On Optimal Ergodic Interference Alignment

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***Abstract*—The original ergodic interference alignment scheme proposed by Nazer et al. requires symmetric channel phase distribution. In this paper, we investigate a new ergodic interference alignment scheme which can achieve one half interference-free degree of freedom (DoF) for arbitrary phase distribution. Even for symmetric phase distributions, the new scheme achieves a better high SNR offset than the original ergodic interference alignment scheme, and depending upon the magnitude distributions it is shown that the SNR offset improvement with the new scheme over the original scheme can be arbitrarily large. The SNR offset optimal ergodic alignment scheme is based on results in majorization theory.**

I. Introduction

Introduced by Nazer et al. in [1], ergodic interference alignment (IA) is the idea of opportunistically pairing complementary channel states, e.g., in the K-user interference channel, such that all interference is automatically aligned and every user is simultaneously able to achieve the same rate as if he had the channel to himself for half the time. An important requirement of the original scheme of Nazer is that the phase distributions must be symmetric, in order to ensure that complementary channel states occur in approximately equal proportions and therefore can be matched evenly. Let us take a 3-user interference channel as an example. In the channel use l_1, the channel state is

$$H(l_1) = \begin{bmatrix} h_{11}(l_1) & h_{12}(l_1) & h_{13}(l_1) \\ h_{21}(l_1) & h_{22}(l_1) & h_{23}(l_1) \\ h_{31}(l_1) & h_{32}(l_1) & h_{33}(l_1) \end{bmatrix} \quad (1)$$

where h_{rt} $t, r \in \{1, 2, 3\}$ is the channel coefficient from transmitter t to receiver r. In another channel use l_2, the complementary channel state should satisfy the property that the direct channel coefficients retain the same value but the cross channel coefficients are the negatives of the original state, i.e., in the original ergodic IA, the complementary channel matrix for $H(l_1)$ is

$$H^c_{\text{orig}} = \begin{bmatrix} h_{11}(l_1) & -h_{12}(l_1) & -h_{13}(l_1) \\ -h_{21}(l_1) & h_{22}(l_1) & -h_{23}(l_1) \\ -h_{31}(l_1) & -h_{32}(l_1) & h_{33}(l_1) \end{bmatrix} \quad (2)$$

In these two complementary states, the transmitters repeat the same symbol and each receiver adds the channel outputs. This causes all interferences to be canceled, $1/2$ DoF is achieved for each user, and at any finite SNR the following rate is achievable for each user [1]

$$R_{\text{orig}} = \frac{1}{2} E[\log_2(1 + 2|h|^2 P)] \quad (3)$$

where P is the transmit power of each user, h is the desired channel (h_{kk} for the k^{th} user), and the expectation is over h. Since additive white Gaussian noise power is normalized to unity, P may be interpreted directly as SNR. Since the ergodic capacity of any user (with uniform power allocation) is

$$C = E[\log_2(1 + |h|^2 P)] \quad (4)$$

the original ergodic IA scheme guarantees to each user the same rate as if he had the channel to himself for half the time. This is clearly a very strong guarantee and shows the remarkable potential of ergodic IA. In [2], Jafar has shown that the ergodic IA scheme is capacity optimal when the channel magnitudes are held fixed.

The success of the ergodic IA relies on a critical assumption that the channel phase distribution is symmetric, which means that the complementary channel states will occur in equal proportions and be matched evenly. Although this assumption applies to a variety of real world channel distributions, like Rayleigh distribution, it is not applicable universally.

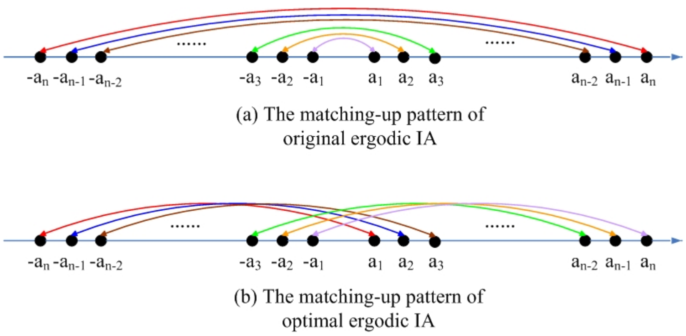
In this paper, we develop a new ergodic IA scheme, which can achieve one half interference-free degree of freedom (DoF) in interference networks with arbitrary phase distribution. In our proposed ergodic IA scheme, we define two channel states as complementary if the direct channel realizations are different and the cross channel realizations are identical. Thus in our proposed scheme, the complementary channel matrix for $H(l_1)$ is

$$H^c_{\text{new}} = \begin{bmatrix} h_{11}(l_2) & h_{12}(l_1) & h_{13}(l_1) \\ h_{21}(l_1) & h_{22}(l_2) & h_{23}(l_1) \\ h_{31}(l_1) & h_{32}(l_1) & h_{33}(l_2) \end{bmatrix} \quad (5)$$

where $h_{11}(l_2) \neq h_{11}(l_1)$, $h_{22}(l_2) \neq h_{22}(l_1)$ and $h_{33}(l_2) \neq h_{33}(l_1)$. Over the two complementary states, we can use a repetition code and subtract one received symbol from another. This new approach can also cancel interference without cancelling the desired signals, thus achieving 1/2 DoF for each receiver without the assumption of symmetric phase distribution.

