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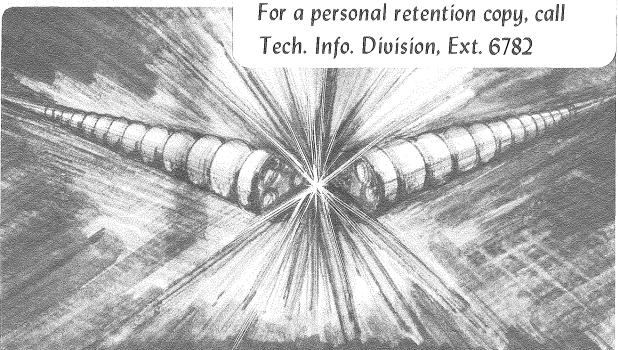## UNIVERSITY OF CALIFORNIA

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PONDEROMOTIVE EFFECTS IN COLLISIONLESS PLASMA:
A LIE TRANSFORM APPROACH

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# PONDEROMOTIVE EFFECTS IN COLLISIONLESS PLASMA:

## A LIE TRANSFORM APPROACH

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## ABSTRACT

A new method for the kinetic analysis of ponderomotive effects in collisionless plasma is presented. This method involves the application of Lie-transform perturbation technique to the Hamiltonian formulation of the Vlasov equation. Basically, a new system, in which the high frequency oscillations are absent, is found. In this system the distribution function evolves according to a ponderomotive Hamiltonian, which is the kinetic generalization of the ponderomotive potential. It is shown that the ponderomotive Hamiltonian can easily be determined from the well-known linear susceptibility. This formalism is used to calculate several new results. Among these results are the general formula for the quasi-static density perturbation produced by a hot magnetoplasma wave, a generalization of previous formulae for the laser-generated quasi-static magnetic field, and the general formula for the ponderomotive gyrofrequency shift produced by an electromagnetic wave propagating at an arbitrary angle.

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# I. INTRODUCTION

Collisionless plasma theory has developed toward the idea that a collisionless plasma consists of two entities, waves and the background through which the waves travel. In the oscillation-center formalism[1-6] the waves consist of the electromagnetic fields plus the oscillating portion of the particle motion. The background consists of pseudo-particles (particles from which the wave motion has been removed) and the equilibrium, slowly varying fields. In this picture, plasma evolution comes from the interaction of waves and the background. Waves are produced by unstable particle distributions or are injected by antennas or lasers. These waves may heat the particles via resonant interactions, or they may redistribute the particles via ponderomotive effects. This changes the background and, hence, the wave evolution. In this paper we discuss one aspect of this interaction, ponderomotive effects, i.e., the change in the background caused by the wave. In a future paper we discuss how the change in the background affects wave propagation.

The ponderomotive interaction deserves special study, because it has had enormous impact on plasma physics. Ponderomotive forces have been proposed as a method for confining or enhancing the confinement of plasma.[7-9] Ponderomotive effects alter the propagation of laser radiation[10] and may affect radiofrequency heating of plasma.[11] Ponderomotive effects also play a role in parametric decay processes.[12]

In the past, some of the analyses of ponderomotive effects have been *ad hoc*. For example, in the derivation of the ponderomotive potential, the cold-plasma limit is used, but then the response to the ponderomotive potential is taken to be given by the Boltzmann factor, a hot plasma result. In part, this is reasonable, as long as the phase velocity of the high-frequency wave is large, and the characteristic velocity of the slow perturbations is small. However, a very basic question remains. Why should the plasma respond to a low-frequency ponderomotive potential via the Boltzmann factor? After all, the ponderomotive potential is not identical to a real potential. In addition, the system, a plasma in the presence of a large amplitude wave, is far from thermodynamic equilibrium.

To answer these questions, it seems necessary to re-address the problem via a rigorous analysis of the Vlasov-Maxwell equations. One method is to average the Vlasov-Maxwell equations order by order.[13,14] The result is that the time-averaged distribution function obeys a diffusion/Vlasov equation. The equations are sufficiently complicated so that solving by inspection is difficult.

In this paper, an alternate method for rigorously analyzing ponderomotive effects in a collisionless plasma is presented. A transformation is introduced which relates the particle distribution, f, to another function, F, the oscillation-center distribution. The transformation is chosen so that F has no rapid variations. The evolution of

F is given by Liouville's equation, in which the Hamiltonian contains a term that depends only on the amplitudes of the high-frequency fields. This term, the ponderomotive Hamiltonian, is the kinetic generalization of the ponderomotive potential. Since the evolution of F is given by Liouville's equation (no diffusion terms are present), solutions can easily be obtained by inspection. Once the solution for F is known, one may use the transformation to determine the physical distribution, f.

The transformations used are Lie transforms, first introduced by Hori.[15] They were later modified by Deprit[16] and Dragt and Finn.[17] Deprit's version of Lie transform theory was further improved by Dewar,[3] (for a review see Ref. 17). In this work, we use the Deprit-Dewar type of Lie transforms, because we have found them well suited to the discovery of new, general theorems.

Specifically, we show how the use of Lie transforms has led to a general relation,[4] Eq. (38), between the ponderomotive Hamiltonian and the linear susceptibility of Vlasov plasma. This relation is very powerful, since it implies the immediate knowledge of a nonlinear quantity, the ponderomotive Hamiltonian, upon performing the linear calculation of the plasma susceptibility. Furthermore, this relation leads to another very general result, Eq. (72), the nonlinear density perturbation produced by a wave packet in Vlasov plasma. (The cold plasma result was reported earlier.[18])

Our use of Lie transforms also allows transparent calculations of complicated results. For example, we show that the laser-generated magnetic fields of Bezzerides et al.[19,20] arise very simply as a result of the momentum dependence of the ponderomotive Hamiltonian and the Lie transform. The ponderomotive gyrofrequency shift, which was previously[21] calculated in the special case of an electrostatic wave propagating across the magnetic field, is generalized here for electromagnetic waves propagating in arbitrary directions by simply taking the derivative of the ponderomotive Hamiltonian with respect to the magnetic moment.

This analysis fits well into the wave-background picture mentioned in the first paragraph. The wave part of this formalism consists of the rapidly varying electromagnetic field and the Lie transform. The background consists of the slowly varying fields, the oscillation-center distribution, and the ponderomotive forces. This interpretation holds even in detail. For example, we show that the wave momentum associated with a given species[22] arises from bilinear, slowly varying terms in the transformation, as it should, since the transformation and wave momentum are objects associated with the waves.

We now give a brief outline of this paper. In Sec. II we discuss the notations used and the canonical formulation of the Vlasov equation. We also summarize the Lie-transform methods which will be used throughout this paper. In Sec. III we discuss very generally the application of

Lie transforms to the Vlasov equation. We show how the linear response is obtained, and we prove a general relation between the linear response and the ponderomotive Hamiltonian. In Sec. IV we discuss the details of ponderomotive theory for unmagnetized particles. We derive the ponderomotive Hamiltonian and use it to derive a general formula for the quasi-static, second-order, density perturbation produced by a wave. In Sec. V we discuss the ponderomotive theory of magnetized particles. We derive the ponderomotive Hamiltonian for magnetized particles, and we use this ponderomotive Hamiltonian to derive the ponderomotive guiding center drifts and the ponderomotive gyrofrequency shift experienced by the particles.

## II. NOTATION AND BACKGROUND

Throughout this paper, we employ units such that $c = 1$, except when certain notable formulae are given. In addition, in most of this paper we use units such that the particle mass m is unity. We will restore the particle mass whenever we are dealing with more than one particle species.

To study the Vlasov equation, we use the Hamiltonian formalism. We denote the six independent phase space variables by $\underset{\sim}{z} \equiv (\underset{\sim}{q}, \underset{\sim}{p})$. In this formulation, the distribution function $f(\underset{\sim}{q}, \underset{\sim}{p}, t)$ gives the particle density in phase space. The evolution of f is determined by Liouville's equation

$$\frac{\partial f}{\partial t} + \{f, h\} = 0 \quad , \tag{1}$$

in which the braces denote the Poisson bracket. The Hamiltonian h is that of a particle in an electromagnetic field described by vector potential $\underset{\sim}{A}$ and scalar potential $\phi$:

$$h(\underset{\sim}{q}, \underset{\sim}{p}, t) = \left( 1 + |\underset{\sim}{p} - e\underset{\sim}{A}(\underset{\sim}{q}, t)|^2 \right)^{\frac{1}{2}} + e\phi(\underset{\sim}{q}, t) \quad . \tag{2}$$

To make the fields self-consistent, we introduce the charge density function, $r(\underset{\sim}{x}; \underset{\sim}{z}, t) = e\delta(\underset{\sim}{x} - \underset{\sim}{q})$ , so that

$$\rho(\underset{\sim}{x}, t) = \int d^6z \; r(\underset{\sim}{x}; \underset{\sim}{z}, t) \; f(\underset{\sim}{z}, t) \quad . \tag{3}$$

Similarly, we introduce the current density function, $\underset{\sim}{\eta}(\underset{\sim}{x}; \underset{\sim}{z}, t) = e\underset{\sim}{v}(\underset{\sim}{q}, \underset{\sim}{p}, t)\delta(\underset{\sim}{x} - \underset{\sim}{q})$ , where

$$\underset{\sim}{v}(\underset{\sim}{q},\underset{\sim}{p},t) = [\underset{\sim}{p} - eA(\underset{\sim}{q},t)]/\gamma(\underset{\sim}{q},\underset{\sim}{p},t) \quad , \tag{4}$$

and

$$\gamma(\underset{\sim}{q},\underset{\sim}{p},t) = [1 + |\underset{\sim}{p} - e\underset{\sim}{A}(\underset{\sim}{q},\underset{\sim}{p},t)|^2]^{1/2} \quad , $$

in order to have

$$\underset{\sim}{j}(\underset{\sim}{x},t) = \int d^6z \; \underset{\sim}{\eta}(\underset{\sim}{x};\underset{\sim}{z},t) \; f(\underset{\sim}{z},t) \quad . \tag{5}$$

In performing perturbative calculations, we introduce an ordering in some small parameter, determined by the specific problem. To denote order, we use subscripts, rather than a formal parameter. As usual, the order of a product of terms is the sum of the orders of the factors. Thus, the distribution and the fields are written

$$f = f_0 + f_1 + f_2 + \ldots \quad , \tag{6a}$$

$$\underset{\sim}{A} = \underset{\sim}{A}_0 + \underset{\sim}{A}_1 + \underset{\sim}{A}_2 + \ldots \quad , \tag{6b}$$

and $\quad \Phi = \Phi_0 + \Phi_1 + \Phi_2 + \ldots \quad . \tag{6c}$

Furthermore, Eqs. (6b) and (6c) determine the ordering of the Hamiltonian and the velocity. In zeroth order we have $h_0 = \gamma_0 + e\Phi_0$ , where $\gamma_0 = \left(1 + |\underset{\sim}{p} - e\underset{\sim}{A}_0|^2\right)^{1/2}$ . The lowest order velocity is $\underset{\sim}{v}_0 = \partial h_0/\partial\underset{\sim}{p}$ . The higher order terms are found by expanding Eqs. (2) and (4):

$$h_1(\underset{\sim}{q},\underset{\sim}{p},t) = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 + e\Phi_1 \quad , \tag{7a}$$

$$h_2(\underset{\sim}{q},\underset{\sim}{p},t) = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2 + e\Phi_2 + \frac{1}{2}e^2\gamma_0^{-1}[\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2] \quad ,$$

$$(7b)$$

$$h_3(\underset{\sim}{q},\underset{\sim}{p},t) = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_3 + e\Phi_3 + e^2\gamma_0^{-1}[\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2 - \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2]$$

$$+ \frac{1}{2}e^2\gamma_0^{-2}[\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 |\underset{\sim}{A}_1|^2 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^3] \quad , \qquad (7c)$$

$$\underset{\sim}{v}_1 = -e\gamma_0^{-1}[\underset{\sim}{A}_1 - \underset{\sim}{v}_0\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1] \quad , \qquad\qquad (8a)$$

$$\underset{\sim}{v}_2 = -e\gamma_0^{-1}[\underset{\sim}{A}_2 - \underset{\sim}{v}_0\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2]$$

$$- \frac{1}{2}e^2\gamma_0^{-2}[\underset{\sim}{v}_0\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 + 2\underset{\sim}{A}_1\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0(\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2], \qquad (8b)$$

$$\underset{\sim}{v}_3 = -e\gamma_0^{-1}(\underset{\sim}{A}_3 - \underset{\sim}{v}_0\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_3) - e^2\gamma_0^{-2}(\underset{\sim}{v}_0\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_2 + \underset{\sim}{A}_1\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2$$

$$+ \underset{\sim}{A}_2\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2) - \frac{1}{2}e^3\gamma_0^{-3}[-\underset{\sim}{A}_1\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1$$

$$+ 3\underset{\sim}{A}_1(\underset{\sim}{v} \cdot \underset{\sim}{A}_1)^2 + 3\underset{\sim}{v}_0\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - 5\underset{\sim}{v}_0(\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^3] \quad . \qquad (8c)$$

