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Bundling under Competition: Duopoly and Oligopoly

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Bundling under Competition: Duopoly and Oligopoly

A dissertation submitted in partial satisfaction of the

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in Management

by

Araz Khodabakhshian

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ABSTRACT OF THE DISSERTATION

Bundling under Competition: Duopoly and Oligopoly

by

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Doctor of Philosophy in Management

University of California, Los Angeles, 2019

Professor Uday S. Karmarkar, Chair

In many markets, bundling, or the offering of two or more products in a package for a single price is a common practice. While most markets are competitive, the majority of research work around bundling has focused on monopolistic markets, which are more tractable for analysis. From a monopolist's perspective, bundling has many benefits such as economies of scope, price discrimination, and expansion of demand. However, competition adds an important dimension to bundling decisions and their results. In this dissertation, the aim is to study the implications of competition for firms' product offering, pricing, and bundle design decisions.

In the first chapter, we study bundling in a duopoly under price competition and show that bundling can serve as a product differentiation tool and moderate competition even when firms are perfectly identical and offer undifferentiated products. In equilibrium, firms have an asymmetric bundling strategy, i.e., if one firm bundles the other does not. The bundling decision depends on the valuations of customer groups for the two products in the market. However, the firm offering the bundle earns a higher profit. This suggests an inherent first-mover advantage to bundling. There are two factors predicting the success of bundling in a price-competition setting. One is being ahead of the competitor in offering the bundle, and the other is the degree of correlation or its lack in customer valuations of bundle components.

In the next chapter, we utilize a quantity competition (Cournot) framework to study the im-

plications of bundling with entry. This model enables the analysis to go beyond duopoly to an oligopolistic market with fixed costs of entry, where firms enter as long as they can recover the fixed cost. We investigate firms' production quantity decisions and profits in equilibrium to determine the number of firms that enter each market. In a two-component setting, we consider examples with two types of offers: a single product, and the product bundled with the other. Then we consider the case of three markets consisting respectively of the first product, the second product, and a matched-quantity bundle of both products. We find that there may not be a unique equilibrium for the number of firms in each market. Moreover, we show that it is possible to construct settings where the number of equilibria can be arbitrarily large. We identify two factors for the success of bundles: one on the demand side and one on the supply side. On the demand side, customers buy bundles as long as both components within the bundle add relatively comparable values to them. On the supply side, firms enter the bundle market if the fixed-cost of entry for the bundle market is lower than the sum of fixed-costs of entry for all components within the bundle by at least a certain amount. We show that these results hold for a single customer group, as well as multiple customer groups.

In the last chapter, we study bundle design, which does not seem to be addressed in the literature. We relax the quantity matching assumption common to most bundling research, and allow the firm to choose the ratio of component quantities within the bundle, i.e. *bundle proportion*, so as to maximize profits. We study four market settings: a monopolist with one type of bundle and one customer group, a monopolist with one type of bundle and two customer groups, two bundling firms with the same bundle design and with multiple customer groups, and two bundling firms in competition with potentially different bundling proportions. We conclude that for a monopolist bundler the optimal bundle proportion depends on the satiation consumption levels of the customer. When there is just one customer group in the market, the bundle proportion has

a unique global maximum. However, with two customer groups the profit function can have two local maxima and it is possible, though unlikely, to have two optimal bundle proportions. When two bundlers offer potentially independent bundles, in equilibrium, the bundle proportion choices converge. The bundling proportions ratio is a function of the aggregate satiation consumption levels of all customers and is the same for simultaneous as well as sequential entry.

The dissertation of Araz Khodabakhshian is approved.

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2019

DEDICATION

To the martyrs and survivors of the Armenian Genocide, the first genocide of the 20th century, and their descendants who make valuable contribution to the betterment of our world.

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Now that my PhD journey has come to successful completion, I want to take a moment to thank all the people who have supported me along this five-year-long journey.

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Chapter 1 Introduction

Firms with multiple products or services can engage in bundling, which is selling more than one product or service as a combination for a single price (Guiltinan 1987, Adams and Yellen 1976). Bundling is prevalent in many industry sectors, including tangible goods (e.g., gift baskets, option bundles for cars, fast-food menu combos), services (e.g., insurance, fast food, telecommunications, retail banking), and digital platforms (e.g., Google, Amazon, Pandora). Bundling strategies range across a spectrum. At one extreme in a "pure component" strategy, products are only offered separately and customers can purchase any mix according to their preferences. At the other extreme of the spectrum, in "pure bundling", products are only offered in pre-defined combinations or bundles. In between are cases that are referred to as "mixed bundling" and "partial mixed bundling", where one or more bundles are offered along with a subset of pure components, hence giving customers more flexibility in their purchase decisions.

The reasons for bundling vary across markets. However, the most common benefits of bundling are seen either on the supply side, on the demand side, or in competition (Venkatesh and Mahajan 2009). On the supply side, bundling can lead to economies of scope by lowering production costs, transaction costs, and administrative costs (Venkatesh and Mahajan 2009). On the demand side, bundling can expand demand by combining complementary products, e.g., a product and a service contract (see

Eppen 1991), and can reduce customer valuation heterogeneities across different market segments (Eppen 1991), enabling firms to extract more customer surplus (Raghunathan and Sarkar 2016). Under competition, bundling serves as a product differentiation strategy to moderate competition and avoid price wars or to increases customer switching costs in recurring sales of services and goods (Eppen 1991).

Competition is an important factor in determining the success or failure of bundling strategies. Competition might come from firms in the same category of bundled offering, from their own offerings which may include single products as well as bundles, and from other categories of firms offering other combinations of bundles and single products. Under competition, bundling can serve as a product differentiation tool. For example, in recent years, the music, TV, and telecom industries have constantly bundled, re-bundled and unbundled their offerings: a practice that is often enabled by technological advancements. Under competition, every new bundle combination serves as a new offering that helps the firm to differentiate its set of offerings from those of competitors. Although bundling is prevalent in many product and service sectors, the number of research papers studying bundling under competition is limited.

In this thesis, the purpose is to study bundling under competition with a focus on three decisions: Bundling strategy, Entry, and Bundle design. Bundling strategy pertains to the optimal choice of bundle combination, e.g., pure bundle or mixed bundle, given the competitive environment. Every bundling strategy creates a certain revenue potential, affects competition intensity and in general requires a different fixed cost of entry. Hence another important factor shaping a firm's decision is whether to enter a particular market or not. Finally, firms can optimize bundle design by adjusting the proportions of the products within the bundle. In each chapter we address one of these

decisions using Bertrand or Cournot competition models.

In the second chapter, we investigate the following questions:

- What is the optimal bundling strategy for two identical firms who compete in undifferentiated markets under price competition?
- Given their symmetry, will firms adopt symmetric bundling strategies?
- How do the results depend on whether firms practice pure or mixed bundling and on whether customers have positively or negatively correlated valuations?

We consider a symmetric Bertrand duopoly where firms have the ability to offer two undifferentiated components (or goods), separately or as a bundle. Given two components A and B, a product consists of any combination of $\{A\}$, $\{B\}$, and the bundle $\{A, B\}$. Firms first choose their offering as a set of products, and then choose the price vector for their offering. Under *pure* bundling, each firm offers at most one product (i.e., $\{A\}$, $\{B\}$, or $\{AB\}$). Whereas under *mixed* bundling, each firm may offer any set of products; e.g., a firm could offer both $\{A\}$ and $\{AB\}$, but not $\{B\}$. On the demand side, we assume that there are two customer groups. Each customer group has a strictly stronger preference for one of the components; these preferences may be positively or negatively correlated across groups. For tractability, we focus on the extreme cases of perfect positive or negative correlation. Customers purchase the products that give them the highest surplus in terms of total value net of total price.

We conclude that even if the structure of demand (i.e., customers' correlation in valuations) suggests that a monopolistic firm should bundle, it may not be optimal for the firm to do so if it faces a competitor offering a bundle. However, in equilibrium, the firm that bundles in a competitive setting captures the largest profit from the market which implies a first-mover advantage to bundling.

In the third chapter we focus on quantity competition in a Cournot framework since it permits the analysis of oligopolistic competition with multiple identical firms and undifferentiated goods. It is also possible to study entry with fixed costs. By contrast price competition (Bertrand) with undifferentiated products results in a collapse even in the duopoly case, so that it is not possible to tackle entry with cost since the entry costs cannot be recovered so that there is no incentive for even a second firm to enter. In practice it is easy to find cases where multiple relatively undifferentiated firms survive in the market, and are also able to recover the fixed costs of entry, which can be quite substantial. Examples include the telecommunication and financial services sectors. Furthermore, the Cournot model potentially allows for many extensions in terms of the number of firms, the number of customer groups, number of products, and multiple bundle designs. Of course, despite this flexibility in modeling, solving formulations and extracting clear general results is not always easy. We will see that even with simplifying assumptions, the solutions are quite complex with multiple equilibria even in small simplified cases.

We present solutions for two special cases. In the first case there are two types of firms. One type offers a single product, the other type offers that product bundled with another. We show that it is possible to have multiple equilibria with both types of firms and that there is no upper bound to the number of equilibria. We then consider the case where there are three types of firms – two that offer each of the two products alone, and one that offers a bundle of the two. We find that bundling can be a viable strategy when the fixed cost of entry for the bundle is lower than the sum of the entry costs for the independent product markets. In order to identify all the equilibria in a given case, we propose enumeration methods to determine all the equilibria in both these scenarios.

In the fourth chapter, we tackle the bundle design issue for several market settings. The majority of papers studying bundling have assumed that firms offer bundles with fixed proportions and their main goal is to maximize profit through pricing and product offering strategies. In this chapter we take a step back and study the effects of bundle design on firm profits. We allow the choice of *bundle proportions* as an option that firms can employ to maximize profits. As an example, Cable TV providers typically offer multiple bundles with different component proportions that match the needs of specific customer segments.

We assume that firms can offer bundles of two components to customers who have quadratic utility functions. We first show that for a monopolist firm offering a bundle of the two products, with a single type of customer, there is always an optimal *bundle proportion*. This proportion is a function of the satiation consumption levels of the customer. We extend this case to an oligopolistic setting by allowing multiple firms to enter the bundle market and determine the number of firms that are able to survive in equilibrium. Next we study a monopolist bundler who sells a single bundle (type) to two customer groups. If customer groups are very heterogeneous, under special circumstances, this firm can have two distinct choices for the optimal *bundle proportion*. Finally, we study two identical firms offering independent bundles to multiple customer groups. In equilibrium, both firms choose the same *bundle proportion*. The optimal *bundle proportion* is again a function of the aggregate satiation consumption levels of all customers.

Chapter 2 Competitive Bundling in a Symmetric Bertrand Duopoly

Abstract

Bundling, i.e., the practice of offering a combination of products or services, for one price, is widely adopted, although not ubiquitous, in competitive industries. Although the game-theoretic literature on bundling in competitive oligopolies predicts uneven adoption of bundling by competing, but asymmetric, firms, it is not clear whether this results from the firms' strategic response or from their inherent asymmetry. In this chapter, we characterize the bundling strategies of two symmetric firms competing in a Bertrand duopoly. We model the bundling and pricing decisions as a two-stage non-cooperative game. In the first stage, firms choose their offerings, which may include any components and the bundle, considering both pure and mixed bundling strategies. In the second stage, they simultaneously choose prices for their offerings. We show that when firms practice *pure* bundling there always exists an equilibrium in which only one firm bundles. This result extends to *mixed* bundling, in the particular case of two firms offering two components to two customer groups with perfectly correlated valuations for the components. We show that irrespective of whether customers' valuations are positively or negatively correlated at most one of two competing firms chooses to bundle

in equilibrium. Moreover the bundling firm earns higher profits. Our results indicate that bundling can serve as a way to soften price competition, and that the bundling firm has a competitive advantage. Hence for both cases of pure and mixed bundling, asymmetric bundling strategies can be expected even in symmetric oligopolies.

2.1 Introduction

Bundling, or tying, is the practice of selling more than one product or service as a combination for a single price (Guiltinan 1987, Adams and Yellen 1976) and is prevalent in many industry sectors, including tangible goods (e..g., gift baskets, option bundles in cars, fast-food menu combos), services (e.g., insurance, fast food, telecommunications, retail banking industries), and digital platforms (e.g., Google, Amazon, Pandora).

Bundling offers several benefits. On the supply side, bundling can lead to economies of scope by lowering production costs, transaction costs, and administrative costs (Venkatesh and Mahajan 2009). On the demand side, bundling can expand demand by combining complementary products, e.g., a product and a service contract (see Eppen 1991), and reduce customer valuation heterogeneities across different market segments (Eppen 1991), enabling firms to extract more customer surplus (Raghunathan and Sarkar 2016). Under competition, bundling serves as a product differentiation tool to soften competition and avoid price wars and increases customer switching costs in recurring sales of services and goods (Eppen 1991).

Although bundling seems to offer multiple benefits, competing firms in any given sector do not appear to employ a uniform bundling strategy. For instance telecom companies compete through bundling, unbundling, and re-bundling of their existing services. In particular cable companies, which provide broadband Internet, TV, and telephone

services, used to offer double-play or triple-play bundles of several sizes and types. Facing intense competition from Internet streaming services (e.g. Netflix, Amazon Video), they have recently changed their bundling strategy by offering smaller targeted "skinny bundles," in which fewer TV channels are bundled for a cheaper price.

Retail banking is another industry where competing firms practice asymmetric bundling strategies. The incumbents, which are mostly large banks, offer bundles including multiple services such as checking and saving accounts, credit cards and mortgages, whereas new entrants, which are mostly digital, often choose a focus strategy on a product or a segment (e.g. TransferWise). While unbundling could be expected in retail banking, not all players will choose to unbundle (Gujran et al. 2019).

As these two examples suggest, firms may adopt different bundling strategies under competition. Consistent with these examples, most of the game-theoretic literature on competitive bundling has suggested uneven adoption of bundling. However,to the best of our knowledge, it this has been under the assumption that firms were *asymmetric* from the start (e.g., different bargaining power, different presence in multiple product markets, different order of market entry). As a result, it has not been clear whether the asymmetric equilibrium outcome in the bundling game is the result of the firms' strategic response or their inherent asymmetry.

In this chapter, we investigate the following questions: 'What is the optimal bundling strategy for two symmetric firms who compete with undifferentiated offers under price competition? Given their symmetry, will they adopt symmetric bundling strategies? How would that result depend on whether firms practice pure or mixed bundling and on whether customers have positively or negatively correlated valuations?'

We consider a symmetric Bertrand duopoly where firms have the ability to offer

two undifferentiated components (or goods), separately or as a bundle. Given two components A and B, a product consists of any combination of $\{A\}$, $\{B\}$, or the bundle $\{A, B\}$. Firms first choose their offering as a set of products, and then choose the price vector for their offering. Under *pure* bundling, each firm offers at most one product (i.e., $\{A\}$, $\{B\}$, or $\{AB\}$). Whereas under *mixed* bundling, each firm may offer any set of products; e.g., Firm 1 could offer both $\{A\}$ and $\{AB\}$, but not $\{B\}$. On the demand side, we assume that there are two customer groups. Each customer group has a strictly stronger preference for one of the components; these preferences may be positively or negatively correlated across groups. For tractability, we focus on the extreme cases of perfect, positive or negative, correlation. Customers purchase the products that give them the highest surplus in terms of total value net of total price.

We obtain the following results. First, under *pure* bundling, regardless of the number of customer groups and their relative valuations for different product, there always exists an equilibrium such that only one firm bundles and the second firm offers either one of the single components. Second, under *mixed* bundling, we find that when customers have perfectly negatively correlated valuations and the difference in their valuation of the components is significant, then in equilibrium just one firm bundles and the other offers a single component. Second, when customers have perfectly positively correlated valuations, or perfectly negatively correlated valuations with a small difference in valuation of the components, then at least one firm bundles. The second firm may bundle, offer a single component, or offer nothing. In other words, multiple equilibria exist for this case. The Pareto-optimal strategy is when each firm focuses on one of the component markets. All equilibria are asymmetric, although firms are symmetric. We thus show that an asymmetric equilibrium outcome in the bundling game typically arises

even when firms are symmetric.

Our results lead to the following managerial implications. Even if the structure of demand (e.g., customers' correlation in valuations) suggests that a monopolistic firm should bundle, it may not be optimal for the firm to do so if it faces a competitor offering a bundle. This may explain why new entrants in media and entertainment (e.g. Netflix) and in retail banking (e.g. TransferWise) compete with large incumbents by staying focused on a niche. Moreover, the firm that bundles in a competitive setting captures the largest profit from the market and there is therefore a first-mover advantage to bundling. An example is the telecom industry, where large incumbent firms are able to earn greater revenues by offering several different bundles of voice, text, data, and content, while smaller firms can only focus on a smaller subset of those products.

This chapter is organized as follows. In the next section we briefly review the literature on bundling. We then model the bundling and pricing game as a two-stage non-cooperative game. We characterize the equilibrium pricing and bundling strategies under pure bundling in Section 2.4 and under mixed bundling in Section 2.5. Section 2.6 summarizes the results and discusses the managerial implications of our work. All proofs appear in the appendices.

2.2 Literature Review

Although bundling has been extensively studied in economics, it has received a lot of recent attention due to the growth of high-technology sectors and increasing competition; see Venkatesh and Mahajan (2009), Bakos and Brynjolfsson (2000), Raghunathan and Sarkar (2016), Kobayashi (2005), and Stigler (1963). Although bundling often arises in competitive settings in practice, most research on bundling has established the optimal-

ity of bundling from a monopolist's point of view, identifying the following advantages of bundling:

- Price discrimination by reducing customer valuation heterogeneities (Stigler 1963, Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989, Salinger 1995, Bakos and Brynjolfsson 2000, Raghunathan and Sarkar 2016),
- Economies of scope in production, distribution, and promotional activities (Raghunathan and Sarkar 2016, Eppen 1991),
- Demand complementarities among the bundle components (Raghunathan and Sarkar 2016, Eppen 1991).

In contrast to the large literature on bundling in monopolistic settings, the literature on bundling under competition is limited (Anderson and Leruth 1993, Bakos and Brynjolfsson 2000, Nalebuff 2004). Typically this literature considers two component-markets with the possibility of either selling the components individually or bundling them (Kobayashi 2005, Shu et al. 2014) and assumes various degrees of asymmetry between firms or different market structures across the components. We distinguish three groups of papers:

- One group of papers (Whinston 1989, Carbajo et al. 1990, Bakos et al. 2005, Venkatesh and Mahajan 2009, Nalebuff 2004, Vamosiu 2018) assumes firms compete in only one of the product markets and that one of the firms has monopolistic power in the other product market. The monopolist firm can increase its competitive advantage in the competitive market by bundling the common product with the product for which it has monopolistic market power. This effect, known as leverage theory, usually hinders new entry in the competitive market (Whinston

1989, Carbajo et al. 1990, Nalebuff 2004).

- A second group of papers (Schmalensee 1984, Greenlee et al. 2008, Chen et al. 1997, Evans and Salinger 2005) assumes perfect competition in one or both of the markets. In other words these papers assume price-taking behavior on some or all of the bundle components.

- A third group of papers (Anderson and Leruth 1993, Liao and Tauman 2002, Matutes and Regibeau 1992, Economides 1993, Raghunathan and Sarkar 2016, Shu et al. 2014), including ours, assumes that both products are offered in oligopolistic markets.

We contribute to that latter stream by considering the most generic model of oligopolistic competition, namely a Bertrand duopoly. In particular, we consider a simultaneous-move game in contrast to Shu et. al. (2014) who consider a Stackelberg leader-follower framework. Also, in contrast to Anderson and Leruth (1993), Liao and Tauman (2002), Matutes and Ragibeau (1992), Economides (1993), and Raghunathan and Sarkar (2016) we do not make any assumptions of perfect complementarity or substitutability between the two products in our model.

Although we assume perfectly symmetric firms competing on symmetric market structures, we obtain asymmetric equilibria for the game. Given that our model relies on very generic assumptions, we clearly demonstrate the optimality of bundling under competition as a way to differentiate offerings and soften price competition, and that an asymmetric equilibrium outcome is that the result of the firms' strategic behavior, and not of their inherent difference.

Moreover, we study the full set of possible strategies, i.e., Pure Component, Unbundled Offering, Pure Bundling, Mixed Bundling and Partial Mixed Bundling (Adams and

Yellen 1976, Bhargava 2013). Previous papers have found that even for a monopolist firm and a small number of products, the computational burden for comparing Pure Bundling, Mixed Bundling and Individual Pricing is quite large (Vamosiu 2018). Therefore several papers do not consider one or more of the bundling strategies (Bhargava 2013, Bakos and Brynjolfsson 2000, Geng et al. 2005). To the best of our knowledge, this dissertation is the first attempt to identify analytical closed form solutions for the the full scope of potential equilibria, including mixed bundling in a competitive setting (Venkatesh and Kamakura 2003, Wilson 1993, Bhargava 2013, Schmalensee 1984).

2.3 Model

In this section we introduce the model and the equilibrium concept. We consider a duopoly with two symmetric firms operating in two independent component-markets. Each firm has the ability to offer three products, namely Product 1 consisting of Component 1, Product 2 consisting of Component 2, and Product 3 consisting of Components 1 and 2. Similar to Bakos and Brynjolfsson (2000), we assume that products have zero marginal cost of production.[1]

Bundling decisions are usually considered irreversible: once a firm makes a product offering decision, it usually needs to stay committed to that decision for a significant amount of time. Accordingly, we model the game as a two-stage game. In the first stage, which we call the *bundling game*, firms choose their product offering (or bundling) strategy simultaneously and non-cooperatively. In the second stage, which we call the

[1] The effect of a uniform non-zero production cost will essentially be similar to a lower customer valuation. However, if the production costs are non-uniform across the firms, the lowest-cost firm has an advantage and the Bertrand paradox disappears (Fudenberg and Tirole 1991). Since our focus is on demonstrating the existence of asymmetric equilibria even when firms are symmetric, this case lies outside the scope of this chapter. We also note that this level of asymmetry would further amplify the combinatorial nature of the problem. Although an equilibrium characterization is in principle feasible in this case, it would most likely be cumbersome to express – making it very difficult to generate insights.

pricing game, each firm chooses its pricing decisions.

The demand side consists of two homogeneous segments of surplus-maximizing customers. For simplicity, we assume that both segments have the same size, which is normalized to one. A customer's valuation for the bundle is assumed to be the sum of her valuations for each component inside the bundle (Guiltinan 1987, Schmalensee 1984, Adams and Yellen 1976, McAfee et al. 1989), i.e., there is no complementarity or substitutability in customer valuations (Venkatesh and Kamakura 2003). We assume that each customer has zero reservation utility and buys at most one unit of each component, i.e., there is full satiation in consumption.

Tie-Breaking Rules

We consider the following tie-breaking rules:

1- More is better: If two products yield the same surplus to a customer, the customer chooses the product that has the largest number of components.

2- Even split: If two products yield the same surplus to a customer and they both have the same number of components, the customer will randomly choose one product with equal probability.

Notation

We adopt the following notations.

Indices:

i	indexes firms $i = 1, 2$,
j	indexes customers $j = 1, 2$,
k	indexes components $k = 1, 2$,
l	indexes products $l = 1, 2, 3,$ where $l = 1$ refers to a product consisting of only Component 1, $l = 2$ refers to a product consisting of only Component 2, and $l = 3$ refers to the bundle of Components 1 and 2.

Parameters:

u_{jk}	Customer j's valuation for Component k,
v_{jl}	Customer j's valuation for Product l. In the absence of complementarity effect $v_{jl} = u_{jk}$ for $k = 1, 2,$ and $v_{j3} = u_{j1} + u_{j2}$.

Decision Variables:

z_{il}	Firm i's decision to include Product l in its offering, $z_{il} \in \{0, 1\}$,
$\mathbf{z}_i = (z_{i1}, z_{i2}, z_{i3})$	
$\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$	
p_{il}	Firm i's price of Product l, $p_{il} \geq 0$,
$\mathbf{p}_i = (p_{i1}, p_{i2}, p_{i3})$	
$x_{ijl}(\mathbf{p}, \mathbf{z})$	indicator variable for Customer j's purchase of Product l from Firm i at p_{il}

$$\begin{cases} x_{ijl}(\mathbf{p}, \mathbf{z}) = 1 \text{ if Customer } j \text{ buys Product } l \text{ from Firm } i \text{ at price } p_{il}, \\ x_{ijl}(\mathbf{p}, \mathbf{z}) = \frac{1}{2} \text{ if Customer } j \text{ randomizes the purchase between the two firms,} \\ x_{ijl}(\mathbf{p}, \mathbf{z}) = 0 \text{ otherwise,} \end{cases}$$

$F_i(\mathbf{p}_i)$	Firm i's pricing cumulative distribution function (CDF),
$\mathcal{F}$	Set of all feasible price distributions,
	$\mathcal{F} = \{F(p) \mid F(0) = 0, \lim_{p\to\infty} F(p) = 1, \text{ and } F(p) \text{ non-decreasing}\}$,
$\mathcal{Z}$	Set of all feasible product offerings.

Functions:

$\Pi_i(\mathbf{p}_i; \mathbf{p}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i})$	Firm i's profit function,
$\hat{\mathbf{p}}_i(\mathbf{p}_{-i}; \mathbf{z}_i, \mathbf{z}_{-i})$	Firm i's best-response correspondence for given $p_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}$,
$\hat{F}_i(F_{-i}; \mathbf{z}_i, \mathbf{z}_{-i})$	Firm i's best response to the pricing mixed-strategy of its opponent,
$s_{ijl}(\mathbf{p})$	$= (v_{jl} - p_{il})x_{ijl}(\mathbf{p}, \mathbf{z})$ Customer j's surplus from consuming Product l at Firm i's p

To simplify the notation, we sometimes omit the dependence of x_{ijl} on $\mathbf{p}$ and $\mathbf{z}$. Since for any $j = 1, 2$, Customer j buys at most one unit of each component, we have:

$$\sum_i x_{ij1} + x_{ij3} \leq 1 \text{ and } \sum_i x_{ij2} + x_{ij3} \leq 1 \quad \forall j.$$

Similarly, since a firm needs to offer a product for a consumer to buy it, we have:

$$x_{ijl}(\mathbf{p}, \mathbf{z}) \leq z_{il}, \forall i, j, l.$$

$$
\begin{aligned}
x_{ij3} = 1 \quad &\Longleftrightarrow \quad z_{i3} = 1 \text{ and } v_{j3} - p_{i3} \geq \max(0, (v_{jl'} - p_{i'l'})z_{i'l'}) \quad \forall i' \text{ and } l' = 1, 2 \\
&\qquad \text{and } v_{j3} - p_{i3} > (v_{j3} - p_{-i3})z_{-i3}. \\
x_{ijl} = 1 \quad &\Longleftrightarrow \quad z_{il} = 1 \text{ and } v_{jl} - p_{il} > \max(0, (v_{jl} - p_{-i,l})z_{-i,l}), (v_{j3} - p_{i3})z_{i3}, (v_{j3} - p_{-i,3})z_{-i,3}) \text{ for } l = 1, \\
x_{ij3} = \frac{1}{2} \quad &\Longleftrightarrow \quad z_{i3} = z_{-i,3} = 1 \text{ and} \\
&\qquad (v_{j3} - p_{i3})z_{i3} = (v_{j3} - p_{-i,3})z_{-i3} \geq \max(0, (v_{jl'} - p_{i'l'})z_{i'l'}) \quad \forall i' \text{ and } l' = 1, 2. \\
x_{ijl} = \frac{1}{2} \quad &\Longleftrightarrow \quad z_{il} = z_{-i,l} = 1 \text{ and} \\
&\qquad (v_{jl} - p_{il})z_{il} = (v_{jl} - p_{-i,l})z_{-il} > \max(0, (v_{j3} - p_{i3})z_{i3}, (v_{j3} - p_{-i,3})z_{-i3}).
\end{aligned}
\tag{2.1}
$$

Pure and Mixed Pricing Strategies

In the pricing game, we assume firms play in pure strategies if there exists such an equilibrium, and adopt otherwise a mixed strategy. For a given bundling strategy, each firm's profit function is a piecewise linear function of its own price:

$$
\Pi_i(\mathbf{p}_i; \mathbf{p}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = \sum_{j,l} p_{il} x_{ijl}(\mathbf{p}, \mathbf{z}) \text{ for } i = 1, 2.
$$

Each firm randomizes its price according to the cumulative distribution function $F_i \in \mathcal{F}$, $i = 1, 2$ and has the following expected profit function:

$$
\Pi_i(F_i; F_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = \int_0^\infty \int_0^\infty \Pi_i(p_i; p_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) dF_{-i}(p_{-i}) dF_i(p_i).
$$

Firm i's best response to the mixed strategy of its opponent is identified by the

cumulative distribution function $\hat{F}_i$:

$$\hat{F}_i(F_{-i}; \mathbf{z}_i, \mathbf{z}_{-i}) = \arg\max_{F_i \in \mathcal{F}} \Pi_i(F_i; F_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}), \quad \forall i = 1, 2. \tag{2.2}$$

Although for any $i = 1, 2$ the profit function of Firm i is not continuous in $\mathbf{p}_i$ over its entire strategy space, there always exists a mixed-strategy Nash equilibrium (Fudenberg and Tirole 1991), and we construct such an equilibrium (see Proposition 2.5).

The game can have a pure-strategy Nash equilibrium if the mixed-strategy best-response correspondences of the firms are singletons; in that case, we select the pure-strategy Nash equilibrium and denote it as $(\hat{\mathbf{p}}_i(\mathbf{z}), \hat{\mathbf{p}}_{-i}(\mathbf{z}))$. If the pair of CDFs $\hat{F}_i(\hat{F}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i})$ and $\hat{F}_{-i}(\hat{F}_i, \mathbf{z}_i, \mathbf{z}_{-i})$ is a mixed-strategy Nash equilibrium for each choice of offering $\mathbf{z} = (\mathbf{z}_1, \mathbf{z}_2)$, then we have the following for $i = 1, 2$:

$$\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \Pi_i(\hat{F}_i; \hat{F}_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}), \tag{2.3}$$

Bundling Strategies

We assume firms choose their bundling strategy simultaneously and non-cooperatively. The bundling game is a finite game and, according to Nash (1950), it always has a mixed-strategy Nash equilibrium, $(\hat{\mathbf{z}}_i, \hat{\mathbf{z}}_{-i})$, such that for $i = 1, 2$:

$$\hat{\mathbf{z}}_i(\mathbf{z}_{-i}) = \arg\max_{\mathbf{z}_i \in \mathcal{Z}} \hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}). \tag{2.4}$$

However, this game might have multiple equilibria. In Section 2.6, we discuss how the indeterminacy associated with multiple equilibria can be addressed by employing equilibrium selection rules (Harsanyi et al. 1988).

We use the general rule that every firm on the market can choose to capture either none, one, or both customers for each product they offer. Profit-maximizing firms can indeed choose to serve only the higher-value customers if doing so enables them to charge higher prices and extract more revenue even if the lower-value consumers remain unserved (Armstrong 1996).

In this chapter we model two types of bundling games: pure and mixed bundling:

Pure Bundling $\mathcal{Z} = \{\mathbf{z}_i | \mathbf{z}_i^T \mathbf{z}_i \leq 1, \mathbf{z}_i \in \{0,1\}^3\}$. Each firm may offer only one product, either one of the individual components or the bundle,

Mixed Bundling $\mathcal{Z} = \{\mathbf{z}_i \in \{0,1\}^3\}$. Each firm may offer any number of products, i.e., any subset of the individual components and the bundle. In the literature, Mixed Bundling sometimes refers to an offerings consisting of the bundle and all of the single components. Here, we use Mixed Bundling for a larger set of offerings including partial mixed-bundling, i.e., offering the bundle along with only one of the single components (Bhargava 2013).

Customer Valuations For tractability we consider the following two special structures of correlation of customer valuations across groups:

Perfectly Negatively Correlated (PPC) valuations between customers, i.e., $v_{11} = v_{22} = \overline{v}$, $v_{12} = v_{21} = \underline{v}$,

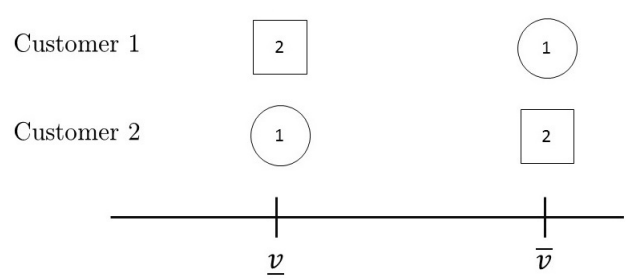


Figure 2.1: Perfectly Negatively Correlated customer valuations

Perfectly Positively Correlated (PPC) valuations between customers, i.e., $v_{12} = v_{22} = \underline{v}$, $v_{21} = v_{11} = \overline{v}$.

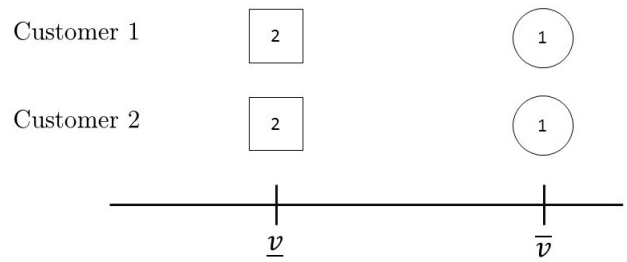


Figure 2.2: Perfectly Positively Correlated customer valuations

In both cases without loss of generality, we assume $\overline{v} > \underline{v} > 0$.

We first show that, when both firms offer the same offering, they engage in a price war, which eventually drives down profits to zero.

Lemma 2.1. *When both firms have identical offerings, i.e.,* $\mathbf{z}_1 = \mathbf{z}_2$*, they both make zero profit, in equilibrium.*

The results of Lemma 2.1 holds for both cases of Pure Bundling and Mixed Bundling under both assumptions of PPC or PNC.

We analyze the pure bundling game in the next section and the mixed bundling game in Section 2.5, considering the cases of PNC and PPC valuations.

2.4 Pure Bundling

We first consider the case where each firm offers only one product, i.e., either the bundle or one of the individual components, for $i = 1, 2$. Hence, each firm faces four possible strategies, namely (0,0,0), (1,0,0), (0,1,0), (0,0,1).[2] We find the set of the potential equilibria always contains four equilibria in which one firm bundles and the

[2] All results in this section are robust to the number of customer groups (i.e., can be greater than 2), their respective size (i.e., can be unequal), and their valuations (i.e., can be arbitrary).

other does not; i.e., for any $i = 1, 2$, $\mathbf{z}_i = (0, 0, 1)$ and $\mathbf{z}_{-i} = (1, 0, 0)$, and $\mathbf{z}_i = (0, 0, 1)$ and $\mathbf{z}_{-i} = (0, 1, 0)$. For certain values of customer reservation prices, the equilibrium set can be larger. In all cases, the firm that offers the bundle makes more profit in equilibrium.

Proposition 2.1. *Under pure bundling, the game always has at least four equilibria where one firm offers the bundle and the other one offers a single component. The firm that offers the bundle earns more profit than the other.*

2.5 Mixed Bundling

In this section, we consider mixed bundling and separately analyze the case of PNC and PPC customer valuations. We solve the game presented in 2.2,2.3, and 2.4 by backward induction. First, we identify the mixed-strategy Nash equilibrium for the pricing game for each possible outcome of bundling game. Subsequently, we identify the equilibrium strategies in the bundling game, incorporating the equilibrium outcomes of the pricing sub-games.

The sixty-four possible offerings $\mathbf{z}$ can be categorized as follows:

- No firm bundles, i.e., $z_{i3} = z_{-i,3} = 0$
- Only one firm bundles, i.e., $z_{i3} + z_{-i,3} = 1$
- Both firms bundle, i.e., $z_{i3} + z_{-i,3} = 2$

We first consider the case of Perfectly Negatively Correlated (PNC) valuations.

2.5.1 Perfectly Negatively Correlated Valuations

2.5.1.1 The Pricing Game

2.5.1.1.1 No firm bundles

Proposition 2.2. *Under PNC valuations, when no firm offers a bundle and when firms have non-overlapping offerings, for $i = 1, 2$:*

- *If* $\mathbf{z}_i = (1, 0, 0)$ *and* $\mathbf{z}_{-i} = (0, 0, 0)$, *then* $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \max\{2\underline{v}, \overline{v}\}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,
- *If* $\mathbf{z}_i = (0, 1, 0)$ *and* $\mathbf{z}_{-i} = (0, 0, 0)$, *then* $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \max\{2\underline{v}, \overline{v}\}$, *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,
- *If* $\mathbf{z}_i = (1, 1, 0)$ *and* $\mathbf{z}_{-i} = (0, 0, 0)$, *then* $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = \max\{2\overline{v}, 4\underline{v}\}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,
- *If* $\mathbf{z}_i = (1, 0, 0)$ *and* $\mathbf{z}_{-i} = (0, 1, 0)$, *then* $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = \max\{2\underline{v}, \overline{v}\}$.

Proposition 2.3. *Under PNC valuations, if* $\mathbf{z}_i = (1, 1, 0)$ *and either* $\mathbf{z}_{-i} = (1, 0, 0)$ *or* $\mathbf{z}_{-i} = (0, 1, 0)$, *then* $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \max\{\overline{v}, 2\underline{v}\}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$.

2.5.1.1.2 Only one firm bundles We break down this category of offerings into several sub-cases and show the equilibrium profits of the firms in each case.

Proposition 2.4. *Under PNC valuations, when one firm offers the bundle, i.e.,* $z_{i3} = 1$, *and the other firms stays out of the business, i.e.,* $\mathbf{z}_{-i} = (0, 0, 0)$, $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = 2(\overline{v} + \underline{v})$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$ *for* $i = 1, 2$.

