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James Hamilton Crichton
(Ph. D. Thesis)

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ABSTRACT

We investigate conditions imposed on the S matrix by the requirement that interactions between elementary particles be of short range. We will consider, in particular, two distinct properties of elementary-particle interactions, which we call cluster properties: (1) that interactions between particles proceed independently of the presence of other particles far away in space and time; (2) that multiparticle scattering processes occur predominantly by means of a freely-propagating particle connecting, in a causal way, successive scattering processes which involve fewer particles. We formulate these properties as limiting equations involving plane-wave S-matrix elements. These equations imply a structure in the S matrix which we call the cluster structure. We demonstrate the independence of the cluster properties from the usually-postulated properties of the S matrix, Lorentz invariance and unitarity. We show that the S operators obtained from the Feynman perturbation theory and from the LSZ formalism satisfy both cluster properties. We claim that <u>any</u> realistic theory of elementary-particle interactions must satisfy these properties.

I. INTRODUCTION

This study is concerned with a class of properties which _any_ realistic theory of elementary particle interactions must satisfy. The properties which concern us are those which derive from the short-range, or approximately local, nature of elementary particle interactions. We choose to discuss these properties and their consequences in terms of the S matrix.

Let us review briefly the S-matrix program proposed by Heisenberg.[1] In the early 1940's, he attempted to isolate from the quantum field theory of that time, beset as it was with divergence difficulties, those concepts which are generally valid, and which would occur in any future theory (possibly not beset by divergence difficulties). He studied in particular the observable quantities in scattering processes: the asymptotic behavior in coordinate space of the wave functions of the participating particles. To discuss these quantities he introduced a matrix S, whose plane-wave momentum-space matrix elements give the transition probability amplitudes for scattering and production processes for two or more particles. That is, the plane-wave S-matrix elements determine the transition probability amplitude between an initial state of non-interacting particles and a final state of non-interacting particles. The observable quantities in coordinate space are obtained by Fourier transformation. A detailed description of the interaction was not attempted by Heisenberg, and he was thus able to bypass the divergence difficulties. In his formulation Heisenberg proposed in particular that S be a unitary operator, and that it commute with Lorentz transformations. He also indicated a "connectedness structure" for the S matrix which implicitly suggests the ideas to be developed in the present study.

We may interpret the spirit of Heisenberg's program to be the following: In any reasonable theory of elementary-particle interactions we must be able to define an S matrix. Ultimately, of course, the theory must reproduce all experimental results in detail. However, even if we do not know the correct, detailed theory, we can nevertheless say that any candidate for the S matrix must satisfy such general conditions as unitarity and Lorentz invariance, or be excluded from consideration. It is thus a meaningful task to inquire what further conditions on the S matrix would limit the choice of theories. It is this task to which we address ourselves. The further conditions on the S matrix which we propose are, as we have said before, those which derive from the short-range of the interactions.

By "short-range" force, we have in mind a definite notion, although we do not formulate it rigorously. In a naive and intuitive sense, we mean that two particles interact only when they are "close" to each other, but we do not specify in detail what is meant by "close." Rather, our notion is that for very early times and for very late times, when the particles participating in a scattering process tend to be more and more separated, they tend to behave as free particles because the interaction ceases to be effective. In other words, we mean that "short-range" forces are of such a nature that asymptotically a multiparticle system can be described by free-particle states. Of course, this assumption about the forces is necessary if S-matrix elements taken between free-particle states are to be the observable quantities in a theory which avoids the specific details of the interaction. It is this notion of "short range" which is referred to hereafter. To avoid misunderstanding, it should be pointed out that we are not restricting the interactions to be derivable from potentials which vanish outside a finite region or which have an exponential decrease. The notion of short range used here is much more general. For simplicity, however, Coulomb-type forces are excluded from consideration.[2]

We turn now to the terminology to be employed. In a multiparticle system any given particle will interact, we assume, only with particles in its neighborhood, only with particles which are clustered about it. Or, in a negative sense, a particle will not interact with particles which are not clustered about it. Hence we call the properties to be discussed here cluster properties. Of course, the word cluster has been used extensively in both classical and quantum statistical mechanics, but here we do not want to assume any knowledge of results from those areas; we use the word in an elementary sense. The cluster properties on which we will focus, to be stated below, will be shown to have the consequence that the S matrix must have a certain structure, which we accordingly call the cluster decomposition of the S matrix.

We will further use the word cluster to designate any normalizable single- or multi-particle asymptotic state. Here we have in mind the negative connotation of the word. A normalizable momentum-space wave function defines for a system of freely-moving particles a region in coordinate space and in time, outside of which, the probability for finding any particle in a given finite volume is vanishingly small. One may even designate some four-vector $(\underset{\sim}{R}, T)$ such that far enough away from the point $\underset{\sim}{R}$, or much earlier or much later than T , or both, such a probability is arbitrarily small. Of course, the probability in the neighborhood of $(\underset{\sim}{R}, T)$ need not be large either. In this very vague and negative sense, the particles are "clustered" about the point $\underset{\sim}{R}$ at about the time T . We wish to use the word cluster for a single- or multi-particle system to emphasize this feature of all normalizable wave-functions, i.e., all bona fide wave functions.

We now state those cluster properties which we will study in detail. They are:

(1) Two clusters initially described by free-particle wave functions characterized by space-time parameters $(\underset{\sim}{R}_1, T_1)$ and $(\underset{\sim}{R}_2, T_2)$, in the sense described above, tend not to interact with each other in the limit that either $|\underset{\sim}{R}_1 - \underset{\sim}{R}_2|$ tends to infinity, or $|T_1 - T_2|$ tends to infinity, or both.

(2) For two clusters initially described by free-particle wave functions characterized by space-time parameters $(\underset{\sim}{R}_1, T_1)$ and $(\underset{\sim}{R}_2, T_2)$, such that the latter is within the forward light cone of the former, the predominant interaction between the clusters is by means of a single particle propagating freely from the "earlier" cluster to the "later" cluster in the limit that the time-like separation of the clusters, $|T_1 - T_2|^2 - |\underset{\sim}{R}_1 - \underset{\sim}{R}_2|^2$, tends to infinity.

What other properties, suggested by (1) and (2), it would be useful to study will be discussed later. We now begin an intuitive physical discussion of the first and second cluster properties.

What we have in mind for the first cluster property is the idea that when two particles scatter off each other, that event is not affected by the presence of other particles very far away at the time of the interaction. The first cluster property is thus obvious in the case of classical particles interacting by means of a potential which decreases rapidly with distance. It is also obvious in the quantum-mechanical case with short-range interactions. Two particles effectively do not interact as long as the amount of overlap of their wave functions in any finite region is vanishingly small. Of course, this fact insures the possibility of describing the asymptotic states as free-particle states. But, furthermore, if during the entire duration of a collision process involving two or more particles, the amount of overlap of the wave functions of one group of particles with those of the rest of the particles is always vanishingly small, then the two groups of particles, or clusters, evolve separately and do not influence each other.

Let us consider two particular clusters. We contemplate the "rigid" displacement of one cluster with respect to the other, either in a space-like or else in a time-like direction. Let us characterize the two clusters by four-vectors $(\underset{\sim}{R}_1, T_1)$ and $(\underset{\sim}{R}_2, T_2)$, in the sense described above. For simplicity we first set $T_1 = T_2$. There may be a significant overlap between the two clusters. However, as $|\underset{\sim}{R}_1 - \underset{\sim}{R}_2|$ tends to infinity, the amount of overlap in any finite region tends to zero, so that the clusters tend to develop independently of each other in this limit. This is the spatial version of the first cluster property. Now we consider the case where $\underset{\sim}{R}_1 = \underset{\sim}{R}_2$ and $T_1 \neq T_2$. The "earlier" cluster may have significant overlap with the "later" cluster, but as $|T_1 - T_2|$ tends to infinity, the amount of overlap in any finite region tends to zero, so that the "earlier" cluster tends to develop independently.

of the "later" cluster, and vice versa. This is the temporal version of the first cluster property.[3]

Both the spatial and the temporal versions have the consequence that the transition probability amplitude for two separated clusters tends to the product of the transition probability amplitudes for each cluster as the separation between them tends to infinity. Let i_1, i_2 designate the initial non-interacting states of clusters 1 and 2, respectively, and f_1, f_2 designate the final non-interacting states. The initial state of the system as a whole is the tensor product $i_1 i_2$, the final state $f_1 f_2$. Let the transition probability amplitude between an inital state i and a final state f, that is, the S-matrix element between these two states, be designated by $S(f;i)$. Finally, let λ be some parameter characteristic of the separation, either spacelike or timelike, between the two clusters. Then the first cluster property can be represented in the following way:

$$\lim_{\lambda\to\infty} S(f_1 f_2;i_1 i_2) = S(f_1;i_1)S(f_2;i_2) \quad . \tag{I-1}$$

Of course the interactions within a cluster may influence the development of another cluster with a later characteristic time. What is meant by the temporal version of the first cluster property is that the probability for "earlier" and "later" events to be related vanishes as the appropriate time difference tends to infinity. A crude notion of causality may be introduced here: the probability for "later" events to influence "earlier" ones must vanish faster, as the time separation tends to infinity, than the probability for "earlier" events to influence "later" ones. It is an interesting problem, then, to find the dominant processes which contribute to causally-connected events and to find the rate at which the probability for them vanishes as a function of the time separation, because this information will give some macroscopic causality properties of the S matrix. This information is contained in the second cluster property.

