mathbf{M}|\mathbf{e}_i\rangle = \lambda_i|\mathbf{e}_i\rangle$$

(Note that a subscript on **e** labels different eigenvectors, not components of a single eigenvector.) For a symplectic matrix, the eigenvectors and eigenvalues satisfy:

$$\lambda_i\lambda_j = 1 \quad \text{if} \quad \langle \mathbf{e}_i|\mathbf{S}|\mathbf{e}_j\rangle \neq 0$$

where **S** is the symplectic form:

$$\mathbf{S} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

To obtain the correct beam distribution, it is important to normalize the eigenvectors so that:

$$\langle \mathbf{e}_i|\mathbf{S}|\mathbf{e}_j\rangle = \begin{cases} \pm\mathrm{i} & \lambda_i\lambda_j = 1 \\ 0 & \lambda_i\lambda_j \neq 1 \end{cases} \tag{9}$$

M has four eigenvectors; two associated with each of the tunes $\nu_{\mathrm{I,II}}$. The four eigenvalues are:

$$\lambda = \mathrm{e}^{\pm 2\pi\mathrm{i}\nu_{\mathrm{I,II}}}$$

Let us define a symmetric matrix **B** with components given by:

$$B_{ij} = \langle \mathbf{e}_i|\mathbf{A}|\mathbf{e}_j\rangle$$

From equation (8), it follows that:

$$\langle \mathbf{e}_i | \mathbf{M}^{\mathrm{T}} \mathbf{A} \mathbf{M} | \mathbf{e}_j \rangle = \lambda_i \lambda_j \langle \mathbf{e}_i | \mathbf{A} | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \mathbf{A} | \mathbf{e}_j \rangle$$

and hence:

$$\lambda_i \lambda_j B_{ij} = B_{ij}$$

Let us select a suitable ordering of the eigenvectors, so that $|\mathbf{e}_1\rangle$ and $|\mathbf{e}_2\rangle$ are associated with one eigenvalue (tune ν_{I}), and $|\mathbf{e}_3\rangle$ and $|\mathbf{e}_4\rangle$ are associated with the other eigenvalue (tune ν_{II}). Now we can use the above properties to write down two linearly independent solutions for the matrix **B**:

$$\mathbf{B}_{\mathrm{I}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{B}_{\mathrm{II}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{10}$$

and it follows that:

$$\mathbf{A}_\alpha = \mathbf{E}^{\mathrm{T}} \mathbf{B}_\alpha \mathbf{E} \qquad \alpha = \mathrm{I}, \mathrm{II} \tag{11}$$

where **E** is the *inverse* of the matrix constructed from the eigenvectors $|\mathbf{e}_i\rangle$ of the single-turn matrix **M**:

$$\mathbf{E}^{-1} = \begin{pmatrix} \\ |\mathbf{e}_1\rangle & |\mathbf{e}_2\rangle & |\mathbf{e}_3\rangle & |\mathbf{e}_4\rangle \\ \\ \end{pmatrix}$$

From equation (11), we can write down two linearly independent quadratic invariants of the single-turn matrix **M**:

$$2J_\alpha = \langle \mathbf{x} | \mathbf{A}_\alpha | \mathbf{x} \rangle \tag{12}$$

which are the actions in the general (either coupled or uncoupled) case.

We define emittances ε_α corresponding to these invariants (and in turn to each of the two tunes) as simply the average action of all particles in the beam:

$$\varepsilon_\alpha = \langle J_\alpha \rangle$$

Given values for these emittances it is straightforward to generalize equation (7) for the beam distribution:

$$\mathbf{\Sigma} = \left(\frac{\mathbf{A}_{\mathrm{I}}}{\varepsilon_{\mathrm{I}}} + \frac{\mathbf{A}_{\mathrm{II}}}{\varepsilon_{\mathrm{II}}} \right)^{-1} \tag{13}$$

2.3 Invariance of the Emittances Along a Beamline

Let $|\mathbf{x}_1\rangle$ be the phase space vector of a particle at a point s_1 in the lattice, and let $\mathbf{M}_{21}$ be the transfer matrix from s_1 to some other point s_2, so that:

$$|\mathbf{x}_2\rangle = \mathbf{M}_{21}|\mathbf{x}_1\rangle$$

Let $\mathbf{A}_1$ be a symmetric matrix defining an invariant of the single-turn matrix $\mathbf{M}_1$ at s_1, that is:

$$\langle\mathbf{x}|\mathbf{M}_1^{\mathrm{T}}\mathbf{A}_1\mathbf{M}_1|\mathbf{x}\rangle = \langle\mathbf{x}|\mathbf{A}_1|\mathbf{x}\rangle \qquad \text{for all } |\mathbf{x}\rangle \tag{14}$$

We can write the single-turn matrix $\mathbf{M}_2$ at s_2 as:

$$\mathbf{M}_2 = \mathbf{M}_{21}\mathbf{M}_1\mathbf{M}_{21}^{-1}$$

Substituting for $\mathbf{M}_1$ in (14), and writing $|\mathbf{x}'\rangle = \mathbf{M}_{21}|\mathbf{x}\rangle$, we find:

$$\langle\mathbf{x}'|\mathbf{M}_2^{\mathrm{T}}\left(\mathbf{M}_{21}^{-1}\right)^{\mathrm{T}}\mathbf{A}_1\mathbf{M}_{21}^{-1}\mathbf{M}_2|\mathbf{x}'\rangle = \langle\mathbf{x}'|\left(\mathbf{M}_{21}^{-1}\right)^{\mathrm{T}}\mathbf{A}_1\mathbf{M}_{21}^{-1}|\mathbf{x}'\rangle \qquad \text{for all } |\mathbf{x}'\rangle$$

and hence the symmetric matrix $\mathbf{A}_2$ given by:

$$\mathbf{A}_1 = \mathbf{M}_{21}^{\mathrm{T}}\mathbf{A}_2\mathbf{M}_{21}$$

defines an invariant of the single-turn matrix $\mathbf{M}_2$ at the point s_2. Now we observe that:

$$\begin{aligned}
2J(s_2) &= \langle\mathbf{x}_2|\mathbf{A}_2|\mathbf{x}_2\rangle \\
&= \langle\mathbf{x}_1|\mathbf{M}_{21}^{\mathrm{T}}\mathbf{A}_2\mathbf{M}_{21}|\mathbf{x}_1\rangle \\
&= \langle\mathbf{x}_1|\mathbf{M}_{21}^{\mathrm{T}}\left(\mathbf{M}_{21}^{-1}\right)^{\mathrm{T}}\mathbf{A}_1\mathbf{M}_{21}^{-1}\mathbf{M}_{21}|\mathbf{x}_1\rangle \\
&= \langle\mathbf{x}_1|\mathbf{A}_1|\mathbf{x}_1\rangle \\
&= 2J(s_1)
\end{aligned}$$

It follows that for a given particle, each of the two actions is invariant along the beamline, and hence the emittances (the mean actions over all particles in the beam) are also invariant along the beamline.

3 Betatron Trajectory in Uncoupled and Coupled Lattices

The beam distribution describes averages over all particles in the beam. A full characterization of the dynamics in a lattice includes a description of the trajectories of individual particles through the lattice.

