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Affine motion of 2d incompressible fluids and flows in $\mathrm{SL}(2,\mathbb{R})$

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Abstract

The affine motion of two-dimensional (2d) incompressible fluids can be reduced to a completely integrable and globally solvable Hamiltonian system of ordinary differential equations for the deformation gradient in $\mathrm{SL}(2,\mathbb{R})$. In the case of perfect fluids, the motion is given by geodesic flow in $\mathrm{SL}(2,\mathbb{R})$ with the Euclidean metric, while for magnetically conducting fluids (MHD), the motion is governed by a harmonic oscillator in $\mathrm{SL}(2,\mathbb{R})$. A complete description of the dynamics is given including rigid motions, rotating eddies with stable and unstable manifolds, and solutions with vanishing pressure. For perfect fluids, the displacement generically becomes unbounded, as $t \to \pm\infty$. For MHD, solutions are bounded and generically multiply periodic and recurrent.

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0. Introduction

In this article we explore the affine motion of incompressible planar fluids surrounded by vacuum. An affine motion is one whose deformation and velocity gradients depend only on time. Under this assumption, the fluid equations reduce to a completely integrable and globally solvable Hamiltonian system of ordinary differentiable equations for the deformation gradient in $\mathrm{SL}(2,\mathbb{R})$. The natural phase space is the 6-dimensional tangent bundle of $\mathrm{SL}(2,\mathbb{R})$, which we regard as being embedded in $\mathbb{R}^8$ with the Euclidean metric. There are three integrals of the motion corresponding to conservation of energy and invariance under the left and right action of $\mathrm{SO}(2,\mathbb{R})$. We shall provide a complete description of all such motions in terms of the values of the invariants in two cases: incompressible perfect fluids (Euler equation) and incompressible magnetically conducting fluids (MHD).

Taking the unit disk as the reference domain, the time-dependent fluid domains arising from incompressible affine motion are ellipses of constant area. The principle axes of these fluid ellipses are determined by the eigendirections and eigenvalues of the stretch tensor. Incompressible affine motion allows for compression along one axis and expansion along the other, combined with rigid rotation.

Once the deformation gradient and fluid domains are known, the pressure, velocity, and magnetic field, if present, are recovered through explicit formulae, taking into account the boundary conditions. The sign of the pressure is preserved by the motion. There exist special solutions whose pressure vanishes identically. In this case, the equations of motion are linear, and solutions may be found explicitly.

The affine motion of incompressible perfect fluids in the plane is described by geodesic flow for the deformation gradient in $\mathrm{SL}(2,\mathbb{R})$, echoing the classic result of Arnold on geodesic flow in the space of volume preserving diffeomorphims [1]. By energy conservation, the material velocity is bounded. However generically, solutions are unbounded, and in this case, the diameter of the fluid domains grow linearly in time while approaching maximal eccentricity. Additionally, there is an invariant manifold of initial data leading to rigidly rotating fluid disks (eddies) of arbitrary angular velocity. These solutions are represented by curves in $\mathrm{SO}(2,\mathbb{R})$. The manifold of rotational solutions is hyperbolic, possessing both stable and unstable invariant manifolds. Solutions on these manifolds are semi-bounded, and they decay exponentially to a rotating disk, as $t \to \infty$ in the stable case and as $t \to -\infty$ in the unstable case. Unbounded solutions are asymptotic to straight lines in the space of 2×2 matrices. In the special case of vanishing pressure, solutions coincide with straight lines in $\mathrm{SL}(2,\mathbb{R})$.

The affine motion of incompressible magnetically conducting planar fluids can be viewed as a simple harmonic oscillator, constrained to $\mathrm{SL}(2,\mathbb{R})$, through the addition of a one-parameter restoring force to the equations of geodesic motion. Here, all solutions are bounded and, generically, multiply periodic and recurrent. There exist rigid solutions with fluid ellipses of arbitrary eccentricity. Included among these are rotating disks of arbitrary angular velocity. For sufficiently large angular velocities, the rotating disk solutions possess a stable/unstable manifold. Solutions on this manifold are homoclinic to a rotational solution, with an exponential decay rate, as $t \to \pm\infty$.

The main results for MHD and perfect fluids are given in Sections 12 and 13, respectively.

Up to that point, the exposition for the two cases is presented in parallel. The initial sections are devoted to the algebraic and geometric properties of the phase space for the system of ordinary equations describing the motion of the deformation gradient. In Section 3, these equations are derived from the fundamental fluid equations, and the global existence theorem is presented. The integral invariants of this system are discussed in the following section. We then devote considerable time discussing the sets in phase space with fixed values of the invariants, and Sections 5 through 8 are thus independent of the dynamics. The point of this discussion becomes apparent in Section 9. The equations of motion can be projected onto the phase plane, and the essential features of the dynamics can be extracted from the orbits of the projected system. Perhaps the most technical portion of the paper involves the reconstruction, in Section 11, of the general flow from the solution in the phase plane. Here we see the decisive influence of the rotation group $\mathrm{SO}(2,\mathbb{R})$ which induces a monodromy for solution trajectories which pass through it. Notation will be introduced as needed throughout the paper. For the convenience of the reader, a glossary appears in the final section.

For initial data satisfying the Rayleigh-Taylor sign condition, local well-posedness for the incompressible free boundary Euler equations with bulk vorticity was established in [2], [6], [3], [7], [11], [4] and for the incompressible free boundary MHD problem in [5], [10]. The use of affine deformations is a well-established tool in continuum mechanics, first introduced in the context of the vacuum free boundary incompressible Euler system in [8], [9].

1. Matrix inner product space and groups

Definition 1.1. By $\mathbb{M}^2$, we denote the set of 2×2 matrices over $\mathbb{R}$ with the Euclidean inner product

$$\langle A, B\rangle = \sum_{i,j} A_{ij}B_{ij} = \operatorname{tr} A^\top B$$

and norm

$$|A|^2 = \langle A, A\rangle \,.$$

Lemma 1.2. *For all $A, B, C \in \mathbb{M}^2$,*

$$\langle AB, C\rangle = \left\langle B, A^\top C\right\rangle = \left\langle A, CB^\top\right\rangle .$$

Definition 1.3. Define the following vectors in $\mathbb{M}^2$:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Lemma 1.4. *The vectors I, K, M, Z are an orthogonal basis for $\mathbb{M}^2$. In particular, the set of symmetric matrices is orthogonal to the set of anti-symmetric matrices.*

Definition 1.5. The special linear group is given by

$$\mathrm{SL}(2,\mathbb{R}) = \{A \in \mathbb{M}^2 : \det A = 1\}$$

and the special orthogonal group is

$$\mathrm{SO}(2,\mathbb{R}) = \{U \in \mathrm{SL}(2,\mathbb{R}) : U^{-1} = U^\top\}.$$

Lemma 1.6. *For all $A \in \mathbb{M}^2$ and $U, V \in \mathrm{SO}(2,\mathbb{R})$,*

$$|UAV| = |A|.$$

The left and right action of $\mathrm{SO}(2,\mathbb{R})$ on $\mathbb{M}^2$ and on $\mathrm{SL}(2,\mathbb{R})$ is an isometry.

The subgroup $\mathrm{SO}(2,\mathbb{R})$ will play a special role in the sequel. Here is the first of several characterizations that we shall repeatedly use.

Lemma 1.7. *Elements of $\mathrm{SO}(2,\mathbb{R})$ are norm minimizers in $\mathrm{SL}(2,\mathbb{R})$. There holds*

$$\min\{|A|^2 : A \in \mathrm{SL}(2,\mathbb{R})\} = 2$$

and

$$\mathrm{SO}(2,\mathbb{R}) = \{A \in \mathrm{SL}(2,\mathbb{R}) : |A|^2 = 2\}.$$

Proof. Given $A \in \mathrm{SL}(2,\mathbb{R})$, we have that

$$\det A^\top A = 1,$$

and so the eigenvalues of the positive definite symmetric matrix $A^\top A$ satisfy

$$\lambda_1 \geq \lambda_2 > 0 \quad \text{and} \quad \lambda_1\lambda_2 = 1.$$

Therefore,

$$|A|^2 = \operatorname{tr} A^\top A = \lambda_1 + \lambda_2 \geq 2,$$

with equality if and only if $A^\top A = I$. Using the polar decomposition, there exists $U \in \mathrm{SO}(2,\mathbb{R})$ such that

$$A = U(A^\top A)^{1/2}.$$

So by Lemma 1.6, $|A|^2 = 2$ if and only if $A = U$. $\square$

Definition 1.8. Define the one-parameter family of rotations

$$U(\sigma) = \exp(\sigma Z) = \begin{bmatrix} \cos\sigma & -\sin\sigma \\ \sin\sigma & \cos\sigma \end{bmatrix}, \quad \sigma \in \mathbb{R}.$$

Lemma 1.9. *With the notation of Definition 1.8, we have*

$$\mathrm{SO}(2,\mathbb{R}) = \{U(\sigma) : \sigma \in \mathbb{R}\}.$$

Lemma 1.10.

$$\mathrm{SO}(2,\mathbb{R}) = \{A \in \mathrm{SL}(2,\mathbb{R}) : [A, Z] = 0\}.$$

Proof. A matrix in $\mathrm{SL}(2,\mathbb{R})$ commutes with Z if and only if it has the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad \text{with} \quad a^2+b^2=1.$$

□

Lemma 1.11. *The cofactor map* $\mathrm{cof} : \mathbb{M}^2 \to \mathbb{M}^2$ *satisfies*

$$\mathrm{cof}\, A = ZAZ^\top.$$

It is symmetric and unitary.

Proof. The identity is easily verified. By Lemma 1.2, we have that

$$\langle \mathrm{cof}\, A, B\rangle = \left\langle ZAZ^\top, B\right\rangle = \left\langle A, Z^\top BZ\right\rangle = \left\langle A, ZBZ^\top\right\rangle = \langle A, \mathrm{cof}\, B\rangle .$$

and

$$\langle \mathrm{cof}\, A, \mathrm{cof}\, B\rangle = \langle A, \mathrm{cof}\,\mathrm{cof}\, B\rangle = \langle A, B\rangle .$$

□

Lemma 1.12. *For any* $A \in \mathbb{M}^2$*, the vectors* $\{AZ, ZA\}$ *and the vectors* $\{A, \mathrm{cof}\, A\}$ *are orthogonal.*

Proof. By Lemmas 1.2 and 1.4, we have

$$\langle A, ZA\rangle = \left\langle AA^\top, Z\right\rangle = 0 \quad \text{and} \quad \langle A, AZ\rangle = \left\langle A^\top A, Z\right\rangle = 0,$$

for any $A \in \mathbb{M}^2$. By Lemmas 1.2 and 1.11 together with what we have just proven, we have

$$\langle \mathrm{cof}\, A, ZA\rangle = -\langle Z\,\mathrm{cof}\, A, A\rangle = -\langle AZ, A\rangle = 0$$

and

$$\langle \mathrm{cof}\, A, AZ\rangle = -\langle \mathrm{cof}\, AZ, A\rangle = -\langle ZA, A\rangle = 0.$$

□

Lemma 1.13. *For any* $A \in \mathbb{M}^2$*, there holds*

$$A\,\mathrm{cof}\, A^\top = A^\top\,\mathrm{cof}\, A = \det A\, I.$$

In particular, we have

$$A^{-\top} = \mathrm{cof}\, A, \quad \textit{for all} \quad A \in \mathrm{SL}(2,\mathbb{R}),$$

and

$$\mathrm{SO}(2,\mathbb{R}) = \{A \in \mathrm{SL}(2,\mathbb{R}) : A = \mathrm{cof}\, A\}.$$

Remark 1.14. We use the notation $A^{-\top} = (A^{-1})^\top$.

Lemma 1.15. *For any* $A \in \mathbb{M}^2$*, we have*

$$2\det A = \langle A, \mathrm{cof}\, A\rangle .$$

Lemma 1.16. *The determinant map* $\det : \mathbb{M} \to \mathbb{R}$ *is* C^∞ *and*

$$\frac{\partial}{\partial A}\det A = \mathrm{cof}\, A.$$

2. The geometry of $\mathrm{SL}(2,\mathbb{R})$

Lemma 2.1. *The vector* $\operatorname{cof} A$ *is normal to* $\mathrm{SL}(2,\mathbb{R})$ *at a point* $A \in \mathrm{SL}(2,\mathbb{R})$*, and the tangent space to* $\mathrm{SL}(2,\mathbb{R})$ *at* A *is*

$$T_A\mathrm{SL}(2,\mathbb{R}) = \{B \in \mathbb{M}^2 : \langle B, \operatorname{cof} A\rangle = \operatorname{tr} BA^{-1} = 0\}.$$

$T_A\mathrm{SL}(2,\mathbb{R})$ *is a three dimensional Euclidean inner product space in* $\mathbb{M}^2$.

Proof. This follows from Lemma 1.16. □

Definition 2.2. Define the special linear Lie algebra

$$\mathfrak{sl}(2,\mathbb{R}) = T_I\mathrm{SL}(2,\mathbb{R}) = \{L \in \mathbb{M}^2 : \operatorname{tr} L = 0\} = \operatorname{span}\{K, M, Z\}.$$

Definition 2.3. Given $A \in \mathrm{SL}(2,\mathbb{R})$, we define the unit normal vector field

$$N(A) = |A|^{-1} \operatorname{cof} A = |A|^{-1} A^{-\top}.$$

Lemma 2.4. $A \in \mathrm{SL}(2,\mathbb{R})$ *is normal to* $T_A\mathrm{SL}(2,\mathbb{R})$ *if and only if* $A \in \mathrm{SO}(2,\mathbb{R})$.

Proof. This follows from Lemmas 1.13 and 2.1. □

Definition 2.5. Define the tangent bundle

$$\mathcal{D} = \{(A,B) \in \mathbb{M}^2 \times \mathbb{M}^2 : A \in \mathrm{SL}(2,\mathbb{R}),\ B \in T_A\mathrm{SL}(2,\mathbb{R})\}.$$

Lemma 2.6. $\mathcal{D}$ *is a smooth 6-dimensional embedded submanifold of* $\mathbb{M}^2 \times \mathbb{M}^2$.

Definition 2.7. Given $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$ and Z as in Definition 1.3, define

$$\begin{aligned}
\tau_1(A) &= ZA + AZ,\\
\tau_2(A) &= ZA - AZ,\\
\tau_3(A) &= \frac{|A|^2}{|A|^4 - 4}\left(A - \frac{2}{|A|^2}\operatorname{cof} A\right).
\end{aligned}$$

We also define $\hat{\tau}_i(A) = \tau_i(A)/|\tau_i(A)|$.

Remark 2.8. The choice of normalization for $\tau_3(A)$ is motivated by Lemma 2.14 below.

Remark 2.9. In situations when the base point $A \in \mathrm{SL}(2,\mathbb{R})$ is fixed, we shall occasionally find it convenient to write simply τ_i instead of $\tau_i(A)$.

Lemma 2.10. *The functions* $\tau_i(A)$ *are smooth tangent vector fields, i.e.*

$$\tau_i : \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}) \to T_A\mathrm{SL}(2,\mathbb{R}), \quad i = 1,2,3.$$

Lemma 2.11. *If $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then $g_{ij}(A) = \langle \tau_i(A), \tau_j(A) \rangle$ defines the metric on $T_A\mathrm{SL}(2,\mathbb{R})$ in local coordinates relative to the basis $\{\tau_i(A)\}$. Explicitly, $g(A)$ is given by*

$$g(A) = \operatorname{diag}\left[2|A|^2 + 4, \quad 2|A|^2 - 4, \quad \frac{|A|^2}{|A|^4 - 4}\right].$$

We also have that

$$\langle A, \tau_i(A) \rangle = 0, \ i = 1, 2 \quad \text{and} \quad \langle A, \tau_3(A) \rangle = 1.$$

Lemma 2.12. *If $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then the set $\{\tau_i(A)\}_{i=1}^3$ spans $T_A\mathrm{SL}(2,\mathbb{R})$, and for any $B \in T_A\mathrm{SL}(2,\mathbb{R})$, we have*

$$B = \sum_i c_i \, \tau_i(A), \qquad \text{with} \qquad c_i = \langle B, \tau_i(A) \rangle / g_{ii}(A).$$

Corollary 2.13. *If $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then the set $\{\hat{\tau}_i(A)\}_{i=1}^3$ is an orthonormal frame in $T_A\mathrm{SL}(2,\mathbb{R})$.*

The next lemma gives a convenient set of local coordinates for $\mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$.

Lemma 2.14. *Define a mapping $\mathcal{A} : \mathbb{R}^2 \times [1, \infty) \to \mathbb{M}^2$ by*

$$\mathcal{A}(s) = U(s_1 + s_2) \, H(s_3) \, U(s_1 - s_2), \quad s = (s_1, s_2, s_3) \in \mathbb{R}^2 \times [1, \infty)$$

with $U(\sigma)$ as in Definition 1.8 and

$$H(\sigma) = \frac{1}{\sqrt{2}} \begin{bmatrix} (\sigma + 1)^{1/2} & (\sigma - 1)^{1/2} \\ (\sigma - 1)^{1/2} & (\sigma + 1)^{1/2} \end{bmatrix}.$$

Then

i. the range of $\mathcal{A}$ is equal to $\mathrm{SL}(2,\mathbb{R})$,

ii. $\frac{1}{2}|\mathcal{A}(s)|^2 = s_3$,

iii. $\mathcal{A}(s) \in \mathrm{SO}(2,\mathbb{R})$ if and only if $s_3 = 1$, and

iv. the restriction
$$\mathcal{A} : \mathbb{R}^2 \times (1, \infty) \to \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$$
is a local diffeomorphism with
$$\partial_i \, \mathcal{A}(s) = \tau_i(\mathcal{A}(s)), \quad i = 1, 2, 3.$$

Proof. Since $\det \mathcal{A}(s) = \det H(s_3) = 1$, we see that $\mathcal{A}(s) \in \mathrm{SL}(2,\mathbb{R})$, for every $s \in \mathbb{R}^2 \times [1, \infty)$. Moreover, by Lemma 1.6, $|\mathcal{A}(s)|^2 = |H(s_3)|^2 = 2s_3$, so $\mathcal{A}(s) \in \mathrm{SO}(2,\mathbb{R})$ if and only if $s_3 = 1$, by Lemma 1.7.

Let $A \in \mathrm{SL}(2,\mathbb{R})$. Using the polar decomposition, we can find a $U \in \mathrm{SO}(2,\mathbb{R})$ such that $A = U\sqrt{A^\top A}$. Since $\sqrt{A^\top A}$ is a symmetric matrix in $\mathrm{SL}(2,\mathbb{R})$, there exists $V \in \mathrm{SO}(2,\mathbb{R})$ such that

$$V(\sqrt{A^\top A})V^\top = \operatorname{diag}\left[\alpha, \quad 1/\alpha\right] = D, \quad \text{with} \quad \alpha \geq 1.$$

Finally, taking

$$W = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

we have $W \in \mathrm{SO}(2,\mathbb{R})$ and $WDW^\top = H(\sigma)$, for $\sigma = (\alpha^2 + \alpha^{-2})/2 \in [1,\infty)$. Thus, we see that

$$A = (UVW)H(\sigma)(W^\top V^\top),$$

with $UVW,\ W^\top V^\top \in \mathrm{SO}(2,\mathbb{R})$. By Lemma 1.9, this shows that the mapping

$$\mathcal{A} : \mathbb{R}^2 \times [1,\infty) \to \mathrm{SL}(2,\mathbb{R})$$

is surjective.

We next verify the formulas for the derivatives. Since

$$U'(\sigma) = ZU(\sigma) = U(\sigma)Z,$$

we find that

$$\partial_i \mathcal{A}(s) = \tau_i(\mathcal{A}(s)), \quad i = 1, 2.$$

A simple calculation yields

$$H'(\sigma) = \tau_3(H(\sigma)).$$

Therefore, by Lemmas 1.6, 1.10 and 1.11, we have that

$$\partial_3 \mathcal{A}(s) = U(s_1 + s_2)\tau_3(H(s_3))U(s_1 - s_2) = \tau_3(\mathcal{A}(s)).$$

Finally, by Lemma 2.11, $\{\tau_i(A)\}_{i=1}^3$ is a frame in $T_A\mathrm{SL}(2,\mathbb{R})$, if $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$. Thus, we see that the mapping

$$\mathcal{A} : \mathbb{R}^2 \times (1,\infty) \to \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$$

is locally invertible. □

Remark 2.15. The matrix $H(\sigma)$ is a hyperbolic rotation. It will emerge as the main term in the description of the motion in local coordinates.

Corollary 2.16. *In local coordinates, $\frac{1}{2}|\mathcal{A}(s)|^2 = s_3$, and hence, the metric $g(\mathcal{A}(s))$ is a function only of s_3. It has the form*

$$g(\mathcal{A}(s)) = \operatorname{diag}\left[4(s_3+1), \quad 4(s_3-1), \quad \frac{s_3}{2(s_3^2-1)}\right].$$

Remark 2.17. With abuse of notation, we shall sometimes write $g(s_3)$ instead of $g(\mathcal{A}(s))$.

Lemma 2.18. *For $s \in \mathbb{R}^2 \times [1,\infty)$, the coordinate map can also be expressed as*

$$\mathcal{A}(s) = \left(\frac{s_3+1}{2}\right)^{1/2} U(2s_1) + \left(\frac{s_3-1}{2}\right)^{1/2} U(2s_2)\, M,$$

and the normalized tangent vector fields have the form

$$\begin{aligned}
\hat{\tau}_1(\mathcal{A}(s)) &= \tfrac{1}{\sqrt{2}} U(2s_1) Z,\\
\hat{\tau}_2(\mathcal{A}(s)) &= -\tfrac{1}{\sqrt{2}} U(2s_2) K,\\
\hat{\tau}_3(\mathcal{A}(s)) &= \frac{1}{2}\left[\left(\frac{s_3-1}{s_3}\right)^{1/2} U(2s_1) + \left(\frac{s_3+1}{s_3}\right)^{1/2} U(2s_2) M\right].
\end{aligned}$$

Proof. The first statement follows from Lemma 2.14 by writing

$$H(\sigma) = \left(\frac{\sigma+1}{2}\right)^{1/2} I + \left(\frac{\sigma-1}{2}\right)^{1/2} M,$$

and then using the fact that

$$M\, U(\theta) = U(-\theta)\, M.$$

Differentiating the new expression for $\mathcal{A}(s)$ with respect to s yields alternate expressions for $\tau_i(\mathcal{A}(s))$. The formulas for $\hat{\tau}_i(\mathcal{A}(s))$ follow after normalization. □

Remark 2.19. The rotation group $\mathrm{SO}(2,\mathbb{R})$ can be identified with the circle S^1, and the spheres $|\mathcal{A}(s)|^2 = s_3$, with $s_3 > 1$, can be identified with the 2-torus $S^1 \times S^1$.