In considering the second cluster property, we are thinking of, for example, a three-particle process in which, first, two particles scatter off each other and then one of these scatters off the third particle. The probability for this process is, of course, less than that in which two particles scatter, with the third particle passing by without interaction. Suppose, for example, that there are two extremely massive particles fixed in space a distance λ apart. Quantum mechanically, of course, this situation can persist only for a finite time. Suppose furthermore that a wave packet for a lighter particle is directed toward one of the stationary particles. There is a certain probability that a scattered wave will be produced. The amplitude of the scattered wave will fall off as the inverse of the distance from the scattering center, in order that probability be conserved, so that the probability for an interaction involving the second fixed particle is proportional to λ^{-2}. Thus the probability for an interaction involving all three particles is λ^{-2} times smaller than the probability for just two of the particles interacting, as λ tends to infinity. But for large enough λ, the second fixed particle's wave function will spread out over an arbitrarily large volume by the time that the scattered particle arrives to be scattered a second time. Of course the wave packet for the fixed particle can be prepared so that it is concentrated at roughly the arrival time of the scattered wave. However, the scattered particle's wave packet will also have spread, so that the situation can no longer be as simply described as with classical particles. Nevertheless, we still expect a decrease in the amplitude as a function of the separation λ . There is, in fact a faster decrease than is called for by the naive geometrical considerations of the classical case.

Quantum mechanically, we could discuss this example in terms of concentrated, for example, Gaussian, wave packets.[4] Suppose there are defined six single-particle wave packets, three for the initial state and three for the final state of the three-particle system. Let packets ϕ_1 , ϕ_2 , and ϕ_3 all be characterized by a four-vector $(\underset{\sim}{x}_1,t_1)$ and packets ϕ_4,ϕ_5 , and ϕ_6 by $(\underset{\sim}{x}_2,t_2)$, with $t_2 > t_1$ and $(t_2 - t_1)^2 - (\underset{\sim}{x}_2 - \underset{\sim}{x}_1)^2 > 0$. First, packets 1 and 2 interact in the neighborhood of $\underset{\sim}{x}_1$ at about the time t_1 , while packet 4 passes by. Packet 4, however, can interact with a particle scattered from the first interaction which arrives in the neighborhood of $\underset{\sim}{x}_2$ at about the time t_2 . The result of this second scattering is the appearance of the packets 5 and 6, the other final state packet being packet 3, resulting directly from the first interaction. The S-matrix element is of the form

$$S(f;i) = \prod_{i=1}^{6} \int d^3\underset{\sim}{p}_i \phi_5^*(\underset{\sim}{p}_5)\phi_6^*(\underset{\sim}{p}_6)\phi_3^*(\underset{\sim}{p}_3)\phi_1(\underset{\sim}{p}_1)\phi_2(\underset{\sim}{p}_2)\phi_4(\underset{\sim}{p}_4)$$

$$\times S_{33}(\underset{\sim}{p}_5\underset{\sim}{p}_6\underset{\sim}{p}_3;\underset{\sim}{p}_1\underset{\sim}{p}_2\underset{\sim}{p}_4) \qquad \text{(I-2)}$$

where the action of the S operator is represented by the integral kernel, $S_{33}(\underset{\sim}{p}_5\underset{\sim}{p}_6\underset{\sim}{p}_3;\underset{\sim}{p}_1\underset{\sim}{p}_2\underset{\sim}{p}_4)$. This expression depends on the characteristic four-vectors $(\underset{\sim}{x}_1,t_1)$, $(\underset{\sim}{x}_2,t_2)$ only in the phases of the wave functions, as would be made explicit by specifying the wave packets in more detail. Explicitly, the dependence would appear as a factor

$$\exp\,[i(p_1\cdot x_1 + p_2\cdot x_1 + p_4\cdot x_2 - p_5\cdot x_2 - p_6\cdot x_2 - p_3\cdot x_1)]$$

but since energy and momentum are conserved, i.e., $p_1 + p_2 + p_4 = p_3 + p_5 + p_6$, only the difference $x_2 - x_1$ appears, e.g.,

$$\exp\,[-i(p_5 + p_6 - p_4)\cdot(x_2 - x_1)] \quad .$$

As $|x_2 - x_1|$ becomes larger and larger, the final state of the first pair of wave packets (the first cluster) tends more and more to a non-interacting state, independent of any "later" interaction with the third packet. Thus it is meaningful to describe the interaction of particles 1 and 2 by a two-particle S-matrix element. Similarly, because the so-called intermediate particle does tend to a free particle state as $|x_2 - x_1|$ tends to infinity, the initial state for the second scattering is a free particle state and it is meaningful to describe the second scattering as well by a two-particle S-matrix element. It should be emphasized that it is the short range of the interaction which permits the intermediate state to propagate freely. On these grounds we expect the S-matrix element, Eq. (I-2), to tend to factor in the form

$S(f;i) \rightarrow$

$$\sum_{\nu} \left[\int d^3\underset{\sim}{p}_5 d^3\underset{\sim}{p}_6 d^3\underset{\sim}{p}_4 d^3\underset{\sim}{k}_2 \phi_5^*(\underset{\sim}{p}_5)\phi_6^*(\underset{\sim}{p}_6) S_{22}(\underset{\sim}{p}_5\underset{\sim}{p}_6;\underset{\sim}{p}_4\underset{\sim}{k}_2)\phi_4(\underset{\sim}{p}_4)\psi_\nu(\underset{\sim}{k}_2)\right]$$

$$\times \left[\int d^3\underset{\sim}{p}_3 d^3\underset{\sim}{p}_2 d^3\underset{\sim}{p}_1 d^3\underset{\sim}{k}_1 \phi_3^*(\underset{\sim}{p}_3)\psi_\nu^*(\underset{\sim}{k}_1) S_{22}(\underset{\sim}{p}_3 k_1;\underset{\sim}{p}_1\underset{\sim}{p}_2)\phi_1(\underset{\sim}{p}_1)\phi_2(\underset{\sim}{p}_2)\right]$$

$$\text{as} \quad |x_2 - x_1| \rightarrow \infty \quad . \qquad \text{(I-3a)}$$

In Eq. (I-3a), the $S_{22}(\underset{\sim}{p}_5\underset{\sim}{p}_6;\underset{\sim}{p}_4\underset{\sim}{k}_2)$ represents the action of the S operator on two particles. We have introduced intermediate states $\psi_\nu(\underset{\sim}{k})$ and summed over all of them. They have a fictitious existence, of course, because they are never "measured." By the usual assumption that such a set is a complete set of one-particle states, mathematically expressed by

$$\sum_{\nu} \psi_\nu(\underset{\sim}{k}_1)\psi_\nu^*(\underset{\sim}{k}_2) = \delta_3(\underset{\sim}{k}_1 - \underset{\sim}{k}_2)$$

we obtain an equally transparent expression, without unobserved quantities:

$S(f;i)$

$$\int d^3\underset{\sim}{p}_5 d^3\underset{\sim}{p}_6 d^3\underset{\sim}{p}_4 d^3\underset{\sim}{k} \phi_5^*(\underset{\sim}{p}_5)\phi_6^*(\underset{\sim}{p}_6) S_{22}(\underset{\sim}{p}_5\underset{\sim}{p}_6;\underset{\sim}{p}_4\underset{\sim}{k})\phi_4(\underset{\sim}{p}_4)$$

$$\times \left[\int d^3\underset{\sim}{p}_3 d^3\underset{\sim}{p}_2 d^3\underset{\sim}{p}_1 \phi_3^*(\underset{\sim}{p}_3) S_{22}(\underset{\sim}{k}\underset{\sim}{p}_3;\underset{\sim}{p}_1\underset{\sim}{p}_2)\phi_1(\underset{\sim}{p}_1)\phi_2(\underset{\sim}{p}_2)\right]$$

$$\text{as} \quad |x_2 - x_1| \rightarrow \infty \quad . \qquad \text{(I-3b)}$$

The quantity in the square brackets is, in a sense, the non-interacting intermediate-particle wave packet. Since, by four-momentum conservation,

$k = p_5 + p_6 - p_4$, the dependence of $S(f;i)$ on $x_2 - x_1$ appears (implicitly) only in a phase factor, $\exp[-ik\cdot(x_2 - x_1)]$. This is the phase which would occur if the intermediate state were translated back to the point x_1, so that it would be spread out by the time it interacts with wave packet 4. Thus the rate at which $S(f;i)$ vanishes is determined by the rate of spreading of a single-particle wave packet. It is a well-known result in quantum mechanics that normalizable, free single-particle time-dependent coordinate-space wave functions tend to zero as $|t|^{-3/2}$ for large $|t|$.[5] In the same way, $S(f;i)$ tends to zero as fast as $|x_2 - x_1|^{-3/2}$ as $|x_2 - x_1|$ tends to infinity. (The mathematical details involved are reviewed in Appendix A.)