3.1 Betatron Trajectory in an Uncoupled Lattice

Let us begin by considering the trajectory of a single particle through a storage ring lattice. We continue to consider only transverse motion. At some position s_1 the horizontal phase space co-ordinates may be written in terms of action-angle variables J_x and φ_x:

$$\begin{aligned}
x &= \sqrt{2\beta_x J_x}\cos(\varphi_x) \\
p_x &= -\sqrt{\frac{2J_x}{\beta_x}}\left[\sin(\varphi_x) + \alpha_x\cos(\varphi_x)\right]
\end{aligned} \tag{15}$$

Note that x and p_x satisfy equation (4); in other words, J_x is just the invariant action defined by (6). Equations (15) may be written:

$$|\mathbf{x}\rangle = \mathbf{N}|\mathbf{J}_x\rangle$$

where

$$\mathbf{N} = \begin{pmatrix} \sqrt{\beta_x} & 0 \\ -\alpha_x/\sqrt{\beta_x} & 1/\sqrt{\beta_x} \end{pmatrix} \tag{16}$$

and

$$|\mathbf{J}_x\rangle = \begin{pmatrix} \sqrt{2J_x}\cos(\varphi_x) \\ -\sqrt{2J_x}\sin(\varphi_x) \end{pmatrix}$$

Any symplectic transformation applied to the phase space co-ordinates must preserve the area of the ellipse defined by (4). It follows immediately that the action J_x is a constant of any single particle trajectory. The matrix **N** is defined in terms of the local properties of the lattice, i.e. it is constructed from the Twiss parameters that describe the shape of the ellipse mapped out by the phase space co-ordinates on successive turns around the lattice. **N** is the transformation that puts the single-turn matrix into "normal form".

$$\mathbf{N}^{-1}\mathbf{M}\mathbf{N} = \mathbf{R}(\mu_x)$$

where $\mathbf{R}(\theta)$ is just a rotation matrix:

$$\mathbf{R}(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

We can write:

$$\mathbf{M} = \mathbf{N}\mathbf{R}(\mu_x)\mathbf{N}^{-1}$$

which decomposes the single-turn matrix into matrices depending on the local properties (the Twiss parameters contained in **N**) and the global properties (the phase advance μ_x). We shall see that the normalizing transformation for the single-turn matrix in a coupled storage ring may be used to provide a convenient definition of the beta functions.

Under the single-turn transformation:

$$|\mathbf{x}\rangle \to \mathbf{M}|\mathbf{x}\rangle$$

we have:

$$\mathbf{N}|\mathbf{J}_x\rangle \to \mathbf{M}\mathbf{N}|\mathbf{J}_x\rangle$$

or:

$$|\mathbf{J}_x\rangle \to \mathbf{N}^{-1}\mathbf{M}\mathbf{N}|\mathbf{J}_x\rangle = \mathbf{R}(\mu_x)|\mathbf{J}_x\rangle$$

Since

$$\mathbf{R}(\mu_x)|\mathbf{J}_x\rangle = \begin{pmatrix} \sqrt{2J_x}\cos(\varphi_x + \mu_x) \\ -\sqrt{2J_x}\sin(\varphi_x + \mu_x) \end{pmatrix}$$

the effect of the single-turn transformation is simply to advance the betatron phase angle φ_x by the phase advance of the lattice.

One further result will be useful. We can show that, for the transfer matrix $\mathbf{M}_{21}$ from lattice position s_1 to s_2:

$$\mathbf{N}_2^{-1}\mathbf{M}_{21}\mathbf{N}_1 = \mathbf{R}(\Delta\mu_x) \tag{17}$$

where $\mathbf{N}_1$ and $\mathbf{N}_2$ are the normalizing transformations of the single-turn matrices at s_1 and s_2 respectively, and $\Delta\mu_x$ is the phase advance from s_1 to s_2. To prove this, let $|\mathbf{x}_1\rangle$ be the phase space vector at position s_1, and $|\mathbf{x}_2\rangle$ be the phase space vector at s_2 obtained by transporting $|\mathbf{x}_1\rangle$ along the beamline. The action-angle variables are related to the Cartesian variables by:

$$|\mathbf{J}_{x1}\rangle = \mathbf{N}_1^{-1}|\mathbf{x}_1\rangle$$
$$|\mathbf{J}_{x2}\rangle = \mathbf{N}_2^{-1}|\mathbf{x}_2\rangle$$

Since for symplectic transport the action J_x is conserved along the beamline, the transportation from s_1 to s_2 must simply be a rotation by the phase advance:

$$|\mathbf{J}_{x2}\rangle = \mathbf{R}(\Delta\mu_x)|\mathbf{J}_{x1}\rangle$$

The Cartesian variables at the two points are related by:

$$|\mathbf{x}_2\rangle = \mathbf{M}_{21}|\mathbf{x}_1\rangle$$

Combining the above equations gives:

$$\mathbf{N}_2^{-1}\mathbf{M}_{21}|\mathbf{x}_1\rangle = \mathbf{R}(\Delta\mu_x)\mathbf{N}_1^{-1}|\mathbf{x}_1\rangle$$

Since this must be true for any phase space vector $|\mathbf{x}_1\rangle$, equation (17) follows at once.

3.2 Betatron Trajectory in a Coupled Lattice

We now generalize the results of the previous section to the case of a coupled storage ring lattice. The basic formula is again:

$$|\mathbf{x}\rangle = \mathbf{N}|\mathbf{J}\rangle \tag{18}$$

where now:

$$|\mathbf{x}\rangle = \begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} \qquad |\mathbf{J}\rangle = \begin{pmatrix} \sqrt{2J_\mathrm{I}}\cos(\varphi_\mathrm{I}) \\ -\sqrt{2J_\mathrm{I}}\sin(\varphi_\mathrm{I}) \\ \sqrt{2J_\mathrm{II}}\cos(\varphi_\mathrm{II}) \\ -\sqrt{2J_\mathrm{II}}\sin(\varphi_\mathrm{II}) \end{pmatrix} \tag{19}$$

and the transformation $\mathbf{N}$ applied to the single-turn matrix $\mathbf{M}$ gives:

$$\mathbf{N}^{-1}\mathbf{M}\mathbf{N} = \mathbf{R}(\mu_{\mathrm{I}}, \mu_{\mathrm{II}}) = \begin{pmatrix} \cos(\mu_{\mathrm{I}}) & \sin(\mu_{\mathrm{I}}) & 0 & 0 \\ -\sin(\mu_{\mathrm{I}}) & \cos(\mu_{\mathrm{I}}) & 0 & 0 \\ 0 & 0 & \cos(\mu_{\mathrm{II}}) & \sin(\mu_{\mathrm{II}}) \\ 0 & 0 & -\sin(\mu_{\mathrm{II}}) & \cos(\mu_{\mathrm{II}}) \end{pmatrix} \tag{20}$$

N is readily constructed from the eigenvectors of **M**:

$$\mathbf{M}|\mathbf{e}_i\rangle = \lambda_i |\mathbf{e}_i\rangle$$

where, as in section 2.2, the index i runs from 1 to 4, and labels different eigenvectors, not different components of a single eigenvector. As before, we order the eigenvectors so that $|\mathbf{e}_1\rangle$ and $|\mathbf{e}_2\rangle$ are associated with the eigenvalues $\exp(\pm i\mu_{\mathrm{I}})$, and $|\mathbf{e}_3\rangle$ and $|\mathbf{e}_4\rangle$ are associated with the eigenvalues $\exp(\pm i\mu_{\mathrm{II}})$. We also use the normalization:

$$\langle \mathbf{e}_i|\mathbf{S}|\mathbf{e}_j\rangle = \begin{cases} \pm \mathrm{i} & \lambda_i\lambda_j = 1 \\ 0 & \lambda_i\lambda_j \neq 1 \end{cases} \tag{21}$$

Then we have:

$$\mathbf{N} = \frac{1}{\sqrt{2}}\begin{pmatrix} |\mathbf{e}_1\rangle + |\mathbf{e}_2\rangle & \dfrac{|\mathbf{e}_1\rangle - |\mathbf{e}_2\rangle}{\mathrm{i}} & |\mathbf{e}_3\rangle + |\mathbf{e}_4\rangle & \dfrac{|\mathbf{e}_3\rangle - |\mathbf{e}_4\rangle}{\mathrm{i}} \end{pmatrix} \tag{22}$$

