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Author
Li, Penghui

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by

Penghui Li

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Committee in charge:

Professor David Nadler, Chair
Professor Constantine Teleman
Professor Kam-Biu Luk

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Abstract

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Let G be a connected reductive complex algebraic group, and E a complex elliptic curve. Let G_E denote the connected component of the trivial bundle in the stack of semistable G-bundles on E. We introduce a complex analytic uniformization of G_E by adjoint quotients of reductive subgroups of the loop group of G. This can be viewed as a nonabelian version of the classical complex analytic uniformization $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$. We similarly construct a complex analytic uniformization of G itself via the exponential map, providing a nonabelian version of the standard isomorphism $\mathbb{C}^* \simeq \mathbb{C}/\mathbb{Z}$, and a complex analytic uniformization of G_E generalizing the standard presentation $E = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. Finally, we apply these results to the study of sheaves with nilpotent singular support.

Contents

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Chapter 1

Introduction

Let G be a connected complex reductive algebraic group, and E a complex elliptic curve. The moduli of G-bundles on E plays a distinguished role in representation theory, gauge theory and algebraic combinatorics (for example [8, 29] as the setting of the elliptic Hall algebra), and its geometry has been the subject of a long and fruitful study. Atiyah [2] classified vector bundles on E in terms of line bundles and their extensions. In particular, he showed rank n vector bundles with trivial Jordan-Holder factors are in bijection with unipotent adjoint orbits in $GL(n)$, with the unique irreducible such vector bundle corresponding to the regular unipotent orbit. This initiated the organizing viewpoint that vector bundles on E form an analogue of the adjoint quotient of $GL(n)$, where the "eigenvalues" of a vector bundle are the line bundles appearing as its Jordan-Holder factors. In a beautiful series of papers, Friedman, Morgan and Witten [11, 9, 10] extended this to any G, definitively describing the Jordan-Holder patterns and the geometry of the coarse moduli of semistable bundles. Our focus here is the moduli stack of semistable bundles, and specifically the construction of an analytic uniformization of it by finite-dimensional subvarieties of the loop group of G. We discuss motivations and applications at the end of the introduction.

Thanks to complex function theory, the uniformization $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$, with $|q| < 1$, has been known since the 19th century. Let $\mathrm{Jac}(E)$ be the Jacobian variety parameterizing degree zero line bundles on E. (Thanks to Serre's GAGA, one can equivalently consider algebraic or holomorphic bundles.) The Abel-Jacobi map $E \to \mathrm{Jac}(E)$, $x \mapsto \mathcal{O}_E(x - x_0)$ is an isomorphism, inducing a similar uniformization $\mathrm{Jac}(E) \simeq \mathbb{C}^*/q^{\mathbb{Z}}$. This isomorphism also results from the following geometric observations. By the uniformization $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$, holomorphic line bundles on E are equivalent to equivariant holomorphic line bundles on $\mathbb{C}^*$. Since every holomorphic line bundle on $\mathbb{C}^*$ is trivializable, equivariant holomorphic line bundles are encoded by their equivariance up to gauge. Such data can be represented by elements of the holomorphic loop group $L_{hol}\mathbb{C}^*$ up to q-twisted conjugacy. Within this identification, one finds the uniformization of $\mathrm{Jac}(E)$ by the constant loops $\mathbb{C}^* \subset L_{hol}\mathbb{C}^*$ up to q-twisted conjugacy by the coweights $\mathbb{Z} \simeq \mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) \subset L_{hol}\mathbb{C}^*$.

Now let G_E denote the connected component of the trivial bundle in the stack of semistable G-bundles on E. By the uniformization $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$, isomorphism classes of G-

bundles on E are in bijection with q-twisted conjugacy classes in the holomorphic loop group $L_{hol}G$ (see for example [4] where this is attributed to Looijenga). We would like to enhance this to an analytic uniformization of G_E by finite-dimensional subvarieties of $L_{hol}G$. As a first attempt, we could take the constant loops $G \subset L_{hol}G$, but unfortunately, in general, the natural map $G/G \to G_E$ from the adjoint-quotient is neither surjective nor étale.

We correct for both of these shortcomings in the following theorem. Let $T \subset G$ be a maximal torus of G. For each $s \in T$, we define a reductive group $G_s \subset L_{hol}G$ and a connected adjoint-invariant open analytic subvariety $G_s^{0,et} \subset G_s$. By construction, the element s lies in both G_s^{et} and also the center of G_s, so in particular G_s is the automorphism group of the semisimple G-bundle on E defined by s.

Theorem 1.1. *There is a finite set $S \subset T$, such that the natural map*

$$\coprod_{s \in S} G_s^{0,et}/G_s \longrightarrow G_E$$

is surjective and étale.

Note that the uniformization $E \simeq \mathbb{C}^*/q^{\mathbb{Z}}$ presents E as a quotient of $\mathbb{C}^*$ by the q-translation action of the group $\mathbb{Z}$. A further challenge of the nonabelian setting is the complicated gluing of the above cover, in particular the fact that it is far from the quotient map of a group action. In Corollary 3.32, we describe the fiber products

$$G_{s_1}^{0,et}/G_{s_1} \times_{G_E} G_{s_2}^{0,et}/G_{s_2}, \qquad s_1, s_2 \in T$$

which again turn out to be disjoint unions of open substacks of adjoint-quotients of reductive groups. Furthermore, it is straightforward to generalize this to the fiber product of any finite number of terms. To organize this canonically involves some care, in particular since the terms and their fiber products are stacks not schemes. To achieve this, we introduce a simplified version of the Čech diagram of the cover and prove that G_E is indeed its colimit in the setting of stacks. We record this extension of Theorem 1.1 in the following abstract form and refer the reader to Theorem 3.28 for a more explicit version.

Theorem 1.2. *There is a category $\mathscr{C}$, a functor $\mathscr{G}_E : \mathscr{C} \to \mathrm{Stacks}$, and an equivalence*

$$colim_{\mathscr{C}} \mathscr{G}_E \xrightarrow{\sim} G_E$$

such that $\mathscr{G}_E$ assigns open analytic substacks of adjoint-quotients of reductive groups to objects of $\mathscr{C}$.

Before discussing motivations for and applications of the above results, it is worth highlighting that the methods also provide a useful "exponential uniformization" of the adjoint-quotient G/G itself. Whereas we previously started with the natural map $G/G \to G_E$ and then refined and extended it in the subsequent theorems, here we start with the exponential map $\mathfrak{g}/G \to G/G$ and then refine and extend it accordingly. What results is an explicit description of G/G as the colimit of a diagram of disjoint unions of open substacks of adjoint-quotients of reductive Lie algebras. We also construct an explicit presentation of G_E in similar terms, which can be thought of as an iteration of exponential uniformization and Theorem 1.2. In these cases, spaces of connections up to gauge transformation play the role of holomorphic loop group up to q-twisted conjugation.

At this point, the reader may jump ahead to 1.1 for a concrete example of the exponential uniformization of G/G. We continue the introduction here.

To put the above results in further context, recall the well-known trichotomy of plane cubic curves. Let $B \subset G$ be a Borel subgroup, and let $H = B/[B,B]$ be the universal Cartan group with Lie algebra $\mathfrak{h}$. Let $\Lambda_H = \mathrm{Hom}(\mathbb{G}_m, H)$ denote the coweight lattice, note the canonical isomorphisms $\mathfrak{h} = (\mathbb{G}_a \otimes \Lambda_H)$, $H = (\mathbb{G}_m \otimes \Lambda_H)$, and let $H_E = (E \otimes \Lambda_H)$ be the "elliptic Cartan group".

The following chart organizes the idea that semistable bundles on plane cubic curves behave like adjoint-quotients. (In the chart, $\Delta = -16(4a^3 + 27b^2)$ is the usual discriminant.) Our results allow us to systematically reduce questions about the second and third columns to the far simpler linear geometry of the first column.

Plane cubic curve	cuspidal curve	nodal curve	elliptic curve
Local equation	$y^2 = x^3$	$y^2 = x^2(x+1)$	$y^2 = x^3 + ax + b, \Delta \neq 0$
π_1(curve)	$\{1\}$	$\mathbb{Z}$	$\mathbb{Z}^2$
Semistable bundles	$\mathfrak{g}/G$	G/G	G_E
Coarse moduli	$\mathfrak{h}/W$	H/W	H_E/W

Finally, let us mention some specific motivations for and applications of the above results. One of our primary goals is to understand the derived category $D^\flat_{\mathcal{N}}(G_E)$ of constructible sheaves with singular support lying in the global nilpotent cone (the zero-fiber of the Hitchin map). Our approach also gives a new understanding of the (derived) category of character sheaves $D^\flat_{\mathcal{N}}(G/G)$. We record the theorems on sheaves in the following abstract form and refer the reader to Theorem 5.25 and Theorem 6.3 for a more explicit version.

Theorem 1.3. *There are categories $\mathscr{C}_0^*, \mathscr{C}^*$, and functors $\mathscr{G}, \mathscr{G}_E^* : \mathscr{C}_0^*, \mathscr{C}^* \to \infty$-**Cat**, and equivalences*

$$\lim_{\mathscr{C}_0^*} \mathscr{G}^* \xrightarrow{\sim} D^\flat_{\mathcal{N}}(G/G)$$

$$\lim_{\mathscr{C}^*} \mathscr{G}_E^* \xrightarrow{\sim} D^\flat_{\mathcal{N}}(G_E)$$

such that $\mathscr{G}^*, \mathscr{G}_E^*$ assigns derived categories of character sheaves on adjoint-quotients of reductive Lie algebras to objects of $\mathscr{C}^*$.

Comparing to Theorem 1.2, Lie groups are replaced by Lie algebras, and the phrase "open substacks of" is removed. An application of the above result is the topological nature of $D^\flat_{\mathcal{N}}(G_E)$.

Corollary 1.4. *The category $D^\flat_{\mathcal{N}}(G_E)$ is locally constant in the parameter q.*

With modest further effort, and similar applications of the above results, one can extend the corollary to the derived category $D^\flat_{\mathcal{N}}(\mathrm{Bun}_G(E))$ of constructible sheaves with nilpotent singular support on the entire moduli of all G-bundles on E. This category contains the Hecke eigensheaves of the geometric Langlands program, and we expect it to offer also a theory of affine character sheaves. Furthermore, under Langlands duality/mirror symmetry, it is expected to correspond to a derived category of coherent sheaves on the commuting stack. (Note that the commuting stack, and hence its coherent sheaves as well, is evidently a topological invariant, only depending on the fundamental group of the elliptic curve.) This is in turn the subject of beautiful recent developments (Schiffmann-Vasserot [30, 32, 31] on Macdonald polynomials and double affine Hecke algebras; Ginzburg [14] on Cherednik algebras and the Harish Chandra system) and in particular its role as affine character sheaves was established in [6].

Another application of the results of this thesis is the construction of a horocycle correspondence for loop groups. The traditional horocycle correspondence relates character sheaves and the Hecke categories, and in particular is the fundamental geometry used to construct character sheaves. Due to the infinite-dimensional nature of the loop group, the analogous horocycle correspondence, or for that matter the adjoint-quotient of the loop group, is difficult to work with. But from the results of this thesis one can construct an "elliptic horocycle correspondence" over the semistable locus. We expect this will play an important technical role in the structure theory and classification of objects of $D^\flat_{\mathcal{N}}(\mathrm{Bun}_G(E))$.

1.1 A first example

To illustrate the ideas, we give a first example in its most plain form. It will be revisited in Example 5.28 in the form compactible with the general discussion.

Let $G = SL_2$, $\mathfrak{g} = \mathfrak{sl}_2$, $T, \mathfrak{t}$ the diagonal matrices in $G, \mathfrak{g}$. Let $U := \{X \in \mathfrak{g} : |Re(\lambda(X))| < 1/2\}$, for $\lambda(X)$ an eigenvalue of X. Let V be another copy of U. We have $U \to G$ by $X \mapsto \exp(2\pi i X)$ and $V \to G$ by $Y \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \exp(2\pi i Y)$. Let $D = \{H \in \mathfrak{t} : 0 < \lambda_1(H) < 1/2\}$, where $\lambda_1(H)$ is the first eigenvalue of H. We have $D \times G/T \to U$ by $(H, g) \mapsto gHg^{-1}$ and $D \times G/T \to V$ by $(H, g) \mapsto g(H - \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix})g^{-1}$. Notice that $[H, \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}] = 0$ We have the commutative diagram

$$\begin{tikzcd} & D \times G/T \arrow[dl] \arrow[dr] & \\ U \arrow[dr] \arrow[rr, phantom, "\square"] & & V \arrow[dl] \\ & G & \end{tikzcd}$$

which is a fiber diagram. All arrows are open embeddings and $U \coprod V \to G$ is surjective. The diagram is G-equivariant, passing to the quotient, we have

$$\begin{tikzcd} & D/T \arrow[dl] \arrow[dr] & \\ U/G \arrow[dr] & \square & V/G \arrow[dl] \\ & G/G & \end{tikzcd} \tag{1.5}$$

with all actions being adjoint actions. Passing to the sheaves, we have all the pullback preserve nilpotent singular support. Hence we have

$$Sh_{\mathcal{N}}(G/G) = \lim \begin{tikzcd} & Sh_{\mathcal{N}}(D/T) & \\ Sh_{\mathcal{N}}(U/G) \arrow[ur] & & Sh_{\mathcal{N}}(V/G) \arrow[ul] \end{tikzcd} \tag{1.6}$$

The above diagram can be identified with

$$\begin{tikzcd} & Sh_{\mathcal{N}}(\mathfrak{t}/T) & \\ Sh_{\mathcal{N}}(\mathfrak{g}/G) \arrow[ur] & & Sh_{\mathcal{N}}(\mathfrak{g}/G) \arrow[ul] \end{tikzcd} \tag{1.7}$$

with the pullback identified with (parabolic/hyperbolic) restriction. Hence this gives a description of the category of character sheaves on a Lie group in terms of categories of character sheaves on the Lie algebras. We have analogous statement in elliptic situation. It turns out that the charts $U/G, V/G$ and D/T we defined above naturally sits in side the infinity dimension uniformizations we shall consider.

1.2 Outline of the strategies

The main content of this thesis can be divided into two parts. The first part is to define some covers of the stacks mentioned in the introduction, and the second part is to use those covers, by the general theory of descent, to describe the category of sheaves with certain singular support condition. We shall outline the main strategies respectively as below.

1.8. First part: at the level of spaces.

We shall start with some general construction. Let $X \in \mathbf{Sp}$, where $\mathbf{Sp}$ = the category **Mfld** of smooth manifolds, **Cplx** of complex manifolds, or **Sch** of schemes. We have tautologically $X = \operatorname{colim}_{Y \in \mathbf{Sp}_{/X}} Y$ in the presheaf category (valued in ∞-groupoids). Now if $\mathscr{X}$

is a 1-stack over **Sp**, (see Appendix A for a brief account of the theory the stacks we shall use) we have similarly $\mathscr{X} = \operatorname{colim}_{\mathscr{Y}\in \mathbf{1\text{-}stack}_{/\mathscr{X}}} \mathscr{Y}$. Notice that $\mathbf{1\text{-}stack}_{/\mathscr{X}}$ is a (2,1)-category, i,e a weak 2-category where all 2-morphisms are invertible.

For each functor $F : \mathscr{C} \to \mathbf{1\text{-}stack}_{/\mathscr{X}}$, we have a natural morphism $\operatorname{colim} F \to \mathscr{X}$. We could understand $\mathscr{X}$ in terms of F. E.g, if $\mathscr{Y} \to \mathscr{X}$ is (representable) smooth surjective, let $\mathscr{C} = \Delta^{op}$ the opposite of simplex category and $F : \mathscr{C} \to \mathbf{1\text{-}stack}_{/\mathscr{X}}$ by $F([n]) = \mathscr{Y}^n_{\mathscr{X}}$. Then smooth descent implies that $\operatorname{colim} F \to \mathscr{X}$ is an isomorphism.

Now suppose $\mathscr{X} = \mathcal{X}/\mathcal{G}$, we can define a (2,1) category $\mathscr{S}$ and a functor $S : \mathscr{S} \to \mathbf{1\text{-}stack}_{/\mathscr{X}}$, as follows: The object of $\mathscr{S}$ are of the form (X, G), for $X \subset \mathcal{X}$, $G \subset \mathcal{G}$ such that G acts on X. $Mor_{\mathscr{S}}((X_1, G_1), (X_2, G_2)) = \{g \in \mathcal{G} : g(X_1) \subset X_2, gG_1g^{-1} \subset G_2\}$. And a 2-morphism for each triangle

$$\begin{tikzcd} (X_1,G_1) \arrow[r, "g"'] \arrow[rr, bend left, "f"] & (X_2,G_2) \arrow[r, "h"'] \arrow[u, phantom, "\Uparrow\eta"] & (X_3,G_3) \end{tikzcd} \tag{1.9}$$

Then for $g : (X_1, G_1) \to (X_2, G_2)$ in $\mathscr{S}$, then g induces $S_g : X_1 \to X_2, x_1 \mapsto g \cdot x_1$ and a group homomorphism $\phi_g : G_1 \to G_2, g_1 \mapsto gg_1g^{-1}$. The two maps (S_g, ϕ_g) are compactible in the sense that $S_g(g_1x_1) = \phi_g(g_1)S_g(x_1)$. Hence they induces a map of quotient stacks $S_g : X_1/G_1 \to X_2/G_2$. Similarly, we have $S_h : (X_2, G_2) \to (X_3, G_3)$ and $S_f : X_1/G_1 \to X_3/G_3$. We have an natural (invertible) 2-morphism

$$S_f^{h,g} : S_h \circ S_g \Rightarrow S_f \tag{1.10}$$

Now there is a functor $S : \mathscr{S} \to \mathbf{1\text{-}stack}_{/\mathscr{X}}$. By taking (1.9) to

$$\begin{tikzcd} X_1/G_1 \arrow[r, "S_g"'] \arrow[rr, bend left, "S_f"] & X_2/G_2 \arrow[r, "S_h"'] \arrow[u, phantom, "\Uparrow S_f^{h,g}"] & X_3/G_3 \end{tikzcd}$$

Now for any $J : \mathscr{C} \to \mathscr{S}$, we have $\operatorname{colim} F \to \mathscr{X}$, where $F = S \circ J$. In practice, $\mathscr{X}$ is assumed to be finite dimensional, but $\mathcal{X}, \mathcal{G}$ could be infinite dimensional. And all objects (X, G) in $\mathscr{S}$ are taken to be finite dimensional.

We are mainly interested in the situation $\mathscr{X}$ is (a) $\operatorname{Loc}_G(S^1) = \Omega^1(S^1, \mathfrak{g})/C^\infty(S^1, G)$, (b) $\operatorname{Bun}_G(E)^0 = \Omega^{0,1}(E, \mathfrak{g})/C^\infty(E, G)$, or (c) $\operatorname{Bun}_G(E) = L_{hol}G/^q L_{hol}G$, We shall construct some $\mathscr{C}, F$ as above such that $\operatorname{colim} F \to \mathscr{X}'$ is an isomorphism, where $\mathscr{X}' = \mathscr{X}$ in case (a), and $\mathscr{X}' = G_E \subset \mathscr{X}$ in case (b),(c). They are discussed in the following places of the thesis:
(1) The category $\mathscr{C}$ is defined in chapter 2. It is of the form $\Delta^{op}_{-,-}$ as in 2.9.
(2) The functor F is defined for case (a) in 5.11, (b) in 6.1 and (c) in 3.3. The covers $F(c), c \in \mathscr{C}$ are isomorphic to U/K where K is some a reductive Lie group and U is an invariant open subset of $\mathfrak{k} = Lie(K)$ for (a),(b) / of K for (c). See also Remark 5.7 to make sense of the above discussion at the level of S-point.
(3) We prove the fact that $\operatorname{colim} F \simeq \mathscr{X}'$ in case (c) in Theorem 3.28. This is the main theorem of the first part. The proof relies on the particular geometry of adjoint quotient.

And the corresponding statement is stated in (5.12) for (a) and in (6.1) for (b). There are various versions of $\mathscr{C}, F$ we will use, c.f Remark 3.33. The diagram (1.5) will be a partiuclar example.

1.11. Second part: at the level of sheaves.

In this part, the linear geometry on Lie algebras is crucial for explicit calculations, so we shall only consider case (a),(b). The functor $F : \mathscr{C} \to \mathbf{Prestk}$, induces pullback on sheaves: $F^* : \mathscr{C}^{op} \to \infty\textbf{-Cat}$. And colim $F \simeq \mathscr{X}'$ together with descent gives $\lim F^* \simeq Sh(\mathscr{X}')$. We show that the sheaves with nilpotent singular support correspond. Hence we have $Sh_{\mathcal{N}}(\mathscr{X}') \simeq \lim_{c \in \mathscr{C}} Sh_{\mathcal{N}}(F(c))$ as in (1.6). To calculate this, we show in that the restriction $Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(U/K)$ is an equivalence in section 4.1, c.f Corollary 4.5. In case (a), under the above equivalence, we show that the pullback $F(\varphi)^*$ induced by 1-morphism $\varphi : c \to c'$ in $\mathscr{C}$ is identified with (i) hyperbolic restriction or (ii) based restriction between reductive Lie algebras. The notion of based restriction is introduced in 4.2. And the fact that based restriction is the same as hyperbolic restriction is shown in 4.3. Using these, we define various functors $\mathscr{C} \to \mathbf{Prestk}$ in 5.2. The fact of they are all isomorphic to F^* (with nilpotent singular support) and hence can be used to calculate $Sh_{\mathcal{N}}(\mathscr{X}')$ is our main theorem, Theorem 5.25. One of the functors will give (1.7). We have similar statememts for case (b), but in this case, we could only use based restriction rather than hyperbolic restriction.

1.12. Structure of the thesis.

The logical order amount chapters are as follows:

$$\begin{tikzcd}
\boxed{\text{Chapter } 2} \arrow[d] & & \\
\boxed{\text{Chapter } 3} \arrow[dr] & & \boxed{\text{Chapter } 4} \arrow[dl] \\
& \boxed{\text{Chapter } 5} \arrow[d] & \\
& \boxed{\text{Chapter } 6} &
\end{tikzcd}$$

The first part mainly consists of Chapter 2,3, and the second part mainly consists of Chapter 4,5,6. We study case (c) in Chapter 3; the adjoint-quotient of Lie algebra $\mathfrak{g}/G$ in Chapter 4; case (a) in Chapter 5; and case (b) in Chapter 6.

Chapter 2

Category preliminaries

In this chapter, we introduce the combinatoric that organize the diagram we will use later. The main object is the category $\Delta^{op}_{-,-}$ defined in (2.9).

Let $F : \mathscr{D} \to \mathscr{T}$ be a functor between two categories, assume $\mathscr{D}$ is small. Let $\mathscr{D}_+$ be the category obtained by adding a final object pt to $\mathscr{D}$. The category $\mathscr{D}_+$ has a final object. Then giving a cocone of F in $\mathscr{T}$ is same as giving a functor $F_+ : \mathscr{D}_+ \to \mathscr{T}$.

Definition 2.1. Let $\mathscr{D}$ be a category, a functor $F_+ : \mathscr{D}_+ \to \mathscr{T}$ is called a *colimit diagram* if the natural map $colim(F) \to F_+(pt)$ is an isomorphism.

2.2. *The category $\mathscr{D}_J$.* Given a functor $J : \mathscr{D} \to \mathbf{Set}$ (assume $\mathscr{D}$ is small), we can associate it a category $\mathscr{D}_J$ by $Ob(\mathscr{D}_J) \coloneqq \coprod_{d \in \mathscr{D}} J(d)$, for $c, c' \in Ob(\mathscr{D}_J)$, we have $c \in J(d), c' \in J(d')$ for unique $d, d' \in Ob(\mathscr{D})$, define $Mor_{\mathscr{D}_J}(c, c') \coloneqq \{f \in Mor_{\mathscr{D}}(d, d') | J(f)(c) = c'\}$. with the identity and composition induced from $\mathscr{D}$. $\mathscr{D}_J$ is indeed a category.

Now let $\mathscr{T}$ be a category with coproduct. Let $K : \mathscr{D}_J \to \mathscr{T}$ a functor. We define a functor $K_J : \mathscr{D} \to \mathscr{T}$. $K_J(d) \coloneqq \coprod_{c \in J(d)} K(c)$, and for $f : d \to d'$, we have $J(f) : J(d) \to J(d')$, for each $c \in J(d)$, then $J(f)c \in J(d')$, and $f \in Mor_{\mathscr{D}_J}(c, J(f)c)$. We have $K(f) : K(c) \to K(J(f)c)$, we have natural morphism $K(J(f)c) \to \coprod_{c' \in J(d')} K(c') = K_J(d')$ in $\mathscr{T}$, by composing above two morphism, we have a morphism, still denoted by $K(f) : K(c) \to \coprod_{c' \in J(d')} K(c') = K_J(d')$. Hence, we have an induced morphism $K_J(f) : K_J(d) = \coprod_{c \in J(d)} K(c) \to K_J(d')$. K_J is indeed a functor.

Example 2.3. Take $S \in \mathbf{Set}$, $\underline{S} : \mathscr{D} \to \mathbf{Set}$ be the constant functor mapping to S. Then $\mathscr{D}_{\underline{S}} \simeq \coprod_S \mathscr{D}$. In particular, $\mathscr{D}_{\underline{pt}} \simeq \mathscr{D}$. For any $K : \mathscr{D}_{\underline{S}} \to \mathscr{T}, K_{\underline{S}} \simeq \coprod_S K : \mathscr{D} \to \mathscr{T}$.

Given a cocone in $\mathscr{T}$ for K, it is naturally a cocone for K_J. So we have an induced morphism (in the category of presheaves on $\mathscr{T}$) $colim(K_J) \to colim(K)$, which is an isomorphism. Let $J_+ : \mathscr{D}_+ \to \mathbf{Set}, J_+(pt) = pt$ Then $\mathscr{D}_{+J_+} \simeq \mathscr{D}_{J+}$, Let $K_+ : \mathscr{D}_{J_+} \to \mathscr{T}$ a functor, then

$$K_+ \text{ is an colimit diagram if and only if } K_{+J_+} \text{ is so.} \tag{2.4}$$

Let $\mathscr{D}$ be the category of a commutative square. $J : \mathscr{D} \to \mathbf{Set}$ any functor.

$$\begin{tikzcd} w \arrow[r, "h"] \arrow[d, "k"] & x \arrow[d, "f"] \\ y \arrow[r, "g"] & z \end{tikzcd}$$

$\mathscr{D}_J, \mathscr{T}, K, K_J$ as above. For $b \in J(x), c \in J(y), d \in J(z)$, such that $J(f)b = J(g)c = d$, Let $F_{b,c} := \{a \in J(w) : J(h)a = b, J(k)a = c\}$. We have the following diagram in $\mathscr{T}$:

$$\begin{tikzcd} \coprod_{a \in F_{b,c}} K(a) \arrow[r, "K(h)"] \arrow[d, "K(k)"] & K(b) \arrow[d, "K(f)"] \\ K(c) \arrow[r, "K(g)"] & K(d) \end{tikzcd} \tag{2.5}$$

Proposition 2.6. *$K_J(\mathscr{D})$ is a fiber diagram if and only if (2.5) is a fiber diagram for all such triple (b,c,d).* □

Let Δ_+^{op} be the opposite category of the augmented simplex category, with objects (n) for each $n \in \mathbb{N} \cup \{-1\}$. Let $\tau_i^n : (n) \to (n-1)$, where τ_i^n be the (opposite of) face maps They satisfies the relation $\tau_i^{n-1} \circ \tau_j^n = \tau_j^{n-1} \circ \tau_{i+1}^n$, for $i < j$ we call such commutative diagram an *elementary square.*

Let $\mathscr{T}$ be a category. Let $U \to X$ in $\mathscr{T}$, assume the n−tuple fiber product U_X^n exist. Then we have a natural functor $Cov_X^U : \Delta_+^{op} \to \mathscr{T}$ by $Cov_X^U(n) := U_X^{n+1}, Cov_X^U(-1) := X$ with natural morphisms.