If a bundling firm faces competition in one of the component markets, it might prefer to capture only the customer that has the highest valuation for that particular

component, or both customers, or randomize between these two strategies. The firm's choice depends on the relative magnitude of customers' valuations for both products.

Proposition 2.5. *Under PNC valuations, when* $\mathbf{z}_i = (0,0,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$ *or when* $\mathbf{z}_i = (0,0,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$ *there exists a pure-strategy Nash equilibrium in the pricing game if and only if* $\overline{v} \leq 2\underline{v}$*. Otherwise, when* $\overline{v} \geq 2\underline{v}$*, there only exists a unique mixed-strategy Nash equilibrium in the pricing game, constructed in Appendix B. With these equilibria, the payoffs are:*

- *For* $\overline{v} > 3\underline{v}$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = \overline{v} + \underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = \frac{\overline{v}-\underline{v}}{2}$*.*
- *For* $2\underline{v} < \overline{v} \leq 3\underline{v}$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2(\overline{v} - \underline{v})$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = \overline{v} - 2\underline{v}$*.*
- *For* $\overline{v} \leq 2\underline{v}$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Proposition 2.6. *Under PNC valuations, when* $\mathbf{z}_i = (1,1,0)$ *and* $\mathbf{z}_{-i} = (0,0,1)$*, then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 0$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 2\underline{v}$*.*

Proposition 2.7. *Under PNC valuations, when* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$ *or when* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$*,*

- For $\overline{v} > 2\underline{v}, \hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = \overline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$
- For $\overline{v} \leq 2\underline{v}, \hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$

Proposition 2.8. *Under PNC valuations, when* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$ *or when* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$*, Firm* i *sets the price of its component so high that no customer buys it. Hence, the equilibrium is the same as in Proposition* 2.5*.*

Proposition 2.9. *Under PNC valuations, when* $\mathbf{z}_i = (1,0,1)$ *or* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (1,1,0)$*, then* $\hat{\Pi}_i = 2\underline{v}$ *and* $\hat{\Pi}_{-i} = 0$*.*

Corollary 2.1. *Under PNC valuations, when* $\mathbf{z}_i = (1,1,1)$ *and when either* $\mathbf{z}_{-i} = (1,0,0)$ *or* $\mathbf{z}_{-i} = (0,1,0)$,

- *For* $\overline{v} > 3\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = \overline{v} + \underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = \frac{\overline{v}-\underline{v}}{2}$.
- *For* $2\underline{v} < \overline{v} \leq 3\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2(\overline{v} - \underline{v})$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = \overline{v} - 2\underline{v}$.
- *For* $\overline{v} \leq 2\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$.

Corollary 2.2. *Under PNC valuations, when* $\mathbf{z}_i = (1,1,1)$ *and* $\mathbf{z}_{-i} = (1,1,0)$, *then* $\hat{\Pi}_i = 2\underline{v}$, *and* $\hat{\Pi}_{-i} = 0$.

2.5.1.1.3 Both firms bundle

Proposition 2.10. *Under PNC valuations, when both firms offer the bundle as part of their offering, then* $\hat{\Pi}_i = \hat{\Pi}_{-i} = 0$.

In Lemma 2.1 we showed an extension of a classic result, i.e., two firms make zero profit under Bertrand competition with identical offerings. Proposition 2.10 extends this result to the case of differentiated offerings, as long as they both contain the bundle.

2.5.1.2 The Bundling Game

Next, we consider the bundling game. Tables 2.1, 2.2, and 2.3 summarize the equilibrium payoffs for each combination of offerings, under different customer valuations. The tables represent finite strategic-form games in which each player has eight strategies to choose from. According to Nash (1950), each game has at least one mixed-strategy Nash equilibrium. In fact, we find that there always exists a pure-strategy equilibrium in each of the bundling games, but that equilibrium may not be unique. All pure-strategy equilibria are highlighted in the tables.

		Firm 2's offering							
		$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
Firm 1's offering	$(0,0,0)$	$0,0$	$0,\overline{v}$	$0,\overline{v}$	$0,2\overline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
	$(1,0,0)$	$\overline{v},0$	$0,0$	$\overline{v},\overline{v}$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$
	$(0,1,0)$	$\overline{v},0$	$\overline{v},\overline{v}$	$0,0$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$	$0,\overline{v}$	$\frac{\overline{v}-\underline{v}}{2},\overline{v}+\underline{v}$
	$(1,1,0)$	$2\overline{v},0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$\overline{v}+\underline{v},\frac{\overline{v}-\underline{v}}{2}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$

Table 2.1: Firms' payoff matrix in the mixed bundling game under PNC valuations when $\overline{v} > 3\underline{v}$

		Firm 2's offering							
		$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
Firm 1's offering	$(0,0,0)$	$0,0$	$0,\overline{v}$	$0,\overline{v}$	$0,2\overline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
	$(1,0,0)$	$\overline{v},0$	$0,0$	$\overline{v},\overline{v}$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$
	$(0,1,0)$	$\overline{v},0$	$\overline{v},\overline{v}$	$0,0$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$	$0,\overline{v}$	$\overline{v}-2\underline{v},2(\overline{v}-\underline{v})$
	$(1,1,0)$	$2\overline{v},0$	$\overline{v},0$	$\overline{v},0$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$\overline{v},0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2(\overline{v}-\underline{v}),\overline{v}-2\underline{v}$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$

Table 2.2: Firms' payoff matrix in the mixed bundling game under PNC valuations when $2\underline{v} < \overline{v} \leq 3\underline{v}$

From Tables 2.1 and 2.2, it appears that only one firm offers the bundle as part of its offering and the other firm offers only a single component. In each case, two symmetric sets of six payoff-equivalent equilibria are possible. The firm offering the bundle earns more profit and thus there is an inherent first-mover advantage to bundling.

In our model we did not associate any cost with offering additional products, but in reality firms will avoid adding components with zero marginal revenue. Thus, in the presence of infinitesimal costs for offering a product, the surviving equilibria for $\overline{v} \geq 2\underline{v}$ will be $\mathbf{z}_i = (0,0,1)$ and either $\mathbf{z}_{-i} = (0,1,0)$ or $\mathbf{z}_{-i} = (1,0,0)$ for $i = 1,2$. This result if consistent with the pure bundling case analyzed in Section 2.4.

		Firm 2's offering							
		$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
Firm 1's offering	$(0,0,0)$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,4\underline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
	$(1,0,0)$	$2\underline{v},0$	$0,0$	$2\underline{v},2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,1,0)$	$2\underline{v},0$	$2\underline{v},2\underline{v}$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(1,1,0)$	$4\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\underline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$

Table 2.3: Firms' payoff matrix in the mixed bundling game under PNC valuations when $\overline{v} \leq 2\underline{v}$

Note that all equilibria under the PNC valuations are payoff-equivalent. Therefore, the bundler does not gain anything (or does not make the other firm lose anything) from offering more than the bundle . The set of equilibria is the same when $\overline{v} > 3\underline{v}$ and when $2\underline{v} < \overline{v} \leq 3\underline{v}$.

When $\overline{v} \leq 2\underline{v}$, the game has multiple equilibria with different payoffs, as shown in Table 2.3. However, the only equilibria where both firms earn non-zero payoffs are when each firm focuses on offering only one type of component. All other equilibria result in zero payoffs for one or both of the firms. Thus the Pareto-optimal equilibria of this game are $\mathbf{z} = ((1,0,0),(0,1,0))$, and $\mathbf{z} = ((0,1,0),(1,0,0))$.

We conclude that under PNC valuations, when customers have a strictly stronger

preference for one of the components (i.e., $\overline{v} > 2\underline{v}$), then one firm bundles in equilibrium. However, when customers have close valuations of the components (i.e., $\overline{v} \leq 2\underline{v}$), then the Pareto-optimal equilibrium is the focus strategy, i.e., firms offer distinct components.

2.5.2 Perfectly Positively Correlated Valuations

2.5.2.1 The Pricing Game

2.5.2.1.1 No firm bundles

Proposition 2.11. *Under PPC valuations, when no firm bundles and when firms have non-overlapping offering, for $i = 1, 2$:*

- *If $\mathbf{z}_i = (1, 0, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\overline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,*
- *If $\mathbf{z}_i = (0, 1, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\underline{v}$, and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,*
- *If $\mathbf{z}_i = (1, 1, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = 2(\overline{v} + \underline{v})$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$,*
- *If $\mathbf{z}_i = (1, 0, 0)$ and $\mathbf{z}_{-i} = (0, 1, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\overline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 2\underline{v}$.*

Proposition 2.12. *Under PPC valuations, if $\boldsymbol{z}_i = (1, 1, 0)$ and $\boldsymbol{z}_{-i} = (1, 0, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\underline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$.*

Proposition 2.13. *Under PPC valuations, if $\boldsymbol{z}_i = (1, 1, 0)$ and $\boldsymbol{z}_{-i} = (0, 1, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\overline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$.*

2.5.2.1.2 Only one firm bundles

Proposition 2.14. *Under PPC valuations, when one firm offers the bundle, i.e., $z_{i3} = 1$, and the other firm stays out of the business, i.e., $\mathbf{z}_{-i} = (0, 0, 0)$, then $\hat{\Pi}_i(\mathbf{z}_i, \mathbf{z}_{-i}) = 2(\overline{v} + \underline{v})$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$ for $i = 1, 2$.*

Proposition 2.15. *Under PPC valuations, when* $\boldsymbol{z}_i = (0,0,1)$ *and* $\boldsymbol{z}_{-i} = (1,0,0)$*, then* $\hat{\Pi}_i = 2\underline{v}$ *and* $\hat{\Pi}_{-i} = 0$*.*

Proposition 2.16. *Under PPC valuations, when* $\boldsymbol{z}_i = (0,0,1)$ *and* $\boldsymbol{z}_{-i} = (0,1,0)$*, then* $\hat{\Pi}_i = 2\overline{v}$ *and* $\hat{\Pi}_{-i} = 0$*.*

Proposition 2.17. *Under PPC valuations, when* $\mathbf{z}_i = (0,0,1)$ *and* $\mathbf{z}_{-i} = (1,1,0)$ *then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Proposition 2.18. *Under PPC valuations, when* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$*, then* $\hat{\Pi}_i = 2\underline{v}$ *and* $\hat{\Pi}_{-i} = 0$*.*

Proposition 2.19. *Under PPC valuations, when* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$*, then* $\hat{\Pi}_i = 2\overline{v}$ *and* $\hat{\Pi}_{-i} = 0$*.*

Proposition 2.20. *Under PPC valuations, when* $\mathbf{z}_i = (1,0,1)$ *and* $\mathbf{z}_{-i} = (0,1,0)$*, then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\overline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Proposition 2.21. *Under PPC valuations, when* $\mathbf{z}_i = (0,1,1)$ *and* $\mathbf{z}_{-i} = (1,0,0)$*, then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Proposition 2.22. *Under PPC valuations, when* $\mathbf{z}_i = (1,0,1)$ *or* $\mathbf{z}_i = (0,1,1)$*, and* $\mathbf{z}_{-i} = (1,1,0)$*, then* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

Corollary 2.3. *Under PPC valuations, when* $\mathbf{z}_i = (1,1,1)$

- *If* $\mathbf{z}_{-i} = (1,0,0)$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*,*
- *If* $\mathbf{z}_{-i} = (0,1,0)$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\overline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*
- *If* $\mathbf{z}_{-i} = (1,1,0)$*,* $\hat{\Pi}_i(\mathbf{z}_i;\mathbf{z}_{-i}) = 2\underline{v}$ *and* $\hat{\Pi}_{-i}(\mathbf{z}_{-i};\mathbf{z}_i) = 0$*.*

2.5.2.1.3 Both firms bundle

Proposition 2.23. *Under PPC valuations, when both firms offer the bundle as part of their offering, then* $\hat{\Pi}_i = \hat{\Pi}_{-i} = 0$.

2.5.2.2 The Bundling Game

		Firm 2's offering							
		$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(1,1,0)$	$(0,0,1)$	$(1,0,1)$	$(0,1,1)$	$(1,1,1)$
Firm 1's offering	$(0,0,0)$	$0,0$	$0,2\overline{v}$	$0,2\underline{v}$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$	$0,2(\overline{v}+\underline{v})$
	$(1,0,0)$	$2\overline{v},0$	$0,0$	$2\overline{v},2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,1,0)$	$2\underline{v},0$	$2\underline{v},2\overline{v}$	$0,0$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$	$0,2\overline{v}$
	$(1,1,0)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$0,0$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$	$0,2\underline{v}$
	$(0,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,0,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(0,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$
	$(1,1,1)$	$2(\overline{v}+\underline{v}),0$	$2\underline{v},0$	$2\overline{v},0$	$2\underline{v},0$	$0,0$	$0,0$	$0,0$	$0,0$

Table 2.4: Firms' payoff matrix in the mixed bundling game under PPC valuations

Considering next the bundling game, we face a finite strategic-form game in which each player has eight strategies to choose from. Table 2.4 summarizes the equilibrium payoffs. Each cell in this table corresponds to the equilibrium payoffs of both firms for each combination of offerings. Similar to the PNC case, we find that there are multiple pure-strategy Nash equilibria in this game. All pure-strategy equilibria are highlighted

in the table. Note that all equilibria have the same payoff as in the PNC case with $\overline{v} < 2\underline{v}$ (see Table 2.3). The equilibrium consists of three sets of strategies: a) both firms offer the bundle that results in zero payoff for both firms, b) only one firm offers the bundle that results in zero payoff for the firm that does not offer the bundle, c) each firm focuses on one of the single component markets, which has non-zero payoff for both firms. The last set, i.e., the focus strategy, is the Pareto-optimal equilibrium of this game.

2.5.3 Stackelberg Game

We can also use a leader-follower approach to analyze the strategic form games presented in tables $2.1, 2.2, 2.3$ and 2.4 .

2.5.3.1 Perfectly Negatively Correlated Valuations

- When $\overline{v} > 2\underline{v}$, the payoff-equivalent subgame perfect Nash equilibria are
  - {(0,0,1),(1,0,0)} and {(0,0,1),(0,1,0)},
  - {(1,0,1),(0,1,0)},
  - {(0,1,1),(1,0,0)},
  - {(1,1,1),(1,0,0)} and {(1,1,1),(0,1,0)}.
- When $\overline{v} < \overline{v} \leq 2\underline{v}$, the game does not have a Weak Stackelberg equilibrium but the Strong Stackelberg equilibria (Breton et al. 1988) are
  - {(1,0,0),(0,1,0)},
  - {(0,1,0),(1,0,0)}.

2.5.3.2 Perfectly Positively Correlated Valuations

The game does not have a Weak Stackelberg equilibrium but the Strong Stackelberg equilibria are

- {(1,0,0),(0,1,0)},
- {(0,1,0),(1,0,0)}.

We conclude that when customers have negatively correlated valuations and each one has a strictly preferred component, the leader bundles and earns larger profits while the follower offers a single component only. However, when customers have perfectly positively correlated valuations or they do not have a strictly preferred component in the PNC case, the Strong Stackelberg Equilibrium for both firms is a focus strategy, i.e., offering an independent component only.

2.6 Conclusions

In this chapter we propose a stylized model of competitive bundling in a Bertrand duopoly. We consider two symmetric firms with perfect information that make simultaneously but non-cooperatively offering decisions followed by pricing decisions. Within that setting, we characterize the equilibrium bundling strategies in the cases of both pure and mixed bundling.

With pure bundling, we show that there always exists an equilibrium when one firm bundles and the other firm offers a single component, to avoid head-to-head price competition. Moreover, this result is robust to the number, size, and preferences of customer groups.

With mixed bundling, we consider two heterogeneous groups of customers in terms

of their valuations for the components and we consider the specific case of perfect positive or negative correlation. We find that there always exists an equilibrium in the bundling game, though multiple equilibria are possible. If customers have relatively similar valuations for both components, i.e., if their valuations are PPC or PNC with small discrepancies, most equilibria are such that only one firm makes positive profit. However, there is always one Pareto-optimal equilibrium where firms operate in independent component markets without bundling and avoid competition. On the other hand, if customers have strong preferences for distinct components, i.e., if their valuations are PNC with medium to large discrepancies, the game has multiple payoff-equivalent Nash equilibria, in which both firms make non-zero profit, one firm bundles, the other firm offers a single component, and the bundler captures a larger profit.

Our study leads to the following managerial implication: Under competition, the firm that offers a bundle earns more profit compared to the firm that does not. However, when an incumbent firm already bundles, it is optimal for a new entrant not to offer the bundle to avoid head-to-head price competition. This suggests a first-mover advantage for bundling. Also, firms who face the decision of whether to offer a bundle or to focus on a single component market should refer to the relative valuations of the heterogeneous customer segments in the market.

Chapter 3 Bundling and Competition in a Cournot Oligopoly

3.1 Introduction

Bundling is the practice of offering some combination of products and services in fixed proportions with one price for the bundle that is typically discounted below the sum of the prices of the components of the bundle. This has been a common strategy for vendors in many product and service industries. Product examples range from gift baskets and meal combinations in restaurants to large tool sets. Service bundles are common in education, telecommunications and cable services. Post sales services such as maintenance and repair are sometimes bundled with products. There are many incentives for vendors to bundle, including cost savings due to economies of scope, expansion of demand due to a reduction in customer heterogeneity, increased barriers to entry, differentiation of the offer, risk pooling with uncertain demand, and increased customer switching costs. One or more of these advantages might be relevant in a specific market setting.

Competition is an important factor in determining the success or failure of bundling where competition might come from firms in the same category of bundled offering, from their own offerings which may include single products as well as bundles, and from other categories of firms offering other combinations of bundles and single products. Although bundling is prevalent in many product and service sectors, the number of

research papers studying bundling under competition is limited. In this chapter we focus on quantity competition in a Cournot framework, since it permits the analysis of oligopolistic competition with multiple identical firms and undifferentiated goods. It is also possible to study entry with fixed costs. By contrast, price competition (Bertrand) with undifferentiated products results in a collapse even in the duopoly case, so that it is not possible to tackle entry with cost since the entry costs cannot be recovered. Hence there is no incentive for even a second firm to enter. In practice it is easy to find cases where multiple relatively undifferentiated firms survive in the market, and are also able to recover the fixed costs of entry, and make profits, which can be quite substantial. Examples include the telecommunications sector and financial services. Furthermore, the Cournot model potentially allows for many extensions in terms of the number of firms, the number of customer groups, number of products, multiple bundle designs and different customer groups. Of course, despite this flexibility in modeling, solving formulations and extracting clear general results is less easy. We will also see that even with simplifying assumptions, the solutions are quite complex with multiple equilibria even in small cases.

This chapter studies a market which may have multiple customer segments that have differing preferences. Firms may offer independent (single) products or services and bundles of products and/or services, in some mix. There may be multiple symmetric firms of each type. Entry in any category incurs a fixed cost. We address the following research questions:

- What are the production decisions made by each type of firm, and the resulting prices formed in the market?
- For each type of offering, consisting of some combination of single products or

services, and bundles, what is the number of firms that can enter and survive?

- How do these outcomes (quantities and prices) depend on the types of firms that compete, and on customer preferences (i.e., utility function parameters)?

We find that there are generally multiple equilibria in terms of the number of firms of each category that enter and survive, with no further entry possible. We then consider the extension to the case of three types of firms and to two customer groups. In general, both firms with independent products, as well as firms that bundle can enter and survive if the fixed costs of entry are not excessive. The presence of bundled offer has the effect of coupling markets for independent products.

In the next section we present a brief review of the literature on bundling and competition. We present a general model formulation in the third section. In subsequent sections we analyze specific examples using both analytical and numerical methods. We conclude that when firms bundle, the equilibrium is not necessarily unique and multiple (arbitrarily large) equilibria are possible. The degree of differentiation between preference for the standalone products and the relative values of the standalone products are two predictors of the prevalence of bundling.

3.2 Literature Review

The majority of papers in the bundling literature study monopoly bundling (Stigler 1963, Adams and Yellen 1976, Schmalensee 1984, McAfee et al. 1989, Salinger 1995, Varian 1995, Bakos and Brynjolfsson 1999, Venkatesh and Kamakura 2003, Geng et al. 2005, Prasad et al. 2010, Chu et al. 2011). However, in this chapter, our focus is on competitive bundling, which is studied in a few previous papers (Anderson and Leruth 1993, Bakos and Brynjolfsson 2000, Nalebuff 2004). In analyzing bundling behavior,

competition type and product differentiation should be taken into consideration (Chung et al. 2013). Among the papers that study competitive bundling the majority consider competition *a la Bertrand* (Whinston 1989, Economides 1993). Cournot equilibrium formulations are largely absent from the literature. Table 1 summarizes the major contributions in this area. The traditional bundling literature, assumes that customers have unit demand for each of the components. However here, customers may purchase any quantity of any of the products or bundles similar to non-linear pricing, where customers demand can be any quantity of each of the offers available (Wilson 1993).

	Bertrand	Cournot
Monopoly in one market and duopoly in the second market	Whinston (1989),Carbajo et al. (1990) Le Blanc (2001) , Nalebuff (2004) Bakos et al. (2005), Peitz (2008) Chung et al. (2013)	Carbajo et al. (1990), Martin (1999) Le Blanc (2001), Hinloopen (2005) Chung et al. (2013)
Monopoly in one market and perfect competition in the second market	Schmalensee (1982) Chen et al. (1997) Evans and Salinger (2005) Greenlee et al. (2008)	
Duopoly in both markets with complementarity or substitutability assumption	Matutes and Regibeau (1992) Anderson and Leruth (1993) Economides (1993) Liao and Tauman (2002) Mantovani (2013) Raghunathan and Sarkar (2016)	Matutes and Regibeau (1992) Anderson and Leruth (1993) Mantovani (2013)
Duopoly in both markets with independent products	Chapter 2	
Oligopoly with undifferentiated products		Chapter 3

Table 3.1: A summary of the bundling literature.

Only a few papers have studied bundling under quantity competition (Martin 1999, Hinloopen 2005). Some papers study bundling under Cournot competition in parallel with Bertrand competition to contrast the results for some special cases (Carbajo et al. 1990, Matutes and Regibeau 1992, Anderson and Leruth 1993). Most other papers assume differentiated products or assume firms have no strategic choice for one of the goods (Orbay 2009); for example they are price-takers under perfect competition (Chen et al. 1997), or one of the firms is a monopolist in one of the goods (Whinston 1989, Martin 1999, Carbajo et al. 1990). Matutes and Regibeau (1992), Anderson and Leruth (1993) and Mantovani (2013) are some papers that consider duopolistic competition with two components but they assume the goods are perfect complements, which of course favors bundling.

Almost all the papers studying bundling, consider a two-firm, two-product model. In our analysis, we allow for multiple firms for each type of offer. We are able to investigate the number of firms that offer each type of bundle in equilibrium, as a function of the fixed costs of entry into each market.

3.3 Model Formulation

In this section, we present a general formulation of the market model. In the subsequent section, we obtain solutions for a specific instance of this model.

3.3.1 Notation

We examine an oligopoly with I firms where all firms have the ability to sell one of K bundles. Each bundle consists of a subset of T standalone components in various fixed proportions. At entry, each firm i makes a decision to enter either one of the

bundle markets where positive profits are attainable post-entry. Profits in each market k depends on the total quantity offered of bundle k and all other bundles that have any overlap with k. The entry game ends when new firms stop entering any bundle markets and all existing firms make non-negative profits.

We use the following notation:

i indexes firms, $i = 1, ..., I$,

j indexes customers, $j = 1, ..., J$,

k indexes offers or bundles, $k = 1, ..., K$,

t indexes standalone products or components included in bundles, $t = 1, ..., T$,

$\mathbf{B}_k$ $= (a_{k1}, a_{k2}, ..., a_{kT}), a_{kt} \geq 0$ the quantity of each product $t = 1, ..., T$ in the kth bundle,

$\mathcal{B}$ $\equiv \{\mathbf{B}_k | k = 1, ..., K\}$ set of all bundles or offers ,

$\mathbf{n}$ $= (n_k | k = 1, ..., K)$ the number of firms in all K markets, n_k is an integer for $k = 1, ..., K$,

r_{ik} Firm i entry decision, $r_{ik} = \begin{cases} 1 & \text{if } i \text{ enters market } k, \\ 0 & \text{else} \end{cases}$

q_{ik} Quantity of the k'th offer by firm i, $q_{ik} \geq 0$,

Q_k Total quantity of the k'th offering in the market, $Q_k = \sum_i q_{ik}$,

$\mathbf{Q}_{-i}$ The vector of the total quantities produced by all firms except firm i,

F_k Fixed cost of entering the k'th market, $F_k \geq 0$,

v_k	Variable cost of producing each unit of the k'th offer, $v_k \geq 0$,
$\Pi_{ik}(q_{ik}; \mathbf{Q}_{-i})$	Profit of firm i from selling q_{ik} units of the bundle k, given Q_{-i},
$\hat{\Pi}_k(\mathbf{n})$	Equilibrium profit of firms in the k'th market,
$\mathbf{y}_j$	$\equiv (y_{jk} \vert k = 1, ..., K)$ The vector of customer j's purchase levels for the k offers,
y_k	$\equiv \sum_j y_{jk}$ the aggregte purchased quantity of offer k
$\mathbf{y}$	$\equiv (\sum_j y_{jk} \vert k = 1, ..., K)$ The vector of all customers' aggregate purchase levels,
$\mathcal{Y}$	The set of all possible purchase vectors $\mathbf{y}_j$,
$U_j(\mathbf{y}_j)$	Utility function of customer j for consumption $\mathbf{y}_j$, in monetary units
y'_{jt}	$= \sum_k a_{kt} y_{jk}$ Customer j's consumption levels for each component product t,
$U'_{jt}(y'_{jt})$	Customer j's utility function for standalone product t,
$S_j(\mathbf{y}_j, \mathbf{p})$	$\equiv U_j(\mathbf{y}_j) - \mathbf{p} \cdot \mathbf{y}_j$, Customer j's net value for $\mathbf{y}_j$ at prices $\mathbf{p}$.

3.3.2 The Walrasian Equilibrium

Two Walrasian equilibrium conditions determine the prices and the quantities in equilibrium:

a) Customers choose utility maximizing purchase quantities:

We assume customers do not have any budget constraints, however they always have an alternative opportunity to spend each dollar to gain a utility of one. In other words, the marginal utility of saving each dollar equals one. Thus, given any price vector $\mathbf{p}$,

each customer j chooses the consumption levels $\mathbf{y}_j$ that maximize total net value S_j.

$$\begin{aligned} \max_{\mathbf{y}_j \in \mathcal{Y}} \quad & S_j(\mathbf{y}_j; \mathbf{p}) \\ \text{subject to} \quad & \mathbf{y}_j \geq 0, \end{aligned} \tag{3.1}$$

where,

$$\bar{\mathbf{y}}_j(\mathbf{p}) = \arg\max_{\mathcal{Y}} S_j(\mathbf{y}_j; \mathbf{p}). \tag{3.2}$$

In our formulation we do not consider any budget constraints.

b) Prices are set to clear the market:

Let $D_k(\mathbf{p}) = \sum_j \bar{y}_{jk}(\mathbf{p})$ represent the aggregate demand for offer k. Prices are chosen so that $D_k(\mathbf{p}) = Q_k$ for $k = 1, ..., K$. Therefore, under market clearing conditions, $\mathbf{p}$ is such that $p_k \cdot (Q_k - D_k(\mathbf{p})) = 0 \ \forall k$. This implies that markets clear for all products with positive prices and any product in excess supply has a price equal to zero.

Let $\mathbf{Q}$ be the vector of the production quantities in all markets. Let $\mathbf{D}$ be a multivariate function such that $\mathbf{Q} = \mathbf{D}(\mathbf{p})$. If $\mathbf{D}$ is an invertible function, the inverse demand function for all offers is[1]:

$$\mathbf{p} = \mathbf{D}^{-1}(\mathbf{Q}) \tag{3.3}$$

c) Firms choose optimal production levels:

Next, firms in each market k choose the optimal production levels in a one-period game, given the price vector $\mathbf{p}$. Each firm can sell in only one market k. Each firm i

[1]Orbay (2009) proposes a method for finding the optimal prices even if the inverse function is not computable.

producing offer k solves for the following profit maximizing problem.

$$\begin{aligned} \max_{q_{ik}} \quad & \Pi_{ik}(q_{ik}; \mathbf{Q}_{-i}) \\ \text{subject to} \quad & q_{ik} \geq 0, \end{aligned} \tag{3.4}$$

where $\Pi_{ik}(q_{ik}; \mathbf{Q}_{-i}) = q_{ik}[p_k(q_{ik}, \mathbf{Q}_{-i}) - v_k]$. Thus the optimal production levels and the profit of the firm i are the following:

$$\hat{\Pi}_{ik}(\mathbf{Q}_{-i}) = \max_{q_{ik}} q_{ik}[p_k(q_{ik}, \mathbf{Q}_{-i}) - v_k]; q_{ik} \geq 0 \tag{3.5}$$

$$\hat{q}_{ik}(\mathbf{Q}_{-i}) = \arg\max_{q_{ik}} q_{ik}[p_k(q_{ik}, \mathbf{Q}_{-i}) - v_k] \tag{3.6}$$

In equilibrium, due to symmetry, all firms in the market k produce the same amount $\hat{q}_k$, and the amount is a function of the number of firms present in all markets, thus, $Q_k = n_k \hat{q}_k(\mathbf{n})$. Therefore, for symmetric firms the equilibrium profits are a function of the number of firms present in all markets, i.e., $\hat{\Pi}_k(\mathbf{n})$.

3.3.2.1 Firm Entry Decisions and Equilibrium Characterization

Firm i has the binary choice of entering the k'th offering market or not. Then we define $\mathbf{n}_{k+} \equiv (n_1, n_2, ..., n_k + 1, n_{k+1}, n_{k+2}, ...)$ and:

$$r_{ik}(\mathbf{n}) = \begin{cases} 1 & \text{If} \quad \hat{\Pi}_k(\mathbf{n}_{k+}) \geq F_k \\ 0 & \text{Otherwise} \end{cases} \tag{3.7}$$

We assume that each firm can enter at most one market k to enable comparison. So $\sum_k r_{ik}(\mathbf{n}) = 1, \forall i$. Also I is assumed to be large enough to let new firms enter as long as they make positive profits.

To identify the equilibrium in this game, we define the following two sets:

$$
\begin{aligned}
S &\equiv \{\mathbf{n} | \hat{\Pi}_k(\mathbf{n}) \geq \min\{F_k, n_k F_k\}, \forall k\}, \quad S \in \mathbb{N}^K, \\
N &\equiv \{\mathbf{n} | \hat{\Pi}_k(\mathbf{n}_{k+}) < F_k, \forall k\}, \quad N \in \mathbb{N}^K.
\end{aligned}
\tag{3.8}
$$

Where set S contains all $\mathbf{n}$ such that every firms earns nonnegative net profit. Set S also contains all the vectors $\mathbf{n}$ with one or more zero components, i.e., markets with no entrants. Set N contains vectors $\mathbf{n}$ representing situation where no new firm is able to make a profitable entry. We define $E \equiv S \cap N, S, N, E \in \mathbb{I}^K$.

Definition 3.1. *The vector $\boldsymbol{n}$ is an equilibrium if and only if $\boldsymbol{n} \in E$.*

satisfies $\Pi_k(\mathbf{n}_{k+}) \leq F_k \leq \Pi_k(\mathbf{n}), \quad \forall k$ and contains all equilibria.

The profit functions $\hat{\Pi}_k(\mathbf{n})$ are defined over integers n_k, $\forall k$. We relax integrality of n, to define $\hat{\Pi}'_k(\mathbf{x})$ over continuous variables x_k. We define corresponding regions N', S' and, E' such that:

$$
\begin{aligned}
S' &\equiv \{\mathbf{x} | \hat{\Pi}'_k(\mathbf{x}) \geq \min\{F_k, x_k F_k\}, \forall k\}, \quad S' \in \mathbb{R}^K_{\geq 0}, \\
N' &\equiv \{\mathbf{x} | \hat{\Pi}'_k(\mathbf{x}_{k+}) < F_k, \exists k\}, \quad N' \in \mathbb{R}^K_{\geq 0}, \\
E' &\equiv S' \cap N', \quad E' \in \mathbb{R}^K_{\geq 0}.
\end{aligned}
\tag{3.9}
$$

The region E' contains the set E. In the analysis that follows we use the relaxed profit functions to identify E'.

3.4 Quadratic Utility Functions and Two Products with Matched-Quantity Bundles

In this section we analyze an example commonly studied in the literature. We examine an oligopoly where firms have the ability to sell one of the 3 bundles consisting of a subset of 2 standalone components A and B. The three bundles are $\{A\}$, $\{B\}$, and a bundle with fixed proportions $\{AB\}$. In a sequential entry game, each firm i makes a decision to enter either one of the bundle markets where positive profits are attainable post-entry. We assume firms in each market have identical cost structures and hence are identical in each market. The entry game stops when there is no further entry.

We focus on two scenarios that have been extensively studied in the literature:

- Exclusive Bundling: One of the goods is only included in the bundle and not sold separately.
- Mixed bundling: Both products are sold independently, as well as in a bundle.

On the demand side, customers have quadratic and strictly concave utility functions, expressed in monetary terms. We study two variations of the exclusive bundling and mixed bundling scenarios:

- With one customer group.
- With multiple customer groups.

In the following sections we study four different cases, i.e., Exclusive Bundling and one customer group, Exclusive Bundling and multiple customer groups, Mixed Bundling and one customer group, and Mixed Bundling and multiple customer groups. We present analytical and numerical results for each case.

3.4.1 Exclusive Bundling and One Customer Group

The majority of previous research on competitive bundling considers two firms: a monopolist who offers a monopoly product tied to a competitive product and competes with a second firm who offers the competitive product only. We first study the following oligopolistic markets,

Case 3.1.

Supply Side: There are two types of firms in two sub-markets: type 1 that offers the competitive product $\{A\}$ in the first sub-market and type 2 that offers a matched-quantity bundle of the competitive product and an exclusive product $\{AB\}$ in the second sub-market.

Demand Side: There is one customer group with quadratic utility function.

The customer chooses consumption vector $\mathbf{y}$ to maximize net value, S:[2]

$$U(\mathbf{y}) = U_1'(y_1') + U_2'(y_2')$$
$$= \begin{cases} a_1(y_1+y_3) + a_2y_3 - \frac{b_1}{2}(y_1+y_3)^2 - \frac{b_2}{2}y_3^2 & \text{if } 0 \le y_3 \le \frac{a_2}{b_2} \text{ and } 0 \le y_1+y_3 \le \frac{a_1}{b_1} \\ a_1(y_1+y_3) - \frac{b_1}{2}(y_1+y_3)^2 + \frac{a_2^2}{2b_2} & \text{if } y_3 > \frac{a_2}{b_2} \text{ and } 0 \le y_1+y_3 \le \frac{a_1}{b_1} \\ a_2y_3 - \frac{b_2}{2}y_3^2 + \frac{a_1^2}{2b_1} & \text{if } 0 \le y_3 \le \frac{a_2}{b_2} \text{ and } y_1+y_3 > \frac{a_1}{b_1} \\ \frac{a_1^2}{2b_1} + \frac{a_2^2}{2b_2} & \text{if } y_3 > \frac{a_2}{b_2} \text{ and } y_1+y_3 > \frac{a_1}{b_1} \end{cases} \tag{3.10}$$

$$S(\mathbf{y}, \mathbf{p}) = U(\mathbf{y}) - p_1y_1 - p_3y_3 \tag{3.11}$$

[2]Since there is only one customer group, we remove the index j from the expression, also we normalize the size of the customer group to one. We can scale the results for a different size of the customer group.