When the channel phase is asymmetric, the original ergodic IA cannot guarantee 1/2 DoF per user due to the occurrence of unmatched states with probability bounded away from zero. Since the new ergodic IA scheme presented in this paper can achieve one half interference-free DoF for each user, *the improvement is in DoF, if the phase distribution is asymmetric*. If the phase distribution is symmetric, both schemes guarantee 1/2 DoF, but what about the high SNR offset (coding gain)? Is the original ergodic IA optimal for high SNR offset?

When the phase distribution is symmetric, the original ergodic IA can be seen as a special case of our proposed scheme. We can achieve the same rate as the original ergodic IA scheme, by requiring $h_{kk}(l_2) = -h_{kk}(l_1)$. However, if the



(a) The matching-up pattern of original ergodic IA

(b) The matching-up pattern of optimal ergodic IA

Fig. 1. The matching-up patterns for the original and optimal ergodic IA schemes when the channel phase distribution is symmetric and the amplitude has $2n$ discrete values $\pm a_1, \pm a_2, ..., \pm a_n$, with equal probability.

phase is indeed symmetric, we ask three questions: 1) Is this the best pairing of complementary states? 2) If not, then what is the optimal pairing? 3) How much improvement can it offer relative to the original ergodic IA? These are the questions that we will answer in this work. The optimality metric that we will use is the high SNR offset.

A. *Overview of the Main Results*

When channel phase distribution is symmetric, for question 1), we find that the original achievable rate in (3) is *not* optimal for SNR offset, unless the channel magnitude is constant (in which case it is capacity optimal). The difference between the optimal pairing and original pairing is illustrated by a toy example in Section III. Then for question 2), we find how to match up direct channel values in complementary channel states to achieve optimal ergodic IA. Here, we give a simple example to explain the optimal matching-up pattern. For instance, suppose the channel amplitude has $2n$ discrete values, which are $\pm a_1, \pm a_2, ..., \pm a_n$, with equal probability. In the original ergodic IA, we match a_i with $-a_i$, $\forall i \in \{1, 2, ..., n\}$. However, we prove that to achieve the optimal SNR offset, we should pair a_i with $-a_{n-i+1}$. The comparison of original and optimal matching-up patterns is shown in Fig. 1. The related proof for more general cases is given in Lemma 2 in Section V, which is mainly based on majorization theory [3]. Then we develop the achievable rate for the optimal ergodic IA scheme, which is stated in Theorem 1 in Section V. Finally for question 3), we demonstrate the improvement of the SNR offset gained by the optimal ergodic IA can be unbounded by an example in Section VI.

II. System Model

We consider the general K-user ergodic fading Gaussian interference network, in which each node is equipped with one antenna. At the r^{th} receiver, the received signal in the l^{th} channel use is

$$Y_r(l) = \sum_{t=1}^{K} h_{rt}(l) e^{j\theta_{rt}(l)} X_t(l) + Z_r(l) \quad t, r \in \{1, 2, ..., K\} \tag{6}$$

where $X_t(l)$ is the transmitted symbol of transmitter t, $Z_r(l)$ is the additive white Gaussian noise at receiver r, and $h_{rt}(l)e^{j\theta_{rt}(l)}$ is the channel coefficient between transmitter t and receiver r. In this paper, the local noise power is normalized at each receiver, and the transmit power of each transmitter is denoted as P, i.e, $E[|X_t|^2] \leq P$, $\forall t \in \{1, 2, ..., K\}$. We assume that all channel coefficient values are time-varying, i.i.d., drawn from a continuous distribution, and the channel phase $\theta_{rt}(l)$ and strength $h_{rt}(l)$ are independent. Throughout this paper, the symmetric phase distribution implies $p(\theta) = p(\theta + \pi)$ for all phase term θ. Global channel knowledge is available at all transmitters and receivers.

In the K-user interference network, there are K independent messages W_1, W_2,...W_K, and the message W_k transmitted by transmitter k is intended for receiver k only. Let $R_k(P)$ denote the achievable rate for message W_k. Then the DoF of message W_k is defined as $d_k = \lim_{P\to\infty} R_k(P)/\log_2(P)$.