Since the eigenvectors come in complex conjugate pairs, the components of **N** are real. With the normalization (21), the matrix **N** is symplectic. There is a degeneracy in the normalization, in that if the matrix **N** satisfies (20), then so does the matrix $\mathbf{NR}(\theta_1, \theta_2)$ for any angles θ_1 and θ_2. We choose these angles such that $n_{12} = n_{34} = 0$, and (for convenience) $n_{11}>0$ and $n_{33}>0$. Then equation (18) can be written:

$$\begin{pmatrix} x \\ p_x \\ y \\ p_y \end{pmatrix} = \begin{pmatrix} n_{11} & 0 & n_{13} & n_{14} \\ n_{21} & n_{22} & n_{23} & n_{24} \\ n_{31} & n_{32} & n_{33} & 0 \\ n_{41} & n_{42} & n_{43} & n_{44} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2J_{\mathrm{I}}}\cos(\varphi_{\mathrm{I}}) \\ -\sqrt{2J_{\mathrm{I}}}\sin(\varphi_{\mathrm{I}}) \\ \sqrt{2J_{\mathrm{II}}}\cos(\varphi_{\mathrm{II}}) \\ -\sqrt{2J_{\mathrm{II}}}\sin(\varphi_{\mathrm{II}}) \end{pmatrix} \tag{23}$$

where the components of **N**, n_{ij}, are all known from the eigenvectors of the single-turn matrix **M**. At a point in the lattice where there is no coupling, the four upper right and four lower left components of **N** are zero, $n_{11} = \sqrt{\beta_x}$, and $n_{33} = \sqrt{\beta_y}$. We extend these definitions to the coupled case, and introduce the complex quantities ζ_x and ζ_y:

$$\begin{aligned}\sqrt{\beta_x} &= n_{11}\\ \sqrt{\beta_y} &= n_{33}\\ \zeta_x &= n_{31} + \mathrm{i}n_{32}\\ \zeta_y &= n_{13} + \mathrm{i}n_{14}\end{aligned} \tag{24}$$

With these definitions, all quantities have clear physical meanings. A particle trajectory is characterized by constant actions J_{I} and J_{II}, associated with the two tunes of the lattice. If J_{II} is zero, the local horizontal amplitude is $\sqrt{\beta_x}$, and the local vertical amplitude is $|\zeta_x|$. Similarly, if J_{I} is zero, the local vertical amplitude is $\sqrt{\beta_y}$, and the local horizontal amplitude is $|\zeta_y|$.

The beta functions defined in this way are always finite and greater than zero. If there is no local coupling, then the zeta functions ζ_x and ζ_y are zero. The beta functions are also simply related to the beam size. If we define the emittances in the usual way as averages over the action of all particles in the beam:

$$\varepsilon_{\mathrm{I}} = \langle J_{\mathrm{I}} \rangle \qquad \varepsilon_{\mathrm{II}} = \langle J_{\mathrm{II}} \rangle$$

If the lattice is not close to a coupling resonance, then the phase angles satisfy:

$$\langle \cos(\varphi_{\mathrm{I}}) \cos(\varphi_{\mathrm{II}}) \rangle = 0$$

etc. for all combinations of cos and sin, and, from (23), the beam sizes may be written:

$$\begin{aligned}\langle x^2 \rangle &= \beta_x \varepsilon_{\mathrm{I}} + |\zeta_y|^2 \varepsilon_{\mathrm{II}}\\ \langle y^2 \rangle &= \beta_y \varepsilon_{\mathrm{II}} + |\zeta_x|^2 \varepsilon_{\mathrm{I}}\\ \langle xy \rangle &= \sqrt{\beta_x}\, \mathrm{Re}(\zeta_x) \varepsilon_{\mathrm{I}} + \sqrt{\beta_y}\, \mathrm{Re}(\zeta_y) \varepsilon_{\mathrm{II}}\end{aligned} \tag{1}$$

The horizontal and vertical beam sizes are necessarily positive and finite; the beam tilt may be positive or negative. The observed beam sizes will include the effects of dispersion and energy spread in the beam: these effects will add in the usual way (in quadrature) to the effects of the range of betatron amplitudes of particles in the beam.

We can also generalize the Twiss alpha functions to the coupled case in a straightforward fashion. Again working from equation (23), we find:

$$\begin{aligned}\langle x p_x \rangle &= -\alpha_x \varepsilon_{\mathrm{I}} + (n_{13} n_{23} + n_{14} n_{24}) \varepsilon_{\mathrm{II}}\\ \langle y p_y \rangle &= -\alpha_y \varepsilon_{\mathrm{II}} + (n_{31} n_{41} + n_{32} n_{42}) \varepsilon_{\mathrm{I}}\end{aligned}$$

where we have defined:

$$\begin{aligned}\alpha_x &= -n_{11} n_{21}\\ \alpha_y &= -n_{33} n_{43}\end{aligned}$$

Note that in the uncoupled case - by inspection of equation (16) - these definitions reduce to the usual definitions of the Twiss alpha functions. A general expression for a component of the covariance matrix is:

$$\langle x_i x_j \rangle = \left(n_{i1}n_{j1} + n_{i2}n_{j2}\right)\varepsilon_{\mathrm{I}} + \left(n_{i3}n_{j3} + n_{i4}n_{j4}\right)\varepsilon_{\mathrm{II}}$$

where i and j index the four components of the vector **x** defined in (19).

Finally, note that as in the uncoupled case, the normalizing matrices at two different points in the lattice may be used to normalize the coupled transfer matrix between those two points:

$$\mathbf{N}_2^{-1}\mathbf{M}_{21}\mathbf{N}_1 = \mathbf{R}\left(\Delta\mu_{\mathrm{I}}, \Delta\mu_{\mathrm{II}}\right)$$

We see that at the same time, we obtain the phase advance between the two points in the lattice.

3.3 Lattice Functions for General Phase of the Normalizing Transformation

Forest has pointed out [5,8] that it must be possible to express the lattice functions in terms of components of the normalizing transformation, using expressions that are invariant under a change of phase $\mathbf{N}\rightarrow\mathbf{NR}(\theta_1, \theta_2)$. In other words, it is not necessary to make the choice for the components of **N**, $n_{12} = n_{34} = 0$ (or, indeed, any choice at all).

For example, it follows from (18) that:

$$\begin{aligned}
\langle x^2 \rangle &= \left(n_{11}^2 + n_{12}^2\right)\varepsilon_{\mathrm{I}} + \left(n_{13}^2 + n_{14}^2\right)\varepsilon_{\mathrm{II}} \\
\langle y^2 \rangle &= \left(n_{31}^2 + n_{32}^2\right)\varepsilon_{\mathrm{I}} + \left(n_{33}^2 + n_{34}^2\right)\varepsilon_{\mathrm{II}} \\
\langle xy \rangle &= \left(n_{11}n_{31} + n_{12}n_{32}\right)\varepsilon_{\mathrm{I}} + \left(n_{13}n_{33} + n_{14}n_{34}\right)\varepsilon_{\mathrm{II}}
\end{aligned}$$

These expressions suggest definitions for the beta functions:

$$\begin{aligned}
\beta_x &= n_{11} + \mathrm{i}n_{12} \\
\beta_y &= n_{31} + \mathrm{i}n_{32}
\end{aligned} \qquad (25)$$

in terms of which, we can write:

$$\begin{aligned}
\langle x^2 \rangle &= \left|\beta_x\right|^2 \varepsilon_{\mathrm{I}} + \left|\zeta_y\right|^2 \varepsilon_{\mathrm{II}} \\
\langle y^2 \rangle &= \left|\zeta_x\right|^2 \varepsilon_{\mathrm{I}} + \left|\beta_y\right|^2 \varepsilon_{\mathrm{II}} \\
\langle xy \rangle &= \mathrm{Re}\left(\beta_x^* \zeta_x\right)\varepsilon_{\mathrm{I}} + \mathrm{Re}\left(\beta_y^* \zeta_y\right)\varepsilon_{\mathrm{II}}
\end{aligned}$$

Clearly, there are many possibilities for the definitions of the lattice functions. Although the definitions (25) lead to simple expressions for the beam parameters that have a pleasing symmetry, they do not reduce to the conventional definitions in the case of an uncoupled lattice. We therefore continue to use the definitions (24), which assume an appropriate phase has been chosen for the normalizing transformation.