Remark 2.20. Note that Lemma 2.18 provides an extension of the normalized tangent vectors $\hat{\tau}_i(A)$ to $T_A\mathrm{SL}(2,\mathbb{R})$ for $A \in \mathrm{SO}(2,\mathbb{R})$.

Lemma 2.21. *In the local coordinates of Lemma 2.14, the Christoffel symbols depend only on s_3, and they have the form*

$$\Gamma^i_{jk}(s_3) = \tfrac{1}{2} g^{ii}(s_3)[\delta_{j3} g'_{ki}(s_3) + \delta_{k3} g'_{ij}(s_3) - \delta_{i3} g'_{jk}(s_3)], \quad s_3 > 1,$$

where $g'(s_3)$ indicates the derivative in s_3.

Proof. Since the metric is diagonal and depends only on s_3, the result follows from the general formula

$$\begin{aligned}
\Gamma^i_{jk}(s_3) &= \tfrac{1}{2} \sum_{\ell} g^{i\ell}(s_3)[\partial_j g_{k\ell}(s_3) + \partial_k g_{\ell j}(s_3) - \partial_\ell g_{jk}(s_3)]\\
&= \tfrac{1}{2} g^{ii}(s_3)[\delta_{j3} g'_{ki}(s_3) + \delta_{k3} g'_{ij}(s_3) - \delta_{i3} g'_{jk}(s_3)].
\end{aligned}$$

□

Lemma 2.22. *The cofactor map acts on the tangent basis as follows:*

$$\begin{aligned}
\operatorname{cof} \tau_1(A) &= \tau_1(A),\\
\operatorname{cof} \tau_2(A) &= -\tau_2(A),\\
\operatorname{cof} \tau_3(A) &= -\frac{2}{|A|^2}\tau_3(A) + \frac{1}{|A|} N(A).
\end{aligned}$$

Lemma 2.23. *The orthogonal projection of* $\mathbb{M}^2$ *onto* $T_A\mathrm{SL}(2,\mathbb{R})$ *is given by*

$$P(A) = I - N(A) \otimes N(A).$$

Definition 2.24. Given $(A, B) \in \mathcal{D}$, we define the shape operator

$$S(A)B = -dN(A)B = -\sum_{a,b} B_{ab} \frac{\partial}{\partial A_{ab}} N(A).$$

Lemma 2.25. *The shape operator may be expressed in the form*

$$S(A)B = -\frac{1}{|A|} P(A) \operatorname{cof} B.$$

Moreover, for each $A \in \mathrm{SL}(2,\mathbb{R})$*, the shape operator is symmetric on* $T_A\mathrm{SL}(2,\mathbb{R})$.

Proof. By direct computation and Lemma 1.11, we have

$$\begin{aligned}
S(A)B &= -\sum_{ab} B_{ab} \frac{\partial}{\partial A_{ab}} \frac{\operatorname{cof} A}{|A|} \\
&= -\frac{\operatorname{cof} B}{|A|} + \frac{\operatorname{cof} A}{|A|^3} \langle A, B \rangle \\
&= -\frac{1}{|A|} \left(\operatorname{cof} B - \left\langle \frac{A}{|A|}, B \right\rangle \frac{\operatorname{cof} A}{|A|} \right) \\
&= -\frac{1}{|A|} \left(\operatorname{cof} B - \langle N(A), \operatorname{cof} B \rangle N(A) \right) \\
&= -\frac{1}{|A|} P(A) \operatorname{cof} B.
\end{aligned}$$

From this formula, we see that $S(A)$ maps into $T_A\mathrm{SL}(2,\mathbb{R})$, and by Lemma 1.11, the verification of symmetry is immediate. □

Lemma 2.26. *If* $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$*, then the vectors* $\{\tau_i(A)\}$ *are principal directions in* $T_A\mathrm{SL}(2,\mathbb{R})$ *with corresponding principal curvatures*

$$-\frac{1}{|A|}, \quad \frac{1}{|A|}, \quad \frac{2}{|A|^3}.$$

Proof. The principal curvatures and directions are the eigenvalues and eigenvectors of the shape operator. So this is an immediate consequence of Lemmas 2.22 and 2.25. □

Definition 2.27. The second fundamental form

$$\Pi(A) : T_A\mathrm{SL}(2,\mathbb{R}) \times T_A\mathrm{SL}(2,\mathbb{R}) \to \mathbb{R}$$

is defined by

$$\Pi(A)[B_1, B_2] = \langle S(A)B_1, B_2 \rangle.$$

Lemma 2.28. *For vector fields $V(A)$, $W(A)$, the Riemannian connection ∇ is given by*

$$\nabla_{V(A)}W(A) = P(A)\sum_{a,b} V_{ab}(A)\frac{\partial}{\partial A_{ab}}W(A).$$

Lemma 2.29. *For vector fields $V(A)$, $W(A)$, $Y(A)$, the curvature tensor is the map given by*

$$\begin{aligned} Y(A) &\mapsto R[V(A),W(A)]Y(A) \\ &= \Pi(A)[W(A),Y(A)]S(A)V(A) - \Pi(A)[V(A),Y(A)]S(A)W(A). \end{aligned}$$

Corollary 2.30. *Relative to the orthonormal basis $\{\hat{\tau}_i(A)\}$, the curvature tensor has the coordinates*

$$\langle R(\hat{\tau}_i,\hat{\tau}_j)\hat{\tau}_k,\hat{\tau}_\ell\rangle = R_{ijk\ell} = \lambda_i\lambda_j(\delta_{jk}\delta_{i\ell} - \delta_{ik}\delta_{j\ell}),$$

where $\{\lambda_i\}$ are the principal curvatures.

3. The equations of affine motion

Definition 3.1. An incompressible affine motion defined on the unit ball $\mathcal{B} \subset \mathbb{R}^2$ is a one-parameter family of volume preserving diffeomorphisms of the form

$$x(t,y) = A(t)y, \quad y \in \mathcal{B},\ t \in \mathbb{R},$$

with

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2).$$

Here, $\mathcal{B}$ is the reference domain, and the domain occupied by the material (fluid) at time t is

$$\Omega(t) = A(t)\mathcal{B} = \{x \in \mathbb{R}^2 : |A(t)^{-1}x|^2 \le 1\}.$$

Note that the fluid domain $\Omega(t)$ is an ellipse centered at the origin with principal axes determined by the eigendirections and eigenvalues of the positive definite symmetric (stretch) matrix $(A(t)A(t)^\top)^{1/2}$.

Remark 3.2. By Lemma 2.14, $(A(t)A(t)^\top)^{1/2}$ is similar to $H(\sigma)$ where $\sigma = \frac{1}{2}|A(t)|^2$, and so the eigenvalues are given by $\left(\frac{\sigma+1}{2}\right)^{1/2} \pm \left(\frac{\sigma-1}{2}\right)^{1/2}$.

The spatial velocity field associated to an affine motion is defined by

$$u(t,x(t,y)) = \partial_t x(t,y) = A'(t)y, \quad y \in \mathcal{B},$$

or equivalently

$$u(t,x) = A'(t)A(t)^{-1}x, \quad x \in \Omega(t).$$

Lemma 3.3. *If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^1(\mathbb{R}, \mathbb{M}^2)$, then $(A, A') \in C^0(\mathbb{R}, \mathcal{D})$. In particular, $A'A^{-1} \in C^0(\mathbb{R}, \mathfrak{sl}(2,\mathbb{R}))$.*

This leads to the following definition.

Definition 3.4. Define the mapping $L : \mathcal{D} \to \mathfrak{sl}(2, \mathbb{R})$ by

$$L(A, B) = BA^{-1}.$$

Remark 3.5. The spatial velocity gradient of an affine motion $x(t, y) = A(t)y$ is given by $\nabla u(t, x) = L(A(t), A'(t))$.

Lemma 3.3 suggests that the tangent bundle $\mathcal{D}$ is the natural phase space for affine incompressible motion.

The equations of motion for 2d incompressible MHD surrounded by vacuum are

$$\begin{aligned}
&D_t u = -\nabla p + b \cdot \nabla b \\
&D_t b = b \cdot \nabla u \\
&\nabla \cdot u = \nabla \cdot b = 0,
\end{aligned}$$

in a space-time domain $x \in \Omega(t)$, $t \in \mathbb{R}$. Here, $D_t = \partial_t + u \cdot \nabla$ is the material time derivative. The equations are supplemented by the boundary conditions

$$p = 0 \quad \text{and} \quad b \cdot n = 0 \quad \text{on} \quad \partial\Omega(t),$$

where dot " $\cdot$ " denotes the inner product on $\mathbb{R}^2$. The free boundary is also assumed to move with the fluid. We do not impose the Taylor sign condition, although that plays a role in the general theory of local well-posedness.

Remark 3.6. When the magnetic field b vanishes identically, the system reduces to the incompressible Euler equations.

Let us now assume that the velocity $u(t, x)$ and the fluid domains $\Omega(t)$ arise from an incompressible affine motion $x(t, y) = A(t)y$, as described above.

By Lemma 3.3, the velocity field is divergence free:

$$\nabla \cdot u(t, x) = \operatorname{tr} A'(t)A(t)^{-1} = 0, \quad t \in \mathbb{R}.$$

Let us write

$$b(t, x) = \beta(t)A(t)^{-1}x \quad \text{and} \quad -\nabla p(t, x) = A(t)^{-\top}\varpi(t)A(t)^{-1}x,$$

with $\beta, \varpi \in C^2(\mathbb{R}, \mathbb{M}^2)$. (As motivation, note that if we assume the other unknowns b and p are also spatially homogeneous, then the PDEs imply that $\nabla p(t, x)$ and $b(t, x)$ should be homogeneous of degree one in the variable x.)

The equation for the magnetic field implies that

$$\beta' = A'A^{-1}\beta$$

from which it follows that

$$\beta(t) = A(t)A_0^{-1}\beta_0, \quad \text{where} \quad A_0 = A(0), \quad \beta_0 = \beta(0).$$

The normal vector to $\partial\Omega(t)$ at a point $x \in \partial\Omega(t)$ has the direction of the vector $A(t)^{-\top}A(t)^{-1}x$. So the boundary condition implies that for all $|y| = 1$,

$$0 = b(t, x(t, y)) \cdot n(t, x(t, y)) = \beta(t)y \cdot A(t)^{-\top}y = A_0^{-1}\beta_0 y \cdot y.$$

It follows that $A_0^{-1}\beta_0$ is anti-symmetric, and so there exists a constant c_0 such that

$$A_0^{-1}\beta_0 = c_0 Z.$$

Thus, we have shown that

$$b(t, x) = c_0 A(t)ZA(t)^{-1}x.$$

As a consequence,

$$\nabla \cdot b(t, x) = c_0 \operatorname{tr} A(t)ZA(t)^{-1} = c_0 \operatorname{tr} Z = 0,$$

so that b is divergence free.

Since $A(t)^{-\top}\varpi(t)A(t)^{-1}x$ is a gradient, the matrix $A(t)^{-\top}\varpi(t)A(t)^{-1}$ must be symmetric. Thus,

$$\nabla p(t, x) = -\tfrac{1}{2}\nabla[A(t)^{-\top}\varpi(t)A(t)^{-1}x \cdot x].$$

We find that

$$p(t, x) = \tfrac{1}{2}\left[\lambda(t) - \varpi(t)A(t)^{-1}x \cdot A(t)^{-1}x\right],$$

for some scalar function $\lambda(t)$. The other boundary condition implies that

$$0 = p(t, x(t, y)) = \tfrac{1}{2}[\lambda(t) - \varpi(t)y \cdot y],$$

for all $|y| = 1$. This forces

$$\varpi(t) = \lambda(t)I,$$

and so

$$p(t, x) = \tfrac{1}{2}\lambda(t)[1 - |A(t)^{-1}x|^2].$$

Finally, from the velocity equation, we derive

$$A''(t) = \lambda(t)A(t)^{-\top} + A(t)(c_0 Z)^2 = \lambda(t)A(t)^{-\top} - c_0^2 A(t).$$

Thus, we have proven

Lemma 3.7. *Suppose that*

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2).$$

Define

$$\begin{aligned}
u(t, x) &= A'(t)A(t)^{-1}x, \\
b(t, x) &= c_0 A(t)ZA(t)^{-1}x, \quad c_0 \in \mathbb{R}, \\
p(t, x) &= \tfrac{1}{2}\lambda(t)\left[1 - |A(t)^{-1}x|^2\right], \quad \lambda \in C^0(\mathbb{R}, \mathbb{R}),
\end{aligned}$$

and

$$\Omega(t) = A(t)\mathcal{B}.$$

Then $u(t, x)$, $b(t, x)$, $p(t, x)$ solve the MHD system in $\Omega(t)$ if and only if

$$A''(t) + c_0^2 A(t) = \lambda(t)A(t)^{-\top}. \tag{3.1}$$

Remark 3.8. An affine solution satisfies the Taylor sign condition if and only if $\lambda(t) > 0$. We shall show later on (in Corollary 6.2) that the sign of this function is preserved under the motion.

Remark 3.9. The equations of motion (3.1) are the Euler-Lagrange equations associated to the Lagrangian $\mathfrak{L} : \mathbb{M}^2 \times \mathbb{M}^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$\mathfrak{L}(A, A', \lambda) = \tfrac{1}{2}|A'|^2 - \tfrac{1}{2}c_0^2|A|^2 + \lambda(\det A - 1).$$

The scalar function $\lambda(t)$ in (3.1) is a Lagrange multiplier which will now be identified.

Definition 3.10. Given a parameter value $\kappa \geq 0$, define the Lagrange multiplier map $\Lambda_\kappa : \mathcal{D} \to \mathbb{R}$ by

$$\Lambda_\kappa(A, B) = \frac{\operatorname{tr}[L(A,B)^2] + 2\kappa}{\operatorname{tr} A^{-\top}A^{-1}} = \frac{\operatorname{tr}[(BA^{-1})^2] + 2\kappa}{\operatorname{tr} A^{-\top}A^{-1}}.$$

Lemma 3.11. *Fix $\kappa \geq 0$. If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ satisfies*

$$A''(t) + \kappa A(t) = \lambda(t)\ A(t)^{-\top}, \quad t \in \mathbb{R},$$

for some function $\lambda \in C^0(\mathbb{R}, \mathbb{R})$, then

$$\lambda(t) = \Lambda_\kappa(A(t), A'(t)).$$

Proof. Since

$$A'(t) = L(A(t), A'(t))A(t),$$

we have

$$A''(t) = L(A(t), A'(t))'\ A(t) + L(A(t), A'(t))\ A'(t).$$

It follows that

$$\begin{aligned} L(A(t), A'(t))' &= A''(t)A(t)^{-1} - L(A(t), A'(t))\ A'(t)A(t)^{-1} \\ &= \left(-\kappa A(t) + \lambda(t)\ A(t)^{-\top}\right) A(t)^{-1} - L(A(t), A'(t))^2 \\ &= -\kappa I + \lambda(t)\ A(t)^{-\top}A(t)^{-1} - L(A(t), A'(t))^2. \end{aligned}$$

Taking the trace, we obtain

$$\operatorname{tr} L(A(t), A'(t))' = \operatorname{tr} A(t)^{-\top}A(t)^{-1}\ \left[\lambda(t) - \Lambda_\kappa(A(t), A'(t))\right].$$

By Lemma 3.3, we have $\operatorname{tr} L(A(t), A'(t)) = 0$, which implies the result. □

Definition 3.12. Given a parameter value $\kappa \geq 0$, define the energy map $E_\kappa : \mathcal{D} \to [\kappa, \infty)$ by

$$E_\kappa(A, B) = \tfrac{1}{2}|B|^2 + \tfrac{\kappa}{2}|A|^2.$$

Remark 3.13. Note that by Lemma 1.7, $E_\kappa(A, B) \geq \frac{\kappa}{2}|A|^2 \geq \kappa$, for all $(A, B) \in \mathcal{D}$.

Theorem 3.14. *Given a parameter value $\kappa \geq 0$ and initial data*

$$(A_0, B_0) \in \mathcal{D}, \tag{3.2}$$

the initial value problem

$$A''(t) + \kappa A(t) = \Lambda_\kappa(A(t), A'(t))\ A(t)^{-\top}, \tag{3.3}$$

$$(A(0), A'(0)) = (A_0, B_0) \tag{3.4}$$

has a unique global solution $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$. Additionally, $\mathcal{D}$ is invariant:

$$(A(t), A'(t)) \in \mathcal{D}, \quad \textit{for all} \quad t \in \mathbb{R},$$

and the energy is conserved:

$$E_\kappa(A(t), A'(t)) = E_\kappa(A_0, B_0), \quad \textit{for all} \quad t \in \mathbb{R}.$$

Proof. When $\kappa = 0$, this result was proven in Lemma 4 of [9]. The proof easily generalizes to the case $\kappa > 0$, and it will be omitted. □

Corollary 3.15. *A curve $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a geodesic in* $\mathrm{SL}(2,\mathbb{R})$ *with the (induced) Euclidean metric if and only if it satisfies* (3.3) *with $\kappa = 0$.*

Remark 3.16. We include constant solutions as geodesics.

Proof. A geodesic curve is one for which $A'(t)$ is parallel along $A(t)$. That is

$$\frac{D_A}{dt} A'(t) = 0,$$

in which the covariant derivative along $A(t)$ is

$$\frac{D_A}{dt} = \nabla_{A'(t)} = P(A(t))\ \frac{d}{dt}.$$

Thus, $A(t)$ is a geodesic if and only if

$$A''(t) = \lambda(t) N(A(t)),$$

for some scalar $\lambda(t)$. The result follows from Lemma 3.11. □

4. Conserved quantities

Definition 4.1. Define the maps $X_i : \mathcal{D} \to \mathbb{R}$, $i = 1, 2$, by

$$X_1(A, B) = \langle ZA, B \rangle \quad \text{and} \quad X_2(A, B) = \langle AZ, B \rangle\,.$$

Remark 4.2. We shall frequently write $X(A, B)$ for the ordered pair $(X_1(A, B), X_2(A, B))$.

Lemma 4.3.

$$\mathrm{SO}(2,\mathbb{R}) = \{A \in \mathrm{SL}(2,\mathbb{R}) : X_1(A,B) = X_2(A,B), \text{ for all } B \in T_A\mathrm{SL}(2,\mathbb{R})\}.$$

Proof. Notice that

$$\begin{aligned} S &\equiv \{A \in \mathrm{SL}(2,\mathbb{R}) : X_1(A,B) = X_2(A,B), \text{ for all } B \in T_A\mathrm{SL}(2,\mathbb{R})\} \\ &= \{A \in \mathrm{SL}(2,\mathbb{R}) : \langle [A,Z], B\rangle = 0, \text{ for all } B \in T_A\mathrm{SL}(2,\mathbb{R})\}. \end{aligned}$$

Lemma 1.12 implies that $[A,Z] \in T_A\mathrm{SL}(2,\mathbb{R})$, so

$$S = \{A \in \mathrm{SL}(2,\mathbb{R}) : [A,Z] = 0\}.$$

Lemma 1.10 says this is equal to $\mathrm{SO}(2,\mathbb{R})$. □

Theorem 4.4. *If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ is a solution of the IVP* (3.2), (3.3), (3.4)*, then the quantities*

$$E_\kappa(A(t), A'(t)) \quad \text{and} \quad X_i(A(t), A'(t)), \quad i = 1, 2,$$

are invariant.

Proof. The first statement is just conservation of energy which was already noted in Theorem 3.14.

Given Lemma 1.12, we can easily compute the derivatives:

$$\begin{aligned} \frac{d}{dt} X_1(A(t), A'(t)) &= \frac{d}{dt}\left\langle A'(t), ZA(t)\right\rangle \\ &= \left\langle A''(t), ZA(t)\right\rangle + \left\langle A'(t), ZA'(t)\right\rangle \\ &= -\kappa \left\langle A(t), ZA(t)\right\rangle + \Lambda_\kappa(A(t), A'(t)) \left\langle \operatorname{cof} A(t), ZA(t)\right\rangle \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt} X_2(A(t), A'(t)) &= \frac{d}{dt}\left\langle A'(t), A(t)Z\right\rangle \\ &= \left\langle A''(t), A(t)Z\right\rangle + \left\langle A'(t), A'(t)Z\right\rangle \\ &= -\kappa \left\langle A(t), A(t)Z\right\rangle + \Lambda_\kappa(A(t), A'(t)) \left\langle \operatorname{cof} A(t), A(t)Z\right\rangle \\ &= 0. \end{aligned}$$

□

Remark 4.5. Observe that by Lemma 1.6, the Lagrangian defined in Remark 3.9 is invariant under the left and right action of $\mathrm{SO}(2,\mathbb{R})$:

$$\mathfrak{L}(A,B,\lambda) = \mathfrak{L}(UA,UB,\lambda) = \mathfrak{L}(AU,BU,\lambda),$$

for all $(A,B,\lambda) \in \mathbb{M}^2 \times \mathbb{M}^2 \times \mathbb{R}$ and $U \in \mathrm{SO}(2,\mathbb{R})$. Therefore, we can deduce the invariants $X_i(A,B)$ from Noether's theorem. For example, we have

$$X_1(A,B) = \frac{\partial}{\partial\sigma}\left\langle \frac{\partial \mathfrak{L}(A,B)}{\partial B}, U(\sigma)A\right\rangle\Big|_{\sigma=0}.$$

Corollary 4.6. *If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$, then its vorticity $W(t)$ satisfies*

$$W(t) \equiv \tfrac{1}{2}[L(A(t), A'(t)) - L(A(t), A'(t))^\top] = \tfrac{1}{2} X_2(A(t), A'(t))Z.$$

If $A(t)$ is also solution of the system (3.3)*, then its vorticity is invariant.*

Proof. Since $W(t)$ is anti-symmetric, we can write $W(t) = \omega(t)Z$, for some scalar function $\omega(t)$. Then by Lemmas 1.4, 1.2, and 1.11, we have

$$\begin{aligned}
2\,\omega(t) &= \omega(t)\,\langle Z, Z\rangle \\
&= \langle W(t), Z\rangle \\
&= \tfrac{1}{2}\left\langle L(A(t), A'(t)) - L(A(t), A'(t))^\top, Z\right\rangle \\
&= \left\langle L(A(t), A'(t)), Z\right\rangle \\
&= \left\langle A'(t)A(t)^{-1}, Z\right\rangle \\
&= \left\langle A'(t), ZA(t)^{-\top}\right\rangle \\
&= \left\langle A'(t), Z \operatorname{cof} A(t)\right\rangle \\
&= \left\langle A'(t), A(t)Z\right\rangle \\
&= X_2(A(t), A'(t)).
\end{aligned}$$

This is invariant by Theorem 4.4, if $A(t)$ solves (3.3). ◻

Remark 4.7. For 2d incompressible perfect fluids ($\kappa = 0$), the material vorticity, i.e. curl $u(t, x(t, y))$, is independent of time in general, however this does not hold in general for MHD ($\kappa > 0$).