The optimal consumption choice given price vector $\mathbf{p}$ is:

$$\bar{\mathbf{y}}(\mathbf{p}) = (\bar{y}_1, \bar{y}_3) = \begin{cases} (\frac{a_1-p_1}{b_1} - \frac{a_2+p_1-p_3}{b_2}, \frac{a_2+p_1-p_3}{b_2}) & \text{if } 0 \leq \frac{a_2+p_1-p_3}{b_2} \leq \frac{a_1-p_1}{b_1} \\ (\frac{a_1-p_1}{b_1}, 0) & \text{if } \frac{a_1-p_1}{b_1} \geq 0 \text{ and } \frac{a_2+p_1-p_3}{b_2} < 0 \\ (0, \frac{a_1+a_2-p_3}{b_1+b_2}) & \text{if } \frac{a_2+p_1-p_3}{b_2} \geq 0 \text{ and } \frac{a_2+p_1-p_3}{b_2} > \frac{a_1-p_1}{b_1} \\ (0, 0) & \text{if } \frac{a_2+p_1-p_3}{b_2} < 0 \text{ and } \frac{a_1-p_1}{b_1} < 0 \end{cases} \tag{3.12}$$

It is clear that there will be non-zero consumption for all components only if $a_1b_2 - a_2b_1 \geq 0$, which we assume throughout this chapter. Next, prices are set such that both markets clear. We find the following inverse-demand functions which characterize the Walrasian equilibrium prices, p_1 and p_3.

$$p_1(Q_1, Q_3) = \begin{cases} a_1 - b_1(Q_1 + Q_3) & \text{if} \quad Q_1 + Q_3 < \frac{a_1}{b_1} \\ 0 & \text{otherwise} \end{cases}$$

$$p_3(Q_1, Q_3) = \begin{cases} a_1 + a_2 - (b_1 + b_2)Q_3 - b_1Q_1 & \text{if} \quad b_1Q_1 + (b_1 + b_2)Q_3 < a_1 + a_2 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

The profit function of each type of firm is the following:

$$\begin{aligned}\Pi_{i1}(q_{i1};\mathbf{Q}_{-i},\mathbf{p}) &= p_1(q_{i1}+Q_{-i,1},Q_3)\cdot q_{i1}\\ &= \begin{cases}(a_1-b_1(q_{i1}+Q_{-i1})-b_1Q_3)q_{i1} & \text{if } q_{i1}<\frac{a_1}{b_1}-Q_{-i1}-Q_3,\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\Pi_{i3}(q_{i3};\mathbf{Q}_{-i},\mathbf{p}) &= p_3(Q_1,q_{i3}+Q_{-i,3})\cdot q_{i3}\\ &= \begin{cases}(a_1+a_2-b_1Q_1-(b_1+b_2)(q_{i3}+Q_{-i3}))q_{i3} & \text{if } q_{i3}<\frac{a_1+a_2}{b_1+b_2}-Q_{-i3}-\frac{b_1}{b_1+b_2}Q_1\\ 0 & \text{otherwise}\end{cases}\end{aligned} \tag{3.15}$$

Thus the optimal production level for each firm is the following:

$$\hat{q}_{i1}(Q_{-i,1},Q_3)=\begin{cases}\frac{1}{2}\left(\frac{a_1}{b_1}-Q_{-i1}-Q_3\right) & \text{if } 0<\frac{a_1}{b_1}-Q_{-i1}-Q_3,\\ 0 & \text{otherwise.}\end{cases} \tag{3.16}$$

$$\hat{q}_{i3}(Q_{-i,3},Q_1)=\begin{cases}\frac{1}{2}\left(\frac{a_1+a_2}{b_1+b_2}-Q_{-i3}-\frac{b_1}{b_1+b_2}Q_1\right) & \text{if } 0<\frac{a_1+a_2}{b_1+b_2}-Q_{-i3}-\frac{b_1}{b_1+b_2}Q_1\\ 0 & \text{otherwise}\end{cases} \tag{3.17}$$

We assume that all firms in each market are symmetric, and have the same optimal production level, i.e. $q_{ik}=q_k$ for any firm i selling product k. These levels only depend

on the number of players in each market and are:

$$\hat{q}_1 = \begin{cases} \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)+(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)}\right) & \text{if } n_1 \geq 1, \\ 0 & \text{if } n_1 = 0, \end{cases} \tag{3.18}$$

$$\hat{q}_3 = \begin{cases} \frac{a_1+a_2+a_2n_1}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)} & \text{if } n_3 \geq 1, \\ 0 & \text{if } n_3 = 0. \end{cases} \tag{3.19}$$

Similarly, the equilibrium prices will only depend on the number of players in each market:

$$p_1(n_1, n_3) = \begin{cases} \frac{a_1(b_1+b_2)+(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)} & \text{if } n_1 \geq 1, \\ 0 & \text{if } n_1 = 0, \end{cases} \tag{3.20}$$

$$p_3(n_1, n_3) = \begin{cases} (b_1+b_2)\left(\frac{a_1+a_2+a_2n_1}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)}\right) & \text{if } n_3 \geq 1, \\ 0 & \text{if } n_3 = 0, \end{cases} \tag{3.21}$$

Result 3.1. *The equilibrium profit for each type of the firms given the total number of players in each market is the following:*

$$\hat{\Pi}_1(n_1, n_3) = \begin{cases} \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)+(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)}\right)^2 & \textit{if } n_1 \geq 1, \\ 0 & \textit{if } n_1 = 0, \end{cases} \tag{3.22}$$

$$\hat{\Pi}_3(n_1, n_3) = \begin{cases} (b_1+b_2)\left(\frac{a_1+a_2+a_2n_1}{b_2(1+n_1)(1+n_3)+b_1(1+n_1+n_3)}\right)^2 & \textit{if } n_3 \geq 1, \\ 0 & \textit{if } n_3 = 0, \end{cases} \tag{3.23}$$

The continuous relaxations of the profit functions are defined as

$$\begin{aligned}\hat{\Pi}_1'(x_1,x_3) &= \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)+(a_1b_2-a_2b_1)x_3}{b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3)}\right)^2,\\ \hat{\Pi}_3'(x_1,x_3) &= (b_1+b_2)\left(\frac{a_1+a_2+a_2x_1}{b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3)}\right)^2.\end{aligned} \tag{3.24}$$

Our goal here is to determine the number of players in each market in equilibrium, i.e., where every firm on the market makes non-negative profits but no new firm can enter and make a profit in any of the markets. The number of firms who can successfully enter each market is a function of the fixed cost of entry in that market.

Result 3.2. *Given that $a_1b_2-a_2b_1 \geq 0$, the continuous relaxations of the profit functions of both types of firms are strictly monotonically decreasing in each x_k, $k=1,3$.*

$$\begin{aligned}\frac{\partial\hat{\Pi}_1'}{\partial x_1} &= -\frac{2(b_1+b_2+b_2x_3)(a_2b_1x_3-a_1(b_1+b_2+b_2x_3))^2}{b_1(b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3))^3} < 0,\\ \frac{\partial\hat{\Pi}_1'}{\partial x_3} &= -\frac{2(b_1+b_2)(a_1+a_2+a_2x_1)(-a_2b_1x_3+a_1(b_1+b_2+b_2x_3))}{(b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3))^3} < 0,\\ \frac{\partial\hat{\Pi}_3'}{\partial x_1} &= -\frac{2(b_1+b_2)(a_1+a_2+a_2x_1)(-a_2b_1x_3+a_1(b_1+b_2+b_2x_3))}{(b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3))^3} < 0,\\ \frac{\partial\hat{\Pi}_3'}{\partial x_3} &= -\frac{2(b_1+b_2)(a_1+a_2+a_2x_1)^2(b_1+b_2+b_2x_1)}{(b_2(1+x_1)(1+x_3)+b_1(1+x_1+x_3))^3} < 0.\end{aligned} \tag{3.25}$$

Result 3.3. *The maximum values of the functions $\hat{\Pi}_k'$ occur at $\boldsymbol{x}=(0,0)$, for $k=1,2$ thus*

$$\begin{aligned}\hat{\Pi}_1'(x_1,x_3) &\leq \hat{\Pi}_1'(0,0) = \frac{a_1^2}{b_1} \quad \forall x_1,x_2 \geq 0\\ \hat{\Pi}_3'(x_1,x_3) &\leq \hat{\Pi}_3'(0,0) = \frac{(a_1+a_2)^2}{b_1+b_2} \quad \forall x_1,x_2 \geq 0\end{aligned} \tag{3.26}$$

It is clear that for any $F_1 > \frac{a_1^2}{b_1}$ the firms entering the first market will not be able to recover the fixed cost of entry, thus there will be no entrant in the first market.

Similarly, for any $F_3 > \frac{(a_1+a_2)^2}{b_1+b_2}$ there will be no entrant in the bundle market.

We can identify the boundary of the Survival region S', i.e. the region that contains all x_1, x_2 such that $\hat{\Pi}'_1(x_1, x_3) \geq F_1$ and $\hat{\Pi}'_3(x_1, x_3) \geq F_3$. For example, for the market described in Figure 3.1, there are two curves limiting the Survival region.

Result 3.4. *The Survival region S' is defined by $\hat{\Pi}'_1(x_1, x_3) \geq F_1$ and $\hat{\Pi}'_3(x_1, x_3) \geq F_3$. Monotonicity implies:*

$$\begin{cases} x_1 \leq -1 + \sqrt{\frac{b_1}{F_1}}\frac{a_1}{b_1} - x_3\left(\frac{b_1+a_2\sqrt{\frac{b_1}{F_1}}}{b_1+b_2(1+x_3)}\right) \\ x_3 \leq -1 + \sqrt{\frac{b_1+b_2}{F_3}}\frac{a_2}{b_2} + \frac{\sqrt{\frac{b_1+b_2}{F_3}}(a_1b_2-a_2b_1)}{b_2(b_1+b_2(1+x_1))} - \frac{b_1x_1}{b_1+b_2(1+x_1)} \end{cases} \tag{3.27}$$

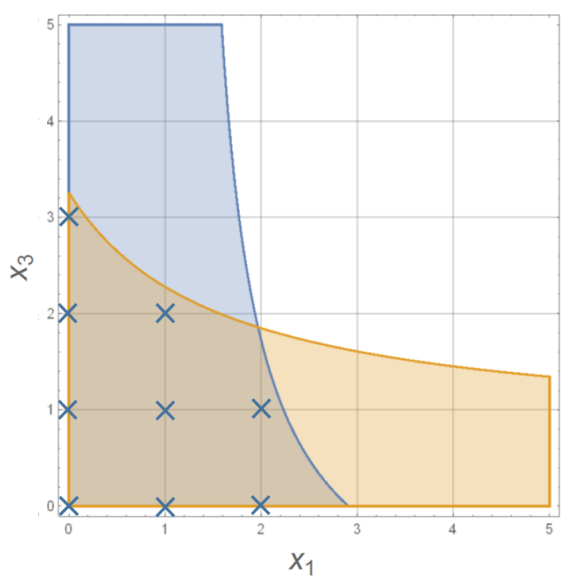


Figure 3.1: The Survival region S' and the Survival Set S when $a1 = 65.6, a2 = 34.9, b1 = 34.9, b2 = 68.7, F1 = 8.45$, and $F3 = 5.3$.

Figure 3.1 shows an example of the Survival region S'. In this example the Survival set S contains the points $\{(0,0),(1,0),(2,0),(0,1),(1,1),(2,1),(0,2),(1,2)\}$. Figure 3.2 shows the No-Entry region N' for the same example. The No-Entry set N contains the

points $\{(i,j)|i \geq 2 \text{ and } j \geq 1, \text{ or } i \geq 1 \text{ and } j \geq 2\}$. For every point in the N' regions, no new firm will enter either market, since it would not be able to recover the fixed cost of entry.

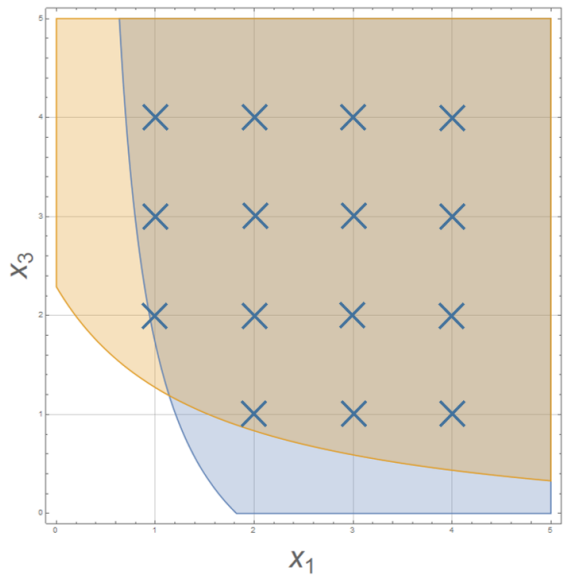


Figure 3.2: The No-Entry region N' and the No-Entry Set N when $a1 = 65.6, a2 = 34.9, b1 = 34.9, b2 = 68.7, F1 = 8.45$, and $F3 = 5.3$.

All equilibria for the game in Case 3.1 lie in the intersection of sets S and N. Figure 3.3 illsustrates the set E and the region E' for an example of Case 3.1. The set E contains two equilibrium points, which establishes that for Case 3.1 there can be multiple equilibria.

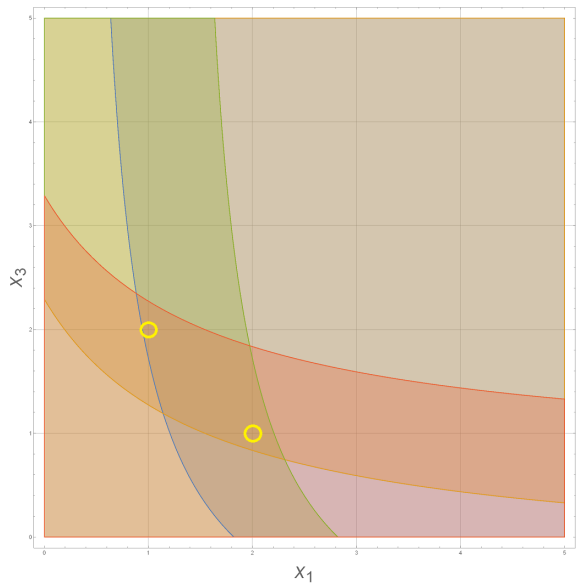


Figure 3.3: The Equilibrium region E' and the Equilibrium set E when $a1 = 65.6, a2 = 34.9, b1 = 34.9, b2 = 68.7, F1 = 8.45,$ and $F3 = 5.3$.

Next we show that there can be markets with many equilibria. For Case 3.1, define $m(a_1, a_2, b_1, b_2, F_1, F_3)$ as the number of equilibrium solutions for the parameters set.

Result 3.5.

$$\textit{For any } M > 0 \exists a_1, a_2, b_1, b_2, F_1, F_3 \textit{ such that } m(a_1, a_2, b_1, b_2, F_1, F_3) \geq M. \tag{3.28}$$

Proof. Proof We can choose the parameters such that the boundaries of S' have similar slopes. We can rewrite Equation (3.30) as:

$$\begin{cases} x_1 + x_3 \leq -1 + \sqrt{\frac{b_1}{F_1}}\frac{a_1}{b_1} - x_3 \left(\frac{a_2\sqrt{\frac{b_1}{F_1}} - b_2(1+x_3)}{b_1+b_2(1+x_3)} \right) \\ x_1 + x_3 \leq -1 + \sqrt{\frac{b_1+b_2}{F_3}}\frac{a_2}{b_2} + \frac{\sqrt{\frac{b_1+b_2}{F_3}}(a_1b_2 - a_2b_1)}{b_2(b_1+b_2(1+x_1))} + x_1 \left(\frac{b_2(1+x_1)}{b_1+b_2(1+x_1)} \right) \end{cases} \tag{3.29}$$

$\forall \epsilon > 0, \exists a_1, a_2, b_1, b_2, F_1, F_3$ such that $\left| \frac{a_2\sqrt{\frac{b_1}{F_1}} - b_2(1+x_3)}{b_1+b_2(1+x_3)} \right| < \epsilon$, $\left| \frac{b_2(1+x_1)}{b_1+b_2(1+x_1)} \right| < \epsilon$, and

$$\frac{d\left(\frac{\sqrt{\frac{b_1+b_2}{F_3}}(a_1b_2-a_2b_1)}{b_2(b_1+b_2(1+x_1))}\right)}{dx_1} < \epsilon.$$

For infinitesimally small b_2, $\left\{\begin{array}{l} \left|\frac{a_2\sqrt{\frac{b_1}{F_1}}-b_2(1+x_3)}{b_1+b_2(1+x_3)}\right| < \left|\frac{a_2\sqrt{\frac{b_1}{F_1}}}{b_1}\right| < \epsilon \\ \left|\frac{b_2(1+x_1)}{b_1+b_2(1+x_1)}\right| \simeq 0 \\ \frac{d\left(\frac{\sqrt{\frac{b_1+b_2}{F_3}}(a_1b_2-a_2b_1)}{b_2(b_1+b_2(1+x_1))}\right)}{dx_1} < \frac{\sqrt{\frac{b_1+b_2}{F_3}}(a_1b_2-a_2b_1)}{(b_1)^2} < \epsilon \end{array}\right\}$

To fulfill the inequalities and also to ensure that $a_1b_2 - a_2b_1 \geq 0$, we need $a_2 \simeq 0$. Therefore, the boundaries become:

$$\begin{aligned} x_1 + x_3 &\approx -1 + \frac{a_1}{\sqrt{b_1F_1}} \\ x_1 + x_3 &\approx -1 + \frac{a_1}{\sqrt{b_1F_3}} is \end{aligned} \tag{3.30}$$

Assuming that $F_3 \geq F_1$, both Horizontal and vertical distances of these two parallel lines are $\Delta = \frac{a_1}{\sqrt{b_1F_1}} - \frac{a_1}{\sqrt{b_1F_3}}$. For $\Delta < 1$ there are $m(a_1, a_2, b_1, b_2, F_1, F_3) = \lfloor \frac{a_1}{\sqrt{b_1F_3}} \rfloor$ equilibria points along the parallel lines (Figure 3.4).

$$\left\{\begin{array}{l} \lfloor \frac{a_1}{\sqrt{b_1F_3}} \rfloor \geq M \\ \frac{a_1}{\sqrt{b_1F_1}} - \frac{a_1}{\sqrt{b_1F_3}} < 1 \end{array}\right\} \implies F_3 < \frac{1}{b_1}\left(\frac{a_1}{M+1}\right)^2 \text{and}$$

$$F_3\left(\frac{a_1}{a_1+\sqrt{b_1F_1}}\right)^2 < F_1 < F_3.$$

□

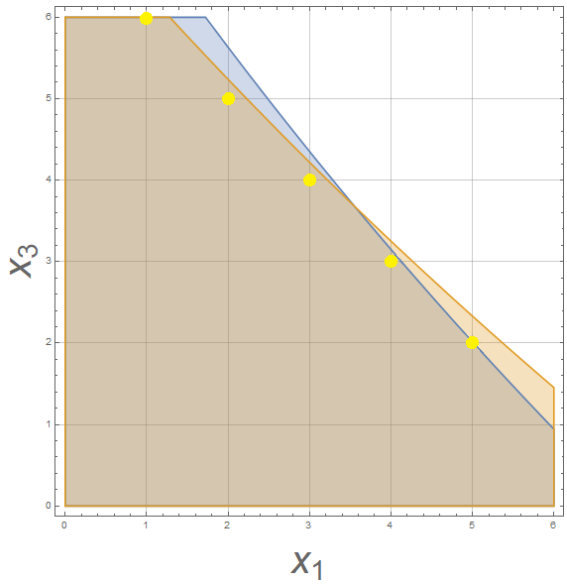


Figure 3.4: The Survival region S' and the Equilibirum Set E when $a1 = 10, a2 = 0.06, b1 = 1.06, b2 = 1.06, F1 = 1.5,$ and $F3 = 1.3$.

Next, we introduce two enumeration methods to find the equilibria of this game under any market setting.

3.4.1.1 First Enumeration Method

We know from (3.25) that each function $\Pi'_k(x_k)$ is monotonic decreasing in every x_k, i.e., $\frac{\partial \Pi'_1}{\partial x_1} < 0$, $\frac{\partial \Pi'_1}{\partial x_3} < 0$, $\frac{\partial \Pi'_3}{\partial x_3} < 0$, and $\frac{\partial \Pi'_3}{\partial x_1} < 0$, which implies that the number of firms entering each market will be capped by a function of the fixed cost of entry in that respective market.

Next, we find the bounds on the solution space to improve computational efficiency. We identify the maximum number of entrants possible n_k in each market. We solve the following problem for each market k, $k = 1, 3$ using the relaxed form of the profit

functions (3.24).

$$\begin{aligned} \bar{x}_k = \max_{x_k} \quad & x_k \\ \text{s.t.} \quad & \hat{\Pi}'_k(x_1, x_3) \geq F_k \\ & x_k \geq 0 \text{ for } k = 1, 3 \end{aligned} \tag{3.31}$$

Finally, we define the solution space as $SS = \{\mathbf{n} | 0 \leq n_k \leq \lceil \bar{x}_k \rceil, n_k \in \mathbb{N} \text{ for } k = 1, 3\}$. The set contains all possible equilibria. We enumerate all integer points in the set and test to ensure that they are in E.

3.4.1.2 Second Enumeration Method

In order to find the equilibria more efficiently we can further limit the solution space. We already know that the solution space is bounded by the boundaries of the Survival region as defined in Equation (3.30). Next, we define another set of boundaries for the No-Entry region. We can express this region as the intersection of the following regions: $N_{i1} = \{\mathbf{x} | \hat{\Pi}'_i(x_1 + 1, x_3) < F_i\}$ and $N_{i2} = \{\mathbf{x} | \hat{\Pi}'_i(x_1, x_3 + 1) < F_i\}$ for $i = 1, 2$. The No-Entry region is defined as $N' = N_{11} \cap N_{22}$. In order to utilize an enumeration approach, we introduce a well-defined alternative for region N' using N_{11} and N_{22}.

Result 3.6. *Define* $\bar{N} = \{\boldsymbol{x} | \hat{\Pi}'_i(x_1 + 1, x_2 + 1) < F_i\}$ *for* $i = 1, 3$*, then* $N \subseteq \bar{N}$*.*

Proof. Proof Suppose $\tilde{x} = (x_1, x_3)$ and $\tilde{x} \in N'$, therefore $\tilde{x} \in N_{11}$ and $\tilde{x} \in N_{22}$. $\tilde{x} \in N_{11}$ implies that $\hat{\Pi}'_1(x_1 + 1, x_3) < F_1$. Due to monotonicity of the continuous profit function we can conclude that, $\hat{\Pi}'_1(x_1 + 1, x_3 + 1) \leq \hat{\Pi}'_1(x_1 + 1, x_3) < F_1$, and $\hat{\Pi}'_1(x_1 + 1, x_3 + 1) \leq \hat{\Pi}'_1(x_1, x_3 + 1) < F_3$. Thus, $\tilde{x} \in \bar{N}$ and $N' \subseteq \bar{N}$. □

Next, we find the boundaries of the $\bar{N}$ region, i.e., $\hat{\Pi}'_1(x_1 + 1, x_3 + 1) = F_1$ and $\hat{\Pi}'_3(x_1 + 1, x_3 + 1) = F_3$

Result 3.7. *$\bar{N}$ is defined by the inequalities*

$$\begin{aligned} x_1 &\leq -2 - \frac{b_1(1+x_3)}{b_1+b_2(2+x_3)} + \frac{a_1}{\sqrt{b_1F_1}} - \frac{(a_2\sqrt{b_1F_1})(1+x_3)}{F_1(b_1+2b_2+b_2x_3)} \\ x_3 &\leq -2 - \frac{b_1(1+x_1)}{b_1+b_2(2+x_1)} + \frac{(a_1+a_2(2+x_1))\sqrt{(b_1+b_2)F_3}}{F_3(b_1+b_2(2+x_1))} \end{aligned} \tag{3.32}$$

We use the four boundaries obtained in Equations (3.30) and (3.32), to construct the following enumeration steps for any set of parameters.

1. Find the values of the four intersection points of the Survival and No-Entry boundaries, i.e., $s_i = (x_{1si}, x_{3si})$ for $i = 1, ..., 4$.

2. Use the integer values $\lfloor \min_i x_{1si} \rfloor \leq n_1 \leq \lceil \max_i x_{1si} \rceil$ and $\lfloor \min_i x_{3si} \rfloor \leq n_3 \leq \lceil \max_i x_{3si} \rceil$ as the solution space for the enumeration method.

3. Eliminate parts of the solution space for which $\hat{\Pi}'_k(x_1, x_3) < F_k$, $\hat{\Pi}'_k(x_1+1, x_3) > F_k$ or $\hat{\Pi}'_k(x_1, x_3+1) > F_k$ for $k = 1, 3$.

3.4.2 Exclusive Bundling and Multiple Customer Groups

Case 3.2.

Supply Side: There are two types of firms in two sub-markets: type 1 that offers the competitive product $\{A\}$ in the first sub-market and type 2 that offers a matched-quantity bundle of the competitive product and an exclusive product $\{AB\}$ in the second sub-market.

Demand Side: We now allow for multiple customer groups in the market, with the following heterogeneous utility functions.

$$U_j(\mathbf{y}_j) = U'_{j1}(y'_{j1}) + U'_{j2}(y'_{j2}) = a_{j1}(y_{j1} + y_{j3}) + a_{j2}y_{j3} - \frac{b_{j1}}{2}(y_{j1} + y_{j3})^2 - \frac{b_{j2}}{2}y_{j3}^2 \tag{3.33}$$

Where $U_j(\mathbf{y}_j)$ is the total utility of the customer j from consuming y_{j1} unit of good 1 and y_{j3} unit of the bundle, $j = 1, 2, ..., J$.

As before, customer choose consumption levels y_{jk} given price vector $\mathbf{p}$, to maximize their net value, $S_j(\mathbf{y}_j, \mathbf{p}_j) = U_j(\mathbf{y}_j) - \mathbf{p} \cdot \mathbf{y}_j$. Customers' optimal consumption levels given the price vector $\mathbf{p}$ is

$$\bar{\mathbf{y}}_j(\mathbf{p}) = (\bar{y}_{j1}, \bar{y}_{j3}) = \begin{cases} (\frac{a_{j1}-p_1}{b_{j1}} - \frac{a_{j2}+p_1-p_3}{b_{j2}}, \frac{a_{j2}+p_1-p_3}{b_{j2}}) & \text{if } 0 \leq \frac{a_{j2}+p_1-p_3}{b_{j2}} \leq \frac{a_{j1}-p_1}{b_{j1}} \\ (\frac{a_{j1}-p_1}{b_{j1}}, 0) & \text{if } \frac{a_{j1}-p_1}{b_{j1}} \geq 0 \text{ and } \frac{a_{j2}+p_1-p_3}{b_{j2}} < 0 \\ (0, \frac{a_{j1}+a_{j2}-p_3}{b_{j1}+b_{j2}}) & \text{if } \frac{a_{j2}+p_1-p_3}{b_{j2}} \geq 0 \text{ and } \frac{a_{j2}+p_1-p_3}{b_{j2}} > \frac{a_{j1}-p_1}{b_{j1}} \\ (0, 0) & \text{if } \frac{a_{j2}+p_1-p_3}{b_{j2}} < 0 \text{ and } \frac{a_{j1}-p_1}{b_{j1}} < 0 \end{cases} \tag{3.34}$$

For tractability, we assume that for each customer j the inequality $a_{j1}b_{j2} - a_{j2}b_{j1} \geq 0$ holds. The aggregate consumption levels of all customers of product 1 and the bundle is:

$$\bar{\mathbf{y}}(\mathbf{p}) = (\bar{y}_1, \bar{y}_3) = (\sum_j \frac{a_{j1} - p_1}{b_{j1}} - \frac{a_{j2} + p_1 - p_3}{b_{j2}}, \sum_j \frac{a_{j2} + p_1 - p_3}{b_{j2}}) \tag{3.35}$$

Next, prices are set such that both markets clear. We find the following inverse-demand

functions which characterize the Walrasian equilibrium prices, p_1 and p_3.

$$p_1(Q_1, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j1}}{b_{j1}} - Q_1 - Q_3}{\sum_j \frac{1}{b_{j1}}} & \text{if} \quad Q_1 + Q_3 < \sum_j \frac{a_{j1}}{b_{j1}} \\ 0 & \text{otherwise} \end{cases}$$

$$p_3(Q_1, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j1}}{b_{j1}} - Q_1 - Q_3}{\sum_j \frac{1}{b_{j1}}} + \frac{\sum_j \frac{a_{j2}}{b_{j2}} - Q_3}{\sum_j \frac{1}{b_{j2}}} & \text{if} \quad \frac{Q_1}{\sum_j \frac{1}{b_{j1}}} + Q_3(\frac{1}{\sum_j \frac{1}{b_{j1}}} + \frac{1}{\sum_j \frac{1}{b_{j2}}}) < \frac{\sum_j \frac{a_{j1}}{b_{j1}}}{\sum_j \frac{1}{b_{j1}}} + \frac{\sum_j \frac{a_{j2}}{b_{j2}}}{\sum_j \frac{1}{b_{j2}}} \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

The aggregate demand of J customer groups can be defined as the demand of one representative customer group with the following parameters.

$$a_1 = \frac{\sum_j \frac{a_{j1}}{b_{j1}}}{\sum_j \frac{1}{b_{j1}}}, \qquad a_2 = \frac{\sum_j \frac{a_{j2}}{b_{j2}}}{\sum_j \frac{1}{b_{j2}}}, \tag{3.37}$$
$$b_1 = \frac{1}{\sum_j \frac{1}{b_{j1}}}, \qquad b_2 = \frac{1}{\sum_j \frac{1}{b_{j2}}}.$$

For example for the case of two customer groups, we have:

$$a_1 = \frac{a_{11}b_{21} + a_{21}b_{11}}{b_{11} + b_{21}}, \qquad a_2 = \frac{a_{12}b_{22} + a_{22}b_{12}}{b_{12} + b_{22}}, \tag{3.38}$$
$$b_1 = \frac{b_{11}b_{21}}{b_{11} + b_{21}}, \qquad b_2 = \frac{b_{12}b_{22}}{b_{12} + b_{12}}.$$

All the results we derived in the one customer group case can be extended to multiple customer groups by replacing a_1, a_2, b_1, and b_2 with their respective equivalents in the multiple customer groups case.

3.4.3 Mixed Bundling and One Customer Group

In this section, we extend the model to allow for mixed bundling.

Case 3.3.

Supply Side: There are three types of firms in three sub-markets: type 1 that offers the product $\{A\}$ in the first sub-market, type 2 that offers the product $\{B\}$ in the second sub-market, and type 3 that offers the bundle of two products $\{AB\}$ in the third sub-market.

Demand Side: There is one customer group with the following quadratic utility function.

$$U(\mathbf{y}) = U_1'(y_1') + U_2'(y_2') = a_1(y_1 + y_3) + a_2(y_2 + y_3) - \frac{b_1}{2}(y_1 + y_3)^2 - \frac{b_2}{2}(y_2 + y_3)^2. \tag{3.39}$$

The utility functions are concave in the consumption levels. Thus, the first order conditions can reveal the optimal consumption levels for any given set of p_1, p_2, p_3.[3]

$$\bar{\mathbf{y}}(\mathbf{p}) = (\bar{y}_1, \bar{y}_2, \bar{y}_3) = \begin{cases} (\frac{a_1-p_1}{b_1} - \bar{y}_3, \frac{a_2-p_2}{b_2} - \bar{y}_3, \bar{y}_3) & \text{if } p_1 + p_2 = p_3 \text{ and } p_1 \leq a_1 \text{ and } p_2 \leq a_2, \\ (\frac{a_1-p_1}{b_1}, \frac{a_2-p_2}{b_2}, 0) & \text{if } p_1 + p_2 < p_3 \text{ and } p_1 \leq a_1 \text{ and } p_2 \leq a_2, \\ (0, \frac{a_2-p_2}{b_2} - \frac{a_1+p_2-p_3}{b_1}, \frac{a_1+p_2-p_3}{b_1}) & \text{if } p_1 + p_2 > p_3 \text{ and } \frac{a_2-p_2}{b_2} \geq \frac{a_1+p_2-p_3}{b_1} \geq 0, \\ (\frac{a_1-p_1}{b_1} - \frac{a_2+p_1-p_3}{b_2}, 0, \frac{a_2+p_1-p_3}{b_2}) & \text{if } p_1 + p_2 > p_3 \text{ and } \frac{a_1-p_1}{b_1} \geq \frac{a_2+p_1-p_3}{b_2} \geq 0. \end{cases} \tag{3.40}$$

Prices p_1, p_2 and p_3 will be such as to ensure non-negative demand for the offerings. Thus in equilibrium $p_3 = p_1 + p_2$.

[3]Since there is only one customer group, we removed the index j from the expression, also we normalized the size of the customer group to one. We can scale the results for a different size of the customer group.

Next, prices are set such that both markets clear. We find the following inverse-demand functions which characterize the Walrasian equilibrium prices, p_1 and p_3. and from there we derive the following inverse demand functions.

$$
\begin{aligned}
p_1(Q_1, Q_2, Q_3) &= \begin{cases} a_1 - b_1(Q_1 + Q_3) & \text{if} \quad Q_1 + Q_3 \leq \frac{a_1}{b_1} \\ 0 & \text{otherwise} \end{cases} \\
p_2(Q_1, Q_2, Q_3) &= \begin{cases} a_2 - b_2(Q_2 + Q_3) & \text{if} \quad Q_2 + Q_3 \leq \frac{a_2}{b_2} \\ 0 & \text{otherwise} \end{cases} \\
p_3(Q_1, Q_2, Q_3) &= p_1(Q_1, Q_2, Q_3) + p_2(Q_1, Q_2, Q_3)
\end{aligned}
\tag{3.41}
$$

Prices such that $p_1 + p_2 = p_3$, ensure market clearing and also prevent resale among the customers.

We obtain the following profit functions for each type of the firms:

$$\Pi_{i1}(q_{i1}; \mathbf{Q}_{-i}, \mathbf{p}) = p_1(q_{i1} + Q_{-i,1}, Q_2, Q_3) \cdot q_{i1} \tag{3.42}$$

$$= \begin{cases} (a_1 - b_1(q_{i1} + Q_{-i1}) - b_1 Q_3) q_{i1} & \text{if } q_{i1} < \frac{a_1}{b_1} - Q_{-i1} - Q_3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Pi_{i2}(q_{i2}; \mathbf{Q}_{-i}, \mathbf{p}) = p_2(Q_1, q_{i2} + Q_{-i,2}, Q_3) \cdot q_{i2} \tag{3.43}$$

$$= \begin{cases} (a_2 - b_2(q_{i2} + Q_{-i2}) - b_2 Q_3) q_{i2} & \text{if } q_{i2} < \frac{a_2}{b_2} - Q_{-i2} - Q_3, \\ 0 & \text{otherwise.} \end{cases}$$

$$\Pi_{i3}(q_{i3}; \mathbf{Q}_{-i}, \mathbf{p}) = p_3(Q_1, Q_2, q_{i3} + Q_{-i,3}) \cdot q_{i3} \tag{3.44}$$

$$= \begin{cases} (a_1 + a_2 - b_1 Q_1 - b_2 Q_2 - (b_1 + b_2)(q_{i3} + Q_{-i3})) q_{i3} & \text{if } q_{i1} < \frac{a_1}{b_1} - Q_{-i3} - Q_1 \\ & \text{and } q_{i2} < \frac{a_2}{b_2} - Q_{-i3} - Q_2 \\ (a_1 - b_1 Q_1 - b_1(q_{i3} + Q_{-i3})) q_{i3} & \text{if } q_{i1} < \frac{a_1}{b_1} - Q_{-i3} - Q_1 \\ & \text{and } q_{i2} \geq \frac{a_2}{b_2} - Q_{-i3} - Q_2 \\ (a_2 - b_2 Q_2 - b_2(q_{i3} + Q_{-i3})) q_{i3} & \text{if } q_{i1} \geq \frac{a_1}{b_1} - Q_{-i3} - Q_1 \\ & \text{and } q_{i2} < \frac{a_2}{b_2} - Q_{-i3} - Q_2 \\ 0 & \text{otherwise} \end{cases}$$

Thus the optimal production level for each firm in each of the markets is the follow-

ing:

$$\hat{q}_{i1}(Q_{-i,1}, Q_2, Q_3) = \begin{cases} \frac{1}{2}\left(\frac{a_1}{b_1} - Q_{-i1} - Q_3\right) & \text{if } 0 < \frac{a_1}{b_1} - Q_{-i1} - Q_3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.45}$$

$$\hat{q}_{i2}(Q_1, Q_{-i,2}, Q_3) = \begin{cases} \frac{1}{2}\left(\frac{a_2}{b_2} - Q_{-i2} - Q_3\right) & \text{if } 0 < \frac{a_1}{b_1} - Q_{-i1} - Q_3, \\ 0 & \text{otherwise.} \end{cases} \tag{3.46}$$

$$\hat{q}_{i3}(Q_1, Q_2, Q_{-i,3}) = \begin{cases} \frac{1}{2}\left(\frac{a_1+a_2}{b_1+b_2} - Q_{-i3} - \frac{b_1}{b_1+b_2}Q_1 - \frac{b_2}{b_1+b_2}Q_2\right) & \text{if } q_{i1} < \frac{a_1}{b_1} - Q_{-i3} - Q_1, \\ & \text{and } q_{i2} < \frac{a_2}{b_2} - Q_{-i3} - Q_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.47}$$

Assuming symmetry between the firms competing in each market, we can find the following optimal production quantities for each producer in each market

$$\hat{q}_1(n_1, n_2, n_3) = \begin{cases} \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)(1+n_2)+(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)}\right) & \text{if } n_1 \geq 1, \\ 0 & \text{if } n_1 = 0, \end{cases} \tag{3.48}$$

$$\hat{q}_2(n_1, n_2, n_3) = \begin{cases} \frac{1}{b_2}\left(\frac{a_2(b_1+b_2)(1+n_1)-(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)}\right) & \text{if } n_2 \geq 1, \\ 0 & \text{if } n_2 = 0, \end{cases} \tag{3.49}$$

$$\hat{q}_3(n_1, n_2, n_3) = \begin{cases} \frac{a_1(1+n_2)+a_2(1+n_1)}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)} & \text{if } n_3 \geq 1 \\ 0 & \text{if } n_3 = 0. \end{cases} \tag{3.50}$$

Result 3.8. *The equilibrium profit for each type of firm given the number of players in each market is the following:*

$$\hat{\Pi}_1(n_1,n_2,n_3) = \begin{cases} \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)(1+n_2)+(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)}\right)^2 & \textit{if } n_1 \geq 1, \\ 0 & \textit{if } n_1 = 0, \end{cases} \tag{3.51}$$

$$\hat{\Pi}_2(n_1,n_2,n_3) = \begin{cases} \frac{1}{b_2}\left(\frac{a_2(b_1+b_2)(1+n_1)-(a_1b_2-a_2b_1)n_3}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)}\right)^2 & \textit{if } n_2 \geq 1, \\ 0 & \textit{if } n_2 = 0, \end{cases} \tag{3.52}$$

$$\hat{\Pi}_3(n_1,n_2,n_3) = \begin{cases} (b_1+b_2)\left(\frac{a_1(1+n_2)+a_2(1+n_1)}{b_2(1+n_1)(1+n_2+n_3)+b_1(1+n_2)(1+n_1+n_3)}\right)^2 & \textit{if } n_3 \geq 1 \\ 0 & \textit{if } n_3 = 0. \end{cases} \tag{3.53}$$

Similar to 3.24, the continuous relaxations of the profit functions are defined as

$$\hat{\Pi}'_1(x_1,x_2,x_3) = \frac{1}{b_1}\left(\frac{a_1(b_1+b_2)(1+n_2)+(a_1b_2-a_2b_1)x_3}{b_2(1+x_1)(1+n_2+x_3)+b_1(1+n_2)(1+x_1+x_3)}\right)^2 \tag{3.54}$$

$$\hat{\Pi}'_2(x_1,x_2,x_3) = \frac{1}{b_2}\left(\frac{a_2(b_1+b_2)(1+x_1)-(a_1b_2-a_2b_1)x_3}{b_2(1+x_1)(1+n_2+x_3)+b_1(1+n_2)(1+x_1+x_3)}\right)^2 \tag{3.55}$$

$$\hat{\Pi}'_3(x_1,x_2,x_3) = (b_1+b_2)\left(\frac{a_1(1+n_2)+a_2(1+x_1)}{b_2(1+x_1)(1+n_2+x_3)+b_1(1+n_2)(1+x_1+x_3)}\right)^2 \quad \textit{if } x_3 \geq 1 \tag{3.56}$$

We observe from the profit functions that $\Pi_{i1} + \Pi_{i2} > \Pi_{i3}$. In a Cournot model, because of quantity matching the bundling firm is in a disadvantaged position, unless it can benefit from a significant fixed cost reduction. We assumed the variable costs are negligible, thus bundling is viable only if it involves a reduction in the fixed cost of entry.