3.4 Independent Components of the Normalizing Transformation

We noted above that following from equations (21) and (22), that the normalizing transformation **N** is symplectic. We have used the degeneracy in **N** to fix two of the components to zero; the symplectic condition then leaves only 8 remaining independent components. For practical purposes, it can be convenient to specify **N** by giving only a minimal set of values. There is some freedom in choosing which of the components are taken to be independent. One additional complication, however, is that situations can occur when particular members out of the set of 8 selected independent components are zero. In such cases, the values of some of the dependent components may not be determined. For this reason, it is desirable to "overspecify" the matrix **N** by giving 10 of the components, rather than the minimal 8. If the set of 10 is chosen appropriately, then the remaining components of the matrix can be found by simple, well behaved expressions.

For example, let us assume that the components n_{22}, n_{23}, n_{24}, n_{44} of **N** are *not* specified. Representing known components by $\circ$ and unknown components by $\times$, **N** may be written:

$$\mathbf{N} = \begin{pmatrix} \circ & 0 & \circ & \circ \\ \circ & \times & \times & \times \\ \circ & \circ & \circ & 0 \\ \circ & \circ & \circ & \times \end{pmatrix}$$

By inspection of (16), we see that in the uncoupled case, the beta and alpha functions will be directly associated with the independent components. The zeta functions defined by (24) are also given by independent components. Applying the symplectic condition yields the following expressions, from which the dependent components on the diagonal may be found:

$$n_{11}n_{22} = n_{33}n_{44} = 1 + n_{32}n_{41} - n_{31}n_{42} \tag{26a}$$

The off-diagonal dependent components may then be found from:

$$\begin{aligned} n_{11}n_{23} &= n_{13}n_{21} - n_{31}n_{43} + n_{33}n_{41} \\ n_{11}n_{24} &= n_{14}n_{21} - n_{31}n_{44} \end{aligned} \tag{26b}$$

Since n_{11} and n_{33} are usually greater than zero, these expressions determine the unknown components in most cases. There may be pathological situations in coupled lattices where either of the components n_{11} or n_{33} is zero, in which case alternative expressions will need to be used.

Equations (26a-b) follow from the fact that the single turn transfer matrix is symplectic, and are in this sense analogous to the familiar expression for uncoupled motion (and not generally true for coupled motion):

$$\beta\gamma - \alpha^2 = 1$$

There are many similar relations to equations for a symplectic matrix; the choice of which ones to use depends on which components of the original matrix are chosen as the independent variables.

In the limit of zero coupling, any components not in the two block diagonal sectors are zero, and we have for the dependent components on the diagonal:

$$n_{11}n_{22} \to 1 \qquad\qquad n_{33}n_{44} \to 1$$

as expected from (16).

3.5 The Normalizing Transformation and the Covariance Matrix

It should not surprise the reader that there is a simple connection between the normalizing transformation of the single-turn map, and the covariance matrix. To make this explicit, note that with **J** defined by (19), we can write:

$$\langle \mathbf{J} | \mathbf{B}_{\mathrm{I}}^2 | \mathbf{J} \rangle = 2J_{\mathrm{I}} \qquad\qquad \langle \mathbf{J} | \mathbf{B}_{\mathrm{II}}^2 | \mathbf{J} \rangle = 2J_{\mathrm{II}}$$

where $\mathbf{B}_{\mathrm{I}}$ and $\mathbf{B}_{\mathrm{II}}$ are defined in (10), and:

$$\mathbf{B}_{\mathrm{I}}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad\qquad \mathbf{B}_{\mathrm{II}}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then, substituting for **J** from (18), we see at once:

$$\mathbf{A}_{\mathrm{I}} = \left(\mathbf{N}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{I}}^2 \mathbf{N}^{-1} \qquad\qquad \mathbf{A}_{\mathrm{II}} = \left(\mathbf{N}^{-1}\right)^{\mathrm{T}} \mathbf{B}_{\mathrm{II}}^2 \mathbf{N}^{-1}$$

And we have the same expression for the covariance matrix:

$$\mathbf{\Sigma} = \left(\frac{\mathbf{A}_{\mathrm{I}}}{\varepsilon_{\mathrm{I}}} + \frac{\mathbf{A}_{\mathrm{II}}}{\varepsilon_{\mathrm{II}}} \right)^{-1} \qquad (13)$$

This provides a simple alternative to the expressions in section 2.2 for calculating the covariance matrix.

4 Example: Beam Sizes and Tilt in PEP-II LER

We have presented two methods for calculating the beam sizes and tilt in a coupled lattice, given the beam emittances. The first method, using equation (13), is based on the "matched" distribution, i.e. finding a distribution of particles in the beam that remains invariant under a full turn through the lattice. The matrices required for the calculation are constructed in a straightforward fashion from the eigenvectors of the single-turn matrix at the required point, and since the eigenvectors are associated with the betatron tunes through the eigenvalues, there is no ambiguity between the planes. The second method, using equation (1), uses the lattice functions calculated from elements of the matrix that puts the single-turn matrix into normal form. These lattice functions are also calculated using the eigenvectors of the single-turn matrix, and the correspondence between the eigenvectors and the eigenvalues again directly associates the lattice functions with the tunes.

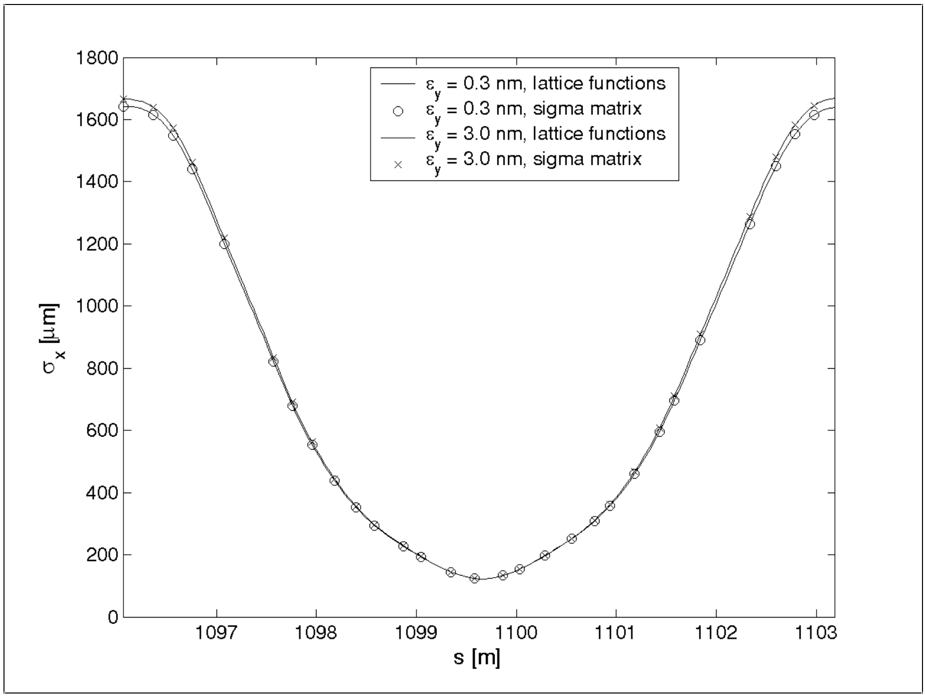

Figure 1

Horizontal beam size through the PEP-II LER interaction region.