Corollary 4.8. *Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a solution of the IVP* (3.2)*,* (3.3)*,* (3.4)*. Set $X_i = X_i(A_0, B_0)$, $i = 1, 2$. Then $A(t)$ is irrotational if and only if $X_2 = 0$.*

5. Invariant sets

Our goal in this section will be to characterize the data in $\mathcal{D}$ with given values of the invariants. For this it is convenient to introduce the following notation.

Definition 5.1. Given parameter values $\kappa \geq 0$ and

$$(E, X) = (E, X_1, X_2) \in [\kappa, \infty) \times \mathbb{R}^2,$$

define

$$\mathcal{D}(X) = \{(A, B) \in \mathcal{D} : X_i(A, B) = X_i,\ i = 1, 2\}.$$

and

$$\mathcal{D}_\kappa(E, X) = \{(A, B) \in \mathcal{D} : E_\kappa(A, B) = E,\ X_i(A, B) = X_i,\ i = 1, 2\}.$$

Remark 5.2. By Theorem 4.4, the sets $\mathcal{D}(X)$ and $\mathcal{D}_\kappa(E, X)$ are invariant under the flow of (3.3).

Remark 5.3. The family $\{\mathcal{D}(X) : X \in \mathbb{R}^2\}$ foliates $\mathcal{D}$, and for each $\kappa \geq 0$, $X \in \mathbb{R}^2$ the family $\{\mathcal{D}_\kappa(E, X) : E \geq \kappa\}$ foliates $\mathcal{D}(X)$.

Lemma 5.4. *Fix $\kappa \geq 0$. Let $(E, X) \in [\kappa, \infty) \times \mathbb{R}^2$. Suppose that $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$ and $(A, B) \in \mathcal{D}$.*

Then $(A, B) \in \mathcal{D}(X)$ if and only if

$$B = \sum_{i=1}^{3} c_i \tau_i(A), \quad \textit{with} \quad c_1 = \frac{X_1 + X_2}{g_{11}(A)},\ c_2 = \frac{X_1 - X_2}{g_{22}(A)},\ c_3 \in \mathbb{R}.$$

In this case, $c_3 = \langle A, B\rangle$.

Moreover, $(A, B) \in \mathcal{D}_\kappa(E, X)$ if and only if $(A, B) \in \mathcal{D}(X)$ and

$$E = \tfrac{\kappa}{2}|A|^2 + \tfrac{1}{2}\sum_{i=1}^{3} g_{ii}(A) c_i^2. \tag{5.1}$$

with c_i defined above.

Proof. The first statement follows immediately from Lemmas 2.12 and 2.11. The second statement is equally simple

$$E = E_\kappa(A, B) = \tfrac{1}{2}|B|^2 + \tfrac{\kappa}{2}|A|^2 = \tfrac{1}{2}\sum_{i=1}^{3} g_{ii}(A) c_i^2 + \tfrac{\kappa}{2}|A|^2.$$

□

Lemma 5.5. *Fix $\kappa \geq 0$. Let $(E, X) \in [\kappa, \infty) \times \mathbb{R}^2$. Suppose that $A \in \mathrm{SO}(2,\mathbb{R})$ and $(A, B) \in \mathcal{D}$.*

Then $(A, B) \in \mathcal{D}(X)$ if and only if $X_1 = X_2$ and

$$A = U(2s_1), \quad B = (\tfrac{1}{2}X_1)\, U(2s_1)\, Z + \beta\, U(2s_2)\, M, \tag{5.2}$$

with $\beta \geq 0$, $s_1, s_2 \in \mathbb{R}$.

Moreover, $(A, B) \in \mathcal{D}_\kappa(E, X)$, if and only if $(A, B) \in \mathcal{D}(X)$ and

$$E = \kappa + \tfrac{1}{4}X_1^2 + \beta^2.$$

Proof. Using Lemma 2.14, write $A = \mathcal{A}(s_1, 0, 1) = U(2s_1)$. By Lemma 2.18, for $B \in T_A\mathrm{SL}(2,\mathbb{R})$, we have

$$B = c_1 U(2s_1) Z - c_2 K + c_3 M.$$

Note that $-K = ZM$ and take $(c_3, c_2) = \beta(\cos 2s_2, \sin 2s_2)$. Then

$$-c_2 K + c_3 M = \beta(\sin 2s_2 Z + \cos 2s_2 I)M = \beta U(2s_2)M,$$

so that

$$B = c_1 U(2s_1) Z + \beta U(2s_2) M.$$

Now since $A \in \mathrm{SO}(2,\mathbb{R})$ and $(A,B) \in \mathcal{D}(X)$, we have

$$X_1 = X_2 = X_2(A,B) = \langle AZ, B\rangle = 2c_1,$$

which yields the formula (5.2).

By (5.2), we have

$$|B|^2 = \tfrac{1}{4}X_1^2|ZU(\sigma_1)|^2 + \beta^2|MU(\sigma_2)|^2 = 2(\tfrac{1}{4}X_1^2 + \beta^2),$$

and so if $(A,B) \in \mathcal{D}_\kappa(E,X)$, then

$$E = E_\kappa(A,B) = \tfrac{1}{2}|B|^2 + \tfrac{\kappa}{2}|A|^2 = \kappa + \tfrac{1}{4}X_1^2 + \beta^2.$$

The converse statements are easily verified. □

Corollary 5.6. *For every $X \in \mathbb{R}^2$, there exists $(A,B) \in \mathcal{D}(X)$ such that $B \neq 0$.*

Proof. This follows from Lemmas 5.4 and 5.5. □

6. The nonlinearity, revisited

Lemma 6.1. *Fix $\kappa \geq 0$. If $(A,B) \in \mathcal{D}_\kappa(E,X)$, then*

$$\Lambda_\kappa(A,B) = \frac{2(\kappa - \det B)}{|A|^2} = \frac{4E - 2X_1X_2}{|A|^4}.$$

Proof. Let $(A,B) \in \mathcal{D}_\kappa(E,X) \subset \mathcal{D}$, and put $L = L(A,B) = BA^{-1}$. Then by Lemma 2.1, $\operatorname{tr} L = 0$, and so the Cayley-Hamilton Theorem implies that

$$L^2 + (\det L)I = 0.$$

Taking the trace yields

$$\operatorname{tr} L^2 = -2\det L = -2\det B \det A^{-1} = -2\det B.$$

Also note that by Lemmas 1.13 and 1.11, we have

$$\operatorname{tr} A^{-\top}A^{-1} = |A^{-\top}|^2 = |\operatorname{cof} A|^2 = |A|^2.$$

According to Definition 3.10, this shows that

$$\Lambda_\kappa(A,B) = \frac{2(\kappa - \det B)}{|A|^2},$$

which is the first statement.

Therefore, the result will follow if we can verify that

$$(\kappa - \det B) = \frac{2E - X_1X_2}{|A|^2}, \quad \text{for} \quad (A,B) \in \mathcal{D}_\kappa(E,X). \tag{6.1}$$

To proceed, we temporarily assume that $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$. Using Lemmas 5.4 and 2.22, we have

$$\operatorname{cof} B = c_1\tau_1(A) - c_2\tau_2(A) - \frac{2}{|A|^2}c_3\tau_3(A) + \frac{1}{|A|}N(A).$$

Therefore, by Lemmas 1.15 and 5.4, we find that

$$2\det B = \langle \operatorname{cof} B, B\rangle = g_{11}(A)c_1^2 - g_{22}(A)c_2^2 - \frac{2}{|A|^2}g_{33}(A)c_3^2.$$

Combining this with (5.1) to eliminate the term with c_3, we obtain

$$|A|^2(\kappa - \det B) = 2E - \tfrac{1}{4}g_{11}(A)^2c_1^2 + \tfrac{1}{4}g_{22}(A)^2c_2^2 = 2E - X_1X_2,$$

as desired.

We now establish the identity (6.1) for $A \in \mathrm{SO}(2,\mathbb{R})$. In this case, we have that $|A|^2 = 2$, by Lemma 1.7, and $X_1 = X_2$ by Lemma 4.3, so we aim to show that

$$\kappa - \det B = E - \tfrac{1}{2}X_1^2.$$

This now follows from Lemma 5.5 since

$$\begin{aligned} 2\det B &= \langle B, \operatorname{cof} B\rangle \\ &= \left\langle \tfrac{1}{2}X_1U(2s_1)Z + \beta U(2s_2)M, \tfrac{1}{2}X_1U(2s_1)Z - \beta U(2s_2)M\right\rangle \\ &= \tfrac{1}{4}X_1^2|U(2s_1)Z|^2 - \beta^2|U(2s_2)M|^2 \\ &= \tfrac{1}{2}X_1^2 - 2\beta^2 \\ &= -2E + 2\kappa + X_1^2. \end{aligned}$$

□

Corollary 6.2. *For each $\kappa \geq 0$, the set*

$$\{(A,B) \in \mathcal{D} : \Lambda_\kappa(A,B) = 0\}$$

is invariant under the flow of (3.3).

The next lemma gives another geometric interpretation of the nonlinear term.

Corollary 6.3. *There holds*

$$\Lambda_\kappa(A,B)A^{-\top} = \left(\frac{2\kappa}{|A|} + \langle S(A)B, B\rangle\right)N(A), \quad (A,B) \in \mathcal{D}.$$

Proof. By Lemmas 1.15 and 2.25, we have

$$-2\det B = -\langle \operatorname{cof} B, B\rangle = |A|\,\langle S(A)B, B\rangle\,.$$

The claim follows from Lemma 6.1 and Definition 2.3. □

7. Energy minimization in $\mathcal{D}(X)$

Definition 7.1. By Corollary 5.6, $\mathcal{D}(X) \neq \emptyset$, so for $X \in \mathbb{R}^2$ and $\kappa \geq 0$, we may define

$$e_\kappa(X) = \inf\{E_\kappa(A,B) : (A,B) \in \mathcal{D}(X)\}.$$

Lemma 7.2. *If $\kappa = 0$, then $e_0(X) = 0$, for any $X \in \mathbb{R}^2$. If $X \neq 0$, then the image of $\mathcal{D}(X)$ under E_0 is the open interval $(0,\infty)$. If $X = 0$, then $E_0(\mathcal{D}(X)) = [0,\infty)$.*

Proof. Fix $X \in \mathbb{R}^2$. Let A_j be a sequence in $\mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $|A_j| \to \infty$. Fix c_1, c_2 and take $c_3 = 0$ in (5.1). We obtain a sequence $(A_j, B_j) \in \mathcal{D}(X)$ such that $E_0(A_j, B_j) \to 0$. Thus, we see that $e_0(X) = 0$.

Now, letting c_3 range over all values in $\mathbb{R}$, we observe that

$$(E_0(A_j, B_j), \infty) \subset E_0(\mathcal{D}(X)),$$

for each j. This shows that $(0,\infty) \subset E_0(\mathcal{D}(X))$.

If $X \neq 0$, then for all $(A,B) \in \mathcal{D}(X)$, $B \neq 0$ and thus $E_0(A,B) \neq 0$. This means that $E_0(\mathcal{D}(X)) = (0,\infty)$.

Finally, take $X = 0$. Since $(I,0) \in \mathcal{D}(0)$, we see that $0 = E_0(I,0) \in E_0(\mathcal{D}(0))$, and we conclude $E_0(\mathcal{D}(X)) = [0,\infty)$. □

Lemma 7.3. *If $\kappa > 0$, then for any $X \in \mathbb{R}^2$, $e_\kappa(X) \geq \kappa$ with equality if and only if $X = 0$. Moreover, $E_\kappa(\mathcal{D}(X)) = [e_\kappa(X), \infty)$.*

Proof. Since $E_\kappa(A,B) \geq \kappa$, for all $(A,B) \in \mathcal{D}$, we have that $e_\kappa(X) \geq \kappa$.

If $e_\kappa(X) = \kappa$, then for any $\varepsilon > 0$, there exists $(A,B) \in \mathcal{D}(X)$ such that $0 \leq E_\kappa(A,B) - \kappa < \varepsilon$. It follows that

$$|B|^2 \leq 2\varepsilon \quad \text{and} \quad |A|^2 \leq 2 + 2\varepsilon/\kappa.$$

Therefore, we see that for $i = 1, 2$,

$$|X_i| = |X_i(A,B)| \leq |A||B| \leq \varepsilon^{1/2}(2 + 2\varepsilon/\kappa)^{1/2}.$$

Since $\varepsilon > 0$ is arbitrary, we get that $X = 0$.

On the other hand, if $X = 0$, then for any $A \in \mathrm{SO}(2,\mathbb{R})$, we have $(A,0) \in \mathcal{D}(X)$. Thus, we see that

$$\kappa \leq e_\kappa(0) \leq E_\kappa(A,0) = \kappa.$$

We have shown that $e_\kappa(X) = \kappa$ if and only if $X = 0$.

Take a sequence $(A_j, B_j) \in \mathcal{D}(X)$ with $E_\kappa(A_j, B_j) \searrow e_\kappa(X)$. By Lemmas 5.4 and 5.5, we may assume without loss of generality that for each j, B_j lies in the span of $\tau_i(A_j)$, $i = 1, 2$. Since $\kappa > 0$, this sequence is bounded in $\mathcal{D}(X)$. By compactness we obtain an energy minimizer $(A,B) \in \mathcal{D}(X)$ where B lies in the span of $\tau_i(A)$, $i = 1, 2$. By considering the family $(A, B + B_1) \in \mathcal{D}(X)$, where $\langle \tau_i(A), B_1 \rangle = 0$, $i = 1, 2$, we see that that $E_\kappa(\mathcal{D}(X)) = [e_\kappa(X), \infty)$. □

Lemma 7.4. *If $\kappa > 0$ and $X_1 = X_2$, then*

$$e_\kappa(X) = \begin{cases} \kappa + \frac{1}{4}X_1^2, & \frac{1}{8}X_1^2 \leq \kappa \\ (2\kappa)^{1/2}|X_1| - \kappa, & \frac{1}{8}X_1^2 \geq \kappa. \end{cases}$$

Proof. If $A \in \mathrm{SO}(2,\mathbb{R})$, then by Lemma 5.5, we see that

$$\min\{E_\kappa(A,B) : B \in T_A\mathrm{SL}(2,\mathbb{R})\} = \kappa + \tfrac{1}{4}X_1^2.$$

If $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, then by Lemma 5.4,

$$\min\{E_\kappa(A,B) : B \in T_A\mathrm{SL}(2,\mathbb{R})\} = \tfrac{\kappa}{2}|A|^2 + \frac{X_1^2}{|A|^2+2} \equiv f(|A|^2).$$

Taking the infimum over $\xi = |A|^2 > 2$, we obtain

$$\begin{aligned} \inf\{E_\kappa(A,B) &: A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}),\ B \in T_A\mathrm{SL}(2,\mathbb{R})\} \\ &= \inf_{\xi>2} f(\xi) = \begin{cases} \kappa + \frac{1}{4}X_1^2, & \frac{1}{8}X_1^2 \leq \kappa \\ \sqrt{2\kappa}|X_1| - \kappa, & \frac{1}{8}X_1^2 \geq \kappa. \end{cases} \end{aligned}$$

□

Remark 7.5. The function defined in Lemma 7.4 is C^1 in X_1.

Remark 7.6. When $\kappa > 0$ and $X_1 = X_2$, we have

$$\kappa + \tfrac{1}{4}X_1^2 \geq e_\kappa(X),$$

with equality if and only if $\frac{1}{8}X_1^2 \leq \kappa$.

8. The reduced Hamiltonian

In this preparatory section we introduce the reduced Hamiltonian and investigate its level curves in the phase plane. The connection with the dynamics will be made in Section 9.

Definition 8.1. Given values $\kappa \geq 0$ and $(E,X) \in [0,\infty) \times \mathbb{R}^2$, define the polynomials

$$\begin{aligned} P_\kappa(x;E,X) = &- 4\kappa x(x^2-1) + 4E(x^2-1) \\ &- \tfrac{1}{2}(X_1-X_2)^2(x+1) - \tfrac{1}{2}(X_1+X_2)^2(x-1), \quad x \in \mathbb{R}, \end{aligned}$$

and

$$\Phi_\kappa(x,y;E,X) = \frac{y^2}{2} - \frac{P_\kappa(x;E,X)}{2x}, \quad (x,y) \in \mathbb{R}^2.$$

Remark 8.2. The reader is cautioned that from now on x and y shall represent real numbers, and not spatial and material points, as in Section 3.

Remark 8.3. We will see momentarily that if $(A,B) \in \mathcal{D}_\kappa(E,X)$, then the point $(x,y) = (\frac{1}{2}|A|^2, \langle A,B\rangle)$ satsifies $\Phi_\kappa(x,y;E,X) = 0$.

Definition 8.4. Define the mapping $\mathcal{P} : \mathcal{D} \to \mathbb{R}^2$ by

$$\mathcal{P}(A, B) = \left(\tfrac{1}{2}|A|^2, \langle A, B\rangle\right).$$

Lemma 8.5. *The range of $\mathcal{P}$ is given by*

$$\mathcal{P}(\mathcal{D}) = \{(x, y) \in \mathbb{R}^2 : x > 1\} \cup \{(1, 0)\},$$

and for all $(A, B) \in \mathcal{D}$, $\mathcal{P}(A, B) = (1, 0)$ if and only if $A \in \mathrm{SO}(2, \mathbb{R})$.

Proof. This follows by Lemmas 1.7 and 2.4. □

Definition 8.6. For fixed parameter values $\kappa \geq 0$ and $(E, X) \in [e_\kappa(X), \infty) \times \mathbb{R}^2$, define

$$\begin{aligned} C_\kappa(E, X) &= \mathcal{P}(\mathcal{D}_\kappa(E, X)) \\ &= \{(x, y) = \mathcal{P}(A, B) \in \mathbb{R}^2 : (A, B) \in \mathcal{D}_\kappa(E, X)\}. \end{aligned}$$

Lemma 8.7. *Fix values $\kappa \geq 0$ and $(E, X) \in [e_\kappa(X), \infty) \times \mathbb{R}^2$. There holds*

$$C_\kappa(E, X) \subset \mathcal{P}(\mathcal{D}).$$

A point (x, y) with $x > 1$ belongs to $C_\kappa(E, X)$ if and only if

$$\Phi_\kappa(x, y; E, X) = 0.$$

The point $(1, 0)$ belongs to $C_\kappa(E, X)$ if and only if

$$\Phi_\kappa(1, 0; E, X) = 0 \quad \textit{and} \quad \partial_x \Phi_\kappa(1, 0; E, X) \leq 0 \tag{8.1}$$

if and only if

$$X_1 = X_2, \quad \textit{and} \quad E \geq \kappa + \tfrac{1}{4}X_1^2. \tag{8.2}$$

Proof. The first statement is a consequence of Definition 5.1.

Select any point $(x, y) = \mathcal{P}(A, B) \in \mathcal{P}(\mathcal{D})$, with $x > 1$. By definition, $(x, y) \in C_\kappa(E, X)$ if and only if $(A, B) \in \mathcal{D}_\kappa(E, X)$. By Lemma 2.11, we find that

$$g(A) = \operatorname{diag}\left[4(x+1), \quad 4(x-1), \quad \frac{x}{2(x^2-1)}\right].$$

The third coordinate c_3 defined in Lemma 5.4 satisfies

$$c_3 = \langle A, B\rangle = y.$$

Therefore, Lemma 5.4 implies that $(A, B) \in \mathcal{D}_\kappa(E, X)$ if and only if

$$E = \kappa x + \frac{(X_1 + X_2)^2}{8(x+1)} + \frac{(X_1 - X_2)^2}{8(x-1)} + \frac{xy^2}{4(x^2-1)},$$

which is in turn equivalent to the desired result $\Phi_\kappa(x, y; E, X) = 0$.

Now suppose that $(1,0) = \mathcal{P}(A,B) \in C_\kappa(E,X)$. Then $A \in \mathrm{SO}(2,\mathbb{R})$ and $X_1 = X_2$, by Lemmas 4.3 and 8.5. By the Cauchy-Schwarz inequality and Lemmas 1.6, 1.7, we have

$$X_1^2 = \langle ZA, B\rangle^2 \leq |ZA|^2|B|^2 = |A|^2|B|^2 = 2|B|^2 = 4(E_\kappa(A,B) - \tfrac{\kappa}{2}|A|^2) = 4(E-\kappa).$$

Thus, (8.2) is true.

Next, suppose that (8.2) holds. Choose $A \in \mathrm{SO}(2,\mathbb{R})$ and using Lemma 5.5 set

$$B = (\tfrac{1}{2}X_1)\, ZA + B_1,$$

with

$$\langle A, B_1\rangle = \langle ZA, B_1\rangle = 0 \quad \text{and} \quad \tfrac{1}{2}|B_1|^2 = E - \kappa - \tfrac{1}{4}X_1^2.$$

By Lemma 5.5, $(A,B) \in \mathcal{D}_\kappa(E,X)$, and so $(1,0) = \mathcal{P}(A,B) \in C_\kappa(E,X)$.

It is immediate to verify the equivalence of (8.1) and (8.2). □

Lemma 8.8. *A point $(x_0,y_0) \in C_\kappa(E,X)$ is a critical point of the Hamiltonian $\Phi_\kappa(x,y;E,X)$ if and only if*

$$x_0 \geq 1, \quad y_0 = 0, \quad \textit{and} \quad P_\kappa(x_0;E,X) = P'_\kappa(x_0;E,X) = 0. \tag{8.3}$$

Proof. Suppose that $(x_0,y_0) \in C_\kappa(E,X)$ is a critical point of $\Phi_\kappa(x,y;E,X)$. Then $(x_0,y_0) \in \mathcal{P}(\mathcal{D})$, so $x_0 \geq 1$. By Lemma 8.7,

$$\Phi_\kappa(x_0,y_0;E,X) = y_0^2/2 - P_\kappa(x_0,E,X)/2x_0 = 0.$$

Critical points are characterized by the equations

$$\partial_x\Phi_\kappa(x_0,y_0;E,X) = (P_\kappa(x_0;E,X) - x_0P'_\kappa(x_0;E,X))/2x_0^2 = 0$$

and

$$\partial_y\Phi_\kappa(x_0,y_0;E,X) = y_0 = 0.$$

Thus, we see that (8.3) holds.