Result 3.9. *Bundling is viable if and only if*

$$F_1 + F_2 - F_3 > \frac{b_1+b_2}{b_1b_2}\left(\frac{a_2b_1(1+n_1+n_3)-a_1b_2(1+n_2+n_3)}{b_1(1+n_2)(1+n_1+n_3)+b_2(1+n_1)(1+n_2+n_3)}\right)^2 \tag{3.57}$$

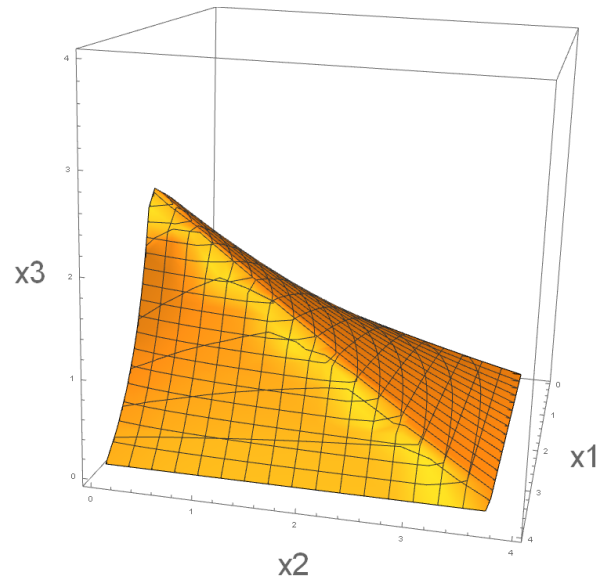


Figure 3.5: The Survival region S' when $a1 = 9.94, a2 = 9.84, b1 = 0.42, b2 = 1.2, F1 = 10.6, F2 = 3.6,$ and $F3 = 8.4$.

We can identify the Survival region S', the No-Entry region N' and the Equilibrium region E' for the mixed bundling case. Figure 3.5 illustrate the Survival region S' for the relaxed profit function $\hat{\Pi}'$ for $k = 1, 2, 3$.

Figure 3.6 illustrates N' the No-Entry region for the example. The intersection of regions S' and N' is the region E' (Figure 3.7). As illustrated in Figure 3.7, the Equilibrium region may have discontinuities. We are able to construct examples where this game has no equilibrium or multiple equilibria.

Next, we develop an enumeration technique to find all the equilibria of this market in terms of number of firms in each market.

3.4.3.1 Enumeration Method

To find all the equilibria of this market we need to identify the set E, i.e., find all $\mathbf{n} = (n_1, n_2, n_3)$ such that $\hat{\Pi}_k(\mathbf{n}) > \min\{F_k, n_k F_k\}$ and $\hat{\Pi}_k(\mathbf{n}_{k+}) < F_k$ for $k = 1, 2, 3$.

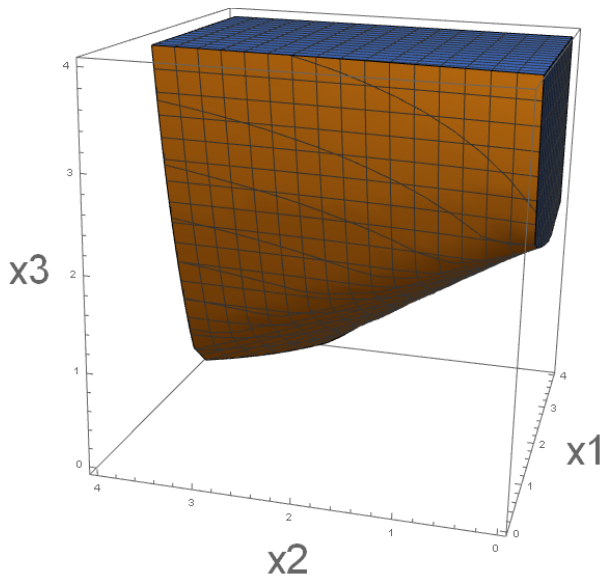


Figure 3.6: The No-Entry region N' when $a1 = 9.94, a2 = 9.84, b1 = 0.42, b2 = 1.2, F1 = 10.6, F2 = 3.6,$ and $F3 = 8.4$.

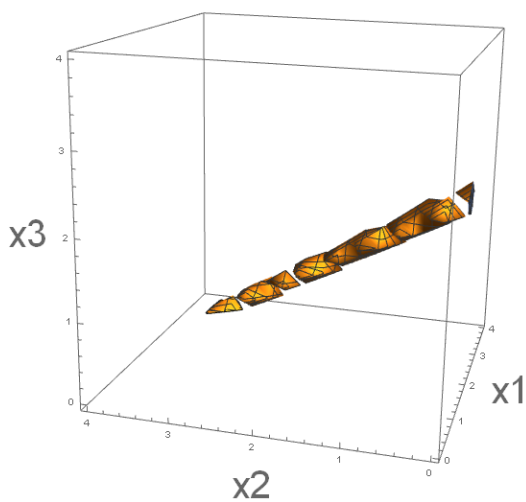


Figure 3.7: The Equilibrium region E' when $a1 = 9.94, a2 = 9.84, b1 = 0.42, b2 = 1.2, F1 = 10.6, F2 = 3.6,$ and $F3 = 8.4$.

First, we show that the solution space is bounded, due to monotonicity of the profit functions.

Proposition 3.1. *The continuous relaxations of the profit functions of all three types of firms are strictly monotonically decreasing in the number of firms in their respective markets.*

$$\begin{aligned}
\frac{\partial \hat{\Pi}_1'}{\partial x_1} &= -\frac{2((b_1+b_2)(1+x_2)+b_2x_3)(a_1(b_1+b_2)(1+x_2)+a_1b_2x_3-a_2b_1x_3)^2}{b_1(b_2(1+x_1)(1+x_2+x_3)+b_1(1+x_2)(1+x_1+x_3))^3} < 0, \\
\frac{\partial \hat{\Pi}_2'}{\partial x_2} &= -\frac{2((b_1+b_2)(1+x_1)+b_1x_3)(a_2(b_1+b_2)(1+x_1)+a_2b_1x_3-a_1b_2x_3)^2}{b_2(b_2(1+x_1)(1+x_2+x_3)+b_1(1+x_2)(1+x_1+x_3))^3} < 0, \\
\frac{\partial \hat{\Pi}_3'}{\partial x_3} &= -\frac{2(b_1+b_2)(a_1+a_2+a_2x_1+a_1x_2)^2(b_1+b_2+b_2x_1+b_1x_2)}{(b_1(b_2(1+x_1)(1+x_2+x_3)+b_1(1+x_2)(1+x_1+x_3))^3} < 0.
\end{aligned} \tag{3.58}$$

This implies that the number of firms entering each market will be capped by a function of the fixed cost of entry in that respective market.

Next, we find bounds on the solution space to improve computational efficiency. We identify the maximum n_k for each market. This is the maximum number of firms that are able to enter each market and make positive profit. We solve the following maximization problem for each market k, $k = 1, 2, 3$ using the relaxed profit functions.

$$\begin{aligned}
\bar{x}_k = \max_{\mathbf{x}} \quad & x_k \\
\text{s.t.} \quad & \hat{\Pi}_k(x_1, x_2, x_3) \geq F_k \\
& x_k \geq 0 \text{ for } k = 1, 2, 3
\end{aligned} \tag{3.59}$$

Finally, we define the enumeration space as $\{\mathbf{n} | 0 \leq n_k \leq \lceil \bar{x}_k \rceil, n_k \in \mathbb{N} \text{ for } k = 1, 2, 3\}$.

3.4.4 Mixed Bundling and Multiple Customer Groups

Similar to the pure bundling case, we can find the parameters representing multiple customer groups, to find the answers to this scenario.

$$U_j(\mathbf{y}_j)$$
$$= U'_{j1}(y'_{j1}) + U'_{j2}(y'_{j2}) = a_{j1}(y_{j1} + y_{j3}) + a_{j2}(y_{j1} + y_{j3}) - \frac{b_{j1}}{2}(y_{j1} + y_{j3})^2 - \frac{b_{j2}}{2}(y_{j1} + y_{j3})^2 \tag{3.60}$$

Total utility of the customer j from consuming y_{j1} unit of product 1, y_{j2} unit of product 2 and y_{j3} unit of the bundle, $j = 1, 2, ..., J$.

Each customer group j chooses optimal consumption levels according to its utility function.

$$\bar{\mathbf{y}}_j(\mathbf{p}) = (\bar{y}_{j1}, \bar{y}_{j2}, \bar{y}_{j3}) = \begin{cases} (\frac{a_{j1}-p_1}{b_{j1}} - \bar{y}_{j3}, \frac{a_{j2}-p_2}{b_{j2}} - \bar{y}_{j3}, \bar{y}_{j3}) & \text{if } p_1 + p_2 = p_3 \text{ and } p_1 \leq a_{j1} \text{ and } p_2 \leq a_{j2}, \\ (\frac{a_{j1}-p_1}{b_{j1}}, \frac{a_{j2}-p_2}{b_{j2}}, 0) & \text{if } p_1 + p_2 < p_3 \text{ and } p_1 \leq a_{j1} \text{ and } p_2 \leq a_{j2}, \\ (0, \frac{a_{j2}-p_2}{b_{j2}} - \frac{a_{j1}+p_2-p_3}{b_{j1}}, \frac{a_{j1}+p_2-p_3}{b_{j1}}) & \text{if } p_1 + p_2 > p_3 \text{ and } \frac{a_{j2}-p_2}{b_{j2}} \geq \frac{a_{j1}+p_2-p_3}{b_{j1}} \geq 0, \\ (\frac{a_{j1}-p_1}{b_{j1}} - \frac{a_{j2}+p_1-p_3}{b_{j2}}, 0, \frac{a_{j2}+p_1-p_3}{b_{j2}}) & \text{if } p_1 + p_2 > p_3 \text{ and } \frac{a_{j1}-p_1}{b_{j1}} \geq \frac{a_{j2}+p_1-p_3}{b_{j2}} \geq 0. \end{cases} \tag{3.61}$$

The Walrasian equilibrium prices are p_1, p_2 and p_3.

$$p_1(Q_1, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j1}}{b_{j1}} - Q_1 - Q_3}{\sum_j \frac{1}{b_{j1}}} & \text{if} \quad Q_1 + Q_3 < \sum_j \frac{a_{j1}}{b_{j1}} \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(Q_2, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j2}}{b_{j2}} - Q_2 - Q_3}{\sum_j \frac{1}{b_{j2}}} & \text{if} \quad Q_2 + Q_3 < \sum_j \frac{a_{j2}}{b_{j2}} \\ 0 & \text{otherwise} \end{cases} \tag{3.62}$$

$$p_3(Q_1, Q_3) = p_1(Q_1, Q_3) + p_2(Q_2, Q_3)$$

$$p_1(Q_1, Q_2, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j1}}{b_{j1}} - Q_1 - Q_3}{\sum_j \frac{1}{b_{j1}}} & \text{if} \quad Q_1 + Q_3 \leq \sum_j \frac{a_{j1}}{b_{j1}} \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(Q_1, Q_2, Q_3) = \begin{cases} \frac{\sum_j \frac{a_{j2}}{b_{j2}} - Q_2 - Q_3}{\sum_j \frac{1}{b_{j2}}} & \text{if} \quad Q_2 + Q_3 \leq \sum_j \frac{a_{j2}}{b_{j2}} \\ 0 & \text{otherwise} \end{cases} \tag{3.63}$$

$$p_3(Q_1, Q_2, Q_3) = p_1(Q_1, Q_2, Q_3) + p_2(Q_1, Q_2, Q_3)$$

We can associate the aggregate demand to a representative customer with the following parameters:

$$a_1 = \frac{\sum_j \frac{a_{j1}}{b_{j1}}}{\sum_j \frac{1}{b_{j1}}}, \qquad a_2 = \frac{\sum_j \frac{a_{j2}}{b_{j2}}}{\sum_j \frac{1}{b_{j2}}}, \tag{3.64}$$

$$b_1 = \frac{1}{\sum_j \frac{1}{b_{j1}}}, \qquad b_2 = \frac{1}{\sum_j \frac{1}{b_{j2}}}. \tag{3.65}$$

Similar to Section 3.4.2 we can extended all the results obtained in a single costumer market to the multiple customer groups case.

3.4.4.1 Numerical Experiments

To ensure that the enumeration methods are efficient, we analyzed multiple instances of the game under different values of parameters to identify major patterns in the market structure.

Market	Number of instances	Average computation time per instance
Exclusive Bundling and One Customer Group	5000	4s
Exclusive Bundling and Multiple Customer Group	5000	4s
Mixed Bundling and One Customer Group	1000	10s
Mixed Bundling and Multiple Customer Group	1000	10s

Table 3.2: A summay of the enumeration results.

The main driver of the computation time is the optimization function that finds the maximum values of n_k for $k = 1, 2, 3$. These games can potentially have multiple equilibria.

Result 3.10. *We observed that the ratio of* $\frac{a_1 b_2}{a_2 b_1}$ *can be a factor determining the maximum number of equilibria for Case* 3.3. *Figure* 3.8 *shows an example.*

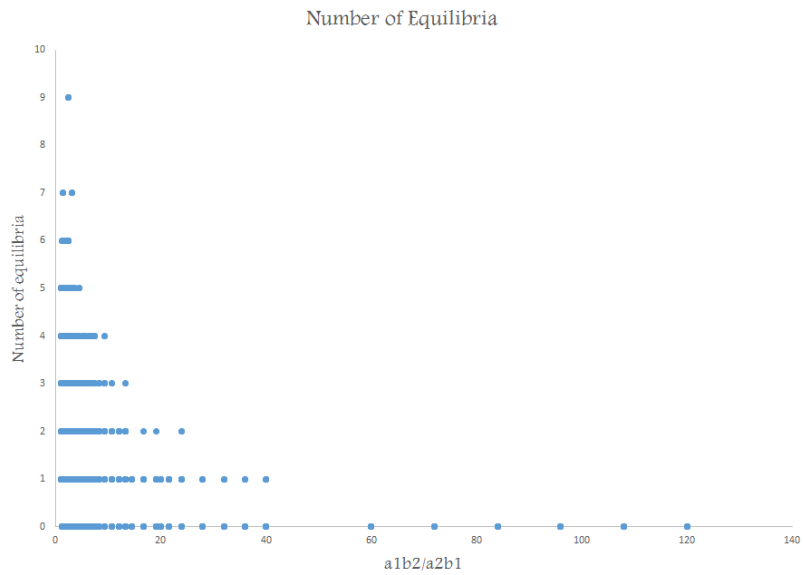


Figure 3.8: Number of equilibria for Case 3.3 as a function of $\frac{a_1 b_2}{a_2 b_1}$.

Result 3.11. *In Case* 3.3*, the number of firms in markets* $\{A\}$ *and* $\{B\}$ *grow in the same direction while the number of* $\{AB\}$ *firms shrinks in the opposite direction, and vice-versa.*

In other words more entry into the bundle market drives firms out of the single product markets.

3.5 Conclusions

In this chapter we studied bundling under competition in a more general setting than the traditional literature. We used a Cournot framework in which multiple firms can enter and offer different combinations of independent products and bundles composed of those products. Firms with the same offers are undifferentiated and engage in quantity competition. The fixed cost for entering the market for each offer, which may consist of some combination of the independent components and bundles, varies depending on the combination. On the demand side, customers buy some mix of the products and bundles available in the market so as to maximize their utility evaluated at the level of the components (products or scenarios).

We presented a general formulation for the competitive multi-product, multi-offer, multi customer group, multi-firm problem. We then investigated certain specific cases, in which we assumed that customer utility functions are quadratic and additive across the components. In our first example, firms are of two types. One type offers a single product, the other type offers that product bundled with another. We showed that it is possible to have multiple equilibria with both types of firms and that there is no upper bound to the number of equilibria that may occur. We then considered the case where there are three types of firms – two that offer each of the two products, and one that offers a bundle of the two. We found that bundling can be a viable strategy when the fixed cost of entry for the bundle is lower than the sum of the entry costs for the independent product markets.

In order to identify all the equilibria in a given case, we proposed enumeration methods that efficiently determine all the equilibria in both these scenarios. Once

again, multiple equilibria can occur.

The main managerial implications about the drivers of bundling are the following:

- Bundles create value when each additional component in the bundle adds value that is comparable to the other components in the bundle.
- Bundling is viable when the cost of entry to the bundled market is less than the aggregate cost of entering separate markets.

The Cournot model that we presented in this chapter has considerable potential. The most general formulation is not very tractable; however, we were able to obtain results for certain special cases that provide insights about entry costs, and oligopolistic competition that considerably extend the existing literature.

Chapter 4 Design of Bundles

4.1 Introduction

In this chapter we study the effects of bundle design on firm profits. We take the choice of *bundle proportions* as a decision option that firms can employ to maximize profits. We have not found previous research addressing the topic of bundle design. In all previous papers the bundle is usually assumed to be pre-designed with given product mix proportions (Venkatesh and Mahajan 2009, Raghunathan and Sarkar 2016).

Telecom companies have been choosing alternative bundle designs for a long time to address specific market segments. For example, some tourist phone plans include very few minutes of phone call time but high data limits, while others allow very low data limits and more call minutes. Each design is suitable for a particular customer group with specific preferences. Another example is cable TV companies who offer skinny bundles and customized bundles to give customers more flexibility in choosing their desired mix of channels (Abdallah et al. 2017). Similarly, we see multiple bundle designs in many products cases such as gift baskets, tool dets, and car options.

We assume that firms can offer bundles of two components to customers who have quadratic utility functions. We show that for a monopolist firm offering a bundle of the two products there is always an optimal *bundle proportion*. This proportion is a function of the satiation consumption levels of the customer. We extend this case to the

oligopolistic setting by allowing multiple firms to enter the bundle market and characterize number of firms that are able to survive in equilibrium. Next we study a monopolist bundler selling to two customer groups. If customer groups are heterogeneous, the firm can have two distinct choices for the *bundle proportion* that maximize profit. This can extend to multiple customer groups as well. Finally, we study two symmetric firms offering independent bundles to multiple customer groups. In equilibrium both firms choose the same *bundle proportion*. The optimal *bundle proportion* is a function of the aggregate satiation consumption levels of all customers.

4.2 Model

In this section we study three different market settings with firms offering bundles with arbitrary proportions of products $\{A\}$ and $\{B\}$.

4.2.1 One Bundle and One Customer Group

4.2.1.1 Monopoly Bundling

Case 4.1.

Supply Side: There is one firm offering a bundle of two products $\{AB\}$. Each unit of the bundle consists of α units of product $\{A\}$ and $1-\alpha$ units of product $\{B\}$, $0 \leq \alpha \leq 1$.

Demand Side: There is one customer group with the following quadratic utility and surplus functions for consumption of $\{A\}$ and $\{B\}$.

$$
\begin{aligned}
U(y) &= a_1(\alpha y) + a_2((1-\alpha)y) - \frac{b_1}{2}(\alpha y)^2 - \frac{b_2}{2}((1-\alpha)y)^2 \\
&= (a_1\alpha + a_2(1-\alpha))y - \frac{b_1\alpha^2 + b_2(1-\alpha)^2}{2}y^2, \qquad (4.1) \\
S(y,p) &= U(y) - py. \qquad (4.2)
\end{aligned}
$$

For $b_1, b_2 > 0$, the surplus function is concave in the consumption level. Thus, the first order conditions reveals the optimal consumption level for any p.

$$
\bar{y}(p) = \begin{cases} \frac{a_1\alpha + a_2(1-\alpha) - p}{b_2 - 2\alpha b_2 + \alpha^2(b_1+b_2)} & \text{if } p \leq a_1\alpha + a_2(1-\alpha) \\ 0 & \text{if } p > a_1\alpha + a_2(1-\alpha) \end{cases} \qquad (4.3)
$$

Next, the price p is set such that the market clears. We find the following inverse-demand function which characterizes the Walrasian equilibrium price p.

$$
p(Q) = \begin{cases} a_1\alpha + a_2(1-\alpha) - (b_2 - 2\alpha b_2 + \alpha^2(b_1+b_2))Q & \text{if} \quad Q \leq \frac{a_1\alpha + a_2(1-\alpha)}{b_2 - 2\alpha b_2 + \alpha^2(b_1+b_2)} \\ 0 & \text{otherwise} \end{cases} \qquad (4.4)
$$

The monopolist selling the bundle chooses the optimal production level $\hat{Q}$ to maximize profit. The optimal $\hat{Q}$ and its corresponding profit level $\hat{\Pi}$ are:

$$
\begin{aligned}
\hat{Q}(\alpha) &= \frac{a_1\alpha + a_2(1-\alpha)}{2(\alpha^2 b_1 + b_2(1-\alpha)^2)} \\
\hat{\Pi}(\alpha) &= \frac{(a_1\alpha + a_2(1-\alpha))^2}{4(\alpha^2 b_1 + b_2(1-\alpha)^2)}
\end{aligned} \qquad (4.5)
$$

Result 4.1. *In Case 4.1, the monopolist chooses α^* as follows.*

$$\alpha^* = \frac{a_1 b_2}{a_2 b_1 + a_1 b_2} \; for \; a_1, a_2 \geq 0, b_1, b_2 > 0 \tag{4.6}$$

Proof. The profit function $\hat{\Pi}$ is not concave; however it is unimodal (Figure 4.1). The first derivative is given by:

For $a_1, a_2 \geq 0, b_1, b_2 > 0$,

$$\frac{\partial \hat{\Pi}}{\partial \alpha} = -\frac{(a_1\alpha + a_2(1-\alpha))(a_2 b_1 \alpha - a_1 b_2 (1-\alpha))}{2(\alpha^2 b_1 + b_2(1-\alpha)^2)^2} \begin{cases} > 0 \text{ for } 0 \leq \alpha < \frac{a_1 b_2}{a_2 b_1 + a_1 b_2}, \\ = 0 \text{ for } \alpha = \frac{a_1 b_2}{a_2 b_1 + a_1 b_2}, \\ < 0 \text{ for } \frac{a_1 b_2}{a_2 b_1 + a_1 b_2} < \alpha \leq 1. \end{cases} \tag{4.7}$$

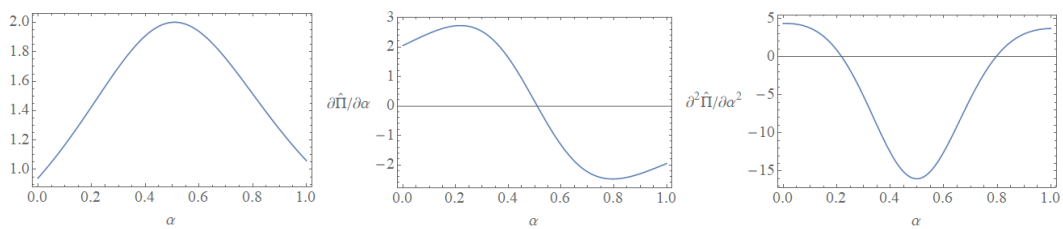


Figure 4.1: a) The profit function $\hat{\Pi}$, b) The first derivative of the profit function $\hat{\Pi}$ with respect to α, c) The second derivative of the profit function $\hat{\Pi}$ with respect to α.

□

The optimal *bundle proportion* α^* represents the ratio of the magnitudes of customer's satiation points for both products.

For $a_1, a_2 \geq 0, b_1, b_2 > 0$ the bundler's profit is maximized at $\alpha^* = \frac{a_1 b_2}{a_2 b_1 + a_1 b_2}$ which gives a profit of $\hat{\Pi}(\alpha^*) = \frac{a_1^2 b_2 + a_2^2 b_1}{4 b_1 b_2}$. In traditional bundling models the bundle proportion is assumed to be $\alpha = 0.5$, which gives a profit of $\hat{\Pi}(0.5) = \frac{(a_1+a_2)^2}{4(b_1+b_2)} \leq \hat{\Pi}(\alpha^*)$.

4.2.1.2 Oligopolistic Bundling

Case 4.2.

Supply Side: There are n symmetric firms each offering an identical bundle of two products $\{AB\}$. Each unit of the bundle consists of α units of product $\{A\}$ and $1-\alpha$ units of product $\{B\}$, $0 \leq \alpha \leq 1$.

Demand Side: There is one customer group with the same quadratic utility and surplus functions as in Equations (4.1) and (4.2).

The same results as in Equations (4.3) and (4.4) hold for customer's optimal consumption level and market clearing prices. However, each symmetric firm's optimal production level, $\hat{q}$, and equilibrium profit, $\hat{\pi}$, is a function of the total number of firms in the market, n:

$$\begin{aligned} \hat{q}(\alpha; n) &= \frac{a_1\alpha + a_2(1-\alpha)}{(n+1)(\alpha^2 b_1 + b_2(1-\alpha)^2)} \\ \hat{\pi}(\alpha; n) &= \frac{(a_1\alpha + a_2(1-\alpha))^2}{(n+1)^2(\alpha^2 b_1 + b_2(1-\alpha)^2)} \end{aligned} \tag{4.8}$$

Note that $\hat{\Pi}(\alpha)$ equals $\hat{\pi}(\alpha; n)$ when $n = 1$. Therefore firms will choose the same α^* under 4.1 as in Equation ((4.6)).

Result 4.2. *For Case 4.2, in equilibrium, all n firms choose α^* as follows.*

$$\alpha^* = \frac{a_1 b_2}{a_2 b_1 + a_1 b_2} \text{ for } a_1, a_2 \geq 0, b_1, b_2 > 0, \tag{4.9}$$

Now suppose that each entering firm incurs a fixed cost F.

Result 4.3. *The number of firms that can enter and survive in Case 4.2 with a bundle proportion α is*

$$n(\alpha) = \lfloor \frac{a_1\alpha + a_2(1-\alpha)}{\sqrt{F(\alpha^2 b_1 + (1-\alpha)^2 b_2)}} \rfloor - 1 \tag{4.10}$$

Result 4.4. *For $a_1, a_2 \geq 0, b_1, b_2 > 0$ the maximum number of firms that can enter with the optimal bundle proportion α^* and survive in a market with a fixed cost of F is*

$$n(\alpha^*) = \lfloor \frac{a_1\alpha^* + a_2(1-\alpha^*)}{\sqrt{F(\alpha^{*2}b_1 + (1-\alpha^*)^2 b_2)}} \rfloor - 1 = \lfloor \frac{a_1^2 b_2 + a_2^2 b_1}{\sqrt{F b_1 b_2}} \rfloor - 1 \tag{4.11}$$

Note that $n(\alpha^*) \geq n(\alpha)$ for any $0 \leq \alpha \leq 1$.

4.2.2 One Bundle and Two Customer Groups

4.2.2.1 Monopoly Bundling

Case 4.3.

Supply Side: There is one firm offering a bundle of two products $\{AB\}$. Each unit of the bundle consists of α units of product $\{A\}$ and $1-\alpha$ units of product $\{B\}$, $0 \leq \alpha \leq 1$.

Demand Side: There are two customer groups, $j = 1, 2$, with the following quadratic utility and surplus functions for consumption of $\{A\}$ and $\{B\}$.

$$U_j(y_j) = (a_{j1}\alpha + a_{j2}(1-\alpha))y_j - \frac{b_{j1}\alpha^2 + b_{j2}(1-\alpha)^2}{2} y_j^2, \text{ for } j = 1, 2, \tag{4.12}$$

$$S_j(y_j, p) = U(y_j) - py_j \tag{4.13}$$

We can find the optimal consumption level for every customer given the price p similar

to (4.3).

$$\bar{y}_j(p) = \begin{cases} \frac{a_{j1}\alpha + a_{j2}(1-\alpha) - p}{b_{j1}\alpha^2 + b_{j2}(1-\alpha)^2} & \text{if } p \leq a_{j1}\alpha + a_{j2}(1-\alpha), \\ 0 & \text{if } p > a_{j1}\alpha + a_{j2}(1-\alpha). \end{cases} \tag{4.14}$$

We assume that $p \leq \min_j a_{j1}\alpha + a_{j2}(1-\alpha)$, for $j = 1, 2$ for simplicity. The aggregate consumption level for all customer groups is given by $\bar{y}(p) = \sum_j \frac{A_j - p}{B_j} = \frac{A_1 - p}{B_1} + \frac{A_2 - p}{B_2}$ where $A_1 \doteq a_{11}\alpha + a_{12}(1-\alpha)$, $A_2 \doteq a_{21}\alpha + a_{22}(1-\alpha)$, $B_1 \doteq b_{11}\alpha^2 + b_{12}(1-\alpha)^2$, and $B_2 \doteq b_{21}\alpha^2 + b_{22}(1-\alpha)^2$.

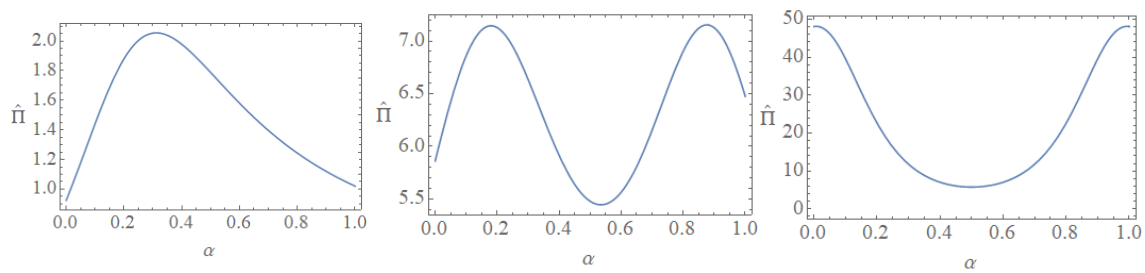
Next, the price p is set such that the markets clear. We find the following inverse-demand function which characterizes the Walrasian equilibrium price p.

$$p(Q) = \begin{cases} \frac{A_1B_2 + A_2B_1 - B_1B_2Q}{B_1 + B_2} & \text{if} \quad Q \leq \frac{A_1B_2 + A_2B_1}{B_1B_2} \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

The monopolist firm chooses the optimal production level.

$$\begin{aligned} \hat{Q}(\alpha) &= \frac{A_1B_2 + A_2B_1}{2B_1B_2} \\ \hat{\Pi}(\alpha) &= \frac{(A_1B_2 + A_2B_1)^2}{4(B_1B_2)(B_1 + B_2)} \end{aligned} \tag{4.16}$$

Our ultimate goal is to find the value of α that maximizes the profit function $\hat{\Pi}(\alpha)$. The profit function need not have a unique maxima as we show in Figure 4.2.



(a) (b) (c)

Figure 4.2: a)Unimodal profit function, b)Bi-modal profit function c)Bi-modal profit function

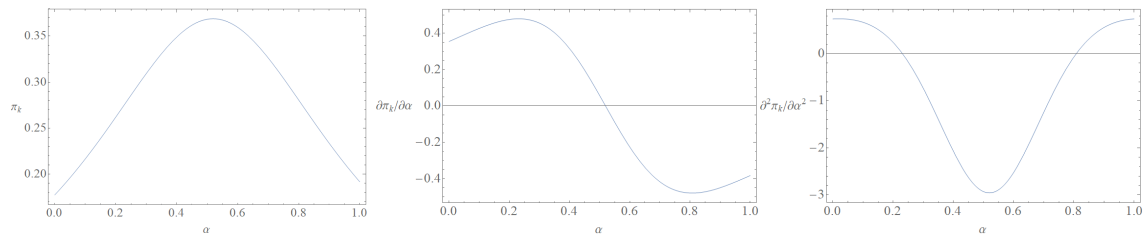
Figure 4.2 shows that the profit function may be bimodal. There can be two local maxima for cases where the customers are quite heterogeneous. If only one bundle may be offered, the profit of each local maximum can be evaluated. The parameters used in Figure 4.2 are the following:

$$
\begin{aligned}
(A_1, B_1, A_2, B_2) =& (a)\left(5\alpha + 1.6(1-\alpha), 4.6\alpha^2 + 0.8(1-\alpha)^2, 1.2\alpha + 1.1(1-\alpha), 4.1\alpha^2 + 1.4(1-\alpha)^2\right) \\
& (b)\left(3.6\alpha + 3.6(1-\alpha), 3.1\alpha^2 + 0.6(1-\alpha)^2, 3.1\alpha + 2.2(1-\alpha), 0.4\alpha^2 + 3(1-\alpha)^2\right) \\
& (c)\left(\alpha + 10(1-\alpha), 10\alpha^2 + 0.5(1-\alpha)^2, 10\alpha + (1-\alpha), 0.5\alpha^2 + 10(1-\alpha)^2\right)
\end{aligned}
$$

Example 4.1. *For $a_{11} = a_{12} = a_{21} = a_{22} = A$ the profit function in Equation (4.16) simplifies to*

$$
\hat{\Pi}(\alpha) = \frac{A^2}{4}\left(\frac{1}{b_{12}(1-\alpha)^2 + b_{11}\alpha^2} + \frac{1}{b_{22}(1-\alpha)^2 + b_{21}\alpha^2}\right) \tag{4.17}
$$

$\hat{\Pi}(\alpha) = \frac{A^2}{4}(\pi_1 + \pi_2)$ where $\pi_j \doteq \frac{1}{b_{j2}(1-\alpha)^2 + b_{j1}\alpha^2}$ for $j = 1, 2$. The functions π_1 and π_2 are not concave; however they are both unimodal (Figure 4.3). The first derivative of π_j for $j = 1, 2$ is:



(a) (b) (c)

Figure 4.3: a) The profit function π_j, b) The first derivative of the profit function π_j with respect to α, c) The second derivative of the profit function π_j with respect to α.

For $b_1, b_2 > 0$,

$$\frac{\partial \pi_j}{\partial \alpha} = \frac{2b_{j2}(1-\alpha) - 2b_{j1}\alpha}{b_{j2}(1-\alpha)^2 + b_{j1}\alpha^2} \begin{cases} > 0 \text{ for } 0 \leq \alpha < \frac{b_{j2}}{b_{j1}+b_{j2}}, \\ = 0 \text{ for } \alpha = \frac{b_{j2}}{b_{j1}+b_{j2}}, \\ < 0 \text{ for } \frac{b_{j2}}{b_{j1}+b_{j2}} < \alpha \leq 1. \end{cases} \tag{4.18}$$

$\pi_j(\alpha)$ has only one maxima at $\alpha_j^* = \frac{b_{j2}}{b_{j1}+b_{j2}}$ and its maximum value equals to $\pi(\alpha_j^*) = \frac{1}{b_{j1}} + \frac{1}{b_{j2}}$.

For $b_{j1} = 0$ and $b_{j2} > 0$,

$$\frac{\partial \pi_j}{\partial \alpha} = \frac{2}{1-\alpha} > 0 \tag{4.19}$$

$\pi_j(\alpha)$ has only one maxima at $\alpha_j^* = 1$ and its maximum value equals to $\pi(\alpha_j^*) = \frac{1}{b_{j1}}$.

For $b_{j2} = 0$ and $b_{j1} > 0$,

$$\frac{\partial \pi_j}{\partial \alpha} = \frac{-2}{\alpha} < 0 \tag{4.20}$$

$\pi_j(\alpha)$ has only one maximum at $\alpha_j^* = 0$ and its value equals to $\pi(\alpha_j^*) = \frac{1}{b_{j2}}$.

The bundler's profit function $\hat{\Pi}(\alpha)$ can potentially have two maxima at $\alpha_1^* = \frac{b_{12}}{b_{11}+b_{12}}$ and $\alpha_2^* = \frac{b_{22}}{b_{21}+b_{22}}$ only if the magnitudes of $\pi(\alpha_1^*)$ and $\pi(\alpha_2^*)$ are comparable. Otherwise, the function will have only one global maximum at the α_j^* corresponding to the j with

the largest $\pi_j(\alpha_j^*)$.

For the general case of J customer groups, such that $A_j = A$ for $j = 1, ..., J$, the profit function is in the form of $\hat{\Pi}(\alpha) = \frac{A^2}{4} \sum_j \frac{1}{b_j 1(1-\alpha)^2 + b_{j1}\alpha^2}$ and can potentially have J modes (Figure 4.4).