Let us discuss the quantum-mechanical case in more generality. Again consider two clusters, characterized by four-vectors $(\underset{\sim}{R}_1, \underset{\sim}{T}_1)$ and $(\underset{\sim}{R}_2, T_2)$ respectively, such that the latter is within the forward light cone of the former, or anyway that a significant neighborhood of the latter is within the forward light cone of a significant neighborhood of the former. We wish to allow for the possibility that one or more particles produced in the interactions of the first cluster can take part in the interactions of the second cluster. Again we may see that the S-matrix element involving, say n particles, propagating freely from the region of the first to the second interaction, depends on the separation $|R_2 - R_1|$ only through the phases of the n intermediate particles and decreases to zero as fast as $|R_2 - R_1|^{-3n/2}$ as $|R_2 - R_1|$ tends to infinity. It is as if a wave packet representing each intermediate particle is translated backwards in time so that by the time it arrives at the second cluster, it is spread out. We choose to call these processes in which n

particles leave the first cluster and interact with the second, n-particle transfers. It is seen from these arguments that the one-particle transfer processes dominate the transfer processes. In fact, the corrections to the asymptotic limit for one-particle transfers are larger than the two-particle transfer contributions for large enough time-like separations. This dominance of the one-particle transfers is of course true also in the classical case. That macroscopic causality is involved can be seen as follows: Suppose a particle produced in the interactions of cluster two proceeds to the region of cluster one. By the time it reaches this region its wave-packet amplitude is smaller by a factor of $\lambda^{-3/2}$. The amplitude for each particle in the first cluster will also be reduced by at least a factor $\lambda^{-3/2}$ at that time because there has been roughly a period of time λ since the characteristic time of the cluster. Thus the amplitude for an anti-casual transfer, one from the "later" cluster to the "earlier" one, is of order $\lambda^{-3/2}$, for large λ, compared with the amplitude for a causal transfer, one from an "earlier" cluster to a "later" one.

In view of the discussion of the previous paragraphs, it is possible to represent symbolically and somewhat imprecisely the consequences of the second cluster property on the structure of the S matrix, just as Eq. (I-1) represents the consequences of the first. Again let i_1, i_2 designate the initial non-interacting states of clusters one and two, respectively, and let f_1, f_2 designate the final states. Let $S(f;i\ \underset{\sim}{p})$ be the function of $\underset{\sim}{p}$ given by

$$S(f;\, i\, \underset{\sim}{p}) \equiv \int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}''_{n_f}\, d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}'_{n_i}\, \phi_f{}^*(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}''_{n_f})$$

$$\times \quad S_{n_f, n_i+1}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}''_{n_i}; \underset{\sim}{p}'_1, \cdots, \underset{\sim}{p}'_{n_i}, \underset{\sim}{p})\phi_i(\underset{\sim}{p}'_1 \cdots \underset{\sim}{p}'_{n_i})$$

where $S_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}''; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}')$ represents the action of the S matrix in describing the development of an n-particle initial state in an m-particle final state. Analogously we define $S(f\, \underset{\sim}{p}; i)$ by

$$S(f\, \underset{\sim}{p}; i) \equiv \int d^3\underset{\sim}{p}_1{}'' \cdots d^2\underset{\sim}{p}''_{n_f}\, d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}'_{n_i}\, \phi_f{}^*(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}''_{n_f})$$

$$\times \quad S_{n_{f+1}, n_i}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}''_{n_f}, \underset{\sim}{p}; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}'_{n_i})\phi_i(\underset{\sim}{p}_1{}' \cdots \underset{\sim}{p}'_{n_i}) \ .$$

We suppose the characteristic four-vectors x_1, x_2 are such that $x_2 - x_1$ is forward timelike, and $\lambda \equiv +[(x_2 - x_1)\cdot(x_2 - x_1)]^{1/2}$. Then, from what has been said above, we expect that

$$\lim_{\lambda\to\infty} \lambda^{3/2} e^{im\lambda}\{S(f_1 f_2; i_1 i_2) - S(f_1; i_1)S(f_2; i_2) - \int d^3\underset{\sim}{p}\, S(f_2; i_2\underset{\sim}{p})S(f_1\underset{\sim}{p}; i_1)\} = 0 \ . \quad \text{(I-4a)}$$

We furthermore expect that the macroscopic causality condition for this configuration of particles be expressed by the limit

$$\lim_{\lambda\to\infty} \lambda^{3/2} \int d^3\underset{\sim}{p}\; S(f_2\underset{\sim}{p}; i_2)S(f_1; i_1\underset{\sim}{p}) = 0 \ . \quad \text{(I-4b)}$$

We have neglected some trivial spreading effects in writing down Eqs. (I-4a) and (I-4b). These will be studied in more detail in

Section IV. They do not affect our conclusion about the importance of the one-particle transfer term.

From the results of the discussions about the first and second cluster properties, it is clear that "higher order cluster properties" could be formulated. For example, in the case of three clusters with characteristic four-vectors x_1, x_2, x_3 such that both $x_2 - x_1$ and $x_3 - x_2$ are forward time-like, a possible event is one in which a particle produced by the interactions in the first cluster interacts in the second cluster, and a particle produced from the second cluster interacts in the third. Then one could argue, as above, that the amplitudes for these processes are the largest of any in which there are interactions among all three clusters, and that these amplitudes are of order $(\lambda_1 \lambda_2)^{-3/2}$ for large $\lambda_1 = +[(x_2 - x_1)\cdot(x_2 - x_1)]^{1/2}$ and large $\lambda_2 = +[(x_3 - x_2)\cdot(x_3 - x_2)]^{1/2}$. We will not discuss these higher order cluster properties, however, because they are seen to follow from the first and second properties, and add nothing new.

The cluster properties which we have discussed above will be given a more precise, mathematical statement in the following sections. We have sought to emphasize the fundamental physical concepts from which our conclusions will be drawn. Of course, these conclusions, as requirements on the structure of the S matrix, have already been incorporated into many theories. Hence, we shall indicate some theories which fulfill these requirements and give examples of some which do not. Our main objectives are to draw attention to the underlying physical principles and to state precisely what their consequences are for the structure of the S matrix.

For the sake of completeness, we sketch briefly the treatment which these requirements on the S matrix have received in the past.

The cluster properties of vacuum expectation values of time-ordered products of field operators, or τ-functions, were investigated by Watanabe,[6] Symanzik,[7] Kristensen,[8] and Freese[9] (for possible application in the quantum theory of fields). With the introduction of asymptotic free-field operators in the Lehmann-Symanzik-Zimmermann formalism, the τ-functions could be related directly to plane-wave S-matrix elements.[10] Zimmermann proved that the τ-functions have a certain singularity structure, and, as will be shown later, this structure implies that the S matrix given by the τ-functions fulfills the requirements of our first and second cluster properties.[11]

Cluster properties have also been studied in the so-called axiomatic quantum field theory. The fundamental objects in the theory are vacuum expectation values of products of field operators, the Wightman functions. In this approach the asymptotic condition is not assumed, but derived from the axioms. This has been accomplished by assuming a spatial cluster property for Wightman functions in work by Haag, Coester, Araki, Hepp, and Ruelle.[12-15] The existence of an asymptotic condition allows for an S matrix and it has been shown by Hepp that such an S matrix ultimately obtained from the axioms would satisfy the requirements of the spatial version of the first cluster property.[16]

In the so-called S-matrix theory, which is based on certain analyticity properties of the S-matrix elements together with Lorentz invariance and unitarity, Stapp and others adopt the cluster decomposition of the S matrix as a postulate.[17] Olive has derived from a set of S-matrix postulates, including this "connectedness structure," a singularity structure in multiparticle matrix elements similar to the requirements of the second cluster property.[18]

We conclude this section with a survey of the contents of our study. In Section II we develop a mathematical framework suitable for a more precise statement of the cluster properties. Particular attention will be paid to plane-wave S-matrix elements, treated as tempered distributions. In Section III, the first cluster property is stated in these terms. The possibilities for a diagrammatic representation are discussed. The relationship between the first cluster property and unitarity and Lorentz invariance is investigated. Then, as examples, we discuss the Feynman perturbation theory, Zimmermann's work on the τ-functions, and an S matrix given by a Hermitian phase matrix. A counter-example is given which violates the first cluster property. Much the same treatment is given the second cluster property in Section IV. Section V is reserved for some concluding remarks.