If (8.3) holds, then

$$\Phi_\kappa(x_0,y_0;E,X) = 0 \quad \text{and} \quad \nabla_{x,y}\Phi_\kappa(x_0,y_0;E,X) = 0.$$

So (x_0,y_0) is a critical point of Φ_κ, and by Lemma 8.7, $(x_0,y_0) \in C_\kappa(E,X)$. □

Corollary 8.9. *If $(\kappa,E,X) \neq 0$, then the set $C_\kappa(E,X)$ can contain at most one critical point of $\Phi_\kappa(x,y;E,X)$.*

Proof. By Lemma 8.8, critical points in $C_\kappa(E,X)$ correspond to double roots of $P_\kappa(x;E,X)$, a nonzero polynomial of degree at most 3, so there can be at most one critical point. □

Lemma 8.10. *Fix $\kappa > 0$, $X \in \mathbb{R}^2$. The set $C_\kappa(E,X)$ is a singleton if and only if $E = e_\kappa(X)$. In this case, $C_\kappa(e_\kappa(X),X) = \{(x_0,0)\}$, where $(x_0,0)$ is a critical point of $\Phi_\kappa(x,y;e_\kappa(X),X)$ and a minimum in $\mathcal{P}(\mathcal{D})$.*

Proof. Suppose that $E < e_\kappa(X)$. Then $C_\kappa(E,X) = \emptyset$, and so by Lemma 8.7

$$\Phi_\kappa(x,y;E,X) \neq 0, \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

Since $\kappa > 0$, we have $\Phi_\kappa(x,0;E,X) \to +\infty$, as $x \to +\infty$, and as a consequence

$$\Phi_\kappa(x,y;E,X) > 0 \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

By continuity, we obtain

$$\Phi_\kappa(x,y;e_\kappa(X),X) \geq 0, \quad \text{for all} \quad (x,y) \in \mathcal{P}(\mathcal{D}).$$

Since

$$\Phi_\kappa(x,y;e_\kappa(X),X) = \tfrac{1}{2}y^2 + \Phi_\kappa(x,0;e_\kappa(X),X),$$

we see that

$$\Phi_\kappa(x,y;e_\kappa(X),X) > 0, \quad \text{for all} \quad (x,y) \in \mathcal{P}(\mathcal{D}),\ y \neq 0.$$

Thus, by Lemma 8.7 we have that

$$C_\kappa(e_\kappa(X),X) \subset \{(x,0) : x \geq 1\}.$$

Additionally, Lemma 7.3 assures us that $C_\kappa(e_\kappa(X),X) = \mathcal{P}(\mathcal{D}_k(e_\kappa(X),X)) \neq \emptyset$.

If $(x_0,0) \in C_\kappa(e_\kappa(X),X)$ for some $x_0 \geq 1$, then

$$0 = \Phi_\kappa(x_0,0;e_\kappa(X),X) \leq \Phi_\kappa(x,y;e_\kappa(X),X), \quad (x,y) \in \mathcal{P}(\mathcal{D}).$$

This says that $(x_0,0)$ is a minimum value for $\Phi_\kappa(x,y;e_\kappa(X),X)$. It follows that

$$\partial_x \Phi_\kappa(x_0,0;e_\kappa(X),X) = 0, \quad \text{if} \quad x_0 > 1$$

and

$$\partial_x \Phi_\kappa(x_0,0;e_\kappa(X),X) \geq 0, \quad \text{if} \quad x_0 = 1.$$

On the other hand, if $x_0 = 1$, then by Lemma 8.7,

$$\partial_x \Phi_\kappa(x_0,0;e_\kappa(X),X) \leq 0.$$

We conclude that

$$\partial_x \Phi_\kappa(x_0,0;e_\kappa(X),X) = 0, \quad \text{for all} \quad x_0 \geq 1.$$

This shows that $(x_0,0)$ must be a critical point of $\Phi_\kappa(x,y;e_\kappa(X),X)$. Since $\kappa > 0$, there can be only one critical point in $C_\kappa(e_\kappa(X),X)$ by Corollary 8.9, and so this set is a singleton.

Now suppose that $C_\kappa(E,X)$ is a singleton. By Lemma 8.7 and definition of $\Phi_\kappa(x,y;E,X)$, if $(x,y) \in C_\kappa(E,X)$, then $(x,-y) \in C_\kappa(E,X)$. So it must be that

$$C_\kappa(E,X) = \{(x_0,0)\}, \quad \text{for some} \quad x_0 \geq 1.$$

By Lemma 8.7, we have that

$$\{x > 1 : \Phi_\kappa(x,y;E,X) = 0\} \subset C_\kappa(E,X) = \{(x_0,0)\}.$$

This implies that $\Phi_\kappa(x, y; E, X)$ does not vanish on the connected open set

$$\{(x, y) : x > 1,\ x \neq x_0\}.$$

Using the fact that $\lim_{x\to\infty} \Phi_\kappa(x, 0; E, X) = \infty$, we conclude that

$$\Phi_\kappa(x, y; E, X) > 0 \quad \text{for all} \quad x > 1,\ x \neq x_0,\ y \in \mathbb{R}. \tag{8.4}$$

Since $C_\kappa(E, X) \neq \emptyset$, we have that $E \geq e_\kappa(X)$. We claim that

$$\bar{E} < E \quad \text{implies} \quad C_\kappa(\bar{E}, X) = \emptyset. \tag{8.5}$$

Given this claim, we would have $\bar{E} < e_\kappa(X)$ so that

$$E = \sup\{\bar{E} : \bar{E} < E\} \leq e_\kappa(X),$$

thereby showing that $E = e_\kappa(X)$, as desired.

Assume $\bar{E} < E$, and let us now proceed to verify (8.5). Write

$$\Phi_\kappa(x, y; \bar{E}, X) = 4(E - \bar{E})(x^2 - 1)/2x + \Phi_\kappa(x, y; E, X). \tag{8.6}$$

By (8.4), the equation (8.6) implies that

$$\Phi_\kappa(x, y; \bar{E}, X) > 0, \quad \text{for all} \quad x > 1,\ y \in \mathbb{R}.$$

Thus, by Lemma 8.7, we discover that

$$C_\kappa(\bar{E}, X) \subset \{(1, 0)\}.$$

If $X_1 \neq X_2$, then $\Phi_\kappa(1, 0; \bar{E}, X) = (X_1 - X_2)^2 > 0$ so that $(1, 0) \notin C_\kappa(\bar{E}, X)$ and (8.5) holds in this case.

Next, suppose that $X_1 = X_2$. Then

$$\Phi_\kappa(1, 0; \bar{E}, X) = \Phi_\kappa(1, 0; E, X) = 0,$$

and by (8.4), we see that

$$\partial_x \Phi_\kappa(1, 0; E, X) \geq 0.$$

Thus, by (8.6), we find that

$$\partial_x \Phi_\kappa(1, 0; \bar{E}, X) = 4(E - \bar{E}) + \partial_x \Phi_\kappa(1, 0; E, X) > 0.$$

By Lemma 8.7, we conclude that $(1, 0) \notin C_\kappa(\bar{E}, X)$, and again (8.5) holds.

□

For convenience, we summarize the relationship between the exceptional point $(1, 0)$ and the sets $C_\kappa(E, X)$.

Corollary 8.11. *Fix* $X \in \mathbb{R}^2$.

i. $(1,0) \in C_\kappa(E,X)$ *if and only if* $X_1 = X_2$ *and* $E \geq \kappa + \frac{1}{4}X_1^2$.

ii. $(1,0) \in C_\kappa(E,X)$ *is a critical point of* $\Phi_\kappa(x,y;E,X)$ *if and only if* $X_1 = X_2$ *and* $E = \kappa + \frac{1}{4}X_1^2$.

iii. $\{(1,0)\} = C_\kappa(E,X)$ *if and only if* $\kappa > 0$, $X_1 = X_2$ *and* $E = \kappa + \frac{1}{4}X_1^2 = e_\kappa(X)$.

Proof. The statement (i) was shown in Lemma 8.7, and (ii) follows from Lemma 8.8. In the next result we shall see that $C_0(E,X)$ is either empty or unbounded. Thus, (iii) is just Lemma 8.10. □

Lemma 8.12. *At each point* $(x,y) \in C_\kappa(E,X)$ *such that* $\nabla\Phi_\kappa(x,y;E,X) \neq 0$, *the set* $C_\kappa(E,X)$ *is a locally smooth curve.*

The sets $C_\kappa(E,X)$ *are closed and connected subsets of* $\mathcal{P}(\mathcal{D})$.

If $C_0(E,X) \neq \emptyset$, *then it is unbounded.*

If $\kappa > 0$, *then* $C_\kappa(e_\kappa(X),X)$ *is a singleton, and for* $E > \bar{E} \geq e_\kappa(X)$, $C_\kappa(E,X)$ *is a closed curve enclosing* $C_\kappa(\bar{E},X) \setminus \{(1,0)\}$.

Proof. By Lemma 8.7,

$$C_\kappa(E,X) \setminus \{(1,0)\} \subset \{(x,y) : \Phi_\kappa(x,y;E,X) = 0,\ x > 1\},$$

so the smoothness of $C_\kappa(E,X)$ away from critical points of $\Phi_\kappa(x,y;E,X)$ in the region $\{x > 1\}$ follows by the implicit function theorem. If $(1,0) \in C_\kappa(E,X)$ is not a critical point of $\Phi_\kappa(x,y;E,X)$, then by Lemma 8.7, $\partial_x\Phi_\kappa(1,0;E,X) < 0$, and the implicit function theorem provides a smooth local parameterization of $C_\kappa(E,X)$ of the form $(x(y),y)$, $|y| \ll 1$, with

$$x(0) = 1, \quad x'(0) = 0, \quad x''(0) = -1/\partial_x\Phi_\kappa(1,0;E,X) > 0,$$

describing a curve contained in $\mathcal{P}(\mathcal{D})$.

To prove the other statements, we consider the cases $\kappa = 0$ and $\kappa > 0$ separately.

Suppose that $\kappa = 0$. If $E = 0$ and $C_0(0,X) \neq \emptyset$, then by Lemma 7.2, we have $X = 0$. By definition, $\Phi_0(x,y;0,0) = \frac{1}{2}y^2$, and so by Lemma 8.7,

$$C_0(0,0) = \{(x,0) : x \geq 1\},$$

which is a closed, connected, and unbounded set. If $E > 0$, then $P_0(x;E,X) \to \infty$, as $|x| \to \infty$. By Lemma 8.7,

$$C_0(E,X) \subset \{(x,y) : \Phi_0(x,y;E,X) = 0,\ x \geq 1\},$$

so $P_0(x;E,X)$ must have real roots $x_1(E,X) \leq x_2(E,X)$. Since

$$P_0(1;E,X) = -(X_1 - X_2)^2 \leq 0,$$

it follows that $x_1(E,X) \leq 1 \leq x_2(E,X)$. If $x_1(E,X) = 1 < x_2(E,X)$, then $\partial_x\Phi_0(1,y;E,X) < 0$, and by Lemma 8.7, $(1,0) \notin C_0(E,X)$ and so

$$C_0(E,X) = \{(x,y) : y^2 = P_0(x;E,X)/x,\ x \geq x_2(E,X)\}.$$

This also holds when $x_1(E,X) < 1 \leq x_2(E,X)$ or when $x_1(E,X) = x_2(E,X) = 1$. Thus, $C_0(E,X)$ again is a closed, connected, and unbounded set.

Now suppose that $\kappa > 0$. Note that $P_\kappa(x;E,X) \to \mp\infty$, as $x \to \pm\infty$ and $P_\kappa(1;E,X) = -(X_1 - X_2)^2 \leq 0$. So if $C_\kappa(E,X) \neq \emptyset$, then $P_\kappa(x;E,X)$ must have three real roots (counting multiplicity) with

$$x_1(E,X) \leq 1 \leq x_2(E,X) \leq x_3(E,X).$$

By Lemma 8.7, $(1,y) \in C_\kappa(E,X)$ if and only if

$$y = 0 \quad \text{and} \quad \partial_x \Phi_\kappa(1,0;E,X) = -\tfrac{1}{2} P'_\kappa(1;E,X) \leq 0.$$

It follows that

$$C_\kappa(E,X) = \{(x,y) : y^2 = P_\kappa(x;E,X)/x,\ x_2(E,X) \leq x \leq x_3(E,X)\}. \tag{8.7}$$

Thus, $C_\kappa(E,X)$, $\kappa > 0$, is a simple closed closed curve and a closed, bounded, and connected set.

We note that for $E > \bar{E} \geq e_\kappa(X)$, we have

$$P_\kappa(x;E,X) - P_\kappa(x;\bar{E},X) = 4(E - \bar{E})(x^2 - 1) > 0, \quad x > 1.$$

Thus, the enclosure claim is a consequence of (8.7).

The fact that $C_\kappa(e_\kappa(X),X)$ is a singleton was shown in Lemma 8.10. □

Remark 8.13. Observe that (with the exception of Corollary 6.2) the results in Sections 5 through 8 are purely algebraic. We now make the connection with the dynamics of the system (3.3).

9. Reduction to the phase plane

If $A \in C^0(\mathbb{R};\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R};\mathbb{M}^2)$, then $(x(t),y(t)) = \mathcal{P}(A(t),A'(t))$ is a C^1 planar curve. We now show that given a solution $A(t)$ of the system (3.3), its phase plane curve $\mathcal{P}(A(t),A'(t))$ satisfies a Hamiltonian system.

Theorem 9.1. *Fix $\kappa \geq 0$ and $(E,X) \in [e_\kappa(X),\infty) \times \mathbb{R}^2$. Suppose that $A \in C^0(\mathbb{R},\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R},\mathbb{M}^2)$ is a solution of* (3.2), (3.3), (3.4) *with initial data in $\mathcal{D}_\kappa(E,X)$. Put*

$$(x(t),y(t)) = \mathcal{P}(A(t),A'(t)).$$

Then

$$\begin{aligned} x'(t) &= \partial_y \Phi_\kappa(x(t),y(t);E,X) = y(t) \\ y'(t) &= -\partial_x \Phi_\kappa(x(t),y(t);E,X), \end{aligned} \tag{9.1}$$

and the solution orbit satisfies

$$(x(t),y(t)) \in C_\kappa(E,X), \tag{9.2}$$

for all $t \in \mathbb{R}$.

Proof. By Theorem 4.4, we have $(A(t), A'(t)) \in \mathcal{D}_\kappa(E, X)$, for all $t \in \mathbb{R}$, and thus, (9.2) follows by definition of $C_\kappa(E, X)$.

The first equation of (9.1) holds because

$$x'(t) = (\tfrac{1}{2}|A(t)|^2)' = \big\langle A(t), A'(t)\big\rangle = y(t)$$

and $\partial_y \Phi_\kappa(x, y; E, X) = y$.

To verify the second, we compute using (3.3), Definition 3.12, and Lemma 6.1

$$\begin{aligned} y'(t) &= x''(t) \\ &= \big\langle A''(t), A(t)\big\rangle + |A'(t)|^2 \\ &= \Big\langle -\kappa A(t) + \Lambda_\kappa(A(t), A'(t))A(t)^{-\top}, A(t)\Big\rangle + |A'(t)|^2 \\ &= -\kappa|A(t)|^2 + 2\Lambda_\kappa(A(t), A'(t)) + |A'(t)|^2 \\ &= -2\kappa|A(t)|^2 + 2E_\kappa(A(t), A'(t)) + 2\Lambda_\kappa(A(t), A'(t)) \\ &= -4\kappa x(t) + 2E + \frac{2E - X_1X_2}{x(t)^2}. \end{aligned} \tag{9.3}$$

A short algebraic manipulation using Defnition 8.1 confirms that

$$\begin{aligned} -4\kappa x + 2E + \frac{2E - X_1X_2}{x^2} &= \frac{xP_\kappa'(x; E, X) - P_\kappa(x; E, X)}{2x^2} \\ &= -\partial_x \Phi_\kappa(x, y; E, X), \end{aligned}$$

for all (x, y) with $x \geq 1$, which completes the verification of (9.1). □

Remark 9.2. Observe that (9.1) has a Hamiltonian structure. The key result (9.2) will allow us understand the behavior of the orbits $(x(t), y(t))$ of (9.1) corresponding to solutions of (3.3) by studying the sets $C_\kappa(E, X)$.

Remark 9.3. When $\kappa = 0$,

$$x'' = \frac{2E(x-1)^2 + (4Ex - X_1X_2)}{x^2} \geq 0$$

with equality if and only if $x = 1$, $X_1 = X_2$, $4E = X_1^2$, (cf. Lemma 13.1).

Corollary 9.4. *Fix $\kappa \geq 0$ and $(E, X) \in [e_\kappa(X), \infty) \times \mathbb{R}^2$. For any initial data $(x(0), y(0)) \in C_\kappa(E, X)$, the IVP for (9.1) has a unique global solution $(x, y) \in C^1(\mathbb{R}, \mathbb{R}^2)$ with $(x(t), y(t)) \in C_\kappa(E, X)$, for all $t \in \mathbb{R}$.*

Proof. Given $(x(0), y(0)) \in C_\kappa(E, X)$, use Lemma 8.7 to find data $(A_0, B_0) \in \mathcal{D}_\kappa(E, X)$ such that

$$\mathcal{P}(A_0, B_0) = (x(0), y(0)).$$

Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2, \mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be the solution of (3.2), (3.3), (3.4) with this data. By Theorem 9.1, $(x(t), y(t)) = \mathcal{P}(A(t), A'(t))$ is the desired solution. □

Remark 9.5. While the quantity $\Phi_\kappa(x(t), x'(t); E, X)$ is conserved along all solutions of the reduced system (9.1), we emphasize that only the portion of the zero level set in $C_\kappa(E, X)$ corresponds to solutions of the full system (3.3).

Lemma 9.6. *Fix $\kappa \geq 0$, $(E, X) \in [e_\kappa(X), \infty) \times \mathbb{R}^2$, with $(\kappa, E, X) \neq 0$.*

If $C_\kappa(E, X)$ does not contain a critical point of $\Phi_\kappa(x, y; E, X)$, then it is a smooth curve in $\mathbb{R}^2$ consisting of a single orbit of (9.1)*.*

If $C_\kappa(E, X)$ contains a single critical point p of $\Phi_\kappa(x, y; E, X)$, then each component of $C_\kappa(E, X) \setminus \{p\}$ is a smooth curve in $\mathbb{R}^2$ consisting of a single orbit of (9.1)*.*

If γ is a nontrivial orbit of (9.1) *in $C_\kappa(E, X)$, then either γ is a closed orbit or its alpha- and omega-limit sets are subsets of $\{p\}$, where p is a critical point in $C_\kappa(E, X)$.*

Proof. Since $(\kappa, E, X) \neq 0$, $C_\kappa(E, X)$ can contain at most one critical point of $\Phi_\kappa(x, y; E, X)$, by Corollary 8.9.

Suppose that $C_\kappa(E, X)$ contains no critical points of $\Phi_\kappa(x, y; E, X)$. Then the orbit through each point of $C_\kappa(E, X)$ is open in $C_\kappa(E, X)$. Since $C_\kappa(E, X)$ is connected, it can contain only one orbit. If $C_\kappa(E, X)$ contains a critical point p of $\Phi_\kappa(x, y; E, X)$, then the same argument is valid on each component of $C_\kappa(E, X) \setminus \{p\}$. These nontrivial orbits are C^1 curves, by (9.1).

Let γ be a nontrivial orbit in $C_\kappa(E, X)$. Since $C_\kappa(E, X)$ is a closed set, it contains $\omega(\gamma)$, the omega-limit set of γ. If $\omega(\gamma) \neq \emptyset$, then it is an invariant set for (9.1). If $\gamma \cap \omega(\gamma) \neq \emptyset$, then $\gamma \subset \omega(\gamma)$. In this case, γ must be a closed orbit. Here's the proof: We can write

$$\gamma = \{\varphi(t) = (x(t), y(t)) : t \in \mathbb{R}\},$$

for some solution $(x(t), y(t))$ of (9.1). $\varphi(0)$ is not a critical point, so by the implicit function theorem, there exists an $\varepsilon > 0$ and a neighborhood N of $\varphi(0)$ such that

$$\{\varphi(t) : t \in (-\varepsilon, \varepsilon)\} = \{(x, y) \in \mathbb{R}^2 : \Phi_\kappa(x, y; E, X) = 0\} \cap N.$$

Since $\varphi(0) \in \omega(\gamma)$, there exists a sequence $t_j \to \infty$ such that $\varphi(t_j) \to \varphi(0)$. Thus, since $\varphi(t) \in C_\kappa(E, X)$ for all $t \in \mathbb{R}$, there exists a $t_j > \varepsilon$ such that

$$\varphi(t_j) \in \{(x, y) \in \mathbb{R}^2 : \Phi_\kappa(x, y; E, X) = 0\} \cap N.$$

It follows that there exists $\tau \in (-\varepsilon, \varepsilon)$ such that $\gamma(t_j) = \gamma(\tau)$. This proves that γ is a closed orbit.

If $q \in C_\kappa(E, X) \setminus \gamma$ is not a critical point, then its orbit, call it η, is an open subset of $C_\kappa(E, X)$. This implies that $\eta \cap \omega(\gamma) = \emptyset$, and so $q \notin \omega(\gamma)$. Therefore, we have either $\gamma \cap \omega(\gamma) \neq \emptyset$, in which case γ is closed, or $\omega(\gamma) \cap \gamma = \emptyset$, in which case $\omega(\gamma)$ can only contain critical points of $\Phi_\kappa(x, y; E, X)$ in $C_\kappa(E, X)$. The same argument applies for $\alpha(\gamma)$. □

10. Special solutions

Equilibria

Lemma 10.1. *Fix $\kappa \geq 0$. A solution of* (3.2)*,* (3.3)*,* (3.4) *is an equilibrium if and only the initial data satisfies $(A_0, B_0) \in \mathcal{D}_\kappa(\kappa, 0)$.*

Proof. First, we note that

$$\mathcal{D}_0(0,0) = \{(A,B) \in \mathcal{D} : A \in \mathrm{SL}(2,\mathbb{R}),\ B = 0\},$$

and for $\kappa > 0$,

$$\mathcal{D}_\kappa(\kappa,0) = \{(A,B) \in \mathcal{D} : A \in \mathrm{SO}(2,\mathbb{R}),\ B = 0\}.$$

Moreover, if $A(t) = A_0$ is an equilibrium solution, then $A'(t) = 0 = B_0$.

Suppose first that $\kappa = 0$. If $A(t)$ is an equilibrium solution, then $(A_0, B_0) = (A_0, 0) \in \mathcal{D}_0(0,0)$. Conversely, if $(A_0, B_0) \in \mathcal{D}_0(0,0)$, then $B_0 = 0$ implies that $\Lambda_0(A_0, B_0) = 0$, and so $A(t) = A_0$ is an equilibrium solution of (3.3).

Now suppose that $\kappa > 0$. Then $A(t) = A_0$ is an equilibrium solution of (3.3) if and only if

$$\kappa A_0 = \Lambda_\kappa(A_0, 0) \operatorname{cof} A_0.$$

By Lemma 6.1, this is equivalent to

$$\kappa A_0 = \frac{4E_\kappa(A_0,0)}{|A_0|^4} \operatorname{cof} A_0 = \frac{2\kappa}{|A_0|^2} \operatorname{cof} A_0.$$

Taking the norm of both sides gives $|A_0|^2 = 2$, so that $A_0 \in \mathrm{SO}(2,\mathbb{R})$. Conversely, by Lemma 1.13, we see that A_0 is an equilibrium solution if $A_0 \in \mathrm{SO}(2,\mathbb{R})$. Thus, when $\kappa > 0$, all equilibrium solutions correspond to initial data $(A_0, 0)$ with $A_0 \in \mathrm{SO}(2,\mathbb{R})$, i.e. $(A_0, B_0) \in \mathcal{D}_\kappa(\kappa, 0)$. □

Remark 10.2. By Lemmas 7.2 and 7.3, equilibrium solutions of (3.3) are those which minimize the energy over $\mathcal{D}$.