III. Rate Optimization Problem of Ergodic Interference Alignment - A Toy Example

For symmetric phase distribution, when P is relatively large, the achievable rate of original ergodic IA can be written as

$$R_{\text{orig}} = \frac{1}{2}E[\log_2(1 + 2|h|^2P)] \approx \frac{1}{2}E[\log_2(2|h|^2P)] \tag{7}$$

As it turns out, while 1/2 DoF is achieved per user, the above achievable rate is not optimal for SNR offset. It is possible to achieve 1/2 DoF per user *and a better SNR offset* through a different matching rule for desired channels. We explain the intuition through a toy example below.

We assume the phase distribution is symmetric, and the channel amplitude has 4 discrete values h_1, h_2, h_3 and h_4, each with probability $1/4$. When P is large enough, in order to achieve $1/2$ DoF no values would be matched up with itself. x_1, x_2, x_3, x_4, x_5 and x_6, which take the value of 1 or 0, denote whether h_1 is matched up with h_2, h_1 with h_3, h_1 with h_4, h_2 with h_3, h_2 with h_4, and h_3 with h_4, respectively. For example, if $x_1 = 1$, it means that h_1 is matched up with h_2, and $x_1 = 0$ represents that h_1 is not paired with h_2. The achievable rate R of ergodic IA can be optimized by binary linear programming (BLP) in (8) on the top of next page.

We can use some classic BLP methods, like linear programming-based branch-and-bound algorithm, to maximize R. For instance, when $h_1 = -2$, $h_2 = -1$, $h_3 = 1$, and $h_4 = 2$, we can get the optimal rate is $\frac{1}{2}\log_2(\frac{9}{2}P)$. The original achievable rate in (7) is only $\frac{1}{2}\log_2(4P)$. The difference between the optimal and original solution lies in the matching-up procedure of direct channel values in the complementary channel matrices. In the original solution, 1 is matched up -1 and 2 is matched up with -2. However, in the optimal solution, 1 is matched up with -2 and 2 is matched up with -1.

Remark 1: We can extend the toy example to general cases easily. For instance, if the channel has $2N$ discrete channel values with equal probability, when P is relatively large the achievable rate is

$$R = \frac{1}{2}\log_2\frac{P}{2} + \frac{1}{2N}\log_2(\prod_{i=1}^{N} y_i^2) = \frac{1}{2}\log_2\frac{P}{2} + \frac{1}{N}\log_2(\prod_{i=1}^{N} y_i) \tag{9}$$

$$
\begin{aligned}
\max R =&\frac{1}{4}x_1\log_2(1+\frac{(|h_1-h_2|)^2}{2}P)+\frac{1}{4}x_2\log_2(1+\frac{(|h_1-h_3|)^2}{2}P)+\frac{1}{4}x_3\log_2(1+\frac{(|h_1-h_4|)^2}{2}P)\\
&+\frac{1}{4}x_4\log_2(1+\frac{(|h_2-h_3|)^2}{2}P)+\frac{1}{4}x_5\log_2(1+\frac{(|h_2-h_4|)^2}{2}P)+\frac{1}{4}x_6\log_2(1+\frac{(|h_3-h_4|)^2}{2}P)\\
\approx&\frac{1}{4}x_1\log_2(\frac{(|h_1-h_2|)^2}{2}P)+\frac{1}{4}x_2\log_2(\frac{(|h_1-h_3|)^2}{2}P)+\frac{1}{4}x_3\log_2(\frac{(|h_1-h_4|)^2}{2}P)\\
&+\frac{1}{4}x_4\log_2(\frac{(|h_2-h_3|)^2}{2}P)+\frac{1}{4}x_5\log_2(\frac{(|h_2-h_4|)^2}{2}P)+\frac{1}{4}x_6\log_2(\frac{(|h_3-h_4|)^2}{2}P)\\
=&\frac{1}{4}[\sum_{i=1}^{6}x_i\log_2\frac{P}{2}+2x_1\log_2(|h_2-h_1|)+2x_2\log_2(|h_3-h_1|)+2x_3\log_2(|h_4-h_1|)+2x_4\log_2(|h_3-h_2|)\\
&+2x_5\log_2(|h_4-h_2|)+2x_6\log_2(|h_4-h_3|)]\\
=&\frac{1}{2}[\log_2\frac{P}{2}+x_1\log_2(|h_2-h_1|)+x_2\log_2(|h_3-h_1|)+x_3\log_2(|h_4-h_1|)+x_4\log_2(|h_3-h_2|)\\
&+x_5\log_2(|h_4-h_2|)+x_6\log_2(|h_4-h_3|)]\\
s.t.\quad & x_1+x_2+x_3=1 \quad x_1+x_4+x_5=1 \quad x_2+x_4+x_6=1 \quad x_3+x_5+x_6=1\\
& x_1,x_2,x_3,x_4,x_5,x_6\in\{0,1\}
\end{aligned}
\tag{8}
$$

where y_i is the Euclidean distance between the matched up direct channel coefficients in the complex plane.