From these expressions, we find the current densities of various orders:

$$\underset{\sim}{\eta} \equiv \sum_{n=0}^{\infty} \underset{\sim}{\eta}_n = \sum_{n=0}^{\infty} e\underset{\sim}{v}_n \, \delta(\underset{\sim}{x}-\underset{\sim}{q}) \quad . \qquad (9)$$

The higher order quantities in Eqs. (7) and (8) may be separated into two types of terms. For example, $h_2$ may be separated into terms which are linear ($\lambda$) in the electromagnetic potentials

$$h_{2\lambda} = - e \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_2 + e \Phi_2 \ , \tag{10}$$

and a portion which is nonlinear ($\nu$) in the electromagnetic potentials of lower order

$$h_{2\nu} = \frac{1}{2} e^2 \gamma_0^{-1} [\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2] \ . \tag{11}$$

Similarly, we can talk about the linear and nonlinear parts of the current operator and the velocity.

An additional numerical subscript that a quantity may have denotes the harmonic number. This arises in the final sections where we assume the first-order fields have a distinguishable frequency. Then, the higher order quantities, such as $h_2$, may have terms at the zeroth harmonic (i.e., they are slowly varying), which are denoted by the second numerical subscript 0, and terms at twice the fundamental. Thus, we write $h_2 = h_{20} + h_{22}$ .

Two objects central to Hamiltonian theory are Poisson brackets and canonical transformations. When using these, we employ the operator notation of Dewar[3] (with some minor modifications). The symbol $L_g$, where g is a function of phase space, denotes an operator whose action on another phase function f is given by $L_g f \equiv \{g,f\}$ . $L_g$ is called

the Lie operator associated with g. Using this notation, Jacobi's identity is $[L_g, L_f] = L_{\{g,f\}}$, where the brackets denote the commutator. Now consider canonical transformations, which can be described by a set of functions $\underset{\sim}{Z}(\underset{\sim}{z},t)$ that give the new functions in terms of the old. Corresponding to this transformation, we introduce a canonical transformation operator, T, whose action on a phase function f is given by $(Tf)(\underset{\sim}{Z},t) = f(\underset{\sim}{Z}(\underset{\sim}{z},t),t)$.

Another object of importance to Hamiltonian theory is the time development transformation. Suppose we solve Hamilton's equations of motion to find the trajectories $\underset{\sim}{z}(\underset{\sim}{z},t,t')$, i.e., the function $\underset{\sim}{z}$ gives the location at time t of a particle which was at $\underset{\sim}{z}$ at time $t'$. Since $\underset{\sim}{z}(\underset{\sim}{z},t,t')$ is a canonical transformation, we can define a corresponding operator $S(t,t')$, whose action on a function $g(\underset{\sim}{z})$ is given by

$$[S(t,t')g](\underset{\sim}{z},t,t') \equiv g[\underset{\sim}{z}(\underset{\sim}{z},t,t')] . \tag{12}$$

From continuity we note that $\underset{\sim}{z}(\underset{\sim}{z},t,t) = \underset{\sim}{z}$, and $S(t,t) = I$, the identity operator. To find the relationship between this operator and the Hamiltonian $h(\underset{\sim}{z},t)$, one can differentiate Eq. (12) to obtain $\partial S(t,t')/\partial t = - S(t,t')L_{h(t')}$.

Knowledge of the trajectories or, equivalently, the time development operator, S, allows one to immediately solve the inhomogeneous Liouville equation

$$\frac{\partial f}{\partial t} + \{f,h\} = g , \tag{13}$$

-11-

which occurs again and again in Hamiltonian perturbation theory. The solution to Eq. (13) is found by introducing the auxiliary function $f'$ defined by $f(t) \equiv S(t',t) f'(t,t')$. Insertion of this expression into Eq. (13) yields a differential equation for $f'$ which can be integrated immediately. The result is

$$f(t) = \int_{t_0}^{t} dt_1 \, S^{-1}(t,t_1) g(t_1) + S^{-1}(t,t_0) f(t_0) \quad . \tag{14}$$

This method of solution is known as finding $f$ by integrating $g$ along the trajectories of $h$.

In the important special case where the Hamiltonian is time-independent, $S(t,t')$ depends only on the time difference: $S(t,t') = S(t-t',0)$. In this case we define $S(t) \equiv S(t,0)$.

A prerequisite to performing perturbative calculations is a detailed knowledge of the unperturbed system. Thus, we assume a detailed knowledge of the unperturbed Hamiltonian $h_0$, its time development operator $S_0(t,t')$, and the unperturbed distribution function $f_0$, which satisfies the homogeneous Liouville equation, $\partial f_0 / \partial t + \{f_0, h_0\} = 0$. According to Eq. (8), the time development of $f_0$ is given by

$$f_0(t) = S_0^{-1}(t,t_0) f_0(t_0) \quad .$$

As mentioned in the introduction, we will use canonical transformations to find a new system where the Hamiltonian is more easily solved. Thus, in addition to the physical distribution $f$ and Hamiltonian $h$, we will have a transformed oscillation-center distribution $F$, whose evolution is given

by the oscillation-center Hamiltonian, $K$, via

$$\frac{\partial F}{\partial t} + \{F, K\} = 0 \ . \tag{15}$$

To find the transformation law for the distribution, we use the fact that the new distribution evaluated at the new point equals the old distribution evaluated at the old point: $F[\underset{\sim}{Z}(\underset{\sim}{z})] = f(\underset{\sim}{z})$. In operator theory this statement is

$$f = TF \quad . \tag{16}$$

To distinguish physical quantities from oscillation center quantities, we adopt the following convention: quantities associated with the physical particles are denoted by lower case letters, quantities associated with the oscillation centers are denoted by upper case letters.

The transform $T$ is a Lie transform, which can be written as an ordered series.

$$T = I + T_1 + T_2 + \ldots \tag{17}$$

Furthermore, the terms of this series are composed of the Lie operators associated with an infinite series of functions, $w_1(\underset{\sim}{z}, t)$, $w_2(\underset{\sim}{z}, t) \ldots$ . For convenience, we denote the Lie operator of $w_n$ as $L_n$. Through third order, the relations between the $T_n$'s and the $L_n$'s are[17]

$$T_1 = - L_1 \tag{18a}$$

$$T_2 = - \frac{1}{2} L_2 + \frac{1}{2} L_1^2 \tag{18b}$$

$$T_3 = -\frac{1}{3} L_3 + \frac{1}{6} L_2 L_1 + \frac{1}{3} L_1 L_2 - \frac{1}{6} L_1^3 \quad , \tag{18c}$$

Hence, the new variable $Z_i$ and the old variable $z_i$ are related by

$$Z_i = z_i - \{w_1(\underset{\sim}{z},t),z_i\} - \frac{1}{2} \{w_2(\underset{\sim}{z},t),z_i\} + \frac{1}{2} \{w_1(\underset{\sim}{z},t),\{w_1(\underset{\sim}{z},t),z_i\}\}$$

$$\tag{19}$$

through second order. [One can easily verify via Poisson bracket calculation that Eq. (19) gives a canonical transformation through second order for arbitrary functions $w_1$ and $w_2$.]

In order to transform back to the original system, we will also need to know the inverse transformation:

$$T^{-1} = I + T_1^{-1} + T_2^{-1} + \ldots \quad . \tag{20}$$

We find $T_n^{-1}$ from $T_n$ by replacing $L_m$ by $- L_m$ and inverting the order of application:

$$T_1^{-1} = L_1 \quad , \tag{21a}$$

$$T_2^{-1} = \frac{1}{2} L_2 + \frac{1}{2} L_1^2 \quad , \tag{21b}$$

$$T_3^{-1} = \frac{1}{3} L_3 + \frac{1}{6} L_1 L_2 + \frac{1}{3} L_2 L_1 + \frac{1}{6} L_1^3 \quad . \tag{21c}$$

The final ingredient needed for the implementation of Lie transform theory is the Hamiltonian K for the new system. Through third order, the equations are

$$K_0 = h_0 \quad, \tag{22a}$$

$$K_1 - \left( \frac{\partial w_1}{\partial t} + \{w_1, h_0\} \right) = h_1 \quad, \tag{22b}$$

$$K_2 - \frac{1}{2} \left( \frac{\partial w_2}{\partial t} + \{w_2, h_0\} \right) = h_2 + \frac{1}{2} L_1 (h_1 + K_1) \quad, \tag{22c}$$

$$K_3 - \frac{1}{3} \left( \frac{\partial w_3}{\partial t} + \{w_3, h_0\} \right) = h_3 + \frac{1}{3} L_1 (2h_2 + K_2)$$

$$+ \frac{1}{3} L_2 \left( \frac{1}{2} h_1 + K_1 \right) + \frac{1}{6} L_1^2 h_1 \quad. \tag{22d}$$

## III. LINEAR SUSCEPTIBILITY OF VLASOV PLASMA AND ITS

### RELATION TO THE PONDEROMOTIVE HAMILTONIAN

The ponderomotive Hamiltonian (the kinetic generalization of the ponderomotive potential), together with the slowly varying fields, determines the time-averaged motion of particles in a high frequency wave. The linear susceptibility of the plasma gives the rapidly varying response of these particles to the wave. In the past, these two quantities, the linear susceptibility and the ponderomotive potential, have been regarded as unrelated, and as a result, these quantities were calculated separately. In this section we prove a relation between the ponderomotive Hamiltonian and the linear susceptibility. The importance of this relation is that it allows the immediate deduction of one quantity from the knowledge of the other.

The calculation of the linear response using Lie transform methods is the first step in proving the theorem. The calculation in any order proceeds in a standard manner:

1) Examine Eq. (22b) to choose the transformation.

2) Solve the evolution equation (15) for F. 3) Obtain the distribution function f by applying the transformation to F. 4) Use f to calculate the current density and charge density.

In choosing the transformation we assume that the perturbation is a rapid oscillation, and thus, it may be entirely transformed away. That is, we choose $K_1 = 0$, reducing Eq. (22b) to $\partial w_1/\partial t + \{w_1, h_o\} = - h_1$ . This equation may be

solved for $w_1$ by using the method of integration along orbits (see Sec. II)

$$w_1(t) = - \int_{-\infty}^{t} dt' \, S_0^{-1}(t,t') h_1(t') \quad . \tag{23}$$

In obtaining Eq. (1) we have assumed that the perturbation $h_1$ and, hence, the transformation function $w_1$ vanish at $t = -\infty$.