Rigid motion

Definition 10.3. A solution $A(t)$ of the system (3.3) shall be called *rigid* if $(x(t), y(t)) = \mathcal{P}(A(t), A'(t))$ is an equilibrium solution of (9.1), or equivalently, if $\mathcal{P}(A(t), A'(t)) = (x, 0)$ for some constant $x \geq 1$.

Remark 10.4. Equilibrium solutions of (3.3) are also rigid solutions.

Remark 10.5. If $A(t)$ is rigid with $\frac{1}{2}|A(t)|^2 = x$, for some constant $x \geq 1$, then the fluid domains are ellipses with principal axes of fixed lengths, i.e. $z \mapsto A(t)z$ is a rigid motion (cf. Remark 3.2).

Lemma 10.6. *A solution of the IVP* (3.2)*,* (3.3)*,* (3.4)

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

with initial data $(A_0, B_0) \in \mathcal{D}_\kappa(E, X)$ is rigid if and only if $\mathcal{P}(A_0, B_0)$ is a critical point of $\Phi_\kappa(x, y; E, X)$ in $C_\kappa(E, X)$.

In particular, initial data $(A_0, B_0) \in \mathcal{D}_\kappa(e_\kappa(X), X)$, $\kappa > 0$, yields a rigid solution.

Proof. If $A(t)$ is rigid, then

$$\mathcal{P}(A(t), A'(t)) = (x_0, 0) = \mathcal{P}(A_0, B_0) \in C_\kappa(E, X)$$

is an equilibrium solution of (9.1). Thus, $(x_0, 0)$ is a critical point of Φ_κ.

Next suppose that $(x_0, 0) = \mathcal{P}(A_0, B_0)$ is a critical point of Φ_κ in $C_\kappa(E, X)$. By Corollary 9.4, $(x(t), y(t)) = (x_0, 0)$ is the unique solution of (9.1) with data $(x_0, 0)$. Let $A(t)$ be the solution of the IVP (3.2), (3.3), (3.4) with initial data (A_0, B_0). By Theorem 9.1, $(x(t), y(t)) = \mathcal{P}(A(t), A'(t))$ solves (9.1) with data $(x_0, 0)$. Thus, $\mathcal{P}(A(t), A'(t)) = (x_0, 0)$, and so $A(t)$ is rigid.

The final statement is a consequence of Lemma 8.10. □

Next, we consider the special case of rigid motion in $\mathrm{SO}(2,\mathbb{R})$ which will play a special role in what follows.

Lemma 10.7. *Fix $\kappa \geq 0$. The following statements are equivalent:*

i. The function $A(t)$ is a solution of (3.3) *in*

$$C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

whose initial data satisfies

$$(A_0, B_0) \in \mathcal{D}_\kappa(E, X), \quad \text{with} \quad X_1 = X_2, \quad E = \kappa + \tfrac{1}{4} X_1^2,$$

and $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, for some $t_0 \in \mathbb{R}$.

ii. The function $A(t)$ is a rigid solution of (3.3) *in*

$$C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

with $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$ for some $t_0 \in \mathbb{R}$.

iii. The function $A(t)$ is a solution of (3.3) *in*

$$C^0(\mathbb{R}, \mathrm{SO}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2).$$

iv. The function $A(t)$ is given by

$$A(t) = U\left(\tfrac{1}{2} X_1 t + \theta\right) = \exp\left[\left(\tfrac{1}{2} X_1 t + \theta\right) Z\right],$$

for some $\theta,\ X_1 \in \mathbb{R}$.

Proof. We shall prove the implications cyclically.

Suppose that (i) holds. The conditions on the invariants (E, X) imply that $P_\kappa(1; E, X) = P'_\kappa(1; E, X) = 0$. By Lemma 8.8, $(1, 0)$ is a critical point of $\Phi_\kappa(x, y; E, X)$ in $C_\kappa(E, X)$, and so it is an equilibrium solution of (9.1). Since $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, Lemma 8.5 says that $\mathcal{P}(A(t_0), A'(t_0)) = (1, 0)$. By Theorem 9.1, $\mathcal{P}(A(t), A'(t))$ is a solution of (9.1), and by uniqueness, it must be equal to the equilibrium solution $(1, 0)$. Thus, $A(t)$ is rigid.

Suppose next that (ii) holds. Since $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, Lemma 8.5 says that $\mathcal{P}(A(t_0), A'(t_0)) = (1,0)$. Since $A(t)$ is rigid, we have

$$\mathcal{P}(A(t), A'(t)) = \mathcal{P}(A(t_0), A'(t_0)) = (1,0),$$

for all $t \in \mathbb{R}$. Thus, $A(t) \in \mathrm{SO}(2,\mathbb{R})$ for all $t \in \mathbb{R}$, Lemma 8.5.

Suppose next that (iii) holds. Differentiating the identity

$$A(t)A(t)^\top = I,$$

we find that $A'(t)A(t)^\top$ is anti-symmetric. Note that since $A(t) \in \mathrm{SO}(2,\mathbb{R})$, we have

$$L(A(t), A'(t)) = A'(t)A(t)^{-1} = A'(t)A(t)^\top$$

is anti-symmetric. Thus, using Corollary 4.6 and Lemma 4.3, we obtain

$$L(A(t), A'(t)) = \tfrac{1}{2}X_2 Z = \tfrac{1}{2}X_1 Z,$$

and so, we see that

$$A'(t) = \tfrac{1}{2}X_1 Z A(t).$$

The explicit solution is

$$A(t) = \exp\left[\tfrac{1}{2}X_1(t-t_0)\, Z\right] A(t_0) = U\left(\tfrac{1}{2}X_1 t\right) U\left(-\tfrac{1}{2}X_1 t_0\right) A(t_0).$$

Since $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, we may use Lemma 1.9 to write

$$U\left(-\tfrac{1}{2}X_1 t_0\right) A(t_0) = U(\theta),$$

for some $\theta \in \mathbb{R}$. This leads to the desired formula.

If (iv) statement holds, then (i) follows by direct calculation using the explicit formula for $A(t)$. $\square$

Remark 10.8. Observe that solutions in $\mathrm{SO}(2,\mathbb{R})$ are periodic.

Remark 10.9. Even and odd dimensions are fundamentally different. There are no nontrivial solutions of the equation (3.1) in the form $A(t) = \exp(Wt)$ with W anti-symmetric in odd dimensions.

Definition 10.10. For each $X_1 \in \mathbb{R}$, define

$$\mathcal{R}(X_1) = \{(U, \tfrac{1}{2}X_1 Z U) : U \in \mathrm{SO}(2,\mathbb{R})\}.$$

Lemma 10.11. *For each $\kappa \geq 0$ and $X_1 \in \mathbb{R}$, $\mathcal{R}(X_1)$ coincides with the orbit of a rigid rotational solution of* (3.3)*.*

If $X = (X_1, X_1)$ and $E = \kappa + \frac{1}{4}X_1^2$, then $\mathcal{R}(X_1) \subset \mathcal{D}_\kappa(E, X)$.

Additionally, $\mathcal{R}(X_1) = \mathcal{D}_\kappa(E, X)$ if and only if $E = e_\kappa(X)$.

Proof. Let $\kappa \geq 0$ and $X_1 \in \mathbb{R}$. Set $A(t) = U\left(\frac{1}{2}X_1 t\right)$, $t \in \mathbb{R}$. By Lemma 10.7 (iv), $A(t)$ is a rigid rotational solution of (3.3). Its orbit

$$(A(t), A'(t)) = \left(A(t), \tfrac{1}{2}X_1 A(t)\right)$$

is equal to $\mathcal{R}(X_1)$, since, by Lemma 1.9, $A(t)$ parameterizes $\mathrm{SO}(2,\mathbb{R})$.

The second statement follows from Lemma 10.7 (i).

Finally, we show that the inclusion is an equality if and only if $E = e_\kappa(X)$.

Note that $\mathcal{P}(\mathcal{R}(X_1)) = \{(1,0)\}$. Thus, we have

$$\{(1,0)\} = \mathcal{P}(\mathcal{R}(X_1)) \subset \mathcal{P}(\mathcal{D}_\kappa(E,X)) = C_\kappa(E,X). \tag{10.1}$$

If $E > e_\kappa(X)$, then $C_\kappa(E,X)$ is not a singleton, by Lemma 8.10, and we see that $\mathcal{D}_\kappa(E,X) \setminus \mathcal{R}(X_1) \neq \emptyset$.

If, on the other hand, $E = e_\kappa(X)$, then $C_\kappa(E,X)$ is a singleton, and (10.1) implies that $C_\kappa(E,X) = \{(1,0)\}$. If $(A,B) \in \mathcal{D}_\kappa(E,X)$, then $\mathcal{P}(A,B) = (1,0)$. By Lemma 8.5, $A \in \mathrm{SO}(2,\mathbb{R})$, and by Lemma 5.5, $B = \frac{1}{2}X_1 Z A$, since $E = e_\kappa(X)$. This shows that $(A,B) \in \mathcal{R}(X_1)$, and so $\mathcal{D}_\kappa(E,X) \subset \mathcal{R}(X_1)$. □

Remark 10.12. Later, we shall see that the invariant manifolds $\mathcal{R}(X_1)$ are hyperbolic, (cf. Corollaries 12.6 and 13.9).

Solutions with vanishing pressure

Lemma 10.13. *Let $(A_0, B_0) \in \mathcal{D}_\kappa(E,X)$ with $2E - X_1X_2 = 0$.*

If $\kappa = 0$, then

$$A(t) = B_0 t + A_0$$

is the solution of (3.2)*,* (3.3)*,* (3.4) *in* $C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$.

If $\kappa > 0$, then

$$A(t) = (\cos\sqrt{\kappa}t)A_0 + \tfrac{1}{\sqrt{\kappa}}(\sin\sqrt{\kappa}t)B_0$$

is the solution of (3.2)*,* (3.3)*,* (3.4) *in* $C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$.

Proof. Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be the solution of (3.3) with data in $\mathcal{D}_\kappa(E,X)$. If $2E - X_1X_2 = 0$, then by Lemmas 4.4 and 6.1, we have that

$$\Lambda_\kappa(A(t), A'(t)) = 0, \quad \text{for all} \quad t \in \mathbb{R}. \tag{10.2}$$

The formulas follow directly by solving the IVP for the linear equation resulting from (3.3)

$$A'' + \kappa A = 0.$$

□

Remark 10.14. The condition $2E - X_1X_2 = 0$ implies equation (10.2) which in turn implies that the pressure vanishes, by Lemmas 3.7 and 3.11.

Remark 10.15. When $\kappa > 0$, the solution $A(t)$ is T-periodic, with $2T = \pi/\sqrt{\kappa}$, while

$$|A(t)|^2 = \tfrac{1}{2}(1+\cos 2\sqrt{\kappa}t)|A_0|^2 + \tfrac{1}{\sqrt{\kappa}}\sin 2\sqrt{\kappa}t\,\langle A_0, B_0\rangle + \tfrac{1}{2\kappa}(1-\cos 2\sqrt{\kappa}t)|B_0|^2$$

is T-periodic. $C_\kappa(E,X)$ is an ellipse given by:

$$y^2 + 4\kappa(x - E/2\kappa)^2 = 4\kappa + E^2/\kappa - X_1^2 - X_2^2.$$

By the Cauchy-Schwarz inequality and the condition $2E - X_1X_2 = 0$, we have $|X_i| \leq E/\sqrt{\kappa} = X_1X_2/(2\sqrt{\kappa})$. We see that $|X_i| \geq 2\sqrt{k}$, and therefore the right-hand side is nonnegative:

$$\frac{1}{4\kappa}(X_1^2 - 4\kappa)(X_2^2 - 4\kappa) \geq 0.$$

Remark 10.16. When $\kappa = 0$, $A(t)$ is a line in $\mathrm{SL}(2,\mathbb{R})$. $C_0(E,X)$ is a parabola given by:

$$y^2 - 2X_1X_2x + X_1^2 + X_2^2 = 0.$$

11. Reconstruction

We now show that solutions $A(t)$ of the system (3.3) can be recovered from knowledge of its phase plane curve $\mathcal{P}(A(t), A'(t))$ and its initial data (3.2), using local coordinates. The proof is complicated by the coordinate singularity on $\mathrm{SO}(2,\mathbb{R})$.

In order to avoid repetition, we enforce the following standing assumption throughout this section:

(**A**) The parameter $\kappa \geq 0$ and the invariants $(E,X) \in [e_\kappa(X),\infty)\times\mathbb{R}^2$ are fixed, and the initial data satisfies $(A_0,B_0) \in \mathcal{D}_\kappa(E,X)$.

Lemma 11.1. *Suppose that* (**A**) *holds. If* $A_0 \in \mathrm{SL}(2,\mathbb{R})\setminus\mathrm{SO}(2,\mathbb{R})$*, then there exists*

$$s(0) = (s_1(0), s_2(0), s_3(0)) \in \mathbb{R}^2 \times [1,\infty)$$

such that

$$A_0 = \mathcal{A}(s(0)) \quad \text{and} \quad \tfrac{1}{2}|A_0|^2 = s_3(0),$$

where $\mathcal{A}(s)$ *was defined in Lemma 2.14. Moreover, there holds*

$$B_0 = \frac{X_1+X_2}{g_{11}(A_0)}\tau_1(A_0) + \frac{X_1-X_2}{g_{22}(A_0)}\tau_2(A_0) + \langle A_0, B_0\rangle\,\tau_3(A_0).$$

If $A_0 \in \mathrm{SO}(2,\mathbb{R})$*, then*

$$X_1 = X_2, \quad E \geq \kappa + \tfrac{1}{4}X_1^2,$$

and there exists

$$s(0) = (s_1(0), s_2(0), s_3(0)) \in \mathbb{R}^2 \times [1,\infty)$$

such that

$$A_0 = U(2s_1(0)), \quad \tfrac{1}{2}|A_0|^2 = 1 = s_3(0),$$

and

$$B_0 = \tfrac{1}{2}X_1\, U(2s_1(0))\, Z + \beta\, U(2s_2(0))\, M,$$

with

$$\beta = \left(E - \kappa - \tfrac{1}{4}X_1^2\right)^{1/2}.$$

Proof. This is just a summary of the results contained in Lemmas 2.12, 5.4, and 5.5. □

Lemma 11.2. *Suppose that* (**A**) *holds. There exists a unique curve* $s = (s_1, s_2, s_3) \in C^2(\mathbb{R}, \mathbb{R}^2 \times [1,\infty))$ *such that* $(s_3(t), s_3'(t))$ *solves* (9.1) *with initial data* $(s_3(0), s_3'(0)) = \mathcal{P}(A_0, B_0)$ *and* $(s_1(t), s_2(t))$ *solves*

$$\begin{aligned} s_1'(t) &= \frac{X_1 + X_2}{4(s_3(t)+1)} \\ s_2'(t) &= \begin{cases} \dfrac{X_1 - X_2}{4(s_3(t)-1)}, & \text{if } X_1 \neq X_2 \\ 0, & \text{if } X_1 = X_2, \end{cases} \end{aligned} \tag{11.1}$$

with initial data $(s_1(0), s_2(0))$ *defined by Lemma 11.1.*

If $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ *solves the IVP* (3.2), (3.3), (3.4) *with initial data* (A_0, B_0)*, then* $\mathcal{P}(A(t), A'(t)) = (s_3(t), s_3'(t))$.

Proof. The existence and uniqueness of a solution $(x, y) \in C^0(\mathbb{R}, \mathbb{R}^2)$ to (9.1) with initial data $\mathcal{P}(A_0, B_0)$ is given Corollary 9.4. Since the first equation of (9.1) says that $x'(t) = y(t)$, we can label the solution as (s_3, s_3'). The proof of the corollary also shows that $\mathcal{P}(A(t), A'(t)) = (s_3(t), s_3'(t))$.

We know that $(s_3(t), s_3'(t)) \in C_\kappa(E, X)$, for all $t \in \mathbb{R}$. If $X_1 \neq X_2$, then $s_3(t) > 1$, by Lemma 8.7, so the right-hand sides of (11.1) are well-defined known functions. The solutions s_1 and s_2 are obtained by integration. □

Remark 11.3. The value of $s_3(0)$ is consistent in Lemmas 11.1 and 11.2.

Lemma 11.4. *Suppose that* (**A**) *holds. Let* $s \in C^0(\mathbb{R}, \mathbb{R}^2 \times [1,\infty))$ *be the curve constructed in Lemma 11.2. If, on some open interval* I*, there holds* $s_3(t) > 1$*, then* $\mathcal{A} \circ s(t)$ *solves* (3.3) *on* I.

Proof. Define

$$\bar{A}(t) = \mathcal{A} \circ s(t).$$

Since $s \in C^2$ and $s_3(t) > 1$ on I, we see by the definition given in Lemma 2.14 that

$$\bar{A} \in C^0(I, \mathrm{SL}(2,\mathbb{R})) \cap C^2(I, \mathbb{M}^2) \quad \text{and} \quad s_3(t) = \tfrac{1}{2}|\bar{A}(t)|^2.$$

Thus, we have that

$$\bar{A}(t) \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R}), \quad t \in I,$$

by Lemma 1.7. It also follows that

$$s_3'(t) = \left\langle \bar{A}(t), \bar{A}'(t) \right\rangle, \quad t \in I,$$

and so by definition, we have

$$\mathcal{P}(\bar{A}(t), \bar{A}'(t)) = (s_3(t), s_3'(t)), \quad t \in I.$$

By Corollary 9.4, $(s_3(t), s_3'(t)) \in C_\kappa(E, X)$, and so

$$(\bar{A}(t), \bar{A}'(t)) \in \mathcal{D}_\kappa(E, X) = \mathcal{P}^{-1}(C_\kappa(E, X)), \quad t \in I. \tag{11.2}$$

In the following calculation, we suppress the dependence of functions upon the independent variable t in order to simplify the formulas. All calculations are valid on the interval I where we have assumed that $s_3 > 1$. Since the metric $g(\bar{A})$ depends only on $s_3 = \frac{1}{2}|\bar{A}|^2$, we shall write $g(s_3)$ for $g(\bar{A}) = g(\mathcal{A} \circ s)$, with abuse of notation.

Using the Christoffel symbols from Lemma 2.21 and the second fundamental form of Definition 2.27, a standard geometric calculation yields

$$\bar{A}'' = \sum_{i=1}^{3} \left[s_i'' + \sum_{j,k=1}^{3} \Gamma^i_{jk}(\bar{A}) s_j' s_k' \right] \tau_i(\bar{A}) + \sum_{j,k=1}^{3} \Pi[\tau_j(\bar{A}), \tau_k(\bar{A})] s_j' s_k' N(\bar{A}).$$

By Definitions 2.7 and 2.3, we have that for any $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$

$$A = \frac{1}{g_{33}(A)} \tau_3(A) + \frac{2}{|A|} N(A) \quad \text{and} \quad \operatorname{cof} A = |A| N(A).$$

It follows that

$$\begin{aligned}
&\bar{A}'' + \kappa \bar{A} - \Lambda_\kappa(\bar{A}, \bar{A}') \operatorname{cof} \bar{A} \\
&\quad = \sum_{i=1}^{3} \left[s_i'' + \sum_{j,k=1}^{3} \Gamma^i_{jk}(\bar{A}) s_j' s_k' \right] \tau_i(\bar{A}) + \frac{\kappa}{g_{33}(\bar{A})} \tau_3(\bar{A}) \\
&\qquad + \left[\sum_{j,k=1}^{3} \Pi[\tau_j(\bar{A}), \tau_k(\bar{A})] s_j' s_k' + \frac{2\kappa}{|\bar{A}|} - |\bar{A}| \Lambda_\kappa(\bar{A}, \bar{A}') \right] N(\bar{A}).
\end{aligned}$$

From this we see that $\bar{A}$ satisfies (3.3) on I if and only if the system

$$\begin{aligned}
&s_i'' + \sum_{j,k=1}^{3} \Gamma^i_{jk}(\bar{A}) s_j' s_k' = 0, \quad i = 1, 2 \\
&s_3'' + \sum_{j,k=1}^{3} \Gamma^3_{jk}(\bar{A}) s_j' s_k' + \frac{\kappa}{g_{33}(\bar{A})} = 0 \\
&\sum_{j,k=1}^{3} \Pi[\tau_j(\bar{A}), \tau_k(\bar{A})] s_j' s_k' + \frac{2\kappa}{|\bar{A}|} - |\bar{A}| \Lambda_\kappa(\bar{A}, \bar{A}') = 0
\end{aligned}$$

holds on I.

By (11.2), $(\bar{A}, \bar{A}') \in \mathcal{D}_\kappa(E, X)$, so Lemma 6.1 tells us that

$$\Lambda_\kappa(\bar{A}, \bar{A}') = \frac{2E - X_1 X_2}{2s_3^2}.$$

Using Lemmas 2.21 and 2.26, we find that our system is equivalent to

$$s_i'' + \frac{g_{ii}'(s_3)}{g_{ii}(s_3)} s_i' s_3' = 0, \quad i = 1, 2 \tag{11.3}$$

$$s_3'' - \frac{g_{11}'(s_3)}{2g_{33}(s_3)}(s_1')^2 - \frac{g_{22}'(s_3)}{2g_{33}(s_3)}(s_2')^2 + \frac{g_{33}'(s_3)}{2g_{33}(s_3)}(s_3')^2 + \frac{\kappa}{g_{33}(s_3)} = 0 \tag{11.4}$$

$$-\frac{g_{11}(s_3)}{2s_3}(s_1')^2 + \frac{g_{22}(s_3)}{2s_3}(s_2')^2 + \frac{g_{33}(s_3)}{2s_3^2}(s_3')^2 + \frac{\kappa}{s_3} - \frac{2E - X_1 X_2}{2s_3^2} = 0, \tag{11.5}$$

where as mentioned above $g(s_3) = g(\bar{A}) = g(\mathcal{A} \circ s)$ and $g'(s_3)$ indicates the derivative in s_3.