Lemma 2.7. *Let $X : \Delta_+^{op} \to \mathscr{T}$ be a functor. $X_i := X(i)$. Assume that for each elementary square D, $X(D)$ is an fiber diagram. Then ${X_0}_{X_{-1}}^n$ exist, and there is unique natural isomorphism of functor $\varphi : X \to Cov_{X_{-1}}^{X_0}$, such that $\varphi(i) : X_i \to X_i$ is identity for $i = -1, 0$.*

Proof: $\kappa_i^n := Cov_{X_0}^{X_1}(\tau_i^n), \sigma_i^n := X(\tau_i^n)$, $\varphi_i := \varphi(i)$, We shall drop the indices if there is no confusion.
We have $\kappa_1^1 = \sigma_1^1$. We define φ inductively. For $n = 2$, we have unique isomorphism φ_2 making the following diagram commutative.

$$\begin{tikzcd} X_2 \arrow[dr, dotted, "\varphi_2"] \arrow[drr, bend left, "\sigma_1^2"] \arrow[ddr, bend right, "\sigma_2^2"'] & & \\ & X_1^2 \arrow[r, "\kappa_1^2"] \arrow[d, "\kappa_2^2"] & X_1 \arrow[d, "\sigma_1^1"] \\ & X_1 \arrow[r, "\sigma_1^1"] & X_0 \end{tikzcd}$$

Suppose φ_k is uniquely defined for $k \le n$. Then there is unique isomorphism φ_{n+1} making the following diagram commute.

$$\begin{tikzcd}
& X_1^{n+1} \arrow[rr, "\kappa_1^{n+1}"] \arrow[dd, dotted, "\kappa_2^{n+1}" near start] & & X_1^n \arrow[dd, "\kappa_1^n"] \\
X_{n+1} \arrow[ur, "\varphi_n"] \arrow[rr, "\sigma_1^{n+1}" near start] \arrow[dd, "\sigma_2^{n+1}"] & & X_n \arrow[ur, "\varphi_n"] \arrow[dd, "\sigma_1^n" near start] & \\
& X_1^n \arrow[rr, dotted, "\kappa_1^n" near start] & & X_1^{n-1} \\
X_n \arrow[ur, dotted, "\varphi_n"] \arrow[rr, "\sigma_1^n"] & & X_{n-1} \arrow[ur, "\varphi_{n-1}"'] &
\end{tikzcd}$$

We need to show the following diagram commute for all i,

$$\begin{tikzcd}
X_{n+1} \arrow[r, "\varphi_{n+1}"] \arrow[d, "\sigma_j"] & X_1^{n+1} \arrow[d, "\kappa_j"] \\
X_n \arrow[r, "\varphi_n"] & X_1^n
\end{tikzcd}$$

For $j = 1,2$, this is the previous diagram (top face and left face). For $j \ge 3$, Let $\kappa'_j := \varphi \circ \sigma_j \circ \varphi^{-1} : X_{n+1} \to X_n$, to show $\kappa'_j = \kappa_j$, it is equivalent to show $\kappa_i \circ \kappa'_j = \kappa_i \circ \kappa_j$ for $i = 1,2$. Indeed, for $i = 1$, $\kappa_1 \circ \kappa'_j = \kappa_1 \circ \varphi \circ \sigma_j \circ \varphi^{-1} = \varphi \circ \sigma_1 \circ \sigma_j \circ \varphi^{-1} = \varphi \circ \sigma_{j-1} \circ \sigma_1 \circ \varphi^{-1} = \kappa_{j-1} \circ \varphi \circ \sigma_1 \circ \varphi^{-1} = \kappa_{j-1} \circ \kappa_1 = \kappa_1 \circ \kappa_j$. Same is true for $i = 2$. □

Let $\tau_0^0 : (0) \to (-1)$ in Δ_+^{op}. Let **Prestk** the category of functors from **Sp** to **∞-Grpd**. Combining the above lemma with etale descent, we have:

Proposition 2.8. *Let $X : \Delta_+^{op} \to$ **Prestk** be a functor, such that $X(i)$ is an analytic stack for all $i \in \Delta_+^{op}$, $X(D)$ is a fiber diagram for every elementary square D, and $X(\tau_0^0)$ is representable and etale surjective. Then X is a colimit diagram.* □

2.9. *The category $\Delta_{S,\Gamma}^{op}$.* Let S be a set, Γ a discrete group acting on S, then Γ acts on S^n diagonally. Let $S, S//\Gamma : \Delta^{op} \to$ **Set**, be simplicial set defined by $S(n) := S^{n+1}, S//\Gamma(n) := S^{n+1}//\Gamma$ with the natural morphisms. We have two categories $\Delta_S^{op}, \Delta_{S//\Gamma}^{op}$ (notation as in 2.2) and a functor $\mathbf{P} : \Delta_S^{op} \to \Delta_{S//\Gamma}^{op}$ induced by $S \to S//\Gamma$. Let $\Delta_{S,\Gamma}^{op}$ be the (2,1)-category defined by:
$Ob(\Delta_{S,\Gamma}^{op}) := Ob(\Delta_S^{op}) = \coprod_{k\in\mathbb{N}} S^k$,
For $\mathbf{s} \in S^k, \mathbf{s}' \in S^l$, $Mor_{\Delta_{S,\Gamma}^{op}}(\mathbf{s}, \mathbf{s}') := \{(\delta, a) \in Mor_{\Delta^{op}}([k],[l]) \times \Gamma : \delta(a(\mathbf{s}')) = \mathbf{s}\}$
$= \{(\delta, a) \in Mor_{\Delta^{op}}([k],[l]) \times \Gamma : \delta \in Mor_{\Delta_S^{op}}(\mathbf{s}', a^{-1}(\mathbf{s}))\}$.
For $(\delta, a), (\delta', a') \in Mor_{\Delta_{S,\Gamma}^{op}}(\mathbf{s}, \mathbf{s}')$,

$$2 - Mor((\delta, a), (\delta', a')) := \begin{cases} pt & \delta = \delta' \\ \varnothing & \delta \ne \delta' \end{cases}$$

Then there are natural commutative diagram of functors

$$\begin{tikzcd} & \Delta^{op}_{S,\Gamma} \arrow[dr, "\mathbf{F}"] \arrow[dr, phantom, "\simeq"', near start] & \\ \Delta^{op}_{S} \arrow[ur] \arrow[rr, "\mathbf{P}"] & & \Delta^{op}_{S//\Gamma} \end{tikzcd} \tag{2.10}$$

Where $\mathbf{F} : \Delta^{op}_{S,\Gamma} \to \Delta^{op}_{S//\Gamma}$ by $\mathbf{F}(\mathbf{s}) := [\mathbf{s}]$, $\mathbf{F}(\delta, a) := [\delta]$, $\mathbf{F}(2{-}morphisms) := id$ is an equivalence of $(2,1)$ categories. It can be viewed as a canonical way to glue up the "fibers" of $\mathbf{P}$.

$\Delta^{op}_{+,S,\Gamma}$ is defined analogously, replacing Δ^{op} by Δ^{op}_{+} in the above definition. We have $\Delta^{op}_{+,S,\Gamma} \simeq (\Delta^{op}_{S,\Gamma})_{+}$. Let $\Delta_{S,\Gamma} := (\Delta^{op}_{S,\Gamma})^{op}, \Delta_{+,S,\Gamma} := (\Delta^{op}_{+,S,\Gamma})^{op}$

Notation 2.11. Let $F : \Delta^{op}_{S,\Gamma}(resp.\ \ \Delta^{op}_{+,S,\Gamma}, \Delta_{S,\Gamma}, \Delta_{+,S,\Gamma}) \to \mathscr{T}$ be a functor. We shall denote $F = (F(\mathbf{s}), F(\epsilon), F(\eta))$, where ϵ is a general 1-morphism in $\Delta^{op}_{S,\Gamma}$, and η is a general 2-morphism in $\Delta^{op}_{S,\Gamma}$. Actually, all the 2-morphism $F(\eta)$ we are going to use is of the form $F^{h,g}_{f}$ as in (1.10). If no confusion arise, we shall suppress the 2-morphisms and denote $F = (F(\mathbf{s}), F(\epsilon))$.

Let U be a topological space, Γ a discrete group acting freely on U. Γ_1, Γ_2 two subgroup of Γ. U_1, U_2 two open subset of U stable under the action of Γ_1, Γ_2 respectively. For $U' \subset U, U^x := x(U), \Gamma' < \Gamma, \Gamma'^x := x\Gamma' x^{-1} < \Gamma$.
Pick a set of representative R for the double coset $\Gamma\backslash\backslash(\Gamma\times\Gamma)//(\Gamma_1\times\Gamma_2)$, for each $(x,y) \in R$, we have a commutative diagram in **Top**:

$$\begin{tikzcd} (U_1^x \cap U_2^y)/(\Gamma_1^x \cap \Gamma_2^y) \arrow[r] \arrow[d] & U_1/\Gamma_1 \arrow[d] \\ U_2/\Gamma_2 \arrow[r] & U/\Gamma \end{tikzcd}$$

Let $U_1 = U_2 = U$, we have

Lemma 2.12.

$$\begin{tikzcd} \coprod_{(x,y)\in R} U/(\Gamma_1^x \cap \Gamma_2^y) \arrow[r] \arrow[d] & U/\Gamma_1 \arrow[d] \\ U/\Gamma_2 \arrow[r] & U/\Gamma \end{tikzcd}$$

is a fiber diagram.

Proof: $U/\Gamma_1 \times_{U/\Gamma} U/\Gamma_2 \simeq (U \times_{U/\Gamma} U)/(\Gamma_1 \times \Gamma_2) = (\coprod_{\gamma\in\Gamma} U_\gamma)/(\Gamma_1\times\Gamma_2)$. Where $U_\gamma := \{(x, \gamma x) : x \in U\} \subset U \times U$. If $\beta = \gamma_2 \alpha \gamma_1^{-1}$, for $\alpha, \beta \in \Gamma, \gamma_i \in \Gamma_i$, then the action of (α, β) identifies U_α and U_β. And $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ stablize U_γ if and only if $\gamma\gamma_1 = \gamma_2\gamma$, all such elements can be identified with $\gamma\Gamma_1\gamma^{-1} \cap \Gamma_2$. Hence $(\coprod_{\gamma\in\Gamma} U_\gamma)/(\Gamma_1 \times \Gamma_2) \simeq \coprod_{\gamma\in\Gamma_1\backslash\Gamma/\Gamma_2} U/(\gamma\Gamma_1\gamma^{-1} \cap \Gamma_2) \simeq \coprod_{(x,y)\in R} U/(\Gamma_1^x \cap \Gamma_2^y)$. $\square$

Example 2.13. The following example is a prototype of the diagram we will define in next chapter. It explains the kind of combinatorics in $\Delta^{op}_{S,\Gamma}$. Let X be a smooth manifold, Γ a discrete group acting on X properly discontinuously. Let $X^{reg} \subset X$ be the open locus where the action of Γ is free. Let $\Gamma_x :=$ the stablizer of Γ at x. Then by the above Lemma, we have the fiber diagram:

$$\begin{tikzcd} \coprod_{(a,b)\in R} X^{reg}/(\Gamma_x^a \cap \Gamma_y^b) \arrow[r] \arrow[d] & X^{reg}/\Gamma_x \arrow[d] \\ X^{reg}/\Gamma_y \arrow[r] & X^{reg}/\Gamma \end{tikzcd} \tag{2.14}$$

Γ has induced action on X^n. For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X^n$, $\Gamma_{\mathbf{x}} :=$ the stablizer at $\mathbf{x}$. Then we have $\Gamma_{\mathbf{x}} = \bigcap_{i=1}^n \Gamma_{x_i}$, and $\Gamma_{a\mathbf{x}} = \Gamma_{\mathbf{x}}^a$.

Let $|X|$ be the underlining set of X, There is a natural functor $\mathbf{X}^{reg}_+ : \Delta^{op}_{+,|X|,\Gamma} \to$**Prestk** that canonically organize the above fiber diagram with higher descent data. And it is given by
$\mathbf{X}^{reg}_+(\mathbf{x}) := X^{reg}/\Gamma_{\mathbf{x}}, \mathbf{X}^{reg}_+(\epsilon = (\delta, a))$:=action by $a, Ac_a : X^{reg}/\Gamma_{\mathbf{x}} \to X^{reg}/\Gamma_{\epsilon(\mathbf{x})}$, For the unique 2-morphism η between $\epsilon = (\delta, a), \epsilon' = (\delta, a') \in Mor(\mathbf{x}_1, \mathbf{x}_2)$, we have $a'(\delta(\mathbf{x}_1)) = \delta(a'(\mathbf{x}_1)) = \mathbf{x}_2 = \delta(a(\mathbf{x}_1)) = a(\delta(\mathbf{x}_1))$, hence $aa'^{-1} \in \Gamma_{\mathbf{x}_2}$. It induces a natural isomoprhism between id and $Ac_{aa'^{-1}}$: $\mathbf{X}^{reg}_+(\mathbf{x}_2) \to \mathbf{X}^{reg}_+(\mathbf{x}_2)$. Composing with the 1-morphism $Ac_{a'} : \mathbf{X}^{reg}_+(\mathbf{x}_1) \to \mathbf{X}^{reg}_+(\mathbf{x}_2)$ gives the desired natural isomorphism $\mathbf{X}^{reg}_+(\eta) : \mathbf{X}^{reg}_+(\delta, a) \to \mathbf{X}^{reg}_+(\delta, a')$. In view of Notation 2.11, we defined a functor $\mathbf{X}^{reg}_+ = (X^{reg}/\Gamma_{\mathbf{x}}, Ac_a, Ac_\eta) : \Delta^{op}_{+,|X|,\Gamma} \to$ **Prestk**, then $\mathbf{X}^{reg}_+$ is a colimit diagram.

We can strength this functor to give a presentation of X/Γ. For $x \in |X|$, let $X^r_{\mathbf{x}} := \{y \in X : \Gamma_y \subset \Gamma_{\mathbf{x}}\}$. Then we have $Ac_a : X^r_{\mathbf{x}}/\Gamma_{\mathbf{x}} \to X^r_{\epsilon(\mathbf{x})}/\Gamma_{\epsilon(\mathbf{x})}$. Define $\mathbf{X}^r_+ = (X^r_{\mathbf{x}}, Ac_a) : \Delta^{op}_{+,|X|,\Gamma} \to$ **Prestk**, then we have similarly:

$$\mathbf{X}^r_+ \text{ is a colimit diagram.} \tag{2.15}$$

Note that $|X|$ could be replace by any Γ invariant subset S, such that $\bigcup_{s\in S} X^r_s = X$. In reasonable situation, we should expect to find such S being discrete in X, and $S//\Gamma$ is finite.

Remark: In fact, in this example, one could just take the whole X instead of $X^r_{\mathbf{x}}$ in the above definition, and one would still get a colimit diagram. However, this is no longer the case for the actual situation we shall consider in next chapter.

Chapter 3

Twisted conjugation of loop group

In this chapter, we shall study G_E for $E = \mathbb{C}^*/q^{\mathbb{Z}}$ using the uniformization $\mathrm{Bun}_G(E) = L_{hol}G/{}^qL_{hol}G$. We shall work in the context of complex analytic stacks, see Appendix A for the facts we will use.

3.1 Automorphism groups

In this section, we calculate automorphism groups of semisimple bundles. The result is given in Corollary 3.10. The result was also obtained in [3, Theorem 5.6]. The approaches on computing connected components are a bit different.

Let $q \in \mathbb{C}^*$ with $|q| < 1, E = \mathbb{C}^*/q^{\mathbb{Z}}$ be an elliptic curve, $0 \in E$ be the identity. Let G a connected reductive algebraic group, with Lie algebra $\mathfrak{g}$, T maximal torus with Lie algebra $\mathfrak{t}.X_*(T) = Hom(\mathbb{C}^*, T), \Lambda = \{x(q)|x \in X_*(T)\} \subset T$, Let G_E be the connected component of the stack of semistable $G-$bundles on E. Let $L_{hol}G$ be the holomorphic loop group of G, $L_{hol}G$ acts on itself by twisted conjugation $Ad^q_{k(z)}g(z) \coloneqq k(qz)g(z)k(z)^{-1}$, unless otherwise mentioned, we will use this twisted conjugation of $L_{hol}G$ instead of the usual conjugation through out Chapter 3.

For any $g(z) \in L_{hol}G$, we can define a G bundle $\mathcal{P}_{g(z)} = \mathbb{C} \times_{q^{\mathbb{Z}}} G$ on E, where the action of $q^{\mathbb{Z}}$ is given by $q(z,x) = (qz, g(z)x)$. Then $Aut(\mathcal{P}_{g(z)})$ can be identified as the subgroup $\{k(z)|k(qz) = g(z)k(z)g(z)^{-1}\} = C_{L_{hol}G}(g(z))$ of $L_{hol}G$. I,e we can pullback the automorphism of $\mathcal{P}_{g(z)}$ to a automorphism of trivial bundle on $\mathbb{C}^*$ compatible with $q^{\mathbb{Z}}$ action, and it is given by the above formula. In another word, $L_{hol}G/L_{hol}G = \mathrm{Bun}_G(E)(\mathbb{C})$ as a groupoid.

For $s \in T$, Let $G_s \coloneqq C_{L_{hol}G}(s)$. It has a twisted conjugation on itself. Indeed, for $f(z), g(z) \in G_s, f(qz)g(z)f(z)^{-1} \in G_s$ by direct calculation. For the moment, G_s is just a set. In the rest of this chapter, we will calculate G_s explicitly and gives it a geometric structure. We shall do $G = GL_N$ first. Let $T_N \subset GL_N$ be the invertible diagonal matrices.

Lemma 3.1. *Let $f : \mathbb{C}^* \to \mathbb{C}$ be a holomorphic function. Assume that $f(qz) = af(z)$ for some $a \in \mathbb{C}$. If $a \in q^{\mathbb{Z}}$, then $f(z) = cz^n$, for some $c \in \mathbb{C}$ and $n = log_q(a)$; Otherwise, $f(z) \equiv 0$.*

Proof: Use the Laurant expansion of f and compare the coefficients. □

Proposition 3.2. *Let $s = diag(\lambda_1, \lambda_2, \dots, \lambda_N) \in T_N$, under the embedding $GL_N \to \mathfrak{gl}_N$, $GL_{N,s}$ consists of those invertible matrices in $\mathfrak{gl}_{N,s} := \bigoplus_{(i,j)} \mathbb{C}z^{n_{ij}}E_{ij}$, for those (i,j) such that $\lambda_i/\lambda_j \in q^{\mathbb{Z}}$, where $n_{ij} = \log_q(\lambda_i/\lambda_j)$, E_{ij} is the elementary matrix of (i,j) entry.*

Proof: Let $g(z) \in GL_{N,s}$, embed GL_N into $\mathfrak{gl}_N$ and write $g(z)$ in terms of coordinate function $g_{ij}(z)$, then $g(qz) = s \cdot g(z) \cdot s^{-1}$ is equivalent to $g_{ij}(qz) = (\lambda_i/\lambda_j)g_{ij}(z)$ for all (i,j). $g_{ij}(z)$ is holomorphic on $\mathbb{C}^*$. By last Lemma, $g_{ij}(z) = cz^{n_{ij}}$ if $\lambda_i/\lambda_j = q^{n_{ij}}$ and $g_{ij}(z) = 0$ otherwise. In particular, $g_{ii}(z) \in \mathbb{C}$ is some constant. □

Corollary 3.3. *$GL_{N,s}$ is Zariski closed in some finite dimensional stratum of $L_{poly}GL_N$. With the reduced subscheme structure, $GL_{N,s}$ is an reductive algebraic group, the twisted conjugation on itself is an algebraic action, and the evaluation map*

$$ev_1 : GL_{N,s} \to GL_N$$

$$g(z) \mapsto g(1)$$

is injective homomorphism of algebraic groups. Furthermore, the Lie algebra of $GL_{N,s}$ is naturally identified with $\mathfrak{gl}_{N,s} \subset L_{poly}\mathfrak{gl}_N$.

Now, for general reductive algebraic group with maximal torus (G,T), pick an embedding $i : (G,T) \to (GL_N, T_N)$. Then $LG \subset LGL_N$ and $G_s \subset GL_{N,s}$. We have $G_s = L_{hol}G \cap GL_{N,s} = L_{poly}G \cap GL_{N,s}$. So G_s is Zariski closed both in $GL_{N,s}$ and $L_{poly}G$.

Proposition 3.4. *G_s is Zariski closed in some finite dimensional stratum of $L_{poly}G$. With reduced subscheme structure, G_s is an reductive algebraic group with maximal torus T, the twisted conjugation on itself is an algebraic action. And the evaluation map $ev_1 : G_s \to G$ is an injective homomorphism of algebraic groups, and $p : G_s \to Aut(\mathcal{P}_s)$ is an isomorphism of algebraic groups.*

View G_s as a algebraic subgroup of $GL_{N,s}$. The Lie algebra $\mathfrak{g}_s = L_{poly}\mathfrak{g} \cap \mathfrak{gl}_{N,s}$. More explicitly, it is given by:

Proposition 3.5. *Let $s \in T$, then $\mathfrak{g}_s = \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha z^{n_\alpha}$, for those root α such that $\alpha(s) \in q^{\mathbb{Z}}$, where $n_\alpha = \log_q(\alpha(s))$.*

Proof: $\mathfrak{gl}_{N,s}$ is a finite dimensional subalgebra of $L_{poly}\mathfrak{gl}_N$ satisfying $\{X(z) \in L\mathfrak{gl}_N | X(qz) = Ad(s)X(z)\}$. So $\mathfrak{g}_s = L_{poly}\mathfrak{g} \cap \mathfrak{gl}_{N,s} = \{X(z) \in L\mathfrak{g} | X(qz) = Ad(s)X(z)\}$. Write $X(z) = h(z) + \sum_{\alpha \in \Phi} f_\alpha(z)$, with respect to the root decomposition of $\mathfrak{t}$, i.e $h(z) : \mathbb{C}^* \to \mathfrak{t}$, $f_\alpha(z) : \mathbb{C}^* \to \mathfrak{g}_\alpha$. Now the condition $X(qz) = Ad(s)X(z)$ is equivalent to $h(qz) = h(z)$, and $f_\alpha(qz) = \alpha(s)f_\alpha(z)$. Let $f_\alpha(z) = \sum_{n=-K}^{K} a_n z^n X_\alpha$, where X_α is the root vector of α. the above condition become $\sum_{n=-K}^{K} a_n q^n z^n = \sum_{n=-K}^{K} a_n \alpha(s) z^n$ compare coefficients, get the only nonvanishing f_α are those with $\alpha(s) = q^{n_\alpha}$ for some $n_\alpha \in \mathbb{Z}$ and in this case $f_\alpha = z^{n_\alpha} X_\alpha$. $h(z)$ corresponds to the case $\alpha(s) = 1$ so $h(z)$ is constant function. □

Corollary 3.6. *G_s^0 is generated by $T, exp(\mathbb{C}X_\alpha z^{n_\alpha})$, for those α and n_α as above.*

Example 3.7. $G = SL_2, T$ diagonal matrices, with roots $\{\alpha, -\alpha\}$ take $s = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & \sqrt{q}^{-1} \end{pmatrix} \in T$, hence $\alpha(s) = \sqrt{q}/(\sqrt{q}^{-1}) = q$, so $n_\alpha = 1$, similarly $n_{-\alpha} = -1$,
we have $X_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, then by the proposition, G_s^0 is generated by $T, \exp\begin{pmatrix} 0 & bz \\ 0 & 0 \end{pmatrix}$, $\exp\begin{pmatrix} 0 & 0 \\ cz^{-1} & 0 \end{pmatrix}$ which equal to the subgroup $\begin{pmatrix} a & bz \\ cz^{-1} & d \end{pmatrix}$ of LG. In fact, in this case we have $G_s^0 = G_s = C_{LG}(s)$. □

Let M, N be C^∞/complex manifolds, let $\mathcal{O}(M,N)$ be the set of C^∞/holomorphic maps between M, N.

Lemma 3.8. *Assume M is connected, $T \subset G \subset \mathcal{O}(M,G)$ be the constant maps, then $N_{\mathcal{O}(M,G)}(T) = \mathcal{O}(M,T) \cdot N_G(T)$ as subgroups of $\mathcal{O}(M,G)$.*

Proof: Let $f(x) \in N_{\mathcal{O}(M,G)}(T)$, then $f(x)Tf(x)^{-1} = T$ for any $x \in M$, hence $f(x) \in \mathcal{O}(M, N_G(T))$. We get $N_{\mathcal{O}(M,G)}(T) \subset \mathcal{O}(M, N_G(T))$. Since M is connected, $\mathcal{O}(M, N_G(T)) = \mathcal{O}(M,T) \cdot N_G(T)$. Now $\mathcal{O}(M,T) \subset N_{\mathcal{O}(M,G)}(T)$, and $N_G(T) \subset N_{\mathcal{O}(M,G)}(T)$. Hence $\mathcal{O}(M,T) \cdot N_G(T) \subset N_{\mathcal{O}(M,G)}(T)$. □

Pick a lifting of W to $N_G(T) \subset G$ as a subset, For $w \in \widetilde{W}$, there is unique way to write $w = u\mu$, for $u \in W, \mu \in X_*(T)$. So we have a lifting of $\widetilde{W}$ to $L_{hol}G$. Denote $\dot{w}$ the lifting of $w \in \widetilde{W}$.

Lemma 3.9. $(L_{hol}T \cdot N_G(T)) \cap G_s = T \cdot C_{\widetilde{W}}(s)$ *as subgroups of* $L_{hol}G$.

Proof: The right hand side does not depend on the lifting and equals $(X_*(T) \cdot N_G(T)) \cap G_s$ which naturally sits inside the left hand side. Need to show that $(L_{hol}T \cdot N_G(T)) \cap G_s \subset X_*(T) \cdot N_G(T)$. Now suppose $fw \in (L_{hol}T \cdot N_G(T)) \cap G_s$, for $f \in L_{hol}T, w \in N_G(T)$. we have $s = f \cdot (w(s))$. However, $w(s), s \in T$, this implies $f \in X_*(T)$ by Lemma 3.1. □

Let $W_s := C_{\widetilde{W}}(s)$, by Lemma 3.8, 3.9, we have $N_{G_s}(T) = N_{L_{hol}G}(T) \cap G_s = (L_{hol}T \cdot N_G(T)) \cap G_s = T \cdot W_s$ and the weyl group $W(G_s, T) := N_{G_s}(T)/T \simeq W_s$. now $T \subset G_s$ is a maximal torus, hence we have

Corollary 3.10. $G_s = G_s^0 \cdot N_{G_s}(T) =< G_s^0, \dot{w} : w \in W_s >=< T, exp(\mathbb{C}X_\alpha z^{n_\alpha}), \dot{w} : \alpha(s) = q^{n_\alpha}, n_\alpha \in \mathbb{Z}, w \in W_s >$ □

Example 3.11. $G = PGL_2, T$ diagonal matrices, with roots $\{\alpha, -\alpha\}$ take $s = \begin{pmatrix} \sqrt{q} & 0 \\ 0 & 1 \end{pmatrix}$ We have $G_s^0 = T$, $G_s = < T, w > \simeq T \rtimes \mathbb{Z}/2$, where $w = \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$. □

Remark: Same method can be use to calculate automorphism group of all semisimple bundles. We will not use them in this thesis.

The following proposition can be check directly.

Proposition 3.12. *The left multiplication by $s^{-1} : G_s \to G_s$ is a G_s-equivariant isomorphism of algebraic varieties, where the first action is q-twisted conjugation, and second action is usual conjugation. Hence we have an isomorphism of stacks $s^{-1} : G_s/G_s \to G_s/_{ad}G_s$.* □

G_s^0 is stable under the twisted conjugation of G_s. Hence G_s^0/G_s has the usual properties of adjoint quotient (on its identity component.). Hence we get the following corollaries.

Corollary 3.13. $\mathbb{C}[G_s^0]^{G_s} = \mathbb{C}[T]^{W_s}$ □

So there is a map $\chi_s : G_s^0 \to T//W_s$. Let $U \subset T$ be an W_s invariant open subset, let $V := \chi_s^{-1}(U//W_s)$

Definition 3.14. Let S be a topological space, $A \subset S$ a subset, A is *abundant* (in S) if the only open subset of S containing A is S.

Examples of abundant subsets we will use is given in Corollary 3.15. It's follows that $A \subset S$ is abundant if and only if for any $s \in S$, $\overline{\{s\}} \cap A$ is nonempty.

For a groupoid $\mathscr{C}$, let $|\mathscr{C}|$ be the set of isomorphism classes in $\mathscr{C}$. For an analytic stack $\mathscr{X}$, $|\mathscr{X}| := |\mathscr{X}(\mathbb{C})|$ has a natural topology comming from (representable) smooth morphisms, i.e, the image of a (representable) smooth map from complex analytic space are open.