II. MATHEMATICAL FORMULATION

In this section we present the mathematical framework on which our study is based.

A complete, realistic theory of elementary-particle interactions would include many different types of particles, classified according to mass, spin, statistics, and internal quantum numbers. The cluster properties must hold, we claim, in any realistic theory, no matter how rich the variety of particles. They do not depend on the internal structure of the particles. This being the case, the inclusion in the discussion of an arbitrary number of different types of particles would complicate the discussion without leading to any further insight into the cluster properties. Hence, for simplicity, we shall limit our discussion to the case of theories with only one kind of particle, namely, a neutral, scalar boson with a finite mass, m. The problem of extending the results to theories with many types of particles is essentially a problem of developing a suitable notation, and we shall not discuss this problem here.

A. General Properties of the S Matrix

We review briefly the general properties of S . The details have been discussed extensively elsewhere.[19] The S operator, being defined by matrix elements between normalizable multiparticle non-interacting states, thus has an action on the Hilbert space $\mathcal{H}$ appropriate to the description of any number of identical non-interacting particles, in particular, S maps $\mathcal{H}$ onto itself. The n-particle subspace $\mathcal{H}_n$, of $\mathcal{H}$, is thus spanned by vectors of the form

$$\int d^3p_1 \cdots d^3p_n \psi(p_1 \cdots p_n) a^\dagger(p_1) \cdots a^\dagger(p_n)|vac\rangle \quad ,$$

where the functions $\psi(p_1 \cdots p_n)$ are square-integrable symmetric "momentum-space wave functions," and the $a^\dagger(p_i)$ are plane-wave boson creation operators. The familiar algebra involving the $a^\dagger(p)$ and their hermitian adjoints, the annihilation operators $a(p)$, is defined as follows

$$[a(p),a(p')] = [a^\dagger(p),a^\dagger(p')] = 0 \; , \; [a(p),a^\dagger(p')] = \delta_3(p - p') \quad . \tag{II-1}$$

We assume the existence of a unique vacuum state, $|vac\rangle$, with the properties

$$a(p)|vac\rangle = 0 \quad , \qquad \langle vac|vac\rangle = 1 \quad . \tag{II-2}$$

We require S to be a unitary operator in accordance with the customary probability interpretation in quantum mechanics.

The Lorentz invariance of the description of a scattering process by an S-matrix element imposes the requirement that the S operator commute with the unitary operator $U(M,z)$ representing an inhomogeneous proper Lorentz transformation. Here M is a four-by-four matrix representing a homogeneous Lorentz transformation and z is a four-vector representing a translation. On a position variable x in four-space such a transformation gives $x' = Mx + z$, and on a four-momentum variable, p , $p' = Mp$. The action of $U(M,z)$ on the one-particle subspace, which thus provides an irreducible representation for the Lorentz group, is

$$U(M,z)a^{\dagger}(\underset{\sim}{p})U^{-1}(M,z) = [\omega(\underset{\sim}{p}')/\omega(\underset{\sim}{p})]^{1/2}e^{iz\cdot p'}a^{\dagger}(\underset{\sim}{p}') \qquad \text{(II-3)}$$

where $p' = \left(\underset{\sim}{p}', \omega(\underset{\sim}{p}')\right)$, and $\omega(\underset{\sim}{p}) = +(\underset{\sim}{p}^2 + m^2)^{1/2}$. The metric here is such that if z is the four-vector $(\underset{\sim}{z},t)$, then $z\cdot p' = \omega(\underset{\sim}{p}')t - \underset{\sim}{z}\cdot\underset{\sim}{p}'$. The Lorentz group also acts irreducibly on the no-particle subspace but the corresponding representation is the trivial one:

$$U(M,z)|vac\rangle = |vac\rangle \quad . \qquad \text{(II-4)}$$

That the S operator commute with the unitary operators $U(M,z)$ implies that S act like a constant of modulus unity on the vacuum state vector, and on the one-particle subspace. Without loss of generality, this constant can be chosen to equal one. In summary, S maps $\mathcal{H}$ onto itself and satisfies the following requirements:

$$SS^{\dagger} = S^{\dagger}S = I \quad . \qquad \text{(II-5a)}$$

$$SU(M,z) = U(M,z)S \quad . \qquad \text{(II-5b)}$$

$$S|vac\rangle = |vac\rangle \quad . \qquad \text{(II-5c)}$$

$$Sa^{\dagger}(\underset{\sim}{p})|vac\rangle = a^{\dagger}(\underset{\sim}{p})|vac\rangle \quad . \qquad \text{(II-5d)}$$

B. Plane-Wave S-Matrix Elements as Tempered Distributions

The quantities of physical interest in scattering theory are the S-matrix elements between initial states

$$|\psi_i\rangle = \int d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_{n_i}' \psi_i(\underset{\sim}{p}_1' \cdots \underset{\sim}{p}_{n_i}')a^{\dagger}(\underset{\sim}{p}_1') \cdots a^{\dagger}(\underset{\sim}{p}_{n_i}')|vac\rangle \quad ,$$

$$\langle\psi_i|\psi_i\rangle < \infty \quad ,$$

and final states

$$|\psi_f\rangle = \int d^3p_1'' \cdots d^3p_{n_f}'' \psi_f(p_1'' \cdots p_{n_f}'') a^\dagger(p_1'') \cdots a^\dagger(p_{n_f}'') |vac\rangle \quad ,$$

$$\langle\psi_f|\psi_f\rangle < \infty \quad ,$$

which are therefore of the form

$$\langle\psi_f|S|\psi_i\rangle = \int d^3p_1'' \cdots d^3p_{n_f}'' d^3p_1' \cdots d^3p_{n_i}' \psi_f^*(p_1'' \cdots p_{n_f}'')$$
$$S_{n_f n_i}(p_1'', \cdots, p_{n_f}''; p_1', \cdots, p_{n_i}') \psi_i(p_1' \cdots p_{n_i}') \quad .$$

The kernels $S_{n_f n_i}$ are the appropriate plane-wave S-matrix elements, which are defined by

$$S_{mn}(p_1'', \cdots, p_m''; p_1' \cdots p_n') \equiv \langle vac|a(p_1'') \cdots a(p_m'') \cdots$$
$$\times S a^\dagger(p_1') \cdots a^\dagger(p_n')|vac\rangle \quad . \qquad \text{(II-6)}$$

Thus the quantities of physical interest are bilinear functionals, on the space of square-integrable "momentum-space wave functions," defined by the plane-wave S-matrix elements. The mathematical statement of the cluster properties can thus be made as statements about the S_{mn}, <u>as objects which define functionals on the space of square-integrable functions</u>. Weaker statements could be made by restricting the space of functions on which the S_{mn} define a functional. However, the subspace must be dense in the larger space to insure that the physical quantities, the S-matrix elements, can be represented by the functionals defined on the subspace.

For our purposes we choose a particular subspace dense in the space of square-integrable functions, namely the space of testing functions with respect to which tempered distributions are defined.[20] These functions may be characterized as infinitely differentiable and rapidly decreasing. We make this choice because tempered distributions have received much attention in elementary particle theories in recent times; for example, in axiomatic quantum field theory.[21] Of course other choices could be made, but, for definiteness, we restrict the mathematical statement of the cluster properties to be statements about the S_{mn} as tempered distributions.

The appropriate space of testing functions is defined as follows: Let $\mathcal{S}(P^{3n})$ be the set of all complex-valued functions $\phi(\underset{\sim}{p}_1 \cdots \underset{\sim}{p}_n)$ of the n three-momentum variables $\underset{\sim}{p}_1, \cdots, \underset{\sim}{p}_n$ such that

(a) ϕ is infinitely differentiable.

(b) If $P = (\underset{\sim}{p}_1{}^2 + \underset{\sim}{p}_2{}^2 + \cdots + \underset{\sim}{p}_n{}^2)^{1/2}$, then

$$\lim_{P\to\infty} P^k \frac{\partial^m \phi}{\partial p_{1x}^{\alpha_1} \cdots \partial p_{nz}^{\alpha_{3n}}} = 0$$

for any k,m and any choice of the indices α_i such that $\sum_{i=1}^{3n} \alpha_i = m$. Furthermore, let $\mathcal{S}(P^{3n})^*$ be the subset of all functions in $\mathcal{S}(P^{3n})$ which satisfy the additional conditions that

(c) $\phi(\underset{\sim}{p}_1 \cdots \underset{\sim}{p}_n)$ is a symmetric function of the momentum variables $\underset{\sim}{p}_1, \cdots \underset{\sim}{p}_n$.