IV. Channel Quantization

Since in real world the channel distribution is continuous, we need to quantize the channel coefficients first and then pair up matrices based on the quantized channel values. Hereafter, for notational convenience, we denote the quantized value of $h_{rt}(l)$ as $\hat{h}_{rt}(l)$. We only focus on the case of symmetric phase distribution.

First, we choose a threshold value h_{MAX} for channel coefficients. If any channel amplitude in channel state matrix is larger than h_{MAX}, we declare an error. The complex plane up to distance h_{MAX} from the origin is divided into $2N$ arches with equal probability according to the channel amplitude distribution (If the channel phase is distributed between $[0, 2\pi)$, the complex plane would be divided into N rings with equal probability). Then these arches are further divided into M equal segments based on phase distribution. To ensure that both $\hat{h}_{rt}$ (the quantized value with phase θ) and $-\hat{h}_{rt}$ (the quantized value with phase $\theta+\pi$) correspond to valid quantization cells, we constraint the number of angles M to be even. Therefore all the segments in the quantization have equal probability. Each segment is called a quantized cell which is represented by its centroid. The maximum distance between any two points in one quantized cell is denoted as δ. The quantization scheme is illustrated in Fig. 2.

Remark 2: The quantization procedure is quite similar with that in [1] except one difference: in [1], each quantized cell has the same probability of occurring as any other cell within the same ring. Here we require that all the quantized cells have the same probability in the whole complex plane.

When h_{MAX} is sufficiently large, by taking larger and larger values for N and M, we can achieve the target rate in the limit with an infinitesimal probability of error using a good code [1]. After quantization, we also know that when the number of channel uses is large enough, the sequence of quantized channel matrices is strong typical with high probability [1], which implies that the empirical distribution of channel matrices is close to the probability distribution. Therefore, nearly all the channel matrices can be matched. Due to limited space, interested readers may refer to [1] for details.

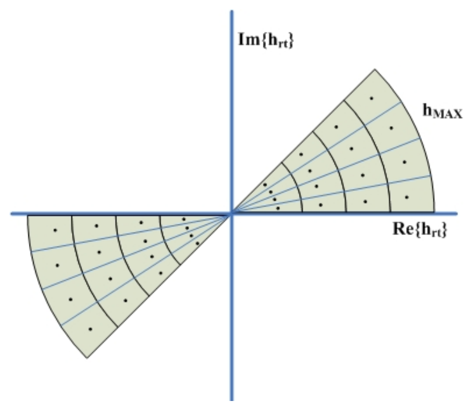


Fig. 2. Quantization of channel coefficients with amplitude less than h_{MAX} when the phase distribution is symmetric, i.e., $p(\theta) = p(\theta + \pi)$. Here each quantized cell has equal probability of occurring and is represented by its centroid.

The effects of quantization errors can be bounded by the following lemma in [1].

Lemma 1 (Lemma 1 in [1]:) Let $h[l_1]$, $h[l_2]$,..., $h[l_n]$ be channel coefficients with magnitude less than h_{MAX}. For any $a_n \in \mathbb{C}$,

$$
\begin{aligned}
&|\sum_{n=1}^{N}a_n h[l_n]| \le |\sum_{n=1}^{N}a_n\hat{h}[l_n]| + \sigma\sum_{n=1}^{N}|a_n|\\
&|\sum_{n=1}^{N}a_n h[l_n]| \ge max\{0, |\sum_{n=1}^{N}a_n\hat{h}[l_n]| - \sigma\sum_{n=1}^{N}|a_n|\}
\end{aligned}
\tag{10}
$$

V. Optimal Ergodic Interference Alignment for Symmetric Phase Distribution

In order to achieve the optimal ergodic IA when the channel phase is symmetric, first we need to find the optimal matching-up pattern for complementary channel matrices to get the optimal achievable rate for high SNR offset.

From (9) we know that the achievable rate of ergodic IA is dependent on the product of distances between the matched-up direct channel values in complementary states. We can match the quantized cell $\hat{h}_\alpha$ with phase θ up with a cell $\hat{h}_\beta$ with any phase in the complex plane. However, when the phase is symmetric, we can always find a quantized cell $\hat{h}_\gamma$ with the phase $\theta+\pi$, which has the same absolute value as that of $\hat{h}_\beta$, guaranteeing that the distance between $\hat{h}_\alpha$ and $\hat{h}_\gamma$ is not shorter than that between $\hat{h}_\alpha$ and $\hat{h}_\beta$. Therefore, the optimal matching-up pattern requires that all the quantized channel values with phase θ should be paired with (subtracted from) the ones with phase $\theta+\pi$. Then we also know that if the channel phase is symmetric, the specific phase distribution would not affect the achievable rate of the optimal ergodic IA scheme. Hence hereafter we only consider one pair of specific channel phases: θ and $\theta+\pi$.