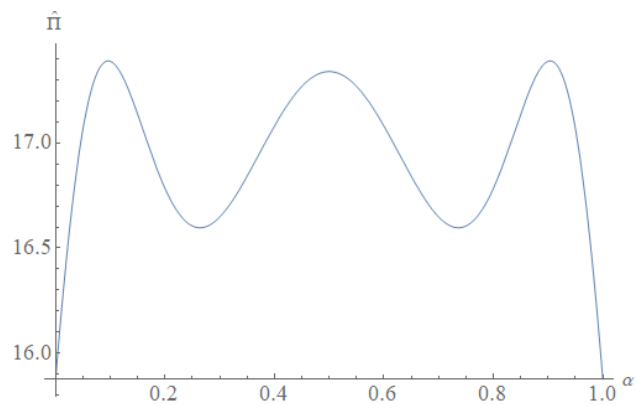


Figure 4.4: Profit function with three modes

4.2.2.2 Duopoly Bundling

Case 4.4.

Supply Side: There are two symmetric firms offering a bundle of two products $\{AB\}$. Each unit of the bundle consists of α units of product $\{A\}$ and $1-\alpha$ units of product $\{B\}$, $0 \leq \alpha \leq 1$.

Demand Side: There are two customer groups, $j = 1, 2$, with the following quadratic utility and surplus functions for consumption of $\{A\}$ and $\{B\}$.

$$U_j(y_j) = (a_{j1}\alpha + a_{j2}(1-\alpha))y_j - \frac{b_{j1}\alpha^2 + b_{j2}(1-\alpha)^2}{2} y_j^2, \text{ for } j = 1, 2, \tag{4.21}$$

$$S_j(y_j, p) = U(y_j) - py_j \tag{4.22}$$

The same results as in Equations (4.14) and (4.15) hold for the customers' optimal consumption levels and market clearing prices. However, each symmetric firm's optimal production level, $\hat{q}$, and equilibrium profit, $\hat{\pi}$, is:

$$\begin{aligned} \hat{q}(\alpha) &= \frac{A_1B_2 + A_2B_1}{3B_1B_2} \\ \hat{\pi}(\alpha) &= \frac{(A_1B_2 + A_2B_1)^2}{9(B_1B_2)(B_1 + B_2)} \end{aligned} \tag{4.23}$$

Which is a scaled version of the monopoly profit and production quantity. Therefore all monopoly results also hold for the duopoly case.

4.2.3 Two Bundling Firms with Distinct Bundles and Multiple Customer Groups

In the general case, this is a three-stage game. In the first stage, each firm i decides to enter or stay away from the bundle market.

$$r_i(n) = \begin{cases} 1 & \text{If} \quad \hat{\Pi}(n+1) \geq F \\ 0 & \text{Otherwise} \end{cases} \tag{4.24}$$

For tractability we skip the first stage and assume that there are two firms in the market, i.e., $n = 2$.

In the second stage, each firm chooses the optimal bundle proportion, given the competitors choice of bundle proportion.

$$\begin{aligned} \hat{\alpha}(\beta) &= \arg\max \hat{\Pi}_1(\alpha; \beta) \\ \hat{\beta}(\alpha) &= \arg\max \hat{\Pi}_2(\beta; \alpha) \end{aligned} \tag{4.25}$$

Finally, in the last stage, firms choose their production quantities $\hat{Q}_1$ and $\hat{Q}_2$.

$$\begin{aligned}\hat{Q}_1(Q_2) &= \arg\max \Pi_1(Q_1; Q_2, \alpha, \beta) \\ \hat{Q}_2(Q_1) &= \arg\max \Pi_2(Q_2; Q_1, \alpha, \beta)\end{aligned} \tag{4.26}$$

We can formalize this game as follows:

Case 4.5.

Supply Side: There are two firms offering bundles of two products $\{AB\}$. The bundle offered by the first firm consists of α units of product $\{A\}$ and $1-\alpha$ units of product $\{B\}$, $\alpha \geq 0$, and the bundle offered by the second firm consists of β units of product $\{A\}$ and $1-\beta$ units of product $\{B\}$, $\alpha, \beta \geq 0$.

Demand Side: There are multiple customer groups, $j = 1, ..., J$, with the following quadratic utility and surplus functions for consumption of $\{A\}$ and $\{B\}$.

$$\begin{aligned}U_j(\mathbf{y}_j) &= a_{j1}(\alpha y_{j1} + \beta y_{j2}) + a_{j2}((1-\alpha)y_{j1} + (1-\beta)y_{j2}) \\ &\quad - \frac{b_{j1}}{2}(\alpha y_{j1} + \beta y_{j2})^2 - \frac{b_{j2}}{2}((1-\alpha)y_{j1} + (1-\beta)y_{j2})^2, \end{aligned} \tag{4.27}$$

$$S_j(\mathbf{y}_j, \mathbf{p}) = U_j(\mathbf{y}_j) - p_1 y_{j1} - p_2 y_{j2}. \tag{4.28}$$

Without loss of generality we assume that $\alpha \geq \beta$. The surplus function is concave in the consumption levels. Thus, the first order conditions reveal the optimal consumption levels for any p_1 and p_2.

$$\bar{y}_{j1} + \bar{y}_{j2} = \frac{a_{j1}}{b_{j1}} + \frac{a_{j2}}{b_{j2}} + p_1 \frac{\beta b_{j1} - (1-\beta) b_{j2}}{(\alpha - \beta) b_{j1} b_{j2}} - p_2 \frac{\alpha b_{j1} - (1-\alpha) b_{j2}}{(\alpha - \beta) b_{j1} b_{j2}} \tag{4.29}$$

$$\bar{\mathbf{y}}_j(\mathbf{p}) = \begin{cases} \Big(\frac{\beta(a_{j2}(\beta-\alpha)-\beta p_1+\alpha p_2)}{(\alpha-\beta)^2 b_{j2}} + \frac{(1-\beta)(a_{j1}(\alpha-\beta)-(1-\beta)p_1+(1-\alpha)p_2)}{(\alpha-\beta)^2 b_{j1}}, & \\ \frac{\alpha(a_{j2}(\alpha-\beta)+\beta p_1-\alpha p_2)}{(\alpha-\beta)^2 b_{j2}} + \frac{(1-\alpha)(a_{j1}(\beta-\alpha)+(1-\beta)p_1-(1-\alpha)p_2)}{(\alpha-\beta)^2 b_{j1}}\Big) & \text{if } \alpha \neq \beta \\ (\frac{a_{j1}}{b_{j1}} + \frac{a_{j2}}{b_{j2}}, 0) & \text{if } \alpha = \beta \text{ and } p_1 < p_2 \\ (0, \frac{a_{j1}}{b_{j1}} + \frac{a_{j2}}{b_{j2}}) & \text{if } \alpha = \beta \text{ and } p_1 > p_2 \\ (y_{j1}, \frac{a_{j1}}{b_{j1}} + \frac{a_{j2}}{b_{j2}} - y_{j1}) & \text{if } \alpha = \beta \text{ and } p_1 = p_2 \end{cases}$$

Thus the aggregate consumption levels of all customers are:

$$\bar{\mathbf{y}}(\mathbf{p}) = \begin{cases} (-\frac{\beta}{(\alpha-\beta)} \sum_j \frac{a_{j2}}{b_{j2}} + \frac{1-\beta}{(\alpha-\beta)} \sum_j \frac{a_{j1}}{b_{j1}} + \frac{\beta(\alpha p_2-\beta p_1)}{(\alpha-\beta)^2} \sum_j \frac{1}{b_{j2}} + \frac{(1-\beta)((1-\alpha)p_2-(1-\beta)p_1)}{(\alpha-\beta)^2} \sum_j \frac{1}{b_{j1}}, & \\ -\frac{1-\alpha}{(\alpha-\beta)} \sum_j \frac{a_{j1}}{b_{j1}} + \frac{\alpha}{(\alpha-\beta)} \sum_j \frac{a_{j2}}{b_{j2}} - \frac{\alpha(\alpha p_2-\beta p_1)}{(\alpha-\beta)^2} \sum_j \frac{1}{b_{j2}} - \frac{(1-\alpha)((1-\alpha)p_2-(1-\beta)p_1)}{(\alpha-\beta)^2} \sum_j \frac{1}{b_{j1}}) & \\ & \text{if } \alpha \neq \beta \\ (\sum_j \frac{a_{j1}}{b_{j1}} + \sum_j \frac{a_{j2}}{b_{j2}}, 0) & \text{if } \alpha = \beta \text{ and } p_1 < p_2 \\ (0, \sum_j \frac{a_{j1}}{b_{j1}} + \sum_j \frac{a_{j2}}{b_{j2}}) & \text{if } \alpha = \beta \text{ and } p_1 > p_2 \\ (\bar{y}_1, \sum_j \frac{a_{j1}}{b_{j1}} + \sum_j \frac{a_{j2}}{b_{j2}}) & \text{if } \alpha = \beta \text{ and } p_1 = p_2 \end{cases}$$

Next, the price vector $\mathbf{p}$ is set such that markets clear. We find the following inverse-demand function which characterizes the Walrasian equilibrium price vector $\mathbf{p}(\mathbf{Q})$ as follows:

$$\begin{aligned} \mathbf{p}(\mathbf{Q}) = \Big(&\alpha\frac{M_1}{N_1} + (1-\alpha)\frac{M_2}{N_2} - Q_1(\frac{(1-\alpha)^2}{N_2} + \frac{\alpha^2}{N_1}) - Q_2(\frac{(1-\alpha)(1-\beta)}{N_2} + \frac{\alpha\beta}{N_1}), \\ &\beta\frac{M_1}{N_1} + (1-\beta)\frac{M_2}{N_2} - Q_1(\frac{(1-\alpha)(1-\beta)}{N_2} + \frac{\alpha\beta}{N_1}) - Q_2(\frac{(1-\beta)^2}{N_2} + \frac{\beta^2}{N_1})\Big) \end{aligned} \tag{4.30}$$

Where $M_t \doteq \sum_j \frac{a_{jt}}{b_{jt}}$ and $N_t \doteq \sum_j \frac{1}{b_{jt}}$ for $t = 1, 2$.

4.2.3.1 Simultaneous Entry

Both firms make the decision for their optimal production level $\hat{Q}_i$ simultaneously:

$$\begin{aligned}\hat{Q}_1(Q_2) &= \frac{(1-\alpha)M_2N_1 + \alpha M_1N_2 - \big((1-\alpha)(1-\beta)N_1 + \alpha\beta N_2\big)Q_2}{2\big((1-\alpha)^2N_1 + \alpha^2N_2\big)},\\ \hat{Q}_2(Q_1) &= \frac{(1-\beta)M_2N_1 + \beta M_1N_2 - ((1-\alpha)(1-\beta)N_1 + \alpha\beta N_2)Q_1}{2\big((1-\beta)^2N_1 + \beta^2N_2\big)}.\end{aligned} \tag{4.31}$$

The optimal production levels in equilibrium is given by:

$$\hat{Q}_1 = \frac{M_1N_2((-1+\beta)(\alpha(-2+\beta)+\beta)N_1+\alpha\beta^2N2)-M_2N_1((-1+\alpha)(-1+\beta)^2N_1+\beta(\alpha+(-2+\alpha)\beta)N_2)}{3(1-\alpha)^2(1-\beta)^2N_1^2+2(2\beta^2-\alpha\beta(1+3\beta)+\alpha^2(2+3(-1+\beta)\beta))N_1N_2+3\alpha^2\beta^2N_2^2}, \tag{4.32}$$

$$\hat{Q}_2 = \frac{M_1N_2((-1+\alpha)(\alpha+(-2+\alpha)\beta)N_1+\alpha^2\beta N2)-M_2N_1((-1+\alpha)^2(-1+\beta)N_1+\alpha(\alpha(-2+\beta)+\beta)N_2)}{3(1-\alpha)^2(1-\beta)^2N_1^2+2(2\beta^2-\alpha\beta(1+3\beta)+\alpha^2(2+3(-1+\beta)\beta))N_1N_2+3\alpha^2\beta^2N_2^2}. \tag{4.33}$$

And the corresponding profits are:

$$\begin{aligned}\hat{\Pi}_1(\alpha;\beta) &= (M_1N_2((1-\beta)(\alpha(-2+\beta)+\beta)N_1-\alpha\beta^2N_2)+M_2N_1((-1+\alpha)(1-\beta)^2N_1+\beta(\alpha+(\alpha-2)\beta)N_2))^2\\ &\times \frac{(1-\alpha)^2N_1+\alpha^2N_2}{N_1N_2\Big(3(\alpha\beta N_2+(1-\alpha)(1-\beta)N_1)^2+4N_1N_2(\alpha-\beta)^2\Big)^2},\end{aligned} \tag{4.34}$$

$$\begin{aligned}\hat{\Pi}_2(\beta;\alpha) &= (M_1N_2((1-\alpha)(\beta(-2+\alpha)+\alpha)N_1-\alpha^2\beta N_2)+M_2N_1((-1+\alpha)^2(-1+\beta)N_1+\alpha(\alpha(\beta-2)+\beta)N_2))^2\\ &\times \frac{(1-\beta)^2N_1+\beta^2N_2}{N_1N_2\Big(3(\alpha\beta N_2+(1-\alpha)(1-\beta)N_1)^2+4N_1N_2(\alpha-\beta)^2\Big)^2}.\end{aligned} \tag{4.35}$$

Result 4.5. *The profit function $\hat{\Pi}_1(\alpha;\beta)$ always has a maximum value within the range of $\alpha \in [0,1]$. Similarly, the profit function $\hat{\Pi}_2(\beta;\alpha)$ always has a maximum value within*

the range of $\beta \in [0, 1]$.

Proof. The profit functions have a quartic numerators that have only two double roots. The denominator is always positive and non-zero, thus it is always continuous. Although, we have not been able to analytically identify the coordinates of the maximum point for each profit function, we are able to show that such a point always exists. It either falls at the boundaries, i.e. $\alpha = 0$ or $\alpha = 1$, or is a single maximal point for $\alpha \in [0, 1]$. □

We define $\hat{\alpha}(\beta)$ as the function of Firm 1's best response to Firm 2's choice of bundle design. Similarly, $\hat{\beta}(\alpha)$ is the function of Firm 2's best response to Firm 1's choice of bundle design.

For a simultaneous game, since the market is symmetric for both firms, the equilibrium has to be symmetric, i.e., either $\alpha = \beta$ or even number of pay-off equivalent symmetric strategies.

Result 4.6. *In equilibrium,* $\hat{\alpha} = \hat{\beta} = \frac{M_1}{M_1+M_2}$.

Proof. First, we show that $\frac{M_1}{M_1+M_2}$ maximizes the profit functions 4.34 and 4.34 for both firms when $\alpha = \beta$. The profit function for $\alpha = \beta$ equals:

$$\hat{\Pi}_1(\alpha; \alpha) = \frac{(\alpha M_1 N_2 + (1-\alpha) M_2 N_1)^2}{9 N_1 N_2 (\alpha^2 N_2 + (1-\alpha)^2 N_1)} \tag{4.36}$$

This function is continuous on $\alpha \in [0, 1]$. The first derivative of $\hat{\Pi}_1$ is

$$\frac{\partial \hat{\Pi}_1}{\partial \alpha} = -\frac{2(\alpha M_2 - (1-\alpha) M_1)(M_2(N_1 - \alpha N_1) + \alpha M_1 N_2)}{9((1-\alpha)^2 N_1 + \alpha^2 N_2)^2} \tag{4.37}$$

has two roots at $\bar{\alpha}_1$ and $\bar{\alpha}_2$.

$$\begin{aligned} \bar{\alpha}_1 &= \frac{M_1}{M_1 + M_2} \\ \bar{\alpha}_2 &= \frac{M_2 N_1}{M_2 N_1 - M_1 N_2} \end{aligned} \tag{4.38}$$

$\bar{\alpha}_2$ always lies outside the interval $[0, 1]$. However, we can show that $\begin{cases} \frac{\partial \hat{\Pi}_1}{\partial \alpha} > 0 \text{ for } 0 \leq \alpha < \bar{\alpha}_1 \\ \frac{\partial \hat{\Pi}_1}{\partial \alpha} = 0 \text{ for } \alpha = \bar{\alpha}_1 \\ \frac{\partial \hat{\Pi}_1}{\partial \alpha} < 0 \text{ for } \bar{\alpha}_1 < \alpha \leq 1 \end{cases}$

Therefore, $\bar{\alpha}_2$ will maximize the profit function of the first firm. Due to symmetry, an identical argument holds for the second firm. □

We present a few numerical examples to verify this result numerically.

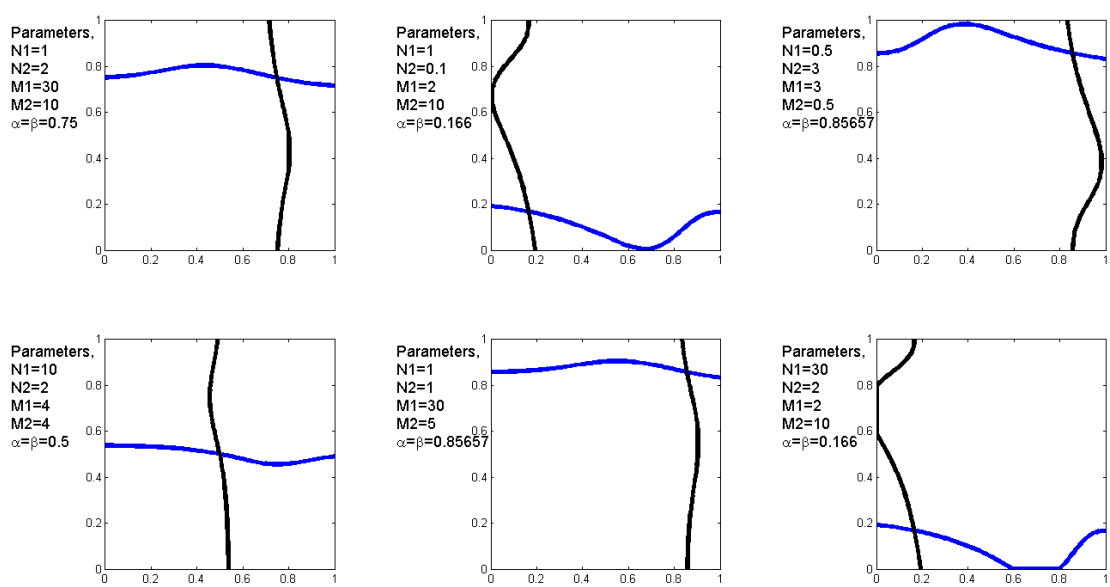
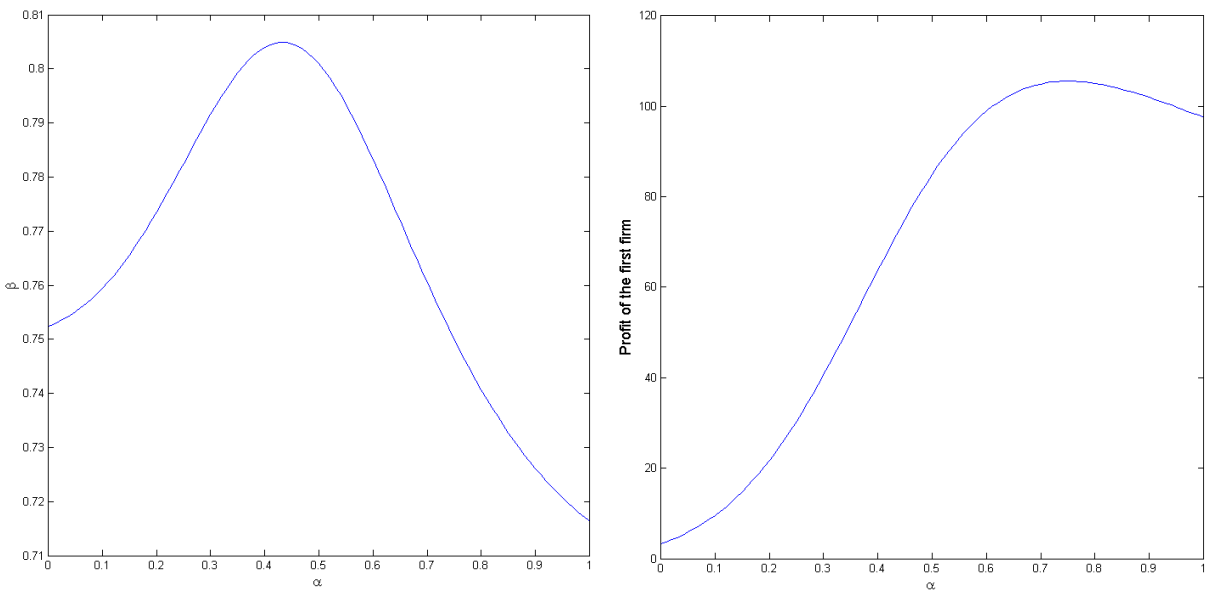


Figure 4.5: Six numerical examples to show the bundle proportions in equilibrium.

4.2.3.2 Sequential Entry

Result 4.7. *In equilibrium, both firms choose the same bundle proportions as they did in simultaneous game, i.e.* $\hat{\alpha} = \hat{\beta} = \frac{M_1}{M_1+M_2}$.

The following results were obtain for an example of this game through numerical analysis. The first firm chooses the *bundle proportion*, α, such that its profit given the second firm's response is maximized (Figure 4.6).



(a) (b)

Figure 4.6: a) The best response of the second firm to the choice of α by the first firm, b) Profit of the first firm as a function of α.

We repeated the same numerical analysis for multiple scenarios and in all cases the sequential equilibrium is identical to the simultaneous equilibrium.

4.3 Conclusions

In this chapter we addressed the issue of bundle design, where the proportions of the products in a bundle can be chosen. We first considered the case of a monopolist

bundler selling a bundle of two products to a single customer type, and showed that the optimal ratio for the quantities of the two products in the bundle is given by the ratio of the satiation points for the products. Next, we showed that the same result holds if multiple symmetric firms are able to offer the same bundle to one customer group. Next we considered a monopolist offering one type of bundle to two customer groups. We showed that the bundler's profit as a function of the bundle proportion can be bi-modal for certain customer valuations. Generally one mode will be preferred. However, if neither mode dominates the other, the firm can choose to cater to either one of the customer segments. Again, we showed that for a symmetric oligopoly identical results hold. Finally we considered a case of two types of firms that can have different bundling proportions, with two distinct customer groups. We considered the response of each type of firm to the bundle design of the other firm. For the game in which there are two firms which choose their bundle proportions simultaneously, there is a symmetric equilibrium in which both firms choose the same bundle proportions. In the sequential form of the game, where one firm chooses the bundle proportion first, with the other firm following, we again found that in equilibrium both firms play symmetric strategies.

Chapter 5 Conclusions

This thesis has studied competitive bundling from three perspectives. In the first chapter, we studied a symmetric Bertrand duopoly for the full scope of bundling strategies. We were able to show that even in the absence of factors such as market power or complementarity between the bundle components, bundling by itself can help the bundler firm to differentiate and earn higher profits. We are able to identify asymmetric equilibrium strategies for two perfectly symmetric firms. The equilibrium strategies depend on customers' valuations of the bundle component. A managerial implication is that the firm that bundles first can earn a higher profit and is able to discourage its competitor from bundling. Also, bundling is only successful if customers are heterogeneous and each customer has a strictly preferred product within the bundle.

In Chapter two, we examined the bundling game in a Cournot, quantity competition framework. We showed that given a fixed cost of entry to enter the market, we will in general observe oligopolies in each submarket, i.e., the bundled product market and the single component markets. The equilibrium is not always unique and it is possible to construct cases with an arbitrarily large number of equilibria. The following factors can determine the success of bundles, which depends on two factors. On the demand side, customers buy bundles as long as both components within the bundle add relatively comparable values to the customer. On the supply side, firms enter the bundle market

if the fixed-cost of entry for the bundle market is lower than the sum of fixed-costs of entry for each component within the bundle by a sufficient amount. These results hold for a single customer group, as well as for multiple customer groups.

In Chapter three, we studied the problem of bundle design. The optimal bundle proportion is a function of customers' satiation levels. For both simultaneous and sequential games, symmetric firms in a duopoly choose symmetric bundle design strategies.

In summary, the equilibrium in the Bertrand model is determined by the customers' valuations and bundling is a Pareto-optimal Nash equilibrium if customers are heterogeneous and each one of them has a strictly preferred product. However, in the Cournot model, bundling is a viable strategy only if the fixed cost of entry to the bundled market is smaller than the sum of the fixed costs of entry to each of the component markets. In equilibrium, unlike the Bertrand model, multiple entry in each market is possible.

Appendices

Appendix A Chapter 2 Proofs

Proof. Proof of Lemma 2.1

Since $p_{il} > 0$ and $\sum_j x_{ijl} > 0$. In order to have $\sum_j x_{ijl} > 0$, we need $p_{-i,l} \geq p_{il}$. But then, Firm $-i$ could lower its price below $p_{i,l}$, steal market share from Firm i, and increase its profit, a contradiction. Hence, there exists no equilibrium such that $p_{il} > 0$ and $\sum_j x_{ijl} > 0$, for any l.

Also, if $p_{il} = 0$, then $\Pi_{-i}(\mathbf{p}_{-i}; \mathbf{p}_i, \mathbf{z}) = 0$ for $p_{-i,l} \geq p_{il}$. In particular, $p_{-i,l} \geq 0$ is a best response to $p_{il} = 0$. As a result, all Nash equilibria of this game are such that $\Pi_i(\mathbf{p}_{il}; \mathbf{p}_{-il}, \mathbf{z}) = \Pi_{-i}(\mathbf{p}_{-il}; \mathbf{p}_{il}, \mathbf{z}) = 0$. □

Proof. Proof of Proposition 2.1

We analyze the best response of Firm 1 to the bundling strategy of Firm 2. Due to symmetry we can extend the results of the analysis to the reverse case. We use the short-hand notation p_1 and p_2 to refer to the prices of the products offered by Firms 1 and 2, respectively.

Let $\hat{\mathbf{z}}_1(\mathbf{z}_2)$ be the set of Firm 1's best response to $\mathbf{z}_2$.

We first show that if Firm 2 offers nothing or a single component, a weakly dominant strategy for Firm 1 is to offer the bundle. We will then show that if Firm 2 offers the bundle, a weakly dominant strategy for Firm 1 is to offer a single component.

- *When $\boldsymbol{z}_2 = (0, 0, 0)$, we consider four different responses by Firm 1:*

1. $\mathbf{z}_1 = (0,0,0) \implies \Pi_1(p_1; p_2, \mathbf{z}) = 0, \forall p_1$.

2. $\mathbf{z}_1 = (1,0,0) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j1}$.

3. $\mathbf{z}_1 = (0,1,0) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j2}$.

4. $\mathbf{z}_1 = (0,0,1) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_1} p_1 \sum_j x_{1j3}$.

   Since each customer's valuation of the bundle is the sum of her valuations for each component, i.e., $v_{j3} = v_{j1} + v_{j2}$, for a fixed p_1, we always have that $\sum_j x_{ij3} \geq \sum_j x_{ij1}$ and $\sum_j x_{ij3} \geq \sum_j x_{ij2}$.

Therefore, $(0,0,1) \in \hat{\mathbf{z}}_1(0,0,0)$

- *When $\boldsymbol{z}_2 = (1,0,0)$, we consider four different responses by Firm 1:*

1. $\mathbf{z}_1 = (0,0,0) \implies \Pi_1(p_1; p_2, \mathbf{z}) = 0, \forall p_1$.

2. $\mathbf{z}_1 = (1,0,0) \implies \hat{\Pi}_1(\hat{p}_1; p_2, \mathbf{z}) = \hat{\Pi}_2(\mathbf{z}_1; \mathbf{z}_2) = 0$ by Lemma 2.1.

3. $\mathbf{z}_1 = (0,1,0) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j2} p_1$, where by (2.1) for $j = 1, 2$,

$$x_{1j2} = 1 \iff v_{j2} - p_1 \geq 0. \tag{A.1}$$

4. $\mathbf{z}_1 = (0,0,1) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j3} p_1$, where by (2.1), for $j = 1, 2$,

$$x_{1j3} = 1 \iff v_{j1} + v_{j2} - p_1 \geq \max\{v_{j1} - p_2, 0\}. \tag{A.2}$$

The first and the second strategies are always dominated by the third and fourth strategies.

From (A.1) and (A.2) we conclude that:

$$x_{1j2} = 1 \iff v_{j2} - p_1 \geq 0 \implies \begin{cases} v_{j1} + v_{j2} - p_1 \geq v_{j1} \geq v_{j1} - p_2, \\ \\ v_{j1} + v_{j2} - p_1 \geq 0 \end{cases} \implies x_{1j3} = 1,$$

thus $\sum_j x_{1j3} p_1 \geq \sum_j x_{1j2} p_1, \forall p_1, p_2$, which means that for a fixed price of p_1, Firm 1 can capture more customers by offering the bundle.

By definition of price equilibrium, we have

$$\begin{aligned} &\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \Pi_i(\hat{p}_i; \hat{p}_{-i}, \mathbf{z}) \text{ such that} \\ &\Pi_i(\hat{p}_i; \hat{p}_{-i}, \mathbf{z}) \geq \Pi_i(p_i; \hat{p}_{-i}, \mathbf{z}) \quad \forall p_i \neq \hat{p}_i; \forall \mathbf{z}, i = 1, 2. \end{aligned} \tag{A.3}$$

Therefore:

$$\hat{\Pi}_1((0,0,1);(1,0,0)) = \sum_j x_{1j3} \hat{p}_1 \geq \sum_j x_{1j3} \tilde{p}_1 \geq \sum_j x_{1j2} \tilde{p}_1 = \hat{\Pi}_1((0,1,0);(1,0,0)) \geq 0,$$

where

$$\hat{p}_i = \text{argmax}_{p_i} \Pi_i(p_i; \hat{p}_{-i}, (0,0,1), (1,0,0)), \forall i = 1, 2$$

$$\tilde{p}_i = \text{argmax}_{p_i} \Pi_i(p_i; \hat{p}_{-i}, (0,1,0), (1,0,0)), \forall i = 1, 2.$$

Therefore $(0,0,1) \in \hat{\mathbf{z}}_1(1,0,0)$.

- *When $\boldsymbol{z}_2 = (0,1,0)$, using a symmetric argument, we obtain that $(0,0,1) \in \hat{\mathbf{z}}_1(0,1,0)$.*
- *When $\boldsymbol{z}_2 = (0,0,1)$, we consider four different responses by Firm 1:*

  1. $\mathbf{z}_1 = (0,0,0) \implies \Pi_1(p_1; p_2, \mathbf{z}) = 0, \forall p_1$.

2. $\mathbf{z}_1 = (1,0,0) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j1} p_1$, in which by (2.1),

$$x_{1j1} = 1 \iff v_{j1} - p_1 > v_{j1} + v_{j2} - p_2 \text{ and } v_{j1} - p_1 \geq 0.$$

3. $\mathbf{z}_1 = (0,1,0) \implies \Pi_1(\hat{p}_1; p_2, \mathbf{z}) = \max_{p_2} \sum_j x_{1j2} p_1$, in which by (2.1),

$$x_{1j2} = 1 \iff v_{j2} - p_1 > v_{j1} + v_{j2} - p_2 \text{ and } v_{j2} - p_1 \geq 0.$$

4. $\mathbf{z}_1 = (0,0,1) \implies \hat{\Pi}_1(\mathbf{z}_1, \mathbf{z}_2) = \hat{\Pi}_2(\mathbf{z}_1, \mathbf{z}_2) = 0$ by Lemma 2.1.

The first and the fourth strategies are always weakly dominated by the second and third strategies. Both $\mathbf{z}_1 = (1,0,0)$ and $\mathbf{z}_1 = (0,1,0)$ can potentially be the dominant strategies depending on the relative sizes of customer segments and their valuations for the products.

Hence, either $(1,0,0) \in \hat{\mathbf{z}}_1(0,0,1)$ or $(0,1,0) \in \hat{\mathbf{z}}_1(0,0,1)$. □

Proof. Proof of Proposition 2.2

With non-overlapping offerings, there is no competition in any of the component markets. Therefore Firm i who offers Product l, $l = 1,2$, can capture Customer j, i.e., $x_{ijl} = 1$, by setting $p_{il} \leq v_{jl}$.

- If $\mathbf{z}_i = (1,0,0)$ and $\mathbf{z}_{-i} = (0,0,0)$, it is clear that $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$. Moreover, Firm i can choose to capture the highest-valuation customer only, i.e., $v_{j1} = \overline{v}$, or both customers. For the former case $\hat{p}_i = \overline{v}$ and $\hat{\Pi}_i = \overline{v}$, and for the latter case $\hat{p}_i = \underline{v}$ and $\hat{\Pi}_i = 2\underline{v}$. Thus, in equilibrium the payoff of Firm i is equal to $\hat{\Pi}_i = \max\{\overline{v}, 2\underline{v}\}$.
- If $\mathbf{z}_i = (0,1,0)$ and $\mathbf{z}_{-i} = (0,0,0)$, due to symmetry, this case is identical to the

previous case.

- If $\mathbf{z}_i = (1,1,0)$ and $\mathbf{z}_{-i} = (0,0,0)$, since Components 1 and 2 have separate markets, Firm i can maximize its profit in each of the markets separately. Following the analysis of the first two cases, $\hat{\Pi}_i = \max\{\overline{v}, 2\underline{v}\} + \max\{\overline{v}, 2\underline{v}\} = \max\{2\overline{v}, 4\underline{v}\}$.

- If $\mathbf{z}_i = (1,0,0)$ and $\mathbf{z}_{-i} = (0,1,0)$, each firm sells in a separate market and thus each one maximizes its profit in its own respective market. Following the analysis in the first two cases, $\hat{\Pi}_i = \hat{\Pi}_{-i} = \max\{\overline{v}, 2\underline{v}\}$.

□

Proof. Proof of Proposition 2.3

We focus on the case where $\mathbf{z}_i = (1,1,0)$ and $\mathbf{z}_{-i} = (1,0,0)$. Due to symmetry, a similar proof holds for the other case and we omit the details for brevity. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline five possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$ ignoring for brevity all situations of ties. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij2} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$.

According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(2\text{-a})} = \hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$.

(b) $\sum_j x_{ij2} = 1$ and $\sum_j x_{ij1} = 0$. Under PNC valuations, Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{i22} = 1$ and $x_{i12} = 0$. According to (2.1), we need $p_{i2} \leq \overline{v}$. Thus, $\hat{p}_{i2} = \overline{v}$, and $\hat{\Pi}_i^{(2\text{-b})} = \hat{p}_{i2} = \overline{v}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

(a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, p_{-i,1} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\hat{p}_{i1} = 2\min\{\underline{v}, p_{-i,1} - \delta\}$.

(b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij2} = 2$. According to (2.1), we need $p_{i2} \leq \underline{v}$. Thus, $\hat{p}_{i2} = \underline{v}$ and $\hat{\Pi}_i^{(3\text{-b})} = 2\hat{p}_{i2} = 2\underline{v}$.

(c) $x_{i11} = 1$, $x_{i21} = 0$, $x_{i22} = 1$, and $x_{i12} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and Product 2 to Customer 2. Hence, we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$ and $p_{i2} \leq \overline{v}$. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i2} = \overline{v}$, and $\hat{\Pi}_i^{(3\text{-c})} = \hat{p}_{i1} + \hat{p}_{i2} = \overline{v} + \min\{\overline{v}, p_{-i,1} - \delta\}$.

4. To sell both products to one customer and one product to the other customer, which can happen in one of the following two cases:

(a) $\sum_j x_{ij1} = 2$, $x_{i22} = 1$, and $x_{i12} = 0$. Under PNC valuations, we need $p_{i1} \leq \min\{\underline{v}, p_{-i,1} - \delta\}$ and $p_{i2} \leq \overline{v}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i2} = \overline{v}$, and $\hat{\Pi}_i^{(4\text{-a})} = 2\hat{p}_{i1} + \hat{p}_{i2} = 2\min\{\underline{v}, p_{-i,1} - \delta\} + \overline{v}$.

(b) $x_{i11} = 1$, $x_{i21} = 0$, and $\sum_j x_{ij2} = 2$. Under PNC valuations, we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$ and $p_{i2} \leq \underline{v}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i2} = \underline{v}$, and $\hat{\Pi}_i^{(4\text{-b})} = \hat{p}_{i1} + 2\hat{p}_{i2} = \min\{\overline{v}, p_{-i,1} - \delta\} + 2\underline{v}$.

5. To sell both products to both customers. Under PNC valuations, we need $p_{i1} \leq \min\{\underline{v}, p_{-i,1} - \delta\}$ and $p_{i2} \leq \underline{v}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i2} = \underline{v}$, and $\hat{\Pi}_i^{(5)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 2\min\{\underline{v}, p_{-i,1} - \delta\} + 2\underline{v}$.

Comparing profits, we obtain that always $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-c})}$, $\hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(3\text{-c})}$, and $\hat{\Pi}_i^{(3\text{-a})} < \hat{\Pi}_i^{(4\text{-c})}$. Moreover,

- when $p_{-i,1} > 0$, $\hat{\Pi}_i^{(2\text{-b})} < \hat{\Pi}_i^{(3\text{-c})}$, and $\hat{\Pi}_i^{(3\text{-b})} < \hat{\Pi}_i^{(4\text{-b})}$, also
  - when $\overline{v} \leq 2\underline{v}$, $\hat{\Pi}_i^{(3\text{-c})} \leq \hat{\Pi}_i^{(4\text{-b})}$, $\hat{\Pi}_i^{(4\text{-a})} \leq \hat{\Pi}_i^{(5)}$, and $\hat{\Pi}_i^{(4\text{-b})} \leq \hat{\Pi}_i^{(5)}$.