Next, we solve the evolution equation for F. Since $K_1 = 0$, we have

$$\frac{\partial F}{\partial t} + \{F, h_0\} = 0 \quad , \tag{24}$$

to first order. We see that F may be any solution of the unperturbed system. To determine F, we look ahead to the transformation. Through first order Eqs. (16), (17), and (18) give $f = F - \{w_1, F\}$.

In order to have $f = f_0$ when $w_1 = 0$, we must choose the particular solution $F = f_0$ of Eq. (24). Hence, the first order part of the distribution is given by

$$f_1 = - \{w_1, f_0\} \quad . \tag{25}$$

Finally, we calculate the current density and charge density. From Eqs. (3), (5), and (18) we find

$$\rho_1(\underset{\sim}{x}, t) = - \int d^6z \, r(\underset{\sim}{x}; \underset{\sim}{z}, t) \{w_1, f_0\} \quad , \tag{26}$$

and

$$\underset{\sim}{j}_1(x,t) = \int d^6z \left[ -\underset{\sim}{\eta}_0(\underset{\sim}{x};\underset{\sim}{z},t)\{w_1,f_1\} + \underset{\sim}{\eta}_1(\underset{\sim}{x};\underset{\sim}{z},t)f_0(\underset{\sim}{z},t) \right] \quad ,$$

(27)

Now, we discuss the ponderomotive Hamiltonian, which arises from Eq. (22b). For $K_1 = 0$, this equation reduces to

$$\frac{\partial w_2}{\partial t} + \{w_2,h_0\} - 2K_2 = -2h_2 - \{w_1,h_1\} \quad .$$ (28)

We break up this equation by transforming away the rapidly varying terms and keeping the slowly varying terms in $K_2$. In particular,

$$K_2(\underset{\sim}{z},t) = \left\langle S(t_2,t)\left[ h_2(t_2) + \frac{1}{2}\{w_1(t_2),h_1(t_2)\} \right] \right\rangle_{t_2} \quad ,$$

(29)

where the angular brackets and the subscript $t_2$ denote an average, possibly local, over the variable $t_2$. Hence, $K_2$ is an orbit average of the terms on the right side of Eq. (28) As a result, $w_2$ must be the orbit integral of the fluctuating part of the right side of Eq. (28).

As explained in Sec. II, the second-order part of the oscillation-center Hamiltonian, can be written as the sum of two terms, $K_2 = K_{2\lambda} + K_{2\nu}$. The second term, $K_{2\nu}$, is non-linear in the lower-order potentials.

$$K_{2\nu} = \left\langle S(t_2,t)\left[ h_{2\nu}(t_2) + \frac{1}{2}\{w_1(t_2),h_1(t_2)\} \right] \right\rangle_{t_2}, \quad (30)$$

where $h_{2\nu} = \frac{1}{2} e^2 \gamma_0^{-1} [\underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2]$ . Both $K_{2\lambda}$ and $K_{2\nu}$

affect the slow motion of the particles. In particular,

from $K_{2\nu}$ we can determine how the rapidly varying first-

order fields affect the average motion of the particles.

Thus, $K_{2\nu}$ is called the ponderomotive Hamiltonian.

To find the connection between the ponderomotive Hamil-

tonian and the linear response, we consider the quantity

$$W \equiv \left\langle \int d^3x [\rho_1(\underset{\sim}{x},t) \Phi_1(\underset{\sim}{x},t) - \underset{\sim}{j}_1(\underset{\sim}{x},t) \cdot \underset{\sim}{A}_1(\underset{\sim}{x},t)] \right\rangle_t . \quad (31)$$

Obviously, W can be calculated if the linear response is known.

However, W can also be calculated from the ponderomotive

Hamiltonian.[4,6] This can be seen by inserting the linear

response (26) and (27) into Eq. (29) and changing the

variable of integration over phase space via the time-

development transformation. Doing this we obtain

$$\int d^6z \, f_0(\underset{\sim}{z},t') K_{2\nu}(\underset{\sim}{z},t')$$

$$= \frac{1}{2} \left\langle \int d^3x [\rho_1(\underset{\sim}{x},t) \Phi_1(\underset{\sim}{x},t) - \underset{\sim}{j}_1(\underset{\sim}{x},t) \cdot \underset{\sim}{A}_1(\underset{\sim}{x},t)] \right\rangle_t , \quad (32)$$

a relation between the ponderomotive Hamiltonian and the

linear response. Indeed, by functional differentiation one

can use the following equation to obtain the ponderomotive

Hamiltonian from the linear response

$$K_{2\nu}(z,t') = \frac{\delta}{\delta f_0(z,t')} \left\langle \int d^3x [\rho_1(\underset{\sim}{x},t) \Phi_1(\underset{\sim}{x},t) \right.$$

$$\left. - \underset{\sim}{j_1}(\underset{\sim}{x},t) \cdot \underset{\sim}{A_1}(\underset{\sim}{x},t)] \right\rangle_t \quad . \tag{33}$$

This relation is known as the $K$-$\underset{\approx}{\chi}$ theorem, since it relates the ponderomotive Hamiltonian, $K_{2\nu}$, to the linear response, which is given by the linear susceptibility $\underset{\approx}{\chi}$. This theorem was first presented by Cary and Kaufman.[4] More recently, Johnston and Kaufman[6] have shown that this relation is one of a family of similar relations which connect $K_n$ to the (n-1)th response.

A particularly useful special case of this result is the case of a time-independent, homogeneous background state in the presence of an oscillation with a slowly varying amplitude

$$\underset{\sim}{E_1}(\underset{\sim}{x},t) = \underset{\sim}{E_1}(\underset{\sim}{x},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}-i\omega t) + c.c. \tag{34}$$

We choose the radiation gauge, $\Phi_1 = 0$ and

$$\underset{\sim}{A_1}(\underset{\sim}{x},t) = \underset{\sim}{A_1}(\underset{\sim}{x},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}-i\omega t) + c.c. \quad , \tag{35}$$

where $A_1 \cong E_1/i\omega$ . Since the background is time-independent and homogeneous, we know that the current is given by

$$\underset{\sim}{j_1}(\underset{\sim}{x},t) = \underset{\sim}{J_1}(\underset{\sim}{x},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{x}-i\omega t) + c.c. \quad , \tag{36}$$

where $\underset{\sim}{J_1} \cong \omega\underset{\approx}{\chi}(\underset{\sim}{k},\omega)\cdot\underset{\sim}{E_1}/4\pi i$ . Combining these equations we obtain

$$\int d^6 z \ f_0(\underset{\sim}{z}) K_{2\nu}(\underset{\sim}{z},t) = -\frac{1}{4\pi} \int d^3 x \underset{\sim 1}{E}^*(\underset{\sim}{x},t) \cdot \underset{\approx h}{\chi}(\underset{\sim}{k},\omega) \cdot \underset{\sim 1}{E}(\underset{\sim}{x},t) \quad ,$$

(37)

where $\underset{\approx h}{\chi}$ is the Hermitian part of $\underset{\approx}{\chi}$.

## IV.  PONDEROMOTIVE EFFECTS IN UNMAGNETIZED PLASMA

### A.  The Unperturbed System

The medium under study is a relativistic, homogeneous, unmagnetized plasma.  Thus, the zeroth-order potentials, $\underset{\sim}{A}_0$ and $\Phi_0$, vanish.  The zeroth-order Hamiltonian is $h_0 = (1 + p^2)^{1/2}$, and the zeroth-order velocity is $\underset{\sim}{v}_0 = \underset{\sim}{p}/(1 + p^2)^{1/2}$.  As is well known, in this case the action of the time-development operator is given by

$$[S_0(t)g](\underset{\sim}{q},\underset{\sim}{p}) = g(\underset{\sim}{q} + \underset{\sim}{v}_0 t,\underset{\sim}{p}) \quad . \tag{38}$$

### B.  Linear Theory via Lie Transforms

Although the linear theory of unmagnetized plasma is well known, we present a brief discussion of it for three reasons.  First of all, the linear theory for this formalism must be known before the nonlinear theory can be done.  Second , this discussion allows us to introduce Lie transforms on a familiar problem.  Finally, in discussing the linear theory we quantify our assumptions.  For example, we say specifically what is meant by the exclusion of resonant particles.

The main assumption of this work is that the system is dominated by a single high-frequency wave with slowly varying amplitude.  That is, the first-order vector potential has the form, (35) .  We use the radiation gauge for the first-order fields, so that $\Phi_1$ vanishes.

To find the response to this potential, we follow the standard method of the last section.  We first analyze the

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transformation equation (22b), in which the first-order Hamiltonian

$$h_1(\underset{\sim}{q},\underset{\sim}{p},t) = - e\underset{\sim}{v}_0(\underset{\sim}{p}) \cdot \underset{\sim}{A}_1(\underset{\sim}{q},t)\exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t) + c.c.$$

(39)

appears. We note that $h_1$ is rapidly varying and, hence, may be transformed away. Thus, we insert $K_1 = 0$ and $h_1$ from Eq. (39) into Eq. (22b). We solve the resulting equation via integration along a trajectory, i.e., we use Eq. (14) with the time development operator of Eq. (38). We obtain

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = - e \exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t)$$

$$\int_{-\infty}^{0} d\tau \; \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1(\underset{\sim}{q}+\underset{\sim}{p}\tau,t+\tau)\exp[i(\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega)\tau] + c.c.$$

(40)

(The integral is regularized in the usual way by giving $\omega$ a small positive imaginary part.) To calculate the integral of Eq. (40), we repeatedly integrate by parts.

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = - ie \frac{\exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t)}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} \sum_{n=0}^{\infty} \left[ \frac{i}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0-\omega} \left( \frac{\partial}{\partial t} - \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial q} \right) \right]^n$$

$$\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1(\underset{\sim}{q},t) + c.c.$$

(41)

At this point in the calculation, we see what approximations are involved. First, we neglect the terms in the sum in Eq. (41) which are of higher-order in $(\partial/\partial t + \underset{\sim}{v}_0\cdot\partial/\partial\underset{\sim}{q})/(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)$, thereby obtaining the simpler result

$$w_1(\underset{\sim}{q},\underset{\sim}{p},t) = -ie \frac{\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 \exp(i\underset{\sim}{k} \cdot \underset{\sim}{q} - i\omega t)}{\underset{\sim}{k} \cdot \underset{\sim}{v}_0 - \omega} + c.c. \qquad (42)$$

Thus, we are excluding resonant particles by making the approximation

$$\left| \left( \frac{\partial}{\partial t} + \underset{\sim}{v}_0 \cdot \frac{\partial}{\partial \underset{\sim}{q}} \right) \underset{\sim}{A}_1(\underset{\sim}{q},t) \right| \ll \left| (\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0) \underset{\sim}{A}_1 \right| . \qquad (43)$$

A more basic approximation is also made as soon as we undertake the perturbation calculation; we are assuming that the higher-order terms are small. Specifically, this means that a second-order term, such as $\{w_1, h_1\}$, must be small compared with a corresponding first-order term, such as $h_1$. In the present case, we find that $\{w_1, h_1\}$ is smaller than $h_1$ by the factor $\left| e\underset{\sim}{k} \cdot \underset{\sim}{E}/(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0)^2 \right|$. Hence, we are assuming

$$\left| e\underset{\sim}{k} \cdot \underset{\sim}{E}/(\omega - \underset{\sim}{k} \cdot \underset{\sim}{v}_0)^2 \right| \ll 1 . \qquad (44)$$

Physically, Eq. (44) implies that the position oscillation of a particle about its unperturbed trajectory is much smaller than a scale length of the oscillation.