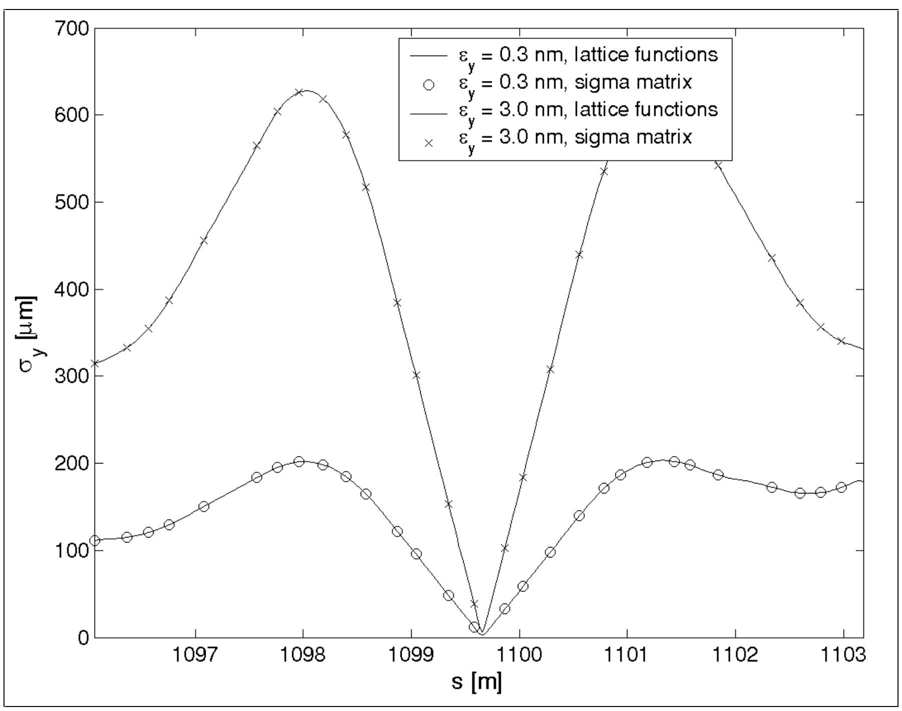

Figure 2

Vertical beam size through the PEP-II LER interaction region.

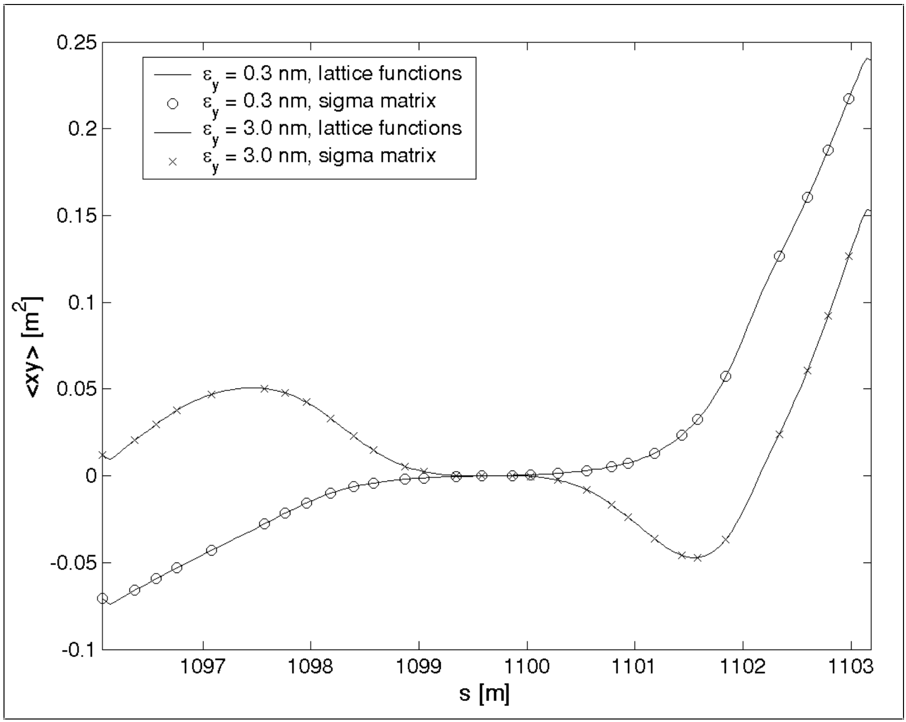

Figure 3

Beam tilt through the PEP-II LER interaction region.

Equations (13) and (1) can be easily implemented in any program with capability of calculating the eigenvalues and eigenvectors of a matrix. It is convenient to use the AT accelerator code [9] running under Matlab. This allows direct calculation of the single-turn matrix and the necessary analysis of this matrix in a single package. As an example, we show here the results of calculations of the beam sizes and tilt through the interaction region of the PEP-II LER. The presence of the solenoid in this region means that the design lattice is necessarily coupled. We assume a horizontal beam emittance of 30 nm, and consider two different values for the vertical emittance, 0.3 nm and 3.0 nm. We calculate the beam sizes and tilt using both equations (13) and (1). The methods and the calculations are somewhat different, but the results should clearly be the same. Figure 1 shows the horizontal beam size, Figure 2 shows the vertical beam size, and Figure 3 shows the beam tilt (the horizontal-vertical correlation). The two methods give identical results. Note that the full lattice functions are shown in Figure 4 and Figure 5. In practice, (1) is rather more convenient, since the lattice functions need only be calculated once. Equation (13) can be useful where all the beam correlations are required.

5 Measurement of Coupled Lattice Parameters

If it is possible to excite a bunch of particles to perform coherent betatron motion associated with just one of the two lattice tunes, then the phase advance and the ratios of beta and zeta functions at different points in the lattice may be measured in a straightforward way. The excitation may be achieved, for example, by “shaking” the beam at a frequency close to the betatron frequency, as is often done for tune measurement. Collecting sets of horizontal and vertical BPM readings then allows the lattice parameters to be determined. The idea is to perform measurements analogous to the phase advance measurements already carried out for uncoupled lattices [2].

Consider, for example, if coherent oscillations are excited such that J_{I} has a non-zero value, while J_{II} is zero. Equation (23) gives us:

$$\begin{pmatrix} x \\ y \end{pmatrix}_{s_1} = \begin{pmatrix} n_{11} & 0 \\ n_{31} & n_{31} \end{pmatrix}_{s_1} \cdot \begin{pmatrix} \sqrt{2J_{\mathrm{I}}}\cos(\varphi_{\mathrm{I}}) \\ -\sqrt{2J_{\mathrm{I}}}\cos(\varphi_{\mathrm{I}}) \end{pmatrix}$$

at the first BPM, and

$$\begin{pmatrix} x \\ y \end{pmatrix}_{s_2} = \begin{pmatrix} n_{11} & 0 \\ n_{31} & n_{31} \end{pmatrix}_{s_2} \cdot \begin{pmatrix} \sqrt{2J_{\mathrm{I}}}\cos(\varphi_{\mathrm{I}} + \Delta\mu_{\mathrm{I}}) \\ -\sqrt{2J_{\mathrm{I}}}\cos(\varphi_{\mathrm{I}} + \Delta\mu_{\mathrm{I}}) \end{pmatrix}$$

Suppose we have a large number of readings for $x_1 = x(s_1)$ and $x_2 = x(s_2)$. We see that if we plot x_2 vs x_1, all the points lie on an ellipse with area $2\tilde{J}$ and with shape defined by the parameters $\tilde{\alpha}$, $\tilde{\beta}$:

$$x_1 = \sqrt{2\tilde{\beta}\tilde{J}}\cos(\tilde{\varphi})$$

$$x_2 = -\sqrt{\frac{2\tilde{J}}{\tilde{\beta}}}[\sin(\tilde{\varphi}) + \tilde{\alpha}\cos(\tilde{\varphi})]$$

where

$$\tilde{J}^2 = \langle x_1^2 \rangle \langle x_2^2 \rangle - \langle x_1 x_2 \rangle^2$$

and

$$\frac{\cos(\Delta\mu_{\mathrm{I}})}{|\sin(\Delta\mu_{\mathrm{I}})|} = -\tilde{\alpha}$$

$$\frac{n_{11}(s_1)}{n_{11}(s_2)} = \sqrt{\frac{\beta_x(s_1)}{\beta_x(s_2)}} = \sqrt{\frac{\tilde{\beta}}{\tilde{\gamma}}}$$

and as usual,

$$\tilde{\beta}\tilde{\gamma} - \tilde{\alpha}^2 = 1$$

Hence, the shape of the ellipse gives the phase advance between the BPMs, and the ratio of the beta functions at the BPMs.