(d) The function ϕ is normalized to unity in the sense that

$$\int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n |\phi(\underset{\sim}{p}_1 \cdots \underset{\sim}{p}_n)|^2 = 1 \quad .$$

With each function ϕ in $\mathcal{S}(P^{3n})^*$ let us associate an operator

$$A^\dagger\{\phi\} \equiv (n!)^{-1/2}\int d^3p_1 \cdots d^3p_n \phi(p_1 \cdots p_n) a^\dagger(p_1) \cdots a^\dagger(p_n) \qquad \text{(II-7)}$$

and a vector $A^\dagger\{\phi\}|vac\rangle$. We may designate $A^\dagger\{\phi\}$ as the creation operator for the ϕ cluster, keeping in mind the discussion of Section I. This designation is meaningful, of course, not only for the testing functions ϕ, but for all normalizable momentum-space wave functions. This vector, $A^\dagger\{\phi\}|vac\rangle$, is a unit vector in $\mathcal{H}_n$. Furthermore, vectors of the form $cA^\dagger\{\phi\}|vac\rangle$, where c is any constant and ϕ is any function in $\mathcal{S}(P^{3n})^*$, are dense in $\mathcal{H}_n$. Thus, all physical S-matrix elements, say between an m-particle final state and an n-particle initial state, can be represented arbitrarily closely by matrix elements of the form $\langle vac|A\{\phi_m\}S\, A^\dagger\{\phi_n\}|vac\rangle$, where ϕ_m, ϕ_n belong to $\mathcal{S}(P^{3m})^*$, $\mathcal{S}(P^{3n})^*$, respectively. (We define $A\{\phi\}$ to be the Hermitian adjoint of $A^\dagger\{\phi\}$.)

The general properties of the S operator, Eqs. (II-5), are, in terms of the tempered distributions S_{mn},

$$\sum_{k=0} (k!)^{-1}\int d^3p_1 \cdots d^3p_k S_{mk}(p_1{}'', \cdots, p_m{}''; p_1, \cdots, p_k)$$

$$\times\; S_{nk}{}^*(p_1{}', \cdots, p_n{}'; p_1, \cdots, p_k)$$

$$= \delta_{mn} \sum_{\begin{pmatrix} 1 & \cdots & m \\ \alpha_1 & \cdots & \alpha_m \end{pmatrix}} \delta_3(p_1{}'' - p'_{\alpha_1}) \cdots \delta_3(p_m{}'' - p'_{\alpha_m}) \;, \qquad \text{(II-8a)}$$

where $\sum_{\begin{pmatrix}1 & \cdots & m\\ \alpha_1 & \cdots & \alpha_m\end{pmatrix}}$ denotes the sum over all permutations of the indices $1, \cdots, m$;

$$\prod_{i=1}^{m} [\omega(p_i")]^{1/2} \prod_{j=1}^{n} [\omega(p_j')]^{1/2} S_{mn}(p_1", \cdots, p_m"; p_1', \cdots, p_n')$$

$$= \prod_{i=1}^{m} [\omega(\hat{p}_i")]^{1/2} \prod_{j=1}^{n} [\omega(\hat{p}_j')]^{1/2} S_{mn}(\hat{p}_1", \cdots, \hat{p}_m"; \hat{p}_1', \cdots, \hat{p}_n') ,$$

(II-8b)

where $(\hat{p}, \omega(\hat{p})) = M(p, \omega(p))$, M defining a homogeneous Lorentz transformation,

$$S_{mn}(p_1", \cdots, p_m"; p_1', \cdots, p_n') = \exp\left[iz\cdot\left(\sum_{j=1}^{n} p_j' - \sum_{i=1}^{m} p_i"\right)\right]$$

$$\times S_{mn}(p_1", \cdots, p_m"; p_1', \cdots, p_n') ,$$

(II-8c)

z defining a space-time translation;

$$S_{00} = 1 , \quad S_{0n} = S_{n0} = 0 \quad \text{for} \quad n > 0 \quad ;$$

$$S_{11}(p_1"; p_1') = \delta_3(p_1" - p_1') , \quad S_{1n} = S_{n1} = 0 \quad \text{for} \quad n > 1 \quad .$$

(II-8e)

Of course, by their definition, the $S_{mn}(p_1", \cdots, p_m"; p_1', \cdots, p_n')$ are symmetric in the primed variables and in the double-primed variables separately.

We consider further the "cluster" creation operators. It is evident that two such operators acting on the vacuum also define a physically acceptable state. Suppose ϕ_a is an m-particle testing function and ϕ_b is an n-particle testing function. Then we have

$$A^{\dagger}\{\phi_a\}A^{\dagger}\{\phi_b\}|vac\rangle = (m!n!)^{-1/2}[(m+n)!]^{-1} \sum_P \int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_{m+n}$$

$$\times\, \phi_a(\underset{\sim}{p}_{\alpha_1} \cdots \underset{\sim}{p}_{\alpha_n})\phi_b(\underset{\sim}{p}_{\alpha_{n+1}} \cdots \underset{\sim}{p}_{\alpha_{m+n}})a^{\dagger}(\underset{\sim}{p}_1) \cdots a^{\dagger}(\underset{\sim}{p}_{m+n})|vac\rangle \tag{II-9}$$

where $\sum_P$ is the sum over the permutations $\begin{pmatrix} 1 & \cdots & m+n \\ \alpha_1 & \cdots & \alpha_{m+n} \end{pmatrix}$ of the $m + n$ momentum variables. Except for normalization, $\sum_P \phi_a(\underset{\sim}{p}_{\alpha_1} \cdots \underset{\sim}{p}_{\alpha_n})$ $\times\phi_b(\underset{\sim}{p}_{\alpha_{n+1}} \cdots \underset{\sim}{p}_{\alpha_{m+n}})$ belongs to $\mathcal{S}(P^{3(m+n)})^*$.

To give meaning to the notion of "far away" in space and time, we must study the effects of space-time translations on the free-particle states, including the subspace spanned by the ϕ functions. Using Eq. (II-3), we obtain

$$U(I,z)A^{\dagger}\{\phi\}U^{-1}(I,z) = A^{\dagger}\{\phi'\} \quad ,$$

$$\phi'(\underset{\sim}{p}_1 \cdots \underset{\sim}{p}_n) = \exp\,(iz\cdot \sum_{i=1}^{n} \underset{\sim}{p}_i)\phi(\underset{\sim}{p}_1 \cdots \underset{\sim}{p}_n) \quad . \tag{II-10b}$$

If one identifies the ϕ with a momentum-space wave function, the physical interpretation of these equations is obvious. The discussion in Section I is repeated here for emphasis. Associated with the free-particle state described by ϕ is a region in coordinate space and in time such that, far enough away from this region, the probability for finding any particle in a given finite volume is arbitrarily small. This region may be loosely identified as some neighborhood of a four-vector R, which is therefore a functional of $\emptyset$. The space-time translation of amount z on the state $A^{\dagger}\{\emptyset\}|vac\rangle$ shifts this characteristic coordinate-space four-vector by amount z.

Suppose that the functions $\emptyset_a$, $\emptyset_b$ have characteristic four-vectors R_a, R_b respectively. The state $A^{\dagger}\{\emptyset_a\}A^{\dagger}\{\emptyset_b\}|vac\rangle$ is thus a state with two clusters, $\emptyset_a$ and $\emptyset_b$, "separated," in time and in coordinate space, by the four-vector $R_a - R_b$. Now let $A^{\dagger}\{\emptyset_b\}$ be translated by amount z. The new state is

$$A^{\dagger}\{\emptyset_a\}A^{\dagger}\{\emptyset_b'\}|vac\rangle = A^{\dagger}\{\emptyset_a\}U(I,z)A^{\dagger}\{\emptyset_b\}|vac\rangle .$$

The two clusters are now "separated" in time and in coordinate space by the four-vector $R_a - R_b - z$. The expectation is that, as z tends to infinity in either a spacelike or a timelike direction, regardless of the fixed vector $R_a - R_b$, there tends to be no overlap between the functions $\emptyset_a$ and $\emptyset_b$. Using the Riemann-Lebesque lemma, it can be shown that, indeed,

$$\lim_{|z|\to\infty} \|A^{\dagger}\{\emptyset_a\}U(I,z)A^{\dagger}\{\emptyset_b\}|vac\rangle\| = \|A^{\dagger}\{\emptyset_a\}|vac\rangle\| \cdot \|A^{\dagger}\{\emptyset_b\}|vac\rangle\|$$

where $\||\emptyset\rangle\| = \langle\emptyset|\emptyset\rangle^{1/2}$, and $|z| = +(|z|^2 + z_0{}^2)^{1/2}$ for fixed $\emptyset_a$ and $\emptyset_b$. Thus we can give a well-defined meaning to the notion of two clusters being "far away" from each other in space and time: the clusters can be given an arbitrarily large relative space-time displacement such that the overlap between them is arbitrarily small.