The second step in the calculation is the determination of the oscillation-center response. Since $K_1 = 0$, there is no oscillation center force, and the oscillation centers follow the unperturbed orbits. This implies $F(\underset{\sim}{q},\underset{\sim}{p},t) = g(\underset{\sim}{q} - \underset{\sim}{v}_0 t, \underset{\sim}{p})$, where g is any arbitrary function. In the present work, we specialize to the case where the unperturbed

distribution is uniform and, hence, $F = f_0(\underset{\sim}{p})$.

The third step in the calculation is the application of the Lie transform to the oscillation-center distribution to obtain the physical distribution as in Eq. (41). We find

$$f_1(\underset{\sim}{q},\underset{\sim}{p},t) = - e \exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t) \frac{\partial f_0}{\partial \underset{\sim}{p}} \cdot (\underset{\sim}{k} - i\frac{\partial}{\partial \underset{\sim}{q}}) \frac{\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega}$$

$$+ \text{ c.c.} \tag{45}$$

We note that we have kept terms in the gradient of the amplitude in this equation, whereas we neglected such terms in Eq. (42). This is valid for the following reason. In neglecting terms containing amplitude gradients in Eq. (42), we were making the approximation (43). To neglect such terms in Eq. (45), we would have to make the additional approximation

$$\left| \frac{\partial \ln|\underset{\sim}{A}_1|}{\partial \underset{\sim}{q}} \right| \ll |\underset{\sim}{k}| \tag{46}$$

Finally, we insert $f_1$ into Eq. (5) to obtain the first-order current. Integrating over $\underset{\sim}{q}$, we obtain

$$\underset{\sim}{j}_1(\underset{\sim}{x},t) = - e^2 \exp(i\underset{\sim}{k}\cdot\underset{\sim}{q}-i\omega t) \int d^3p \left[ \frac{\partial f_0}{\partial \underset{\sim}{p}} \cdot (\underset{\sim}{k} - i\underset{\sim}{\nabla}) \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} \right.$$

$$\left. + f_0(\underset{\sim}{A}_1 - \underset{\sim}{v}_0\underset{\sim}{v}_0\cdot\underset{\sim}{A}_1)/\gamma_0 \right] + \text{ c.c.} \tag{47}$$

To find the linear susceptibility, $\underset{\approx}{\chi}(\underset{\sim}{k},\omega) \equiv 4\pi i \underset{\approx}{\sigma}(\underset{\sim}{k},\omega)/\omega$, ($\underset{\approx}{\sigma}$ is the conductivity tensor) from Eq. (47), we specialize to plane waves, for which $\nabla \underset{\sim}{A}_1 = \partial \underset{\sim}{A}_1/\partial t = 0$, and the electric

field amplitude $\underset{\sim}{E}_1$ is given by $\underset{\sim}{E}_1 = i\omega\underset{\sim}{A}_1$. Restoring species labels and ordinary units, and integrating by parts, we obtain

$$
\underset{\approx}{\chi}^s(\underset{\sim}{k},\omega) = -\frac{4\pi e_s^2}{m_s\omega^2} \int d^3p \, \frac{f_0}{\gamma_0} \left( \underset{\approx}{I} + \frac{\underset{\sim}{k}\underset{\sim}{v}_0 + \underset{\sim}{v}_0\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0} \right.
$$
$$
\left. + \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2} \right) . \tag{48}
$$

To obtain the better known nonrelativistic result (Ref. 23, p. 45), we must set $\gamma_0 = 1$ and $\underset{\sim}{v}_0 = \underset{\sim}{p}/m_s$, and we must neglect the quantity $\omega^2/c^2$ in the last term, which arises from the relativistic dynamics. A more detailed discussion of this point is given in Ref. 24, Chap. 8.

C.   Oscillation Center Evolution Equations

Through second order, we can describe the oscillation center picture as follows.  The ponderomotive Hamiltonian and the slowly varying background electromagnetic fields determine the evolution of the oscillation-center distribution. The oscillation-center distribution and the Lie transform determine the physical particle distribution.  The physical particle distribution determines the charge and current distributions.  Finally, the charge and current densities determine the evolution of the slowly varying fields, thus completing the loop.

In order to analyze the transformation equation   (22c), we must first know the structure of the second-order, self-

consistent, electromagnetic potentials which appear in Eq. (22c) through $h_2$, as in Eq. (7b). Since the last term, $\{w_1, h_1\}$ of Eq. (22c) is bilinear in two quantities with phase $i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)$, it follows that this term can be written as a sum of two terms, one, which is slowly varying, and another, which varies at twice the frequency. Thus, we anticipate that the second-order, self-consistent, electromagnetic potentials have the same structure:

$$\underset{\sim}{A}_2(\underset{\sim}{x},t) = \underset{\sim}{A}_{20}(\underset{\sim}{x},t) + [\underset{\sim}{A}_{22}(\underset{\sim}{x},t)\exp(2i\underset{\sim}{k}\cdot\underset{\sim}{x}-2i\omega t) + c.c.], \quad (49a)$$

$$\Phi_2(\underset{\sim}{x},t) = \Phi_{20}(\underset{\sim}{x},t) + [\varphi_{22}(\underset{\sim}{x},t)\exp(2i\underset{\sim}{k}\cdot\underset{\sim}{x}-2i\omega t) + c.c.]. \quad (49b)$$

Of the two types of terms on the right-side of Eq. (22c), those varying with twice the frequency may be transformed away, i.e., equated to the terms containing $w_2$ on the left side of Eq. (22c). The remaining terms are equated to the second-order, oscillation-center Hamiltonian, $K_{20} = h_{20} - \frac{1}{2}\{w_1,h_1\}_0$ . (The subscript 0 on the Poisson brackets means that only the zeroth harmonic is kept.)

This oscillation-center Hamiltonian may be separated into two terms, $K_{20} = K_{2\lambda0} + K_{2\nu0}$, as discussed in Sec. II. The first term depends linearly ($\lambda$) on the second-order, electromagnetic potentials: $K_{2\lambda0} = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_{20} + e\Phi_{20}$ . The second term depends nonlinearly ($\nu$) on the lower order electromagnetic potentials.

$$K_{2\nu 0} = \frac{1}{2} e^2 \gamma_0^{-1} \left[ \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 - (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2 \right]_0 - \frac{1}{2} \{w_1, h_1\}_0 \quad . \tag{50}$$

This second term is called the ponderomotive Hamiltonian, since it determines the influence of the rapidly varying fields on the slow evolution of the background. We defer calculation of $K_{2\nu 0}$ until the next section.

The oscillation-center Hamiltonian determines the evolution of the oscillation-center distribution via the second-order Liouville equation

$$\frac{\partial F}{\partial t} + \{F, h_0 + K_{20}\} = 0 \quad . \tag{51}$$

This equation cannot be solved in general. We will discuss some special solutions in later subsections. However, note that the rapid variation has been removed, and that there is no fake diffusion in this equation.

To determine the distribution in physical space, we transform back. Inserting F into Eq. (16), we obtain

$$f = F - \{w_1, F\} + \frac{1}{2} \{w_1, \{w_1, F\}\} - \frac{1}{2} \{w_2, F\} \quad . \tag{52}$$

Several physical phenomena are present in this equation. For example, upon time averaging we find that the slowly varying part of f,

$$(f)_0 = F + \frac{1}{2} \{w_1, \{w_1, F\}\}_0 \quad , \tag{53}$$

has two contributions, the oscillation-center distribution, plus the time average of the transformation. As we shall see in Sec. IVE, the last term in Eq. (53) leads to fake diffusion. The term $\{w_1, F\}$ in Eq. (52) contains the modification in the linear response due to the change in the oscillation-center distribution. Finally, f contains terms, $(f)_2 = \frac{1}{2}\{w_1, \{w_1, F\}\}_2 - \frac{1}{2}\{w_2, F\}$, which vary at twice the phase.

From the distribution in physical space, we determine the slowly varying charge and current densities

$$<\rho(\underset{\sim}{x}, t)> = \rho_0 + \rho_{20} = e \int d^3p \left\langle f(\underset{\sim}{x}, \underset{\sim}{p}, t) \right\rangle , \qquad (54a)$$

$$<\underset{\sim}{j}(\underset{\sim}{x}, t)> = \underset{\sim}{j}_0 + \underset{\sim}{j}_{20} = e \int d^3p \left\langle \underset{\sim}{v}(\underset{\sim}{x}, \underset{\sim}{p}, t) f(\underset{\sim}{x}, \underset{\sim}{p}, t) \right\rangle . \qquad (54b)$$

Inserting Eqs. (8), (9), and (52) into Eqs. (54), we obtain

$$\rho_{20} = e \int d^3p [(F - f_0) + \frac{1}{2}\{w_1, \{w_1, F\}\}_0] , \qquad (55a)$$

and

$$\underset{\sim}{j}_{20} = e \int d^3p \left( \underset{\sim}{v}_0 [(F - f_0) + \frac{1}{2}\{w_1, \{w_1, F\}\}_0] \right.$$
$$\left. - \left\langle \underset{\sim}{v}_1 \{w_1, F\} \right\rangle + \underset{\sim}{v}_{20} F \right) . \qquad (55b)$$

In a multispecies plasma, we would have to sum the charge and current density contributions from each of the species.

Finally, we complete the loop with Maxwell's equations, which, in the Coulomb gauge, are

$$- \nabla^2 \phi_{20} = 4\pi \rho_{20} \quad , \tag{56a}$$

$$\frac{\partial^2 \underset{\sim}{A}_{20}}{\partial t^2} - \nabla^2 \underset{\sim}{A}_{20} = 4\pi \underset{\sim}{j}_{20} - \underset{\sim}{\nabla} \frac{\partial \phi_{20}}{\partial t} \quad . \tag{56b}$$

Thus, we have a complete set of equations describing the slowly varying evolution of the background. In fact, upon combining these equations with the zeroth-order equations, we obtain a unified theory of the evolution of the background plasma including ponderomotive effects.

D. Explicit Form of the Ponderomotive Hamiltonian

There are two ways to calculate the ponderomotive Hamiltonian. The first way is to deduce the expression for the ponderomotive Hamiltonian from the linear susceptibility and the $K$-$\underset{\approx}{\chi}$ theorem. The second way is to compute the Poisson bracket and perform the average as in Eq. (50). In this section we show that the first method gives the ponderomotive Hamiltonian quickly and with little effort. However, the second method allows one to see the approximations involved.

Given our usual form for the first-order electric field, Eq. (34), we note that the linear susceptibility, Eq. (48), and the $K$-$\underset{\approx}{\chi}$ theorem, Eq. (37), combine to produce the relation

$$\int d^3q \, d^3p \, f_0(\underset{\sim}{p}) \, K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{m\omega^2} \int d^3x \, d^3p \, f_0(\underset{\sim}{p}) \, \underset{\sim}{E}_1^*(\underset{\sim}{x},t)$$

$$\cdot \, \gamma_0^{-1} \left( \underset{\approx}{I} + \frac{\underset{\sim}{k}\underset{\sim}{v}_0 + \underset{\sim}{v}_0\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0} + \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2} \right) \cdot \underset{\sim}{E}_1(\underset{\sim}{x},t) \ .$$

$$(57)$$

By functional differentiation with respect to $f_0(p)$, we can reduce this expression to

$$\int d^3q \, K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{m\omega^2} \int d^3x \, \underset{\sim}{E}_1^*(\underset{\sim}{x},t)$$

$$\cdot \, \gamma_0^{-1} \left( \underset{\approx}{I} + \frac{\underset{\sim}{k}\underset{\sim}{v}_0 + \underset{\sim}{v}_0\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0} + \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2} \right) \cdot \underset{\sim}{E}_1(\underset{\sim}{x},t) \ .$$

$$(58)$$

In this expression, we have two quantities with equal spatial integrals. Therefore, the difference of these two quantities is a function with vanishing integral, i.e., a derivative. Thus, we deduce the formula

$$K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2}{m\omega^2} \underset{\sim}{E}_1^*(\underset{\sim}{x},t) \cdot \gamma_0^{-1} \left( \underset{\approx}{I} + \frac{\underset{\sim}{k}\underset{\sim}{v}_0 + \underset{\sim}{v}_0\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0} \right.$$

$$\left. + \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2} \right) \cdot \underset{\sim}{E}_1(\underset{\sim}{x},t) + \mathcal{O}(\nabla) \qquad (59)$$

for the ponderomotive Hamiltonian.