    Strategy (5) is the only strategy that is not weakly dominated. Therefore, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$ and $\hat{p}_{i2} = \underline{v}$, which results in the same profit as any other payoff-equivalent strategy, is the unique best response.
  - when $\overline{v} > 2\underline{v}$, $\hat{\Pi}_i^{(4\text{-b})} < \hat{\Pi}_i^{(3\text{-c})}$, and $\hat{\Pi}_i^{(5)} < \hat{\Pi}_i^{(4\text{-a})}$.
    * When $0 < p_{-i,1} \leq 2\underline{v}$, $\hat{\Pi}_i^{(3\text{-c})} < \hat{\Pi}_i^{(4\text{-a})}$. Strategy (4-a) is the only strategy that is not dominated. Therefore, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$ and $\hat{p}_{i2} = \overline{v}$, is the unique best response.
    * When $2\underline{v} < p_{-i,1}$, $\hat{\Pi}_i^{(4\text{-a})} < \hat{\Pi}_i^{(3\text{-c})}$. Strategy (3-c) is the only strategy that is not dominated. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{p}_{i2} = \overline{v}$, is the unique best response.
- when $p_{-i,1} = 0$, $\hat{\Pi}_i^{(2\text{-b})} = \hat{\Pi}_i^{(3\text{-c})} = \hat{\Pi}_i^{(4\text{-a})} = \overline{v}$, and $\hat{\Pi}_i^{(3\text{-b})} = \hat{\Pi}_i^{(4\text{-b})} = \hat{\Pi}_i^{(5)} = 2\underline{v}$.
  - when $\overline{v} \leq 2\underline{v}$, Strategies (3-b), (4-b), and (5) are three payoff-equivalent best responses. Thus, Firm i sets $\hat{p}_{i1} = 0$ and $\hat{p}_{i2} = \underline{v}$ and earns $\hat{\Pi}_i = 2\underline{v}$.
  - when $\overline{v} > 2\underline{v}$, Strategies (2-b), (3-c), and (4-a) are three payoff-equivalent best responses. Thus, Firm i sets $\hat{p}_{i1} = 0$ and $\hat{p}_{i2} = \overline{v}$ and earns $\hat{\Pi}_i = \overline{v}$.

Firm $-i$. We next outline three possible strategies for Firm $-i$ in response to its competitor's prices $p_{i,1} > 0$ and $p_{i,2}$ ignoring for brevity all situations of ties. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\mathrm{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,j1} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell to one customer, i.e., $\sum_{j,l} x_{-i,jl} = 1$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's valuation for Product 1. Hence, $x_{-i,11} = 1$ and $x_{-i,21} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta\}$ and $\hat{\Pi}_{-i}^{(2)} = \hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta\}$.

3. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$. Under PNC valuations, we need $p_{-i,1} \leq \min\{\underline{v}, p_{i1} - \delta\}$. Thus, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta\}$ and $\hat{\Pi}_{-i}^{(3)} = 2\min\{\underline{v}, p_{i1} - \delta\}$.

Comparing profits, we obtain that

- when $p_{i1} > 0$, $\hat{\Pi}_{-i}^{(1)} < \hat{\Pi}_{-i}^{(2)}$, also,
  - when $\overline{v} \leq 2\underline{v}$, $\hat{\Pi}_{-i}^{(2)} \leq \hat{\Pi}_{-i}^{(3)}$

    Strategy (3) is the only strategy that is not weakly dominated. Therefore, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta\}$, which results in the same profit as any other payoff-equivalent strategy, is the unique best response.
  - when $\overline{v} > 2\underline{v}$,
    * When $0 < p_{i1} \leq 2\underline{v}$, $\hat{\Pi}_{-i}^{(2)} < \hat{\Pi}_{-i}^{(3)}$. Strategy (3) is the only strategy that is not dominated. Therefore, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta\}$, is the unique best response.
    * When $2\underline{v} < p_{i1}$, $\hat{\Pi}_{-i}^{(2)} < \hat{\Pi}_{-i}^{(3)}$. Strategy (2) is the only strategy that is not dominated. Therefore, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta\}$, is the unique best response.
- when $p_{i1} = 0$, $\hat{\Pi}_{-i}^{(1)} = \hat{\Pi}_{-i}^{(2)} = \hat{\Pi}_{-i}^{(3)} = 0$ are all payoff-equivalent best response

strategies.

Comparison. Comparing the best responses of the two firms (in the situations where there is no tie), it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, the firms will engage in a price war on Component 1, undercutting each other's price. Hence, there is no equilibrium where the market of Component 1 is not shared. Applying a similar argument to situations of ties shows that price undercutting remains a best response if firms have the opportunity to do so. Hence, the unique equilibrium is $\hat{p}_{i1} = \hat{p}_{-i,1} = 0$. On the other hand, $\hat{p}_{i2} = \underline{v}$ if $2\underline{v} \leq \overline{v}$ and $\hat{p}_{i2} = \overline{v}$ otherwise. As a result $\hat{\Pi}_i = \max\{2\underline{v}, \overline{v}\}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.4

Suppose that $z_{i3} = 1$, and $\mathbf{z}_{-i} = (0, 0, 0)$, then one of the following four cases are possible:

1. $\mathbf{z}_i = (0, 0, 1)$ then $\Pi_i(p_i; p_{-i}, \mathbf{z}) = p_{i3} \sum_j x_{ij3} = 2p_{i3}$ if $p_{i3} \leq \overline{v}+\underline{v}$, and $\Pi_i(p_i; p_{-i}, \mathbf{z}) = 0$ if $p_{i3} > \overline{v} + \underline{v}$. Thus $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

2. $\mathbf{z}_i = (1, 0, 1)$ then by selling Product 1, Firm i can earn $\overline{v}$ if it sells Product 1 to one of the customers, and $2\underline{v}$ if it sells Product 1 to both customers. However, if Firm i prices any $p_{i1} < \overline{v}$, it can earn a profit of $2(\overline{v} + \underline{v})$ by selling the bundle to both customers. Thus in equilibrium, $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

3. $\mathbf{z}_i = (0, 1, 1)$ similar to the previous case, in equilibrium, $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

4. $\mathbf{z}_i = (1, 1, 1)$. Firm i offers three products and there are only two customer groups in the market. Firm i has to sell at most two products to the customer groups. If it chooses Product 1 and Product 2, then the equilibrium profit is $\hat{\Pi}_i = \max\{2\overline{v}, 4\underline{v}\}$,

according to Proposition 2.2. If it chooses Product 1 and Product 3 (the bundle), or Product 2 and Product 3, then the equilibrium profit is $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$, similar to the former two cases. Thus Firm i is better off if it sells only the bundle to both customers. Thus in equilibrium, $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

□

Proof. Proof of Proposition 2.5

Since $\mathbf{z}_i$ and $\mathbf{z}_{-i}$ are kept fixed, for simplicity we hereon omit these arguments from the profit functions and best-response correspondences; also we use the simple notation of $p_i = p_{i3}$ and $p_{-i} = p_{-il}$ and $s_{ij} = s_{ij3}$ and $s_{-i,j} = s_{-ijl}$ for $j = 1, 2$ and $l = 1, 2$.

We focus on the case where $\mathbf{z}_i = (0, 0, 1)$ and $\mathbf{z}_{-i} = (0, 1, 0)$. First, we analyze the best-response correspondence of Firm i for different ranges of prices of Firm $-i$, namely, $p_{-i} > \overline{v}$, $\underline{v} \leq p_{-i} \leq \overline{v}$, and $0 \leq p_{-i} < \underline{v}$.

- If $p_{-i} > \overline{v}$, then customers' surplus from Firm $-i$ is negative, i.e., $s_{-i,1}(p_{-i}) = s_{-i,2}(p_{-i}) < 0$. As long as Firm i offers them a nonnegative surplus, i.e., $s_{i1}(p_i) = s_{i2}(p_i) = \overline{v} + \underline{v} - p_i \geq 0$, it can capture both customers and earn $\Pi_i(p_i; p_{-i}) = 2p_i$; hence Firm i's best response is

  $\hat{p}_i(p_{-i}) = \overline{v} + \underline{v}$, and $\hat{\Pi}_i(p_i; p_{-i}) = 2(\underline{v} + \overline{v})$.

- If $\underline{v} \leq p_{-i} \leq \overline{v}$, then $s_{-i,1}(p_{-i}) = \underline{v} - p_{-i} \leq 0$ and $s_{-i,2}(p_{-i}) = \overline{v} - p_{-i} > 0$. For Firm i to capture Customer 1 it is necessary and sufficient to have $s_{i1}(p_i) \geq 0$, which means $p_i \leq \overline{v} + \underline{v}$. Similarly, to capture Customer 2, it is necessary and sufficient that $s_{i2}(p_i) \geq 0$ and $s_{i2}(p_i) \geq s_{-i,2}(p_{-i})$, i.e., $\overline{v} + \underline{v} - p_i \geq \overline{v} - p_{-i} > 0$. Hence, Firm i captures both customers if $p_i \leq \underline{v} + p_{-i}$, it captures Customer 1 only if $\underline{v} + p_{-i} < p_i \leq \overline{v} + \underline{v}$, and it captures no customer if $p_i > \underline{v} + \overline{v}$. Therefore,

$\hat{\Pi}_i(p_i; p_{-i}) = \max\left\{2(p_{-i} + \underline{v}), \overline{v} + \underline{v}\right\}$, and $\hat{p}_i(p_{-i}) = \begin{cases} p_{-i} + \underline{v} & \text{if } p_{-i} \geq \frac{\overline{v}-\underline{v}}{2} \\ \overline{v} + \underline{v} & \text{if } p_{-i} \leq \frac{\overline{v}-\underline{v}}{2} \end{cases}$.

- If $0 \leq p_{-i} < \underline{v}$, then $s_{-i,1}(p_{-i}) = \underline{v} - p_{-i} > 0$ and $s_{-i,2}(p_{-i}) = \overline{v} - p_{-i} > 0$. For Firm i to capture Customer 1 it is necessary and sufficient that $s_{i1}(p_i) \geq s_{-i,1}(p_{-i})$, i.e., $\overline{v} + \underline{v} - p_i \geq \underline{v} - p_{-i}$, i.e., $p_i \leq p_{-i} + \overline{v}$. Similarly, to capture Customer 2, $s_{i2}(p_i) \geq s_{-i,2}(p_{-i})$, i.e., $\overline{v} + \underline{v} - p_i \geq \overline{v} - p_{-i}$, i.e., $p_i \leq p_{-i} + \underline{v}$. Therefore,

  $\hat{\Pi}_i(p_i; p_{-i}) = \max\left\{2(p_{-i} + \underline{v}), p_{-i} + \overline{v}\right\}$, and $\hat{p}_i(p_{-i}) = \begin{cases} p_{-i} + \underline{v} & \text{if } p_{-i} \geq \overline{v} - 2\underline{v} \\ p_{-i} + \overline{v} & \text{if } p_{-i} \leq \overline{v} - 2\underline{v} \end{cases}$.

Next, we consider Firm $-i$'s best response. We need to consider four sets for p_i; namely $p_i > \overline{v} + \underline{v}$, $\overline{v} \leq p_i \leq \overline{v} + \underline{v}$, $\underline{v} \leq p_i < \overline{v}$, and $p_i < \underline{v}$.

- If $p_i > \overline{v} + \underline{v}$, then $s_{i1}(p_i) = s_{i2}(p_i) < 0$. Thus for Firm $-i$ to capture Customer 1 it is necessary and sufficient that $p_{-i} \leq \underline{v}$. Also to capture Customer 2, the requirement is $p_{-i} \leq \overline{v}$. Hence, Firm $-i$ captures both customers if $p_{-i} \leq \underline{v}$, captures only one customer if $\underline{v} < p_{-i} \leq \overline{v}$, and captures no customer if $p_{-i} > \overline{v}$. Therefore,

  $\hat{\Pi}_{-i}(p_{-i}; p_i) = \max\left\{2\underline{v}, \overline{v}\right\}$, and $\hat{p}_{-i}(p_i) = \begin{cases} \underline{v} & \text{if } 2\underline{v} \geq \overline{v} \\ \overline{v} & \text{if } 2\underline{v} \leq \overline{v} \end{cases}$.

- If $\overline{v} \leq p_i \leq \overline{v} + \underline{v}$, then $s_{i1}(p_i) = s_{i2}(p_i) \geq 0$. Thus, to capture Customer 1, Firm $-i$ needs to price such that $\underline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$, i.e., $p_{-i} < p_i - \overline{v}$. Similarly capturing Customer 2 requires $\overline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$, i.e., $p_{-i} < p_i - \underline{v}$. Hence, Firm $-i$ captures both customers if $p_{-i} < p_i - \overline{v}$, captures only one customer if $p_i - \overline{v} \leq p_{-i} < p_i - \underline{v}$, and captures no customer if $p_{-i} \geq p_i - \underline{v}$. Therefore,

  $\hat{\Pi}_{-i}(p_{-i}; p_i) = \sup_{\delta > 0}\{2(p_i - \overline{v} - \delta)^+, (p_i - \underline{v} - \delta)^+\}$, and for some infinitesimal

$$\delta > 0,\ \hat{p}_{-i}(p_i) = \begin{cases} p_i - \overline{v} - \delta & \text{if } p_i > 2\overline{v} - \underline{v} \\ p_i - \underline{v} - \delta & \text{if } p_i \leq 2\overline{v} - \underline{v} \end{cases}.$$

- If $\underline{v} < p_i < \overline{v}$, then $\underline{v} - p_{-i} < \overline{v} + \underline{v} - p_i$, i.e., $s_{-i,1}(p_{-i}) < s_{i1}(p_i)\ \forall p_{-i} \geq 0$. Thus Firm $-i$ can never capture Customer 1 at any nonnegative price $p_{-i} \geq 0$. To capture Customer 2, it is necessary and sufficient to have $s_{-i,2}(p_{-i}) > s_{i2}(p_i)$ and $s_{-i,2}(p_{-i}) \geq 0$, which is equivalent to $\overline{v} - p_{-i} > \overline{v} + \underline{v} - p_i$ and $\overline{v} - p_{-i} \geq 0$, i.e., $p_{-i} < p_i - \underline{v}$. Therefore,

  $\hat{\Pi}_{-i}(p_{-i}; p_i) = \hat{p}_{-i}(p_i) = p_i - \underline{v} - \delta$ for some infinitesimal $\delta > 0$.

- If $0 \leq p_i \leq \underline{v}$, then $s_{-i,1}(p_{-i}) < s_{i1}(p_i)$ and $s_{-i,2}(p_{-i}) < s_{i2}(p_i)$ for any $p_{-i} \geq 0$. Hence Firm $-i$ does not capture any customer at any nonnegative price $p_{-i} \geq 0$. Hence,

  $\hat{\Pi}_{-i}(p_{-i}; p_i) = 0$, and $\hat{p}_{-i}(p_i) = \{p_{-i} | p_{-i} \geq 0\}$.

In conclusion, there are three separate best-response correspondences for both firms depending on the relative values of $\overline{v}$ and $\underline{v}$:

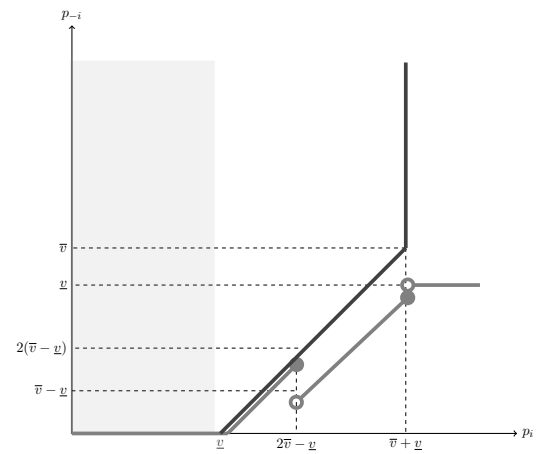
1. When $\overline{v} \leq 2\underline{v}$, assuming that prices have to be set in increments of $\delta > 0$ (e.g. cents)

$$\hat{p}_i(p_{-i}) = \begin{cases} \overline{v} + \underline{v} & \text{if } p_{-i} > \overline{v} \\ p_{-i} + \underline{v} & \text{if } 0 \leq p_{-i} \leq \overline{v} \end{cases} \quad \text{and } \hat{p}_{-i}(p_i) = \begin{cases} \underline{v} & \text{if } p_i > \overline{v} + \underline{v} \\ p_i - \overline{v} - \delta & \text{if } 2\overline{v} - \underline{v} < p_i \leq \overline{v} + \underline{v} \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \leq 2\overline{v} - \underline{v} \\ [0, \infty) & \text{if } 0 \leq p_i \leq \underline{v} \end{cases}.$$

   This game has a pure-strategy Nash equilibrium at $(\underline{v}, 0)$ (Figure A.1), yielding $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. Firm i does not deviate from this strategy since it would

earn strictly less profit by either increasing or decreasing its price. On the other hand, $p_2 = 0$ is a weakly dominant strategy for Firm $-i$, since Firm $-i$ can still earn zero profit for any other p_{-i}. This equilibrium is not unique; however it is payoff-equivalent to any other equilibria for this game.

Figure A.1: Firms' pricing best-response correspondences when $\overline{v} \leq 2\underline{v}$

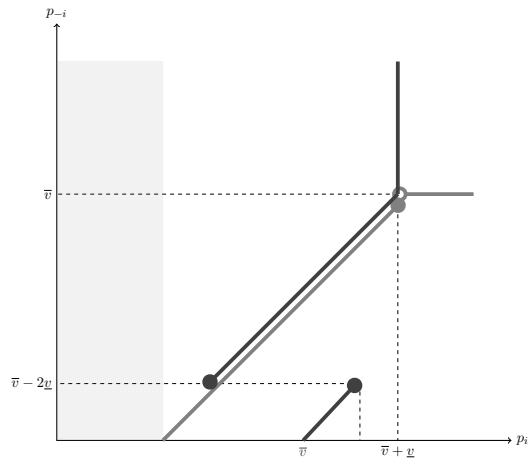


2. When $2\underline{v} < \overline{v} \leq 3\underline{v}$, assuming that prices have to be set in increments of $\delta > 0$,

$$\hat{p}_i(p_{-i}) = \begin{cases} \underline{v} + \overline{v} & \text{if } p_{-i} > \overline{v} \\ p_{-i} + \underline{v} & \text{if } \overline{v} - 2\underline{v} \leq p_{-i} \leq \overline{v} \\ p_{-i} + \overline{v} & \text{if } 0 \leq p_{-i} \leq \overline{v} - 2\underline{v} \end{cases}, \quad \text{and} \quad \hat{p}_{-i}(p_i) = \begin{cases} \overline{v} & \text{if } p_i > \overline{v} + \underline{v} \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \leq \overline{v} + \underline{v} \\ [0, \infty) & \text{if } 0 \leq p_i \leq \underline{v} \end{cases}. \tag{A.4}$$

This game does not have a pure-strategy Nash equilibrium since the best-response correspondences do not intersect.

Figure A.2: Firms' pricing best response correspondences when $2\underline{v} < \overline{v} \leq 3\underline{v}$

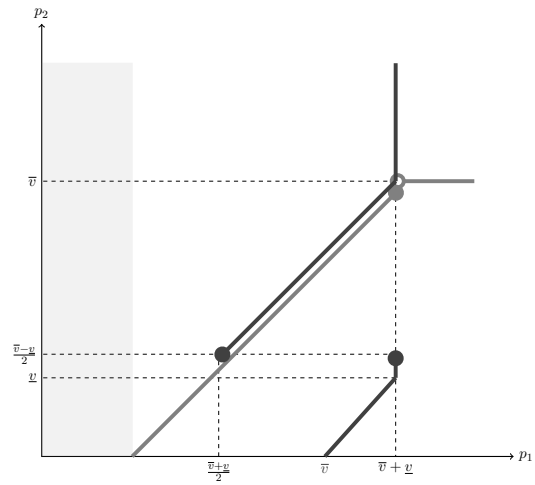


3. When $\overline{v} > 3\underline{v}$, assuming that prices have to be set in increments of $\delta > 0$

$$\hat{p}_i(p_{-i}) = \begin{cases} \underline{v} + \overline{v} & \text{if } p_{-i} > \overline{v} \\ p_{-i} + \underline{v} & \text{if } \frac{\overline{v}-\underline{v}}{2} \leq p_{-i} \leq \overline{v} \\ \underline{v} + \overline{v} & \text{if } \underline{v} \leq p_{-i} \leq \frac{\overline{v}-\underline{v}}{2} \\ p_{-i} + \overline{v} & \text{if } 0 \leq p_{-i} < \underline{v} \end{cases}, \quad \text{and} \quad \hat{p}_{-i}(p_i) = \begin{cases} \overline{v} & \text{if } p_i > \overline{v} + \underline{v} \\ p_i - \underline{v} - \delta & \text{if } \underline{v} < p_i \leq \overline{v} + \underline{v} \\ [0, \infty) & \text{if } 0 \leq p_i \leq \underline{v} \end{cases}. \tag{A.5}$$

It can be verified that this game also does not have a pure-strategy Nash equilibrium.

Figure A.3: Firms' pricing best-response correspondences when $3\underline{v} < \overline{v}$



Assuming that δ is the smallest price deviation that firms make to undercut the competitor's price, the following iteration process of elimination of strictly dominated strategies reveals the strategy supports for games described in $(A.4)$ and $(A.5)$.

- When $2\underline{v} < \overline{v} \leq 3\underline{v}$

  In the first iteration, Firm $-i$ starts off by the full set of pricing strategies, i.e., $\hat{p}_{-i} \in [0, \infty)$. According to $(A.4)$, in the first iteration, $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v}]$ is Firm i's best response to Firm $-i$ prices. In the second iteration, Firm $-i$ responds by $\hat{p}_{-i} \in [\overline{v} - 2\underline{v} - \delta, \overline{v} - \delta]$ and Firm i responds by $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v} - \delta] + \{2(\overline{v} - \underline{v}) - \delta\}$. The best response of Firm i in iteration t is $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v} - (t-1)\delta) + \{2(\overline{v} - \underline{v}) - \delta\}$. However, for $t > 3 + \frac{\underline{v}-\overline{v}}{\delta}$ the single point is included within the range, i.e., $\overline{v} - \underline{v} \leq 2(\overline{v} + \underline{v}) - \delta \leq \overline{v} + \underline{v} - (t-1)\delta$, and the best response of Firm i becomes $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, 2(\overline{v} - \underline{v}) - \delta]$. As a result, Firm $-i$'s best response to $\hat{p}_i \in [\overline{v} - \underline{v}, 2(\overline{v} - \underline{v}) - \delta]$ will be $\hat{p}_{-i}(p_i) \in [\overline{v} - 2\underline{v} - \delta, \overline{v} - 2\delta]$ and as a response Firm i will set $\hat{p}_i(p_{-i}) \in [\overline{v} - \underline{v}, \overline{v} + \underline{v} - 2\delta]$. Each iteration shrinks both firms' range of best responses until the following ranges are achieved, after

which none of the strategies are strictly dominated, hence no further elimination occurs.

$$\begin{aligned} \hat{p}_i(p_{-i}) &\in [\overline{v} - \underline{v}, 2(\overline{v} - \underline{v}) - \delta], \\ \hat{p}_{-i}(p_i) &\in [\overline{v} - 2\underline{v} - \delta, 2\overline{v} - 3\underline{v} - 2\delta]. \end{aligned} \tag{A.6}$$

- When $\overline{v} > 3\underline{v}$ In the first iteration, Firm $-i$ starts off by the full set of pricing strategies, i.e., $\hat{p}_{-i} \in [0, \infty)$. According to $(A.5)$, $\hat{p}_i(p_{-i}) \in [\frac{\overline{v}+\underline{v}}{2}, \overline{v} + \underline{v}]$ is Firm i's best response to Firm $-i$ prices. In the next iteration, Firm $-i$ responds by setting $\hat{p}_{-i}(p_i) \in [\frac{\overline{v}-\underline{v}}{2} - \delta, \overline{v} - \delta]$ and as a response Firm i sets $\hat{p}_i(p_{-i}) \in [\frac{\overline{v}+\underline{v}}{2}, \overline{v} + \underline{v}]$. Any further iteration will not shrink firms' range of best responses, since none of the strategies are strictly dominated, hence no further elimination occurs.

$$\begin{aligned} \hat{p}_i(p_{-i}) &\in \left[\frac{\overline{v} + \underline{v}}{2}, \overline{v} + \underline{v}\right], \\ \hat{p}_{-i}(p_i) &\in \left[\frac{\overline{v} - \underline{v}}{2} - \delta, \overline{v} - \delta\right]. \end{aligned} \tag{A.7}$$

Although there exists no pure-strategy Nash equilibria when $2\underline{v} < \overline{v} \leq 3\underline{v}$ and when $\overline{v} > 3\underline{v}$, we next show that the following CDF functions characterize the mixed-strategy Nash equilibrium of this game:

$$F_i(p_i) = \begin{cases} \frac{p_i - a}{p_i + c - a}, & \text{for } a \leq p_i < b \\ 1, & \text{for } p_i = b \end{cases}$$

$$F_{-i}(p_{-i}) = 2 - \frac{b}{p_{-i} + a - c} \text{ for } c \leq p_{-i} \leq c + b - a$$

where

$$
\begin{aligned}
&a \doteq \overline{v} - \underline{v}, \quad b \doteq 2(\overline{v} - \underline{v}) - \delta, \quad \text{and } c \doteq \overline{v} - 2\underline{v} - \delta \quad \text{when } 2\underline{v} < \overline{v} \leq 3\underline{v}, \\
&a \doteq \frac{\overline{v} + \underline{v}}{2}, \quad b \doteq \overline{v} + \underline{v}, \quad \text{and } c \doteq \frac{\overline{v} - \underline{v}}{2} - \delta \quad \text{when } \overline{v} > 3\underline{v}.
\end{aligned}
\tag{A.8}
$$

Firm i offers a surplus of $s_{ij3}(p_i) = \overline{v} + \underline{v} - p_i$ to both customers, $j = 1, 2$, while Firm $-i$ offers a surplus of $s_{-i,12}(p_{-i}) = \underline{v} - p_{-i}$ to Customer 1 and a surplus of $s_{-i,22}(p_{-i}) = \overline{v} - p_{-i}$ to Customer 2.

- When $\overline{v} > 3\underline{v}$, for any $p_i \in [a, b]$, $s_{i13}(p_i) \geq 0$, and $s_{i13} \geq s_{-i,12}$ over all $p_{-i} \in [c, c + b - a]$,[1] thus Firm i captures Customer 1 for every $p_i \in [a, b]$. Similarly, for any $p_i \in [a, b]$, $s_{i23}(p_i) \geq 0$, while $s_{i23}(p_i) \geq s_{-i,22}(p_{-i})$ only if $p_{-i} \geq p_i - \underline{v}$, thus Firm i captures Customer 2 for every $p_i \leq p_{-i} + \underline{v}$. Therefore, Firm i's market share equals $\Big(1 + \int_{p_i - \underline{v}}^{c+b-a} dF_{-i}(p_{-i})\Big)$. For any $p_i \in [a, b]$, $\Pi_i(p_i; F_{-i})$ is Firm i's profit when Firm $-i$ employs the randomizing profile $F_{-i}(p_{-i}) = 2 - \frac{b}{p_{-i}+a-c}$ over $c \leq p_{-i} \leq c + b - a$. Hence, for any $p_i \in [a, b]$:

$$
\begin{aligned}
\Pi_i(p_i; F_{-i}) =& \lim_{\delta \to 0} \int_c^{c+b-a} \Pi_i(p_i; p_{-i}) dF_{-i}(p_{-i}) = \lim_{\delta \to 0} \left(p_i \Big(1 + \int_{p_i - \underline{v}}^{c+b-a} dF_{-i}(p_{-i}) \Big) \right) \\
=& \lim_{\delta \to 0} \left(p_i \Big(1 + F_{-i}(c + b - a) - F_{-i}(p_i - \underline{v}) \Big) \right) \\
=& \lim_{\delta \to 0} \left(p_i \Big(\frac{b}{p_i + \delta} \Big) \right) = \overline{v} + \underline{v}.
\end{aligned}
\tag{A.9}
$$

Similarly, for any $p_{-i} \in [c, c + b - a]$, $\Pi_{-i}(p_{-i}; F_{-i})$ denotes Firm $-i$'s profit when Firm i employs the randomizing profile $F_i(p_i) = \frac{p_i - a}{p_i + c - a}$ over $a \leq p_i < b$ and $F_i(b) = 1$. Because Firm $-i$ captures Customer 2 if Firm i prices at any $p_i \in (p_{-i} + \underline{v}, b]$,

[1] The requirement for $s_{i13} \geq s_{-i,12}$, is $p_i - p_{-i} \leq \overline{v}$. For $p_i \in [a, b]$ and $p_{-i} \in [c, c + b - a]$, $\max(p_i - p_{-i}) = \overline{v} + \underline{v} - (\frac{\overline{v} - \underline{v}}{2} - \delta) = \frac{\overline{v} + 3\underline{v}}{2} + \delta$. When $\overline{v} > 3\underline{v}$ and δ is infinitesimal, $\frac{\overline{v} + 3\underline{v}}{2} + \delta \leq \overline{v}$

its market share equals $\int_{p_{-i}+\underline{v}}^{b} dF_i(p_i)$. Hence, for any $p_i \in [c, c+b-a]$,

$$\begin{aligned}
\Pi_{-i}(p_{-i}; F_i) &= \lim_{\delta\to 0}\left(\int_a^b \Pi_{-i}(p_{-i}; p_i) dF_i(p_i)\right) \\
&= \lim_{\delta\to 0}\left(p_{-i}\int_{p_{-i}+\underline{v}}^{b} dF_i(p_i)\right) = \lim_{\delta\to 0}\Big(p_{-i}\Big(F_i(b) - F_i(p_{-i}+\underline{v})\Big)\Big) \\
&= \lim_{\delta\to 0}\left(p_{-i}\Big(1 - \frac{p_{-i} - \frac{\overline{v}-\underline{v}}{2}}{p_{-i}-\delta}\Big)\right) = \frac{\overline{v}-\underline{v}}{2}.
\end{aligned} \tag{A.10}$$

- When $2\underline{v} < \overline{v} \leq 3\underline{v}$, for any $p_i \in [a, b]$, $s_{i13}(p_i) \geq 0$, and $s_{i13} \geq s_{-i,12}$ over all $p_{-i} \in [c, c+b-a]$, thus Firm i captures Customer 1 for every $p_i \in [a, b]$. Similarly, for any $p_i \in [a, b]$, $s_{i23}(p_i) \geq 0$, while $s_{i23}(p_i) \geq s_{-i,22}(p_{-i})$ only if $p_{-i} \geq p_i - \underline{v}$, thus Firm i captures Customer 2 for every $p_i \leq p_{-i} + \underline{v}$. Therefore, Firm i's market share equals $\Big(1 + \int_{p_i-\underline{v}}^{c+b-a} dF_{-i}(p_{-i})\Big)$. For any $p_i \in [a, b]$, $\Pi_i(p_i; F_{-i})$ is Firm i's profit when Firm $-i$ employs the randomizing profile $F_{-i}(p_{-i}) = 2 - \frac{b}{p_{-i}+a-c}$ over $c \leq p_{-i} \leq c+b-a$. Hence, for any $p_i \in [a, b]$:

$$\begin{aligned}
\Pi_i(p_i; F_{-i}) =& \lim_{\delta\to 0}\int_c^{c+b-a} \Pi_i(p_i; p_{-i}) dF_{-i}(p_{-i}) = \lim_{\delta\to 0}\left(p_i\Big(1 + \int_{p_i-\underline{v}}^{c+b-a} dF_{-i}(p_{-i})\Big)\right) \\
=& \lim_{\delta\to 0}\Big(p_i\Big(1 + F_{-i}(c+b-a) - F_{-i}(p_i - \underline{v})\Big)\Big) \\
=& \lim_{\delta\to 0}\left(p_i\Big(\frac{b}{p_i+\delta}\Big)\right) = 2(\overline{v} - \underline{v}).
\end{aligned} \tag{A.11}$$

Similarly, for any $p_{-i} \in [c, c+b-a]$, $\Pi_{-i}(p_{-i}; F_{-i})$ denotes Firm $-i$'s profit when Firm i employs the randomizing profile $F_i(p_i) = \frac{p_i-a}{p_i+c-a}$ over $a \leq p_i < b$ and $F_i(b) = 1$. Because Firm $-i$ captures Customer 2 if Firm i prices at any $p_i \in (p_{-i} + \underline{v}, b]$,

its market share equals $\int_{p_{-i}+\underline{v}}^{b} dF_i(p_i)$. Hence, for any $p_i \in [c, c+b-a]$,

$$\begin{aligned}\Pi_{-i}(p_{-i}; F_i) =& \lim_{\delta\to 0}\left(\int_a^b \Pi_{-i}(p_{-i}; p_i) dF_i(p_i)\right) = \lim_{\delta\to 0}\left(p_{-i}\int_{p_{-i}+\underline{v}}^{b} dF_i(p_i)\right) \\ =& \lim_{\delta\to 0}\Big(p_{-i}\Big(F_i(b) - F_i(p_{-i}+\underline{v})\Big)\Big) \\ =& \lim_{\delta\to 0}\left(p_{-i}\Big(1 - \frac{p_{-i} - (\overline{v} - 2\underline{v})}{p_{-i} - \delta}\Big)\right) = \overline{v} - 2\underline{v}.\end{aligned} \tag{A.12}$$

We already showed in (A.6) and (A.7) that the strategy support is obtained through iterated eliminated of dominated strategies, thus $\forall p_i \notin [a, b]$ Firm i's profit is strictly less than its profit at any $p_i \in [a, b]$. Similarly, $\forall p_{-i} \notin [c, c+b-a]$ Firm $-i$'s profit is strictly less than its profit at any $p_{-i} \in [c, c+b-a]$. Therefore, $F_i(p_i)$ and $F_{-i}(p_{-i})$ characterize a mixed strategy Nash equilibrium over $p_i \in [a, b]$ and $p_{-i} \in [c, c+b-a]$.

To summarize, we have the following:

- For $3\underline{v} < \overline{v}$

  Strategy supports of both firms: $\sigma_i = [\frac{\overline{v}+\underline{v}}{2}, \overline{v}+\underline{v}]$, $\sigma_{-i} = [\frac{\overline{v}-\underline{v}}{2} - \delta, \overline{v} - \delta]$,

  CDF of Firm i: $F_i(p_i) = \frac{2p_i - (\overline{v}+\underline{v})}{2(p_i - \underline{v} - \delta)}$ for $\frac{\overline{v}+\underline{v}}{2} \le p_i < \overline{v}+\underline{v}$ and $F_i(\overline{v}+\underline{v}) = 1$,

  CDF of Firm $-i$: $F_{-i}(p_{-i}) = 2 - \frac{\overline{v}+\underline{v}}{p_{-i}+\underline{v}+\delta}$ for $\frac{\overline{v}-\underline{v}}{2} \le p_{-i} \le \overline{v} - \delta$,

  Equilibrium profits: $\hat{\Pi}_i = \overline{v} + \underline{v}$, $\hat{\Pi}_{-i} = \frac{\overline{v}-\underline{v}}{2}$.

- For $2\underline{v} \le \overline{v} \le 3\underline{v}$

  Strategy supports of both firms: $\sigma_i = [\overline{v} - \underline{v}, 2(\overline{v} - \underline{v}) - \delta]$, $\sigma_{-i} = [\overline{v} - 2\underline{v} - \delta, 2\overline{v} - 3\underline{v} - 2\delta]$,

  CDF of Firm i: $F_i(p_i) = \frac{p_i - (\overline{v}-\underline{v})}{p_i - \underline{v} - \delta}for\overline{v} - \underline{v} \le p_i < 2(\overline{v} - \underline{v}) - \delta$and $F_i(2(\overline{v}-\underline{v}) - \delta) = 1$,

CDF of Firm $-i$: $F_{-i}(p_{-i}) = 2 - \frac{2(\overline{v}-\underline{v})}{p_{-i}+\underline{v}+\delta}$ for $\overline{v} - 2\underline{v} \leq p_{-i} \leq 2\overline{v} - 3\underline{v} - 2\delta$,

Equilibrium profits: $\hat{\Pi}_i = 2(\overline{v} - \underline{v})$ and $\hat{\Pi}_{-i} = \overline{v} - 2\underline{v}$.

In Appendix B we characterize this equilibrium by construction and prove that it is unique. □

Proof. Proof of Proposition 2.6

To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline five possible strategies for Firm i in response to its competitor's price $p_{-i,3} > 0$ ignoring all situations of ties for brevity. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij2} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-a})} = \hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$.

   (b) $\sum_j x_{ij2} = 1$ and $\sum_j x_{ij1} = 0$. Under PNC valuations, Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{i22} = 1$ and $x_{i12} = 0$. According to (2.1), we need $p_{i2} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, and $\hat{\Pi}_i^{(\text{2-b})} = \hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\hat{p}_{i1} = 2\min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$.

   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij2} = 2$. According to (2.1), we need $p_{i2} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$. Thus, $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-b})} = 2\min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$.

   (c) $x_{i11} = 1$, $x_{i21} = 0$, $x_{i22} = 1$, and $x_{i12} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and Product 2 to Customer 2. Hence, we need $p_{i1} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$ and $p_{i2} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, $\hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, and $\hat{\Pi}_i^{(3\text{-c})} = \hat{p}_{i1} + \hat{p}_{i2} = 2\min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$.

4. To sell both products to one customer and one product to the other customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 2$, $x_{i22} = 1$, and $x_{i12} = 0$. Under PNC valuations, we need $p_{i1} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $p_{i2} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$, $\hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, and $\hat{\Pi}_i^{(4\text{-a})} = 2\hat{p}_{i1} + \hat{p}_{i2} = 2\min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\} + \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$.