6. Functional Formulation

We can readily anticipate that the consequences of the cluster properties on the structure of the S matrix will be given by an infinite hierarchy of limiting equations involving the tempered distributions S_{mn}. Now, as is well known, the most convenient way of expressing a hierarchy of relations among symmetric functions of any number of variables is by means of a generating functional. Some details of such a functional formulation are given in Appendix B. With the view in mind of thus expressing the consequences of the cluster properties functionally, we introduce the scattering functional, a functional of two independent functions $\alpha(\underset{\sim}{p})$ and $\alpha^{\dagger}(\underset{\sim}{p})$:

$$F\{\alpha^{\dagger}(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} \equiv \exp(-\alpha\cdot\alpha^{\dagger})\langle vac|\exp(\alpha^{\dagger}\cdot a)S\exp(\alpha\cdot a^{\dagger})|vac\rangle, \tag{II-12}$$

with the abbreviations

$$\alpha\cdot\alpha^{\dagger} \equiv \int d^3\underset{\sim}{p}\,\alpha^{\dagger}(\underset{\sim}{p})\alpha(\underset{\sim}{p}) \tag{II.13a}$$

$$\alpha^{\dagger}\cdot a \equiv \int d^3\underset{\sim}{p}\,\alpha^{\dagger}(\underset{\sim}{p})a(\underset{\sim}{p}) \tag{II-13b}$$

$$\alpha\cdot a^{\dagger} \equiv \int d^3\underset{\sim}{p}\,\alpha(\underset{\sim}{p})a^{\dagger}(\underset{\sim}{p}) \quad . \tag{II-13c}$$

As mentioned in Appendix B, the tempered distributions S_{mn} can be recovered by the appropriate functional differentiation of $\exp(\alpha\cdot\alpha^{\dagger})F\{\alpha^{\dagger}(\underset{\sim}{p});\alpha(\underset{\sim}{p})\}$. Furthermore, it is shown there that $F\{\alpha^{\dagger};\alpha\}$ is the generating functional for the expansion coefficients of S if S is expanded in a sum of normal-ordered products of annihilation and creation operators. That is, if

$$F(\alpha^{\dagger};\alpha) \equiv \sum_{m,n=0}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n' \alpha^{\dagger}(\underset{\sim}{p}_1'') \cdots \alpha^{\dagger}(\underset{\sim}{p}_m'')$$

$$\times \alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_n') F_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad , \tag{II-14a}$$

then

$$S = \sum_{m,n=0}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n' a^{\dagger}(\underset{\sim}{p}_1'') \cdots a^{\dagger}(\underset{\sim}{p}_m'')$$

$$\times a(\underset{\sim}{p}_1') \cdots a(\underset{\sim}{p}_n') F_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad . \tag{II-14b}$$

Clearly, the F_{mn} are tempered distributions, because of their relationship to the S_{mn}. Let us introduce the linear mapping $\mathcal{N}$ which maps the function $\alpha(\underset{\sim}{p})$ into $a(\underset{\sim}{p})$ and $\alpha^{\dagger}(\underset{\sim}{p})$ into $a^{\dagger}(\underset{\sim}{p})$, as follows:

$$\mathcal{N}\left(1\right) = 1 \quad , \tag{II-15a}$$

$$\mathcal{N}\left(c_1 P_1 + c_2 P_2\right) = c_1 \mathcal{N}\left(P_1\right) + c_2 \mathcal{N}\left(P_2\right) \quad , \tag{II-15b}$$

$$\mathcal{N}\left(\left[\prod_{r=1}^{m} \alpha^{\dagger}(\underset{\sim}{q}_r)\right]\left[\prod_{s=1}^{n} \alpha(\underset{\sim}{p}_s)\right]\right) = \left[\prod_{r=1}^{m} a^{\dagger}(\underset{\sim}{q}_r)\right]\left[\prod_{s=1}^{n} a(\underset{\sim}{p}_s)\right] \quad , \tag{II-15c}$$

where c_1 qnd c_2 are any two complex numbers, and P_1 and P_2 are any two power-series functionals of α and $\alpha^{\dagger}$. Equations (II-14) give

$$S = \mathcal{N}\left(F(\alpha^{\dagger};\alpha)\right) \tag{II-16}$$

which is an identity following from the definitions of $\mathcal{N}$ and $F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\}$. In Sections III and IV, we will use the scattering functional $F\{\alpha^\dagger;\alpha\}$ to compactly summarize the hierarchy of relations among the plane-wave S-matrix elements.

The properties of S expressed by the Eqs. (II-5) have the following expression in terms of the scattering functional. The unitarity condition, according to Eq. (B-22), is

$$\sum_{n=0}^{\infty} (n!)^{-1} \int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n \frac{\delta^n F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\}}{\delta\alpha(\underset{\sim}{p}_1) \cdots \delta\alpha(\underset{\sim}{p}_n)} \left(\frac{\delta^n F\{\alpha^*(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})\}}{\delta\alpha^{\dagger *}(\underset{\sim}{p}_1) \cdots \delta\alpha^{\dagger *}(\underset{\sim}{p}_n)} \right)^* = 1 . \tag{II-17a}$$

Lorentz invariance is given by

$$F\{[\omega(\underset{\sim}{p})]^{1/2}\alpha^\dagger(\underset{\sim}{p});[\omega(\underset{\sim}{p})]^{1/2}\alpha(\underset{\sim}{p})\} = F\{[\omega(\underset{\sim}{p}')]^{1/2}\alpha^\dagger(\underset{\sim}{p}');[\omega(\underset{\sim}{p}')]^{1/2}\alpha(\underset{\sim}{p}')\} \tag{II-17b}$$

where $\left(\underset{\sim}{p}', \omega(\underset{\sim}{p}')\right) = M\left(\underset{\sim}{p}, \omega(\underset{\sim}{p})\right)$, M defining a proper homogeneous Lorentz transformation; and

$$F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} = F\{e^{-iz\cdot p}\alpha^\dagger(\underset{\sim}{p});e^{iz\cdot p}\alpha(\underset{\sim}{p})\} \tag{II-17c}$$

where z defines a space-time translation. Equations (II-5c) and (II-5d) imply that

$$F_{00} = 1 ; \quad F_{0n} = F_{n0} = 0 \quad \text{for } n > 0 . \tag{II-17d}$$

$$F_{1n} = F_{n1} = 0 \quad \text{for } n > 0 . \tag{II-17e}$$

Thus the scattering functional must have the form:

$$F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} = 1 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(m!n!)^{-1}\int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n'$$

$$\times\, \alpha^\dagger(\underset{\sim}{p}_1'') \cdots \alpha^\dagger(\underset{\sim}{p}_m'')\alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_n')$$

$$\times\, F_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \, . \qquad \text{(II-18)}$$

III. THE FIRST CLUSTER PROPERTY

With the aid of the formalism of the last Section, we can make a more precise statement of the cluster properties. The first cluster property requires a certain expansion of the S matrix in terms of "connected" parts which we call the cluster decomposition of the S matrix. This section is concerned with the mathematical statement of the first cluster property and how it gives the cluster decomposition a well-defined meaning.

A. The First Cluster Property in Terms of Plane-Wave S-Matrix Elements

To discuss the implications of the short range of interactions, we introduce the "initial" state $|\psi_i\rangle$

$$|\psi_i\rangle = A^{\dagger}\{\phi_{i1}\}U(I,z)A^{\dagger}\{\phi_{i2}\}|vac\rangle \qquad \text{(III-1)}$$

where the functions ϕ_{i1}, (ϕ_{i2}) describe n_1, (n_2)-particle (cluster) states. We have said in Section II that, for large $|z|$, the overlap integral of ϕ_{i1} with ϕ_{12} can be made arbitrarily small. Likewise, because of the short range of the interactions among the particles, the amplitude for an interaction between the particles of cluster one with the particles of cluster two can be made arbitrarily small. This is the first cluster property. Let us therefore introduce a "final" state $|\psi_f\rangle$

$$|\psi_f\rangle = A^{\dagger}\{\phi_{f1}\}U(I,z)A^{\dagger}\{\phi_{f2}\}|vac\rangle \qquad \text{(III-2)}$$

where the functions ϕ_{f1}, (ϕ_{f2}) describe m_1, (m_2)-particle cluster states. The overlap integral of ϕ_{f1} and ϕ_{f2} can also be made arbitrarily small for large enough $|z|$.

In light of the discussion in the introduction, we formulate the first cluster property as follows:

$$\lim_{|z|\to\infty} \langle \psi_f|S|\psi_i\rangle = \lim_{|z|\to\infty} \langle vac|A\{\phi_{f2}\}U^\dagger(I,z)A\{\phi_{f1}\}SA^\dagger\{\phi_{i1}\}U(I,z)A^\dagger\{\phi_{i2}\}|vac\rangle$$

$$= \langle vac|A\{\phi_{f1}\}SA^\dagger\{\phi_{i1}\}|vac\rangle \times \langle vac|A\{\phi_{f2}\}SA^\dagger\{\phi_{i2}\}|vac\rangle \quad . \tag{III-3}$$

Equation (III-3) is thus the precise statement of what was indicated by Eq. (I-1). The sense of convergence in Eq. (III-3) is roughly this, supposing that z is either space-like or time-like: Given functions ϕ_{i1}, ϕ_{i2}, ϕ_{f1}, and ϕ_{f2} and an $\varepsilon > 0$, there exists a four-vector Z depending on ε and on the particular ϕ functions, such that for $|z\cdot z| > |Z\cdot Z|$, the absolute value of the difference between the left and right-hand sides of Eq. (III-3) is less than ε.