The equations (11.3) hold thanks to the definitions (11.1). Again using (11.1), we find after some computation that (11.5) is equivalent to the equation $\Phi_\kappa(s_3, s_3'; E, X) = 0$, which holds by Lemma 8.7, since $(s_3, s_3') \in C_\kappa(E, X)$ and $s_3 > 1$. Finally, (11.4) is equivalent to the equation

$$s_3'' + \partial_x \Phi_\kappa(s_3, s_3'; E, X) + \frac{g_{33}'(s_3)}{g_{33}(s_3)} \Phi_\kappa(s_3, s_3'; E, X) = 0, \tag{11.6}$$

which holds by (9.1) and the fact that $\Phi_\kappa(s_3, s_3'; E, X) = 0$. Thus, we have verified that $\bar{A} = \mathcal{A} \circ s$ solves (3.3) on I. □

Remark 11.5. Define the Lagrangian

$$\mathcal{L}(s, s') = \mathcal{L}(s_3, s') = \tfrac{1}{2} \sum_{i=1}^{3} g_{ii}(s_3)(s_i')^2 - \kappa s_3,$$

and the Hamiltonian

$$\mathcal{H}(s, p) = \mathcal{H}(s_3, p) = \tfrac{1}{2} \sum_{i=1}^{3} \frac{p_i^2}{g_{ii}(s_3)} + \kappa s_3.$$

Then (11.3), (11.4) are equivalent to

$$\frac{d}{dt} \mathcal{L}_{s_i'}(s, s') - \mathcal{L}_{s_i}(s_3, s') = 0, \quad i = 1, 2, 3. \tag{11.7}$$

With the Legendre transformation $p_i = g_{ii}(s_3) s_i'$, (11.7) is equivalent to

$$s_i' = \mathcal{H}_{q_i}(s, p), \quad p_i' = -\mathcal{H}_{s_i}(s, p), \quad i = 1, 2, 3.$$

We note that

$$g_{33}(s_3) \Phi_\kappa(s_3, s_3'; E, X) = \mathcal{H}(s_3, X_1 + X_2, X_1 - X_2, g_{33}(s_3) s_3') - E.$$

Thus, the equation $p_3' = -\mathcal{H}_{s_3}(s, q)$, is equivalent to (11.6).

Theorem 11.6. *Suppose that* $(\mathbf{A})$ *holds, and let*

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be the solution of (3.2), (3.3), (3.4) *with initial data* (A_0, B_0). *Let* $s \in C^0(\mathbb{R}, \mathbb{R}^2 \times [1,\infty))$ *be the curve constructed in Lemma 11.2.*

If $(1,0) \notin C_\kappa(E,X)$ *or if* $(1,0) \in C_\kappa(E,X)$ *is a critical point of* $\Phi_\kappa(x,y;E,X)$, *then*

$$A(t) = \mathcal{A} \circ s(t), \quad \textit{for all} \quad t \in \mathbb{R}.$$

Proof. By Corollary 9.4, we have $(s_3(t), s_3'(t)) \in C_\kappa(E,X)$, for all $t \in \mathbb{R}$. If $(1,0) \notin C_\kappa(E,X)$, then $s_3(t) > 1$, for all $t \in \mathbb{R}$.

If $(1,0) \in C_\kappa(E,X)$ is a critical point of $\Phi_\kappa(x,y;E,X)$, then $(1,0)$ is an equilibrium solution of (9.1). Thus, if $(s_3(t_0), s_3'(t_0)) = (1,0)$, at a single time t_0, then $s_3(t) = 1$, for all $t \in \mathbb{R}$. Otherwise, $(s_3(t), s_3'(t)) \in C_\kappa(E,X) \setminus \{(1,0)\}$, and we obtain $s_3(t) > 1$, for all $t \in \mathbb{R}$.

If $s_3(t) = 1$, for all $t \in \mathbb{R}$, then $A(t) \in \mathrm{SO}(2,\mathbb{R})$, for all $t \in \mathbb{R}$, so $X_1 = X_2$, by Lemma 4.3. By Lemma 10.7, we have that

$$A(t) = U(\tfrac{1}{2}X_1 t + \theta),$$

for some $\theta \in \mathbb{R}$. Since

$$U(\theta) = A(0) = U(2s_1(0)),$$

we may take $\theta = 2s_1(0)$.

On the other hand, we can calculate the function $s_1(t)$ directly from (11.1), and we find

$$s_1(t) = \tfrac{1}{4}X_1 t + s_1(0).$$

Since $s_3(t) = 1$, the formula in Lemma 2.14 reduces to

$$\mathcal{A} \circ s(t) = U(s_1(t) + s_2(t))\; I\; U(s_1(t) - s_2(t)) = U(2s_1(t)).$$

This shows that $A(t) = \mathcal{A} \circ s(t)$, for all $t \in \mathbb{R}$, when $s_3(t) = 1$.

Now let us assume that $s_3(t) > 1$, for all $t \in \mathbb{R}$. Define

$$\bar{A}(t) = \mathcal{A} \circ s(t).$$

By applying Lemma 11.4 on the interval $I = \mathbb{R}$, we see that

$$\bar{A} \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

is a solution of (3.3).

We now check the initial data of $\bar{A}$. By Lemma 11.1 we have

$$\bar{A}(0) = \mathcal{A} \circ s(0) = A_0.$$

By Lemmas 2.14 and 11.1, we have

$$\bar{A}'(0) = (\mathcal{A} \circ s)'(0) = \sum_{i=1}^{3} s_i'(0)\tau_i(\mathcal{A} \circ s(0)) = \sum_{i=1}^{3} s_i'(0)\tau_i(A_0).$$

From (11.1), we see that

$$s_1'(0) = \frac{X_1 + X_2}{g_{11}(A_0)} \quad \text{and} \quad s_2'(0) = \frac{X_1 - X_2}{g_{22}(A_0)}.$$

Moreover, $s_3'(0) = \langle A_0, B_0 \rangle$, by definition. Thus, from Lemma 11.1 we find that $\bar{A}'(0) = B_0$

Having shown that $\bar{A}$ solves (3.3) with the same initial data as A, we conclude that $\bar{A} = \mathcal{A} \circ s = A$, by uniqueness of solutions to the IVP. □

Lemma 11.7. *Suppose that* (**A**) *holds. Suppose that* $(1,0) \in C_\kappa(E,X)$ *and* $(1,0)$ *is not a critical point of* $\Phi_\kappa(x,y;E,X)$*. Let*

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be a solution of (3.2)*,* (3.3)*,* (3.4) *with initial data* (A_0, B_0)*.*

If $\kappa = 0$*, then there exists a unique* $t_0 \in \mathbb{R}$ *such that*

$$\mathcal{P}(A(t_0), A'(t_0)) = (1,0).$$

If $\kappa > 0$*, then* $\mathcal{P}(A(t), A'(t))$ *is periodic with minimum period* $T > 0$*. Morevoer, there exists a unique* $t_0 \in \mathbb{R}$ *such that* $0 \in [t_0, t_0 + T)$ *and*

$$\{t \in \mathbb{R} : \mathcal{P}(A(t), A'(t)) = (1,0)\} = \{t_j = t_0 + jT : j \in \mathbb{Z}\}.$$

Proof. By Lemma 9.6, $C_\kappa(E,X)$ consists of a single smooth orbit

$$(x(t), y(t)) = \mathcal{P}(A(t), A'(t)).$$

Thus, by Lemma 8.12, there exists $t_0 \in \mathbb{R}$ such that $(x(t_0), y(t_0)) = (1,0)$. If $\kappa = 0$, this orbit is unbounded, so it is not closed, and t_0 is the unique time with this property. If $\kappa > 0$, then the orbit is closed and therefore periodic with minimal period $T > 0$. Since T is a minimal period, we have that the set $\{t_0 + jT\}$, $j \in \mathbb{Z}$, coincides with the set of times t where $(x(t), y(t)) = (1,0)$. We can redefine t_0, if necessary, so that $0 \in [t_0, t_0 + T)$. □

Theorem 11.8. *Suppose that* (**A**) *holds, and let*

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be the solution of (3.2)*,* (3.3)*,* (3.4) *with initial data* (A_0, B_0)*. Let* $s(t) \in C^0(\mathbb{R}, \mathbb{R}^2 \times [1,\infty))$ *be the curve constructed in Lemma 11.2.*

If $(1,0) \in C_\kappa(E,X)$ *and* $(1,0)$ *is not a critical point of* $\Phi_\kappa(x,y;E,X)$*, then*

$$A(t) = (\mathrm{cof})^{n(t)} \mathcal{A}(s(t)),$$

where $n(t) = 0, 1$ *is the piece-wise constant right continuous function with* $n(0) = 0$ *and jump discontinuities on the set* $\{t_j\}$ *from Lemma 11.7.*

Proof. Define $\bar{A}(t) = (\mathrm{cof})^{n(t)}\mathcal{A}(s(t))$. The goal is to prove that $A = \bar{A}$, using the same uniqueness argument as in the proof of Theorem 11.6.

Note that $s \in C^2(\mathbb{R})$, and so $\mathcal{A} \circ s \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$. Since the cofactor map leaves $\mathrm{SL}(2,\mathbb{R})$ invariant, we see that $\mathrm{cof}\,\mathcal{A} \circ s \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$, as well. Thus, $\bar{A}$ is continuous, except possibly at the points $\{t_j\}$. However, at the points $\{t_j\}$, we have $s_3 = 1$, and so by Lemma 1.7, $\mathcal{A} \circ s(t_j) \in \mathrm{SO}(2,\mathbb{R})$. By Lemma 1.13, $\mathcal{A} \circ s(t_j) = \mathrm{cof}\,\mathcal{A} \circ s(t_j)$, so we see that $\bar{A} \in C^0(\mathbb{R}; \mathrm{SL}(2,\mathbb{R}))$.

Examining the definition of $\mathcal{A} \circ s$, we see that this function could fail to be differentiable at the times t_j when $s_3(t_j) = 1$, because of the term $\sqrt{s_3(t) - 1}$.

Let us suppose first that $\kappa = 0$. Then by Lemma 11.7, there exists a single time $t_0 \in \mathbb{R}$ such that $s(t_0) = 1$. Assume that $0 \in [t_0, \infty)$ so that

$$\bar{A}(t) = \begin{cases} \mathrm{cof}\,\mathcal{A} \circ s(t), & t < t_0 \\ \mathcal{A} \circ s(t), & t \geq t_0. \end{cases}$$

(If $0 \in (-\infty, t_0)$, then the cofactor would be applied on the other interval.) Now $s_3(t_0) = 1$ is a minimum value for s_3, so $s_3'(t_0) = 0$ and $s_3''(t_0) \geq 0$. Since $(1,0)$ is not a critical point of $\Phi_\kappa(x, y; E, X)$, $P_\kappa(x; E, X)$ has a simple root at $x = 1$, by Lemma 8.8. Since $(s_3(t), s_3'(t))$ satisfies (9.1), we have $s_3''(t_0) = -P_\kappa'(1; E, X)/2 \neq 0$. Thus, $s_3''(t_0) > 0$, and we can write

$$s_3(t) - 1 = \alpha(t)(t - t_0)^2, \quad \alpha \in C^2, \quad \alpha(t_0) = \tfrac{1}{2}s''(t_0) > 0. \tag{11.8}$$

Thus, $\alpha(t)$ is strictly positive in a neighborhood of $t = t_0$. From this we see that the function

$$\sqrt{\alpha(t)}(t - t_0) = \begin{cases} -\sqrt{s_3(t) - 1}, & \text{if } t < t_0 \\ \sqrt{s_3(t) - 1}, & \text{if } t \geq t_0 \end{cases}$$

is C^2 in a neighborhood of $t = t_0$. (If $t_0 \in (-\infty, 0)$, then the signs of the two terms above would be reversed.) This shows that

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{s_3(t) + 1} & \sqrt{\alpha(t)}(t - t_0) \\ \sqrt{\alpha(t)}(t - t_0) & \sqrt{s_3(t) + 1} \end{bmatrix} &= \begin{cases} \mathrm{cof}\,H(s_3(t)), & \text{if } t < t_0 \\ H(s_3(t)), & \text{if } t \geq t_0 \end{cases} \\ &= (\mathrm{cof})^{n(t)} H(s_3(t)). \end{aligned} \tag{11.9}$$

This function belongs to C^2 for t near t_0. Finally, we conclude that

$$\bar{A}(t) = U(s_1(t) + s_2(t))\,(\mathrm{cof})^{n(t)} H(s_3(t))\,U(s_1(t) - s_2(t))$$

belongs to $C^2(\mathbb{R}, \mathbb{M}^2)$.

If $\kappa > 0$, then the set $\{t_j\}$ is countable and a repetition of the argument of the previous paragraph near each t_j again shows that $\bar{A} \in C^2(\mathbb{R}, \mathbb{M}^2)$.

Next, we show that $\bar{A}$ solves (3.3).

Suppose that $s_3(t) > 1$ on some open interval I. Then by Lemma 11.4, $\mathcal{A} \circ s \in C^0(I; \mathrm{SL}(2,\mathbb{R})) \cap C^2(I, \mathbb{M}^2)$ solves (3.3) on I. Since the cofactor map leaves solutions of (3.3) invariant, we see that $\mathrm{cof}\, \mathcal{A} \circ s$ also solves (3.3) on I. Therefore, $\bar{A}$ solves (3.3) except on the at most countable set of isolated points $\{t_j\}$. Having shown that $\bar{A} \in C^2(\mathbb{R}, \mathbb{M}^2)$, it follows that $\bar{A}$ solves (3.3) on $\mathbb{R}$.

It remains to verify that A and $\bar{A}$ share the same initial data.

If $A_0 \notin \mathrm{SO}(2,\mathbb{R})$, then according to Lemma 11.1, our choice $s(0)$ gives $\bar{A}(0) = \mathcal{A} \circ s(0) = A_0$. Also, by Lemmas 11.1 and 11.2, we have

$$\bar{A}'(0) = (\mathcal{A} \circ s)'(0) = \sum_{i=1}^{3} s_i'(0) \tau_i(A_0) = B_0.$$

If $A_0 \in \mathrm{SO}(2,\mathbb{R})$, then $\mathcal{P}(A_0, B_0) = (s_3(0), s_3'(0)) = (1, 0)$. By Lemma 11.1, we have

$$A_0 = U(2s_1(0))$$

and

$$B_0 = (\tfrac{1}{2} X_1)\, U(2s_1(0))\, Z + \beta\, U(2s_2(0))\, M,$$

with

$$\beta = \left(E - \kappa - \tfrac{1}{4} X_1^2\right)^{1/2}.$$

Since $s_3(0) = 1$, we have $t_0 = 0$, and so

$$\bar{A}(0) = \mathcal{A} \circ s(0) = U(2s_1(0)) = A_0.$$

Going back to the formula (11.9), we have

$$\bar{A}(t) = \tfrac{1}{\sqrt{2}} U(s_1(t) + s_2(t)) \begin{bmatrix} \sqrt{s_3(t) + 1} & \sqrt{\alpha(t)}\, t \\ \sqrt{\alpha(t)}\, t & \sqrt{s_3(t) + 1} \end{bmatrix} U(s_1(t) - s_2(t)),$$

for $0 \le t < T$, where by (11.8), (11.6)

$$\alpha(0) = \tfrac{1}{2} s_1''(0) = -\tfrac{1}{2} \partial_x \Phi_\kappa(1, 0; E, X) = \tfrac{1}{4} P_\kappa'(1; E, X) = 2\beta^2.$$

Since $X_1 = X_2$, we have $s_2(t) = s_2(0)$, by (11.1). As in Lemma 2.18, this can be written as

$$\bar{A}(t) = \left(\frac{s_3(t) + 1}{2}\right)^{1/2} U(2s_1(t)) + t \left(\frac{\alpha(t)}{2}\right)^{1/2} U(2s_2(0))\, M,$$

$0 \le t < T$. Since $\bar{A}$ is C^2, it is enough to compute its right derivative at $t = 0$:

$$\bar{A}'(0) = U(2s_1(0))\, Z\, 2s_1'(0) + \beta\, U(2s_2(0))\, M = B_0,$$

by (11.1), as desired. □

Corollary 11.9. *A solution $A(t)$ of* (3.3) *is symmetric if and only if $s_1(t) = j\pi/2$, for some $j \in \mathbb{Z}$.*

12. MHD

In this section, we focus on the case where $\kappa > 0$. The next result summarizes the properties of the orbits $C_\kappa(E, X)$ of (9.1) when $\kappa > 0$. Recall that these orbits are contained in the set

$$\mathcal{P}(\mathcal{D}) = \{(x, y) : x > 1\} \cup \{(1, 0)\}.$$

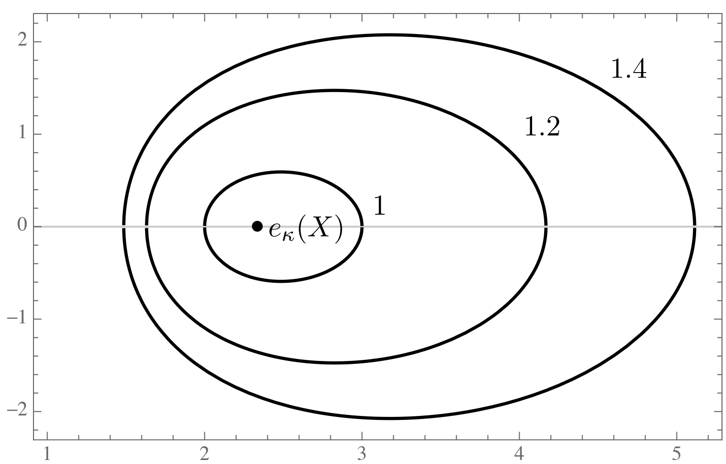
Lemma 12.1. *Fix $\kappa > 0$.*

i. If $X_1 \neq X_2$, then:

a. $C_\kappa(e_\kappa(X), X) = \{(x_0, 0)\}$ where $x_0 > 1$ and $(x_0, 0)$ is a critical point of the Hamiltonian $\Phi_\kappa(x, y; e_\kappa(X), X)$, and

b. for all $E > \bar{E} \geq e_\kappa(X)$, $C_\kappa(E, X)$ is a nontrivial closed orbit of the system (9.1) *in $\mathcal{P}(\mathcal{D}) \setminus \{(1, 0)\}$ enclosing $C_\kappa(\bar{E}, X)$. (See Figure 1)*

Figure 1: Level curves $C_\kappa(E, X)$ in the case $X_1 \neq X_2$, with $\kappa = 1/4$, $X_1 = -X_2 = 1$, $E = e_\kappa(X)$, 1, 1.2, 1.4.



ii. If $X_1 = X_2$ and $\frac{1}{8}X_1^2 \leq \kappa$, then:

a. $e_\kappa(X) = \kappa + \frac{1}{4}X_1^2$,

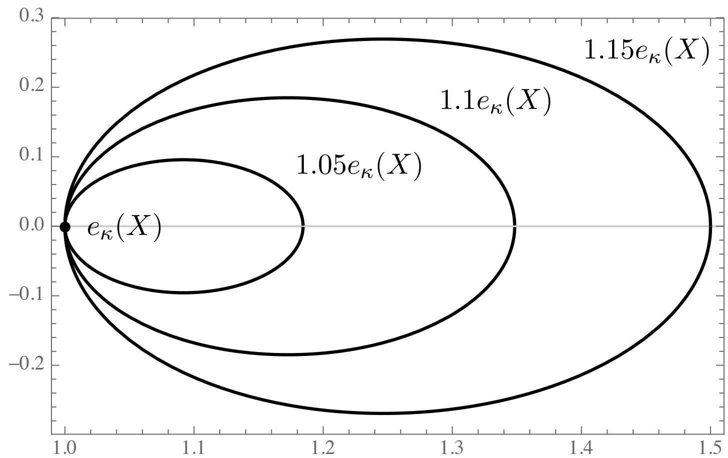
b. $C_\kappa(e_\kappa(X), X) = \{(1, 0)\}$ and $(1, 0)$ is a critical point of the Hamiltonian $\Phi_\kappa(x, y; e_\kappa(X), X)$,

c. for all $E > \bar{E} > e_\kappa(X)$, $C_\kappa(E, X)$ is a nontrivial closed orbit of the system (9.1) *in $\mathcal{P}(\mathcal{D})$ containing $\{(1, 0)\}$ and enclosing $C_\kappa(\bar{E}, X) \setminus \{(1, 0)\}$. (See Figure 2)*

iii. If $X_1 = X_2$ and $\frac{1}{8}X_1^2 > \kappa$, then:

a. $3\kappa \leq e_\kappa(X) = (2\kappa)^{1/2}|X_1| - \kappa < E_* \equiv \kappa + \frac{1}{4}X_1^2$,

Figure 2: Level curves $C_\kappa(E,X)$ in the case $X_1 = X_2$, $\kappa \geq \frac{1}{8}X_1^2$, with $\kappa = 1/4$, $X_1 = X_2 = 1$, $E = e_\kappa(X)$, $1.05\ e_\kappa(X)$, $1.1\ e_\kappa(X)$, $1.15\ e_\kappa(X)$.



b. *$C_\kappa(e_\kappa(X),X) = \{(x_0,0)\}$ where $x_0 > 1$ and $(x_0,0)$ is a critical point of $\Phi_\kappa(x,y;e_\kappa(X),X)$,*

c. *for all $E_* > E > \bar{E} \geq e_\kappa(X)$, $C_\kappa(E,X)$ is a is a nontrivial closed orbit of (9.1) in $\mathcal{P}(\mathcal{D}) \setminus \{(1,0)\}$ enclosing $C_\kappa(\bar{E},X)$,*

d. *$C_\kappa(E_*,X)$ is a nontrivial closed curve in $\mathcal{P}(\mathcal{D})$ containing $\{(1,0)\}$ and enclosing $C_\kappa(\bar{E},X) \setminus \{(1,0)\}$ for $E_* > \bar{E} \geq e_\kappa(X)$, $(1,0)$ is a critical point of the Hamiltonian $\Phi_\kappa(x,y;E_*,X)$, and $C_\kappa(E_*,X) \setminus \{(1,0)\}$ is a homoclinic orbit,*

e. *for all $E > E_*$, $E > \bar{E} \geq e_\kappa(X)$, $C_\kappa(E,X)$ is a nontrivial closed orbit in $\mathcal{P}(\mathcal{D})$ containing $\{(1,0)\}$ and enclosing $C_\kappa(\bar{E},X) \setminus \{(1,0)\}$. (See Figure 3)*

Proof. This is an application of Lemmas 7.4, 8.10, and 8.12. □

Remark 12.2. Cases (ii) and (iii) of Lemma 12.1 can also be characterized by $X_1 = X_2$ and $E > \kappa + \frac{1}{4}X_1^2$ or $E = e_\kappa(X)$, respectively, by Lemma 7.4.