Corollary 3.15. *The image of U in $|V/G_s|$ is abundant.* □

Let T^{s-reg} be the locus where the action of W_s is free.

Corollary 3.16. *Assume further that $U \subset T^{s-reg}$, then $V/G_s \xleftarrow{\simeq} U/N_{G_s}(T) \xrightarrow{\simeq} (U \times BT)/W_s$* □

In particular, let T^{q-reg} be the locus where the action of $\widetilde{W}$ is free. Let $G_s^{0,q-reg} := \chi_s^{-1}(T^{q-reg}//W_s)$. We have,

Corollary 3.17. $G_s^{0,q-reg}/G_s \xleftarrow{\simeq} T^{q-reg}/N_{G_s}(T) \xrightarrow{\simeq} (T^{q-reg} \times BT)/W_s$ □

3.2 Etale charts

In this section, we will define some étale charts of G_E. The main results in this section are Proposition 3.23 and its strengthen version Theorem 3.27. Facts about semistable bundles on elliptic curves are collected in Appendix B.

There are three mutually commuting action of $q^{\mathbb{Z}}, G_s, G$ on $\mathbb{C}^* \times G_s^0 \times G$:

$$\begin{aligned} q \cdot (z, h, g) &:= (qz, h, h(z)g), && q \in q^{\mathbb{Z}} \\ k \cdot (z, h, g) &:= (z, Ad_k^q(h), k(z)g), && k \in G_s \\ g' \cdot (z, h, g) &:= (z, h, gg'^{-1}), && g' \in G \end{aligned}$$

Let $\mathscr{P}_s := (\mathbb{C}^* \times G_s^0 \times G)/q^{\mathbb{Z}}$. $\mathscr{P}_s$ maps naturally to $E \times G_s^0$ which is a G_s^0 equivariant (with respect to the twisted conjugation on G_s^0) principal G-bundle .

1-morphisms. There is a map $p_s : G_s^0/G_s \to \mathrm{Bun}_G(E)$ by: given a S point of G_s^0/G_s

$$\begin{tikzcd} P \arrow[r] \arrow[d] & G_s^0 \\ S & \end{tikzcd}$$

Where P is an G_s-bundles on S, and $P \to G_s^0$ is a G_s equivariant map. We have

$$\begin{tikzcd} Y := (E \times P) \times_{E \times G_s^0} \mathscr{P}_s \arrow[r] \arrow[d] & \mathscr{P}_s \arrow[d] \\ E \times P \arrow[r] \arrow[d] & E \times G_s^0 \\ E \times S & \end{tikzcd}$$

Y has an induced transitive G_s action, such that $Y \to E \times P$ is G_s equivariant, And the induced map $Y/G_s \to E \times P/G_s = E \times S$ is a principal G bundle. So we get an S point of $\mathrm{Bun}_G(E)$.

2-morphisms. The lifting $\dot{w} = \dot{w}(z)$ of $w \in \widetilde{W}$ induces $Ad_{\dot{w}}^q : G_s^0 \to G_{w(s)}^0$. There is commutative diagram:

$$\begin{tikzcd} \mathbb{C} \times G_s^0 \times G \arrow[r, "\varphi_w", "\simeq"'] \arrow[d] & \mathbb{C} \times G_{w(s)}^0 \times G \arrow[d] \\ \mathbb{C} \times G_s^0 \arrow[r, "Id \times Ad_{\dot{w}}^q", "\simeq"'] & \mathbb{C} \times G_{w(s)}^0 \end{tikzcd}$$

Where $\varphi_{\dot{w}}(z,h,g) := (z, Ad^q_{\dot{w}}(h), \dot{w}(z)g)$. The diagram is $q^{\mathbb{Z}}$-equivariant and hence induces:

$$\begin{tikzcd}
\mathscr{P}_s \arrow[r, "\varphi_{\dot{w}}", "\simeq"'] \arrow[d] & \mathscr{P}_{w(s)} \arrow[d] \\
E \times G^0_s \arrow[r, "Id \times Ad^q_{\dot{w}}", "\simeq"'] & E \times G^0_{w(s)}
\end{tikzcd}$$

And induces an isomorphism between the diagrams below, intertwining the G_s action and $G_{w(s)}$ action, factorial in S:

$$\begin{tikzcd}
(E \times P) \times_{E \times G^0_s} \mathscr{P}_s \arrow[r] \arrow[d] & \mathscr{P}_s \arrow[d] & \overset{\varphi_{\dot{w}}}{\longrightarrow} & (E \times P_{\dot{w}}) \times_{E \times G^0_{w(s)}} \mathscr{P}_{w(s)} \arrow[r] \arrow[d] & \mathscr{P}_{w(s)} \arrow[d] \\
E \times P \arrow[r] \arrow[d] & E \times G^0_s & & E \times P_{\dot{w}} \arrow[r] \arrow[d] & E \times G^0_{w(s)} \\
E \times S & & & E \times S &
\end{tikzcd}$$

Hence $\varphi_{\dot{w}} : p_s \Rightarrow p_{w(s)} \circ Ad^q_{\dot{w}} : G^0_s/G_s \to G_E$ an isomorphism between the morphisms of stacks.

Proposition 3.18. *Let $G_E := \mathrm{Bun}_G(E)^{0,ss}$, the image of p_s lies in G_E.*

Proof: Since G^0_s/G_s is connected, the image lies in $\mathrm{Bun}_G(E)^0$. $T \subset G^0_s$ maps to the degree 0 semisimple semistable bundles. By Lemma 3.15, Any other point in G^0_s has closure containing points in T. So by Proposition B.2. G^0_s maps to $G_E \subset \mathrm{Bun}_G(E)^0$. $\square$

Let $G_{E,0}$ be the stack classifying $\{(\mathcal{P}, \beta)\}$, where $\mathcal{P}$ is a semistable G bundle of degree 0, β is an trivialization of $\mathcal{P}$ at $0 \in E$. $G_{E,0}$ is representable by Proposition B.6. There is a natural map $p'_s : G^0_s \to G_{E,0}$ defined by $\mathscr{P}_s$ with the natural trivialization by identifing the fiber over $0 \in E$ with the fiber over $1 \in \mathbb{C}^*$. p'_s is G_s equivariant, where the G_s acts on $G_{E,0}$ via $ev_1 : G_s \to G$ and G acts via change of trivialization. So p'_s induces $\overline{p'_s} : G^0_s/G_s \to G_{E,0}/G_s$. Hence $\overline{p'_s}$ is representable. We have the following commutative diagram of stacks:

$$\begin{tikzcd}
G^0_s/G_s \arrow[r, "p'_s"] \arrow[drr, "p_s"'] & G_{E,0}/G_s \arrow[r] & G_{E,0}/G \arrow[d, "\simeq"'] \\
& & G_E
\end{tikzcd}$$

By Proposition 3.4, $ev_1 : G_s \to G$ is injective, so the top arrows are representable and hence:

Proposition 3.19. *p_s is representable.* $\square$

Let $G = T$ be a torus, $T^0_s = T$, $p_s : T/T \to T_E$ does not depend on s, denoted by p_T. Λ acts freely via twisted conjugation on the fibers of p_T, so we have a map $Ab : (T/T)/\Lambda = T/\Lambda \times BT \to T_E$. By following proposition, we can be view p_T as the uniformization in the abelian case.

Proposition 3.20. *Ab is an isomorphism.*

Proof: It suffices to prove the map Ab' induced by Ab, $Ab' : T/\Lambda \to T_{E,0}$ is an isomorphism. Two T bundles with trivialization are the same if and only if the underline bundles are isomorphic. Any element $s \in T$ is twisted conjugation in LT to 1 if and only if it is in Λ, So we have Ab' is injective, but T/Λ is proper, any injective map to another proper irreducible variety of the same dimension must be surjective. □

Given any G bundle $\mathcal{P}$ on a space X, V a representation of G. Let $V_{\mathcal{P}} := \mathcal{P} \times_G V$ be the associated vector bundle. Then $V_{\mathcal{P}}$ has a natural $Aut(\mathcal{P})$ action (the action preserve fibers). In particular, $Aut(\mathcal{P})$ acts on $H^i(X, V_{\mathcal{P}})$. Let V^* be the dual representation. Then we have an canonical isomorphism $(V_{\mathcal{P}})^\vee \simeq (V^*)_{\mathcal{P}}$ which commute with $Aut(\mathcal{P})$ action on both sides. $H^{-1}(T_{\mathcal{P}}\mathrm{Bun}_G(C)) = H^0(C, \mathfrak{g}_{\mathcal{P}}) = Lie(Aut(\mathcal{P}))$. And $H^0(T_{\mathcal{P}}\mathrm{Bun}_G(C)) = H^1(C, \mathfrak{g}_{\mathcal{P}})$. We have $H^{-1}(T_{\mathcal{P}}\mathrm{Bun}_G(C)) = H^0(C, \mathfrak{g}_{\mathcal{P}})$, $H^0(T_{\mathcal{P}}\mathrm{Bun}_G(C)) = H^1(C, \mathfrak{g}_{\mathcal{P}})$ and the action of $Aut(\mathcal{P})$ is identified as the above action for $V = \mathfrak{g}$.

Let H be an algebraic group, with Lie algebra $\mathfrak{h}$, $\mathcal{P}$ a H bundle on E. We have $H^0(E, \mathfrak{h}_{\mathcal{P}}) \xrightarrow{a} H^0(E, (\mathfrak{h}_{\mathcal{P}})^\vee)^* \xrightarrow{\simeq} H^1(E, \mathfrak{h}_{\mathcal{P}})$, Where the second isomorphism is Serre duality and all arrows are $Aut(\mathcal{P})$ module map. When H is a torus, $\mathfrak{h}_{\mathcal{P}}$ is trivial, and hence a is an isomorphism. Let $b : H^0(E, \mathfrak{h}_{\mathcal{P}}) \to H^1(E, \mathfrak{h}_{\mathcal{P}})$ be the composition of the above two maps.

Let $\varphi : H \to K$ be a homomorphism of algebraic group, we have a morphism between smooth stacks $()_K : H_E \to K_E$ by assigning a H bundle $\mathcal{P}$ its associated K bundle $\mathcal{P}_K = \mathcal{P} \times_H K$, we want to calculate the differential of this map.

On the other hand, we have a natural H module map $d\varphi : \mathfrak{h} \to \mathfrak{k}$, this gives $\varphi_* : \mathfrak{h}_{\mathcal{P}} \to \mathfrak{k}_{\mathcal{P}} \simeq \mathfrak{k}_{\mathcal{P}_K}$ vector bundle map, hence give $H^1(\varphi_*) : H^1(E, \mathfrak{h}_{\mathcal{P}}) \to H^1(E, \mathfrak{k}_{\mathcal{P}_K})$.

Lemma 3.21. *The following diagram commutes.*

$$\begin{tikzcd}
H^0(E, \mathfrak{h}_{\mathcal{P}}) \arrow[r, "H^0(\varphi_*)"] \arrow[d, "b"] & H^0(E, \mathfrak{k}_{\mathcal{P}_K}) \arrow[d, "b"] \\
H^1(E, \mathfrak{h}_{\mathcal{P}}) \arrow[r, "H^1(\varphi_*)"] \arrow[d, "\simeq"] & H^1(E, \mathfrak{k}_{\mathcal{P}_K}) \arrow[d, "\simeq"] \\
H^0(T_{\mathcal{P}} H_E) \arrow[r, "H^0(d()_{K,\mathcal{P}})"] & H^0(T_{\mathcal{P}_K} K_E)
\end{tikzcd}$$

Moreover, it interwine the $Aut(\mathcal{P})$ action on the left and $Aut(\mathcal{P}_K)$ action on the right. And b is isomorphism if H is a torus and $\mathcal{P}$ is an H-bundle of degree 0 . □

Let Let $T_s^{et} := \{t \in T : G_t \subset G_s\}$, then T_s^{et} is an W_s invariant open subset of T, $G_s^{0,et} := \chi_s^{-1}(T_s^{et}//W_s)$.

Proposition 3.22. *$p_s : G_s^0/G_s \to G_E$ is etale at t, for $t \in T_s^{et}$.*

Proof: For $t \in T_s^{et}, G_t \subset G_s$, then $G_t = (C_{LG}(t) \cap G_s) = C_{G_s}(t)$. So p_s induce $Aut(t) = C_{G_s}(t) \to Aut(\mathcal{P}_t)$ which give isomorphism. So $H^{-1}(dp_{s,t}) : H^{-1}(T_t G_s/G_s) \to H^{-1}(T_{\mathcal{P}_t} G_E)$ is an isomorphism. To calculate $H^0(dp_{s,t})$, we use the following diagram:

$$\begin{tikzcd} T/T \arrow[r] \arrow[d, "p_T"] & G_s^0/G_s \arrow[d, "p_s"] \\ T_E \arrow[r] & G_E \end{tikzcd}$$

The induced map on H^0 of tangent groupoid is:

$$\begin{tikzcd} \mathfrak{t} \arrow[r] \arrow[d, "\simeq"] & H^0(T_t(G_s^0/G_s)) \arrow[d, "H^0(dp_{s,t})"] \\ H^0(E, \mathfrak{t}_{\mathcal{P}_t}) \arrow[r] & H^0(E, \mathfrak{g}_{\mathcal{P}_t}) \end{tikzcd}$$

Where the last line is identified via Lemma 3.21. The left arrow is isomorphism due to Proposition 3.20.

The top arrow is injective. The bottom row is inclusion of $\mathfrak{t}$ into $H^0(E, \mathfrak{g}_{\mathcal{P}_t}) = Span\{\mathfrak{t}, z^{n_\alpha} X_\alpha\}$. So the image of $H^0(dp_{s,t})$ contain the Cartan subalgebra $\mathfrak{t}$ of $H^0(E, \mathfrak{g}_{\mathcal{P}_t})$. But the tangent map is Aut equivalent, so we get the image of $H^0(dp_{s,t})$ contains $Aut^0(\mathcal{P}_t)\mathfrak{t} = H^0(E, \mathfrak{g}_{\mathcal{P}_t})$, hence $H^0(dp_{s,t})$ is surjective. The two H^0 of tangent groupoid have the same dimension(both of them have dimension = dimension of G_t), so it is isomorphism. □

Proposition 3.23. *$p_s : G_s^{0,et}/G_s \to G_E$ is etale. And $p : \coprod_{s \in T} G_s^{0,et}/G_s \to G_E$ is surjective.*

Proof: The first assertion follows from the above propsition and T_s^{et} is abundant in $|G_s^{0,et}/G_s|$ and the fact that etale locus is open. For the second assertion, the image of p contains all semisimple bundles, by Proposition B.2 the set of semisimple bundles is abundant in $|G_E|$, and p has open image since it is etale, so we conclude that p is surjective. □

Let X be a algebraic variety, $Aut(K)$ acts on $\mathrm{Bun}_K(X)$ via twisting the action of K on a K bundle $\mathcal{P}$ by group automorphism. The action induced by $Inn(G)$ is canonical ismorphic to identity morphism on $\mathrm{Bun}_G(X)$. Hence $Out(G)$ acts on $\mathrm{Bun}_G(X)$. Let $H < K$, $\mathrm{Bun}_H(X) \to \mathrm{Bun}_K(X)$ is $N_K(H)$ equivariant, and $N_K(H)$ action on $\mathrm{Bun}_K(X)$ is canonical isomorphic to identity since $N_K(H) < K$ acts via inner automorphism. Take $X = E, K = G, H = T$, then $W = N_G(T)$ acts on T_E. Let T_E^{reg} be the locus where this action is free (on the set of points). So we have $T_E^{reg}/W \to G_E$, which factor through $T_E^{reg}/W \to G_E^{reg} \subset G_E$, where G_E^{reg} is the open substack of G_E consisting of regular semisimple bundles with *connected* automorphism group. And $T_E^{reg}/W \to G_E^{reg}$ is an isomorphism.

Using the identification $T_E^{reg}/W \simeq (T^{q-reg}/\Lambda \times BT)/W \simeq (T^{q-reg} \times BT)/\widetilde{W}$, we have the commutative diagram:

$$
\begin{tikzcd}
(T^{q-reg} \times BT)/W_s \arrow[r, "\simeq"] \arrow[d] & G_s^{0,q-reg}/G_s \arrow[d, "p_s"] \\
(T^{q-reg} \times BT)/\widetilde{W} \arrow[r, "\simeq"] & G_E^{reg}
\end{tikzcd}
\tag{3.24}
$$

Definition 3.25. Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism between analytic stacks, we say f is a *local isomorphism* if for any $\mathbb{C}$-point x of $\mathscr{X}$, there is an open substack $\mathscr{U}$ of $\mathscr{X}$, $x \in \mathscr{U}$, such that $f|_{\mathscr{U}} : \mathscr{U} \to \mathscr{Y}$ is an open embedding.

Local isomorphism is a property stable under base change, but it does not satisfy descent. If $f : \mathscr{X} \to \mathscr{Y}$ is local isomorphism, then the induced map $f : |\mathscr{X}| \to |\mathscr{Y}|$ is local homeomorphism, and $f_* : Aut(x) \to Aut(f(x))$ is an isomorphism. The following Lemma will be used in 3.3.

Lemma 3.26. *Let $f : \mathscr{X} \to \mathscr{Z}, g : \mathscr{Y} \to \mathscr{Z}$ be local isomorphisms of analytic stacks, then the natural map $h : |\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}| \to |\mathscr{X}| \times_{|\mathscr{Z}|} |\mathscr{Y}|$ is a homeomorphism.*

Proof: need to show that h is bijective and local homeomorphism.

$$
\begin{tikzcd}
|\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y}| \arrow[dr, dotted, "h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& |\mathscr{X}| \times_{|\mathscr{Z}|} |\mathscr{Y}| \arrow[r] \arrow[d] & |\mathscr{X}| \arrow[d, "f"] \\
& |\mathscr{Y}| \arrow[r, "g"] & |\mathscr{Z}|
\end{tikzcd}
$$

All real arrows are local homeomorphisms, so h is also a local hemeomorphism. Surjectivity is clear, and the fiber of h over (x, y) is identified with
$f_*(Aut(x))\backslash\backslash Isom(f(x), g(y))//g_*(Aut(y))$ is a point if either $f_*(Aut(x)) = Aut(f(x))$ or $g_*(Aut(y)) = Aut(g(y))$, which is satisfied in our situation. □

The following stronger version of Proposition 3.23 will also be used in 3.3.

Theorem 3.27. *$p_s : G_s^{0,et}/G_s \to G_E$ is a local isomorphism. And $p : \coprod_{s \in T} G_s^{0,et}/G_s \to G_E$ is surjective.*

Proof: Let $r : T^{q-reg}/W_s \to T^{q-reg}/\widetilde{W}$. For $t \in T_s^{et}$, we have $W_t \subset W_s$, hence $W_t = C_{\widetilde{W}}(t) = C_{W_s}(t)$, so there is an W_s invariant open neighborhood U of t, such that $r|_{(U \cap T^{q-reg})/W_s}$ maps isomorphically onto image. Take $V = \chi_s^{-1}(U//W_s)$, we have $p_s|_V$ is etale and generically open, so it is open embedding by Lemma A.2. □

For $\mathbf{s} = (s_1, s_2, \ldots, s_k) \in T^k$, let $G_{\mathbf{s}} \coloneqq \bigcap_{i=1}^k G_{s_i}$. $W_{\mathbf{s}} \coloneqq N_{G_{\mathbf{s}}}(T)/T$, define $\chi_{\mathbf{s}}, T_{\mathbf{s}}^{et}, G_{\mathbf{s}}^{et}$ analogously, We have all the above statement for G_s still holds for $G_{\mathbf{s}}$, and $W_{\mathbf{s}} = \bigcap_{i=1}^k W_{s_i}, T_{\mathbf{s}}^{et} = \bigcap_{i=1}^k T_{s_i}^{et}$. $T^{q-reg} \subset T_{\mathbf{s}}^{et}$ for all $\mathbf{s}$ by the connectedness of G.

3.3 Gluing

In this section, we will glue the charts defined in section 3.2, i.e we will calculate the fiber products of the charts. The combinatorics of higher descent data is naturally organized in a similar diagram as in Example 2.13. The main result of this chapter is Theorem 3.28, which is analogous to (2.15).

For $s \in T$, $w = u\lambda \in \widetilde{W} = W \ltimes X_*(T)$. Since $G_{\dot{w}(s)} = C_{LG}(\dot{w}(s)) = Ad_{\dot{w}}C_{LG}(s)$, so we have a isomorphism of algebraic groups: $Ad_{\dot{w}} : G_s \to G_{w(s)}$ where $Ad_{\dot{w}}$ is the *usual* conjugation of $\dot{w}$ in LG.

$Ad^q_{\dot{w}}(G_s) = \dot{u}\lambda(qz)G_s\lambda(z)^{-1}\dot{u}^{-1} = \dot{u}\lambda(z)\lambda(q)G_s\lambda(z)^{-1}\dot{u}^{-1} = Ad_{\dot{w}}(G_s) = G_{w(s)}$ since $\lambda(q) \in T \subset G_s$. Since $Ad^q_{\dot{w}}$ stablize T, we have $Ad^q_{\dot{w}} : G^0_s \to G^0_{w(s)}$ isomorphism of algebraic varieties. The pair of isomorphism of algebraic varieties and algebraic group $(Ad^q_{\dot{w}}, Ad_{\dot{w}}) : (G^0_s, G_s) \to (G^0_{w(s)}, G_{w(s)})$ interwine the twisted conjugation action on both sides. Hence we have an induced isomorphism of quotient stacks, still denoted by $Ad^q_{\dot{w}} : G^0_s/G_s \to G^0_{w(s)}/G_{w(s)}$. It's also easy to see that the above map takes etale locus to etale locus, so we have $Ad^q_{\dot{w}} : G^{0,et}_s/G_s \to G^{0,et}_{w(s)}/G_{w(s)}$. Similarly, we can replace s by any element $\mathbf{s} \in T^k$.

With the notation in 2.9, 2.11, let $\mathbf{G}^{et} := ({G_{\mathbf{s}}}^{0,et}/G_{\mathbf{s}}, Ad^q_{\epsilon}) : \Delta^{op}_{|T|,\widetilde{W}} \to \mathbf{Prestk}$. Where $Ad^q_{\epsilon=(\sigma,w)} := Ad^q_{\dot{w}} : {G_{\mathbf{s}_1}}^{0,et}/G_{\mathbf{s}_1} \to {G_{\mathbf{s}_2}}^{0,et}/G_{\mathbf{s}_2}$. The augment 1-morphisms $p_{\mathbf{s}} : {G_{\mathbf{s}}}^{0,et}/G_{\mathbf{s}} \to G_E$ and augment 2-morphisms $\varphi_{\dot{w}}$ extend $\mathbf{G}^{et}$ to a functor $\mathbf{G}^{et}_+ : \Delta^{op}_{+,|T|,\widetilde{W}} \to \mathbf{Prestk}$.

Theorem 3.28. *$\mathbf{G}^{et}_+$ is a colimit diagram.*

Proof: Let $\mathbf{S}$ be a section of $\mathbf{F} : \Delta^{op}_{+,|T|,\widetilde{W}} \to \Delta^{op}_{+,|T|//\widetilde{W}}$, $\mathbf{K}_+ := \mathbf{G}^{et}_+ \circ \mathbf{S}$. To show $\mathbf{G}^{et}_+$ is a colimit diagram, since $\mathbf{S}$ is an equivalence, it suffices to show $\mathbf{K}_+$ is so. Then by (2.4), it suffices to show $\mathbf{K}_{+,|T|//\widetilde{W}} : \Delta^{op}_+ \to \mathbf{Prestk}$ is a colimit diagram. By Lemma 2.8, it suffices to show $\mathbf{K}_{+,|T|//\widetilde{W}}(D)$ is fiber diagram, and $\mathbf{K}_{+,|T|//\widetilde{W}}(\tau^0_0)$ is etale surjection. The latter one follows from Propsition 3.23. By Proposition 2.6, what remains to show is the following:

Proposition 3.29. *For every triple $(b,c,d) \in T^k//\widetilde{W} \times T^k//\widetilde{W} \times T^{k-1}//\widetilde{W}$, such that $\delta_j(b) = \delta_{i+1}(c) = d$, Let $R_{b,c} := \{a \in T^{k+1}//\widetilde{W} : \delta_i(a) = b, \delta_j(a) = c.\}$, the following (2-)commutative diagram is a fiber diagram:*

$$\begin{tikzcd} \coprod_{a\in R_{b,c}} \mathbf{G}^{et}_+(a) \arrow[r] \arrow[d] & \mathbf{G}^{et}_+(b) \arrow[d] \\ \mathbf{G}^{et}_+(c) \arrow[r] & \mathbf{G}(d) \end{tikzcd} \tag{3.30}$$

For $a \in T^l//\widetilde{W}$, $a^\# := \mathbf{S}(a)$, and for $\mathbf{s} \in T^l, [\mathbf{s}] := \mathbf{F}(\mathbf{s})$. Then $\delta_j(b) = \delta_{i+1}(c) = d$ implies that, there exist $u, v \in \widetilde{W}$ such that $\delta_j(ub^\#) = \delta_{i+1}(vc^\#) = d^\#$, then there is unique $a_1 \in T^{k+1}$ such that, $\delta_i(a_1) = b^\#$, $\delta_j(a_1) = c^\#$. Since $W_{\mathbf{s}} = C_{\widetilde{W}}(\mathbf{s})$ we have a bijection of sets

$$W_{d^\#}\backslash\backslash(W_{d^\#} \times W_{d^\#})//(W_{b^\#} \times W_{c^\#}) \xrightarrow{\sim} R_{b,c}$$

by $(w,w') \mapsto (w,w') \cdot a^{\#}$ where the action of the first factor is by acting on those coordinate coming from $b^{\#}$, the second factor is by acting on those coordinate coming from $c^{\#}$.

Proof of Proposition 3.29: We will only prove it for $k = 1$, other cases are similar. In this case, $R_{b,c} = \widetilde{W}\backslash\backslash(\widetilde{W} \times \widetilde{W})//(W_{b'} \times W_{c'})$ the diagram (3.30) become:

$$\begin{tikzcd}
\coprod_{(w,w')\in\widetilde{W}\backslash\backslash(\widetilde{W}\times\widetilde{W})//(W_{b^{\#}}\times W_{c^{\#}})} G^{0,et}_{[(w,w')a_1]^{\#}}/G_{[(w,w')a_1]^{\#}} \arrow[r] \arrow[d] & G^{0,et}_{b^{\#}}/G_{b^{\#}} \arrow[d] \\
G^{0,et}_{c^{\#}}/G_{c^{\#}} \arrow[r] & G_E
\end{tikzcd}$$

This give a map

$$A : \coprod_{(w,w')\in\widetilde{W}\backslash\backslash(\widetilde{W}\times\widetilde{W})//(W_{b^{\#}}\times W_{c^{\#}})} G^{0,et}_{[(w,w')a_1]^{\#}}/G_{[(w,w')a_1]^{\#}} \to G^{0,et}_{b^{\#}}/G_{b^{\#}} \times_{G_E} G^{0,et}_{c^{\#}}/G_{c^{\#}}$$

By Lemma A.2, to show A is an isomorphism, it is equivalent to show that A is etale, generically open and surjetive on the set of $\mathbb{C}$-points. A is etale since all maps in the diagram are etale. Take the generic q-reg locus in the above diagram:

$$\begin{tikzcd}
\coprod_{(w,w')\in\widetilde{W}\backslash\backslash(\widetilde{W}\times\widetilde{W})//(W_{b^{\#}}\times W_{c^{\#}})} G^{0,q-reg}_{[(w,w')a_1]^{\#}}/G_{[(w,w')a_1]^{\#}} \arrow[r] \arrow[d] & G^{0,q-reg}_{b^{\#}}/G_{b^{\#}} \arrow[d] \\
G^{0,q-reg}_{c^{\#}}/G_{c^{\#}} \arrow[r] & G^{reg}_E
\end{tikzcd}$$

By (3.24), this is same the the diagram

$$\begin{tikzcd}
\coprod_{(w,w')\in\widetilde{W}\backslash\backslash(\widetilde{W}\times\widetilde{W})//(W_{b^{\#}}\times W_{c^{\#}})} T^{q-reg} \times BT/W_{[(w,w')a_1]^{\#}} \arrow[r] \arrow[d] & T^{q-reg} \times BT/W_{b^{\#}} \arrow[d] \\
T^{q-reg} \times BT/W_{c^{\#}} \arrow[r] & T^{q-reg} \times BT/\widetilde{W}
\end{tikzcd}$$

We have $W_{(w,w')a_1} = W(b^{\#})^w \cap W(c^{\#})^{w'}$, So by Lemma 2.12 the above diagram is a fiber diagram. So A is generically open.