Similar results are obtained by plotting combinations of other variables: the various expressions obtained are shown in Table 1. In each case, the phase determined is found from an expression of the form:

$$\frac{\cos(\theta)}{|\sin(\theta)|} = -\tilde{\alpha}$$

Table 1

Phase advances and lattice parameters determined from plots of BPM measurements of coherent betatron oscillations.

Measurements	Phase Determined	$\sqrt{\dfrac{\tilde{\beta}}{\tilde{\gamma}}}$
x_2 vs x_1	$\Delta\mu_{\mathrm{I}}$	$\sqrt{\dfrac{\beta_x(s_1)}{\beta_x(s_2)}}$
y_2 vs x_1	$\Delta\mu_{\mathrm{I}} + \arg[\zeta_x(s_2)]$	$\dfrac{\sqrt{\beta_x(s_1)}}{\lvert\zeta_x(s_2)\rvert}$
x_2 vs y_1	$\Delta\mu_{\mathrm{I}} - \arg[\zeta_x(s_1)]$	$\dfrac{\lvert\zeta_x(s_1)\rvert}{\sqrt{\beta_x(s_2)}}$
y_2 vs y_1	$\Delta\mu_{\mathrm{I}} + \arg[\zeta_x(s_2)] - \arg[\zeta_x(s_1)]$	$\dfrac{\lvert\zeta_x(s_1)\rvert}{\lvert\zeta_x(s_2)\rvert}$

5.1 Simulation example: the PEP-II LER nominal lattice

As an illustration of the technique, we consider the PEP-II LER lattice. Although the technique can in principle be applied to any pair of BPMs, we select two BPMs near the interaction region, where the design lattice has significant coupling. The beta and zeta functions in the lattice are shown in Figure 4 and Figure 5 respectively.

We simulate data by tracking a particle with horizontal action 500 nm about 1000 turns through the lattice, recording the horizontal and vertical positions at two BPMs (at approximate locations 1094 m and 1099 m) at each turn. In fact, it is not necessary that the data be on consecutive turns; only that the action be the same each time, and that a given reading on one BPM is taken on the same turn as the corresponding reading on the other BPM. Knowing the normalizing transformation at the starting point, appropriate initial co-ordinates are easily determined from equation (18), choosing J_{I} = 500 nm, J_{II} = 0, and an arbitrary initial phase.

If we plot different combinations of the horizontal and vertical co-ordinates against one another, we get the plots shown in Figure 6. These plots are readily analyzed as described above, to determine the phase advance between the BPMs, and the ratio of the lattice functions at the two BPMs. The results are given in Table 2.

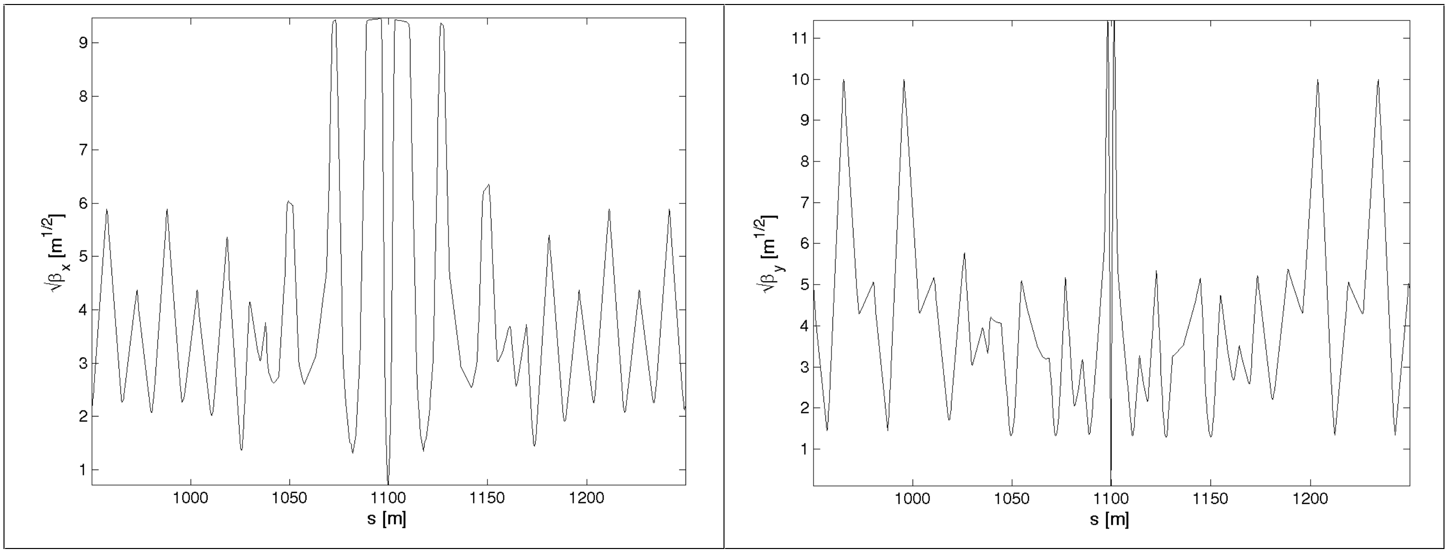

Figure 4

Square roots of the coupled beta functions (left, horizontal; and right, vertical) in the coupled region of the nominal PEP-II LER lattice.