According to the channel quantization procedure in Section IV, for any specific phase pair θ and $\theta+\pi$, we divide the channel amplitude value into $2N$ segments. The quantized value for each segments are denoted as $\{\hat{h}_1, \hat{h}_2, ..., \hat{h}_{2N}\}$. We assume that $Re\{\hat{h}_1\} < Re\{\hat{h}_2\} < ... < Re\{\hat{h}_{2N}\}$ without loss of generality. If we define two vectors $\vec{H_A} = (|\hat{h}_1|, |\hat{h}_2|, ..., |\hat{h}_N|)$ (the phases of $\hat{h}_1$, $\hat{h}_2$,...,$\hat{h}_N$ are all θ), and $\vec{H_B} = (|\hat{h}_{N+1}|, |\hat{h}_{N+2}|, ..., |\hat{h}_{2N}|)$ (the phases of $\hat{h}_{N+1}$, $\hat{h}_{N+2}$,...,$\hat{h}_{2N}$ are all $\theta+\pi$), then the elements in $\vec{H_A}$ should always be paired up with those in $\vec{H_B}$ to achieve the optimal ergodic IA. When we complete the matching-up procedure, we get N positive variables $\{y_1, y_2, ..., y_N\}$. From (9), we know if we intend to maximize the achievable rate R, we need to maximize $y = \prod_{i=1}^{N} y_i$. The following lemma tells us how to match up the direct channel coefficients further to get the optimal rate.

Lemma 2: Vectors A and B include N random positive real numbers $\{a_1, a_2, ..., a_N\}$ and $\{b_1, b_2, ..., b_N\}$ respectively. We assume that $a_1 < a_2 < ... < a_N$ and $b_1 < b_2 < ... < b_N$ without loss of generality. If we add A and B pairwise and then get their product, $(a_1+b_N)(a_2+b_{N-1})...(a_N+b_1)$ is always the largest, i.e. $(a_1+b_N)(a_2+b_{N-1})...(a_N+b_1) \geq (a_{i_1}+b_{j_1})(a_{i_2}+b_{j_2})...(a_{j_N}+b_{j_N})$, where $i_1 \neq i_2 \neq ...i_N$, $j_1 \neq j_2 \neq ...j_N$, and i_1, i_2 ... $i_N \in \{1, 2, ..., N\}$, j_1, j_2 ... $j_N \in \{1, 2, ..., N\}$.

Proof: The proof is based on inequalities via majorization [3], which is shown in the Appendix A. □

Based on the above Lemma, we know that when $y_i = |\hat{h}_{N+i}| + |\hat{h}_i| = |\hat{h}_{N+i} - \hat{h}_i|$ where $i \in \{1, 2, ...N\}$, the product y achieves the largest value. Therefor the matching-up pattern to achieve the optimal rate for SNR offset is: *$\hat{h}_i$ should always be matched up with $\hat{h}_{N+i}$, $\forall i \in \{1, 2, ...N\}$.* The optimal matching-up pattern when $\theta = 0$ is shown in Fig. 3.

Then we can get the achievable rate for the optimal ergodic

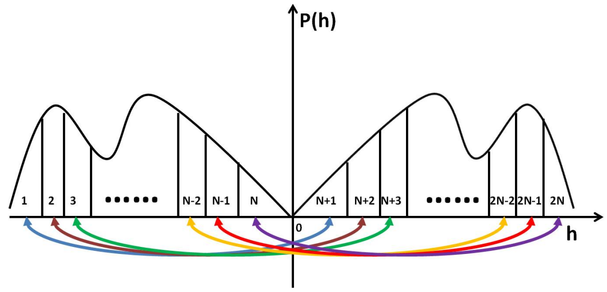


Fig. 3. The optimal matching-up pattern for symmetric phase distribution when $\theta = 0$: $\hat{h}_i$ is matched up with $\hat{h}_{N+i}$, $\forall i \in \{1, 2, ...N\}$. Here each quantized cell in this figure has equal probability and is represented by its centroid.

IA scheme.