To derive the well-known ponderomotive potential[7] from the ponderomotive Hamiltonian, we must perform the following operations on $K_{2\nu 0}$. We must take the nonrelativistic limit, we must assume $\underset{\sim}{k} \cdot \underset{\sim}{v_0}(\underset{\sim}{p}) \ll \omega$, and we must restore units. Thus, we obtain the usual formula for the ponderomotive potential, $\Phi_p(\underset{\sim}{x},t) = e^2 |\underset{\sim}{E}|^2/m\omega^2$. Therefore, the usual results are valid only for particles which move slowly compared with the speed of light and the wave phase velocity. In contrast, the ponderomotive Hamiltonian is also valid for particles moving faster than the phase velocity of the wave as long as the basic approximation (43) holds.

We can also relate our result to that of Vedenov, et al.[25] To do so we first assume that the field is longitudinal, i.e., $\underset{\sim}{E_1}$ and k are parallel. Secondly we take the nonrelativistic limit; we set $m\underset{\sim}{v_0} = \underset{\sim}{p}$ and $\gamma_0 = 1$, and we neglect the $\omega^2/c^2$ term as discussed in Sec. IVB.        This time we obtain

$$K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = \frac{e^2 |\underset{\sim}{E_1}(\underset{\sim}{q},t)|^2}{m(\omega - \underset{\sim}{k} \cdot \underset{\sim}{p}/m)^2} \quad .$$

Vedenov, et al. obtained this expression in Ref. 25. However, they interpreted this quantity as a velocity dependent potential, from which the particle acceleration was determined via $m\underset{\sim}{\dot{v}} = - \underset{\sim}{\nabla} K_{2\nu 0}$. We note that this interpretation cannot be correct since it violates Liouville's theorem; the divergence of the flow, $(\partial/\partial \underset{\sim}{x}) \cdot \underset{\sim}{\dot{x}} + (\partial/\partial \underset{\sim}{v}) \cdot \underset{\sim}{\dot{v}}$, does not vanish.

Finally, we examine the validity of our result, Eq. (59). As we noted in deriving this result, it is valid when the terms

in the gradients of the amplitudes may be neglected. To see exactly what this means, we calculate $K_{2\nu 0}$ via Eq. (50). As this is a long calculation, we present only the result,

$$K_{2\nu 0}(\underset{\sim}{q},\underset{\sim}{p},t) = e^2 \underset{\sim}{A}_1^* \cdot \gamma_0^{-1} \left( \underset{\approx}{I} + \frac{\underset{\sim}{k}\underset{\sim}{v}_0 + \underset{\sim}{v}_0\underset{\sim}{k}}{\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0} + \frac{\underset{\sim}{v}_0\underset{\sim}{v}_0(k^2 - \omega^2/c^2)}{(\omega - \underset{\sim}{k}\cdot\underset{\sim}{v}_0)^2} \right) \cdot \underset{\sim}{A}_1$$

$$+ \frac{e^2}{2} \left[ i\left(\frac{\partial}{\partial \underset{\sim}{q}} \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1\right) \cdot \left( \frac{2}{\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega} \frac{\partial}{\partial \underset{\sim}{p}} (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1^*) \right. \right.$$

$$\left. \left. + \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1^* \frac{\partial}{\partial \underset{\sim}{p}} (\underset{\sim}{k}\cdot\underset{\sim}{v}_0 - \omega)^{-1} \right) + \text{c.c.} \right] \quad . \qquad (60)$$

We note that Eq. (60) does indeed reduce to Eq. (59) if we set $\underset{\sim}{A}_1 = \underset{\sim}{E}_1/i\omega$, and we neglect the gradient of the amplitude. Thus, in order for the result (59) to be valid, we must have $|\partial \underset{\sim}{A}_1/\partial t| \ll |\omega \underset{\sim}{A}_1|$. In addition, we must be able to neglect the second term in Eq. (60). This is certainly possible when the approximation (46) holds. However, suppose the wave has large phase velocity

$$\left| \underset{\sim}{k}\cdot\underset{\sim}{v}_0/\omega \right| \ll 1 \quad , \qquad (61)$$

and the plasma is nonrelativistic, $|\underset{\sim}{v}_0/c| \ll 1$. In this case the approximation (46) need not be invoked, since the second term in Eq. (60) is small by virtue of assumption (43).

## E. Wave Momentum and Fake Diffusion

It is well known that the time-averaged momentum and energy of nonresonant particles increases in the presence of a growing oscillation.[13,26] These increments, which are known

as the wave momentum and wave energy,[22] are caused by fake diffusion.[13,14]

In the present theory, fake diffusion can be singled out by neglecting the response to the ponderomotive Hamiltonian, i.e., setting $F = f_0$. This, for example, implies that the second-order change in the time averaged distribution is given by only $f_{20} = \frac{1}{2}\{w_1,\{w_1,f_0\}\}_0$ . We note that this formula is consistent with the interpretation. As the wave amplitude increases, so does $w_1$ and, hence, $f_{20}$. If the wave dies away, $f_{20}$ vanishes. In this case we see that fake diffusion comes from the bilinear terms in the Lie transform.

Fake diffusion effects also arise because of the difference between canonical momentum and kinetic momentum. For example, if we calculate the local kinetic  momentum density $\underset{\sim}{G}_{20}$ neglecting the second-order, self-consistent fields and $K_2$, we obtain, from Eq. (52),

$$\underset{\sim}{G}_{20} = \int d^3\underset{\sim}{p} \left\langle (\underset{\sim}{p} - e\underset{\sim}{A})f \right\rangle_t$$

$$= \int d^3p \left( \underset{\sim}{p}\, \tfrac{1}{2}\{w_1,\{w_1,f_0\}\}_0 + \left\langle e\underset{\sim}{A}_1\{w_1,f_0\} \right\rangle \right) . \qquad (62)$$

In fact, a calculation of the wave momentum via Eq. (62) gives the well known result:[26]  the momentum calculated via fake diffusion equals the wave momentum which is calculated from a knowledge of the linear susceptibility.[22]

## F.    The Self-Consistent, Quasistatic Response

The evolution of the oscillation center distribution is said to be quasi-static when the terms in the oscillation center equations that contain time derivatives are negligible. Physically, this means that the oscillation centers are in equilibrium with their Hamiltonian. Mathematically, this is valid when the thermal speed of the particles is so great that they can cross the wave packet before it evolves significantly.

In this section we obtain rigorous, quasi-static, self-consistent solutions for the density perturbation and magnetic field produced by the wave. We show that the Boltzmann response[10,11] to the ponderomotive potential is only approximately valid. In fact, when the kinetic corrections are kept, we obtain a simple and general formula for the density perturbation. Secondly, we note that the wave-generated magnetic field, as derived by Bezzerides et al.,[19,20] is simply the response to the momentum dependence of the oscillation-center Hamiltonian and the Lie transform.

Upon neglecting the time derivative in Eq. (51),    we find that the oscillation-center distribution can be any function of the oscillation-center Hamiltonian, $F = g(h_0 + K_{20})$. However, this solution must reduce to the unperturbed distribution, when the wave amplitude vanishes. This requirement implies

$g[h_0(\underset{\sim}{p})] = f_0(\underset{\sim}{p})$. Thus, we deduce that the quasi-static solution for the oscillation-center distribution is

$$F(\underset{\sim}{z},t) = f_0[h_0(\underset{\sim}{p}) + K_{20}(\underset{\sim}{q},\underset{\sim}{p},t)] \quad . \tag{63}$$

We note that the unperturbed distribution must be isotropic for this procedure to be valid.

To understand the physics involved in neglecting the time derivative in Eq. (51), we consider its linearized form

$$\frac{\partial F_2}{\partial t} + \{F_2, h_0\} + \{f_0, K_{20}\} = 0 \quad , \tag{64}$$

where

$$F = f_0 + F_2 \quad . \tag{65}$$

From the approximate solution, (63), we find

$$F_2 = \frac{df_0}{dh_0} K_{20} \quad . \tag{66}$$

If we insert Eq. (66) into Eq. (64), we find that the two Poisson brackets cancel. Hence, Eq. (66) is a good solution when the first term in Eq. (64) is small relative to either of the other terms, i.e., $|(df_0/dh_0)(\partial K_{20}/\partial t)| \ll |(df_0/dh_0) \underset{\sim}{V}_0 \cdot \nabla K_{20}|$. This condition is satisfied for the vast majority of the particles when

$$\left| \frac{\partial \underset{\sim}{A}_1}{\partial t} \right| \ll v_T \left| \underset{\sim}{\nabla} \underset{\sim}{A}_1 \right| \quad , \tag{67}$$

where $v_T$ is the thermal speed of the particles. Physically, this means then the particles must be able to cross the wave packet before it changes significantly.

For simplicity, we will henceforth use the linearized solution (66). However, we note that this linearization does not follow from the ordering introduced in Sec. IVB. There, the approximation is given by Eq. (44). Here, we are making an independent approximation. For Maxwellian $f_0$, the approximation needed for the linear solution to be valid is $K_{20}/T \ll 1$, where T is the temperature of the distribution.

The quasi-static approximation also implies that we can neglect the time-derivatives in Maxwell's equations, since (67) implies that motions are slow compared with the speed of light. Thus, Eqs. (56b) reduces to

$$\nabla \times \underset{\sim}{B}_{20} \equiv - \nabla^2 \underset{\sim}{A}_{20} = 4\pi \underset{\sim}{j}_{20} \quad . \tag{68}$$

1. The Quasi-static Density Perturbation

We are now prepared to calculate the self-consistent density perturbation. We take $f_0$ Maxwellian, since this case gives particularly simple results. Then, Eq. (66) yields $F_2 = - f_0 K_{20}/T$. Inserting this result into Eq. (55a) , we find the density perturbation to be given by

$$n_{20} = \int d^3p \left[ - f_0 (K_{2v0} + e\Phi_{20} - ev_0 \cdot A_{20})/T + \frac{1}{2} \{w_1, \{w_1, f_0\}\}_0 \right] .$$

$$\tag{69}$$

We can neglect the term containing $A_{20}$ since it is an odd moment of an isotropic distribution. Furthermore, we can neglect the last term. A quick calculation shows that if $w_1$ is defined by $w_1 = W_1(q, p, t)\exp(ik \cdot q - i\omega t) + c.c.$, then

$$\frac{1}{2}\{w_1, \{w_1, f_0\}\}_0 = k \cdot \frac{\partial}{\partial p}\left(|W_1|^2 \, k \cdot \frac{\partial f_0}{\partial p}\right) + 0(\nabla A_1) \quad ,$$

where the terms represented by $0(\nabla A)$ can be neglected on the basis of either (61) or (46) as discussed in Sec. IVD. If we now use the K-$\chi$ theorem in the form (59), the evaluation of the remaining terms in (69) is trivial. We obtain

$$n_{20}^s = [E_1^*(x, t) \cdot \chi^s(k, \omega) \cdot E_1(x, t)/4\pi - n_0^s e_s \Phi_{20}]/T_s \quad , \quad (70)$$

upon restoring species labels.