Let us clarify the question about the normalization of the state vectors occuring in Eq. (III-3). For the S-matrix element to give the transition probability amplitude, both the initial and the final states must be normalized to unity. Thus, for Eq. (III-3) to correctly represent transition probability amplitudes, the left-hand side must be multiplied by

$$N(z) = \left(\| A^\dagger\{\phi_{f1}\}U(I,z)A^\dagger\{\phi_{f2}\}|vac\rangle \| \times \| A^\dagger\{\phi_{i1}\}U(I,z)A^\dagger\{\phi_{i2}\}|vac\rangle \| \right)^{-1/2} \tag{III-4}$$

and the right-hand side by

$$N = \left(\| A^{\dagger}\{\phi_{f1}\} | \mathrm{vac} \rangle \| \times \| A^{\dagger}\{\phi_{f2}\} | \mathrm{vac} \rangle \| \times \| A^{\dagger}\{\phi_{i1}\} | \mathrm{vac} \rangle \| \times \| A^{\dagger}\{\phi_{i2}\} | \mathrm{vac} \rangle \| \right)^{-1/2} . \tag{III-5}$$

Since $\lim_{|z|\to\infty} N(z) = N$, however, by Eq. (II-11) the correction terms tend to zero and the limit for the transition probability amplitude is expressed correctly by Eq. (III-3).

Equation (III-3) implies a limiting equation for the tempered distributions S_{mn}. Writing the equation out in detail, we have

$$\lim_{|z|\to\infty} \int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}''_{m_1} d^3\underset{\sim}{q}_1{}'' \cdots d^3\underset{\sim}{q}''_{m_2} d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}'_{n_1} d^3\underset{\sim}{q}_1{}' \cdots d^3\underset{\sim}{q}'_{n_2}$$

$$\times \phi^*_{f1}(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}''_{m_1}) \phi^*_{f_2}(\underset{\sim}{q}_1{}'' \cdots \underset{\sim}{q}''_{m_2}) \phi_{i1}(\underset{\sim}{p}_1{}' \cdots \underset{\sim}{p}'_{n_1}) \phi_{i2}(\underset{\sim}{q}_1{}' \cdots \underset{\sim}{q}'_{n_2})$$

$$\times \exp \left[iz \cdot \left(\sum_{i=1}^{n_2} \underset{\sim}{q}_i{}' - \sum_{j=1}^{m_2} q_j{}'' \right) \right]$$

$$\times S_{m_1+m_2, n_1+n_2}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}''_{m_1}, \underset{\sim}{q}_1{}'', \cdots, \underset{\sim}{q}''_{m_2}; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}'_{n_1}, \underset{\sim}{q}_1{}', \cdots, \underset{\sim}{q}'_{n_2})$$

$$= \int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}''_{m_1} d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}'_{n_1} \phi^*_{f1}(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}''_{m_1}) \phi_{i1}(\underset{\sim}{p}_1{}' \cdots \underset{\sim}{p}'_{n_1})$$

$$\times S_{m_1 n_1}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}''_{m_1}; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}'_{n_1})$$

$$\times \int d^3\underset{\sim}{q}_1{}'' \cdots d^3\underset{\sim}{q}''_{m_2} d^3\underset{\sim}{q}_1{}' \cdots d^3\underset{\sim}{q}'_{n_2} \phi^*_{f2}(\underset{\sim}{q}_1{}'' \cdots \underset{\sim}{q}'_{m_2}) \phi_{i2}(\underset{\sim}{q}_1{}' \cdots \underset{\sim}{q}'_{n_2})$$

$$\times S_{m_2 n_2}(\underset{\sim}{q}_1{}'', \cdots, \underset{\sim}{q}''_{m_2}; \underset{\sim}{q}_1{}', \cdots, \underset{\sim}{q}'_{n_1}) \quad . \tag{III-6}$$

Equation (III-6) implies the following limiting equation for the tempered distributions

$$\lim_{|z|\to\infty} \exp\left[iz\cdot\left(\sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j''\right)\right]$$

$$\times S_{m_1+m_2,n_1+n_2}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}', \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}')$$

$$= S_{m_1n_1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')S_{m_2n_2}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}') . \tag{III-7}$$

Equation (III-7) is not immediately obvious, because the testing functions on the left-hand side of Eq. (III-6) are not arbitrary testing functions in the space appropriate to $S_{m_1+m_2,n_1+n_2}$, but products of testing functions of lower dimensionality. It is true, of course, that these product functions "span" the larger space; i.e., functions in the larger space can be arbitrarily well represented by finite sums of product functions.[22] In the following, we make these abbreviations

$$d\tau_{1+2} \equiv d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_{m_1}'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_{n_1}' d^3\underset{\sim}{q}_1'' \cdots d^3\underset{\sim}{q}_{m_2}'' d^3\underset{\sim}{q}_1' \cdots d^3\underset{\sim}{q}_{n_2}'$$

$$\Delta \equiv \sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j''$$

and we suppress the momentum dependence of the testing functions and the distributions S_{mn}. The problem then is to show that

$$\lim_{|z|\to\infty} \int d\tau_{1+2} \phi_f^* S_{m_1+m_2n_1+n_2} \phi_i e^{iz\cdot\Delta} = \int d\tau_{1+2} \phi_f^* S_{m_1n_1} S_{m_2n_2} \phi_i \tag{III-8}$$

where ϕ_f^* and ϕ_i are arbitrary testing functions in the spaces $\mathcal{S}(P^{3(m_1+m_2)})$ and $\mathcal{S}(P^{3(n_1+n_2)})$ respectively. First we represent these functions exactly by means of infinite sums of product functions

$$\phi_f^* = \sum_\mu \phi^*_{f1,\mu}\phi^*_{f2,\mu} \quad , \quad \phi_i = \sum_\nu \phi_{i1,\nu}\phi_{i2,\nu} \quad .$$

Then the left-hand side of Eq. (III-8), before the limit is taken, is the following infinite sum

$$\sum_\mu \sum_\nu \int d\tau_{1+2}\phi^*_{f1,\mu}\phi^*_{f2,\mu}S_{m_1+m_2 n_1+n_2}\phi_{i1,\nu}\phi_{i2,\nu}e^{iz\cdot\Delta} \quad .$$

Because S is bounded (in fact, unitary), and z appears in a phase factor, this sum converges uniformly in z . Hence the limit $|z| \to \infty$ and the summation over μ,ν can be interchanged. The limit of each term in the summand is given by Eq. (III-6). Thus

$$\lim_{|z|\to\infty} \int d\tau_{1+2}\phi_f^* S_{m_1+m_2,n_1+n_2}\phi_i e^{iz\cdot\Delta}$$

$$= \sum_\mu \sum_\nu \int d\tau_{1+2}\phi^*_{f1,\mu}\phi^*_{f2,\mu}S_{m_1 n_1}S_{m_2 n_2}\phi_{i1,\nu}\phi_{i2,\nu} \quad .$$

But the right-hand side of the above equation is just the right-hand side of Eq. (III-8). Therefore Eq. (III-6) implies Eq. (III-7).

This limiting equation is the statement of the first cluster property in terms of the tempered distributions S_{mn} . Conversely, any theory which has plane-wave S-matrix elements satisfying Eq. (III-7) would necessarily satisfy the first cluster property.