Theorem 1: When the channel phase distribution is symmetric and P is large, the achievable rate for the optimal ergodic IA is

$$R_{\text{opt}} = \frac{1}{2}\log_2\frac{P}{2} + \int_0^1 \log_2(F^{-1}(x) + F^{-1}(1-x))dx \quad (11)$$

where $F(h)$ is the cumulative distribution function (cdf) of the channel amplitude h, and $F^{-1}(x)$ is its inverse function with values between $[H_{min}, H_{max}]$. H_{min} and H_{max} denote the minimal and maximal absolute value of channel amplitude respectively.

Remark 3: If the domain of the probability distribution function (pdf) of channel amplitude $p(h)$ is unbounded, i.e. $H_{max} \to \infty$, the second term in (11) is an improper integral. We can easily prove that if $E(h)$ is bounded, as it is invariably in practice, the second term is convergent. Therefore, for almost all the channel distributions of interest in the real world, (11) is convergent.

Proof of Theorem 1: In the proposed ergodic IA scheme, over the two complementary channel matrices, the cross channel coefficients have the same quantized values, and the quantized values of the direct channel coefficients are different. According to Lemma 1, for direct channels, when the optimal matching-up pattern is applied, we have

$$\begin{aligned}|h_{rr}(l_1) - h_{rr}(l_2)| &\geq max\{0, |\hat{h}_{rr}(l_1) - \hat{h}_{rr}(l_2)| - 2\sigma\} \\ &= |\hat{h}_{N+i}| + |\hat{h}_i| - 2\sigma\end{aligned} \quad (12)$$

For cross channels we have

$$|h_{rt}(l_1) - h_{rt}(l_2)| \leq |\hat{h}_{rt}(l_1) - \hat{h}_{rt}(l_2)| + 2\sigma = 2\sigma \quad \forall t \neq r \quad (13)$$

so the total interference power is at most $4\sigma^2(K-1)P$.

Therefore, the SINR at receiver r is at least

$$\text{SINR}_r \geq \frac{(|\hat{h}_{N+i}| + |\hat{h}_i| - 2\sigma)^2 P}{2 + 4\sigma^2(K-1)P} \quad (14)$$

Then we intend to set up the relationship between the channel quantized value $\hat{h}_i$ and the cdf of channel amplitude $F(h)$. we assume that the channel amplitude h is divide into N segments, and $\tilde{h}_i$ is the centroid of segment $i \in \{1, 2, ..., N\}$, which is utilized to represent the quantized segment. We define $\triangle = \frac{1}{N}$. The relationship between quantized channel amplitude $\tilde{h}_i$ and the cdf $F(h)$ can be expressed as

$$F(\tilde{h}_i) = \frac{2i-1}{2N} = \frac{2i-1}{2}\triangle$$
$$\tilde{h}_i = F^{-1}(\frac{2i-1}{2}\triangle) \quad (15)$$

Due to phase symmetry, we have the following equations for $\forall i \in \{1, 2, ..., N\}$ according to the channel quantization,

$$|\hat{h}_i| = \tilde{h}_{N-i+1}$$
$$|\hat{h}_{N+i}| = \tilde{h}_i \quad (16)$$

Then the achievable rate of optimal ergodic IA can be written as (17) on the top of next page.

When $N \to \infty$ and $M \to \infty$, $\sigma \to 0$, and finally we can obtain

$$R_{\text{opt}} = \frac{1}{2}\log_2\frac{P}{2} + \int_0^1 \log_2(F^{-1}(x) + F^{-1}(1-x))dx \quad (18)$$

□

Remark 4: Compared with the original ergodic IA, the optimal scheme can potentially achieve an unbounded SNR offset improvement. An example (*Example 3*) is given in Section VI.

Then according to Theorem 1, we can obtain the achievable rate for the optimal ergodic IA scheme when both the channel phase and amplitude distributions are symmetric. Here, symmetric amplitude means for any channel coefficient $he^{j\theta}$, there exists a point $h_0e^{j\theta}$, satisfying $h_0 = \frac{1}{2}(H_{min} + H_{max})$ and $p((h_0 - i)e^{j\theta}) = p((h_0 + i)e^{j\theta})$ for any $i \leq h_0 - H_{min}$. A typical example of symmetric phase and amplitude distribution is the uniform distribution $[-H_{max}, -H_{min}] \cup [H_{min}, H_{max}]$.

Corollary 1: For channels with symmetric amplitude and phase distributions, when P is large, the achievable rate for the optimal ergodic IA is

$$R_{\text{opt}} = \frac{1}{2}\log_2(2h_0^2P) \quad (19)$$

Proof: This corollary can be obtained straightforwardly from Theorem 1. When both the channel amplitude and phase are symmetric, for any $x \in [0, 1]$, $\log_2(F^{-1}(x) + F^{-1}(1-x)) = \log_2(H_{max} + H_{min}) = \log_2(2h_0)$, so the optimal rate in this case is

$$R_{\text{opt}} = \frac{1}{2}\log_2\frac{P}{2} + \int_0^1 \log_2(2h_0)dx = \frac{1}{2}\log_2(2h_0^2P) \quad (20)$$

□

Remark 5: Interestingly, it is worthwhile noticing that when the phase and amplitude are both symmetric, the achievable rate for the optimal ergodic IA scheme is only related with the value of h_0, and independent of the specific channel amplitude probability distribution function $p(h)$.