   (b) $x_{i11} = 1$, $x_{i21} = 0$, and $\sum_j x_{ij2} = 2$. Under PNC valuations, we need $p_{i2} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $p_{i1} \leq \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$, and $\hat{\Pi}_i^{(4\text{-b})} = 2\hat{p}_{i2} + \hat{p}_{i1} = 2\min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\} + \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$.

5. To sell both products to both customers. Under PNC valuations, we need $p_{i1} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $p_{i2} \leq \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$,

$\hat{p}_{i2} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$, and $\hat{\Pi}_i^{(5)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 4\min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$.

Comparing profits, we obtain that

- When $p_{-i,3} > \overline{v} + \underline{v}$,

  $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(2-a)} = \hat{\Pi}_i^{(2-b)} < \hat{\Pi}_i^{(3-c)} = 2\overline{v}$ and $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3-a)} = \hat{\Pi}_i^{(3-b)} < \hat{\Pi}_i^{(5)} = 4\underline{v}$.

  Also, $\hat{\Pi}_i^{(4\text{-a})} = \hat{\Pi}_i^{(4\text{-b})}$ is a convex combination of $\hat{\Pi}_i^{(3\text{-c})}$ and $\hat{\Pi}_i^{(5)}$ and it only equals $\hat{\Pi}_i^{(3\text{-c})}$ and $\hat{\Pi}_i^{(5)}$ when $\hat{\Pi}_i^{(3\text{-c})}$=$\hat{\Pi}_i^{(5)}$, i.e. $\overline{v} = 2\underline{v}$.

  - When $\overline{v} \leq 2\underline{v}$, Strategy (5) is the dominant strategy.
  - When $\overline{v} = 2\underline{v}$, both Strategies (3-c) and (5) are payoff equivalent and not dominated.
  - When $\overline{v} > 2\underline{v}$, Strategy (3-c) is the dominant strategy.

- When $\overline{v} < p_{-i,3} \leq \overline{v} + \underline{v}$,

  $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-c})}$, $\hat{\Pi}_i^{(2\text{-a})} = \hat{\Pi}_i^{(2\text{-b})} < \hat{\Pi}_i^{(3\text{-c})}$, $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-a})} = \hat{\Pi}_i^{(3\text{-b})} < \hat{\Pi}_i^{(5)}$, also $\hat{\Pi}_i^{(4\text{-a})} = \hat{\Pi}_i^{(4\text{-b})}$ is a convex combination of $\hat{\Pi}_i^{(3\text{-c})}$ and $\hat{\Pi}_i^{(5)}$ and it only equals $\hat{\Pi}_i^{(3\text{-c})}$ and $\hat{\Pi}_i^{(5)}$ when $\hat{\Pi}_i^{(3\text{-c})}$=$\hat{\Pi}_i^{(5)}$.

  - When $\overline{v} \leq 2\underline{v}$,

    Strategy (3-c), i.e., $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$ and $\hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, is the best response when $\overline{v} < p_{-i,3} \leq 2\overline{v} - \underline{v}$.

    Strategy (5), i.e., $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ and $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$, is the best response when $2\overline{v} - \underline{v} < p_{-i,3} \leq \overline{v} + \underline{v}$.

  - When $\overline{v} \geq 2\underline{v}$, $\hat{\Pi}_i^{(4\text{-a})} = \hat{\Pi}_i^{(4\text{-b})} < \max\{\hat{\Pi}_i^{(3\text{-c})}, \hat{\Pi}_i^{(5)}\}$ and $\hat{\Pi}_i^{(3\text{-c})} < \hat{\Pi}_i^{(5)}$.

    Strategy (3-c), i.e., $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$ and $\hat{p}_{i2} = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$, is the best response.

- When $\underline{v} < p_{-i,3} \leq \overline{v}$,

$\hat{\Pi}_i^{(3\text{-a})} = \hat{\Pi}_i^{(3\text{-b})} < 0 = \hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-c})}$, $\hat{\Pi}_i^{(5)} < 0 = \hat{\Pi}_i^{(1)}$, $\hat{\Pi}_i^{(4\text{-a})} = \hat{\Pi}_i^{(4\text{-b})} < \hat{\Pi}_i^{(2\text{-a})} = \hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(3\text{-c})}$. Thus Strategy (3-c) is the only strategy that is not weakly dominated when.

- When $0 \leq p_{-i,3} \leq \underline{v}$,

  $\hat{\Pi}_i^{(2\text{-a})} = \hat{\Pi}_i^{(2\text{-b})} = \hat{\Pi}_i^{(3\text{-a})} = \hat{\Pi}_i^{(3\text{-b})} = \hat{\Pi}_i^{(3\text{-c})} = \hat{\Pi}_i^{(4\text{-a})} = \hat{\Pi}_i^{(4\text{-b})} = \hat{\Pi}_i^{(5)} < 0 = \hat{\Pi}_i^{(1)}$.

  The Strategy (1), which means staying out of business, is the only one that is not weakly dominated when $0 \leq p_{-i,3} \leq \underline{v}$.

Strategies (1), (3-c), and (5) that survive weakly dominance are symmetric, i.e., $\hat{p}_{i1} = \hat{p}_{i2} = \hat{p}_i$. Accordingly, we restrict our analysis of Firm $-i$'s best response to only symmetric strategies for Firm i.

Firm $-i$. We next outline three possible strategies for Firm $-i$ in response to its competitor's price $p_i > 0$. For any strategy (x), we denote Firm $-i$'s profit as $\hat{\Pi}_{-i}^{(\text{x})}$. Given Firm i's symmetric pricing strategy, Firm $-i$ can sell to either both or no customers.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,j1} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$. We need $\hat{p}_{-i,3} = \min\{\overline{v} + \underline{v}, p_i + \underline{v}\}$, and $\hat{\Pi}_{-i}^{(2)} = 2\min\{\overline{v} + \underline{v}, p_i + \underline{v}\}$

Comparing profits, we obtain that $\hat{\Pi}_{-i}^{(1)} < \hat{\Pi}_{-i}^{(2)}$

Comparison. Comparing the best responses of any two firms it is clear that if $p_i = 0$, then $\hat{p}_{-i,3} = \underline{v}$. And if $p_{-i,3} = \underline{v}$, the best response of Firm i is $\hat{p}_i \in [0, \infty)$; hence $\hat{p}_i = 0$ and $\hat{p}_{-i,3} = \underline{v}$ is an equilibrium. If $0 < p_i < \overline{v}$, then $\hat{p}_{-i,3} = \underline{v} + p_i$. The best response of Firm i is then either $\hat{p}_i = \min\{\overline{v}, p_{-i,3} - \underline{v} - \delta\}$ or $\hat{p}_i = \min\{\underline{v}, p_{-i,3} - \overline{v} - \delta\}$ so when $\hat{p}_{-i,3} = \underline{v} + p_i$, this becomes $\hat{p}_i = \min\{\overline{v}, \hat{p}_i - \delta\} \leq \hat{p}_i - \delta$, or $\hat{p}_i = \min\{\underline{v}, \underline{v} - \overline{v} - \delta + \hat{p}_i\} < \hat{p}_i$

which both yield a contradiction. Similarly, if $p_i > \overline{v}$, then $\hat{p}_{-i,3} = \overline{v} + \underline{v}$ and when $p_{-i,3} = \overline{v} + \underline{v}$, the best response of Firm i is either $\hat{p}_i = p_{-i,3} - \underline{v} - \delta = \overline{v} - \delta$, or $\hat{p}_i = p_{-i,3} - \overline{v} - \delta = \underline{v} - \delta$ which both yield a contradiction. Hence, the only equilibrium is $\hat{p}_{i1} = \hat{p}_{i2} = 0$ and $\hat{p}_{-i,3} = \underline{v}$. As a result $\hat{\Pi}_i = 0$ and $\hat{\Pi}_{-i} = 2\underline{v}$. □

Proof. Proof of Proposition 2.7

We focus on the case where $\mathbf{z}_i = (1, 0, 1)$ and $\mathbf{z}_{-i} = (1, 0, 0)$. Due to symmetry, a similar proof holds for the other case and we omit the details for brevity. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$ ignoring all situations of ties. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij3} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta, p_{i3} - \underline{v} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-a})} = \hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$.

   (b) $\sum_j x_{ij3} = 1$ and $\sum_j x_{ij1} = 0$. Under PNC valuations, Customer 2's valuation for Product 1 is lower than Customer 1's. Hence, $x_{i23} = 1$ and $x_{i13} = 0$.

According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, \overline{v} + \min\{p_{i1}, p_{-i,1}\}\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \overline{v}\}$, and $\hat{\Pi}_i^{(2\text{-b})} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \overline{v}\}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, p_{i3} - \overline{v}, p_{-i,1} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\hat{p}_{i1} = 2\min\{\underline{v}, p_{-i,1} - \delta\}$.

   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, \underline{v} + \min\{p_{i1}, p_{-i,1}\}\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \overline{v}\}$ and $\hat{\Pi}_i^{(3\text{-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}\}$.

   (c) $x_{i11} = 1$, $x_{i21} = 0$, $x_{i23} = 1$, and $x_{i13} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and the bundle to Customer 2. Hence, we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\}$ and $p_{i3} \leq \min\{\overline{v} + \underline{v}, \overline{v} + \min\{p_{i1}, p_{-i,1}\}$. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\}$, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}\}$, and $\hat{\Pi}_i^{(3\text{-c})} = \hat{p}_{i1} + \hat{p}_{i3} = \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\} + \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}\}$.

Comparing profits, we obtain that

- When $p_{-i,1} > \overline{v}$,
  $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-a})} < \hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(3\text{-c})} < \hat{\Pi}_i^{(3\text{-b})}$ and the only strategy that are not dominated is Strategy (3-b).

- When $0 < p_{-i,1} \leq \overline{v}$,
  $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-a})} < \hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(3\text{-c})}$. The only strategies that are not dominated are (3-b) and (3-c). Relative values of $\overline{v}$ and $\underline{v}$ determine the dominant strategy:

- When $\overline{v} \leq 2\underline{v}$,

  Strategy (3-b), i.e., any $\hat{p}_{i1} \geq \overline{v}$ and $\hat{p}_{i3} = \min\{\overline{v}+\underline{v}, p_{-i,1}+\underline{v}\}$, is the best response.

- When $\overline{v} > 2\underline{v}$,

  Strategy (3-c), i.e., $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1}-\delta\}$ and $\hat{p}_{i3} = \min\{\overline{v}+\underline{v}, p_{-i,1}+\overline{v}-\delta\}$, is the best response when $0 < p_{-i,1} < \overline{v} - \underline{v}$.

  Strategy (3-b), i.e., any $\hat{p}_{i1} > \overline{v}$ and $\hat{p}_{i3} = \min\{\overline{v}+\underline{v}, p_{-i,1}+\underline{v}\}$, is the best response when $\overline{v} - \underline{v} \leq p_{-i,1}$.

- When $p_{-i,1} = 0$, $\hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(1)}$, $\hat{\Pi}_i^{(3\text{-a})} < \hat{\Pi}_i^{(1)}$,$\hat{\Pi}_i^{(2\text{-b})} < \hat{\Pi}_i^{(3\text{-c})}$,$\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3\text{-c})}$.Relative values of $\overline{v}$ and $\underline{v}$ determine the dominant strategy:

  - When $\overline{v} \leq 2\underline{v}$,

    Strategy (3-b), i.e., any $\hat{p}_{i1} \geq \overline{v}$ and $\hat{p}_{i3} = \underline{v}$, is the best response.

  - When $\overline{v} > 2\underline{v}$,

    Strategy (3-c), i.e., $\hat{p}_{i1} = 0$ and $\hat{p}_{i3} = \overline{v}$ is the best response when

Firm $-i$. We next outline three possible strategies for Firm $-i$ in response to its competitor's prices $p_{i,1} > 0$ and $p_{i,3} > 0$ ignoring all situations of ties. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\text{x})}$.

1. To sell to no customer, i.e., $\sum_j x_{-i,j1} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell to one customer, i.e., $\sum_j x_{-i,j1} = 1$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's valuation for Product 1. Hence, $x_{-i,11} = 1$ and $x_{-i,21} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, p_{i3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, p_{i3} - \underline{v} - \delta\}$ and $\hat{\Pi}_{-i}^{(2)} = \hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, p_{i3} - \underline{v} - \delta\}$.

3. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$. Under PNC valuations, we need $p_{-i,1} \leq \min\{\underline{v}, p_{i1} - \delta, p_{i3} - \overline{v} - \delta\}$. Thus, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta, p_{i3} - \overline{v} - \delta\}$ and $\hat{\Pi}_{-i}^{(3)} = 2\min\{\underline{v}, p_{i1} - \delta, p_{i3} - \overline{v} - \delta\}$.

Comparing profits, we obtain that if $p_{i1} = 0$ or $p_{i3} < \underline{v}$ then Strategy 1 is the dominant strategy. Otherwise, $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(2)}$ and $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(3)}$. Relative values of $\overline{v}$ and $\underline{v}$ determine the dominant strategy between Strategies (2) and (3).

- Firm i plays Strategy (3-b) and Firm $-i$ plays Strategy (2)
- Firm i plays Strategy (3-c) and Firm $-i$ plays Strategy (2)
- Firm i plays Strategy (3-b) and Firm $-i$ plays Strategy (3)
- Firm i plays Strategy (3-c) and Firm $-i$ plays Strategy (3)

Under the first and third cases the equilibrium prices are $p_{i3} = \underline{v}$ and $p_{-i,1} = 0$. Similarly, under second and fourth cases firms will engage in a price war in the first product market, thus $p_{i1} = p_{-i,1} = 0$ and $p_{i3} = \overline{v}$. Therefore the equilibrium profits are $\hat{\Pi}_i = \max\{2\underline{v}, \overline{v}\}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.8

We focus on the case where $\mathbf{z}_i = (1, 0, 1)$ and $\mathbf{z}_{-i} = (0, 1, 0)$. Due to symmetry, a similar proof holds for the other case and we omit the details for brevity. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's price $p_{-i,2} \geq 0$ ignoring all situations of ties. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij3} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \overline{v}$ and $\hat{\Pi}_i^{(2\text{-a})} = \hat{p}_{i1} = \overline{v}$.

   (b) $\sum_j x_{ij3} = 1$ and $\sum_j x_{ij1} = 0$. Under PNC valuations, we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{i1} + \underline{v}, p_{-i,2} + \overline{v}\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,2} + \overline{v}\}$, and $\hat{\Pi}_i^{(2\text{-b})} = \min\{\overline{v} + \underline{v}, p_{-i,2} + \overline{v}\}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, p_{i3} - \overline{v}\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \underline{v}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\hat{p}_{i1} = 2\underline{v}$.

   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,2} + \underline{v}\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,2} + \underline{v}\}$ and $\hat{\Pi}_i^{(3\text{-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,2} + \underline{v}\}$.

   (c) $x_{i11} = 1$, $x_{i21} = 0$, $x_{i23} = 1$, and $x_{i13} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and the bundle to Customer 2. Hence, we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta\}$ and $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,2} + \underline{v}\}$. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{i3} - \underline{v} - \delta\}$, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,2} + \underline{v}\}$, and $\hat{\Pi}_i^{(3\text{-c})} = \hat{p}_{i1} + \hat{p}_{i3} = \min\{\overline{v}, p_{i3} - \underline{v} - \delta\} + \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,2} + \underline{v}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} < \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(2\text{-b})}$, $\hat{\Pi}_i^{(3\text{-a})} \leq \hat{\Pi}_i^{(3\text{-b})}$, and $\hat{\Pi}_i^{(3\text{-c})} < \hat{\Pi}_i^{(3\text{-b})}$ with equality occurring only for $p_{-i,2} = 0$. Thus the only strategies that are not weakly dominated are Strategies (2-b) and (3-b). Among them, the one that will be adopted depends on the values of $\overline{v}$, $\underline{v}$, and $p_{-i,2}$:

When $p_{-i,2} > 0$, the best response of Firm i, involves offering the bundle only. Thus, Firm i avoids competition in the market of Product 1. Therefore, this game reduces to the game characterized in Proposition 2.5, where $\mathbf{z}_i = (0, 0, 1)$ and $\mathbf{z}_{-i} = (1, 0, 0)$.

When $p_{-i,2} = 0$, Firm $-i$ will deviate to earn non-zero profits similar to Proposition 2.5. □

Proof. Proof of Proposition 2.9

We focus on the case where $\mathbf{z}_i = (1, 0, 1)$ and $\mathbf{z}_{-i} = (1, 1, 0)$. Due to symmetry, a similar proof holds for the other case and we omit the details for brevity. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$ and $p_{-i,2} \geq 0$ ignoring all situations of ties. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij3} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$.

According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(2\text{-a})} = \hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$.

(b) $\sum_j x_{ij3} = 1$ and $\sum_j x_{ij1} = 0$. Under PNC valuations, to ensure $\sum_j x_{ij3} = 1$ Firm i focuses on capturing the customer j for whom the following inequalities could be fulfilled $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{-i,1} + v_{j1}, p_{-i,2} + v_{j2}\}$ for $j = 1$ or $j = 2$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + v_{j1}, p_{-i,2} + v_{j2}\}$ and $\hat{\Pi}_i^{(2\text{-b})} = \hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + v_{j1}, p_{-i,2} + v_{j2}\}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

(a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, p_{i3} - \overline{v}, p_{-i,1} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\hat{p}_{i1} = 2\min\{\underline{v}, p_{-i,1} - \delta\}$.

(b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \underline{v}\}$. Firm i sets $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \underline{v}\}$ and $\hat{\Pi}_i^{(3\text{-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \underline{v}\}$.

(c) $x_{i11} = 1$, $x_{i21} = 0$, $x_{i23} = 1$, and $x_{i13} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and the bundle to Customer 2. Hence, we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\}$ and $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{i1} + \overline{v}, p_{-i,1} + \overline{v}, p_{-i,2} + \underline{v}\}$. Therefore, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \overline{v}, p_{-i,2} + \underline{v}\}$, and $\hat{\Pi}_i^{(3\text{-c})} = \min\{\overline{v}, p_{-i,1} - \delta\} + \min\{\overline{v} + \underline{v}, p_{-i,1} + \overline{v}, p_{-i,2} + \underline{v}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-c})}$, $\hat{\Pi}_i^{(3\text{-a})} < \hat{\Pi}_i^{(3\text{-b})}$, $\hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(3\text{-c})}$, and $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(3\text{-c})}$. Thus the only strategies that are not weakly dominated are Strategies

(3-b) and (3-c).

Firm $-i$. We next outline five possible strategies for Firm i in response to its competitor's prices $p_{-i,1} > 0$, and $p_{-i,3} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following two cases:

   (a) $\sum_j x_{-i,j1} = 1$ and $\sum_j x_{-i,j2} = 0$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{-i,11} = 1$ and $x_{-i,21} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(\text{2-a})} = \hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

   (b) $\sum_j x_{-i,j2} = 1$ and $\sum_j x_{-i,j1} = 0$. Under PNC valuations, Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{-i,22} = 1$ and $x_{-i,12} = 0$. According to (2.1), we need $p_{-i,2} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta\}$. Thus, $\hat{p}_{-i,2} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $\hat{\Pi}_{-i}^{(\text{2-b})} = \hat{p}_{i2} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

3. To sell one product to each customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{-i,j1} = 2$ and $\sum_j x_{-i,j2} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(\text{3-a})} = 2\hat{p}_{-i,1} = 2\min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

   (b) $\sum_j x_{-i,j1} = 0$ and $\sum_j x_{-i,j2} = 2$. According to (2.1), we need $p_{-i,2} \leq$

$\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}$. Thus, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(3\text{-b})} = 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

(c) $x_{-i,11} = 1$, $x_{-i,21} = 0$, $x_{-i,22} = 1$, and $x_{-i,12} = 0$. Under PNC valuations, the firm sells Product 1 to Customer 1 and Product 2 to Customer 2. Hence, we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $p_{-i,2} \leq \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Therefore, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $\hat{\Pi}_{-i}^{(3\text{-c})} = \hat{p}_{-i,1} + \hat{p}_{-i,2} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\} + \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

4. To sell both products to one customer and one product to the other customer, which can happen in one of the following two cases:

(a) $\sum_j x_{-i,j1} = 2$, $x_{-i,22} = 1$, and $x_{-i,12} = 0$. Under PNC valuations, we need $p_{-i,1} \leq \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $p_{-i,2} \leq \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $\hat{\Pi}_{-i}^{(4\text{-a})} = 2\hat{p}_{i1} + \hat{p}_{i2} = 2\min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\} + \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

(b) $x_{-i,11} = 1$, $x_{-i,21} = 0$, and $\sum_j x_{-i,j2} = 2$. Under PNC valuations, we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$, and $\hat{\Pi}_{i}^{(4\text{-b})} = 2\hat{p}_{-i,2} + \hat{p}_{-i,1} = 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\} + \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

5. To sell both products to both customers. Under PNC valuations, we need $p_{-i,1} \leq \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$, and $\hat{\Pi}_{-i}^{(5)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 2\min\{\underline{v}, p_{i1} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\} + 2\min\{\underline{v}, \max\{p_{i3} -$

$\overline{v} - \delta, 0\}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(2\text{-b})}$, $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(4\text{-a})}$, $\hat{\Pi}_i^{(3\text{-a})} \leq \hat{\Pi}_i^{(3\text{-b})}$, and $\hat{\Pi}_i^{(3\text{-b})} \leq \hat{\Pi}_i^{(4\text{-b})}$. Thus the only strategies that are not weakly dominated are Strategies (3-c), (4-a), (4-b), and (5).

Comparison. Comparing the best responses of the two firms, it appears that both firms try to capture both customer groups. Below, we analyze the eight different combinations of best response strategies of the firms.

- Firm i plays (3-b) and Firm $-i$ plays (3-c). On one hand, Firm i only sells the bundle and has to lower $\hat{p}_{i3}$ according to $\hat{p}_{-i,1}$ and $\hat{p}_{-i,2}$. On the other hand, Firm $-i$ has to undercut Firm i's bundle price to capture any customers. Thus, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-b) and Firm $-i$ plays (4-a). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-b) and Firm $-i$ plays (4-b). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-b) and Firm $-i$ plays (5). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-c) and Firm $-i$ plays (3-c). Both firms will engage in a price war on Component 1, undercutting each other's price. Also due to a similar reason to the one above, $p_{-i,2}$ will be priced to zero, in equilibrium. In equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = \underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-c) and Firm $-i$ plays (4-a). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.

- Firm i plays (3-c) and Firm $-i$ plays (4-b). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.
- Firm i plays (3-c) and Firm $-i$ plays (5). Due to a similar reason to the one above, in equilibrium $\hat{p}_{i3} = \underline{v}$, $\hat{p}_{-i,1} = p_{-i,2} = 0$, $\hat{\Pi}_i = 2\underline{v}$, and $\hat{\Pi}_{-i} = 0$.

It appears that irrespective of the strategy each firm plays, in equilibrium, $\hat{p}_{i1} = \hat{p}_{-i,1} = \hat{p}_{-i,2} = 0$, Firm i always sells only the bundle, and thus $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Corollary 2.1

Firm i offers three different products, i.e., Component 1, Component 2, and the bundle, but there are only two customer groups in the market. Firm i can capture at most two customers, thus it has to strategically choose only two products to compete with Firm $-i$. There are three possibilities.

1. Capture customers by Product 1 and Product 2. According to Proposition 2.3, $\hat{\Pi}_i = \max\{\overline{v}, 2\underline{v}\}$ and $\hat{\Pi}_{-i} = 0$.
2. Capture customers by the common product, also offered by Firm $-i$, and the bundle. According to Propositions 2.7, $\hat{\Pi}_i = \max\{\overline{v}, 2\underline{v}\}$ and $\hat{\Pi}_{-i} = 0$.
3. Capture customers by the products that is not offered by Firm $-i$ and the bundle. According to Propositions 2.8,
   - For $\overline{v} \geq 3\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = \overline{v} + \underline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = \frac{\overline{v}-\underline{v}}{2}$.
   - For $2\underline{v} < \overline{v} < 3\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2(\overline{v} - \underline{v})$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = \overline{v} - 2\underline{v}$.
   - For $\overline{v} \leq 2\underline{v}$, $\hat{\Pi}_i(\mathbf{z}_i; \mathbf{z}_{-i}) = 2\underline{v}$ and $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$.

Under any relative valuations $\overline{v}, \underline{v}$, the third strategy, i.e., offering the product not offered by Firm $-i$ and the bundle, yields the maximum profit for Firm i. □

Proof. Proof of Corollary 2.2

Firm i offers three different products, i.e., Component 1, Component 2, and the bundle, but there are only two customer groups in the market. Firm i can capture at most two customers, thus it has to strategically choose only two products to compete with Firm $-i$. There are three possibilities.

1. Capture customers by Product 1 and Product 2. According to Proposition 2.1, both firms make zero profit.

2. Capture customers by Product 1 and the bundle. According to Proposition 2.9, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$.

3. Capture customers by Product 2 and the bundle. According to Proposition 2.9, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$.

Firm i will only price either Product 1 or Product 2 and the bundle competitively to capture both customer groups. In equilibrium $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.10

Both firms offer the bundle, i.e., $z_{i3} = z_{-i3} = 1$. For the sake of simplicity we remove the $(\mathbf{z}_1, \mathbf{z}_2)$ arguments from the profit function expressions. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider the best responses of Firm i.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's prices $p_{-i,k} > 0$ for $k = 1, 2$ and $p_{-i,3} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to one customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{ij1} = 1$, $\sum_j x_{ij3} = 0$, and $\sum_j x_{ij2} = 0$. This strategy is feasible only if $z_{i1} = 1$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, \max\{\min\{p_{i3}, p_{-i3}\} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ and any $p_{i2} > \overline{v}$ to ensure that $\sum_j x_{ij3} = 0$ and $\sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, z_{-i1}(p_{-i1} - \delta), 0\}\}$ and $\hat{\Pi}_i^{(2\text{-a})} = \hat{p}_{i1} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$.

   (b) $\sum_j x_{ij2} = 1$, $\sum_j x_{ij3} = 0$, and $\sum_j x_{ij1} = 0$. This strategy is feasible only if $z_{i2} = 1$. Under PNC valuations, Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{i22} = 1$ and $x_{i12} = 0$. According to (2.1), we need $p_{i2} \leq \min\{\overline{v}, \max\{\min\{p_{i3}, p_{-i3}\} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ and any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij3} = 0$ and $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i2} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$ and $\hat{\Pi}_i^{(2\text{-b})} = \hat{p}_{i2} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$.

   (c) $\sum_j x_{ij3} = 1$, $\sum_j x_{ij1} = 0$, and $\sum_j x_{ij2} = 0$. Under PNC valuations, to ensure $\sum_j x_{ij3} = 1$, Firm i focuses on capturing the Customer j for whom the following inequalities could be fulfilled $v_{(-j)3} - z_{-ik}(v_{(-j)k} - p_{-ik}) \leq p_{i3} \leq \min\{v_{j3}, p_{-i3} - \delta, v_{j3} - z_{-ik}(v_{jk} - p_{-ik})\}$ for $k = 1, 2$. Firm i sets any $p_{i1} = p_{i2} > \overline{v}$ to ensure that $\sum_j x_{ij1} = \sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i3} = \min\{v_{j3}, p_{-i3} - \delta, v_{j3} - z_{-ik}(v_{jk} - p_{-ik})\}$ for $k = 1, 2$, and $\hat{\Pi}_i^{(2\text{-c})} = \hat{p}_{i3} = \min\{v_{j3}, p_{-i3} - \delta, v_{j3} - z_{-i1}(v_{j1} - p_{-i1}), v_{j3} - z_{-i2}(v_{j2} - p_{-i2})\}$ for $j = 1$ or 2.

3. To sell one product to each customer, which can happen in one of the following

three cases:

(a) $\sum_j x_{ij1} = 2$, $\sum_j x_{ij2} = 0$, and $\sum_j x_{ij3} = 0$. This strategy is feasible only if $z_{i1} = 1$. According to (2.1), we need $p_{i1} \leq \min\{\underline{v}, z_{-i1}(p_{-i1}-\delta), \max\{\min\{p_{i3}, p_{-i3}\} - \overline{v} - \delta, 0\}\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$ and $\hat{\Pi}_i^{(3\text{-a})} = 2\min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$.

(b) $\sum_j x_{ij1} = 1$ and $\sum_j x_{ij3} = 1$. This strategy is feasible only if $z_{i1} = 1$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's. Hence, $x_{i11} = 1$ and $x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, z_{-i1}(p_{-i1} - \delta), \max\{\min\{p_{i3}, p_{-i3}\} - \underline{v} - \delta, 0\}\}$, and $p_{i3} \leq \min\{\overline{v} + \underline{v}, \overline{v} + p_{i1}, z_{-i1}(\overline{v} + p_{-i1}), z_{i2}(\underline{v} + p_{i2}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$. Firm i sets $p_{i2} > \overline{v}$ to ensure that $\sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, z_{-i1}(p_{-i1} - \delta), \max\{p_{-i3} - \underline{v} - \delta, 0\}\}$ and $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, \overline{v} + p_{i1}, z_{-i1}(\overline{v} + p_{-i1}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-b})} = \hat{p}_{i1} + \hat{p}_{i3} = \min\{\overline{v}, z_{-i1}(p_{-i1} - \delta), \max\{p_{-i3} - \underline{v} - \delta, 0\}\} + \min\{\overline{v} + \underline{v}, \overline{v} + p_{i1}, z_{-i1}(\overline{v} + p_{-i1}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$.

(c) $\sum_j x_{ij2} = 2$, $\sum_j x_{ij1} = 0$, and $\sum_j x_{ij3} = 0$. This strategy is feasible only if $z_{i2} = 1$. According to (2.1), we need $p_{i2} \leq \min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i2} = \min\{\underline{v}, -\overline{v} + p_{-i3} - \delta, z_{-i2}(p_{-i2} - \delta)\}$ and $\hat{\Pi}_i^{(3\text{-c})} = 2\hat{p}_{i2} = 2\min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$.

(d) $\sum_j x_{ij2} = 1$ and $\sum_j x_{ij3} = 1$. This strategy is feasible only if $z_{i2} = 1$. Under PNC valuations, Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{i22} = 1$ and $x_{i12} = 0$. According to (2.1), we need $p_{i2} \leq \min\{\overline{v}, z_{-i2}(p_{-i2} - \delta), \max\{p_{-i3} - \underline{v} - \delta, 0\}\}$ and $p_{i3} \leq \min\{\overline{v} + \underline{v}, \overline{v} +$

$p_{i2}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$. Firm i sets $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i2} = \min\{\overline{v}, z_{-i2}(p_{-i2} - \delta), \max\{p_{-i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, \overline{v} + p_{i2}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$, and $\hat{\Pi}_i^{(3\text{-d})} = \hat{p}_{i2} + \hat{p}_{i3} = \min\{\overline{v}, z_{-i2}(p_{-i2} - \delta), \max\{p_{-i3} - \underline{v} - \delta, 0\}\} + \min\{\overline{v} + \underline{v}, \overline{v} + p_{i2}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$.

(e) $\sum_j x_{ij1} = 1$, $\sum_j x_{ij2} = 1$, and $\sum_j x_{ij3} = 0$. This strategy is feasible only if $z_{i1} = z_{i2} = 1$. Under PNC valuations, Customer 1's valuation for Product 1 is higher than Customer 2's and Customer 2's valuation for Product 2 is higher than Customer 1's. Hence, $x_{i11} = x_{i22} = 1$ and $x_{i12} = x_{i21} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$ and $p_{i2} \leq \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$, $\hat{p}_{i2} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$ and $\hat{\Pi}_i^{(3\text{-e})} = \hat{p}_{i1} + \hat{p}_{i2} = \min\{\overline{v}, -\underline{v} + p_{-i3} - \delta, z_{-i1}(p_{-i1} - \delta)\} + \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$.

(f) $\sum_j x_{ij3} = 2$, $\sum_j x_{ij1} = 0$, and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$ and $\hat{\Pi}_i^{(3\text{-f})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\underline{v} + p_{-i2}), p_{-i3} - \delta\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(2\text{-c})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(2\text{-c})}$, $\hat{\Pi}_i^{(3\text{-a})} \leq \hat{\Pi}_i^{(3\text{-f})}$, $\hat{\Pi}_i^{(3\text{-c})} \leq \hat{\Pi}_i^{(3\text{-f})}$, $\hat{\Pi}_i^{(3\text{-e})} \leq \hat{\Pi}_i^{(3\text{-b})}$, and $\hat{\Pi}_i^{(3\text{-e})} \leq \hat{\Pi}_i^{(3\text{-d})}$. Thus the only strategies that are not weakly dominated are strategies where Firm i sells the bundle and therefore competes in the bundled product market namely Strategies (2-c), (3-b), (3-d), and (3-f). The common requirement of all such strategies is setting $p_{i3} < p_{-i3}$ for $i = 1, 2$. Firms will thus engage in a price war on the bundle, undercutting each other's price.

In equilibrium $\hat{p}_{i3} = \hat{p}_{-i,3} = 0$. From (2.1) we conclude that in such a market, $x_{ijk} = 0$ $\forall i, j, k \neq 3$. Therefore $\hat{\Pi}_i = \hat{\Pi}_{-i} = 0$ □

Proof. Proof of Proposition 2.11

With non-overlapping offerings, there is no competition in any of the product markets. Therefore Firm i who offers Product l can capture both customers, i.e., $x_{ijl} = 2$, by setting $p_{il} \leq v_{jl}$.

- If $\mathbf{z}_i = (1, 0, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, it is clear that $\hat{\Pi}_{-i}(\mathbf{z}_{-i}; \mathbf{z}_i) = 0$. Under PPC, Firm i can capture both customers by setting $\hat{p}_i = \overline{v}$ and $\hat{\Pi}_i = 2\overline{v}$.
- If $\mathbf{z}_i = (0, 1, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, Firm i can capture both customers under PPC by setting $\hat{p}_i = \underline{v}$, and $\hat{\Pi}_i = 2\underline{v}$.
- If $\mathbf{z}_i = (1, 1, 0)$ and $\mathbf{z}_{-i} = (0, 0, 0)$, Firm i can maximize its profit by setting $p_{i1} = \overline{v}$ and $p_{i2} = \underline{v}$ and thus earn $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.
- If $\mathbf{z}_i = (1, 0, 0)$ and $\mathbf{z}_{-i} = (0, 1, 0)$, firms operate in separate markets. Following the results in the first two cases, $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 2\underline{v}$.

□

Proof. Proof of Proposition 2.12

We focus on the case where $\mathbf{z}_i = (1, 1, 0)$ and $\mathbf{z}_{-i} = (1, 0, 0)$. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to each customer, i.e., $\sum_l x_{i1l} = 1$ and $\sum_l x_{i2l} = 1$, which can happen in one of the following two cases.

   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$. Firm i sets any $p_{i2} > \underline{v}$ to ensure that $\sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(2\text{-a})} = 2\hat{p}_{i1} = 2\min\{\overline{v}, p_{-i,1} - \delta\}$.

   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij2} = 2$. According to (2.1), we need $p_{i2} \leq \underline{v}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i2} = \underline{v}$ and $\hat{\Pi}_i^{(2\text{-b})} = 2\hat{p}_{i2} = 2\underline{v}$.

3. To sell both products to both customers. Under PPC valuations, we need $p_{i1} \leq \min\{\overline{v}, p_{-i,1} - \delta\}$ and $p_{i2} \leq \underline{v}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$, $\hat{p}_{i2} = \underline{v}$, and $\hat{\Pi}_i^{(3)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 2\min\{\overline{v}, p_{-i,1} - \delta\} + 2\underline{v}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(3)}$, and $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(3)}$. Thus the only strategy that is not weakly dominated is Strategy (3).

Firm $-i$. We next outline two possible strategies for Firm $-i$ in response to its competitor's prices $p_{i,1} > 0$ and $p_{i,2} \geq 0$. For any strategy (x), we denote Firm $-i$'s profit as $\hat{\Pi}_{-i}^{(\text{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,j1} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$. Under PPC valuations, we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\min\{\overline{v}, p_{i1} - \delta\}$.

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, the firms will engage in a price war on Component 1, undercutting each other's price. Hence, there is no equilibrium where

the market of Component 1 is not shared. Hence, $\hat{p}_{i1} = \hat{p}_{-i,1} = 0$, and $\hat{p}_{i2} = \underline{v}$. As a result $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.13

We focus on the case where $\mathbf{z}_i = (1, 1, 0)$ and $\mathbf{z}_{-i} = (0, 1, 0)$. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's price $p_{-i,2} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.
2. To sell one product to each customer, i.e., $\sum_l x_{i1l} = 1$ and $\sum_l x_{i2l} = 1$, which can happen in one of the following two cases.
   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i1} \leq \overline{v}$. Firm i sets any $p_{i2} > \underline{v}$ to ensure that $\sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i1} = \overline{v}$ and $\hat{\Pi}_i^{(\text{2-a})} = 2\hat{p}_{i1} = 2\overline{v}$.
   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij2} = 2$. According to (2.1), we need $p_{i2} \leq \min\{\underline{v}, p_{-i,2} - \delta\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,2} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-b})} = 2\hat{p}_{i2} = 2\min\{\underline{v}, p_{-i,2} - \delta\}$.
3. To sell both products to both customers. Under PPC valuations, we need $p_{i1} \leq \overline{v}$ and $p_{i2} \leq \min\{\underline{v}, p_{-i,2} - \delta\}$. Thus, $\hat{p}_{i1} = \overline{v}$, $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,2} - \delta\}$, and $\hat{\Pi}_i^{(3)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 2\overline{v} + 2\min\{\underline{v}, p_{-i,1} - \delta\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(\text{2-a})}$, $\hat{\Pi}_i^{(\text{2-a})} \leq \hat{\Pi}_i^{(3)}$, and $\hat{\Pi}_i^{(\text{2-b})} \leq \hat{\Pi}_i^{(3)}$. Thus the only strategy that is not weakly dominated is Strategy (3).