We now assume a two-species plasma, and that the wave-packet is large compared with a Debye length, implying quasi-neutrality. (These assumptions are not necessary, but their use leads to a more attractive result.) Quasi-neutrality and Eq. (70) together imply

$$n_{20}^i = n_{20}^e = E_1^* \cdot \chi(k, \omega) \cdot E_1/4\pi(T_e + T_i) \quad , \quad (71)$$

where $\chi \equiv \chi^e + \chi^i$. This equation can be further simplified using Faraday's law, $k \times E_1 = \omega B_1/c$, and the linear propagation equation, $\chi \cdot E_1 = -E_1 - k \times B_1/\omega$. The result is

$$n_{20} \equiv n_{20}^i = -\left[|E_1(x, t)|^2 - |B_1(x, t)|^2\right]/4\pi(T_e + T_i) \quad .$$

$$(72)$$

A derivation for cold plasma waves was reported earlier.[18]

As we shall show in Sec. V, this result also applies in Vlasov magnetized plasma. Hence, the use of the $K$-$\underset{\approx}{\chi}$ theorem explains why Eq. (71) and Eq. (72) have appeared again and again in the literature.[10,11,27]

### 2. The Quasi-Static Magnetic Field

To calculate the quasi-static magnetic field produced by the wave, we begin by finding the second-order current density

$$\underset{\sim}{j}_{20} = e \int d^3p \left( \underset{\sim}{v}_0 f_{20} + \langle \underset{\sim}{v}_1 f_1 \rangle + \underset{\sim}{v}_{20} f_0 \right) \; .$$

Combining Eqs. (8), (25), (52), and (66) and keeping terms up to only second order, we obtain

$$\underset{\sim}{j}_{20} = e \int d^3p \left[ \frac{\partial f_0}{\partial \underset{\sim}{p}} K_{2\nu 0} + \underset{\sim}{v}_0 \frac{1}{2} \left\langle \{w_1, \{w_1, f_0\}\} \right\rangle \right.$$

$$- e\gamma_0^{-1} \left\langle (\underset{\sim}{A}_1 - \underset{\sim}{v}_0 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1) \{w_1, f_0\} \right\rangle$$

$$\left. - \frac{1}{2} e \gamma_0^{-2} \left\langle \underset{\sim}{v}_0 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 + 2\underset{\sim}{A}_1 \underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1 - 3\underset{\sim}{v}_0 (\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1)^2 \right\rangle f_0 \right] \; . \qquad (73)$$

For simplicity, we first evaluate the current density in the case where the gradient terms may be neglected, i.e., where either approximation (46) or approximation (61) applies. Since this evaluation is tedious, we only present the result, $j_{20} = o(\nabla \underset{\sim}{A})$, that is, the second-order current vanishes to zeroth order in the gradient of the amplitude. At first glance, this result seems surprising. After all, in the last section

we showed that fake diffusion leads to wave momentum and, hence (at least for nonrelativistic particles), wave current. It seems to be quite a coincidence that the ponderomotive effects and the fake diffusion effects cancel to lowest order.

However, further inspection shows that this fact is implied by charge conservation. We note that the structure of the theory is such that a calculation of $\underset{\sim}{j}_{2\nu 0}$ must result in an expression of the form

$$[\underset{\sim}{j}_{2\nu 0}(\underset{\sim}{x},t)]_{\ell} = \sum_{\substack{m=1 \\ n=1}}^{3} \alpha_{\ell mn}(\underset{\sim}{k},\omega,f_0) A_m(\underset{\sim}{x},t) A_n^*(\underset{\sim}{x},t) \quad .$$

In the quasi-static limit $\underset{\sim}{\nabla} \cdot \underset{\sim}{j} = 0$, which implies

$$\sum_{\ell mn} \alpha_{\ell mn} \partial (A_m A_n^*)/\partial x_{\ell} = 0 \quad ,$$

for any $\underset{\sim}{A}$, since $\underset{\sim}{A}$ is arbitrary. ($\underset{\sim}{A}$ need not be self-consistent; the wave could be externally driven.) Thus, $\alpha$ must vanish, and $\underset{\sim}{j}_{20}$ must be the curl of some quantity in order for its divergence to vanish.

Thus, to find $\underset{\sim}{j}_{20}$, we must re-evaluate ( 73 ) keeping the terms containing amplitude gradients. For simplicity, we use the approximation $\underset{\sim}{k} \cdot \underset{\sim}{v}_0 \ll \omega$ and $|\underset{\sim}{v}_0| \ll 1$ in this calculation. By this procedure, Eq. ( 73 ) reduces to

$$\underset{\sim}{j}_{20}^{s}(\underset{\sim}{x},t) = \frac{n_0 e_s^3}{m_s^2 \omega^3} \nabla \times [-i\underset{\sim}{E}_1^*(\underset{\sim}{x},t) \times \underset{\sim}{E}_1(\underset{\sim}{x},t)] \quad ,$$

upon restoring species labels. We note that $\nabla \cdot \underset{\sim}{j}_{20}$ vanishes identically. We also note that only the electron current

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contributes significantly because of the presence of the mass in the denominator of this expression.

To calculate the induced magnetic field, we use Ampere's law, (68). This yields

$$\nabla \times \left( \underset{\sim}{B}_{20} - \frac{4\pi n_0 e_0^3}{m_e^2 \omega^3 c} i\underset{\sim}{E}_1^* \times \underset{\sim}{E}_1 \right) = 0 \quad,$$

in which $-e_0$ is the charge of the electron. From this we deduce

$$\underset{\sim}{B}_{20} = \frac{4\pi n_0 e_0^S}{m_e^2 \omega^3 c} \left( i\underset{\sim}{E}_1 \times \underset{\sim}{E}_1 - \nabla U \right) \quad,$$

where the quantity $U$ is determined from $\nabla \cdot \underset{\sim}{B}_{20} = 0$ to be

$$U(\underset{\sim}{x},t) = - \frac{i n_0 e_0^3}{m_e^2 \omega^3 c} \int d^3 x' \, \frac{\nabla' \cdot [\underset{\sim}{E}_1^*(\underset{\sim}{x}',t) \times \underset{\sim}{E}_1(\underset{\sim}{x}',t)]}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \quad.$$

This expression for $B_{20}$ is the general three-dimensional result. It reduces to the result of Bezzerides, et al.[19,20] in the one-dimensional case, and to the result of Mora and Pellat,[28,29] which is valid when $U$ vanishes.

Finally, we would like to say how the cold-fluid calculation by Speziale and Catto[30] of the wave-generated magnetic field fits into the present theory. With some work, one can show that the results of Speziale and Catto are obtained by neglecting the ponderomotive Hamiltonian in Eq. (73). Thus, (as pointed out by Mora and Pellat) the results of Speziale and Catto are applicable for the short-time response, before the particles have had time to equilibrate with the ponderomotive force.

## V. PONDEROMOTIVE EFFECTS IN MAGNETIZED PLASMA

### A. The Unperturbed System

The unperturbed, nonrelativistic Hamiltonian for a particle in a uniform magnetic field is $h_0 = \frac{1}{2}[\underset{\sim}{p} - e\underset{\sim}{A}_0(\underset{\sim}{q})]^2$, where the vector potential can be taken to be $\underset{\sim}{A}_0(\underset{\sim}{q}) = q_x B_0 \hat{y}$ for the magnetic field $\underset{\sim}{B}_0 = B_0 \hat{z}$.

For convenience we introduce new canonical pairs, $(y_g, p_g)$ and $(\psi, \mu)$, to describe motion perpendicular to $\underset{\sim}{B}_0$.

$$q_x = p_g/\Omega + (2\mu/\Omega)^{\frac{1}{2}} \sin \psi \quad ,$$

$$p_x = (2\Omega\mu)^{\frac{1}{2}} \cos \psi \quad ,$$

$$q_y = y_g + (2\mu/\Omega)^{\frac{1}{2}} \cos \psi \quad ,$$

$$p_y = p_g \quad .$$

The variable $\Omega \equiv eB_0$ in these equations is the signed gyrofrequency. Note that particles with positive (negative) $\Omega$ must have positive (negative) $\mu$. In terms of these variables, the unperturbed Hamiltonian is $h_0 = \Omega\mu + \frac{1}{2}p_z^2$ .

The interpretation of these variables is well known. The x guiding center position is $x_g \equiv p_g/\Omega$. The y guiding center position is $y_g$. [To denote the vector guiding center we use $\underset{\sim}{r}_g \equiv (x_g, y_g, q_z)$. To denote the vector gyroradius we use $\underset{\sim}{\xi} \equiv \hat{x}(2\mu/\Omega)^{\frac{1}{2}}\sin \psi + \hat{y}(2\mu/\Omega)^{\frac{1}{2}}\cos \psi.$] The gyrophase is $\psi$, and the action (magnetic moment divided by the charge) is $\mu$. For convenience, we will sometimes use the gyrospeed $v \equiv (2\mu\Omega)^{\frac{1}{2}}$. In terms of these variables, the zeroth-order, time-development transformation is given by

$$S_0(t)g(y_g,p_g,\psi,\mu,q_z,p_z) = g(y_g,p_g,\psi + \Omega t,\mu,q_z + p_zt,p_z) \ . \quad (74)$$

To calculate the response, we will need the charge and current density operators. The charge density operator is $r(\underset{\sim}{x};\underset{\sim}{z}) = e\delta(\underset{\sim}{x}-\underset{\sim}{r}_g-\underset{\sim}{\xi})$. For the nonrelativistic Hamiltonian, the current density·operator is

$$\underset{\sim}{\eta}(\underset{\sim}{x};\underset{\sim}{z},t) = \sum_{n=0}^{\infty} e\underset{\sim}{v}_n\delta(\underset{\sim}{x}-\underset{\sim}{r}_g-\underset{\sim}{\xi}) \ , \quad (75)$$

where $\underset{\sim}{v}_n = -e\underset{\sim}{A}_n$ for $n > 0$, and

$$\underset{\sim}{v}_0 = \underset{\sim}{p} - e\underset{\sim}{A}_0 = \frac{v}{\sqrt{2}} e^{i\psi}\hat{u}_+ + \frac{v}{\sqrt{2}} e^{-i\psi}\hat{u}_- + p_z\hat{u}_z \quad (76)$$

in terms of the unit vectors $\hat{u}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}$ and $\hat{u}_z = \hat{z}$ .

## B.   Linear Theory Via Lie Transforms

As in the unmagnetized case, we must give a brief presentation of the linear theory of a magnetized particle in a modulated wave before proceeding to the nonlinear theory. The calculation is analogous to the one in Sec. IV.   We integrate $h_1$ to find $w_1$. From $w_1$ and $f_0$ we compute $f_1$. From $f_1$ we compute the linear response.

Without loss of generality, we take the wave vector of perturbing field to be in the $x - z$ plane: $\underset{\sim}{A}_1(\underset{\sim}{x},t) = \underset{\sim}{A}_1(\underset{\sim}{x},t) \exp(ik_xx + ik_zz - i\omega t) + $ c.c. As before, we use the radiation gauge, $\Phi_1 = 0$, for the first-order fields.

For the given vector potential, the first-order Hamiltonian

$h_1 = -e\underset{\sim}{v}_0 \cdot \underset{\sim}{A}_1$, is

$$h_1 = -e\left(\frac{v}{\sqrt{2}} e^{i\psi}\hat{u}_+ + \frac{v}{\sqrt{2}} e^{-i\psi}\hat{u}_- + p_z\hat{u}_z\right)$$

$$\cdot \; \underset{\sim}{A}_1(\underset{\sim}{r}_g + \underset{\sim}{\xi},t) \; \exp[ik_x x_g + ik_x(v/\Omega)\sin\psi + ik_z q_z - i\omega t] + \text{c.c.}$$

$$= -e\sum_{n=-\infty}^{\infty} \underset{\sim}{U}_n^* \cdot \underset{\sim}{A}_1(\underset{\sim}{r}_g + \xi,t) \; \exp(ik_x x_g + in\psi + ik_z q_z - i\omega t) + \text{c.c.},$$

where the vector $\underset{\sim}{U}_n$ is defined by

$$\underset{\sim}{U}_n \equiv \frac{v}{\sqrt{2}} J_{n+1}(k_x v/\Omega)\hat{u}_+ + \frac{v}{\sqrt{2}} J_{n-1}(k_x v/\Omega)\hat{u}_- + p_z J_n(k_x v/\Omega)\hat{u}_z \quad .$$

We note that $\underset{\sim}{U}_n$ is an invariant of the unperturbed Hamiltonian.