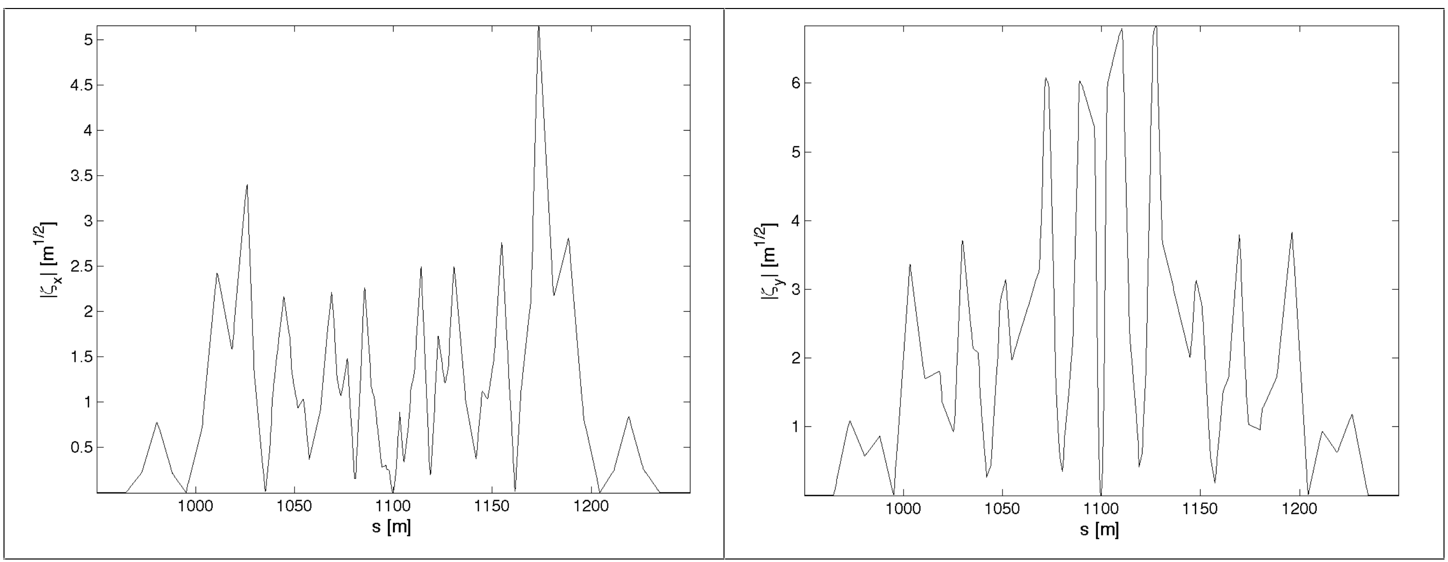

Figure 5

Coupled zeta functions (left, horizontal; and right, vertical) in the coupled region of the nominal PEP-II LER lattice. Outside the coupled region, the zeta functions are zero.

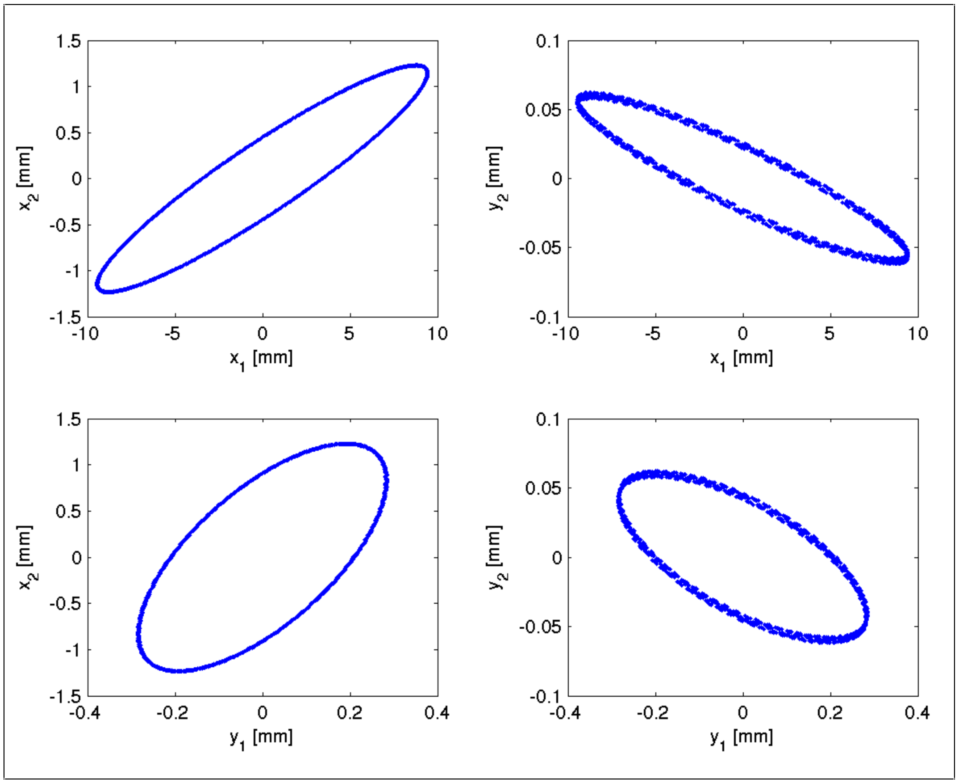

Figure 6

Correlation plots between BPM readings in a tracking simulation for the nominal lattice.

Table 2

Comparison of phase advances and lattice parameters determined from the tracking simulation and from the model. The upper number in each pair is the result of the fit to the simulated tracking data, and the lower number is the value determined from the model. The results are for the nominal lattice model.

Measurements	Phase Determined		$\sqrt{\frac{\tilde{\beta}}{\tilde{\gamma}}}$	
x_2 vs x_1	$\Delta\mu_I$	2.5613	$\sqrt{\frac{\beta_x(s_1)}{\beta_x(s_2)}}$	7.6756
		2.5728		7.6666
y_2 vs x_1	$\Delta\mu_I + \arg[\zeta_x(s_2)]$	-2.3614	$\frac{\sqrt{\beta_x(s_1)}}{\lvert\zeta_x(s_2)\rvert}$	156.33
		-2.3174		155.97
x_2 vs y_1	$\Delta\mu_I - \arg[\zeta_x(s_1)]$	0.9110	$\frac{\lvert\zeta_x(s_1)\rvert}{\sqrt{\beta_x(s_2)}}$	0.2304
		0.9142		0.2304
y_2 vs y_1	$\Delta\mu_I + \arg[\zeta_x(s_2)] - \arg[\zeta_x(s_1)]$	-0.9594	$\frac{\lvert\zeta_x(s_1)\rvert}{\lvert\zeta_x(s_2)\rvert}$	4.6917
		-0.9839		4.6871

5.2 Simulation example: the PEP-II LER "detuned" lattice

Let us compare the above results with those obtained from a "detuned" lattice. In this case, we use a lattice model derived from the actual magnet settings and steering on one date in November 2003. The lattice functions are shown in Figure 7 and Figure 8. Some beta beating is evident, but there is clearly a significant change in the coupling.

We repeat the tracking simulation, again using J_I = 500 nm, J_{II} = 0, and determining the initial co-ordinates from (18) with the appropriate normalizing transformation for the new lattice. The correlation plots (for the same pair of BPMs) are shown in Figure 9, and the numerical results are given in Table 3. We note that the values for the phase advances and the lattice functions calculated from the simulated tracking data are again in excellent agreement with the values calculated directly from the model. The values in the "detuned" lattice are significantly different from those in the nominal lattice.

It is of course possible to take data from adjacent BPMs throughout the lattice, and (assuming a global scale factor) compare the lattice functions derived from the BPM data with those calculated directly from the model.

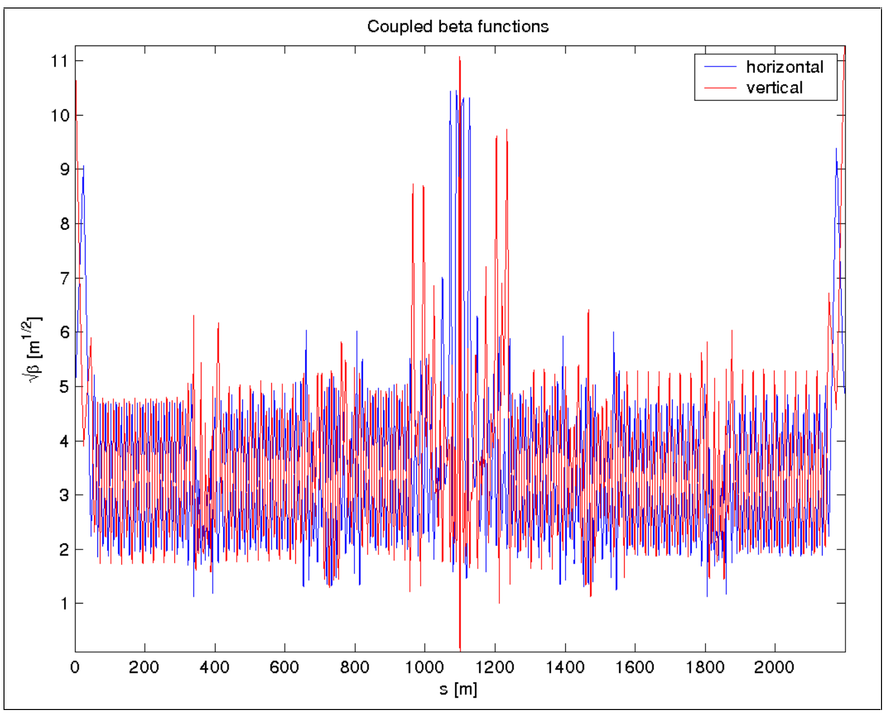

Figure 7

Square roots of the coupled beta functions in the detuned PEP-II LER lattice.