It remains to show that A is surjective on $\mathbb{C}$ points. By Lemma 3.26 and Theorem 3.27, $h : |G^{0,et}_{b^{\#}}/G_{b^{\#}} \times_{G_E} G^{0,et}_{c^{\#}}/G_{c^{\#}}| \xrightarrow{\sim} |G^{0,et}_{b^{\#}}/G_{b^{\#}}| \times_{|G_E|} |G^{0,et}_{c^{\#}}/G_{c^{\#}}|$ is a homeomorphism. Let $A' := h \circ A$, A is etale and therefore A' has open image. Two elements in $s, t \in T$ defines the same bundle in G_E if and only if $\exists w \in \widetilde{W}$, such that $w(s) = t$. Hence the semisimple locus $|T^{et}_{b^{\#}}|//W_{b^{\#}} \times_{|G_E|} |T^{et}_{c^{\#}}|//W_{c^{\#}} \subset |G^{0,et}_{b^{\#}}/G_{b^{\#}}| \times_{|G_E|} |G^{0,et}_{c^{\#}}/G_{c^{\#}}|$ is in the image of A', where $|T^{et}_{b^{\#}}|//W_{b^{\#}} \subset |G^{0,et}_{b^{\#}}/G_{c^{\#}}|$ denote the image of $|T^{et}_{b^{\#}}| \to |G^{0,et}_{b^{\#}}/G_{b^{\#}}|$. To show the surjectivity of A' and hence of A, we are left to show:

Proposition 3.31. *The set* $|T^{et}_{b'}|//W_{b'} \times_{|G_E|} |T^{et}_{c'}|//W_{c'} \subset |G^{0,et}_{b'}/G_{b'}| \times_{|G_E|} |G^{0,et}_{c'}/G_{c'}| := S$ *is abundant.*

Proof: Take $(g,h) \in S$, i,e. $g \in G^{0,et}_{b'}, h \in G^{0,et}_{c'}$,such that $\mathcal{P}_g \simeq \mathcal{P}_h$ then we have $s \in \overline{\{g\}}, t \in \overline{\{h\}}$ semisimple elements. Then $\mathcal{P}_s, \mathcal{P}_t$ is the semisimplification of $\mathcal{P}_g$, $\mathcal{P}_h$ respectively, by Proposition B.2(2b) $\mathcal{P}_s \simeq \mathcal{P}_t$ and hence $(s,t) \in S$. And by product topology, $(s,t) \in \overline{\{(g,h)\}}$. Hence we have prove Proposition 3.29. And therefore finish the prove of Theorem 3.28. □

As an immediate corollary of Propsition 3.29, we gives a explicit description of fiber product of two local charts.

Corollary 3.32. *For $s,t \in T$, Let $B := \{(w,w') \in \widetilde{W}\backslash\backslash(\widetilde{W} \times \widetilde{W})//(W_s \times W_t)\}$*

$$\begin{tikzcd}[column sep=small]
& \coprod_{(w,w')\in B}(G_{w^{-1}(s)} \cap G_{w'^{-1}(t)})^{0,et}/(G_{w^{-1}(s)} \cap G_{w'^{-1}(t)}) \arrow[dl, "Ad^q_w"'] \arrow[dr, "Ad^q_{w'}"] & \\
G^{0,et}_s/G_s \arrow[dr, "p_s"'] & \square & G^{0,et}_t/G_t \arrow[dl, "p_t"] \\
& G_E &
\end{tikzcd}$$

is a fiber diagram. □

Remark 3.33. In fact, the etale covers and glueing hold in more general context. Take $S \subset |T|$ any $\widetilde{W}$-invariant subset. And for given $V_s \subset T^{et}_s$ open W_s-invariant subset for each $s \in S$, such that $w(V_s) = V_{w(s)}$ and $\coprod_{s\in S} V_s = T$. Let $V_{\mathbf{s}} := \cap V_{s_i}, U_{\mathbf{s}} :=^{G_{\mathbf{s}}} V_{\mathbf{s}}$. We have the functor

$$\mathbf{U} := (U_{\mathbf{s}}/G_{\mathbf{s}}, Ad^q_\epsilon) : \Delta^{op}_{S,\widetilde{W}} \to \mathbf{Prestk}.$$

Let $\mathbf{U}_+$ be the augmented functor with $\mathbf{U}_+(-1) = G_E$. Then by the same reasoning:

$$\mathbf{U}_+ \text{ is a colimit diagram.}$$

Besides the one given in Theorem 3.28, there are another two useful choices of such functor as follows:
(1) Let $s \in T$, s is called a *vertex* if G_s is maximal among $G_t, t \in T$ as subset of LG. Let $\Xi := \{s \in T, s \text{ is a vertex}\}$. Then $\Xi/Z(G)$ is discrete in $T/Z(G)$, And $\Xi/(Z(G)\times\widetilde{W})$ is finite. Then

$$\mathbf{G}^{et} := (G^{0,et}_{\mathbf{s}}/G_{\mathbf{s}}, Ad^q_\epsilon) : \Delta^{op}_{\Xi,\widetilde{W}} \to \mathbf{Prestk}$$

satisfies the above property. In this way, G_E is essentially covered by finitely many charts.
(2) Take $S = |T|$, $U_s = D_s$ a small W_s-invariant ball near s, such that the augmentation maps ${}^{G_s}D_s/G_s \to G_E$ are open embedding. We have the functor

$$\mathbf{D} := ({}^{G_{\mathbf{s}}}D_{\mathbf{s}}/G_{\mathbf{s}}, Ad^q_\epsilon) : \Delta^{op}_{|T|,\widetilde{W}} \to \mathbf{Prestk}$$

This functor has the advantage that its local topology is easy to study. A combination of the functor in (1) and (2) in Lie algebra version will be used in Chapter 5 and 6. See (5.12),(6.1).

Chapter 4

Character sheaves on Lie algebras

The theory of character sheaves on a reductive algebraic group G was introduced in a series of papers by Lusztig [17, 18, 19, 20, 21]. In characteristic p, it play important roles in the representation theory of the finite group $G(\mathbb{F}_p)$. In particular, Lusztig show that under Grothendieck sheaf-function correspondence, the (irreducible) character sheaves gives an orthonormal basis for the space of class functions on $G(\mathbb{F}_p)$. In characteristic 0, [24, 12] show that (irreducible) character sheaves agree with (irreducible) adjoint-equivariant perverse sheaves with singular support in nilpotent cone. Parallel theory of character sheaves on reductive Lie algebras was also developed in [22, 25].

In this chapter, we shall focus on the microlocal description of nilpotent singular support, to study the category of character sheaves of a reductive Lie algebra and various restriction functors between them. The viewpoint in [13] is also helpful for us. Results in this chapter serve as local geometry for Chapter 5 and 6.

4.1 A rescaling lemma

We start with some preliminaries on (constructible) sheaves. The reference we use is [15]. Let k be a commutative ring with finite global dimension, X an analytic stack, and k_X be the constant sheaf on X. Let $Sh(X) = Sh(X, k_X)$ be the ∞-category of sheaves of k_X-modules. For any $F \in Sh(X)$, we can associate to it the singular support $SS(F)$ of F, where $SS(F)$ is a conical closed coisotropic subset of $H^0(T^*X)$. Many properties of the sheaves can be seen from the microlocal measurement. For example, F is constructible if and only if $SS(F)$ is Lagrangian subvariety and F_x is perfect for all $x \in X$. And a constructible sheaf F is perverse if and only if the microlocal stalk F_ξ vanish outside degree 0, for ξ in the smooth locus of $SS(F)$.

Let $\mathcal{L} \subset H^0(T^*X)$ a closed conical isotropic subset. Let $Sh_{\mathcal{L}}(X) \coloneqq \{F \in Sh(X) : SS(F) \subset \mathcal{L}\}$, $D^b_{\mathcal{L}}(X) \coloneqq \{F \in Sh_{\mathcal{L}}(X) : F_x \text{ is perfect, for any } x \in X.\}$ the ∞-category of constructible sheaves, and $\mathrm{Perv}_{\mathcal{L}}(X) \coloneqq \{F \in D^b_{\mathcal{L}}(X) : F \text{ is perverse.}\}$ the abelian category of perverse sheaves. Our results in the remaining of the thesis will be stated for $Sh_{\mathcal{L}}(X)$, but they work equally well for $D^b_{\mathcal{L}}(X)$ and $\mathrm{Perv}_{\mathcal{L}}(X)$.

Let $f : U \to X$ a smooth map, $Sh_{\mathcal{L}}(U) := Sh_{df^*(\mathcal{L})}(U)$. By the theory of descent, the functor $Sh_{\mathcal{L}} : X_{sm} \to \infty\textbf{-Cat}$, where $U \mapsto Sh_{\mathcal{L}}(U)$, is a sheaf of categories. In particular, if f is surjective, the natural map $Sh_{\mathcal{L}}(X) \to \lim_{\Delta} Sh_{\mathcal{L}}(U^{\bullet}_X)$ is an equivalence.

Let A be a Lie group acting on a smooth manifold X, $\mathcal{L} \subset T^*X$ be a closed A-invariant conical isotropic subset. Let $\pi : X \to X/A$ and $\mu : T^*X \to \mathfrak{g}^*$ the moment map, then $\pi^*(T^*(X/A)) = \mu^{-1}(0) = \coprod_{x \in X} T^*_{Ax}X$, and

Proposition 4.1. *$\mathcal{L}$ is contained in $\mu^{-1}(0)$.*

Proof: Suffices to check on the smooth locus of $\mathcal{L}$, where it follows from definition of isotropic submanifold. □

Proposition 4.2. *Let $U \subset X$ open subset, $F \in Sh_{\mathcal{L}}(U)$, then $F|_{Ax \cap U}$ is locally constant for all $x \in U$.*

Proof: $SS(F) \subset \mathcal{L} \subset \pi|_U^*(T^*(X/A))$, hence by [15, Prop. 6.6.2], in a neighborhood of $x \in U$, $F = \pi|_U^*(F')$, for some $F' \in Sh(X/A)$. □

Lemma 4.3. *Let X be a smooth manifold with $\mathbb{R}^+$ action, $\mathcal{L}$ be a closed biconical isotropic subset of T^*X. Let $j : U \to X$ open embedding, such that $U \cap \mathbb{R}^+x$ is contractible, for any $x \in X$, then the restriction functor:*

$$j^* : Sh_{\mathcal{L}}(X) \to Sh_{\mathcal{L}}(U)$$

is an equivalence of categories.

Proof: By Proposition 4.2, any $F \in Sh_{\mathcal{L}}(X)$ is conic, i.e. satisfies $F|_{\mathbb{R}^+x}$ is locally constant for all $x \in X$. Hence for $F_1, F_2 \in Sh_{\mathcal{L}}(X)$ by [15, Prop. 3.7.4(iii), Cor. 3.7.3], $Hom_{Sh_{\mathcal{L}}(X)}(F_1, F_2) \xrightarrow{\simeq} Hom_{Sh_{\mathcal{L}}(U)}(j^*F_1, j^*F_2)$ is an isomorphism, hence j^* is fully faithful. Let $F' \in Sh_{\mathcal{L}}(U)$, the by Proposition 4.2, for any $x \in U$, $F'|_{\mathbb{R}^+x \cap U}$ is locally constant. We have natural maps

$$\begin{tikzcd}
U \times \mathbb{R}^+ \arrow[d, "\tilde{j}"] \arrow[dr, "a'"] & U \arrow[l, "i'"'] \arrow[d, "j"] \\
X \times \mathbb{R}^+ \arrow[r, shift left=2, "a"] \arrow[r, shift right=2, "p"'] & X \arrow[l, "i"]
\end{tikzcd}$$

Where $\tilde{j} = j \times Id$, a is the action map, p is the projection, i(*resp.* i') is inclusion to X(*resp.* U) $\times \{1\}$. $a' := a \circ \tilde{j} : U \times \mathbb{R}^+ \to X$, then a' is $\mathbb{R}^+$-equivariant, has contractible fibers and $F' \boxtimes k_{\mathbb{R}^+}$ is constructible along the fibers of a'. $F := a'_*(F' \boxtimes k_{\mathbb{R}^+}) \in Sh(X)$, then F is conic by [15, Prop. 3.7.4(ii)] . There is a chain of isomorphisms

$$F' \simeq i'^*(F' \boxtimes k_{\mathbb{R}^+}) \simeq i'^*a'^*(F) \simeq i'^*\tilde{j}^*a^*(F) \simeq i'^*\tilde{j}^*p^*(F) \simeq j^*(F) \tag{4.4}$$

where the second isomorphism is by [15, Prop. 2.7.8] , the fourth isomorphism is by [15, Prop. 3.7.2] . a' is smooth and surjective, and $SS(a'^*F) = SS(F' \boxtimes k_{\mathbb{R}^+}) \subset \mathcal{L} \times T^*_{\mathbb{R}^+}\mathbb{R}^+ = a'^*(\mathcal{L})$ since $\mathcal{L}$ is biconical. Hence $SS(F) \subset \mathcal{L}$ by descent [15, Prop. 5.4.5] . Combining with (4.4), j^* is essentially surjective. □

4.2 Base restriction

Let K be a connected complex reductive group with $Lie(K) = \mathfrak{k}$, T maximal torus of K, $\mathfrak{t} := Lie(T)$. Let $\mathfrak{c}_{\mathfrak{k}}$ be the center of $\mathfrak{k}$.

A completely invariant open subset U of $\mathfrak{k}$ is by definition a K-invariant open subset such that $U \cap \mathfrak{t}$ is abundant in $|U/K|$. There is one to one correspondence between:

$$\{\text{completely invariant subsets of } \mathfrak{k}\} \longleftrightarrow \{W(K,T) \text{ invariant subsets of } \mathfrak{t}\}$$

$$U \longmapsto U \cap \mathfrak{t}$$

$${}^K V \longleftarrow\!\shortmid V$$

Where ${}^K V :=$ the minimal complete invaraint open subset containing V (such thing exists), the correspondence preserve the property of being star-shape at $C \in \mathfrak{c}_{\mathfrak{k}}$.

Let $\mathcal{N} = \mathcal{N}_{\mathfrak{k}} \subset H^0(T^*(\mathfrak{k}/K))$ be the nilpotent cone, i.e $\mathcal{N}_{\mathfrak{k},X} := \{Y \in \mathfrak{k}^* : [X,Y] = 0, Y \text{ is nilpotent.}\}$, where Y is nilpotent in $\mathfrak{k}^*$ if it is nilpotent under the identifcation $\mathfrak{k}^* \simeq \mathfrak{k}$ given by some/any non-degenerate invariant bilinear form on $\mathfrak{k}$. We will use nilpotent cones lying in different cotangent bundles in the thesis. When the context is understood, we shall drop the indices and write all nilpotent cones as $\mathcal{N}$.

Corollary 4.5. *Let U be a open invariant subset of $\mathfrak{k}$, such that there is some $C \in \mathfrak{c}_{\mathfrak{k}}$ with U being star-shaped centered at C. Then the restriction functor:*

$$j^* : Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(U/K)$$

is an equivalence of categories.

Proof: This follows from Lemma 4.3 since $\mathcal{N}$ is biconical with respect to the $\mathbb{R}^+$ action on $\mathfrak{k}$ centered at C and also the fact that this $\mathbb{R}^+$ action commute with conjugation by K. Note also that the set of all such C is contractible. □

Remark: $Sh_{\mathcal{N}}(\mathfrak{k}/K)$ is by definition the category of character sheaves on $\mathfrak{k}$. Fourier transform gives an equivalence $Fr : Sh_{\mathcal{N}}(\mathfrak{k}/K) \xrightarrow{\simeq} Sh(\mathcal{N}/K)$. The latter category is studied in the generalized Springer theory initiated in [23]. For characteristic 0 coefficient, $Sh(\mathcal{N}/K)$ is explicitly calculated in [27, 28]. For characteristic p coefficient, the abelian category $\mathrm{Perv}(\mathcal{N}/K)$ is the subject of recent developed modular generalized Springer theory [1].

Let L be a Levi subgroup of K containing T and $\mathfrak{l} := Lie(L)$. Let κ be a non-degenerate invariant bilinear form of $\mathfrak{k}$, then $\kappa|_{\mathfrak{l}\times\mathfrak{l}}$ is also a non-degenerate invariant bilinear form on $\mathfrak{l}$.

For any $X \in \mathfrak{k}$, $T_X(\mathfrak{k}/K) = \mathfrak{k} \xrightarrow{[X,-]} \mathfrak{k}$, in degree $-1, 0$. Under $\mathfrak{k}^* \simeq \mathfrak{k}$ by κ, $T^*_X(\mathfrak{k}/K) = \mathfrak{k} \xrightarrow{[X,-]} \mathfrak{k}$ (in degree $0, 1$) $= T_X(\mathfrak{k}/K)[-1]$. Let $f : \mathfrak{l}/L \to \mathfrak{k}/K$, $Y \in \mathfrak{l}$, then $df_Y : T_X(\mathfrak{l}/L) \to T_X(\mathfrak{k}/K)$ and $df^*_Y : T_X(\mathfrak{k}/K)[-1] \to T_X(\mathfrak{l}/L)[-1]$. A direct computation shows that $H^0(df^*_Y)|_{\mathfrak{l}} = Id_{\mathfrak{l}}$. Let $\mathfrak{l}^r = \mathfrak{l}^r_{\mathfrak{k}} := \{Z \in \mathfrak{l} : C_K(Z) = C_L(Z)\}$.

Proposition 4.6. *$f|_{\mathfrak{l}^r/L} : \mathfrak{l}^r/L \to \mathfrak{k}/K$ is etale and $df|^*_{\mathfrak{l}^r/L}$ respects nilpotent singular support. I.e, for $Y \in \mathfrak{l}^r, \xi \in H^0(T_Y(\mathfrak{k}/K))$, we have $H^0(df^*_Y)(\xi) \in \mathcal{N}_{\mathfrak{l}}$ if and only if $\xi \in \mathcal{N}_{\mathfrak{k}}$.*

Proof: Let $Y \in \mathfrak{l}^r$, suffices to prove df_Y is a quasi-isomorphism. I.e $H^{-1}(df_Y)$ and $H^0(df_Y)$ are isomorphisms. $H^{-1}(df_Y)$ is isomorphism because $Y \in \mathfrak{l}^r$ and by definition, $H^{-1}(T_Y(\mathfrak{l}/L)) = \{Z \in \mathfrak{l} : [Y, Z] = 0\} = \{Z \in \mathfrak{k} : [Y, Z] = 0\} = H^{-1}(T_Y(\mathfrak{k}/K))$. $H^0(df_Y)$ being isomorphism is equivalent to $H^0(df^*_Y)$ being isomorphism. Again by $Y \in \mathfrak{l}^r$ we have $H^0(T^*_Y(\mathfrak{k}/K)) \subset \mathfrak{l}$. Then the fact that $H^0(df^*_Y)|_{\mathfrak{l}} = Id$ proves the proposition. □

Definition 4.7. A *base open subset* U of $\mathfrak{l}$ is invariant open subset of $\mathfrak{l}^r$ satisfing the assumption in Corollary 4.5. For such U, define the *base restriction* $R_U : Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(\mathfrak{l}/L)$ by the following commutative diagram:

$$\begin{tikzcd} Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[d, "R_U"] \arrow[dr, "f|^*_U"] & \\ Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, "j^*", "\simeq"'] & Sh_{\mathcal{N}}(U/L) \end{tikzcd} \tag{4.8}$$

For V an W_L-invariant convex open subset of $\mathfrak{t} \cap \mathfrak{l}^r$ with $V \cap \mathfrak{c}_{\mathfrak{l}} \neq \varnothing$, ${}^L V$ is a base open subset of $\mathfrak{l}$, and put $R_V := R_{{}^L V}$ Let $U \subset U'$ both base open subset star-shape centered at C, the natural map $R_{U'} \Rightarrow R_U$ is an isomorphism. For $x \in \mathfrak{c}^r_{\mathfrak{l}} := \mathfrak{c}_{\mathfrak{l}} \cap \mathfrak{l}^r$, put $R_x := \lim_{x \in U} R_U$, where U range over all base open subset star-shape centered at x. Then for any such U, the natural map $R_U \Rightarrow R_x$ is an isomorphism.

Let $\mathfrak{k} \subset \mathfrak{g}$ Levi subalgebra. Let V_1 an W_K-invariant convex open subset of $\mathfrak{t} \cap \mathfrak{k}^r_{\mathfrak{g}}$ with $V_1 \cap \mathfrak{c}_{\mathfrak{k}} \neq \varnothing$, V_2 an W_L-invariant convex open subset of $\mathfrak{t} \cap \mathfrak{l}^r_{\mathfrak{k}}$ with $V_2 \cap \mathfrak{c}_{\mathfrak{l}} \neq \varnothing$. Then we have $R_{V_1} : Sh_{\mathcal{N}}(\mathfrak{g}/G) \to Sh_{\mathcal{N}}(\mathfrak{k}/K)$ and $R_{V_2} : Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(\mathfrak{l}/L)$.

The set $V := V_1 \cap V_2$ is W_L-invraint convex open subset of $\mathfrak{t} \cap \mathfrak{l}^r_{\mathfrak{g}}$, assume further that $V \cap \mathfrak{c}_{\mathfrak{l}} \neq \varnothing$. Then we have $R_V : Sh_{\mathcal{N}}(\mathfrak{g}/G) \to Sh_{\mathcal{N}}(\mathfrak{l}/L)$. There is a natural isomorphism

$$R_{V_2} \circ R_{V_1} \Rightarrow R_V \tag{4.9}$$

Given by the following diagram:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}/G) \arrow[d] \arrow[dr] & & \\
Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[r, "\simeq"'] \arrow[d] \arrow[dr] & Sh_{\mathcal{N}}({}^K V_1/K) \arrow[dr] \arrow[d, phantom, "\circlearrowleft"] & \\
Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, "\simeq"'] & Sh_{\mathcal{N}}({}^L V_2/L) \arrow[r, "\simeq"'] & Sh_{\mathcal{N}}({}^L V/L)
\end{tikzcd}$$

Let P be a parabolic subgroup of K containing L, $\mathfrak{p} := Lie(P)$. We have

$$\begin{tikzcd}
\mathfrak{l}/L \arrow[r, shift left, "i"] & \mathfrak{p}/P \arrow[l, shift left, "q"] \arrow[r, "p"] & \mathfrak{k}/K
\end{tikzcd}$$

Let $\mathrm{Res}_{\mathfrak{p}} := i^! p^*$ be the hyperbolic restriction functor. By similar argument as in Proposition 4.6 the map $i|_{\mathfrak{l}^r_{\mathfrak{k}}/L} : \mathfrak{l}^r/L \to \mathfrak{p}/P$ is also etale, Let $\mathfrak{p}^r \subset \mathfrak{p}$ such that $\mathfrak{p}^r/P = i(\mathfrak{l}^r_{\mathfrak{k}}/L)$. Then we have the commutative diagram, with the top arrows are etale morphisms.

$$\begin{tikzcd}
\mathfrak{l}^r_{\mathfrak{k}}/L \arrow[r, "i", "\simeq"'] \arrow[d] & \mathfrak{p}^r/P \arrow[r, "p"] \arrow[d] & \mathfrak{k}/K \\
\mathfrak{l}/L \arrow[r, "i"] & \mathfrak{p}/P \arrow[ur, "p"] &
\end{tikzcd}$$

Hence we have

$$\mathrm{Res}_{\mathfrak{p}}|_{\mathfrak{l}^r_{\mathfrak{k}}/L} \simeq f^*|_{\mathfrak{l}^r_{\mathfrak{k}}/L} \tag{4.10}$$

$qi = Id_{\mathfrak{l}/L}$, The contraction principle gives: $i^! \simeq q_! i_! i^! \xrightarrow{\simeq} q_!$, See [15, Proposition 3.7.5] hence $\mathrm{Res}_{\mathfrak{p}} \xrightarrow{\simeq} q_! p^*$.

The parabolic restriction $q_! p^*$ commute with the the Fourier transform [25, Lemma 4.2]. And also $q_! p^* : Sh(\mathcal{N}_{\mathfrak{k}}/K) \to Sh(\mathcal{N}_{\mathfrak{l}}/L)$. So we have parabolic restriction preserve nilpotent singular support, we get:

$$\mathrm{Res}_{\mathfrak{p}} : Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(\mathfrak{l}/L)$$

Take any based open subset U of $\mathfrak{l}$, by (4.10) and Corollary 4.5, there is commutative diagram:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[d, "\mathrm{Res}_{\mathfrak{p}}"] \arrow[dr, "f|_U^*"] & \\
Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, "\simeq"] & Sh_{\mathcal{N}}(U/L)
\end{tikzcd} \tag{4.11}$$

Compare with (4.8), we have a naturally commutative diagram:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[r, "id"'] \arrow[d, "\mathrm{Res}_{\mathfrak{p}}"] & Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[d, "R_U"] \\
Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, "id"'] & Sh_{\mathcal{N}}(\mathfrak{l}/L)
\end{tikzcd} \tag{4.12}$$

4.3 Real structure

There is a natural identification of parabolic subalgebras with certain base open subsets. We show that under this identification, hyperbolic restriction and based restriction are isomorphic compactible with higher transitivity.

The identification $X_*(T)\otimes\mathbb{C}\to\mathfrak{t}$, by $(\lambda,c)\mapsto d\lambda(c)$ gives $\mathfrak{t}_{\mathbb{R}}:=X_*(T)\otimes\mathbb{R}\hookrightarrow\mathfrak{t}$. Then all the weights take real value on $\mathfrak{t}_{\mathbb{R}}$. Let $\mathfrak{c}_{\mathfrak{l}}^r=\mathfrak{c}_{\mathfrak{l}\subset\mathfrak{k}}^r:=\mathfrak{c}_{\mathfrak{l}}\cap\mathfrak{l}_{\mathfrak{k}}^r$, and $\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r=\mathfrak{c}_{\mathfrak{l}\subset\mathfrak{k},\mathbb{R}}^r:=\mathfrak{c}_{\mathfrak{l}}\cap\mathfrak{l}_{\mathfrak{k}}^r\cap\mathfrak{t}_{\mathbb{R}}$. We have $\mathfrak{t}_{\mathbb{R}}=\coprod_{\mathfrak{l}\supset\mathfrak{t}}\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r=\coprod_{\mathfrak{l}\supset\mathfrak{t}}\coprod_{C\subset\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r\text{ a component}}C$. This defines a stratification on $\mathfrak{t}_{\mathbb{R}}$ by simplices. This stratification is actually the stratification of $\mathfrak{t}_{\mathbb{R}}$ by facets. For $\alpha\in\Phi(K,T)$ a root, let $\mathfrak{k}_\alpha$ be the corresponding root space.

Proposition 4.13. *The assignment $\mathfrak{p}:\mathfrak{t}_{\mathbb{R}}\to\{\textit{Subalgebras of }\mathfrak{k}\}$,*

$$x\mapsto\mathfrak{p}_x:=\mathfrak{t}\oplus\oplus_{\alpha\in\Phi(K,T),\alpha(x)\geq0}\mathfrak{k}_\alpha$$

satisfying:
(1) $\mathfrak{p}$ is constant on each facet C. Let $\mathfrak{p}_C:=\mathfrak{p}_x$, for some $x\in C$.
(2) If $C\neq C'$, then $\mathfrak{p}_C\neq\mathfrak{p}_{C'}$.
(3) If $C\subset\bar{C}'$, then $\mathfrak{p}_C\supset\mathfrak{p}_{C'}$.
(4) $\mathfrak{p}_C$ is a parabolic subalgebra of $\mathfrak{k}$ with Levi factor $\mathfrak{l}=\mathfrak{c}_{\mathfrak{k}}(C)$.
(5) The induced map $\{\textit{Facets of }\mathfrak{t}_{\mathbb{R}}\}\to\{\textit{Parabolic subalgebra of }\mathfrak{k}\textit{ containing }\mathfrak{t}\}$ is a bijection.

Corollary 4.14. *$\mathfrak{p}$ induces a bijiection of sets between:*
$$\{\textit{Components of }\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r\}\leftrightarrow\mathcal{P}_{\mathfrak{k},\mathfrak{l}}:=\{\textit{Parabolic subalgebras of }\mathfrak{k}\textit{ with Levi }\mathfrak{l}\}$$

This bijection of sets can be thought as homotopy equivalence between $K(\pi,0)$ spaces: $\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r\xrightarrow{\simeq}\mathcal{P}_{\mathfrak{k},\mathfrak{l}}$.