To find the first-order term, $w_1$, of the Lie transform, we must integrate $h_1$ along a trajectory

$$w_1 = -\int_{-\infty}^{0} d\tau \; h_1(x_g,y_g,\psi+\Omega\tau,\mu,q_z+p_z\tau,p_z,t+\tau) \quad .$$

As in the unmagnetized case, we calculate this integral by repeatedly integrating by parts

$$w_1 = -ie\sum_{n=-\infty}^{\infty}\left\{\exp\left(ik_x x_g + in\psi + ik_z q_z - i\omega t\right)\frac{\underset{\sim}{U}_n^*}{n\Omega + k_z p_z - \omega}\right.$$

$$\times \sum_{m=0}^{\infty}\left[\frac{i}{n\Omega + k_z p_z - \omega}\left(\frac{\partial}{\partial t} + p_z\frac{\partial}{\partial q_z} + \Omega\frac{\partial}{\partial\psi}\right)\right]^m$$

$$\left. \cdot \underset{\sim}{A}_1[x_g + (v/\Omega)\sin\psi, y_g + (v/\Omega)\cos\psi, q_z, t]\right\} + \text{c.c.} \quad (77)$$

Here, we see that the basic approxiamtion needed for pondero-motive theory to apply is that the amplitude, $\underset{\sim}{A}$, must vary slowly along an unperturbed trajectory:

$$\left| \left( \frac{\partial}{\partial t} + p_z \frac{\partial}{\partial q_z} + \Omega \frac{\partial}{\partial \psi} \right) \underset{\sim}{A}_1 (\underset{\sim}{r}_g + \underset{\sim}{\xi}, t) \right| << \left| (\omega - k_z p_z - n\Omega) \underset{\sim}{A}_1 \right|$$

$$(78)$$

Since the minimum of $|\omega - k_z p_z - n\Omega|$ for all n is less than $\Omega$, the approximation (78) implies that

$$\left| \left( \frac{\partial}{\partial t} + p_z \frac{\partial}{\partial q_z} + \Omega \frac{\partial}{\partial \psi} \right) \underset{\sim}{A}_1 \right| << \left| \Omega \underset{\sim}{A}_1 \right| \ .$$

$$(79)$$

The reason the approximation (79) must hold is that we are averaging over a gyroperiod. If (79) did not hold, then the gyroperiod would be a long time period. Hence, magnetic effects would be unimportant, and the unmagnetized theory would be used.

Equation (79), which can be written in the form,

$$\left| \left( \frac{\partial}{\partial t} + p_z \frac{\partial}{\partial q_z} + \underset{\sim}{v}_{0\perp} \cdot \underset{\sim}{\nabla}_\perp \right) \underset{\sim}{A}_1 \right| << \left| \Omega \underset{\sim}{A}_1 \right| \ ,$$

$$(80)$$

also implies that the gyroradius, $v/\Omega$, must be small when com-pared with the scale length of the wavepacket. To see this, we note that since $\underset{\sim}{v}_{0\perp}$ oscillates sinusoidally along a trajectory, Eq. (80) can just as well be written

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$$\left| \left( \frac{\partial}{\partial t} + p_z \frac{\partial}{\partial q_z} - \underset{\sim}{v}_{0\perp} \cdot \underset{\sim}{\nabla}_\perp \right) \underset{\sim}{A}_1 \right| \ll |\Omega \underset{\sim}{A}_1| \quad . \tag{81}$$

In combination, Eqs. ( 80 ) and ( 81 ) imply $\left| (\partial/\partial t + p_z \partial/\partial q_z) \underset{\sim}{A}_1 \right| \ll |\Omega \underset{\sim}{A}_1|$ and $|\underset{\sim}{v}_{0\perp} \cdot \underset{\sim}{\nabla}_\perp \underset{\sim}{A}_1| \ll |\Omega \underset{\sim}{A}_1|$ , from which the small gyro-radius assumption follows.

In any case, the approximation, ( 78), allows us to truncate the series in Eq.(77 ) to obtain the expression,

$$w_1 = -ie \sum_{n=-\infty}^{\infty} \exp\left( ik_x x_g + in\psi + ik_z q_z - i\omega t \right) \frac{\underset{\sim}{U}_n^* \cdot \underset{\sim}{A}(q,t)}{n\Omega + k_z p_z - \omega} + c.c. \tag{82}$$

for the first-order term of the Lie transform.

To first-order, the oscillation-center distribution equals the unperturbed distribution. For a homogeneous, time-independent system, the unperturbed distribution is a function of $\mu$ and $p_z$ only. Hence, $F = f_0(p_z, \mu)$. Inserting $f_0$ and $w_1$ into Eq. ( 25 ), we obtain the first-order distribution

$$f_1 = -\{w_1, f_0\} = -\frac{\partial w_1}{\partial \psi} \frac{\partial f_0}{\partial \mu} - \frac{\partial w_1}{\partial q_z} \frac{\partial f_0}{\partial p_z}$$

$$= -e \sum_{n=-\infty}^{\infty} \frac{\exp[ik_x x_g + in\psi + ik_z q_z - i\omega t]}{n\Omega + k_z p_z - \omega} \left[ \frac{\partial f_0}{\partial p_z} \left( k_z - i\frac{\partial}{\partial q_z} \right) \right.$$

$$\left. + \frac{\partial f_0}{\partial \mu} \left( n - i(v/\Omega)\cos\psi \frac{\partial}{\partial q_x} + i(v/\Omega)\sin\psi \frac{\partial}{\partial q_y} \right) \right]$$

$$\times \underset{\sim}{U}_n^* \cdot \underset{\sim}{A}_1(q,t) + c.c.$$

Finally, we insert this expression for $f_1$ and the current operation in Eq. (75) into Eq. (27) to obtain the first-order current density response. Neglecting the amplitude gradients, we obtain, for the linear susceptibility (Ref. 23, p. 51),

$$\underset{\approx}{\chi}^S(\underset{\sim}{k},\omega) = -\frac{4\pi e_s^2}{\omega^2} \int 2\pi\Omega d\mu dp_z f_0 \left[ \underset{\approx}{I} + \sum_{n=-\infty}^{\infty} \left( n \frac{\partial}{\partial\mu} + k_z \frac{\partial}{\partial p_z} \right) \right.$$

$$\left. \frac{\underset{\sim}{U}_n \underset{\sim}{U}_n^*}{\omega - k_z p_z - n\Omega} \right] . \tag{83}$$

## C.    Oscillation-Center Evolution Equations

The oscillation-center picture for magnetized particles is analogous to that of unmagnetized particles except for the existence of non-zero $w_{20}$.

We begin by dividing Eq. (22b) into linear and nonlinear parts. Here we only keep the zeroth harmonic terms, since the second harmonic terms may all be transformed away. We obtain

$K_{2\nu 0} - \frac{1}{2}(\partial w_{2\nu 0}/\partial t + \{w_{2\nu 0}, h_0\}) = h_{2\nu 0} + \frac{1}{2}\{w_1, h_1\}$ and

$K_{2\nu 0} - \frac{1}{2}(\partial w_{2\lambda 0}/\partial t + \{w_{2\lambda 0}, h_0\}) = h_{2\lambda 0}$, where $h_{2\lambda 0} = - e(\underset{\sim}{p} - e\underset{\sim}{A}) \cdot \underset{\sim}{A}_{20}$

$+ e\Phi_{20}$ and $h_{2\nu 0} = \langle \frac{1}{2} e^2 \underset{\sim}{A}_1 \cdot \underset{\sim}{A}_1 \rangle$.

We first analyze the second-order linear terms. Using Eq. (76), we obtain the following expression for $h_{2\lambda 0}$:

$$h_{2\lambda 0} = - \left( \frac{ev}{\sqrt{2}} e^{-i\psi}\hat{u}_- + \frac{ev}{\sqrt{2}} e^{i\psi} \hat{u}_+ + ep_z\hat{u}_z \right) \cdot \underset{\sim}{A}_{20}(\underset{\sim}{q},t) + e\Phi_{20}(\underset{\sim}{q},t) .$$

$$\tag{84}$$

Using the small gyroradius approximation, this becomes

$$h_{2\lambda 0} = -\left( \frac{ev}{\sqrt{2}}\, e^{-i\psi}\hat{u}_- + \frac{ev}{\sqrt{2}}\, e^{i\psi}\hat{u}_+ + ep_z\hat{u}_z \right)\cdot \underset{\sim}{A}_{20}(\underset{\sim}{r}_g,t) + e\Phi_{20}(\underset{\sim}{r}_g,t) \quad .$$

$$( 85 )$$

We use this result to select $K_{2\lambda 0}$ and $w_{2\lambda 0}$. As usual, the terms which vary slowly along an orbit are equated to $K_{2\lambda 0}$,

$$K_{2\lambda 0} = -ep_z\hat{u}_z\cdot \underset{\sim}{A}_{20}(\underset{\sim}{r}_g,t) + e\Phi_{20}(\underset{\sim}{r}_g,t) \quad . \qquad ( 86 )$$

The remaining part of $h_{2\lambda 0}$ is equated to the terms containing $w_{2\lambda 0}$. Integrating along a trajectory, we find

$$\frac{1}{2} w_{2\lambda 0} = \left( \frac{iev}{\Omega\sqrt{2}}\, e^{-i\psi}\hat{u}_- - \frac{iev}{\Omega\sqrt{2}}\, e^{i\psi}\hat{u}_+ \right)\cdot \underset{\sim}{A}_{20}(\underset{\sim}{r}_g,t) \quad . \qquad ( 87 )$$

By a similar procedure we can calculate $K_{2\nu 0}$ and $w_{2\nu 0}$. However, we defer the calculation of $K_{2\nu 0}$ until the next section where we use the $\underset{\approx}{K}$-$\chi$ theorem. We defer the calculation of $w_{20}$ altogether since it will not be needed in this paper. In any case, the point has been made that the basic difference between the magnetized and unmagnetized cases is the presence of nonzero $w_{20}$.

The remaining aspects of the oscillation-center picture for magnetized particles are identical to the unmagnetized case. The oscillation center distribution evolves according to Eq. ( 51 ). The physical distribution is found using Eq. ( 52 ). Of course, now the time-averaged physical distribution is given by

$(f)_0 = F + \frac{1}{2}\{w_1,\{w_1,F\}\}_0 - \frac{1}{2}\{w_{20},F\}$ , since $w_{20}$ does not vanish.