VI. Examples and Discussions

Example 1 (Uniform Distribution): The achievable rates of both optimal and original ergodic IA schemes for uniform amplitude distribution $[-10000 \ \ -10] \cup [10 \ \ 10000]$ are shown in Fig. 4. In this figure, the upper bound derived in [1] is based on the MAC bound.

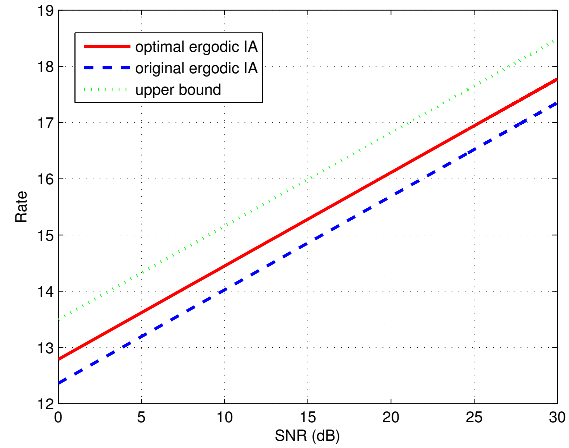


Fig. 4. Achievable rates per user for symmetric phase and uniform amplitude distribution ($H_{max} = 10000, H_{min} = 10$)

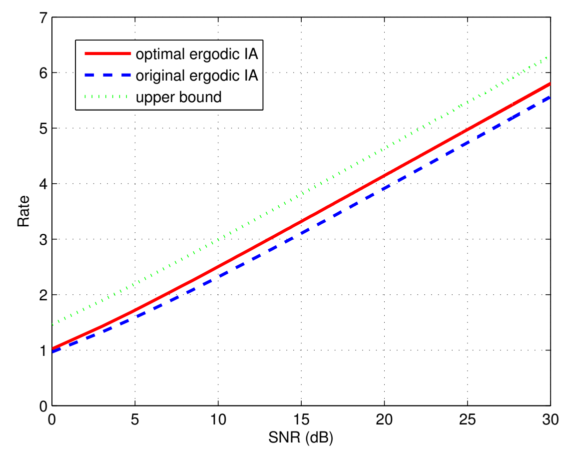


Fig. 5. Achievable rates per user for Rayleigh distribution

Remark 6: Furthermore, when the phase distribution is symmetric and the amplitude distribution is uniform, we find that the SNR offset between the optimal and original ergodic IA is only dependent on the ratio of H_{max} and H_{min}. With H_{max}/H_{min} approaching infinity, the SNR offset between these two schemes can be up to about 2.67 dB. The proof in given in the Appendix B.

Example 2 (Rayleigh Distribution): When the channel distribution is Rayleigh, the achievable rates for different schemes are shown in Fig. 5, from which we can find that when SNR is relatively large, the perform gain of the optimal scheme is more than 1 dB compared with its original counterpart.

Example 3 (Unbounded SNR offset over Original Ergodic IA): We assume that the channel distribution is $p(1) = p(-1) = p(\tau) = p(-\tau) = 1/4$, where τ is an arbitrary large positive real number. The achievable rate of the original ergodic IA is

$$R_{\text{orig}} = \frac{1}{2}\log_2\frac{P}{2} + \frac{1}{2} + \frac{1}{2}\log_2 2\tau \quad (21)$$

$$
\begin{aligned}
R_{\text{opt}} =& \frac{1}{2}\log_2 \frac{P}{2+4\sigma^2(K-1)P} + \sum_{i=1}^{N} \frac{1}{2N}\log_2[(|\hat{h}_{N+i}| + |\hat{h}_i| - 2\sigma)^2] \\
=& \frac{1}{2}\log_2 \frac{P}{2+4\sigma^2(K-1)P} + \sum_{i=1}^{N} \frac{1}{N}\log_2(\tilde{h}_i + \tilde{h}_{N-i+1} - 2\sigma) \\
=& \frac{1}{2}\log_2 \frac{P}{2+4\sigma^2(K-1)P} + \sum_{i=1}^{N} \log_2[F^{-1}(i\triangle - \frac{1}{2}\triangle) + F^{-1}(N\triangle - i\triangle + \frac{1}{2}\triangle) - 2\sigma]\triangle
\end{aligned} \tag{17}
$$

and the achievable rate of the optimal ergodic IA is

$$
R_{\text{opt}} = \frac{1}{2}\log_2 \frac{P}{2} + \log_2(\tau+1) \tag{22}
$$

We assume the optimal and original ergodic IA achieve the same rate at SNR P_1 and P_2 respectively, then

$$
\frac{1}{2}\log_2 \frac{P_2}{P_1} = \log_2(\tau+1) - \frac{1}{2}\log_2 \tau - 1 \approx \frac{1}{2}\log_2 \tau - 1 \tag{23}
$$

As τ approaches infinity, P_2/P_1 also approaches infinity. Therefore, the SNR offset between the optimal and original ergodic IA schemes $10\log_{10}(P_2/P_1)$ is unbounded.