We shall also provide another identification with $\mathcal{P}_{\mathfrak{k},\mathfrak{l}}$: Let $\mathfrak{p}\in\mathcal{P}_{\mathfrak{k},\mathfrak{l}},C_{\mathfrak{p}}$ be the corresponding component in $\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r$. Let $V_{\mathfrak{p},\mathbb{R}}\subset\mathfrak{t}_{\mathbb{R}}$ be the union of all facets in $\mathfrak{t}_{\mathbb{R}}$ whose closure intersect $C_{\mathfrak{p}}$. In formula,

$$V_{\mathfrak{p},\mathbb{R}}=\bigcap_{\alpha\in\Phi(K,T),\alpha(C_{\mathfrak{p}})>0}\mathbb{H}_\alpha=\bigcap_{\alpha\in\Phi(\mathfrak{u}_{\mathfrak{p}})}\mathbb{H}_\alpha \tag{4.15}$$

Where $\mathbb{H}_\alpha:=\{X\in\mathfrak{t}_{\mathbb{R}}:\alpha(X)>0\}$, and $\Phi(\mathfrak{u}_{\mathfrak{p}}):=\{\alpha:\mathfrak{k}_\alpha\subset\mathfrak{u}_{\mathfrak{p}}:=\text{the nilpotent radical of }\mathfrak{p}\}$. The set $V_{\mathfrak{p},\mathbb{R}}$ is W_L-invariant open convex subset of $\mathfrak{t}_{\mathbb{R}}$. The union of $V_{P,\mathbb{R}}$'s in $\mathfrak{t}_{\mathbb{R}}$ are disjoint, and $\mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r\to\coprod_{\mathfrak{p}\in\mathcal{P}_{\mathfrak{k},\mathfrak{l}}}V_{\mathfrak{p},\mathbb{R}}$ is a homotopy equivalence. In summary, we have the following commutative diagram of $K(\pi,0)$-spaces, with horizontal maps being homotopy equivalences:

$$\begin{tikzcd}
\{\mathfrak{p}\} \arrow[d] & C_{\mathfrak{p}} \arrow[l] \arrow[r] \arrow[d] & V_{\mathfrak{p},\mathbb{R}} \arrow[d] \\
\mathcal{P}_{\mathfrak{k},\mathfrak{l}} & \mathfrak{c}_{\mathfrak{l},\mathbb{R}}^r \arrow[l] \arrow[r] & \coprod V_{\mathfrak{p},\mathbb{R}}
\end{tikzcd} \tag{4.16}$$

Example 4.17. When $\mathfrak{l} = \mathfrak{t}$, we have $\mathfrak{c}^r_{\mathfrak{t},\mathbb{R}} = \mathfrak{t}^r_{\mathbb{R}} = \mathfrak{t}^{reg}_{\mathbb{R}}$ and $C_{\mathfrak{p}} = V_{\mathfrak{p},\mathbb{R}}$ is a chamber in $\mathfrak{t}_{\mathbb{R}}$. This gives the correspondence between Borel subalgebras and chambers in $\mathfrak{t}_{\mathbb{R}}$. □

Let $V_{\mathfrak{p}} := V_{\mathfrak{p},\mathbb{R}} \times i\mathfrak{t}_{\mathbb{R}}$ is W_L-invariant open convex subset of $\mathfrak{t}$. By (4.12), we have

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[r, "id"'] \arrow[d, "\mathrm{Res}_{\mathfrak{p}}"] & Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[d, "R_{V_{\mathfrak{p}}}"] \\
Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, "id"'] & Sh_{\mathcal{N}}(\mathfrak{l}/L)
\end{tikzcd} \tag{4.18}$$

This identification is compactible with (higher) transitivity, in the following sense: Let $\mathfrak{l} \subset \mathfrak{k} \subset \mathfrak{g}$ reductive Lie algebras, and $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{k}$ with Levi factor $\mathfrak{l}$, and $\mathfrak{r}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{k}$. Then $V_{\mathfrak{p},\mathbb{R}} \cap V_{\mathfrak{r},\mathbb{R}} = V_{\mathfrak{q},\mathbb{R}}$, for $\mathfrak{q} = \mathfrak{p} \oplus \mathfrak{u}_{\mathfrak{r}}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$. Then $\mathfrak{q} \cap \mathfrak{k} = \mathfrak{p}$, and we have

$$\begin{tikzcd}
 & & \mathfrak{q}/Q \arrow[dr, "\tilde{p}_1"] & & \\
 & \mathfrak{p}/P \arrow[ur, "\tilde{i}_2"] \arrow[dr, "p_1"] \arrow[rr, phantom, "\square"] & & \mathfrak{r}/R \arrow[dr, "p_2"] & \\
\mathfrak{l}/L \arrow[ur, "i_1"] & & \mathfrak{k}/K \arrow[ur, "i_2"] & & \mathfrak{g}/G
\end{tikzcd}$$

The natural map $\tilde{i}_2^! \circ \tilde{p}_1^* \xrightarrow{\simeq} p_1^* \circ i_2^!$ is an isomorphism. This gives

$$\alpha^{\mathfrak{p},\mathfrak{r}} : \mathrm{Res}_{\mathfrak{p}} \circ \mathrm{Res}_{\mathfrak{r}} \xrightarrow{\simeq} \mathrm{Res}_{\mathfrak{q}} \tag{4.19}$$

On the other hand, (4.9) gives a natural isomorphism $\beta^{\mathfrak{p},\mathfrak{r}} : R_{V_{\mathfrak{p}}} \circ R_{V_{\mathfrak{r}}} \xrightarrow{\simeq} R_{V_{\mathfrak{q}}}$.

And the 2-diagrams naturally commutes (there is a natural way to fill the solid):

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}/G) \arrow[rr] \arrow[dd, "\mathrm{Res}_{\mathfrak{q}}"'] \arrow[dr, dotted, "\mathrm{Res}_{\mathfrak{r}}"] & & Sh_{\mathcal{N}}(\mathfrak{g}/G) \arrow[dd, "R_{V_{\mathfrak{q}}}"'] \arrow[dr, "R_{V_{\mathfrak{r}}}"] & \\
\arrow[r, phantom, "\overset{\alpha^{\mathfrak{p},\mathfrak{r}}}{\Longleftarrow}"] & Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[dl, dotted, "\mathrm{Res}_{\mathfrak{p}}"] & \arrow[r, phantom, "\overset{\beta^{\mathfrak{p},\mathfrak{r}}}{\Longleftarrow}"] & Sh_{\mathcal{N}}(\mathfrak{k}/K) \arrow[dl, "R_{V_{\mathfrak{p}}}"] \\
Sh_{\mathcal{N}}(\mathfrak{l}/L) \arrow[rr] & & Sh_{\mathcal{N}}(\mathfrak{l}/L) &
\end{tikzcd} \tag{4.20}$$

There is similar compactibility with higher transitivities. We shall omit the details.

The following proposition is easy to verify.

Proposition 4.21. *Let $C \in \mathfrak{c}_{\mathfrak{k}}$, then $-C^* : Sh_{\mathcal{N}}(\mathfrak{k}/K) \to Sh_{\mathcal{N}}(\mathfrak{k}/K)$ is canonically isomorphic to identity functor.* □

4.4 Dependence on parabolic subgroups

Results in this section will not be used later in the thesis.

Taking a step back to the theory of finite groups. Let $f : A \hookrightarrow B$ be an inclusion of finite groups. Then we have the following commutative diagram of induction/restriction on representations/characters (with horizontal arrows going in the same direction):

$$\begin{tikzcd}
\mathbb{C}[A]-mod_{fd} \arrow[r, shift left, "f_*"] \arrow[d, "ch"] & \mathbb{C}[B]-mod_{fd} \arrow[l, shift left, "f^*"] \arrow[d, "ch"] \\
\mathbb{C}[A/A] \arrow[r, shift left, "f_*"] & \mathbb{C}[B/B] \arrow[l, shift left, "f^*"]
\end{tikzcd}$$

Where the vertical lines are taking characters of a representation.

Now let G be a reductive Lie group, $L \subset G$ a Levi. An useful technique is to reduce the study of G to that of L which is a smaller group. However, it turns out that direct induction/restriction between G and L as in finite group case does not behave well. To correct it, the idea is to use an intermediate parabolic subgroup P. And define the parabolic induction/restriction in various context using the diagram

$$L \xleftarrow{q} P \xrightarrow{p} G$$

It's natural to ask how what is the dependence of resulting restriction/induction on the choice of parabolic subgroups. Indeed, this has been proved to be independent in many context. In the case of character sheaves on Lie algebra,

$$\begin{tikzcd}
\mathfrak{l}/L \arrow[r, shift left, "i"] \arrow[rr, bend left, "f"] & \mathfrak{p}/P \arrow[l, shift left, "q"] \arrow[r, "p"] & \mathfrak{g}/G
\end{tikzcd}$$

f is not étale, and the direct restriction functor f^* does not preserve nilpotent singular support whereas its restriction to $\mathfrak{l}^r/L$ does. Define $res_{\mathfrak{p}} := q_! p^*$.

At the level of abelian categories, the following argument is essentially contained in [13, Theorem 4.1]. using (4.11), we have:

$$\begin{tikzcd}
\mathrm{Perv}_{\mathcal{N}}(\mathfrak{g}/G) \arrow[d, "res_{\mathfrak{p}}"] \arrow[dr, "f^*|_{\mathfrak{l}^r/L}"] \arrow[drr, bend left] & & \\
\mathrm{Perv}_{\mathcal{N}}(\mathfrak{l}/L) \arrow[r, hook, "j_r^*"] \arrow[rr, bend right, "j^*"'] & \mathrm{Perv}_{\mathcal{N}}(\mathfrak{l}^r/L) \arrow[r] & \mathrm{Perv}_{\mathcal{N}}(U/L)
\end{tikzcd}$$

where j_r^* is a faithful embedding since j^* is an equivalence. So on the level of objects, for each $F \in \mathrm{Perv}(\mathfrak{g}/G)$, $res_{\mathfrak{p}}(F)$ can be characterized up to isomorphism by the commutativity of the diagram (also up to natural isomorphism), and hence it's isomorphism class is independent of $\mathfrak{p}$. Similar argument also works for character sheaves on Lie groups.

On the level of derived category, faithful embedding does not make sense. Nevertheless, we could overcome this by taking into account of the higher homotopy: for $x \in \mathfrak{c}^r_{\mathfrak{l}\subset\mathfrak{k}}$ a point, we have defined $R_x : Sh_{\mathcal{N}}(\mathfrak{g}/G) \to Sh_{\mathcal{N}}(\mathfrak{l}/L)$.

And for every path $\gamma : [0,1] \to \mathfrak{c}^r_{\mathfrak{l}\subset\mathfrak{k}}, \gamma(0) = x, \gamma(1) = y$, we have a natural isomorphism of functors $R_\gamma : R_x \Rightarrow R_y$. Similarly for higher cells in $\mathfrak{c}^r_{\mathfrak{l}\subset\mathfrak{g}}$. Let $\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}}$ be the underline homotopy type of $\mathfrak{c}^r_{\mathfrak{l}\subset\mathfrak{g}}$. We have defined a functor:

$$R : \underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}} \to Fun(Sh_{\mathcal{N}}(\mathfrak{g}/G), Sh_{\mathcal{N}}(\mathfrak{l}/L))$$

For a topological space X, Let $\mathcal{C}X$:=the cone of X. $\mathcal{S}X$:=the suspension of X. Let $\mathfrak{p} \supset \mathfrak{l}$ a parabolic subalgebra of $\mathfrak{g}$. Then (4.11) gives

$$R_{\mathfrak{p}} : \mathcal{C}\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}} \to Fun(Sh_{\mathcal{N}}(\mathfrak{g}/G), Sh_{\mathcal{N}}(\mathfrak{l}/L))$$

such that $R_{\mathfrak{p}}(0 \in \mathcal{C}\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}}) = \mathrm{Res}_{\mathfrak{p}}$. For $\mathfrak{q} \supset \mathfrak{l}$ another parabolic subgroup, then we can define $R_{\mathfrak{p},\mathfrak{q}} := R_{\mathfrak{p}} \cup_R R_{\mathfrak{q}} : \mathcal{S}\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}} \to Fun(Sh_{\mathcal{N}}(\mathfrak{g}/G), Sh_{\mathcal{N}}(\mathfrak{l}/L))$, such that $R_{\mathfrak{p},\mathfrak{q}}(0_{\mathfrak{p}}) = \mathrm{Res}_{\mathfrak{p}}, R_{\mathfrak{p},\mathfrak{q}}(0_{\mathfrak{q}}) = \mathrm{Res}_{\mathfrak{q}}$. If $\mathfrak{l} \neq \mathfrak{g}$, then $\mathcal{S}\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}}$ is connected but not contractible, we have $\mathrm{Res}_{\mathfrak{p}} \simeq \mathrm{Res}_{\mathfrak{q}}$, but not canonically. And such isomorphism can be made canonical, e.g, by choosing a point in $\mathfrak{c}^r_{\mathfrak{l}\subset\mathfrak{g}}$. Note that $\underline{\mathfrak{c}}^r_{\mathfrak{l}\subset\mathfrak{g}}$ is $K(\pi,1)$-space. The situation for Lie group at derived level is more complicated due to the topology appearing in Lie group, and we shall study this in the future.

Remark: The independence of parabolic subalgebra at derived level was proved in [25, Theorem 7.3], and the same choice in higher homotopy is implicit in the proof.

Chapter 5

Complex gauge theory on S^1

The gauge theory on S^1 provides the simplest nontrivial example of gauge theories. In this chapter, we study the stack $G/G \simeq \mathrm{Loc}_G(S^1)$ using the gauge uniformization $\Omega^1(S^1,\mathfrak{g})/C^\infty(S^1,G)$. We use complex structure groups which in turns gives nilpotent codirections in $T^*\mathrm{Loc}_G(S^1)$.

The charts in $\Omega^1(S^1,\mathfrak{g})/C^\infty(S^1,G)$ mapping to G/G are a kind of exponential map. The exponential map plays important role in Lie theory, for example in Harish-Chandra's theory of harmonic analysis on (non-compact) semisimple Lie groups. We develop similar method of exponential maps for character sheaves. And one of our main theorem, Theorem 5.25, describing of the category character sheaves on G/G as glued from character sheaves on various Lie algebras, gives a new viewpoint on character sheaves on Lie groups.

5.1 Gauge uniformization on circle

We will establish the results in Chapter 3 in the present situation.

Let G be a connected reductive algebraic group over $\mathbb{C}$. $\mathrm{Conn}_G(S^1)$:= the moduli space of C^∞ G-bundles on S^1 with connection. Any G-bundle on S^1 is trivial, hence $\mathrm{Conn}_G(S^1) = Conn(S^1\times G)/Aut(S^1\times G)$ the space of connection on trivial bundle modulo automorphism group of trivial bundle. We have $Aut(S^1\times G)\simeq C^\infty(S^1,G)$. Fixing the trivial connection on $S^1\times G$ gives $Conn(S^1\times G)\simeq\Omega^1(S^1,\mathfrak{g})$. The action of $C^\infty(S^1,G)$ on $\Omega^1(S^1,\mathfrak{g})$ is identified with the Gauge transformation, i.e,

$$\mathrm{Conn}_G(S^1) = \Omega^1(S^1,\mathfrak{g})/C^\infty(S^1,G)$$

where the gauge action $Gg_g(A) := gAg^{-1} - dg\cdot g^{-1}$ for $g\in C^\infty(S^1,G), A\in\Omega^1(S^1,\mathfrak{g})$. The tangent complex at $A\in\Omega^1(S^1,\mathfrak{g})$ is:

$$T_A\mathrm{Conn}_G(S^1) = \Omega^{\bullet-1}_{\nabla=d+A}(S^1,\mathfrak{g}) = [C^\infty(S^1,\mathfrak{g}) \xrightarrow{d+A} \Omega^1(S^1,\mathfrak{g})]$$

Fix an orientation on S^1, the cup product/Poincare duality paring:

$$\Omega^{\bullet-1}_{\nabla}(S^1,\mathfrak{g})\times\Omega^{\bullet}_{\nabla^*}(S^1,\mathfrak{g}^*)\to\mathbb{C}$$

$$(\alpha, \beta) \to \int_{S^1} \alpha \wedge \beta$$

gives identification $T_A^*\mathrm{Conn}_G(S^1) \simeq \Omega^\bullet_{\nabla^*}(S^1, \mathfrak{g}^*)$, in particular, $H^0(T_A^*\mathrm{Conn}_G(S_1)) \simeq \{h \in C^\infty(S^1, \mathfrak{g}^*) : \int_{S^1} \nabla(f) \wedge h = 0, \forall f \in C^\infty(S^1, \mathfrak{g})\}$. Let $\mathcal{N}_A \subset H^0(T_A^*\mathrm{Conn}_G(S^1))$ be the conical subset consist those h such that $h(x) \in \mathcal{N} \subset \mathfrak{g}^*$ for all $x \in S^1$. We have $g^*(\mathcal{N}_A) = \mathcal{N}_{g\cdot A}$, hence $\mathcal{N} = \mathcal{N}_{S^1} := \bigcup_{A \in \mathrm{Conn}_{S_1}} \mathcal{N}_A \subset H^0(T^*\mathrm{Conn}_G(S^1))$ is well defined.

Remark: Any connection on $S^1 \times G$ is automatically flat. Hence we have equivalences between differentiable stacks: $G/G \xrightarrow{\simeq} \mathrm{Loc}_G(S^1) \xleftarrow{\simeq} \mathrm{Conn}_G(S^1)$, c.f Remark 5.7 to make sense of the stacks over arbitrary points. And $\mathcal{N}_{S^1}$ is identified with $\mathcal{N} \subset T^*(G/G)$. □

Take κ be an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$. It induces $\mathfrak{g} \xrightarrow{\simeq} \mathfrak{g}^*$. We have for $f, h \in C^\infty(S^1, \mathfrak{g})$:

$$\int_{S^1} \nabla(f) \wedge h = -\int_{S^1} f \wedge \nabla(h)$$

For $h^* := \kappa(h, -) \in C^\infty(S^1, \mathfrak{g})$,

$$\int_{S^1} f \wedge \nabla^*(h^*) = -\int_{S^1} \nabla(f) \wedge h^* = -\int_{S^1} \nabla(f) \wedge h = \int_{S^1} f \wedge \nabla(h)$$

Hence, we have a identification

$$T_\nabla^*\mathrm{Conn}_G(S^1) \simeq T_\nabla\mathrm{Conn}_G(S^1)[-1] \tag{5.1}$$

Remark: This is (the de-Rham resolution of) the 1-shifted sympletic structure on $\mathrm{Loc}_G(S^1)$ in [26].

Automorphism groups. Let $A \in \Omega^1(S^1, \mathfrak{t}), G_A := C_{C^\infty(S^1,G)}(A)$, Then $\mathfrak{g}_A := Lie(G_A) = \{X(\theta) \in C^\infty(S^1, \mathfrak{g}) : dX + [A, X] = 0\}$, Let $X = H(\theta) + \sum_{\alpha \in \Phi} f_\alpha(\theta) X_\alpha$, where $H : S^1 \to \mathfrak{t}, f_\alpha : S^1 \to \mathbb{C}$. Then the equation $dX + [A, X] = 0$ is equivalent to $dH(\theta) = 0, df_\alpha = f_\alpha \alpha(A)$. The first equation has solution constant functions and second equation has either trivial solution or 1 dimensional solution, denoted by $\mathbb{C} f_A^\alpha$. Hence

$$G_A^0 = < T, \exp \mathbb{C} f_A^\alpha : \alpha \in \Phi > \tag{5.2}$$

T is a maximal torus of G_A^0, hence

$$G_A = G_A^0 \cdot N_{G_A}(T) = G_A^0 \cdot (N_{LG}(T) \cap G_A) = G_A^0 \cdot ((LT \cdot N_G(T)) \cap G_A) \tag{5.3}$$

where the last equality is by Lemma 3.8.

Since $H^1(X, \mathscr{C}_X^\infty) = 0$, the boundary map in the exponential long exact sequence $C^\infty(S^1, \mathbb{C}^*) \to H^1(X, \mathbb{Z})$ is surjective. Let $f \in C^\infty(S^1, \mathbb{C}^*)$ be a lift of $1 \in H^1(X, \mathbb{Z})$.

Now for concreteness, we shall let $\mathbb{S}$ be the unit circle in $\mathbb{C}^*$, and θ be the angle coordinate on $\mathbb{S}$. Take $f = e^{i\theta}$. And define $\Gamma_{\mathbb{S}} := f^{\mathbb{Z}} \subset C^\infty(\mathbb{S}, \mathbb{C}^*)$. Note that $\mathbb{S}$ is naturally a Lie group, and $\Gamma_{\mathbb{S}} = Hom_{Lie}(\mathbb{S}, \mathbb{C}^*)$.

There is commutative diagram, the normalization $-2\pi i$ is to make it compactible with Gauge transformation:

$$\begin{tikzcd}
\Gamma_{\mathbb{S}} \arrow[r, "\simeq"] \arrow[d] & H^1(\mathbb{S}, \mathbb{Z}) \arrow[dd, "\times(-2\pi i)"] \\
C^\infty(\mathbb{S}, \mathbb{C}^*) \arrow[d, "-d\log(-)"] & \\
\Omega^1(\mathbb{S}, \mathbb{C}) \arrow[r] & H^1_{dR}(\mathbb{S}, \mathbb{C})
\end{tikzcd}$$

Let $\Omega_{\mathbb{S}} := \Gamma_{\mathbb{S}} \otimes \mathbb{C}, \Lambda_{\mathbb{S}} := X_*(T) \otimes \Gamma_{\mathbb{S}}, \mathfrak{t}_{\mathbb{S}} := X_*(T) \otimes \Gamma_{\mathbb{S}} \otimes \mathbb{C} = \Lambda_{\mathbb{S}} \otimes \mathbb{C} = X_*(T) \otimes \Omega_{\mathbb{S}} = \Gamma_{\mathbb{S}} \otimes \mathfrak{t}$. We have, $\Omega_{\mathbb{S}} \xrightarrow{-d\log(-)} \Omega^1(\mathbb{S}, \mathbb{C})$, which is injective onto the space of translation invariant 1-forms, i.e, those of the form $ad\theta$, for $a \in \mathbb{C}$. We have $\mathfrak{t}_{\mathbb{S}} \hookrightarrow \Omega^1(\mathbb{S}, \mathfrak{t}) \hookrightarrow \Omega^1(\mathbb{S}, \mathfrak{g})$. On the other hand, $\Lambda_{\mathbb{S}} \hookrightarrow X_*(T) \otimes C^\infty(\mathbb{S}, \mathbb{C}^*) = C^\infty(\mathbb{S}, T) \hookrightarrow C^\infty(\mathbb{S}, G)$. $C^\infty(\mathbb{S}, G)$ acts on $\Omega^1(\mathbb{S}, \mathfrak{g})$ via Gauge transformation, and under this action, $\Lambda_{\mathbb{S}}$ stablize $\mathfrak{t}_{\mathbb{S}}$ and the action is identified with the canonical action. Similarly, the action of $N_G(T) \subset C^\infty(\mathbb{S}, G)$ stablize $\Omega^1(\mathbb{S}, \mathfrak{t}) \subset \Omega^1(\mathbb{S}, \mathfrak{g})$. T acts trivially, $W = N_G(T)/T$ acts on $\Omega^{0,1}(\mathbb{S}, \mathfrak{t})$ which can be identified with the canonical action of W on $\mathfrak{t}_{\mathbb{S}}$. Let $W_{\mathbb{S}} := \Lambda_{\mathbb{S}} \rtimes W \subset Aff(\mathfrak{t}_{\mathbb{S}})$. Fix a lifting $W \to N_G(T)$ gives a lifting $W_{\mathbb{S}} \to C^\infty(\mathbb{S}, G)$ as a subset. Let $\Phi_{\mathbb{S}}$ be the set $\{nf + \alpha' : n \in \mathbb{Z}, \alpha' \in \Phi\} \subset Map(\mathfrak{t}_{\mathbb{S}}, \Omega_{\mathbb{S}})$. $\Phi_{aff} := \{n + \alpha' : n \in \mathbb{Z}, \alpha' \in \Phi\} \subset Map(\mathfrak{t}, \mathbb{C})$ be the set of affine roots. For any $\alpha = nf + \alpha' \in \Phi_{\mathbb{S}}$, define $U_\alpha := \exp(e^{in\theta}\mathbb{C}X_\alpha) \subset C^\infty(\mathbb{S}, G)$ the corresponding 1-parameter subgroup. Let $\Phi_{\mathbb{S},A} := \{\alpha \in \Phi_{\mathbb{S}} : \alpha(A) = 0\}$. There is a natural map $\Phi_{\mathbb{S}}/\Phi_{aff} \to \Phi$, $\alpha \mapsto \underline{\alpha} := \alpha - \alpha(0)$. By (5.2)

$$G_A^0 =< T, U_\alpha : \alpha(A) = 0 >$$

The kernal of $d\log(-)$ is $\mathbb{C}^*$, so we have $N_{G_A}(T) =< LT, N_G(T) > \cap G_A =< T, C_{W_{\mathbb{S}}}(A) >$, hence by (5.3)

$$G_A :=< G_A^0, C_{W_{\mathbb{S}}}(A) > \tag{5.4}$$

From the explicit expression of G_A^0, the following proposition is easy to verify:

Proposition 5.5. *Let $X(\theta) \in \mathfrak{g}_A$, the following are equivalent:*
(i) $X(\theta)$ is nilpotent in $\mathfrak{g}_{\boldsymbol{A}}$.
(ii) $X(\theta_0)$ is nilpotent in $\mathfrak{g}$, for some $\theta_0 \in \mathbb{S}$.
(iii) $X(\theta_0)$ is nilpotent in $\mathfrak{g}$, for all $\theta_0 \in \mathbb{S}$.
□

Let $W_A := N_{G_A}(T)/T, W_A^0 := N_{G_A^0}(T)/T$, then $W_A^0 \subset W_A = C_{W_{\mathbb{S}}}(A)$. On the other hand, W_A^0 is generated by $\alpha(A)\alpha^\vee w_\alpha \in W_{\mathbb{S}}$, for those $\alpha \in \Phi_{\mathbb{S},A}$. They are affine (complex) reflection on $\mathfrak{t}_{\mathbb{S}}$ with respect to the hyperplane $\alpha(X) = \alpha(A) \in \Gamma_{\mathbb{S}}$.

Let $Q^\vee \subset X_*(T)$ be the coroot lattice, $Q^\vee_{\mathbb{S}} := Q^\vee \otimes \Gamma_{\mathbb{S}} \subset \Lambda_{\mathbb{S}}$, then $Q^\vee_{\mathbb{S}} \rtimes W$ is group generated by reflections whose reflections are of the form $n\alpha^\vee w_\alpha$, for some $n \in \Gamma_{\mathbb{S}}$. Now by the general property of reflection groups, see e.g. [7, V.3.3 Prop.1], $C_{Q^\vee_{\mathbb{S}} \rtimes W}(A)$ is generated by reflections fixing A, (this was stated for A real, but can be extend to complex case in present situation.). Hence we have $W^0_A = C_{Q^\vee_{\mathbb{S}} \rtimes W}(A)$. In particular, when G is simply-connected, $W^0_A = W_A$, i.e, G_A is connected. This gives:

$$G \text{ simply-connected, } s \in G \text{ semisimple, then } C_G(s) \text{ is connected.}$$

To summarize the notation, we have the following identifications given by $f = e^{i\theta}$:

(1)	$\Gamma_{\mathbb{S}}$	$\mathbb{Z}$	$e^{in\theta} \leftrightarrow n$
(2)	$\Omega_{\mathbb{S},\mathbb{R}}$	$\mathbb{R}$	$(1) \otimes \mathbb{R}$
(3)	$\Omega_{\mathbb{S}}$	$\mathbb{C}$	$(1) \otimes \mathbb{C}$
(4)	$\Lambda_{\mathbb{S}}$	$X_*(T)$	$(1) \otimes X_*(T)$
(5)	$Q^\vee_{\mathbb{S}}$	$Q^\vee$	Inside (4)
(6)	$\mathfrak{t}_{\mathbb{S},\mathbb{R}}$	$\mathfrak{t}_{\mathbb{R}}$	$(2) \otimes X_*(T)$
(7)	$\mathfrak{t}_{\mathbb{S}}$	$\mathfrak{t}$	$(3) \otimes X_*(T)$
(8)	$W_{\mathbb{S}}$	$W^{ex}_{aff} := X_*(T) \rtimes W$	$(4) \rtimes W$
(9)	$W^\circ_{\mathbb{S}}$	$W_{aff} := Q^\vee \rtimes W$	$(5) \rtimes W$
(10)	$W_{\mathbb{S}} \curvearrowright \mathfrak{t}_{\mathbb{S}}$	$W^{ex}_{aff} \curvearrowright \mathfrak{t}$	Under (7),(8)
(11)	$\Phi_{\mathbb{S}}$	Φ_{aff}	Under (3),(7)

Etale charts. Let $\mathfrak{g}^\triangle_A := \mathfrak{g}_A \cdot d\log(f) = \mathfrak{g}_A id\theta$. The map $-A : \mathfrak{g}^\triangle_A \to \mathfrak{g}_A$, $X \mapsto \frac{(X-A)}{id\theta}$ intertwine the G_A action, this shows that $\mathfrak{g}^\triangle_A$ is stable under action of G_A, and induces $\mathfrak{g}^\triangle_A/G_A \simeq \mathfrak{g}_A/G_A$. Where the first action is Gauge transformation and the second action is conjugation.