Theorem 12.3. *Let $\kappa > 0$ and $(E,X) \in [e_\kappa(X),\infty) \times \mathbb{R}^2$. Let*

$$A \in C^0(\mathbb{R},\mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R},\mathbb{M}^2)$$

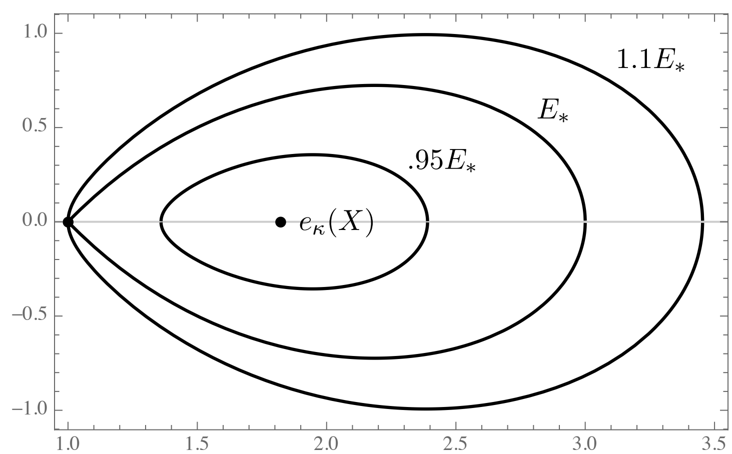
be a solution of (3.2), (3.3), (3.4) with initial data $(A_0,B_0) \in \mathcal{D}_\kappa(E,X)$.

i. *The solution A is constant if and only if $X = 0$ and $E = \kappa$.*

ii. *The solution A is non-constant and rigid if and only if $X \neq 0$ and either $E = e_\kappa(X)$ or $A_0 \in \mathrm{SO}(2,\mathbb{R})$ and $E = \kappa + \frac{1}{4}X_1^2 > e_\kappa(X)$.*

Proof. This is an application of Lemmas 10.1, 10.6, and 10.7. □

Figure 3: Level curves $C_\kappa(E,X)$ in the case $X_1 = X_2$, $\kappa < \frac{1}{8}X_1^2$, for the values $\kappa = 1/4$, $X_1 = X_2 = 2$, $E = e_\kappa(X)$, $.95E_*$, E_*, $1.1E_*$, $E_* = \kappa + \frac{1}{4}X_1^2$.



The next results concern the homoclinic orbit in case (iiid) of Lemma 12.1 .

Theorem 12.4. *Let $\kappa > 0$. Suppose that $(E,X) \in [e_\kappa(X),\infty) \times \mathbb{R}^2$ satisfies*

$$X_1 = X_2 \quad \textit{and} \quad E = \kappa + \tfrac{1}{4}X_1^2 > e_\kappa(X).$$

Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a solution of the IVP (3.2), (3.3), (3.4) with initial data $(A_0, B_0) \in \mathcal{D}_\kappa(E,X) \setminus \mathcal{R}(X_1)$, (cf. Lemma 10.10). Then there exist phases $\theta_\pm$ such that for $0 < \lambda < \frac{1}{2}(X_1^2 - 8\kappa)^{1/2}$, the solution satisfies

$$\lim_{t\to\pm\infty} e^{\lambda|t|} \left| \frac{d^j}{dt^j}\left[A(t) - U\left(\tfrac{1}{2}X_1 t + \theta_\pm\right)\right]\right| = 0, \quad j = 0, 1.$$

Proof. The assumptions on the parameters put us in case (iiid) of Lemma 12.1, and in particular, we have $\kappa < \frac{1}{8}X_1^2$. The Hamiltonian $\Phi_\kappa(x,y;E,X)$ has a critical point at $(1,0)$, the set $C_\kappa(E,X) \setminus \{(1,0)\}$ is a nontrivial homoclinic orbit, and since $A_0 \notin \mathrm{SO}(2,\mathbb{R})$,

$$\{\mathcal{P}(A(t), A'(t)) : t \in \mathbb{R}\} = C_\kappa(E,X) \setminus \{(1,0)\}.$$

Set $(x(t), y(t)) = \mathcal{P}(A(t), A'(t))$. Then we have

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \nearrow 0, \quad \text{as} \quad t \to \infty,$$

and

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \searrow 0, \quad \text{as} \quad t \to -\infty.$$

We shall prove the result for $t \to \infty$, the other case being nearly the same.

Using Definition 8.1, (9.1), and the condition on (E, X), we find that

$$\begin{aligned} x'(t) = y(t) &= -\left(\frac{P_\kappa(x(t); E, X)}{x(t)}\right)^{1/2} \\ &= -\left(\frac{X_1^2 - 4\kappa(x(t)+1)}{x(t)}\right)^{1/2} (x(t)-1), \quad t \gg 1. \end{aligned} \tag{12.1}$$

Fix $0 < \lambda < \frac{1}{2}(X_1^2 - 8\kappa)^{1/2}$ and choose $t_0 \gg 1$ such that

$$\left(\frac{X_1^2 - 4\kappa(x(t)+1)}{x(t)}\right)^{1/2} > 2\lambda, \quad t \geq t_0.$$

Then

$$x'(t) \leq -2\lambda(x(t)-1), \quad t \geq t_0,$$

and we obtain the estimate

$$0 < x(t) - 1 \leq (x(t_0)-1)\exp[-2\lambda(t-t_0)], \quad t \geq t_0.$$

Applying this in (12.1) yields

$$|x'(t)| \lesssim \exp[-2\lambda(t-t_0)], \quad t \geq t_0.$$

Using the notation from Lemma 2.14, it follows that

$$\left|\frac{d^j}{dt^j}\left[H(x(t)) - I\right]\right| = \left|\frac{d^j}{dt^j}\begin{bmatrix} \sqrt{\frac{x(t)+1}{2}} - 1 & \sqrt{\frac{x(t)-1}{2}} \\ \sqrt{\frac{x(t)-1}{2}} & \sqrt{\frac{x(t)+1}{2}} - 1 \end{bmatrix}\right| \lesssim \exp(-\lambda t),$$

for $t \geq t_0$, $j = 0, 1$.

We now use Theorem 11.6 to reconstruct $A(t)$. Define $(s_1(t), s_2(t))$ according to Lemma 11.2. Then since $X_1 = X_2$, we have $s_2(t) = s_2(0)$ and

$$\begin{aligned} s_1(t) &= s_0(0) + \int_0^t \frac{X_1}{2(x(\sigma)+1)} d\sigma \\ &= \tfrac{1}{4}X_1 t + s_0(0) + \tfrac{1}{4}X_1 \int_0^t \frac{-x(\sigma)+1}{x(\sigma)+1} d\sigma \\ &= \tfrac{1}{4}X_1 t + \tfrac{1}{2}\theta_+ + \tfrac{1}{4}X_1 \int_t^\infty \frac{x(\sigma)-1}{x(\sigma)+1} d\sigma, \end{aligned}$$

where

$$\tfrac{1}{2}\theta_+ = s_0(0) + \tfrac{1}{4}X_1 \int_0^\infty \frac{-x(\sigma)+1}{x(\sigma)+1} d\sigma.$$

Thus, we have

$$\left|\frac{d^j}{dt^j}\left[s_1(t) - (\tfrac{1}{4}X_1 t + \tfrac{1}{2}\theta_+)\right]\right| \lesssim \exp(-\lambda t), \quad t \geq t_0, \quad j = 0, 1.$$

It follows that

$$\begin{aligned}
&\left|\frac{d^j}{dt^j}\left[U(2s_1(t)) - U(\tfrac{1}{2}X_1 t + \theta_+)\right]\right| \\
&\qquad = \left|\frac{d^j}{dt^j}\left[U(2s_1(t) - \tfrac{1}{2}X_1 t - \theta_+) - I\right] U(\tfrac{1}{2}X_1 t + \theta_+)\right| \\
&\qquad\qquad \lesssim \exp(-\lambda t), \quad \text{for} \quad t \geq t_0, \quad j = 0, 1.
\end{aligned}$$

By Theorem 11.6, we obtain

$$A(t) = U(s_1(t) + s_2(0))\, H(x(t))\, U(s_1(t) - s_2(0)).$$

The desired estimates follow after writing

$$\begin{aligned}
A(t) - U(\tfrac{1}{2}X_1 t + \theta_+) = U(s_1(t) + s_2(0))\, [H(x(t)) - I]\, U(s_1(t) - s_2(0)) \\
+ U(2s_1(t)) - U(\tfrac{1}{2}X_1 t + \theta_+).
\end{aligned}$$

□

Remark 12.5. The total phase shift is given by the expression

$$\theta_+ - \theta_- = -\tfrac{1}{4}X_1 \int_{-\infty}^{\infty} \frac{x(\sigma) - 1}{x(\sigma) + 1} d\sigma.$$

Corollary 12.6. *Let $\kappa > 0$. Suppose that*

$$X_1 = X_2 \quad \textit{and} \quad E = \kappa + \tfrac{1}{4}X_1^2 > e_\kappa(X).$$

Then $\mathcal{R}(X_1) \subset \mathcal{D}_\kappa(E, X)$ corresponds to the orbit of the rigid rotation $U\left(\tfrac{1}{2}X_1 t\right)$. The set $\mathcal{D}_\kappa(E, X) \setminus \mathcal{R}(X_1)$ is a stable and unstable manifold for $\mathcal{R}(X_1)$. Every solution orbit (A, A') in $\mathcal{D}_\kappa(E, X) \setminus \mathcal{R}(X_1)$ is homoclinic to $\mathcal{R}(X_1)$, that is,

$$\lim_{|t| \to \infty} e^{\lambda |t|} \operatorname{dist}[(A(t), A'(t)), \mathcal{R}(X_1)] = 0,$$

for some $\lambda > 0$.

Proof. This follows from Lemma 10.11 and Theorem 12.4. □

Remark 12.7. $\bigcup_{X_1 \in \mathbb{R}} \mathcal{R}(X_1)$ is a normally hyperbolic invariant manifold.

Theorem 12.8. *If $A(t)$ is a solution of the IVP* (3.2), (3.3), (3.4) *such that the quantity $\frac{1}{2}|A(t)|^2$ is T-periodic for some $T > 0$, then the solution has the form*

$$A(t) = U(\omega_1 t)\hat{A}(t)U(\omega_2 t),$$

where $\hat{A}(t)$ is is T-periodic if $(1, 0) \notin C_\kappa(E, X)$ and $2T$-periodic if $(1, 0) \in C_\kappa(E, X)$. The frequencies are defined by

$$\omega_1 + \omega_2 = \frac{2}{T} \int_0^T \frac{X_1 + X_2}{g_{11}(A(t))} dt$$

and

$$\omega_1 - \omega_2 = \begin{cases} 0, & X_1 = X_2 \\ \dfrac{2}{T}\displaystyle\int_0^T \frac{X_1 - X_2}{g_{22}(A(t))}dt, & X_1 \neq X_2. \end{cases}$$

Proof. Define $(s_1(t), s_2(t))$ as in Lemma 11.2 and ω_1, ω_2 as above. Since $g_{ii}(A(t))$ is T-periodic, $i = 1, 2$, the functions

$$s_1'(t) + s_2'(t) - \omega_1 \quad \text{and} \quad s_1'(t) - s_2'(t) - \omega_2$$

are T-periodic and have mean zero over the interval $[0, T]$. Hence, their antidervatives

$$s_1(t) + s_2(t) - \omega_1 t \quad \text{and} \quad s_1(t) - s_2(t) - \omega_2 t$$

are T-periodic. It follows that

$$U(s_1(t) + s_2(t))\, U(-\omega_1 t) \quad \text{and} \quad U(s_1(t) - s_2(t))\, U(-\omega_2 t)$$

are T-periodic.

Now going back to Theorems 11.6 and 11.8, we find that

$$\hat{A}(t) = U(-\omega_1 t)A(t)U(-\omega_2 t)$$

is T-periodic if $(1,0) \notin C_\kappa(E, X)$ and $2T$-periodic if $(1,0) \in C_\kappa(E, X)$. □

Remark 12.9. The result shows that there is monodromy when the solution $A(t)$ passes through $\mathrm{SL}(2,\mathbb{R})$.

Remark 12.10. Note that the result holds for rigid solutions. In this case, the quantity $|A(t)|$ is constant and thus T-periodic for all $T \geq 0$. Any value of $T > 0$ can be used in computing the frequencies.

Theorem 12.11. *Let $A(t)$ be a solution of the IVP* (3.2)*,* (3.3)*,* (3.4) *such that the quantity $|A(t)|$ is T-periodic for some $T > 0$.*

For every $N \in \mathbb{N}$, there exists $\ell(N) \in \{1, \ldots, N^2\}$ such that

$$|A(2\ell(N)T + t) - A(t)| \leq 8\pi|A(t)|/N, \quad \text{for all} \quad t \in \mathbb{R}.$$

If $A(t)$ is rigid, then either $A(t)$ is periodic or the range of $A(t)$ is dense in the sphere of radius $|A_0|$ in $\mathrm{SL}(2,\mathbb{R})$.

Proof. By Theorem 12.8, we may write

$$A(t) = U(\omega_1 t)\hat{A}(t)U(\omega_2 t),$$

in which $\hat{A}(t)$ is $2T$-periodic. (If $A(t)$ does not pass through $\mathrm{SO}(2,\mathbb{R})$, then we know that $\hat{A}(t)$ is T-periodic.)

For every $x \in \mathbb{R}$, there is a unique $k \in \mathbb{Z}$ such that

$$\{x\} \equiv x - 2\pi k \in [0, 2\pi).$$

Consider the set of $N^2 + 1$ ordered pairs

$$\{\ (\{\omega_1 2jT\}, \{\omega_2 2jT\}) : j = 0, 1, \dots, N^2\ \}$$

contained in the square $[0, 2\pi) \times [0, 2\pi)$. Partition this square into N^2 congruent subsquares of side $2\pi/N$. By the pigeonhole principle, two of these ordered pairs belong to the same subsquare. It follows that there exist $k, \ell(N) \in \mathbb{Z}$ such that $0 \le k < k + \ell(N) \le N^2$ and

$$|\{\omega_i 2kT\} - \{\omega_i 2(k + \ell(N))T\}| \le 2\pi/N, \quad i = 1, 2.$$

Thus, there exist $m_i \in \mathbb{Z}$ such that

$$|\omega_i 2\ell(N)T + 2\pi m_i| \le 2\pi/N, \quad i = 1, 2.$$

Define

$$\tau_i = \omega_i 2\ell(N)T + 2\pi m_i.$$

For $i = 1, 2$ and $t \in \mathbb{R}$, we have using Definition 1.8 and the mean value theorem

$$\begin{aligned}
|U(\omega_i 2\ell(N)T + t) - U(t)| &= |U(\omega_i 2\ell(N)T) - I| \\
&= |U(\tau_i) - I| \\
&= \sqrt{2}[(\cos\tau_i - 1)^2 + \sin^2\tau_i]^{1/2} \\
&= 2(1 - \cos\tau_i)^{1/2} \\
&\le 2|\tau_i| \\
&\le 4\pi/N.
\end{aligned}$$

For any $t \in \mathbb{R}$, we have

$$\begin{aligned}
A(2\ell(N)T + t) &= U(\omega_1(2\ell(N)T + t))\hat{A}(2\ell(N)T + t)U(\omega_2(2\ell(N)T + t)) \\
&= U(\omega_1 2\ell(N)T)U(\omega_1 t)\hat{A}(t)U(\omega_2 t)U(\omega_2 2\ell(N)T) \\
&= U(\tau_1)A(t)U(\tau_2).
\end{aligned}$$

We now estimate as follows

$$\begin{aligned}
|A(2\ell(N)T + t) - A(t)| &= |U(\tau_1)A(t)U(\tau_2) - A(t)| \\
&= |[U(\tau_1) - I]A(t)U(\tau_2) + A(t)[U(\tau_2) - I]| \\
&\le |U(\tau_1) - I||A(t)||U(\tau_2)| + |A(t)||U(\tau_2) - I| \\
&\le 2(4\pi/N)|A(t)|.
\end{aligned}$$

This proves the first statement.

If $A(t)$ is rigid, then $|A(t)| = |A_0|$, and so by Theorem 12.8

$$A(t) = U(\omega_1 t)A_0 U(\omega_2 t) = U(\{\omega_1 t\})A_0 U(\{\omega_2 t\}).$$

If ω_1 and ω_2 are rationally dependent, then $A(t)$ is periodic. The curve

$$t \mapsto (\{\omega_1 t\}, \{\omega_2 t\})$$

represents linear flow on the torus. If ω_1 and ω_2 are rationally independent, then it is well-known that the image of the curve is dense in the square $[0, 2\pi) \times [0, 2\pi)$. By Lemma 2.14, the set

$$\{UA_0V : U, V \in \mathrm{SO}(2,\mathbb{R})\}$$

coincides with the sphere of radius $|A_0|$ in $\mathrm{SL}(2,\mathbb{R})$. Thus, the range of $A(t)$ is dense in this sphere. □

Remark 12.12. The only solutions $A(t)$ for which $|A(t)|$ is not periodic are those which are homoclinic to a rigid rotation. Thus, the result shows that, generically, solutions are recurrent.

Remark 12.13. Since

$$|A(t)| \le \left[\tfrac{2}{\kappa}E_\kappa(A(t), A'(t))\right]^{1/2}$$

and the energy is conserved, Theorem 12.11 shows that

$$|A(2\ell(N)T + t) - A(t)| \lesssim 1/N, \quad \text{for all} \quad t \in \mathbb{R}.$$

13. Perfect fluids

Lemma 13.1. *Fix $\kappa = 0$ and $(E, X) \in (0,\infty) \times \mathbb{R}^2$. Let*

$$A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$$

be a solution of the IVP (3.2), (3.3), (3.4) *with initial data in $\mathcal{D}_0(E, X)$. The quantity $x(t) = \frac{1}{2}|A(t)|^2$ satisfies*

$$x''(t) = \frac{2E(x(t)-1)^2 + 4Ex(t) - X_1X_2}{x(t)^2}. \tag{13.1}$$

Moreover, $x''(t) \ge 0$, for all $t \in \mathbb{R}$, and if there exists $t_0 \in \mathbb{R}$ such that $x''(t_0) = 0$, then $x(t) = 1$ for all $t \in \mathbb{R}$.

Proof. Equation (13.1) is just a restatement of (9.3) in the case $\kappa = 0$. Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|X_1X_2| = |\ X_1(A(t), A'(t))\ X_2(A(t), A'(t))\ | &\le |A(t)|^2|A'(t)|^2 \\ &= 4x(t)E_0(A(t), A'(t)) = 4Ex(t).\end{aligned}$$

From this we see that $x''(t) \geq 0$, for all $t \in \mathbb{R}$.

If $x''(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then $E(x(t_0) - 1)^2 = 0$. Since $E > 0$, we have $x(t_0) = 1$. This implies that $A(t_0) \in \mathrm{SO}(2,\mathbb{R})$, so we must have $X_1 = X_2$ by Lemma 4.3. But then $x''(t_0) = 0$ implies that $4E = X_1^2$. By Lemma 10.7, $A(t)$ is a rigid solution in $\mathrm{SO}(2,\mathbb{R})$ for all $t \in \mathbb{R}$, and therefore, $x(t) \equiv 1$. □

Lemma 13.2. *Fix $\kappa = 0$ and $(E, X) \in [0,\infty) \times \mathbb{R}^2$.*

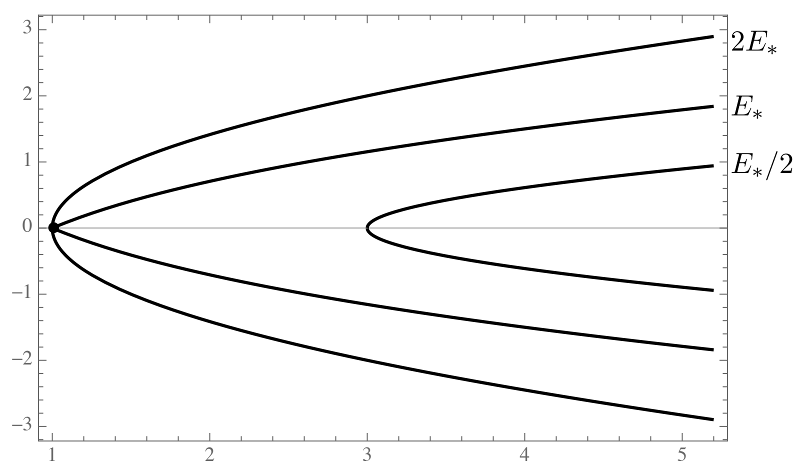
We have $C_0(0,0) = \{(x,0) : 1 \leq x < \infty\}$, and each point $(x,0) \in C_0(0,0)$ corresponds to an equilibrium solution of (9.1)*.*

If $X_1 = X_2 \neq 0$ and $E = \frac{1}{4}X_1^2$, then $C_0(E,X)$ is the union of an equilibrium solution $\{(1,0)\}$ of (9.1) *and two semi-bounded orbits. (See Figure 4)*

In all other cases, $C_0(E,X)$ is a single orbit which is unbounded as $t \to \pm\infty$. (See Figures 4 and 5)

The point $(1,0)$ belongs to $C_0(E,X)$ if and only if $X_1 = X_2$ and $E \geq \frac{1}{4}X_1^2$.

Figure 4: Level curves $C_0(E,X)$ in the case $X_1 = X_2 = 1$, $E = E_*/2$, E_*, $2E_*$, $E_* = \frac{1}{4}X_1^2$.

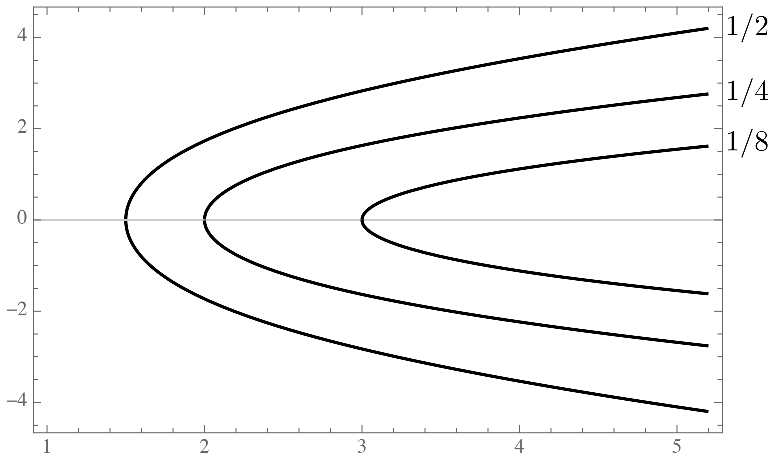


Proof. As already shown in Lemma 8.12, the sets $C_0(E,X)$ are unbounded, and the set $C_0(E,X)$ consists of a single orbit unless it contains a critical point of $\Phi_0(x,y;E,X)$. This occurs when $(E,X) = (0,0)$ and when $X_1 = X_2$, $E = \frac{1}{4}X_1^2$, by Lemma 8.8. Lemma 8.7 gives the condition for $(1,0) \in C_0(E,X)$. □

Theorem 13.3. *Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a solution of the IVP* (3.2)*,* (3.3)*,* (3.4) *with initial data $(A_0, B_0) \in \mathcal{D}_0(E,X)$. If $\sup_{t>0} |A(t)|^2 = \infty$, then there exist A_∞, $B_\infty \in \mathbb{M}^2$ such that for $t > 0$, $j = 0,1,2$,*

$$\left| \frac{d^j}{dt^j} \left[A(t) - (B_\infty t + A_\infty)\right] \right| \lesssim (1+t)^{-1-j}. \tag{13.2}$$

Figure 5: Level curves $C_0(E,X)$ in the case $X_1 = -X_2 = 1$, $E = 1/8,\ 1/4,\ 1/2$.