VII. Conclusion

In this paper, we develop a new ergodic IA scheme which can achieve one half interference-free DoF for each user when the channel phase distribution is either symmetric or asymmetric in K-user interference channel. Further we investigate the matching-up pattern of direct channel values in complementary states in order to achieve optimal ergodic IA, which can obtain the optimal SNR offset over the original scheme. Then the rate of optimal ergodic IA for symmetric channel phase distribution is developed. We also demonstrate that compared with the original ergodic IA scheme, the SNR offset improvement of the optimal scheme can be unbounded.

Appendix

A. Proof of Lemma 2

Proof: For any vector $X = (X_1, X_2, ..., X_N)$ of real components, we denote the vector with decreasing or increasing X_i as

$$
\begin{aligned}
\vec{X}_{\downarrow} &= (X_{[1]}, X_{[2]}, ..., X_{[N]}) \\
\vec{X}_{\uparrow} &= (X_{(1)}, X_{(2)}, ..., X_{(N)})
\end{aligned} \tag{24}
$$

where $X_{[1]} \geq X_{[2]} \geq ... \geq X_{[N]}$ and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(N)}$. We also let $\vec{X}$ represent the vector with any order of elements in X. Based on the majorization theory [3], we know that

$$
\vec{X}_{\downarrow} + \vec{Y}_{\uparrow} \prec \vec{X} + \vec{Y} \tag{25}
$$

where $X \prec Y$ implies that

$$
\begin{aligned}
&\sum_{n=1}^{i} X_{[n]} \leq \sum_{n=1}^{i} Y_{[n]} \quad i = 1, 2, ..., N-1 \\
&\sum_{n=1}^{N} X_{[n]} = \sum_{n=1}^{N} Y_{[n]}
\end{aligned} \tag{26}
$$

Then, according to majorization, we also have

$$
\vec{X} \prec \vec{Y} \Rightarrow \prod_i X_i \geq \prod_i Y_i \tag{27}
$$

Therefore, the proof is completed. □

B. Proof of Remark 6

Proof: We denote H_{max} and H_{min} as a and b for simplicity in the following proof. For symmetric channel phase and uniform amplitude distribution, we assume that when the optimal and original schemes achieve the same rate in high SNR regime, the required SNR for these two schemes are P_1 and P_2 respectively, then

$$
\begin{aligned}
\frac{1}{2}(\log_2 \frac{P_2}{P_1}) &= \log_2 \frac{a+b}{2} - E(\log_2 |h|) \\
&= \log_2 \frac{a+b}{2} - \int_b^a \frac{1}{a-b}\log_2 |h| dh \\
&= \log_2 \frac{a+b}{2} - \frac{1}{a-b}[h\log_2 |h| - \frac{h}{In2}]_b^a \\
&= \log_2(\frac{a+b}{2}\frac{b^{\frac{b}{a-b}}}{a^{\frac{a}{a-b}}}) + (In2)^{-1}
\end{aligned} \tag{28}
$$

we denote $x = \frac{a}{b}$. Through some manipulations, we get

$$
\frac{1}{2}(\log_2 \frac{P_2}{P_1}) = \log_2[\frac{x+1}{2}(\frac{1}{x})^{\frac{x}{x-1}}] + In(2)^{-1} \tag{29}
$$

As $x \rightarrow \infty$,

$$
\begin{aligned}
\lim_{x\rightarrow\infty} \frac{1}{2}(\log_2 \frac{P_1}{P_2}) &= \lim_{x\rightarrow\infty} \log_2[\frac{x+1}{2}(\frac{1}{x})^{\frac{x}{x-1}}] + In(2)^{-1} \\
&= (In2)^{-1} - 1 \\
&\approx 0.443
\end{aligned} \tag{30}
$$

Therefore, when $x \rightarrow \infty$, $\frac{P_2}{P_1} \rightarrow 2^{0.886}$, and the SNR offset between the optimal and original ergodic IA is then

$$
10\log_{10}(\frac{P_2}{P_1}) \approx 2.67 \tag{31}
$$

□

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