For $\mathbf{A} = (A_1, A_2, \dots, A_n) \in \mathfrak{t}^n_{\mathbb{S}}$, we'll say $A_i \in \mathbf{A}$. Define $G_{\mathbf{A}} := \bigcap_{A \in \mathbf{A}} G_A, \mathfrak{g}_{\mathbf{A}} := Lie(G_{\mathbf{A}}) \subset C^\infty(\mathbb{S}, \mathfrak{g}), \mathfrak{g}^\triangle_{\mathbf{A}} := \mathfrak{g}_{\mathbf{A}} id\theta \subset \Omega^1(\mathbb{S}, \mathfrak{g})$. $\mathfrak{c}^\triangle_{\mathbf{A}} := (\mathfrak{g}^\triangle_{\mathbf{A}})^{G_{\mathbf{A}}}, \mathfrak{c}_{\mathbf{A}} := (\mathfrak{g}_{\mathbf{A}})^{G_{\mathbf{A}}} = \mathfrak{c}(\mathfrak{g}_{\mathbf{A}}, G_{\mathbf{A}})$. $\mathfrak{c}^\triangle_{\mathbf{A}}$ is an affine space in $\mathfrak{t}_{\mathbb{S}}$ which contains A_i's, while $\mathfrak{c}_{\mathbf{A}}$ is a linear subspace of $\mathfrak{t}$. Any element C in $\mathfrak{c}^\triangle_{\mathbf{A}}$ induces $-C : \mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}} \xrightarrow{\simeq} \mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}$ which identifies $\mathfrak{c}^\triangle_{\mathbf{A}}$ and $\mathfrak{c}_{\mathbf{A}}$. Let $\mathcal{N} \subset T^*(\mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}})$ corresponds to $\mathcal{N} \subset T^*(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}})$ under this isomorphism. It does not depend on the choice of C.

Hence the results in Chapter 4 can be applied to the situation of Gauge action of $G_{\mathbf{A}}$ on $\mathfrak{g}^\triangle_{\mathbf{A}}$. In particular, Proposition 4.21 implies that the functors $-A_i^* : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \to Sh_{\mathcal{N}}(\mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}})$ are canonically isomorphic. Denote this (contractible space of) functor(s) by $-\mathbf{A}^* : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \to Sh_{\mathcal{N}}(\mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}})$.

For $\epsilon = (\delta, w) : \mathbf{A}' \to \mathbf{A}$ in $\Delta^{op}_{+,\mathfrak{t}_{\mathbb{S}}, W_{\mathbb{S}}}$, Let $Gg_\epsilon := Gg_{\dot{w}} : \Omega^1(\mathbb{S}, \mathfrak{g}) \to \Omega^1(\mathbb{S}, \mathfrak{g})$. For $pt \in \mathfrak{t}^0_{\mathbb{S}} \in \Delta^{op}_{+,\mathfrak{t}_{\mathbb{S}}, W_{\mathbb{S}}}$, set $\mathfrak{g}^\triangle_{pt} := \Omega^1(\mathbb{S}, \mathfrak{g}), G^\triangle_{pt} := C^\infty(\mathbb{S}, G)$. We have $Gg_\epsilon(\mathfrak{g}^\triangle_{\mathbf{A}'}) \subset \mathfrak{g}^\triangle_{\mathbf{A}}$, and $Ad_{\dot{w}}(G_{\mathbf{A}'}) \subset G_{\mathbf{A}}$. This gives $Gg_\epsilon : \mathfrak{g}^\triangle_{\mathbf{A}'}/G_{\mathbf{A}'} \to \mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}}$. See Remark 5.7 below to make sense of the map at S-points for $\mathbf{A} = pt$. let $\mathfrak{g}^{\triangle,r}_{\mathbf{A}',\epsilon} := \{X \in \mathfrak{g}^\triangle_{\mathbf{A}'} : G_{Gg_\epsilon(X)} = Gg_\epsilon(G_X)\}$. Using (5.1) and Prop. 5.5, the following proposition is proved in a similar way as in Proposition 4.6.

Proposition 5.6. *$Gg_\epsilon : \mathfrak{g}^{\triangle,r}_{\boldsymbol{A}',\epsilon} \to \mathfrak{g}^{\triangle}_{\boldsymbol{A}}$ is etale, and nilpotent cones correspond under $(dGg_\epsilon)^*$.* □

Remark 5.7. Let S be a set or a C^∞/complex manifold, we shall make sense of the above morphisms at S-point. It suffices to define the morphism before sheafification. Let the stack (over C^∞/complex manifolds) $\mathfrak{g}^{\triangle}_{pt}/G_{pt} := \mathrm{Loc}_G(S^1) = Map(S^1, BG)$, this stack is (the sheafification of) the functor $S \mapsto \{(\mathcal{P},\beta) : \mathcal{P}$ a G-bundle on S, $\beta : \mathcal{P} \to \mathcal{P}$ an automorphism$\}$. And for $\mathbf{A} \in \Delta^{op}_{\mathrm{t}_\mathbb{S},W_\mathbb{S}}$, $\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}$ is the sheafification of the prestack $(\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}})^{pre} := \{S \mapsto \mathfrak{g}^{\triangle}_{\mathbf{A}}(S)/G_{\mathbf{A}}(S)\}$. Now given $X : S \to \Omega^1(\mathbb{S},\mathfrak{g})$ a set valued point, we get $(G \times S, \beta_X : G \times S \to G \times S)$, where $\beta_X|_{s\in S}$ is the automorphism of the trivial bundle G by solving the equation $dg(\theta) = X(s)g$ on $\mathbb{S}$. hence $(G \times S, \beta_X)$ defines a set-valued point S of $\mathrm{Loc}_G(S^1)$. And for h an set valued S-point of $C^\infty(\mathbb{S}, G)$, multiplication by h gives an isomorphism of sets $\gamma_h : (G \times S, \beta_X) \to (G \times S, \beta_{h(X)})$. Hence we have the map $\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}} \xrightarrow{Gg_{\dot{w}}} \Omega^1(\mathbb{S},\mathfrak{g})/C^\infty(\mathbb{S},G) \to \mathrm{Loc}_G(S^1)$ on set valued points. Now if X, h are C^∞/holomorphic points of $\mathfrak{g}^{\triangle}_{\mathbf{A}}, G_{\mathbf{A}}$, then $\beta_{Gg_{\dot{w}}(X)}, \gamma_{Ad_{\dot{w}}(h)}$ are C^∞/holomorphic. Hence we have the morphism of prestacks $(\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}})^{pre} \to \mathfrak{g}^{\triangle}_{pt}/G_{pt}$ which gives the desired morphism of stacks $Gg_{\dot{w}} : \mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}} \to \mathfrak{g}^{\triangle}_{pt}/G_{pt}$ (on C^∞/holomorphic points).

It is not hard to directly define the C^∞ points for $\Omega^1(\mathbb{S},\mathfrak{g})$ and $C^\infty(\mathbb{S},G)$, we omit the details.

5.2 The functors

For the convenience later, we will denote a solid triangle in $\Delta^{op}_{+,\mathrm{t}_\mathbb{S},W_\mathbb{S}}$ by
$\begin{tikzcd} \mathbf{A}''_2 \arrow[r, "\epsilon_2"'] \arrow[rr, bend left, "\epsilon"] & \mathbf{A}'_1 \arrow[r, "\epsilon_1"'] \arrow[phantom, u, "\Uparrow\eta"] & \mathbf{A} \end{tikzcd}$.
$\epsilon_1 = (\delta_1, w_1), \epsilon_2 = (\delta_2, w_2)$, $\epsilon = (\delta, w)$. Where $\delta_1 : \mathbf{A}_1 \to \mathbf{A}'_1, \mathbf{A}_1 = w_1^{-1}(\mathbf{A})$, $\delta_2 : \mathbf{A}'_2 \to \mathbf{A}''_2, \mathbf{A}'_2 = w_2^{-1}(\mathbf{A}'_1)$, and $\delta : \mathbf{A}_2 \to \mathbf{A}''_2, \mathbf{A}_2 = w^{-1}(\mathbf{A})$. Then by definition, for the two arrow to exist, we have $\delta = \delta_2 \circ \delta_1^{w_2^{-1}}$, and $\mathbf{A}_2 = w^{-1}(\mathbf{A}) = w_2^{-1}(\mathbf{A}_1) = w_2^{-1}(w_1^{-1}(\mathbf{A}))$. By (1.10), we have

$$\begin{tikzcd} \mathfrak{g}^{\triangle}_{\mathbf{A}''_2}/G_{\mathbf{A}''_2} \arrow[r, "Gg_{\dot{w}_2}"'] \arrow[rr, bend left, "Gg_{\dot{w}}"] & \mathfrak{g}^{\triangle}_{\mathbf{A}'_1}/G_{\mathbf{A}'_1} \arrow[r, "Gg_{\dot{w}_1}"'] \arrow[phantom, u, "\Uparrow Gg_\eta"] & \mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}} \end{tikzcd} \tag{5.8}$$

This induces pull back on sheaves:

$$\begin{tikzcd} Sh(\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Gg^*_{\dot{w}_1}"'] \arrow[rr, bend left, "Gg^*_{\dot{w}}"] & Sh(\mathfrak{g}^{\triangle}_{\mathbf{A}'_1}/G_{\mathbf{A}'_1}) \arrow[r, "Gg^*_{\dot{w}_2}"'] \arrow[phantom, u, "\Downarrow"] & Sh(\mathfrak{g}^{\triangle}_{\mathbf{A}''_2}/G_{\mathbf{A}''_2}) \end{tikzcd}$$

For $\epsilon = (\delta, w) : \mathbf{A}' \to \mathbf{A}$ in $\Delta^{op}_{+,\mathrm{t}_\mathbb{S},W_\mathbb{S}}$, by Proposition 5.6:

$$\begin{tikzcd} Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Gg^*_{\dot{w}_1}"'] \arrow[rr, bend left, "Gg^*_{\dot{w}}"] & Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle,r}_{\mathbf{A}'_1,\epsilon_1}/G_{\mathbf{A}'_1}) \arrow[r, "Gg^*_{\dot{w}_2}"'] \arrow[phantom, u, "\Downarrow"] & Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle,r}_{\mathbf{A}''_2,\epsilon}/G_{\mathbf{A}''_2}) \end{tikzcd} \tag{5.9}$$

In particular, we have

$$\begin{tikzcd} Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Gg^*_{\dot{w}_1}"'] \arrow[rr, bend left, "Gg^*_{\dot{w}}"] & Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle}_{\mathbf{A}_1}/G_{\mathbf{A}_1}) \arrow[u, phantom, "\Downarrow"] \arrow[r, "Gg^*_{\dot{w}_2}"'] & Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle}_{\mathbf{A}_2}/G_{\mathbf{A}_2}) \end{tikzcd} \tag{5.10}$$

There are analogous diagrams for $\Delta^{op}_{\mathfrak{t}_{\mathbb{S}},W_{\mathbb{S}}}$, when we replace $\mathfrak{g}^{\triangle}_{\mathbf{A}}$ by $\mathfrak{g}_{\mathbf{A}}$ and $Gg_{\dot{w}}$ by $Ad_{\dot{w}}$.

5.11. *The functor* $\mathbf{D}^{\triangle,*}$.

Now for simplicity assume that G is simply-connected. $\mathfrak{t}_{\mathbb{S},\mathbb{R}} := X_*(T)\otimes\Gamma_{\mathbb{S}}\otimes\mathbb{R}$. By the previous discussion, G_A is determined by the hyperplane structure on $\mathfrak{t}_{\mathbb{S},\mathbb{R}}$. Let $C^{\triangle}_{A,\mathbb{R}} := \bigcup_{A\in\bar{F}} F$ for F facet. Then $C^{\triangle}_{A,\mathbb{R}}$ is W_A-invariant open convex in $\mathfrak{t}_{\mathbb{S},\mathbb{R}}$, and $C^{\triangle}_{A} := V_{A,\mathbb{R}}\times i\mathfrak{t}_{\mathbb{R},\mathbb{S}}$ is W_A-invariant open convex subset of $\mathfrak{t}_{\mathbb{S}}$. Let $D^{\triangle}_{A} :=^{G_A} C^{\triangle}_{A}\subset\mathfrak{g}^{\triangle}_{A}$. Let $\Xi\subset\mathfrak{t}_{\mathbb{S},\mathbb{R}}$ be the set of vertices. Restricting (5.8) to $D^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}$ gives a functor:

$$\mathbf{D}^{\triangle}_{+} = (D^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}, Gg_{\epsilon}, Gg_{\eta}) : \Delta^{\circ,op}_{+,\Xi,W_{\mathbb{S}}} \to \mathbf{Prestk} \tag{5.12}$$

where $\mathbf{D}^{\triangle}_{pt}/G_{pt} := \mathrm{Loc}_G(S^1)$, and $\Delta^{\circ,op}_{+,\Xi,W_{\mathbb{S}}}\subset\Delta^{op}_{+,\Xi,W_{\mathbb{S}}}$ is the full subcategory such that $C^{\triangle}_{\mathbf{A}}\neq\varnothing$, this condition can be describled completely in terms of root data. We have $\bigcup_{A\in\Xi} C^{\triangle}_{A} = \mathfrak{t}_{\mathbb{S}}$, and $Gg_{\dot{w}}$ is etale on $D^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}$. Hence in view of Remark 3.33, the functor $\mathbf{D}^{\triangle}_{+}$ is a colimit diagram. It induces a limit diagram on pull back of sheaves with nilpotent singular support:

$$\mathbf{D}^{\triangle,*}_{+} = (Sh_{\mathcal{N}}(D^{\triangle}_{\mathbf{A}}/G_{\mathbf{A}}), Gg^*_{\epsilon}, Gg^*_{\eta}) : \Delta^{\circ}_{+,\Xi,W_{\mathbb{S}}} \to \infty\text{-}\mathbf{Cat} \tag{5.13}$$

5.14. *The functors* $\mathbf{g}^{\triangle,*}_{\flat}, \mathbf{g}^{*}_{\flat}$. Let $\mathbf{A}\in\Delta^{op}_{\mathfrak{t}_{\mathbb{S},\mathbb{R}}}$, $\alpha\in\Phi_{\mathbb{S}}$, define $\alpha(\mathbf{A})\geq/=0$ if $\alpha(A_i)\geq/=0, \forall A_i\in\mathbf{A}$, and $\alpha(\mathbf{A})\gneq 0$ if $\alpha(\mathbf{A})\geq 0$ but not $\alpha(\mathbf{A})=0$. For $\alpha\in\Phi_{\mathbb{S}}$, let $\mathbb{H}_{\alpha} := \{A\in\mathfrak{t}_{\mathbb{S},\mathbb{R}} : \alpha(A)>0\}$ and for $\alpha'\in\Phi$, let $\mathbb{H}_{\alpha'} := \{X\in\mathfrak{t}_{\mathbb{R}} : \alpha'(X)>0\}$.

Let $\delta:\mathbf{A}'\to\mathbf{A}$ in $\Delta^{op,\circ}_{\Xi}$, define $\Phi^{+}_{\delta} := \{\alpha\in\Phi_{\mathbb{S},\mathbf{A}} : \alpha(\mathbf{A}')\gneq 0\}, \Phi^{-}_{\delta} := -\Phi^{+}_{\delta}$. We have $\Phi_{\mathbb{S},\mathbf{A}} = \Phi^{+}_{\delta}\coprod\Phi_{\mathbb{S},\mathbf{A}'}\coprod\Phi^{-}_{\delta}$. Let $V_{\delta,\mathbb{R}} := \bigcap_{\alpha\in\Phi^{+}_{\delta}}\mathbb{H}_{\underline{\alpha}}\subset\mathfrak{t}_{\mathbb{R}}$ and $V_{\delta} := (V_{\delta,\mathbb{R}}\times i\mathfrak{t}_{\mathbb{R}})\subset\mathfrak{t}$. As in (4.8), put $R_{\delta} := R_{V_{\delta}} : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}})\to Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'})$. For $w\in W_{\mathbb{S}}$, let $\mathbf{A}^w := w(\mathbf{A})$, and $\delta^w : \mathbf{A}'^w\to\mathbf{A}^w$ in $\Delta^{op,\circ}_{\Xi}$ induced by δ. Then $Ad_{\dot{w}}(V_{\delta}) = V_{\delta^w}$, this induces

$$\begin{tikzcd} Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Ad^*_{\dot{w}}"] \arrow[d, "R_{\delta}"] \arrow[dr, phantom, "\Rightarrow"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}^w}/G_{\mathbf{A}^w}) \arrow[d, "R_{\delta^w}"] \\ Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'}) \arrow[r, "Ad^*_{\dot{w}}"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'^w}/G_{\mathbf{A}'^w}) \end{tikzcd} \tag{5.15}$$

For $\mathbf{A}''\xrightarrow{\delta_2}\mathbf{A}'\xrightarrow{\delta_1}\mathbf{A}$ in $\Delta^{op,\circ}_{\Xi}$, let $\delta := \delta_1\circ\delta_2$. Then $\Phi^{+}_{\delta_1}\cup\Phi^{+}_{\delta_2} = \Phi^{+}_{\delta}$, hence $V_{\delta_2}\cap V_{\delta_1} = V_{\delta}$. By (4.9), we have

$$R_{\delta_2}\circ R_{\delta_1}\Rightarrow R_{\delta} \tag{5.16}$$

Now for $\epsilon = (\delta,w) : \mathbf{A}'_1\to\mathbf{A}$ in $\Delta^{op,\circ}_{\Xi,W_{\mathbb{S}}}$, where $\delta : \mathbf{A}'_1\to\mathbf{A}_1$ in $\Delta^{op,\circ}_{\Xi}$ for $\mathbf{A}_1 = w^{-1}(\mathbf{A})$, define R_{ϵ} by:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Ad^*_{\dot w}", "\simeq"'] \arrow[dr, "R_\epsilon"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}_1}/G_{\mathbf{A}_1}) \arrow[d, "R_\delta"] \\
 & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'_1}/G_{\mathbf{A}'_1})
\end{tikzcd}$$

For $\begin{tikzcd} \mathbf{A}''_2 \arrow[r, "\epsilon_2"'] \arrow[rr, bend left, "\epsilon"] & \mathbf{A}'_1 \arrow[u, phantom, "\Uparrow\eta"] \arrow[r, "\epsilon_1"'] & \mathbf{A} \end{tikzcd}$ in $\Delta^{op,\circ}_{\Xi,W_{\mathbb{S}}}$, $\epsilon_1 = (\delta_1, w_1), \epsilon_2 = (\delta_2, w_2)$, $\epsilon = (\delta, w)$. Where $\delta_1 : \mathbf{A}_1 \to \mathbf{A}'_1, \mathbf{A}_1 = w_1^{-1}(\mathbf{A})$, $\delta_2 : \mathbf{A}'_2 \to \mathbf{A}''_2, \mathbf{A}'_2 = w_2^{-1}(\mathbf{A}'_1)$, and $\delta : \mathbf{A}_2 \to \mathbf{A}''_2, \mathbf{A}_2 = w^{-1}(\mathbf{A})$. We define $R_\eta : R_{\epsilon_2} \circ R_{\epsilon_1} \Rightarrow R_\epsilon$ by the diagram:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Ad^*_{\dot w_1}"] \arrow[rr, bend left, "Ad^*_{\dot w}", "(5.10)"'] \arrow[dr, "R_{\epsilon_1}"] \arrow[ddrr, bend right, "R_\epsilon"'] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}_1}/G_{\mathbf{A}_1}) \arrow[r, "Ad^*_{\dot w_2}"] \arrow[d, "R_{\delta_1}"'] \arrow[dr, phantom, "(5.15)"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}_2}/G_{\mathbf{A}_2}) \arrow[d, "R_{(\delta_1 w_2{}^{-1})}"'] \arrow[dd, bend left=60, "R_\delta", "(5.16)"'] \\
 & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'_1}/G_{\mathbf{A}'_1}) \arrow[r, "Ad^*_{\dot w_2}"] \arrow[dr, "R_{\epsilon_2}"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'_2}/G_{\mathbf{A}'_2}) \arrow[d, "R_{\delta_2}"'] \\
 & & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}''_2}/G_{\mathbf{A}''_2})
\end{tikzcd} \tag{5.17}$$

Hence we have defined a functor

$$\mathbf{g}^*_\flat := (Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}), R_\epsilon, R_\eta) : \Delta^\circ_{\Xi,W_{\mathbb{S}}} \to \infty\textbf{-Cat}$$

.

The subscript $\flat$ is to denote the fact that we are using based restriction for 1-morphisms. Analogously, $V^\triangle_{\delta,\mathbb{R}} := \bigcap_{\alpha\in\Phi^+_\delta} \mathbb{H}_\alpha \subset \mathfrak{t}_{\mathbb{S},\mathbb{R}}$. $V^\triangle_\delta := V^\triangle_{\delta,\mathbb{R}} \times i\mathfrak{t}_{\mathbb{S},\mathbb{R}} \subset \mathfrak{t}_{\mathbb{S}}$. We have $R^\triangle_\delta : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \to Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'})$, $R^\triangle_\epsilon := R^\triangle_\delta \circ Gg^*_{\dot w} : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \to Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'_1}/G_{\mathbf{A}'_1})$. And $R^\triangle_\eta : R^\triangle_{\epsilon_2} \circ R^\triangle_{\epsilon_1} \Rightarrow R^\triangle_\epsilon$. We have the functor

$$\mathbf{g}^{\triangle,*}_\flat := (Sh_{\mathcal{N}}(\mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}}), R^\triangle_\epsilon, R^\triangle_\eta) : \Delta^\circ_{\Xi,W_{\mathbb{S}}} \to \infty\textbf{-Cat}$$

$-\mathbf{A}$ identifies $V^\triangle_{\delta,\mathbb{R}}$ with $V_{\delta,\mathbb{R}}$, and induces a natural isomorphism of functors:

$$\mathbf{g}^*_\flat \Rightarrow \mathbf{g}^{\triangle,*}_\flat \tag{5.18}$$

Since $C^\triangle_{\mathbf{A}'} \hookrightarrow V^\triangle_\delta$ is inclusion of $W_{\mathbf{A}}$-invariant convex open subset of $\mathfrak{t}_{\mathbb{S}}$. There is natural isomorphism of functors:

$$\mathbf{g}^{\triangle,*}_\flat \Rightarrow \mathbf{D}^{\triangle,*} := \mathbf{D}^{\triangle,*}_+|_{\Delta^\circ_{\Xi,W_{\mathbb{S}}}} \tag{5.19}$$

given by

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}^\triangle_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "\simeq"] \arrow[d, "R_\epsilon"] & Sh_{\mathcal{N}}(D^\triangle_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[d, "Gg^*_\epsilon"] \\
Sh_{\mathcal{N}}(\mathfrak{g}^\triangle_{\mathbf{A}'}/G_{\mathbf{A}'}) \arrow[r, "\simeq"] & Sh_{\mathcal{N}}(D^\triangle_{\mathbf{A}'}/G_{\mathbf{A}'})
\end{tikzcd}$$

5.20. *The functors* $\mathbf{g}^{\triangle,*}, \mathbf{g}^*$.

For $\delta : \mathbf{A}' \to \mathbf{A}$ in Δ^{op}_{Ξ}, let $F_{\mathbf{A}}$:=the (real) open convex hall of $\{A_i : A_i \in \mathbf{A}\}$. Let $\overrightarrow{\mathbf{AA}'}^{\triangle} := \bigcup_{x \in F_{\mathbf{A}}, y \in F_{\mathbf{A}'}} \overrightarrow{xy} \subset \mathfrak{c}^{\triangle,r}_{\mathbf{A}'} \cap \mathfrak{t}_{\mathbb{S},\mathbb{R}}$, where $\overrightarrow{xy}$:= the ray from x to y. $\overrightarrow{\mathbf{AA}'} := -\mathbf{A}(\overrightarrow{\mathbf{AA}'}^{\triangle}) \subset \mathfrak{c}^r_{\mathbf{A}'} \cap \mathfrak{t}_{\mathbb{R}} =: \mathfrak{c}^r_{\mathbf{A}',\mathbb{R}}$.

$P_\delta :=< T, U_\alpha : \alpha \in \Phi^+_\delta >$. Let $\mathfrak{p}_\delta := Lie P_\delta \subset L\mathfrak{g}$. $\mathfrak{p}^{\triangle}_\delta := \mathfrak{p}_\delta i d\theta \subset \mathfrak{g}^{\triangle}_{\mathbf{A}}$.

Lemma 5.21. *P_δ is a parabolic subgroup of $G_{\mathbf{A}}$ with Levi factor $G_{\mathbf{A}'}$.* $\square$

We have

$$\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'} \xrightarrow{i} \mathfrak{p}_\delta/P_\delta \xrightarrow{p} \mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}$$

Let $\mathrm{Res}_\delta := \mathrm{Res}_{\mathfrak{p}_\delta} = i^! p^*$, Since $Ad_{\dot{w}}(\mathfrak{p}_\delta) = \mathfrak{p}_{\delta^w}$, there is analogous diagram of (5.15):

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[r, "Ad^*_{\dot{w}}"] \arrow[d, "\mathrm{Res}_\delta"] \arrow[dr, phantom, "\Rightarrow"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}^w}/G_{\mathbf{A}^w}) \arrow[d, "\mathrm{Res}_{\delta^w}"] \\
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'}) \arrow[r, "Ad^*_{\dot{w}}"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'^w}/G_{\mathbf{A}'^w})
\end{tikzcd} \tag{5.22}$$

For $\mathbf{A}'' \xrightarrow{\delta_2} \mathbf{A}' \xrightarrow{\delta_1} \mathbf{A}$ in $\Delta^{op,\circ}_{\Xi}$, (4.19) gives

$$\mathrm{Res}_{\delta_2} \circ \mathrm{Res}_{\delta_1} \Rightarrow \mathrm{Res}_{\delta_1 \circ \delta_2} \tag{5.23}$$

For $\epsilon = (\delta, w) : \mathbf{A}'_1 \to \mathbf{A}$ in $\Delta^{op,\circ}_{\Xi,W_{\mathbb{S}}}$, $\mathrm{Res}_\epsilon := \mathrm{Res}_\delta \circ Ad^*_{\dot{w}}$.

For $\begin{tikzcd} \mathbf{A}''_2 \arrow[r, "\epsilon_2"'] \arrow[rr, bend left, "\epsilon"] & \mathbf{A}'_1 \arrow[r, "\epsilon_1"'] \arrow[u, phantom, "\Uparrow\eta"] & \mathbf{A} \end{tikzcd}$ in $\Delta^{\circ}_{\Xi,W_{\mathbb{S}}}$ in $\Delta^{op,\circ}_{\Xi,W_{\mathbb{S}}}$. Similar to (5.17), the diagrams (5.22),(5.23) and (5.10) gives $\mathrm{Res}_\eta : \mathrm{Res}_{\epsilon_2} \circ \mathrm{Res}_{\epsilon_1} \Rightarrow \mathrm{Res}_\epsilon$. We have a functor $\mathbf{g}^* := (Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}), \mathrm{Res}_\epsilon, \mathrm{Res}_\eta)$.