Hence, the slowly varying charges and currents are given by

$$\rho_{20} = \int d^6z \; r(\underset{\sim}{x};\underset{\sim}{z},t) \left[ F - f_0 + \frac{1}{2} \{w_1,\{w_1,F\}\}_0 - \frac{1}{2} \{w_{20},F\} \right] \qquad (\;88\;)$$

and

$$\underset{\sim}{j}_{20} = \int d^6z \left[ \underset{\sim}{\eta}_0(\underset{\sim}{x};\underset{\sim}{z},t) \left( F - f_0 + \frac{1}{2} \{w_1,\{w_1,F\}\}_0 - \frac{1}{2} \{w_{20},F\} \right) \right.$$

$$\left. - \left\langle \underset{\sim}{\eta}_1(\underset{\sim}{x};\underset{\sim}{z},t) \{w_1,F\} \right\rangle + \underset{\sim}{\eta}_{20} F \right] \; . \qquad (\;89\;)$$

## D.   The Ponderomotive Hamiltonian for a Magnetized Particle

Upon inserting the magnetic susceptibility, Eq. ( 83 ), into the K-$\underset{\approx}{\chi}$ theorem, Eq. ( 37 ), we obtain the relation

$$\int dx_g dy_g d\psi \Omega d\mu dq_z dp_z \; f_0(\mu,p_z) K_{2\nu}(x_g,y_g,q_z,\mu,p_z,t)$$

$$= \frac{e^2}{\omega^2} \int d^3x \; \underset{\sim}{E}_1^*(\underset{\sim}{x},t) \int 2\pi\Omega d\mu dp_z \; f_0(\mu,p_z)$$

$$\cdot \left[ \underset{\approx}{I} + \sum_{n=-\infty}^{\infty} \left( n \frac{\partial}{\partial\mu} + k_z \frac{\partial}{\partial p_z} \right) \frac{\underset{\sim}{U}_n \underset{\sim}{U}_n^*}{\omega - k_z p_z - n\Omega} \right] \cdot \underset{\sim}{E}_1(\underset{\sim}{x},t) \quad .$$

$$(\;90\;)$$

In writing this relation we have used the fact that $K_{2\nu}$ cannot depend on $\psi$ since it is an orbit average. By functionally differentiating Eq. ( 90 ) with respect to $f_0$ and integrating with respect to $\psi$, we obtain

$$\int dx_g dy_g dq_z K_{2\nu}(x_g, y_g, q_z, \mu, p_z, t)$$

$$= \frac{e^2}{\omega^2} \int d^3x \, \underset{\sim}{E}_1^*(x,t) \cdot \left[ \underset{\approx}{I} + \sum_{n=-\infty}^{\infty} \left( n \frac{\partial}{\partial \mu} + k_z \frac{\partial}{\partial p_z} \right) \right.$$

$$\left. \frac{\underset{\sim}{U}_n \underset{\sim}{U}_n^*}{\omega - k_z p_z - n\Omega} \right] \cdot \underset{\sim}{E}_1(\underset{\sim}{x}, t) \quad .$$

In this expression we have two quantities with equal integrals. As in the unmagnetized case, we argue that the integrands must be equal except for terms of the order of the amplitude gradient.

$$K_{2\nu}(x_g, y_g, q_z, \mu, p_z, t)$$

$$= \frac{e^2}{\omega^2} \underset{\sim}{E}_1^*(\underset{\sim}{r}_g, t) \cdot \left[ \underset{\approx}{I} + \sum_{n=-\infty}^{\infty} \left( n \frac{\partial}{\partial \mu} + k_z \frac{\partial}{\partial p_z} \right) \right.$$

$$\left. \frac{\underset{\sim}{U}_n \underset{\sim}{U}_n^*}{\omega - k_z p_z - n\Omega} \right] \cdot \underset{\sim}{E}_1(\underset{\sim}{r}_g, t) \quad . \tag{91}$$

Thus, we have obtained the ponderomotive Hamiltonian for a magnetized particle.

From our experience with the unmagnetized case, we know that the validity of Eq. ( 91 ) rests on the approximation ( 46 ) plus the approximations used in deriving the linear response. Specifically, ( 78 ) must hold, and, as shown in Sec. VB, this implies that the gyroradius must be small in comparison with the scale length of the wave packet.

To reduce Eq. ( 91 ) to previously known results, we must take the limits $k_x v/\Omega \ll 1$ and $k_z p_z \ll \omega, |\omega - \Omega|, |\omega + \Omega|$. This gives $K_{2v} = |\hat{u}_z \cdot \underset{\sim}{E}_1|^2/\omega^2 + |\hat{u}_+ \cdot \underset{\sim}{E}_1|^2/\omega(\omega-\Omega) + |\hat{u}_- \cdot \underset{\sim}{E}_1|^2/\omega(\omega+\Omega)$, the usual result for the ponderomotive potential for cold, magnetized particles.[7]

At this point we discuss how the ponderomotive Hamiltonian affects the motion of the particles. We recall that the Lie-transform expressions are functional relationships. Hence, the ponderomotive Hamiltonian for an oscillation center, $(X_g, Y_g, Q_z, \Psi, M, P_z)$, is found by simply substituting those variables for $(x_g, y_g, q_z, \psi, \mu, p_z)$.

The implications of the ponderomotive Hamiltonian for parallel motion are similar to those of the unmagnetized case. The oscillation center feels a force, $\dot{P}_z = -\partial K_{2v}/\partial Q_z$, which is proportional to the parallel gradient of the ponderomotive Hamiltonian. In addition, the parallel velocity and the momentum differ by the momentum derivative of the Hamiltonian, $\dot{Q}_z - P_z = \partial K_{2v}/\partial P_z$. The ponderomotive Hamiltonian also predicts that magnetized guiding centers will experience ponderomotive drifts, $\dot{X}_g = -(\partial K_{2v}/\partial Y_g)/\Omega$ and $\dot{Y}_g = (\partial K_{2v}/\partial X_g)/\Omega$.

Furthermore, from the ponderomotive Hamiltonian, one can compute the gyrofrequency shift of a particle in the presence of a high-frequency wave:

$$\dot{\Psi} - \Omega = \frac{\partial K_{2v}}{\partial M}$$

$$= \frac{e^2}{\omega^2} \underset{\sim}{E}_1^*(\underset{\sim}{R}_g, t) \cdot \sum_{n=-\infty}^{\infty} \left[ \frac{\partial}{\partial M} \left( n \frac{\partial}{\partial M} + k_z \frac{\partial}{\partial p} \right) \frac{\underset{\sim}{U}_n \underset{\sim}{U}_n^*}{\omega - k_z p_z - n\Omega} \right] \cdot \underset{\sim}{E}_1(\underset{\sim}{R}_g, t).$$

$$( 92 )$$

This result generalizes a formula derived previously by Aamodt et al.[21] To reduce Eq. ( 92 ) to their result one must assume that the wave is electrostatic $(\underset{\sim}{k} || \underset{\sim}{E}_1)$, that only one term in the sum contributes, and that $k_z P_z/(\omega - n\Omega) \ll 1$.

Finally, we note that the oscillation center magnetic moment is an invariant, since K does not depend on $\Psi$. Actually M is an adiabatic invariant. Because we invoked approximation ( 79 ), M is conserved only when the time derivative of the amplitude is small. In terms of the physical variables, M is given by $M = \mu - \{w_1, \mu\} - \frac{1}{2}\{w_2, \mu\} + \frac{1}{2}\{w_1, \{w_1, \mu\}\}$. Alternatively, the magnetic moment $\mu$ is given by $\mu = M + \{w_1(\underset{\sim}{Z}, t), M\} + \frac{1}{2}\{w_2(\underset{\sim}{Z}, t), M\} + \frac{1}{2}\{w_1(\underset{\sim}{Z}, t), \{w_1(\underset{\sim}{Z}, t), M\}\}$ in terms of the oscillation-center variables. Note that $\mu$ is not invariant because of the presence of oscillating terms such as $\{w_1, M\}$. However, one can easily verify that the term $\{w_1, M\} = \partial w_1/\partial \Psi$ does vanish in the limit $(k_z p - \omega)/\Omega \to 0$ and $k_x v/\Omega \to 0$ as it should, since then $\mu$ must be an adiabatic invariant.

## E.   The Self-Consistent, Quasistatic Response

If we neglect the time derivatives in Eq. ( 51 ), we find that the oscillation-center distribution can be any function of the invariants of the oscillation-center Hamiltonian, $h_0 + K_{20}$. In the

magnetized case, there are two such invariants, $\mu$ and $\frac{1}{2}p_z^2 + K_{20}$. Furthermore, using the arguments of Sec. IVF, the function must be the unperturbed distribution. Thus, the solution for F is given by $F = f_0(\frac{1}{2} p_z^2 + K_{20}, \mu)$. We note that if F is linearized, as in Eq. ( 65 ), then the change in F is given by

$$F_2 = \frac{df_0}{d \frac{1}{2} p_z^2} K_{20} \quad . \qquad\qquad ( 93 )$$

To determine the validity of this solution, we insert it into Eq. ( 64 ) and require the term $\partial F_2/\partial t$ to be small. We find that the quasi-static approximation is valid when $| \partial \underset{\sim}{A}/\partial t| \ll v_\| |\partial \underset{\sim}{A}/\partial z|$ holds. Physically, this means that the particles must be able to cross the wave packet by moving along field lines before the wave packet changes significantly.

For simplicity, henceforth we will use the linearized solution ( 93 ). In addition, we will assume the unperturbed distribution to be Maxwellian in the parallel energy. In this case, Eq. ( 93 ) becomes

$$F_2 = - \frac{f_0}{T_\|} K_{20} \quad , \qquad\qquad ( 94 )$$

where $T_\|$ is the parallel temperature.

To find the density perturbation, we insert the distribution ( 94 ) into Eq. ( 88 ). We obtain

$$n_{20} = \int d^3x_g d\psi \Omega d\mu dp_z \quad \delta(\underset{\sim}{x} - \underset{\sim}{r}_g - \xi) \left[ -f_0(K_{2\nu 0} + e\phi_{20} - ep_z \hat{u}_z \cdot \underset{\sim}{A}_{20}/T_\parallel \right.$$

$$\left. + \frac{1}{2}\{w_1, \{w_1, f_0\}\}_0 - \frac{1}{2}\{w_{20}, f_0\} \right].$$

In evaluating this expression, we can neglect the term involving $\underset{\sim}{A}_{20}$, since it is an odd moment of an even function. When performing the integral over $d^3r_g$, we can replace $\underset{\sim}{r}_g$ by $\underset{\sim}{x}$ since $\xi$ is small compared with the scale length of the wave packet. Thus we have

$$n_{20} = \int d\psi \Omega d\mu dp_z \left[ -f_0(K_{2\nu 0} + e\phi_{20})/T_\parallel \right.$$

$$\left. + \frac{1}{2}\{w_1, \{w_1, f_0\}\}_0 - \frac{1}{2}\{w_{20}, f_0\} \right]_{\underset{\sim}{r}_g = \underset{\sim}{x}} . \qquad (95)$$

Next, we note that the last two terms in Eq. ( 95 ) vanish to lowest order in the amplitude gradients. The term

$$-\frac{1}{2}\int d\psi \Omega \ d\mu \ dp_z\{w_{20}, f_0\} = -\frac{1}{2}\int d\psi \Omega \ d\mu \ dp_z \frac{\partial w_{20}}{\partial \psi} \frac{\partial f_0}{\partial \mu} ,$$

vanishes since it is the integral over $\psi$ of a $\psi$ derivative. To prove that the other term vanishes, we insert $w_1$ in the form,

$$w_1 = \mathcal{W}_1(\underset{\sim}{q}, \psi, \mu, p_z, t) \exp(ik_x x_g + ik_z \sigma_z - i\omega t) + c.c. ,$$

where

$$\mathcal{W}_1 = -ie \sum_{n=-\infty}^{\infty} \exp(in\psi) \frac{\underset{\sim}{U}_n^* \cdot \underset{\sim}{A}(\underset{\sim}{q}, t)}{n\Omega + k_z p_z - \omega} ,$$

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into Eq. (95). The term is then seen to vanish upon integrating by parts.

Thus, Eq. (95) for the density perturbation reduces to

$$n_{20} = -\left(n_0 e\phi_{20} + \int d\psi\Omega \; d\mu \; dp_z f_0 K_{2\nu 0}\right)/T_\parallel \quad,$$

which can be put in the form, ( 70 ). At this point, we can follow the analysis in Sec. IVF from Eq. ( 70 ) to Eq. ( 72 ). Thus, Eqs. (70) and (72) also apply to a magnetized plasma.

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