If $\bar{A}_\infty,\ \bar{B}_\infty \in \mathbb{M}^2$ *is any pair such that*

$$\lim_{t\to\infty} |A(t) - (\bar{B}_\infty t + \bar{A}_\infty)| = 0, \tag{13.3}$$

then $(\bar{A}_\infty, \bar{B}_\infty) = (A_\infty, B_\infty)$.

The vectors A_∞, B_∞ *satisfy*

$$E_0(A_\infty, B_\infty) = \tfrac{1}{2}|B_\infty|^2 = E > 0, \quad X(A_\infty, B_\infty) = X,$$

and

$$\langle B_\infty, \operatorname{cof} A_\infty\rangle = \det B_\infty = 0, \quad \det A_\infty = \frac{X_1 X_2}{2E}.$$

If $\det B_0 = 0$*, then* $(A_\infty, B_\infty) = (A_0, B_0) \in \mathcal{D}(E,X)$ *and*

$$A(t) = B_0 t + A_0.$$

Proof. Suppose that A is a solution in $\mathcal{D}_\kappa(E,X)$ with $\sup_{t>0}|A(t)|^2 = \infty$. Set $(x(t),y(t)) = \mathcal{P}(A(t),A'(t))$. Then $\sup_{t>0} x(t) = \infty$, and so there exists $t_0 > 0$ such that $x'(t_0) > 0$. Since $x(t)$ is not identically equal to 1, Lemma 13.1 implies $x''(t) > 0$, for all $t \in \mathbb{R}$. It follows that $x'(t) \geq x'(t_0) > 0$, for $t \geq t_0$, and consequently, $x(t) \to \infty$, as $t \to \infty$. From (13.1), there exists $t_1 > 0$ such that

$$x''(t) \geq E > 0, \quad t \geq t_1.$$

After integration, this leads to the lower bound

$$x(t) \geq \tfrac{1}{2}E(t-t_1)^2 + y(0)(t-t_1) + x(0), \quad t \geq t_1,$$

and thus,

$$|A(t)|^2 \gtrsim (1+t)^2, \quad t \geq 0.$$

Since $A(t)$ solves (3.3), we obtain from Lemma 6.1 that

$$|A''(t)| \lesssim |A(t)|^{-3} \lesssim (1+t)^{-3}, \quad t \geq 0.$$

Thus, by Lemma 6 of [9], we can write

$$A(t) = B_\infty t + A_\infty + A_1(t),$$

with

$$\begin{aligned} B_\infty &= B_0 + \int_0^\infty A''(s)ds, \\ A_\infty &= A_0 - \int_0^\infty \int_s^\infty A''(\sigma)d\sigma ds, \\ A_1(t) &= \int_t^\infty \int_s^\infty A''(\sigma)d\sigma ds. \end{aligned}$$

Note that our estimate for $|A''(t)|$ implies that

$$\left| \frac{d^j}{dt^j} A_1(t) \right| \lesssim (1+t)^{-1-j}, \quad t \geq 0, \quad j = 0,\ 1\ ,2,$$

thereby proving (13.2).

If (13.3) holds, then using (13.2), we find that

$$\lim_{t\to\infty} |(B_\infty - \bar{B}_\infty)t + (A_\infty - \bar{A}_\infty)| = 0,$$

and uniqueness of the states (A_∞, B_∞) follows from this.

Applying (13.2), we find

$$E = \tfrac{1}{2}|A'(t)|^2 = \tfrac{1}{2}|B_\infty + A_1'(t)|^2 = \tfrac{1}{2}|B_\infty|^2 + O(t^{-1}), \quad t > 0.$$

Sending $t \to \infty$ shows that $E = \frac{1}{2}|B_\infty|^2$.

For the other invariants, we have

$$\begin{aligned} X = X(A(t), A'(t)) = X(B_\infty t + A_\infty + A_1(t), B_\infty + A_1'(t)) \\ = tX(B_\infty, B_\infty) + X(A_\infty, B_\infty) + O(t^{-1}). \end{aligned}$$

By Lemmas 1.2 and 1.4, we see that $X(B_\infty, B_\infty) = 0$, and so letting $t \to \infty$ we obtain $X = X(A_\infty, B_\infty)$.

Since $A(t) \in \mathrm{SL}(2,\mathbb{R})$, we get from Lemma 1.15

$$\begin{aligned} 2 &= 2 \det A(t) \\ &= \langle A(t), \operatorname{cof} A(t) \rangle \\ &= t^2 \langle B_\infty, \operatorname{cof} B_\infty \rangle + 2t \langle A_\infty, \operatorname{cof} B_\infty \rangle \\ &\quad + \langle A_\infty, \operatorname{cof} A_\infty \rangle + 2t \langle B_\infty, \operatorname{cof} A_1(t) \rangle + O(t^{-1}) \end{aligned}$$

$$
\begin{aligned}
&= 2t^2 \det B_\infty + 2t \langle A_\infty, \operatorname{cof} B_\infty\rangle \\
&\quad + 2\det A_\infty + 2t\langle B_\infty, \operatorname{cof} A_1(t)\rangle + O(t^{-1}).
\end{aligned}
$$

This implies that

$$2\det B_\infty = \langle B_\infty, \operatorname{cof} B_\infty\rangle = 0, \quad \langle A_\infty, \operatorname{cof} B_\infty\rangle = 0,$$

and

$$2\det A_\infty + \lim_{t\to\infty} 2t\langle B_\infty, \operatorname{cof} A_1(t)\rangle = 2.$$

Using the formula for $A_1(t)$, l'Hôpital's rule, (3.3), Lemma 6.1, and (13.2), we find that

$$\lim_{t\to\infty} tA_1(t) = \lim_{t\to\infty} \tfrac{1}{2}t^3 A_1''(t) = \lim_{t\to\infty} \tfrac{1}{2}t^3 A''(t) = (2E - X_1X_2)\frac{\operatorname{cof} B_\infty}{|B_\infty|^4}.$$

From this follows

$$\det A_\infty = 1 - \frac{2E - X_1X_2}{|B_\infty|^2} = \frac{X_1X_2}{2E}.$$

If $\det B_0 = 0$, then by Lemma 6.1, we get $2E - X_1X_2 = 0$. Lemma 6.1 then says that $\Lambda_0(A(t), A'(t)) = 0$, for all $t \in \mathbb{R}$. So the equations of motion simplify dramatically to $A''(t) = 0$, and from this we see that $A(t)$ must be linear in t. $\square$

Remark 13.4. If $(A_0, B_0) \in \mathcal{D}$ and $\det B_0 = 0$, then $A(t) = B_0 t + A_0$ is a geodesic line in $\mathrm{SL}(2,\mathbb{R})$, by Lemma 10.13.

Remark 13.5. In Theorem 13.3, if $\det B_0 \neq 0$, then $A_\infty \notin \mathrm{SL}(2,\mathbb{R})$, and hence $(A_\infty, B_\infty) \notin \mathcal{D}$.

Remark 13.6. An analogous result holds when $\sup_{t<0} |A(t)|^2 = \infty$.

Theorem 13.7. *Let $A \in C^0(\mathbb{R}, \mathrm{SL}(2,\mathbb{R})) \cap C^2(\mathbb{R}, \mathbb{M}^2)$ be a* non-rigid *solution of the IVP (3.2), (3.3), (3.4) in $\mathcal{D}_0(E, X)$.*

If $\sup_{t>0} |A(t)|^2 < \infty$, then $X_1 = X_2 \neq 0$, $E = \frac{1}{4}X_1^2$, the orbit $(A(t), A'(t))$ belongs to the set

$$\mathcal{W}^s(X_1) = \{(A, B) \in \mathcal{D}_\kappa(E, X) : \langle A, B\rangle < 0\},$$

and there exists a phase θ_+ such that for every $0 < \lambda < \frac{1}{2}|X_1|$,

$$\left| \frac{d^j}{dt^j}\left[A(t) - U\left(\tfrac{1}{2}X_1 t + \theta_+\right)\right]\right| \lesssim \exp(-\lambda t),$$

for all $t \geq 0$, $j = 0, 1$.

Proof. Since the solution A is non-rigid and semi-bounded, Lemma 13.2 implies that $X_1 = X_2 \neq 0$ and $E = \frac{1}{4}X_1^2$. By Lemma 13.1, we have

$$x(t) \searrow 1 \quad \text{and} \quad y(t) \nearrow 0, \quad \text{as} \quad t \to \infty.$$

Thus, $y(t) = \langle A(t), A'(t)\rangle < 0$, $t \in \mathbb{R}$, and so the solution orbit $(A(t), A'(t))$ lies in $\mathcal{W}^s(X_1)$. Since the phase plane orbit $(x(t), y(t))$ lies in $C_0(E, X)$, we have $\Phi_0(x(t), y(t); E, X) = 0$, $t \in \mathbb{R}$. Using Definition 8.1, (9.1), and the condition on (E, X), we find that

$$x'(t) = y(t) = -|X_1|\, x(t)^{-1/2}(x(t) - 1), \quad t \in \mathbb{R}.$$

Given $0 < \lambda < |X_1|/2$, choose t_0 large enough so that

$$|X_1|\, x(t)^{-1/2} \geq 2\lambda, \quad t \geq t_0.$$

Then

$$x'(t) \leq -2\lambda(x(t) - 1), \quad t \geq t_0.$$

From this we obtain the estimates

$$0 < x(t) - 1 \leq (x(t_0) - 1)\exp[-2\lambda(t - t_0)],$$

$$|x'(t)| \lesssim \exp[-2\lambda(t - t_0)],$$

for $t \geq t_0$. The rest of the proof proceeds exactly as in Theorem 12.4.

□

Remark 13.8. There is an obvious companion result in the case when $\sup_{t<0} |A(t)|^2 < \infty$ for the set

$$\mathcal{W}^u(X_1) = \{(A, B) \in \mathcal{D}_\kappa(E, X) : \langle A, B\rangle > 0\},$$

with $X_1 = X_2$ and $E = \frac{1}{4}X_1^2$.

Corollary 13.9. *For $0 \neq X_1 \in \mathbb{R}$ and $E = \frac{1}{4}X_1^2$, the sets $\mathcal{W}^s(X_1)$ and $\mathcal{W}^u(X_1)$ are stable and unstable manifolds for $\mathcal{R}(X_1)$.*

14. The picture in $T_A\mathrm{SL}(2,\mathbb{R})$

Several special situations have emerged: the existence of stable and unstable manifolds for $\mathcal{R}(X_1)$, the existence of solutions with vanishing pressure, and the existence of rigid solutions. Here we shall attempt to visualize the corresponding tangent directions in $T_A\mathrm{SL}(2,\mathbb{R})$ for a fixed point $A \in \mathrm{SL}(2,\mathbb{R})$.

Let us first assume that $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$. By Lemma 5.4, we can represent an element $B \in T_A\mathrm{SL}(2,\mathbb{R})$ using the normalized frame $\{\hat{\tau}_i(A)\}$ from Definition 2.7 as

$$B = \sum_i c_i\, \hat{\tau}_i(A),$$

in which

$$c_1 = \frac{X_1 + X_2}{\sqrt{g_{11}}}, \quad c_2 = \frac{X_1 - X_2}{\sqrt{g_{22}}}, \quad \text{with} \quad X_i = X_i(A, B), \quad g_{ii} = g_{ii}(A).$$

The metric g was given in Lemma 2.11. Thus, we have

$$X_1 = \tfrac{1}{2}(\sqrt{g_{11}}\, c_1 + \sqrt{g_{22}}\, c_2) \quad \text{and} \quad X_2 = \tfrac{1}{2}(\sqrt{g_{11}}\, c_1 - \sqrt{g_{22}}\, c_2). \tag{14.1}$$

We also have

$$E = E_\kappa(A,B) = \tfrac{1}{2}|B|^2 + \tfrac{\kappa}{2}|A|^2 = \tfrac{1}{2}\sum_i c_i^2 + \tfrac{\kappa}{2}|A|^2. \tag{14.2}$$

By Lemma 6.1, solutions with vanishing pressure are characterized by the condition $E = \frac{1}{2}X_1X_2$. Using expressions (14.1), we find that

$$E = \tfrac{1}{8}(g_{11}\, c_1^2 - g_{22}\, c_2^2).$$

From (14.2), this leads to the relation

$$c_1^2 - c_2^2 - \frac{2}{|A|^2}c_3^2 = 2\kappa.$$

Thus, the set

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : E_\kappa(A,B) = \tfrac{1}{2}X_1(A,B)X_2(A,B)\}$$

is a two-sheeted hyperboloid when $\kappa > 0$, and a cone when $\kappa = 0$. The region of positive pressure is connected, and the region of negative pressure has two connected components.

The critical point $(1,0)$ for (9.1) corresponds to the family of rotating solutions. The homoclinic orbits produce a stable/unstable manifold characterized by the conditions $X_1 = X_2$ and $E = \kappa + \frac{1}{4}X_1^2$. Here, we have $c_2 = 0$, and so

$$E = \kappa + \tfrac{1}{16}g_{11}\, c_1^2.$$

Thus, in local coordinates, the set

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : E_\kappa(A,B) = \kappa + \tfrac{1}{4}X_1(A,B)^2,\ X_1(A,B) = X_2(A,B)\}$$

is given by

$$\tfrac{1}{8}g_{22}\, c_1^2 - c_3^2 = \tfrac{\kappa}{2}g_{22}.$$

This describes a hyperbola in the $\hat{\tau}_1$, $\hat{\tau}_3$ plane with two branches, each contained within one of the components of negative pressure. The limiting solution is a rotation of the form $U(\frac{1}{2}X_1t + \theta)$, by Lemma 10.7. Since X_1 and c_1 have the same sign, we see that the branch with $c_1 > 0$ corresponds to counterclockwise rotation in the limit. When $\kappa = 0$, the hyperbola degenerates to a pair of lines through the origin. Parameter values $c_3 < 0$ along these lines correspond to stable directions while values $c_3 > 0$ correspond to unstable directions.

By Lemmas 8.8 and 10.6, the set

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : (A,B) \text{ is initial data for a rigid solution of (3.3)}\}$$

is equal to

$$\{B \in T_A\mathrm{SL}(2,\mathbb{R}) : (x,y) = \mathcal{P}(A,B) \text{ satisfies } y=0,\, P_\kappa(x;E,X)=0,\, P'_\kappa(x;E,X)=0\}.$$

The condition $P_\kappa(x;E,X)=0$ is the same as (14.2). Now $y=c_3=0$, so we have

$$E = \tfrac{1}{2}(c_1^2+c_2^2)+\kappa x.$$

The condition $P'_\kappa(x;E,X)=0$ is equivalent to

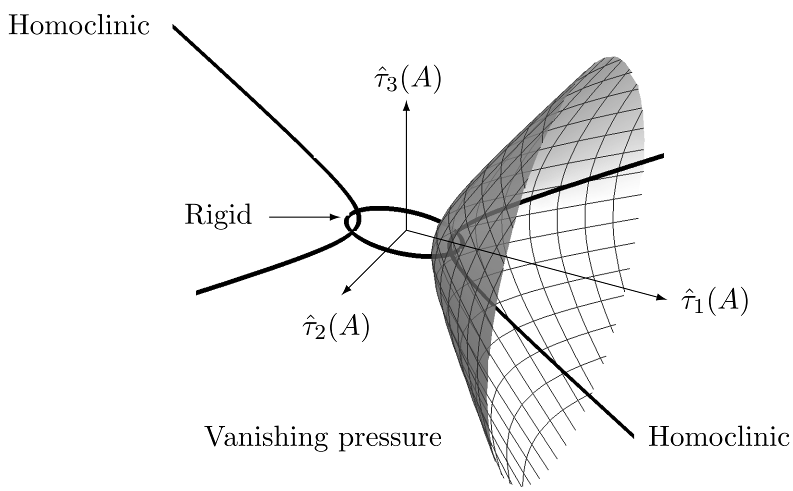
$$8Ex = 4\kappa(3x^2-1)+\tfrac{1}{2}g_{11}c_1^2+\tfrac{1}{2}g_{22}c_2^2.$$

We find that the local coordinates $(c_1,c_2,0)$ of B must lie on the ellipse

$$\frac{c_1^2}{g_{11}}+\frac{c_2^2}{g_{22}}=\kappa/2.$$

This intersects the hyperboloid of data with vanishing pressure at four points. The ellipse shrinks to the origin as $\kappa \to 0$. See Figures 6 and 7.

Figure 6: Distinguished directions in $T_A\mathrm{SL}(2,\mathbb{R})$ for a fixed $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $\kappa=1/2$. The branch of pressureless directions in the half space $c_1<0$ is not shown.



When $A \in \mathrm{SO}(2,\mathbb{R})$, we have

$$B = \sum_{i=1}^{3} c_i\hat{\tau}_i(A) \quad \text{with} \quad c_i = \langle B, \hat{\tau}_i(A)\rangle\,.$$

This yields

$$c_1 = \tfrac{1}{\sqrt{2}}X_1, \quad c_2 = -\tfrac{1}{\sqrt{2}}\langle B, U(2s_2)K\rangle\,, \quad c_3 = \tfrac{1}{\sqrt{2}}\langle B, U(2s_2)M\rangle\,,$$

for an arbitrary $s_2 \in \mathbb{R}$. We have

$$E = \kappa + \tfrac{1}{2}\sum_{i=1}^{3} c_i^2 = \kappa + \tfrac{1}{4}X_1^2 + \tfrac{1}{2}\sum_{i=2}^{3} c_i^2.$$

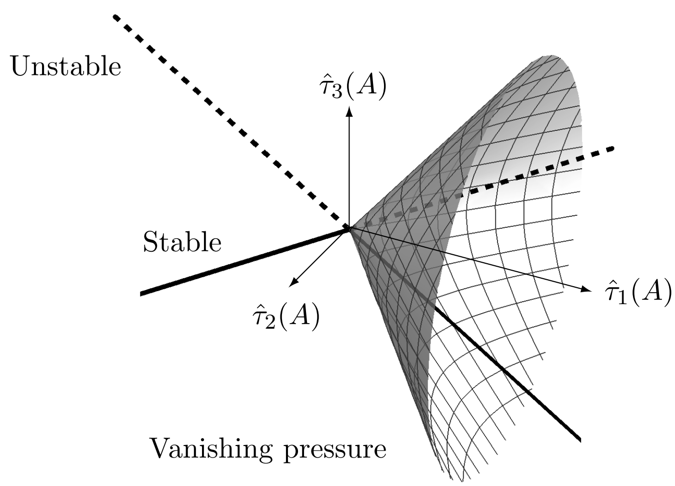
The pressureless solutions are described by the equation

$$c_1^2 - c_2^2 - c_3^2 = 2\kappa,$$

which is consistent with taking the limit as $A \to \mathrm{SO}(2,\mathbb{R})$.

The rigid solutions are given by $E = \kappa + \frac{1}{4}X_1^2$, or equivalently $c_2 = c_3 = 0$. The segment $|c_1| \leq 4\kappa$ along the $\hat{\tau}_1(A)$ axis arises as the limit $A \to \mathrm{SO}(2,\mathbb{R})$. The portion $|c_1| > 4\kappa$ corresponds to the limit set of the homoclinic orbits.

Figure 7: Distinguished directions in $T_A\mathrm{SL}(2,\mathbb{R})$ for a fixed $A \in \mathrm{SL}(2,\mathbb{R}) \setminus \mathrm{SO}(2,\mathbb{R})$, with $\kappa = 0$. The cone of pressureless directions in the half space $c_1 < 0$ is not shown.



15. Glossary of notation

Symbol	Reference	Description
$\mathbb{M}^2$	1.1	vector space of 2×2 matrices over $\mathbb{R}$
$\langle\cdot,\cdot\rangle$	1.1	Euclidean inner product on $\mathbb{M}^2$
$\mathrm{SL}(2,\mathbb{R})$	1.5	special linear group
$\mathrm{SO}(2,\mathbb{R})$	1.5	special orthogonal group
I,K,M,Z	1.3	orthogonal basis vectors in $\mathbb{M}^2$
$U(\sigma)$	1.8	parameterization of $\mathrm{SO}(2,\mathbb{R})$
cof	1.11	cofactor map
$T_A\mathrm{SL}(2,\mathbb{R})$	2.1	tangent space at $A\in\mathrm{SL}(2,\mathbb{R})$
$\mathfrak{sl}(2,\mathbb{R})$	2.2	special linear Lie algebra
$N(A)$	2.3	unit normal to $T_A\mathrm{SL}(2,\mathbb{R})$
$\mathcal{D}$	2.5	tangent bundle / phase space
$\tau_i(A)$	2.7	tangent vectors at $A\in\mathrm{SL}(2,\mathbb{R})\setminus\mathrm{SO}(2,\mathbb{R})$
$\hat{\tau}_i(A)$	2.7, 2.18	unit tangent vectors at $A\in T_A\mathrm{SL}(2,\mathbb{R})$
$g(A)$	2.11	metric on $T_A\mathrm{SL}(2,\mathbb{R})$
$\mathcal{A}(s)$	2.14	local coordinates on $\mathrm{SL}(2,\mathbb{R})\setminus\mathrm{SO}(2,\mathbb{R})$
$\Gamma^i_{jk}(s)$	2.21	Christoffel symbols
$P(A)$	2.23	projection of $\mathbb{M}^2$ onto $T_A\mathrm{SL}(2,\mathbb{R})$
$S(A)$	2.25	shape operator
$\Pi(A)$	2.27	second fundamental form
$\nabla_V W$	2.28	Riemannian connection
$L(A,B)$	3.4	velocity gradient map
$\Lambda_\kappa(A,B)$	3.10	nonlinearity / Lagrange multiplier
$E_\kappa(A,B)$	3.12	energy
$X_i(A,B)$	4.1	invariant quantities
$\mathcal{D}(X)$	5.1	invariant set
$\mathcal{D}_\kappa(E,X)$	5.1	invariant set
$e_\kappa(X)$	7.1	minimum energy
$\Phi_\kappa(x,y;E,X)$	8.1	Hamiltonian for phase plane dynamics
$P_\kappa(x;E,X)$	8.1	used in defining Φ_κ
$\mathcal{P}(A,B)$	8.4	projection of $\mathcal{D}$ into phase plane
$C_\kappa(E,X)$	8.6	level set in phase plane
$\mathcal{R}(X_1)$	10.10	invariant manifold of rotational solutions

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Compliance with Ethical Standards

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