The diagram (4.16) becomes:

$$\begin{tikzcd}
\{\mathfrak{p}_\delta\} \arrow[d] & \overrightarrow{\mathbf{AA}'} \arrow[l] \arrow[r] \arrow[d] & V_{\delta,\mathbb{R}} \arrow[d] \\
\mathcal{P}_{\mathfrak{g}_{\mathbf{A}},\mathfrak{g}_{\mathbf{A}'}} & \mathfrak{c}^r_{\mathbf{A}',\mathbb{R}} \arrow[l] \arrow[r] & \coprod V_{\delta,\mathbb{R}}
\end{tikzcd}$$

(4.20) gives commutative triangles:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[rr] \arrow[dr, "\mathrm{Res}_{\epsilon_1}"] \arrow[dd, "\mathrm{Res}_\epsilon"'] & & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}}/G_{\mathbf{A}}) \arrow[dr, "R_{\epsilon_1}"] \arrow[dd, "R_\epsilon"'] & \\
& Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'}) \arrow[dl, "\mathrm{Res}_{\epsilon_2}"] \arrow[l, phantom, "\Leftarrow \mathrm{Res}_\eta"'] \arrow[r, dotted, no head] & {} & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}'}/G_{\mathbf{A}'}) \arrow[dl, "R_{\epsilon_2}"] \arrow[l, phantom, "\Leftarrow R_\eta"'] \\
Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{A}''}/G_{\mathbf{A}''}) \arrow[rr] & & Sh_{\mathcal{N}}(g_{\mathbf{A}''}/G_{\mathbf{A}''}) &
\end{tikzcd}$$

i.e a natural isomorphism of functors

$$\mathbf{g}^* \Rightarrow \mathbf{g}_\flat^* \tag{5.24}$$

We left to the readers the definition of the functor $\mathbf{g}^{\triangle,*}$, and the natural isomorphism of functors $\mathbf{g}^{\triangle,*} \Rightarrow \mathbf{g}_\flat^{\triangle,*}$ and $\mathbf{g}^* \Rightarrow \mathbf{g}^{\triangle,*}$.

5.3 Main theorem: circle case

So far, we have constructed a diagram of isomorphisms of functors from $\Delta^\circ_{\Xi,W_{\mathbb{S}}}$ to ∞-**Cat**:

$$\begin{tikzcd}
\mathbf{g}^* \arrow[r, "(5.24)"] \arrow[d] & \mathbf{g}_\flat^* \arrow[d, "(5.18)"] & \\
\mathbf{g}^{\triangle,*} \arrow[r] & \mathbf{g}_\flat^{\triangle,*} \arrow[r, "(5.19)"'] & \mathbf{D}^{\triangle,*}
\end{tikzcd}$$

with the square being commutative. After taking the limit, we have obtained the following main theorem on sheaves in the circle case:

Theorem 5.25. *For G simply-connected, the natural isomorphisms defined above gives commutative diagram of equivalences between ∞-categories:*

$$\begin{tikzcd}
\lim \boldsymbol{g}^* \arrow[r] \arrow[d] & \lim \boldsymbol{g}_\flat^* \arrow[d] & & \\
\lim \boldsymbol{g}^{\triangle,*} \arrow[r] & \lim \boldsymbol{g}_\flat^{\triangle,*} \arrow[r] & \lim \boldsymbol{D}^{\triangle,*} & Sh_{\mathcal{N}}(G/G) \arrow[l, "(5.13)"']
\end{tikzcd}$$

□

Remark: (1) It is not hard to define similar diagrams for any reductive group, we shall omit the details.
(2) Use of exponential map on nilpotent cone has appeared in e.g. [25, p. 6.6] to understand cuspidal sheaves. The use of analytic exponential maps enable us to consider all character sheaves at once.

5.26. *The functors* $(?)^{\spadesuit;*}_{\clubsuit;\diamond}$.

Let $\diamond = (\mathscr{C}_\diamond, F_\diamond : \mathscr{C}_\diamond \to \Delta^\circ_{\Xi,W_{S^1}}) \in \mathbf{Cat}_{/\Delta^\circ_{\Xi,W_{S^1}}}$. For each functor $(?)^{\spadesuit,*}_{\clubsuit}$, where $? =$**D** or **g**, $\spadesuit = \triangle$ or $\varnothing$, $\clubsuit = \flat$ or $\varnothing$ whenever defined above, precomposing with $\diamond$ gives: $(?)^{\spadesuit;*}_{\clubsuit;\diamond} := (?)^{\spadesuit,*}_{\clubsuit} \circ F_\diamond : \mathscr{C}_\diamond \to \infty$-**Cat**. There is a canonical map $\lim(?)^{\spadesuit,*}_{\clubsuit} \to \lim(?)^{\spadesuit;*}_{\clubsuit;\diamond}$. We'll consider the question when we could reduce the diagram $\Delta^\circ_{\Xi,W_{S^1}}$ to $C_\diamond$, i.e when the above maps is an isomorphism. It's clear that if one of the five maps is an isomorphism, so the others.

Now assume G is simply-connected, we have a natural choice of $\diamond$ as follows: let C be a chamber in $\mathfrak{t}_{\mathbb{S},\mathbb{R}}$, Let $S_C = \{A_1, A_2, \ldots, A_r\}$ be the set of vertices of C. Let $\mathscr{P}_C$ be the category of non-empty subset of S_C with morphism inclusions. Fixing a total order on S_C gives a functor $\diamond_C : \mathscr{P}_C \to \Delta^{\circ}_{\Xi, W_{\mathbb{S}}}$, Let $(?)^{\spadesuit,*}_{\clubsuit,C} := (?)^{\spadesuit,*}_{\clubsuit,\diamond_C}$. Note that $(?)^{\spadesuit,*}_{\clubsuit,C}$ is actually independent of the choice of the total order. The map $\lim \mathbf{D}^{\triangle,*} \to \lim \mathbf{D}^{\triangle,*}_C$ is induced from the embedding of non-degenerate Cech complex to the degenerate Cech complex, hence it is an isomorphism. So we have the following:

Proposition 5.27. *The natural maps* $\lim(?)^{\spadesuit,*}_{\clubsuit} \to \lim(?)^{\spadesuit,*}_{\clubsuit,C}$ *are isomorphisms.* □

Remark: $\mathscr{P}_C$ is a finite diagram with only 1-morphisms while $\Delta^{\circ}_{\Xi, W_{S^1}}$ is an infinity diagram involving 2-morphisms. This is vaguely similar to the phenomenon that at Langlands dual side, the Tanakian formalism for adjoint group can be essentially reduced adjoint representation plus certain additional data.

Example 5.28. For $G = SL_2$, T=diagonal matrices.
$H := \begin{pmatrix} \frac{id\theta}{2} & 0 \\ 0 & -\frac{id\theta}{2} \end{pmatrix}$, then $\mathfrak{t}_{S^1} = \mathbb{C} \cdot H, \Lambda_{S^1} = \{\begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix} : n \in \mathbb{Z}\}$. For $s \in \mathbb{C}$, $G_s := G_{sH} =$

$$\begin{cases} \{\begin{pmatrix} a & be^{in\theta} \\ ce^{-in\theta} & d \end{pmatrix} : a, b, c, d \in \mathbb{C}, ad - bc = 1\} & s = n \in \mathbb{Z} \\ \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^*\} = T & s \notin \mathbb{Z} \end{cases}$$

$\Xi = \{nH : n \in \mathbb{Z}\}$. And for $\mathbf{n} = (n_1, n_2, \ldots, n_k) \in \mathbb{Z}^k$,

$$V_{\mathbf{n}} := V_{\mathbf{n}H} = \begin{cases} ((n-1, n+1) \times i\mathbb{R})H & n_1 = n_2 = \cdots = n_l = n \\ ((n, n+1) \times i\mathbb{R})H & \{n_1, n_2, \ldots, n_l\} = \{n, n+1\} \\ \varnothing & \exists n_i, n_j, |n_i - n_j| > 1 \end{cases}$$

The nontrivial proper parabolic P_σ are those $\sigma : \mathbf{n} = (n_1, n_2, \ldots, n_l) \to \mathbf{n}' = (n'_1, n'_2, \ldots, n'_k)$, such that $\{n_1, n_2, \ldots, n_l\} = n$, $\{n'_1, n'_2, \ldots, n'_k\} = \{n-1, n\}$ or $\{n, n+1\}$. In the first case, $P_\sigma = \begin{pmatrix} a & be^{in\theta} \\ 0 & a^{-1} \end{pmatrix}$, and in the second case, $P_\sigma = \begin{pmatrix} a & 0 \\ ce^{-in\theta} & a^{-1} \end{pmatrix}$. Let $C = (0, 1)$, Theorem 5.25 and Proposition 5.27 gives

$$Sh_{\mathcal{N}}(G/G) = \lim \begin{tikzcd} & Sh_{\mathcal{N}}(\mathfrak{t}/T) & \\ Sh_{\mathcal{N}}(\mathfrak{g}_0/G_0) \arrow[ur, "\mathrm{Res}_{\mathfrak{p}_{0\to\{0,1\}}}"] & & Sh_{\mathcal{N}}(\mathfrak{g}_1/G_1) \arrow[ul, "\mathrm{Res}_{\mathfrak{p}_{1\to\{0,1\}}}"'] \end{tikzcd}$$

If k is field, $char(k) = 0$, and for perverse sheaves, the above diagram can be explicitly calculated as:

$$\mathrm{Perv}_{\mathcal{N}}(G/G) = \lim \begin{tikzcd} & \mathrm{Vect} & \\ \mathrm{Vect} \oplus k[\mathbb{Z}/2]\text{-mod} \arrow[ur, "0\oplus F"] & & \mathrm{Vect} \oplus k[\mathbb{Z}/2]\text{-mod} \arrow[ul, "0\oplus F"'] \end{tikzcd}$$

$= \mathrm{Vect} \oplus \mathrm{Vect} \oplus k[(\mathbb{Z}/2 * \mathbb{Z}/2)]\text{-mod} = \mathrm{Vect} \oplus \mathrm{Vect} \oplus k[W_{aff}]\text{-mod}$

Where $F : k[\mathbb{Z}/2]\text{-mod} \to \mathrm{Vect}$ is the forgetful (restriction) functor. The two Vect are generated by two cuspidal sheaves, and $k[W_{aff}]$-mod corresponds to sheaves coming from Springer correspondence, see [5].

Chapter 6

Holomorphic gauge theory on elliptic curves

We prove analogous results in last chapter for G_E using the gauge uniformazation $\Omega^{0,1}(E,G)/C^\infty(E,G)$ in 6.1. We get a similar presentation of $Sh_{\mathcal{N}}(G_E)$ as glued from character sheaves on Lie algebras. In addition we also study the situation over $\mathcal{M}_{1,1}$ the moduli space of elliptic curves in 6.2.

6.1 Main theorem: ellitpic case

Let z be the coordinate on $\mathbb{C}$. Let ω_1, ω_2 be two complex numbers not contained in a real line. $\mathbb{E} = \mathbb{C}/(\omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z})$ is an elliptic curve. Let $\mathbb{S}_1, \mathbb{S}_2$ be two unit circles with angle coordinate θ_1, θ_2. We have an isomorphism of Lie groups $\mathbb{S}_1 \times \mathbb{S}_2 \xrightarrow{\simeq} \mathbb{E}$, by $(\theta_1, \theta_2) \mapsto \frac{\omega_1\theta_1 + \omega_2\theta_2}{2\pi}$. Most of Chapter 5 has direct analog with minor changes, we shall not repeat the proof but only state the facts.

This is an natural isomorphism of groupoids $\mathrm{Bun}_G^0(\mathbb{E})(\mathbb{C}) = \Omega^{0,1}(\mathbb{E}, \mathfrak{g})/C^\infty(\mathbb{E}, G)$. $\Gamma_{\mathbb{E}} := Hom_{Lie}(\mathbb{E}, \mathbb{C}^*)$, $\bar{\Omega}_{\mathbb{E}} := \Gamma_{\mathbb{E}} \otimes \mathbb{R}$, $\Lambda_{\mathbb{E}} := X_*(T) \otimes \Gamma_{\mathbb{E}}$, $\mathfrak{t}_{\mathbb{E}} := X_*(T) \otimes \Gamma_{\mathbb{E}} \otimes \mathbb{R} = \Lambda_{\mathbb{E}} \otimes \mathbb{R} = X_*(T) \otimes \bar{\Omega}_{\mathbb{E}} = \Gamma_{\mathbb{E}} \otimes \mathfrak{t}_{\mathbb{R}}$. We have, $\bar{\Omega}_{S^1} \xrightarrow{\bar{\partial}\log(-)} \Omega^{0,1}(E, \mathbb{C})$, which is injective onto the space of antiholomorphic 1-forms, i.e, those of the form $ad\bar{z}$, for $a \in \mathbb{C}$. We have $\mathfrak{t}_{\mathbb{E}} \hookrightarrow \Omega^{0,1}(\mathbb{E}, \mathfrak{t}) \hookrightarrow \Omega^{0,1}(\mathbb{E}, \mathfrak{g})$. $d\bar{z}$ gives an identification $\mathfrak{t} \simeq \mathfrak{t}_{\mathbb{E}}$. On the other hand, $\Lambda_{\mathbb{E}} \hookrightarrow X_*(T) \otimes C^\infty(\mathbb{E}, \mathbb{C}^*) = C^\infty(\mathbb{E}, T) \hookrightarrow C^\infty(\mathbb{E}, G)$. $C^\infty(\mathbb{E}, G)$ acts on $\Omega^{0,1}(\mathbb{E}, \mathfrak{g})$ via Gauge transformation, and under this action, $\Lambda_{\mathbb{E}}$ stablize $\mathfrak{t}_{\mathbb{E}}$ and the action is identified with the canonical action. Similarly, the action of $N_G(T) \subset C^\infty(\mathbb{E}, G)$ stablize $\Omega^{0,1}(\mathbb{E}, \mathfrak{t}) \subset \Omega^{0,1}(\mathbb{E}, \mathfrak{g})$. T acts trivially, $W = N_G(T)/T$ acts on $\Omega^{0,1}(E, \mathfrak{t})$ which can be identified with the canonical action of W on $\mathfrak{t}_{\mathbb{E}}$. Let $W_{\mathbb{E}} := \Lambda_{\mathbb{E}} \rtimes W \subset Aff(\mathfrak{t}_{\mathbb{E}})$. Fix a lifting $W \to N_G(T)$ gives a lifting $W_{\mathbb{E}} \to C^\infty(\mathbb{E}, G)$ as a subset.

Let $\Phi_{\mathbb{E}}$ be the set $\{f + \alpha' : f \in \Gamma_{\mathbb{E}}, \alpha' \in \Phi\} \subset Map(\mathfrak{t}_{\mathbb{E}}, \bar{\Omega}_{\mathbb{E}})$. For any $\alpha = f + \alpha' \in \Phi_{\mathbb{E}}$, define $U_\alpha := \exp(\mathbb{C} f X_\alpha) \subset C^\infty(\mathbb{E}, G)$ the corresponding 1-parameter subgroup. Let $\Phi_{\mathbb{E},B} := \{\alpha \in \Phi_{\mathbb{E}} : \hat{\alpha}(B) = 0\}$.

For $B \in \mathfrak{t}_{\mathbb{E}}$, let $G_B := C_{C^\infty(\mathbb{E},G)}(B)$ and $\mathfrak{g}_B := Lie(G_B) \subset C^\infty(\mathbb{E}, \mathfrak{g})$. Then $\mathfrak{g}_B = \{X(z) \in C^\infty(\mathbb{E}, \mathfrak{g}) : \bar{\partial}X + [B, X] = 0\}$, Let $X = H(z) + \sum_{\alpha\in\Phi} f_\alpha(z)X_\alpha$, where $H : \mathbb{E} \to \mathfrak{t}, f_\alpha : \mathbb{E} \to \mathbb{C}$. Then the equation $\bar{\partial}X + [B, X] = 0$ is equivalent to $\bar{\partial}H(z) = 0, \bar{\partial}f_\alpha = f_\alpha\alpha(B)$. The first equation has only constant solution. The second equation has solution $f(z) = e^{a\bar{z}+h(z)}$, for some holomorphic function $h(z)$ on $\mathbb{C}$, with $ad\bar{z} = \alpha(B) \in \bar{\Omega}_{\mathbb{E}}$ and satisfying the periodic conditions $f(z + \omega_1) = f(z), f(z + \omega_2) = f(z)$. We get $h(z) = cz + d$, for some constants c, d. And solution exist only when $\alpha(B) = \bar{\partial}\log f'$ for some $f' \in \Gamma_{\mathbb{E}}$. In this case, the solution is f' up to a constant. In conclusion, we have:

$$G_B^0 = < T, U_\alpha : \alpha \in \Phi_{\mathbb{E},B} >$$

To calculate to full centralizer, it suffices to show that $(\mathbb{C}^\infty(\mathbb{E}, T) \cdot N_G(T)) \cap G_B \subset \Lambda_{\mathbb{E}} \cdot N_G(T)$. Now suppose $Fw \in (\mathbb{C}^\infty(\mathbb{E}, T) \cdot N_G(T)) \cap G_B$, we have $w(B) \in \mathfrak{t}_E$ and $B = F(w(B)) = w(B) - \bar{\partial}\log(F)$, hence $\bar{\partial}\log(F) \in \mathfrak{t}_{\mathbb{E}}$. By previous argument, $F \in \Lambda_{\mathbb{E}} \cdot T$. We get:

$$G_B = < G_B^0, C_{W_{\mathbb{E}}}(B) >$$

$\mathfrak{g}_{\mathbf{B}}^{\triangle} := \mathfrak{g}_{\mathbf{B}} d\bar{z}$. The map $-B : \mathfrak{g}_B^{\triangle} \to \mathfrak{g}_B$, $X \mapsto \frac{(X-B)}{d\bar{z}}$ intertwine the G_B action, and induces $\mathfrak{g}_B^{\triangle}/G_B \simeq \mathfrak{g}_B/G_B$. Where the first action is Gauge transformation and the second action is conjugation.

Define $G_{\mathbf{B}}, \mathfrak{g}_{\mathbf{B}}^{\triangle}, \mathfrak{g}_{\mathbf{B}}$ as before. We also have the (contractible space of) functor(s) $-\mathbf{B}^* : Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{B}}/G_{\mathbf{B}}) \to Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{B}}^{\triangle}/G_{\mathbf{B}})$

The identification $\mathbb{E} \simeq \mathbb{S}_1 \times \mathbb{S}_2$ gives $\Gamma_{\mathbb{E}} = \Gamma_{\mathbb{S}_1} \times \Gamma_{\mathbb{S}_2}$, $\mathfrak{t} = \mathfrak{t}_{\mathbb{E}} = \mathfrak{t}_{\mathbb{S}_1,\mathbb{R}} \times \mathfrak{t}_{\mathbb{S}_2,\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \times \mathfrak{t}_{\mathbb{R}}$, $\Lambda_{\mathbb{E}} = \Lambda_{\mathbb{S}_1} \times \Lambda_{\mathbb{S}_2}$ acts on $\mathfrak{t}_{\mathbb{E}}$ by product action, and W acts diagonally. We have commutative diagrams:

$$\begin{tikzcd} & W_{\mathbb{E}} \arrow[dl] \arrow[dr] & \\ W_{\mathbb{S}_1} \arrow[dr] & & W_{\mathbb{S}_2} \arrow[dl] \\ & W & \end{tikzcd} \quad \begin{tikzcd} & \Phi_{\mathbb{E}} \arrow[dl] \arrow[dr] & \\ \Phi_{\mathbb{S}_1} \arrow[dr] & & \Phi_{\mathbb{S}_2} \arrow[dl] \\ & \Phi & \end{tikzcd} \quad \begin{tikzcd} & C^\infty(\mathbb{E}, G) \arrow[dl] \arrow[dr] & \\ C^\infty(\mathbb{S}_1, G) \arrow[dr] & & C^\infty(\mathbb{S}_2, G) \arrow[dl] \\ & G & \end{tikzcd}$$

For $\mathbf{B} = (\mathbf{A}_1, \mathbf{A}_2)$ under the identification, we have inside the above diagrams:

$$\begin{tikzcd} & W_{\mathbb{E},\mathbf{B}} \arrow[dl] \arrow[dr] & \\ W_{\mathbb{S}_1,\mathbf{A}_1} \arrow[dr] \arrow[r, phantom, "\square"] & & W_{\mathbb{S}_2,\mathbf{A}_2} \arrow[dl] \\ & W & \end{tikzcd} \quad \begin{tikzcd} & \Phi_{\mathbb{E},\mathbf{B}} \arrow[dl] \arrow[dr] & \\ \Phi_{\mathbb{S}_1,\mathbf{A}_1} \arrow[dr] \arrow[r, phantom, "\square"] & & \Phi_{\mathbb{S}_2,\mathbf{A}_2} \arrow[dl] \\ & \Phi & \end{tikzcd} \quad \begin{tikzcd} & G_{\mathbf{B}} \arrow[dl] \arrow[dr] & \\ G_{\mathbf{A}_1} \arrow[dr] \arrow[r, phantom, "\square"] & & G_{\mathbf{A}_2} \arrow[dl] \\ & G & \end{tikzcd}$$

where all arrows are injections.

Now assume G is simply-connected, Let $\Xi_{\mathbb{E}} := \Xi_{\mathbb{S}_1} \times \Xi_{\mathbb{S}_2}$. For $\mathbf{B} \in \mathfrak{t}_{\mathbb{E}}$, let $C_{\mathbf{B},\mathbb{E}}^{\triangle} := C_{\mathbf{A}_1}^{\triangle} \times C_{\mathbf{A}_2}^{\triangle} \subset \mathfrak{t}_{\mathbb{E}}$, $C_{\mathbf{B},\mathbb{E}}^{\triangle}$ is $W_{\mathbb{E},\mathbf{B}}$-invariant open in $\mathfrak{t}_{\mathbb{E}}$. $D_{\mathbf{B},\mathbb{E}}^{\triangle} :=^{G_{\mathbf{B}}} C_{\mathbf{B},\mathbb{E}}$. We have the functors

$$\mathbf{D}_{\mathbb{E},+}^{\triangle} = ((D_{\mathbf{B},\mathbb{E}}^{\triangle}/G_{\mathbf{B}}), Gg) : \Delta_{+,\Xi_{\mathbb{E}},W_{\mathbb{E}}}^{op,\circ} \to \mathbf{Prestk} \tag{6.1}$$

$$\tilde{\mathbf{D}}^{\triangle}_{\mathbb{E},+} = ((D^{\triangle}_{\mathbf{B},\mathbb{E}}/G_{\mathbf{B}}), Gg) : \Delta^{op,\circ}_{+,\mathfrak{t}_{\mathbb{E}},W_{\mathbb{E}}} \to \mathbf{Prestk} \tag{6.2}$$

are colimit diagrams.

For $\delta = (\delta_1, \delta_2) : \mathbf{B}' = (\mathbf{A}'_1, \mathbf{A}'_2) \to \mathbf{B} = (\mathbf{A}_1, \mathbf{A}_2)$, Let $V^{\triangle}_{\delta,\mathbb{E}} := V^{\triangle}_{\delta_1,\mathbb{R}} \times V^{\triangle}_{\delta_2,\mathbb{R}}, V_{\delta,\mathbb{E}} := V_{\delta_1,\mathbb{R}} \times V_{\delta_2,\mathbb{R}}$, define

$$\mathbf{g}^{*}_{\mathbb{E},\flat} := (Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{B}}/G_{\mathbf{B}}), R_{V_{\delta,\mathbb{E}}} \circ Ad^{*}_{\dot{w}}) : \Delta_{\Xi_{\mathbb{E}},W_{\mathbb{E}}} \to \infty\textbf{-Cat}$$

and $\mathbf{g}^{\triangle,*}_{\mathbb{E},\flat}$ similarly, we have

Theorem 6.3. *For G simply-connected, there is natural isomorphisms of equivalences between ∞-categories:*

$$\begin{tikzcd} \lim \boldsymbol{g}^{*}_{\mathbb{E},\flat} \arrow[d] & & \\ \lim \boldsymbol{g}^{\triangle,*}_{\mathbb{E},\flat} \arrow[r] & \lim \boldsymbol{D}^{\triangle,*}_{\mathbb{E}} & Sh_{\mathcal{N}}(G_{\mathbb{E}}) \arrow[l] \end{tikzcd}$$

□

6.2 Parallel transport over $\mathcal{M}_{1,1}$

Let X be a smooth manifold, $\mathcal{L} \subset T^*X$ be a conical subset, then the functor $Sh_{\mathcal{L}} : X_{sm} \to \infty\textbf{-Cat}$ by $U \to Sh_{\mathcal{L}|_U}(U)$ is a sheaf of categories. Where X_{sm} is the site of manifolds with submersion to X.

Let $\mathcal{W} = \{(\omega_1, \omega_2) \in \mathbb{C}^2 : \omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1 \neq 0\}$. For $\omega = (\omega_1, \omega_2) \in \mathcal{W}$, let $\mathbb{E}_\omega := \mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2)$. The group $\mathbb{C}^* \times GL(2,\mathbb{Z})$ acts on $\mathcal{W}$ via $(\lambda, \gamma) \cdot \omega := \lambda\gamma\omega$. Then $\mathcal{W}/(\mathbb{C}^* \times GL(2,\mathbb{Z})) \to \mathcal{M}_{1,1}$ is an isomorphism of complex analytic stacks. Let $\mathrm{Bun}^{0,ss}_{G,1}$ be the universal family of degree 0 semistable bundles over $\mathcal{M}_{1,1}$. We have $\pi : \mathrm{Bun}^{0,ss}_{G,1} \to \mathcal{M}_{1,1}$, where $\pi^{-1}(E) = G_E$. Let $\mathrm{Bun}^{0,ss}_{G,\mathcal{W}} := \mathrm{Bun}^{0,ss}_{G,1} \times_{\mathcal{M}_{1,1}} \mathcal{W}$.

For $\lambda \in \mathbb{C}^*$, we have $\lambda : \mathbb{E}_\omega \to \mathbb{E}_{\lambda\omega}$. This induces pullback of connections on the trivial bundles $\lambda^* : Conn(\mathbb{E}_{\lambda\omega} \times G) \to Conn(\mathbb{E}_\omega \times G)$. Under the identification $Conn(\mathbb{E}_\omega \times G) \simeq \Omega^{0,1}(\mathbb{E}, \mathfrak{g}) \simeq C^\infty(\mathbb{S}_1 \times \mathbb{S}_2)d\bar{z}$, $(\lambda^*)^{-1} : C^\infty(\mathbb{S}_1 \times \mathbb{S}_2)d\bar{z} \to C^\infty(\mathbb{S}_1 \times \mathbb{S}_2)d\bar{z}$ is identified with $f(\theta_1, \theta_2)d\bar{z} \mapsto \lambda f(\theta_1, \theta_2)d\bar{z}$. For $\gamma \in GL(2,\mathbb{Z})$, we have $\gamma : \mathbb{E}_\omega \to \mathbb{E}_{\gamma\omega}$. Similarly, $(\gamma^*)^{-1}$ is identified with $f(\theta)d\bar{z} \mapsto f(\gamma\theta)d\bar{z}$.

Under the identification $\mathfrak{t}_{\mathbb{E}_\omega} \simeq \mathfrak{t}_{\mathbb{S}_1} \times \mathfrak{t}_{\mathbb{S}_2}$, for $\mathbf{B} = (\mathbf{A}_1, \mathbf{A}_2) \in \mathfrak{t}_{\mathbb{S}_1} \times \mathfrak{t}_{\mathbb{S}_2}$, let $\mathbf{B}_w \in \mathfrak{t}_{\mathbb{E}_\omega}$ the corresponding element. the groups $G_{\mathbf{B}_w}$ (identified as subgroup of $C^\infty(\mathbb{S}_1 \times \mathbb{S}_2, G)$) and hence $\mathfrak{g}_{\mathbf{B}}$ do not depend on ω, denote them by $G_{\mathbf{B}}$ and $\mathfrak{g}_{\mathbf{B}}$. We also regard $\mathfrak{g}^{\triangle}_{\mathbf{B}} = \mathfrak{g}_{\mathbf{B}}d\bar{z}$ as some abstract space does not depend on ω, and for given ω, we have an embedding $\mathfrak{g}^{\triangle}_{\mathbf{B}} \to \Omega^{0,1}(\mathbb{E}_\omega, \mathfrak{g})$, and by definition $\mathfrak{g}^{\triangle}_{\mathbf{B}_w}$ is the image.