Firm $-i$. We next outline two possible strategies for Firm $-i$ in response to its competitor's prices $p_{i,1} \geq 0$ and $p_{i,2} > 0$. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\mathrm{x})}$.

1. To sell to no customer, i.e., $\sum_j x_{-i,j2} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell to both customers, i.e., $\sum_j x_{-ij2} = 2$. Under PPC valuations, we need $p_{-i,2} \leq \min\{\underline{v}, p_{i2} - \delta\}$. Thus, $\hat{p}_{-i,2} = \min\{\underline{v}, p_{i2} - \delta\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\min\{\underline{v}, p_{i2} - \delta\}$.

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, the firms will engage in a price war on Component 2, undercutting each other's price. Hence, there is no equilibrium where the market of Component 2 is not shared. Hence, $\hat{p}_{i2} = \hat{p}_{-i,2} = 0$, and $\hat{p}_{i1} = \overline{v}$. As a result $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.14

1. If $\mathbf{z}_i = (0, 0, 1)$, then $\Pi_i(p_i; p_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = p_{i3} \sum_j x_{ij3} = 2p_{i3}$ if $p_{i3} \leq \overline{v} + \underline{v}$, and $\Pi_i(p_i; p_{-i}, \mathbf{z}_i, \mathbf{z}_{-i}) = 0$ if $p_{i3} > \overline{v} + \underline{v}$. Thus $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

2. If $\mathbf{z}_i = (1, 0, 1)$, then by selling Product 1, Firm i can earn $2\overline{v}$, if it sells its to both customers. However, if Firm i prices $p_{i1} > \overline{v}$, it can earn a profit of $2(\overline{v} + \underline{v})$ by selling the bundle to both customers. Thus in equilibrium, $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

3. If $\mathbf{z}_i = (0, 1, 1)$, similar to the previous case, we obtain that in equilibrium $\hat{p}_{i3} = \overline{v} + \underline{v}$ and $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

4. $\mathbf{z}_i = (1, 1, 1)$, Firm i offers three products and there are only two customer groups in the market. Firm i has to sell at most two products to the customer groups.

If it chooses Product 1 and Product 2, then the equilibrium profit $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$, according to Proposition 2.11. If it chooses Product 1 and the bundle, or Product 2 and the bundle, then the equilibrium profit is again $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$. Thus in equilibrium, $\hat{\Pi}_i = 2(\overline{v} + \underline{v})$.

□

Proof. Proof of Proposition 2.15

To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline two possible strategies for Firm i in response to its competitor's price $p_{-i,1} \geq 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.
2. To sell the bundle to both customers. Under PPC valuations, we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}\}$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}\}$, and $\hat{\Pi}_i^{(3)} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}\}$.

Firm $-i$. We next outline two possible strategies for Firm $-i$ in response to its competitor's price $p_{i3} > 0$. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\mathrm{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,j1} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.
2. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$. Under PPC valuations, we need $p_{-i,1} \leq \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, Firm $-i$ will undercut $p_{i3} - \underline{v}$ and Firm i will lower its bundle price accordingly. In equilibrium, $\hat{p}_{-i,1} = 0$ and $\hat{p}_{i3} = \underline{v}$. As a result $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.16

The proof follows the same logic as in the proof of Proposition 2.15, and is omitted for the sake of brevity. □

Proof. Proof of Proposition 2.17

To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline two possible strategies for Firm i in response to its competitor's prices $p_{-i,1} \geq 0$ and $p_{-i,2} \geq 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.
2. To sell the bundle to both customers. Under PPC valuations, we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \overline{v}\}$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \overline{v}\}$, and $\hat{\Pi}_i^{(3)} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \overline{v}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2)}$. Thus Firm i's best response is Strategy (2), i.e., capturing both customers with the bundle.

Firm $-i$. We next outline three possible strategies for Firm $-i$ in response to its competitor's price $p_{i3} > 0$. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\mathrm{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,jl} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell one product to each customer, i.e., $\sum_l x_{i1l} = 1$ and $\sum_l x_{i2l} = 1$, which can happen in one of the following two cases.

   (a) $\sum_j x_{-i,j1} = 2$ and $\sum_j x_{-i,j2} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Firm $-i$ sets any $p_{-i,2} > \underline{v}$ to ensure that $\sum_j x_{-i,j2} = 0$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2\text{-a})} = 2\hat{p}_{-i,1} = 2\min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$..

   (b) $\sum_j x_{-i,j1} = 0$ and $\sum_j x_{-i,j2} = 2$. According to (2.1), we need $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Firm $-i$ sets any $p_{-i,1} > \overline{v}$ to ensure that $\sum_j x_{-i,j1} = 0$. Thus, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2\text{-b})} = 2\hat{p}_{-i,2} = 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

3. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$ and $\sum_j x_{-ij2} = 2$. Under PPC valuations, we need $p_{-i,1} \leq \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\hat{p}_{-i,1} + 2\hat{p}_{-i,2} = 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\} + 2\min\{\overline{v}, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(3)}$, and $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(3)}$. Thus the only strategy that is not weakly dominated is Strategy (3).

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, Firm $-i$ will undercut p_{i3} and Firm i will lower its bundle price accordingly. In equilibrium, $\hat{p}_{-i,1} = \hat{p}_{-i,2} = 0$, and $\hat{p}_{i3} = \underline{v}$. As a result $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.18

To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline two possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to each customer, i.e., $\sum_j x_{ij1} = 2$ or $\sum_j x_{ij3} = 2$, which can happen in one of the following two cases.

   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-a})} = 2\hat{p}_{i1} = 2\min\{\overline{v}, p_{-i,1} - \delta\}$.

   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v} - \delta\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v} - \delta\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(\text{2-b})}$ and $\hat{\Pi}_i^{(\text{2-a})} < \hat{\Pi}_i^{(\text{2-b})}$. Thus Firm i's best response is Strategy (2-b), i.e., capturing both customers with the bundle.

Firm $-i$. We next outline two possible strategies for Firm $-i$ in response to its competitor's price $p_{i1} > 0$ and $p_{i3} > 0$. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\text{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,jl} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell Product 1 to both customers, i.e., $\sum_j x_{-i,j1} = 2$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} -$

$\delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\hat{p}_{-i,1} = 2\min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2)}$. Thus the best response of Firm $-i$ is Strategy (2).

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, Firm $-i$ will undercut $p_{i3} - \underline{v}$ and p_{i1} and Firm i will lower its bundle price accordingly. In equilibrium, $\hat{p}_{-i,1} = 0$, and $\hat{p}_{i3} = \underline{v}$. As a result $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.19

To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline two possible strategies for Firm i in response to its competitor's price $p_{-i,2} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.
2. To sell one product to each customer, i.e., $\sum_j x_{ij2} = 2$ or $\sum_j x_{ij3} = 2$, which can happen in one of the following two cases.
   (a) $\sum_j x_{ij2} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}, p_{-i,2} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i2} = \min\{\underline{v}, p_{-i,2} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-a})} = 2\hat{p}_{i2} = 2\min\{\underline{v}, p_{-i,2} - \delta\}$.
   (b) $\sum_j x_{ij2} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, p_{-i,2} + \overline{v} - \delta\}$. Firm i sets any $p_{i2} > \underline{v}$ to ensure that $\sum_j x_{ij2} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,2} + \overline{v} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,2} + \overline{v} - \delta\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-b})}$ and $\hat{\Pi}_i^{(2\text{-a})} < \hat{\Pi}_i^{(2\text{-b})}$. Thus Firm i's best response is Strategy (2-b), i.e., capturing both customers with the bundle.

Firm $-i$. We next outline two possible strategies for Firm $-i$ in response to its competitor's price $p_{i2} > 0$ and $p_{i3} > 0$. For any strategy (x), we denote Firm $-i$'s profit, as $\hat{\Pi}_{-i}^{(\text{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,j2} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.
2. To sell Product 2 to both customers, i.e., $\sum_j x_{-i,j2} = 2$. According to (2.1), we need $p_{-i,2} \leq \min\{\underline{v}, p_{i2} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,2} = \min\{\underline{v}, p_{i2} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\hat{p}_{-i,2} = 2\min\{\underline{v}, p_{i2} - \delta, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2)}$. Thus the best response of Firm $-i$ is Strategy (2).

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, Firm $-i$ will undercut $p_{i3} - \overline{v}$ and p_{i2} and Firm i will lower its bundle price accordingly. In equilibrium, $\hat{p}_{-i,2} = 0$, and $\hat{p}_{i3} = \overline{v}$. As a result $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Proposition 2.20

The proof is similar to the proof of Proposition 2.18 and is therefore omitted for brevity. This time Firm $-i$ seeks to price below $p_{i3} - \overline{v}$. □

Proof. Proof of Proposition 2.21

The proof is similar to that of Proposition 2.19 and is therefore omitted for brevity. This time, Firm $-i$ seeks to price below $p_{i3} - \underline{v}$. □

Proof. Proof of Proposition 2.22

We focus on the case where $\mathbf{z}_i = (1, 0, 1)$ and $\mathbf{z}_{-i} = (1, 1, 0)$. Due to symmetry, a similar proof holds for the other case and we omit the details for brevity. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider in turn the best responses of Firm i and Firm $-i$ and then compare them.

Firm i. We first outline two possible strategies for Firm i in response to its competitor's price $p_{-i,1} > 0$ and $p_{-i,2} \geq 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\text{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.
2. To sell one product to each customer, i.e., $\sum_j x_{ij1} = 2$ or $\sum_j x_{ij3} = 2$, which can happen in one of the following two cases.
   (a) $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij3} = 0$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, p_{i3} - \underline{v} - \delta, p_{-i,1} - \delta\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, p_{-i,1} - \delta\}$ and $\hat{\Pi}_i^{(\text{2-a})} = 2\hat{p}_{i1} = 2\min\{\overline{v}, p_{-i,1} - \delta\}$.
   (b) $\sum_j x_{ij1} = 0$ and $\sum_j x_{ij3} = 2$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, \min\{p_{i1}, p_{-i,1}\} + \underline{v}, p_{-i,2} + \overline{v}\}$. Firm i sets any $p_{i1} > \overline{v}$ to ensure that $\sum_j x_{ij1} = 0$. Thus, $\hat{p}_{i3} = \min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \overline{v}\}$ and $\hat{\Pi}_i^{(\text{2-b})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, p_{-i,1} + \underline{v}, p_{-i,2} + \overline{v}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(\text{2-b})}$ and $\hat{\Pi}_i^{(\text{2-a})} < \hat{\Pi}_i^{(\text{2-b})}$. Thus Firm i's best response is Strategy (2-b), i.e., capturing both customers with the bundle.

Firm $-i$. We next outline three possible strategies for Firm $-i$ in response to its competitor's prices $p_{i1} > 0$ and $p_{i3} > 0$. For any strategy (x), we denote Firm $-i$'s

profit, as $\hat{\Pi}_{-i}^{(\mathrm{x})}$.

1. To sell to no customer, i.e., $\sum_{j,l} x_{-i,jl} = 0$. In that case, $\hat{\Pi}_{-i}^{(1)} = 0$.

2. To sell one product to each customer, i.e., $\sum_l x_{i1l} = 1$ and $\sum_l x_{i2l} = 1$, which can happen in one of the following three cases.

   (a) $\sum_j x_{-i,j1} = 2$ and $\sum_j x_{-i,j2} = 0$. According to (2.1), we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$. Firm $-i$ sets any $p_{-i,2} > \underline{v}$ to ensure that $\sum_j x_{-i,j2} = 0$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, p_{i3} - \underline{v} - \delta\}$ and $\hat{\Pi}_{-i}^{(2\text{-a})} = 2\hat{p}_{-i,1} = 2\min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$.

   (b) $\sum_j x_{-i,j1} = 0$ and $\sum_j x_{-i,j2} = 2$. According to (2.1), we need $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Firm $-i$ sets any $p_{-i,1} > \overline{v}$ to ensure that $\sum_j x_{-i,j1} = 0$. Thus, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2\text{-b})} = 2\hat{p}_{-i,2} = 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

3. To sell to both customers, i.e., $\sum_j x_{-ij1} = 2$ and $\sum_j x_{-ij2} = 2$. Under PPC valuations, we need $p_{-i,1} \leq \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $p_{-i,2} \leq \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{-i,1} = \min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{-i,2} = \min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_{-i}^{(2)} = 2\hat{p}_{-i,1} + 2\hat{p}_{-i,2} = 2\min\{\overline{v}, p_{i1} - \delta, \max\{p_{i3} - \underline{v} - \delta, 0\}\} + 2\min\{\underline{v}, \max\{p_{i3} - \overline{v} - \delta, 0\}\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(3)}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(3)}$, and $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(3)}$. Thus the only strategy that is not weakly dominated is Strategy (3).

Comparison. Comparing the best responses of the two firms, it appears that, irrespective of the relative values of $\overline{v}$ and $\underline{v}$, Firm $-i$ will undercut p_{i3} and Firm i will lower its bundle price accordingly. In equilibrium, $\hat{p}_{-i,1} = \hat{p}_{-i,2} = 0$, and $\hat{p}_{i3} = \underline{v}$. As a result $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$. □

Proof. Proof of Corollary 2.3

Firm i offers three different products, i.e., Product 1, Product 2, and the bundle, but there are only two customer groups in the market. Firm i can capture at most two customers, thus it has to strategically choose only two products to compete with Firm $-i$. There are three possibilities.

1. Capture customers with Product 1 and Product 2.
   - If $\mathbf{z}_{-i} = (1, 0, 0)$, according to Proposition 2.12, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (0, 1, 0)$, according to Proposition 2.13, $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (1, 1, 0)$, according to Lemma 2.1, $\hat{\Pi}_i = 0$ and $\hat{\Pi}_{-i} = 0$.
2. Capture customers with Product 1 and the bundle.
   - If $\mathbf{z}_{-i} = (1, 0, 0)$, according to Proposition 2.18, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (0, 1, 0)$, according to Proposition 2.20, $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (1, 1, 0)$, according to Proposition 2.22, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$.
3. Capture customers with Product 2 and the bundle.
   - If $\mathbf{z}_{-i} = (1, 0, 0)$, according to Proposition 2.21, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (0, 1, 0)$, according to Proposition 2.19, $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$,
   - If $\mathbf{z}_{-i} = (1, 1, 0)$, according to Proposition 2.22, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$.

Summarizing the three scenarios above, firms' equilibrium profits are:

- If $\mathbf{z}_{-i} = (1, 0, 0)$, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$,
- If $\mathbf{z}_{-i} = (0, 1, 0)$, $\hat{\Pi}_i = 2\overline{v}$ and $\hat{\Pi}_{-i} = 0$,
- If $\mathbf{z}_{-i} = (1, 1, 0)$, $\hat{\Pi}_i = 2\underline{v}$ and $\hat{\Pi}_{-i} = 0$.

□

Proof. Proof of Proposition 2.23

Both firms offer the bundle, i.e., $z_{i3} = z_{-i3} = 1$. For the sake of simplicity we remove the $(\mathbf{z}_1, \mathbf{z}_2)$ arguments from the profit function expressions. To express best responses, it is useful to think of price sets as discrete grids; let δ be the smallest price increment (e.g., cents). We consider the best responses of Firm i.

Firm i. We first outline three possible strategies for Firm i in response to its competitor's prices, $p_{-i,k} > 0$ for $k = 1, 2$ and $p_{-i,3} > 0$. For any strategy (x), we denote Firm i's profit, as $\hat{\Pi}_i^{(\mathrm{x})}$.

1. To sell to no customer. In that case, $\hat{\Pi}_i^{(1)} = 0$.

2. To sell one product to each customer, which can happen in one of the following three cases:

   (a) $\sum_j x_{ij1} = 2$, $\sum_j x_{ij2} = 0$, and $\sum_j x_{ij3} = 0$. This strategy is feasible only if $z_{i1} = 1$. According to (2.1), we need $p_{i1} \leq \min\{\overline{v}, z_{-i1}(p_{-i1} - \delta), \max\{\min\{p_{i3}, p_{-i3}\} - \underline{v} - \delta, 0\}\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$ and $\hat{\Pi}_i^{(\text{2-a})} = 2\min\{\overline{v}, \max\{p_{-i3} - \underline{v} - \delta, 0\}, z_{-i1}(p_{-i1} - \delta)\}$.

   (b) $\sum_j x_{ij2} = 2$, $\sum_j x_{ij1} = 0$, and $\sum_j x_{ij3} = 0$. This strategy is feasible only if $z_{i2} = 1$. According to (2.1), we need $p_{i2} \leq \min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$. Firm i sets any $p_{i3} > \overline{v} + \underline{v}$ to ensure that $\sum_j x_{ij3} = 0$. Thus, $\hat{p}_{i2} = \min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$ and $\hat{\Pi}_i^{(\text{2-b})} = 2\hat{p}_{i2} = 2\min\{\underline{v}, \max\{p_{-i3} - \overline{v} - \delta, 0\}, z_{-i2}(p_{-i2} - \delta)\}$.

   (c) $\sum_j x_{ij3} = 2$, $\sum_j x_{ij1} = 0$, and $\sum_j x_{ij2} = 0$. According to (2.1), we need $p_{i3} \leq \min\{\overline{v} + \underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$. Thus, $\hat{p}_{i3} = \min\{\overline{v} +$

$\underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$ and $\hat{\Pi}_i^{(2\text{-c})} = 2\hat{p}_{i3} = 2\min\{\overline{v} + \underline{v}, z_{-i1}(\underline{v} + p_{-i1}), z_{-i2}(\overline{v} + p_{-i2}), p_{-i3} - \delta\}$.

3. To sell to both customers, i.e., $\sum_j x_{ij1} = 2$ and $\sum_j x_{ij2} = 2$ and $x_{ij3} = 0$. This strategy is feasible only if $z_{i1} = 1$ and $z_{i2} = 1$. Under PPC valuations, we need $p_{i1} \leq \min\{\overline{v}, z_{-i,1}(p_{-i,1} - \delta), \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, and $p_{i2} \leq \min\{\underline{v}, z_{-i,2}(p_{-i,2} - \delta), \max\{p_{i3} - \overline{v} - \delta, 0\}\}$. Thus, $\hat{p}_{i1} = \min\{\overline{v}, z_{-i,1}(p_{-i,1} - \delta), \max\{p_{i3} - \underline{v} - \delta, 0\}\}$, $\hat{p}_{i2} = \min\{\underline{v}, z_{-i,2}(p_{-i,2} - \delta), \max\{p_{i3} - \overline{v} - \delta, 0\}\}$ and $\hat{\Pi}_i^{(3)} = 2\hat{p}_{i1} + 2\hat{p}_{i2} = 2\min\{\overline{v}, z_{-i,1}(p_{-i,1} - \delta), \max\{p_{i3} - \underline{v} - \delta, 0\}\} + 2\min\{\underline{v}, z_{-i,2}(p_{-i,2} - \delta), p_{i3} - \overline{v} - \delta\}$.

Comparing profits, we obtain that $\hat{\Pi}_i^{(1)} \leq \hat{\Pi}_i^{(2\text{-a})}$, $\hat{\Pi}_i^{(2\text{-a})} \leq \hat{\Pi}_i^{(2\text{-c})}$, $\hat{\Pi}_i^{(2\text{-b})} \leq \hat{\Pi}_i^{(2\text{-c})}$, and $\hat{\Pi}_i^{(2\text{-c})} \leq \hat{\Pi}_i^{(2\text{-d})}$. Thus the only strategy that is not weakly dominated are the Strategies (2-c) and (3), according to which Firm i sells the bundle, which requires $p_{i3} < p_{-i,3}$ for $i = 1, 2$. Firms will engage in a price war on the bundle, undercutting each other's price. In equilibrium $\hat{p}_{i3} = \hat{p}_{-i,3} = 0$. From 2.1 we conclude that in such a market, $x_{ijk} = 0\ \forall i, j, k \neq 3$. Therefore $\hat{\Pi}_i = \hat{\Pi}_{-i} = 0$. $\square$

Appendix B Construction of Mixed Strategy in Chapter 2 Pricing Games

A mixed-strategy Nash equilibrium is constructed by discretization of the action spaces where a pure-strategy Nash equilibrium does not exist, i.e., when the best-response correspondences do not intersect. We study the game where $\mathbf{z}_1 = (0, 0, 1)$ and $\mathbf{z}_2 = (0, 1, 0)$ as an example.

Proposition B.1. *Fix m and let $n \doteq 2^m$. Suppose $\boldsymbol{z}_i = (0, 0, 1)$ and $\boldsymbol{z}_{-i} = (0, 1, 0)$, and let $a, b,$ and c as in (A.8) Suppose that Firm 1 randomizes on $p_{ik} = a + k \cdot \frac{(b-a)}{n}$ for $k = 0, ..., n$ according to the following distribution:*[1]

$$\begin{aligned} F_1^{(n)}(p_{1k}) &= \frac{(p_{1k} - a)n + b - a}{(p_{1k} + c - a)n + b - a} \text{ for } 0 \leq k \leq n-1, \\ F_1^{(n)}(p_{1n}) &= 1. \end{aligned} \tag{B.1}$$

and that Firm 2 randomizes on $p_{2k} = c + k \cdot \frac{(b-a)}{n}$ for $k = 0, ..., n$ according ot the following distribution:

$$F_2^{(n)}(p_{2k}) = 2 - \frac{b}{p_{2k} - c + a} \text{ for } 0 \leq k \leq n. \tag{B.2}$$

Then $\Pi_1(p_{1k}; \boldsymbol{z}, F_2^{(n)}) = b \ \forall k = 0, ..., n$ and $\Pi_2(p_{2k}; \boldsymbol{z}, F_1^{(n)}) = c \ \forall k = 0, ..., n$.

[1] Here we assume that the smallest discrete price increment, $\frac{b-a}{n}$, is larger than the infinitesimal price deviation, δ, in (A.8). Thus, the results hold for $n > \frac{b-a}{\delta}$.

Proof. Proof of Proposition B.1 When $\mathbf{z}_1 = (0,0,1)$ and $\mathbf{z}_2 = (0,1,0)$, the discrete randomization profile of the firms are indicated by two PMFs $f_1^{(n)}(p_{1k})$, and $f_2^{(n)}(p_{2k})$. Similar to the derivation of $(A.9)$, $(A.10)$, $(A.11)$, and $(A.12)$ we obtain:

$$\pi_1\left(p_{1k}; \mathbf{z}_1, \mathbf{z}_2, F_2^{(n)}\right) = \begin{cases} p_{1k}\left(\sum_{l=k+1}^{n} f_2^{(n)}(p_{2l}) + 1\right) & \text{for } k = 0, ..., n-1, \\ p_{1n} = b & \text{for } k = n. \end{cases}$$

$$\pi_2\left(p_{2k}; \mathbf{z}_1, \mathbf{z}_2, F_1^{(n)}\right) = p_{2k}\left(\sum_{l=k}^{n} f_1^{(n)}(p_{1l})\right) \text{ for all } k = 0, ..., n.$$

We next find $f_1^{(n)}(p_{1k})$, and $f_2^{(n)}(p_{2k})$ by construction.

Firm 1:

$$f_2^{(n)}(p_{2k}) = \sum_{l=k}^{n} f_2^{(n)}(p_{2l}) - \sum_{l=k+1}^{n} f_2^{(n)}(p_{2l}) = \frac{\pi_1(p_{1k-1})}{a + \frac{(k-1)(b-a)}{n}} - \frac{\pi_1(p_{1,k})}{a + \frac{(k)(b-a)}{n}}, \text{ for } k = 1, ..., n-1,$$

$$f_2^{(n)}(p_{2n}) = \frac{\pi_1(p_{1,n-1})}{p_{1,n-1}} - 1 = \frac{b-a}{n(b - \frac{b-a}{n})},$$

$$f_2^{(n)}(p_{20}) = 1 - \sum_{l=1}^{n} f_2^{(n)}(p_{2l}) = 2 - \frac{b}{a}.$$

By the definition of mixed strategies, the profit functions have to be equal, i.e.,

$$\pi_1(p_{1k}) = \pi_1(p_{1k-1}) = \pi_1(p_{1n}) = b.$$

As a result, we obtain the following formulation for $f_2^{(n)}(p_{2k})$

$$f_2^{(n)}(p_{2k}) = \frac{b(b-a)}{n(p_{2k} - c + a)(p_{2,k} - c + a - \frac{b-a}{n})} \text{ for } 1 \leq k \leq n,$$

$$f_2^{(n)}(p_{20}) = 1 - \sum_{l=1}^{n} f_2^{(n)}(p_{2k}) = 2 - \frac{b}{a}.$$

And the discrete CDF corresponding to $f_2^{(n)}(p_{2k})$ is

$$F_2^{(n)}(p_{2k}) = 2 - \frac{b}{p_{2k} - c + a} \text{ for } 0 \leq k \leq n.$$

Firm 2:

$$f_1^{(n)}(p_{1k}) = \sum_{l=k}^{n} f_1^{(n)}(p_{1k}) - \sum_{l=k+1}^{n} f_1^{(n)}(p_{1k}) = \frac{\pi_2(p_{2k})}{c + \frac{(k)(b-a)}{n}} - \frac{\pi_2(p_{2k+1})}{c + \frac{(k+1)(b-a)}{n}}, \text{ for } k = 0, ..., n-1,$$
$$f_1^{(n)}(p_{1n}) = 1 - \sum_{l=0}^{n-1} f_1^{(n)}(p_{1l}).$$

By definition of a mixed strategy, all profit functions have to be equal:

$$\pi_2(p_{2k}) = \pi_2(p_{2k+1}) = \pi_2(p_{20}) = p_{20} \sum_{l=0}^{n} f_1^{(n)}(p_{1l}) = p_{20} \cdot 1 = c.$$

Hence, we obtain

$$f_1^{(n)}(p_{1k}) = \frac{c(b-a)}{n(p_{1k} + c - a)(p_{1k} + c - a + \frac{b-a}{n})} \text{ for } 0 \leq k \leq n-1,$$
$$f_1^{(n)}(p_{1n}) = 1 - \sum_{k=0}^{n-1} f_1^{(n)}(p_{1k}) = \frac{c}{c + b - a} \qquad k = n.$$

And the discrete CDF corresponding to $f_1^{(n)}(p_{1k})$ is

$$F_1^{(n)}(p_{1k}) = \frac{(p_{1k} - a)n + b - a}{(p_{1k} + c - a)n + b - a} \text{ for } 0 \leq k \leq n-1,$$
$$F_1^{(n)}(p_{1n}) = 1.$$

□

Corollary B.1. *The discrete CDFs proposed in Proposition B.1 describe a unique discrete mixed strategy Nash equilibrium for the game in (A.8) with $n+1$ equally-distanced*

price points.

Proof. Proof of Corollary B.1 First, we show that every discrete price outside the strategy support gives strictly less profit than the mixed strategy profits:

Given $F_2^{(n)}(p_{2k})$, $\forall p_{2k} \in [c, c+b-a]$,

- If $p_{1k} < a$, Firm 1 captures both customers. Thus $\pi_1(p_{1k}) = 2p_{1k} < 2a$. However, Firm 1's profit under the mixed strategy is $b = 2a$.
- If $p_{1k} > b$
  - When $\overline{v} > 3\underline{v}$, Firm 1 does not capture any customer. Thus $\pi_1(p_{1k}) = 0$ and strictly less than Firm 1's profit under the mixed strategy.
  - When $2\underline{v} \leq \overline{v} \leq 3\underline{v}$, $p_{1k} = b + k\frac{b-a}{n}$ and $\pi_1(p_{1k}) = p_{1k}(1 - F_1(k-1))$. We can prove that $\pi_1(p_{1k}) < b$ by mathematical induction.
    1. For $k = 1$, $p_{11}(1 - F(0)) < b$,
    2. For $k \geq 1$, $\pi_1(p_{1,k+1}) < \pi_1(p_{1k})$.

Given $F_1^{(n)}(p_{1k})$, $\forall p_{1k} \in [a, b]$,

- If $p_{2k} < c$,
  - When $3\underline{v} < \overline{v}$
    - For $a - c \leq p_{2k} < c$, $\pi_2(p_{2k}) = p_{2k} < c$.
    - For $0 \leq p_{2k} < a - c$, $p_{2k} = a - c - k\frac{b-a}{n}$ and $\pi_2(p_{2k}) = p_{2k}(2 - F_1(n-k))$. We can prove that $\pi_2(p_{2k}) < c$ by mathematical induction.
      1. For $k = 1$, $p_{21}(2 - F(n-1)) < c$,
      2. For $k \geq 1$, $\pi_2(p_{2,k+1}) < \pi_2(p_{2k})$.
  - When $2\underline{v} \leq \overline{v} \leq 3\underline{v}$,

* For $0 \leq p_{2k} < c$, $p_{2k} = c - k\frac{b-a}{n}$ and $\pi_2(p_{2k}) = p_{2k}(2 - F_1(n-k))$. We can prove that $\pi_2(p_{2k}) < c$ by mathematical induction.

  1. For $k = 1$, $p_{21}(2 - F(n-1)) < c$,

  2. For $k \geq 1$, $\pi_2(p_{2,k+1}) < \pi_2(p_{2k})$.

- If $p_{2k} > d$, Firm 1 captures no customers. Thus $\pi_2(p_{2k}) = 0$ and strictly less than Firm 2's profit under the mixed strategy.

We conclude that $F_1^{(n)}$ and $F_2^{(n)}$ identify a mixed strategy Nash equilibrium, since the profit for each firm is constant over the strategy support of that firm (Proposition B.1), and its strictly less outside the strategy support. Moreover, in Proposition B.1 we arrived at a unique discrete CDF with $n+1$ equally distanced price points, thus the discrete mixed strategy is unique. □

Next, we use G_m as a refinement of the discretization grid over a continuous strategy space $[0, 1]$.[2]

Definition B.1. *The m th refinement grid, G_m, is of the form $G_m = \{x | x = \frac{k}{2^m}, 0 \leq k \leq 2^m\}$ for $m \geq 0$.*

Corollary B.2. *Each grid G_m contains all the grids G_l for $0 \leq l \leq m$. In other words, as m grows larger, the previous discrete points on the grid are preserved.*

Definition B.2. *For any $p \in [\alpha, \beta]$ and for any m, let $p^m = \alpha + (\beta - \alpha)\frac{k_m}{2^m}$ where $k_m = \arg\min\{k | \alpha + (\beta - \alpha)\frac{k}{2^m} \geq p\}$*

Next, we show that the limiting set of the discrete grid G_n is dense.

[2]Please note that any interval can be normalized to $[0, 1]$, thus the results we obtain here can be extended to any continuous interval.

Lemma B.1.

$$\forall p \in [0,1], \text{ and } \forall \gamma > 0, \exists N \text{ such that } \forall m \geq N, 0 \leq p^m - p < \gamma.$$

Proof. Proof of Lemma B.1 We have that $0 \leq p^m - p$ according to the definition of p^m. Also according to the same definition $p^m - p \leq \frac{1}{2^m}$ thus $\forall m > -\log_2 \gamma$, we obtain $p^m - p < \gamma$. □

Lemma B.2. $\forall \epsilon > 0$, *and* $\forall p \in [a,b], \exists N : \forall m > N, |F_1^{(2^m)}(p^m) - F_1^*(p)| < \epsilon$ *in which* $F_1^*(p) = \frac{p-a}{p+c-a}$ *and* $F_1^{(n)}$ *is defined in* $(B.1)$.

Proof. Proof of Lemma B.2 For any ϵ we set $\gamma < \frac{c\epsilon}{2}$ and $N > \log_2 \frac{2(b-a)}{c\epsilon}$. We choose the largest N such that $0 \leq p^m - p < \gamma$ $\forall m > N$, which is guaranteed to exist by Lemma $B.1$, and also $N > \log_2 \frac{2(b-a)}{c\epsilon}$. Therefore, $\forall m > N$,

$$\begin{aligned}
|F_1^{(2^m)}(p^m) - F_1^*(p)| &= \left| \frac{(p^m - a)2^m + b - a}{(p^m + c - a)2^m + b - a} - \frac{p-a}{p+c-a} \right| = \left| \frac{c(p^m - p) + c\frac{(b-a)}{2^m}}{(p^m - a + c + \frac{b-a}{2^m})(p+c-a)} \right| \\
&< \left| \frac{c(p^m - p)}{(p+c-a)^2} \right| + \left| \frac{c(b-a)}{2^m(p+c-a)^2} \right| < \frac{c\gamma}{c^2} + \frac{c(b-a)}{2^m c^2} = \frac{\gamma}{c} + \frac{b-a}{2^m c} < \epsilon,
\end{aligned}$$

where the first inequality results from the fact that $0 \leq p^m - p < \gamma$ and $0 < (p + c - a) < (p^m + c - a) + \frac{b-a}{2^m}$ and the second inequality follows from $0 \leq p^m - p < \gamma$ and $0 < c \leq p + c - a$, $\forall p \geq a$. □

Lemma B.3. $\forall \epsilon > 0$ *and* $\forall p \in [c, c+b-a)$, $\exists N : \forall m > N, |F_2^{(2^m)}(p^m) - F_2^*(p)| < \epsilon$ *in which* $F_2^*(p) = 2 - \frac{b}{p+a-c}$ *is defined in* $(B.2)$.

Proof. Proof of Lemma B.3 For any ϵ we set $\gamma < \frac{a^2\epsilon}{2b}$ and $N > \log_2 \frac{2b(b-a)}{a^2\epsilon}$. We choose the largest N such that $0 \leq p^m - p < \gamma$ $\forall m > N$, which is guaranteed to exist by Lemma

$B.1$, and also $N > \log_2^{\frac{2b(b-a)}{a^2\epsilon}}$. Therefore, $\forall m > N$,

$$|F_2^{(2^m)}(p^m) - F_2^*(p)| = \left|2 - \frac{b}{p^m - c + a} - \left(2 - \frac{b}{p - c + a}\right)\right| = \left|\frac{p^m - p}{(p^m - c + a)(p - c + a)}\right|$$
$$< \left|\frac{\gamma}{(p - c + a)^2}\right| < \frac{\gamma}{a^2} < \epsilon.$$

Where, the first inequality results from the fact that $0 \leq p^m - p < \gamma$ and $0 < (p + a - c) \leq (p^m + a - c)$ and the second inequality follows from $0 < a \leq p - c + a$, $\forall p \geq c$. $\square$

Appendix C Examples

Below we present through a few examples, how different chapters of this dissertation pertain to different industries and their bundling practices.

	Chapter 2	Chapter 3	Chapter 4
Desktop Computers	Applicable	Applicable	Applicable

- Desktop computers are sold as mixed-bundles. All the computer parts consisting of the main unit, display, input devices, and the printer can be sold separately, or as a bundle.
- Both Cournot and Bertrand models are applicable to this industry. Companies such as HP and Dell offer the full spectrum of the mixed-bundle, while companies such as LG and Samsung focus only on displays, while Epson focuses only on printers.
- The manufacturers can choose different sizes of storage and processing power to customize the design of the bundle.

	Chapter 2	Chapter 3	Chapter 4
Laptops	Not applicable	Applicable	Applicable

- Laptop manufacturers always offer pure bundles of computer parts. The best model to capture their competition is a Cournot oligopoly model with identical firms.
- Manufacturers can choose different sizes of storage and processing power to customize the design of the bundle.

	Chapter 2	Chapter 3	Chapter 4
Fast food combo meals	Applicable	Applicable	Applicable

- Most fast food chains often offer mixed bundles of sandwiches and soft drinks, which is very close to the two-product bundles presented in this thesis.
- The competition among the chains can be modeled using both Bertrand and Cournot frameworks. However, the fast food industry is far from a duopoly and the oligopolistic approach is more applicable.
- Since most customers only consume one unit of the sandwich and one unit of the coke, the design of the bundle is only applicable for family-size meals.

	Chapter 2	Chapter 3	Chapter 4
Gift Baskets	Not applicable	Applicable	Applicable

- Gift basket companies offer an assortment of food and gifts that are priced based on quantity.
- Cournot framework is more applicable to this industry, where there is a separate oligopolistic market for each component inside the bundle.

- Gift basket companies practice several designs to meet the demand of a multitude of customer segments.

	Chapter 2	Chapter 3	Chapter 4
Telecom Bundles	Not applicable	Applicable	Applicable

- Due to high entry costs, telecom is often an oligopoly with very few firms.
- Different offerings are usually prices based on the quantity, e.g. number of minutes, text messages, or size of the data plan.
- Design of multiple bundles with different sizes of date, text, and call is very common among telecom firms.

	Chapter 2	Chapter 3	Chapter 4
Amazon Prime Membership	Applicable	Not applicable	Not applicable

- Amazon Prime offers multiple services including shipping, streaming, reading and other shopping services. It is the only firm that bundles this large set a services in a single bundle.
- Amazon prime faces competition in each of the its component markets. For example, in online retail Ebay is a competitor, or in streaming services Netflix and Spotify are competitors. This setting is captured in the Bertrand model (Chapter 2)
- Amazon Prime bundle only consists of services and thus the bundle proportion design question is not applicable.

	Chapter 2	Chapter 3	Chapter 4
Campus Photo Studio	Not applicable	Not applicable	Applicable

- The UCLA photo studio that offers graduation portrait services is a monopoly.
- None of the Bertrand and Cournot models are applicable to this monopolistic setting.
- They offer multiple bundles with different designs (e.g. the number and size of printed copies and files vary across bundles, or some bundles contain a frame and a tassel).

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