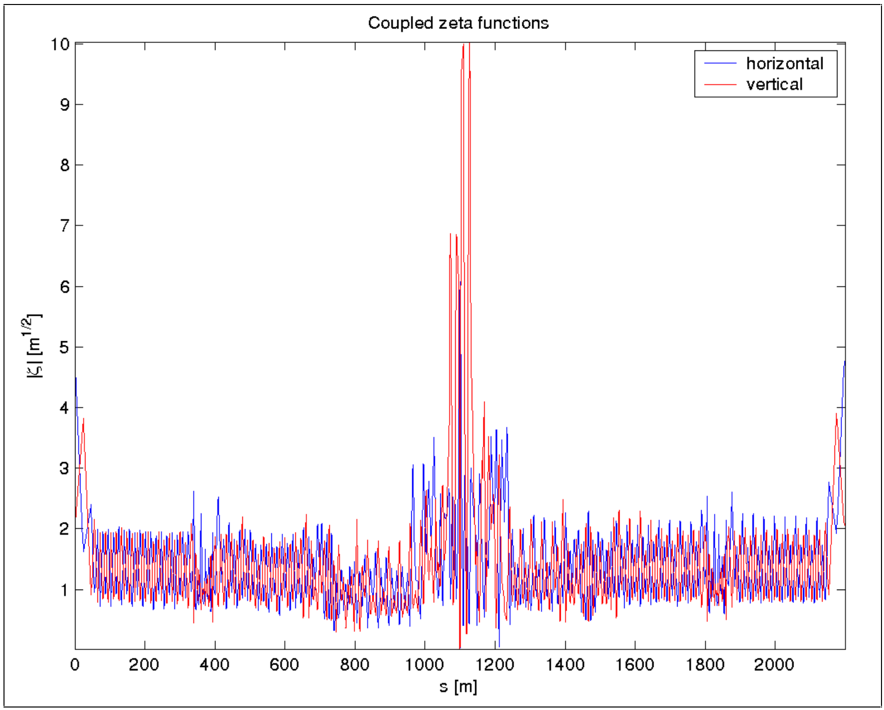

Figure 8

Coupled zeta functions in the detuned PEP-II LER lattice.

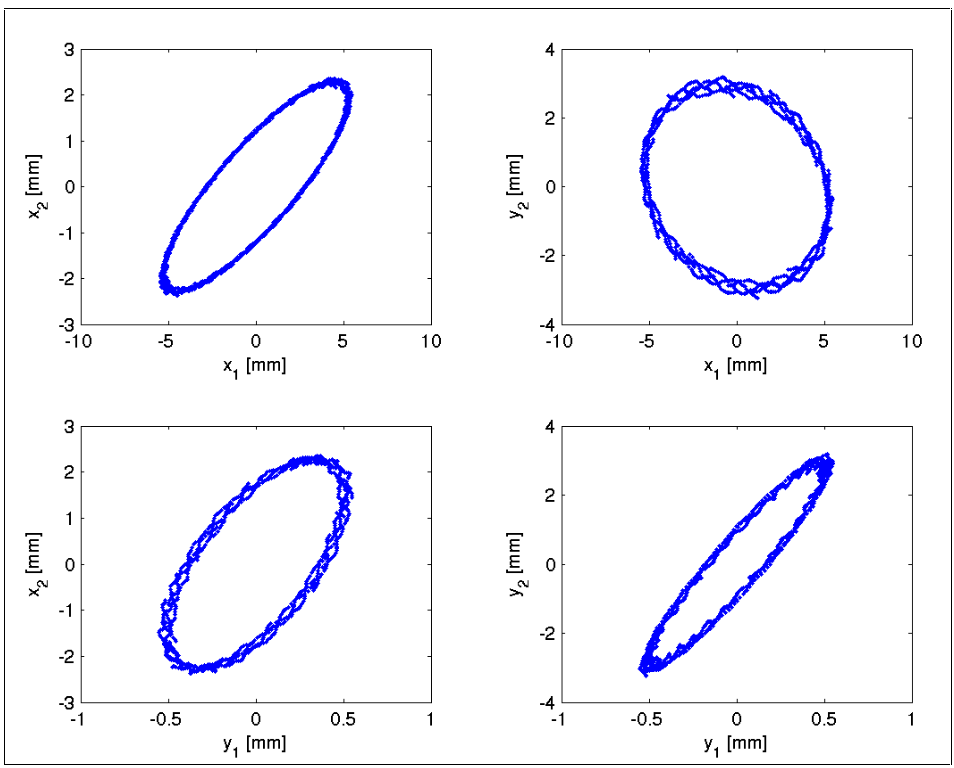

Figure 9

Correlation plots between BPM readings in a tracking simulation for the "detuned" lattice.

Table 3

Comparison of phase advances and lattice parameters determined from the tracking simulation and from the model. The upper number in each pair is the result of the fit to the simulated tracking data, and the lower number is the value determined from the model. The results are for the "detuned" lattice model.

Measurements	Phase Determined		$\sqrt{\tilde{\beta}/\tilde{\gamma}}$	
x_2 vs x_1	$\Delta\mu_I$	1.5867	$\sqrt{\frac{\beta_x(s_1)}{\beta_x(s_2)}}$	2.3196
		1.5889		2.3193
y_2 vs x_1	$\Delta\mu_I + \arg[\zeta_x(s_2)]$	-0.1863	$\frac{\sqrt{\beta_x(s_1)}}{\lvert\zeta_x(s_2)\rvert}$	1.7961
		-0.1887		1.8089
x_2 vs y_1	$\Delta\mu_I - \arg[\zeta_x(s_1)]$	0.8558	$\frac{\lvert\zeta_x(s_1)\rvert}{\sqrt{\beta_x(s_2)}}$	0.2231
		0.8521		0.2217
y_2 vs y_1	$\Delta\mu_I + \arg[\zeta_x(s_2)] - \arg[\zeta_x(s_1)]$	2.8880	$\frac{\lvert\zeta_x(s_1)\rvert}{\lvert\zeta_x(s_2)\rvert}$	0.1727
		2.9147		0.1729

6 Conclusions

We have shown that betatron coupling in a lattice may be characterized in a straightforward way using the existing techniques of normal form analysis. Betatron trajectories are viewed as the sum of modes of oscillation with frequencies given by the tunes of the lattice; the beta functions continue (as in the uncoupled case) to give the local amplitude of oscillation, and are always positive and finite. The beta functions generalized to coupled lattices are readily found from the single-turn matrix. The co-ordinates we use are always the horizontal and vertical co-ordinates of the laboratory frame, so there is no possibility of confusion resulting from geometric transformations of the co-ordinate frames. If the beam emittances are known (or viewed as parameters), the generalized beta functions also give directly the beam distribution. We have also shown, with some illustrations from simulations in the PEP-II LER lattice, that it should be possible to make direct measurements of the significant quantities describing the coupled dynamics.

7 Acknowledgements

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