The functor 6.2 varies over $\mathcal{W}$, and gives a colimit diagram over $\mathcal{W}$:

$$\tilde{\mathbf{D}}^{\triangle}_{\mathcal{W},+} := (D^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}}, Gg) : \Delta^{op,\circ}_{+,\mathfrak{t}_{\mathbb{S}_1}\times\mathfrak{t}_{\mathbb{S}_2},(\Lambda_{\mathbb{S}_1}\times\Lambda_{\mathbb{S}_2})\rtimes W} \to \mathbf{Prestk} \tag{6.4}$$

where $\tilde{\mathbf{D}}^{\triangle}_{\mathcal{W},+}(pt) := \mathrm{Bun}^{0,ss}_{G,\mathcal{W}}$, $D^{\triangle}_{\mathbf{B},\mathcal{W}} := \bigcup_{\omega\in\mathcal{W}} D^{\triangle}_{\mathbf{B},\mathbb{E}_\omega} \times \{\omega\} \subset \bigcup_{w\in\mathcal{W}} \mathfrak{g}^{\triangle}_{\mathbf{B}_w} = \mathfrak{g}^{\triangle}_{\mathbf{B}} \times \mathcal{W} =: \mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{W}}$. The geometric structure on $D^{\triangle}_{\mathbf{B},\mathcal{W}}$ is induced as being open subset of $\mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{W}}$. The $G_{\mathbf{B}}$ action on $D^{\triangle}_{\mathbf{B},\mathcal{W}}$ is induced from the Gauge action of $G_{\mathbf{B}_w}$ on $\mathfrak{g}^{\triangle}_{\mathbf{B}_w}$.

Let $\mathcal{N}_{\mathfrak{c}} := \mathcal{N} \times T^*_{\mathfrak{c}_\mathfrak{g}}\mathfrak{c}_\mathfrak{g} \subset T^*\mathfrak{g} \times T^*\mathfrak{c}_\mathfrak{g} = T^*(\mathfrak{g}\times\mathfrak{c}_\mathfrak{g})$. Let $s : \mathfrak{g}\times\mathfrak{c}_\mathfrak{g} \to \mathfrak{g}\times\mathfrak{c}_\mathfrak{g}$ defined by $s(X,C) := (X+C,C)$. We have:

Lemma 6.5. $s^*(\mathcal{N}_\mathfrak{c}) = \mathcal{N}_\mathfrak{c}$.□

Choose a C^∞ map $C_{\mathcal{W}} : \mathcal{W} \to \mathfrak{c}^{\triangle}_{\mathbf{B},\mathcal{W}} := \bigcup_{w\in\mathcal{W}} \mathfrak{c}^{\triangle}_{\mathbf{B}_w} \subset \mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{W}}$. For example, we could choose $C_{\mathcal{W}}(\omega) = B(\omega) := B_\omega$, for some $B \in \mathbf{B}$. then $-C_{\mathcal{W}} : \mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}} \to \mathfrak{g}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}} := (\mathfrak{g}_{\mathbf{B}}\times\mathcal{W})/G_{\mathbf{B}}$ by $(X,\omega) \mapsto (\frac{X-C_{\mathcal{W}}(\omega)}{d\bar{z}},\omega)$ is an isomorphism. Where the later action of $G_{\mathbf{B}}$ is by conjugation on the factor $\mathfrak{g}_{\mathbf{B}}$. Let $\mathcal{N}_{\mathcal{W}} := \mathcal{N} \times T^*_{\mathcal{W}}\mathcal{W} \subset T^*(\mathfrak{g}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}})$, then by Lemma 6.5, $-C^*_{\mathcal{W}}(\mathcal{N}_{\mathcal{W}}) \subset T^*(\mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}})$ does not depend on the choice of $C_{\mathcal{W}}$. We denote this set still by $\mathcal{N}_{\mathcal{W}}$. We have Gg^* preserve $\mathcal{N}_{\mathcal{W}}$ on $D^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}}$. Then by (6.4), $\mathcal{N}_{\mathcal{W}} \subset T^*(D^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}})$ descent to a subset in $T^*\mathrm{Bun}^{0,ss}_{G,\mathcal{W}}$, which we shall still denote by $\mathcal{N}_{\mathcal{W}}$. It is clear that $\mathcal{N}_{\mathcal{W}}$'s are isotropic. We have the limit diagram $\tilde{\mathbf{D}}^{\triangle,*}_{\mathcal{W},+} := (Sh_{\mathcal{N}_{\mathcal{W}}}(D^{\triangle}_{\mathbf{B},\mathcal{W}}/G_{\mathbf{B}}), Gg^*)$.

Proposition 6.6. *The isotropic subset $\mathcal{N}_{\mathcal{W}}$ of $T^*\mathrm{Bun}^{0,ss}_{G,\mathcal{W}}$ is $\mathbb{C}^*\times GL(2,\mathbb{Z})$ invariant, hence it descent to some $\mathcal{N}_1 \subset T^*\mathrm{Bun}^{0,ss}_{G,1}$.*

Proof: For $\lambda\in\mathbb{C}^*$, under the identification $\mathfrak{t}_{\mathbb{E}_w} \simeq \mathfrak{t}d\bar{z}$, for any $\omega\in\mathcal{W}$. $\lambda(B_\omega) = B_{\lambda\omega} \in \mathfrak{t}d\bar{z}, \lambda^*(D^{\triangle}_{B,\mathbb{E}_{\lambda\omega}}) = D^{\triangle}_{B,\mathbb{E}_\omega} \subset \mathfrak{g}^{\triangle}_B$, and there is a commutative diagrams:

$$\begin{array}{ccccccc}
D^{\triangle}_{B,\mathcal{W}} & \xrightarrow{\lambda^*} & D^{\triangle}_{B,\mathcal{W}} & & D^{\triangle}_{B,\mathcal{W}} & \xrightarrow{\lambda^*} & D^{\triangle}_{B,\mathcal{W}} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathfrak{g}^{\triangle}_{B,\mathcal{W}} & \xrightarrow{\lambda^*} & \mathfrak{g}^{\triangle}_{B,\mathcal{W}} & & \mathrm{Bun}^{0,ss}_{G,\mathcal{W}} & \xrightarrow{\lambda^*} & \mathrm{Bun}^{0,ss}_{G,\mathcal{W}} \\
\downarrow{\scriptstyle -B} & & \downarrow{\scriptstyle -B} & & \downarrow & & \downarrow \\
\mathfrak{g}_{B,\mathcal{W}} & \xrightarrow{\lambda^*} & \mathfrak{g}_{B,\mathcal{W}} & & \mathcal{W} & \xrightarrow{\lambda^{-1}} & \mathcal{W}
\end{array}$$

The bottom row on the left preserve $\mathcal{N}_{\mathcal{W}}$, hence the top rows preserve $\mathcal{N}_{\mathcal{W}}$, and hence the middle row on the right.

For $\gamma\in GL(2,\mathbb{Z})$, We have $\gamma^* : \mathfrak{g}^{\triangle}_{B,\mathcal{W}} \to \mathfrak{g}^{\triangle}_{\gamma B,\mathcal{W}}$. Let $\mathfrak{g}^{\triangle,r}_{B_\omega} := \mathfrak{g}^{\triangle,r}_{B_\omega,\epsilon_{pt}} \subset \mathfrak{g}^{\triangle}_{B_\omega}$ for $\epsilon_{pt} : B_\omega \to pt$ a final morphism, and $\mathfrak{g}^{\triangle,r}_{B,\mathcal{W}} := \bigcup_{w\in\mathcal{W}} \mathfrak{g}^{\triangle,r}_{B_\omega} \subset \mathfrak{g}^{\triangle}_{B,\mathcal{W}}$. We have also $\gamma^* : \mathfrak{g}^{\triangle,r}_{B,\mathcal{W}} \to \mathfrak{g}^{\triangle,r}_{\gamma B,\mathcal{W}}$. There are similar diagrams:

$$\begin{tikzcd}
\mathfrak{g}^{\triangle,r}_{B,\mathcal{W}} \arrow[r, "\gamma^*"] \arrow[d] & \mathfrak{g}^{\triangle,r}_{\gamma B,\mathcal{W}} \arrow[d] & \mathfrak{g}^{\triangle,r}_{B,\mathcal{W}} \arrow[r, "\gamma^*"] \arrow[d] & \mathfrak{g}^{\triangle,r}_{\gamma B,\mathcal{W}} \arrow[d] \\
\mathfrak{g}^{\triangle}_{B,\mathcal{W}} \arrow[r, "\gamma^*"] \arrow[d, "-B"] & \mathfrak{g}^{\triangle}_{\gamma B,\mathcal{W}} \arrow[d, "-\gamma B"] & \mathrm{Bun}^{0,ss}_{G,\mathcal{W}} \arrow[r, "\gamma^*"] \arrow[d] & \mathrm{Bun}^{0,ss}_{G,\mathcal{W}} \arrow[d] \\
\mathfrak{g}_{B,\mathcal{W}} \arrow[r, "\gamma^*"] & \mathfrak{g}_{\gamma B,\mathcal{W}} & \mathcal{W} \arrow[r, "\gamma^{-1}"] & \mathcal{W}
\end{tikzcd}$$

The bottom arrow on the left again preserves $\mathcal{N}_{\mathcal{W}}$. $\square$

We have the sheaf of categories $Sh_{\mathcal{N}_1}$ on $\mathrm{Bun}^{0,ss}_{G,1}$. Let $Sh_{\mathcal{N}_1}(\mathrm{Bun}^{0,ss}_G) := \pi_*(Sh_{\mathcal{N}_1})$ a sheaf of categories on $\mathcal{M}_{1,1}$. For any $f : \mathcal{U} \to \mathcal{M}_{1,1}$ a smooth map, by definition, $Sh_{\mathcal{N}_1}(\mathrm{Bun}^{0,ss}_G)(\mathcal{U}) = Sh_{d\tilde{f}^*(\mathcal{N}_1)}(\mathrm{Bun}^{0,ss}_{G,1} \times_{\mathcal{M}_{1,1}} \mathcal{U})$, where $\tilde{f} : \mathrm{Bun}^{0,ss}_{G,1} \times_{\mathcal{M}_{1,1}} \mathcal{U} \to \mathrm{Bun}^{0,ss}_{G,1}$ is the base change of f.

Theorem 6.7. *$Sh_{\mathcal{N}_1}(\mathrm{Bun}^{0,ss}_G)$ is a locally constant sheaf of categories on $\mathcal{M}_{1,1}$. And for $i : \{E\} \to \mathcal{M}_{1,1}$ a point, the natural map $i^* : Sh_{\mathcal{N}_1}(\mathrm{Bun}^{0,ss}_G)_E \to Sh(G_E)$ factor through $i^* : Sh_{\mathcal{N}_1}(\mathrm{Bun}^{0,ss}_G)_E \to Sh_{\mathcal{N}}(G_E)$ which is an equivalence of categories.*

Proof: Suffices to prove the statements after base change to $\mathcal{W}$. Now assume $\mathcal{U}$ is a contractible open subset of $\mathcal{W}$ and $\omega \in \mathcal{U}$. The notations introduced below are self-evident. Let $i_\omega : G_{\mathbb{E}_\omega} \to \mathrm{Bun}^{0,ss}_{G,\mathcal{U}}$ the closed embedding. There is a natural transformation of functors $\imath_\omega : \tilde{\mathbf{D}}^{\triangle}_{\mathbb{E}_\omega,+} \Rightarrow \tilde{\mathbf{D}}^{\triangle}_{\mathcal{U},+}$ induced by the inclusion $i_{\omega,\mathbf{B}} : D^{\triangle}_{\mathbf{B},\mathbb{E}_\omega} \to D^{\triangle}_{\mathbf{B},\mathcal{U}}$. We have colim $(\imath_\omega) \simeq i_\omega$. The following diagram is commutative:

$$\begin{tikzcd}
D^{\triangle}_{\mathbf{B},\mathbb{E}_\omega}/G_{\mathbf{B}} \arrow[r, "j_\omega"] \arrow[d, "i_\omega"] & \mathfrak{g}^{\triangle}_{\mathbf{B}_\omega}/G_{\mathbf{B}} \arrow[r, "-C_{\mathcal{U}}(\omega)", "\simeq"'] \arrow[d, "i_\omega"] & \mathfrak{g}_{\mathbf{B}}/G_{\mathbf{B}} \arrow[d, "i_\omega"] \\
D^{\triangle}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}} \arrow[r, "j_{\mathcal{U}}"] & \mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}} \arrow[r, "-C_{\mathcal{U}}", "\simeq"'] & \mathfrak{g}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}}
\end{tikzcd}$$

It induces a commutative diagram on sheaves:

$$\begin{tikzcd}
Sh_{\mathcal{N}}(D^{\triangle}_{\mathbf{B},\mathbb{E}_\omega}/G_{\mathbf{B}}) & Sh_{\mathcal{N}}(\mathfrak{g}^{\triangle}_{\mathbf{B}_w}/G_{\mathbf{B}}) \arrow[l, "j^*_\omega"', "\simeq"] & Sh_{\mathcal{N}}(\mathfrak{g}_{\mathbf{B}_\omega}/G_{\mathbf{B}}) \arrow[l, "\simeq"] \\
Sh_{\mathcal{N}_{\mathcal{U}}}(D^{\triangle}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}}) \arrow[u, dashed, "i^*_\omega"'] & Sh_{\mathcal{N}_{\mathcal{U}}}(\mathfrak{g}^{\triangle}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}}) \arrow[l, "j^*_{\mathcal{U}}"', "\simeq"] \arrow[u, dashed, "i^*_\omega"'] & Sh_{\mathcal{N}_{\mathcal{U}}}(\mathfrak{g}_{\mathbf{B},\mathcal{U}}/G_{\mathbf{B}}) \arrow[l, "\simeq"] \arrow[u, "\simeq", "i^*_\omega"']
\end{tikzcd}$$

Where j^*_ω being equivalence by Corollary 4.5, and similarly for $j^*_{\mathcal{U}}$. And i^*_ω on the right is equivalence since $\mathcal{U}$ is contracitble and $\mathcal{N}_{\mathcal{U}} = \mathcal{N} \times T^*_{\mathcal{U}}\mathcal{U} \subset T^*(\mathfrak{g}_{\mathbf{B}}/G_{\mathbf{B}}) \times T^*\mathcal{U}$. We conclude that all the vertical i^*_ω's preverse nilpotent singular support and are equivalences. Hence the natural transformation $\imath_\omega$ induces an isomorphism $\imath^*_\omega : \tilde{\mathbf{D}}^{\triangle,*}_{\mathcal{U},+} \Rightarrow \tilde{\mathbf{D}}^{\triangle,*}_{\mathbb{E}_\omega,+}$. Taking limit, we get the desired results. $\square$

Remark: In fact, i is non-characteristic w.r.t $\mathcal{N}_1$ and $i^*(\mathcal{N}_1) = \mathcal{N}$. This in particular implies that $i^* \simeq i^!$ up to a shift, and both preserve nilpotent singular support.

Appendix A

Analytic stacks

Let **ComSp** be the site of complex spaces, where the coverings are etale surjective maps. A morphism between two complex spaces is *etale* if it is a local biholomorphism. A *prestack* is by definition a functor $\mathscr{X} : \mathbf{ComSp} \to \mathbf{Grpd}$, and a *stack* is a prestack which is a sheaf. An morphism $f : \mathscr{X} \to \mathscr{Y}$ between stacks is representable if for any morphism from a complex space $g : Y \to \mathscr{Y}, Y \times_{\mathscr{Y}} \mathscr{X}$ is representable by complex space.

A morphism between two complex spaces is *smooth* if it is locally isomorphic to $U \times D^n \to U$, where D is the unit disk in $\mathbb{C}$. A representable morphism between two stacks is called *smooth* if it is so after base change from any complex space. Let P be a property of morphism that is stable in smooth topology on **ComSp**, we say a represetable morphism has property P if it has property P after base change by smooth morphism from any complex space. Such properties include being surjective, etale, smooth, closed embedding, open embedding, open dense embedding, isomorphism. It's not hard to see that the two definition of being smooth of a representable morphism agrees.

An analytic stack is by definition a stack $\mathscr{X}$ such that:
(1) $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is representable, and
(2) There is smooth morphism surjective $\pi : X \to \mathscr{X}$, where X is a complex space.

A analytic stack is *smooth* if in the above definition, X is a complex manifold. To study smooth analytic stack $\mathscr{X}, \mathscr{Y}$ and smooth morphism between them, it suffices to study functor $\mathscr{X}, \mathscr{Y}$ after restriction to **Cplx** the site of complex manifolds.

Definition A.1. $f : \mathscr{X} \to \mathscr{Y}$ is *generically open* if there is $\mathscr{U} \subset \mathscr{X}$ open dense embedding, such that $f|_{\mathscr{U}} : \mathscr{U} \to \mathscr{Y}$ is open embedding.

Lemma A.2. *An etale and generically open morphism of analytic stacks is open embedding.*

Let $x \in Ob(\mathscr{X}(\mathbb{C}))$, we say f is *etale at* x if for any base change $Y \to \mathscr{Y}$, $f' : X := \mathscr{X} \times_{\mathscr{Y}} Y \to Y$ is etale at any point $x' \in X$ over x. By definition, etaleness only depends on isomorphism class of x, so we can also speak about f being etale at $\underline{x} \in |\mathscr{X}|$. We have the locus in $|\mathscr{X}|$ where f is etale is open.

It's easy to see that f is etale if and only if it is etale at every $|\mathscr{X}|$.

$T_x\mathscr{X}$ the tangent groupoid at x is defined to be the fiber category of $\sigma : \mathscr{X}(\mathbb{C}[\epsilon]) \to \mathscr{X}(\mathbb{C})$ over x. I.e. $Ob(T_x\mathscr{X}) := \{(v,\phi) : v \in \mathscr{X}(\mathbb{C}[\epsilon]), \phi : \sigma(v) \xrightarrow{\sim} x\}$, and morphism are those induces identity on x. There is an action of $Aut(x)$ on $T_x\mathscr{X}$ via $(v,\phi) \mapsto (v, g\circ\phi)$ We have natural map of groupoid $df_x : T_x\mathscr{X} \to T_{f(x)}\mathscr{Y}$ by postcomposition with f. The map interwine the action of $Aut(x)$ and $Aut(f(x))$. By base change, We have

Proposition A.3. *Assume $\mathscr{X},\mathscr{Y}$ smooth analytic stacks, then f is etale at x if and only if df_x is an equivalence.* □

Tangent groupoid and tangent complex. For smooth analytic stack $\mathscr{X}$, the tangent groupoid $T_x\mathscr{X}$ has a natural structure of category in vector spaces such that the commutativity constraint is trivial. Such datum is equivalent to complex of vector spaces in degree -1,0. The assignment is by associate such a category $\mathscr{C}$ to $H^{-1} \to H^0$ where the differential is trivial and $H^0(\mathscr{C})$:=isomorphism classes of objects in $\mathscr{C}$, and $H^{-1}(\mathscr{C})$:= the automorphisms of identity object. Note that both of them have vector space structures. Under this assignment, $Aut(x)$ acts linearly on $H^i(T_x\mathscr{X})$, and df_x induces an linear map between H^i 's and it is an isomorphism if and only if the original functor between groupoids is an equivalence. We have $H^{-1}(T_x\mathscr{X}) = Lie(Aut(x))$, and the action of $Aut(x)$ is conjugation.

Example A.4. Let $\mathscr{X} = X/G$ be the quotient stack, $T\mathscr{X}$ is represented by the complex $\mathfrak{g}\otimes_{\mathbb{C}}\mathscr{O}_X \to TX$, for $x \in X$, let $\bar{x}$ be the image of x in $\mathscr{X}$, then $T_{\bar{x}}\mathscr{X}$ is quasi-isomorphic to the complex $\mathfrak{g} \to T_xX$, where the arrow is $H \mapsto \frac{d}{dt}|_{t=0}\exp(tH)x$. We have $Aut(\bar{x}) = C_G(x)$ the stablizer at x acts on $T_{\bar{x}}\mathscr{X} = \{\mathfrak{g} \xrightarrow{\delta} T_xX\}$, by conjugation on $\mathfrak{g}$, and for $g \in Aut(\bar{x}), v = \gamma'(0) \in T_xX$, for $\gamma(t)$ a curve through x. $g\cdot v := \frac{d}{dt}|_{t=0}g\gamma(t)$. Then δ is an $Aut(\bar{x})$ module map. Indeed, $\delta(ad_g(H)) = \frac{d}{dt}|_{t=0}\exp(tad_g(H))x = \frac{d}{dt}|_{t=0}g\exp(tH)g^{-1}x = \frac{d}{dt}|_{t=0}g\exp(tH)x = g\cdot\delta(H)$, where $g^{-1}x = x$ because $g \in C_G(x)$. □

Appendix B

Semisimple and semistable bundles

In this chapter, we collected some basic facts about G-bundles on elliptic curve, summarized in Proposition B.2.

Following [4, 9, 10], we make the following

Definition B.1. Let $\mathcal{P}$ be a G bundle on a compact Riemann surface C.
$\mathcal{P}$ is of *degree 0* if it lies in the neutral component $\mathrm{Bun}_G(C)^0$ of $\mathrm{Bun}_G(C)$
$\mathcal{P}$ is *semisimple* if $\mathcal{P}$ has reduction to a maximal torus T.
Let $C = E$ be an elliptic curve.
$\mathcal{P}$ is *semistable* if the associated adjoint bundle $\mathfrak{g}_{\mathcal{P}}$ is semistable.

It's easy to see that semisimple semistable G bundles of degree 0 are exactly those in the image of the map $\mathrm{Bun}_T(C)^0 \to \mathrm{Bun}_G(C)^0$. Let $\mathrm{Bun}_G(E)^{0,ss}$ be the stack of degree 0 semistable G bundles on E. It is an open substack of $\mathrm{Bun}_G(E)^0$. We have the following characterization of degree 0 semistable bundle on E.

Proposition B.2. *Let $\mathcal{P}$ be a G bundle of degree 0 on E, then:*
(1) $\mathcal{P}$ is semisimple semistable if and only if $\mathcal{P}$ is a closed point in $|\mathrm{Bun}_G(E)^{0,ss}|$.
(2a) $\mathcal{P}$ is semistable if and only if the closure of $\mathcal{P}$ in $|\mathrm{Bun}_G(E)^0|$ contains a semisimple semistable bundle. (2b) Moveover, in this case, such semisimple semistable bundle is unique, we shall call it the semisimplification of $\mathcal{P}$.

The propostion is known in the case when G is semisimple and simply connected [3, 10]. For the general case, we need the following lemmas.

Lemma B.3. *Let $H' \to H$ finite cover of algebraic groups, with kernel K central in H'. C a compact Riemann surface. $\mathcal{P}$ an H' bundle C. Then*
(1) $H^1(C,K)$ acts on $|\mathrm{Bun}_{H'}(C)^0|$, with quotient $|\mathrm{Bun}_H(C)^0|$. In particular, $\pi : |\mathrm{Bun}_{H'}(C)^0| \to |\mathrm{Bun}_H(C)^0|$ is open finite surjective.
(2) Assume H (hence H') is reductive, then $\mathcal{P}$ is semisimple if and only if $\mathcal{P}_H$ is semisimple.
(3) Assume $C = E$, H is reductive, then $\mathcal{P}$ is semistable if and only if $\mathcal{P}_H$ is semistable.

Proof: (3) follows directly from definition.
(1) There is a short exact sequence $1 \to \mathscr{K} \to \mathscr{H}' \to \mathscr{H} \to 1$, where $\mathscr{K}$ (*resp.*$\mathscr{H}', \mathscr{H}$) is the sheaf of sections of the constant group scheme K (*resp.*H', H) over C. Notices that $H^0(C, \mathscr{H}') \to H^0(C, \mathscr{H})$ is surjective. We have

$$\begin{tikzcd}
1 \arrow[r] & H^1(C,K) \arrow[r] & {|\mathrm{Bun}_{H'}(C)|} \arrow[r] \arrow[d] & {|\mathrm{Bun}_H(C)|} \arrow[r] \arrow[d] & H^2(C,K) \arrow[d, "\simeq"] & \\
 & 1 \arrow[r] & \pi_1(\mathrm{Bun}_{H'}(C)) \arrow[r] & \pi_1(\mathrm{Bun}_H(C)) \arrow[r] & K \arrow[r] & 1
\end{tikzcd}$$

This gives $1 \to H^1(C,K) \to |\mathrm{Bun}_{H'}(C)^0| \to |\mathrm{Bun}_H(C)^0| \to 1$.

(2) Let T' be a maximal torus of H', then $K \subset T'$ and $T := T'/K$ is a maximal torus in H. We have the commutative diagram of short exact sequences of groups.

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K \arrow[r] & H' \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

Take the corresponding sheaves and long exact sequences on cohomology, (2) is obtained by diagram chasing. □

Lemma B.4. *Let $H' \to H$ be a finite central cover of reductive algebraic groups with kernel K. If Proposition B.2 holds for H', then it holds H.*

Proof: Suppose Proposition B.2 holds for H'. Let $\mathcal{P} \in |\mathrm{Bun}_H(E)^0|$. For (1), assume $\mathcal{P}$ is semisimple semistable. Then by Lemma B.3, $\pi^{-1}(\mathcal{P})$ is a finite union of semisimple semistable H' bundles, hence is closed in $|\mathrm{Bun}_{H'}(E)^0|$. Hence $\{\mathcal{P}\} = \pi((\pi^{-1}(\mathcal{P}))^c)^c$ is closed (π is open, c stands for complement). Conversely if $\mathcal{P}$ is a closed point in $|\mathrm{Bun}_H(E)^{0,ss}|$. $\pi^{-1}(\mathcal{P})$ is a closed set in $|\mathrm{Bun}_{H'}(E)^{0,ss}|$ hence it contains closure of any of its point, hence contains a semisimple semistable H' bundle. Again by Lemma B.3, $\mathcal{P}$ is semisimple semistable. To prove (2a), assume $\mathcal{P}$ is semistable, then there is a semistable H' bundle $\mathcal{P}'$, such that $\pi(\mathcal{P}') = \mathcal{P}$, there is a semisimple semistable H' bundle $\mathcal{R}' \in \overline{\{\mathcal{P}'\}}$, then we have $\mathcal{R} := \pi(\mathcal{R}')$ is semisimple semistable and $\mathcal{R} \in \overline{\{\mathcal{P}\}}$. Conversely, suppose $\overline{\{\mathcal{P}\}}$ contains some semisimple semistable bundle. Let $\pi^{-1}(\mathcal{P}) = \{\mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_n\}$, if any of $\mathcal{P}'_i$ has a semisimple semistable H' bundle in its closure, then we conclude $\mathcal{P}'_i$ is semistable, hence so is $\mathcal{P}$. Suppose none of $\mathcal{P}'_i$ contains a semisimple semistable bundle in its closure, we want to reach a contridiction. Indeed, we'll have $(\pi((\bigcup_{i=1}^n \overline{\{\mathcal{P}_i\}})^c))^c$ is closed, containing $\mathcal{P}$ and does not contain any semisimple semistable H bundle, which contridicts to the assumption. For (2b), let $\mathcal{P}$ be semistable, take $\mathcal{P}' \in \pi^{-1}(\mathcal{P})$, then $\pi(H^1(E,K) \cdot \overline{\{\mathcal{P}'\}})$ is closed containing $\mathcal{P}$ and only contains one closed point which the image of the closed point in $\overline{\{\mathcal{P}'\}}$ under π. □

Proof of Proposition B.2: Proposition holds for all semisimple simply-connected groups, Lemma B.4 implies it holds for all semisimple groups. It's easy to see if proposition holds for H, then it holds for $T \times H$ for T a torus. Then for any connected reductive group G, $G = (Z(G)^0 \times [G,G])/F$ for a finite central F. So by Lemma B.4 again, we conclude proposition holds for G. □

Corollary B.5. *Let $K \subset G$ two reductive algebraic groups. Under the natural induction map $\mathrm{Bun}_K(E) \to \mathrm{Bun}_G(E)$, the image of $\mathrm{Bun}_K(E)^{0,ss}$ lies in $\mathrm{Bun}_G(E)^{0,ss}$.*

Let $G_{E,0}$ be the moduli stack of degree 0 semistable G-bundles with trivialization at $0 \in E$. We have:

Proposition B.6. *$G_{E,0}$ is representable.*

Proof: Suffices to prove that for any degree 0 semistable G-bundles $\mathcal{P}$, the natural map $Aut(\mathcal{P}) \to Aut(\mathcal{P}_0)$ is injective, where $\mathcal{P}_0$ is the fiber of $\mathcal{P}$ over $0 \in E$. By Corollary B.5, we can reduce to the case when $G = GL_n$. Then the above injectivity follows from Atiyah's classification. □

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