B. Representation by the Functional Formulation

Equation (III-7) represents a denumerable infinity of equations, one for each set of numbers n_1, n_2, m_1, m_2. The totality of these equations can be summarized by means of an equation involving the scattering functional, introduced in Section II. Specifically, this is accomplished by multiplying Eq. (III-7), for a given set n_1, n_2, m_1, m_2 by $\alpha^\dagger(\underset{\sim}{p}_1'') \cdots \alpha^\dagger(\underset{\sim}{p}''_{m_1})\alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}'_{n_1})\beta^\dagger(\underset{\sim}{q}_1'') \cdots \beta^\dagger(\underset{\sim}{q}''_{m_2})\beta(\underset{\sim}{q}_1') \cdots \beta(\underset{\sim}{q}'_{n_2})$, where $\alpha^\dagger, \alpha, \beta^\dagger, \beta$ are any four independent functions, and integrating over all the variables. Further, one divides by $n_1! n_2! m_1! m_2!$ and sums n_1, n_2, m_1, m_2 over all non-negative integers. The result is

$$\lim_{|z|\to\infty} \langle vac| e^{(\alpha^\dagger + \beta^\dagger e^{-iz\cdot p})\cdot a} S\, e^{(\alpha + \beta e^{iz\cdot p})\cdot a^\dagger} |vac\rangle$$

$$= \langle vac| e^{\alpha^\dagger\cdot a} S\, e^{\alpha\cdot a^\dagger} |vac\rangle \langle vac| e^{\beta^\dagger\cdot a} S\, e^{\beta\cdot a^\dagger} |vac\rangle \quad . \qquad \text{(III-9)}$$

Equation (III-7), for any choice of n_1, n_2, m_1, m_2 can be recovered from Eq. (III-9) by the appropriate functional differentiation. Equation (III-9) implies for the scattering functional the limit:

$$\lim_{|z|\to\infty} F\{\alpha^\dagger(\underset{\sim}{p}) + \beta^\dagger(\underset{\sim}{p})e^{iz\cdot p}; \alpha(\underset{\sim}{p}) + \beta(\underset{\sim}{p})e^{-iz\cdot p}\} = F\{\alpha^\dagger(\underset{\sim}{p}); \alpha(\underset{\sim}{p})\} F\{\beta^\dagger(\underset{\sim}{p}); \beta(\underset{\sim}{p})\} \quad . \qquad \text{(III-10)}$$

In obtaining the last equation, it is necessary to use the result

$$\lim_{|z|\to\infty} \exp\left[\int d^3\underset{\sim}{p}(\alpha(\underset{\sim}{p})\beta^\dagger(\underset{\sim}{p})e^{-ip\cdot z} + \alpha^\dagger(\underset{\sim}{p})\beta(\underset{\sim}{p})e^{ip\cdot z})\right] = 1 \quad ,$$

a consequence of the Riemann-Lebesque lemma. Any scattering functional satisfying Eq. (III-10) necessarily gives plane-wave S-matrix elements which satisfy Eq. (III-7).

C. The Cluster Decomposition of the S Matrix

The form of Eq. (III-10) immediately suggests a different representation of the scattering functional, namely

$$F\{\alpha^\dagger;\alpha\} = \exp\left(A\{\alpha^\dagger;\alpha\}\right) \quad . \qquad \text{(III-11)}$$

Of course, Eq. (III-11) can be used to define a functional A without any reference to the cluster properties. In light of Eq. (III-10), however, such a definition is particularly advantageous. In terms of the functional A, the first cluster property is the following:

$$\lim_{|z|\to\infty} A\{\alpha^\dagger(\underset{\sim}{p}) + \beta^\dagger(\underset{\sim}{p})e^{-iz\cdot p};\alpha(\underset{\sim}{p}) + \beta(\underset{\sim}{p})e^{iz\cdot p}\}$$

$$= A\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} + A\{\beta^\dagger(\underset{\sim}{p});\beta(\underset{\sim}{p})\} \quad . \qquad \text{(III-12)}$$

According to Eq. (II-18), A has the form

$$A\{\alpha^\dagger;\alpha\} = \sum_{m,n\geq 2}^{\infty} (m!n!)^{-1}\int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m''d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n'\alpha^\dagger(\underset{\sim}{p}_1'') \cdots \alpha^\dagger(\underset{\sim}{p}_m'')$$

$$\times\, \alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_n')A_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m'';\underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad . \qquad \text{(III-13)}$$

The A_{mn} are tempered distributions because of the relation to the F_{mn} through Eq. (III-11), and hence to the S_{mn}. Like the S_{mn}, the $A_{mn}(\underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_m'';\underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')$ are symmetric in the primed and double-primed variables separately.

The generating functional for the S_{mn} is

$$\langle vac|\, e^{\alpha^\dagger\cdot a} S\, e^{\alpha\cdot a^\dagger} |vac\rangle = e^{\alpha\cdot\alpha^\dagger} \exp\left(A\{\alpha^\dagger;\alpha\}\right) \qquad \text{(III-14)}$$

(Cont'd)

$$= \exp\left[\sum_{m,n\geq 1}^{\infty} (m!n!)^{-1}\int d^3p_1'' \cdots d^3p_m'' d^3p_1' \cdots d^3p_n' \right.$$

$$\left. \alpha^{\dagger}(p_1'') \cdots \alpha^{\dagger}(p_m'')\alpha(p_1') \cdots \alpha(p_n')A_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')\right] ,$$

(III-14)

where now we define $A_{11}(p'';p')$ to be $\delta_3(p'' - p')$ and $A_{1m}, A_{m1} = 0$ for $m \geqslant 2$. (We retain, however, the definition of A, Eq. (III-13), which does not include A_{11}.) Hence

$$S_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = A_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$$

$$+ \sum_{\text{part}''} \sum_{\text{part}'} A_{m_i n_k}(p_{i_1}', p_{i_2}', \cdots; p_{k_1}', p_{k_2}', \cdots) A_{m_j n_\ell}(p_{j_1}'', p_{j_2}'', \cdots; p_{\ell_1}', p_{\ell_2}', \cdots) \cdots$$

(III-15)

where $\sum_{\text{part}''}$ represents the sum of partitions of the indices of the double-primed variables into distince classes $i_1, i_2 \cdots; j_1, j_2 \cdots; \cdots$ and $\sum_{\text{part}'}$ represents the sum of partitions of the indices of the single-primed variables into distinct classes $k_1, k_2 \cdots; \ell_1, \ell_2 \cdots; \cdots$. The positive integers $m_i, m_j, \cdots$ sum to m and the positive integers $n_k, n_\ell, \cdots$ sum to n. Some simple examples of the relationship between the S_{mn} and the A_{mn} are given below for illustration.

$$S_{11}(p'';p') = \delta_3(p'' - p') \quad ; \qquad \text{(III-16a)}$$

$$S_{22}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}') = \sum_{\begin{pmatrix}1&2\\i&j\end{pmatrix}} \delta_3(\underset{\sim}{p}_1{}'' - \underset{\sim}{p}_i{}')\delta_3(\underset{\sim}{p}_2{}'' - \underset{\sim}{p}_j{}')$$

$$+ A_{22}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}') \quad ; \qquad \text{(III-16b)}$$

$$S_{33}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'',\underset{\sim}{p}_3{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}',\underset{\sim}{p}_3{}') = \sum_{\begin{pmatrix}1&2&3\\i&j&k\end{pmatrix}} \delta_3(\underset{\sim}{p}_1{}'' - \underset{\sim}{p}_i{}')\delta_3(\underset{\sim}{p}_2{}'' - \underset{\sim}{p}_j{}')\delta_3(\underset{\sim}{p}_3{}'' - \underset{\sim}{p}_k{}')$$

$$+ \sum_{\substack{\text{cycl.}\\ ijk}} \sum_{\substack{\text{cycl.}\\ \ell mn}} \delta_3(\underset{\sim}{p}_i{}'' - \underset{\sim}{p}_\ell{}')A_{22}(\underset{\sim}{p}_j{}'',\underset{\sim}{p}_k{}'';\underset{\sim}{p}_m{}',\underset{\sim}{p}_n{}')$$

$$+ A_{33}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'',\underset{\sim}{p}_3{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}',\underset{\sim}{p}_3{}') \quad . \qquad \text{(III-16c)}$$

These expansions of the plane-wave S-matrix elements are just those given by Heisenberg.[23]

The statement of the first cluster property in terms of the A_{mn} , is, according to Eq. (III-12),

$$\lim_{|\underset{\sim}{z}|\to\infty} e^{i z\cdot\Delta} A_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}'';\underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}') = 0 \quad , \qquad \text{(III-17)}$$

$$\Delta = \sum_i{}' p_i{}' - \sum_j{}'' p_j{}'' \quad ,$$

where Σ' denotes a summation over any subset of the primed variables and Σ'' denotes a summation over any subset of the double-primed variables, such that Δ contains at least one variable, but not all of them. This equation implies that $A_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}'';\underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}')$ cannot contain any δ-functions, or derivatives of δ-functions, which imply conservation of a subset of four-momenta. If there were such δ-functions, the A_{mn} would obviously vanish identically. Of course, the A_{mn} must contain an overall four-momentum conserving δ-function, just as the S_{mn} must, according to Eq. (II-8c). Thus the expansion of the S_{mn} into the A_{mn} is an expansion with respect to four-momentum conserving δ-functions.

It should be noticed that the expansion of the S_{mn} into the A_{mn} is a cluster expansion of the type mentioned in Appendix B, Eqs. (B-25) through (B-27), in two different sets of variables. Thus, we call the A_{mn} <u>cluster amplitudes</u> and the expansion of the S matrix defined by Eqs. (II-16), (III-11), and (III-13), namely,

$$S = \eta\left(\exp\left[\sum_{m,n\geq 2}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n' \alpha^+(\underset{\sim}{p}_1'') \cdots \alpha^+(\underset{\sim}{p}_m'') \right.\right.$$

$$\left.\left. \times\, \alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_n') A_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')\right]\right) \quad , \quad \text{(III-18)}$$

the <u>cluster decomposition of the S matrix</u>.

We have shown that the cluster decomposition is a consequence of the first cluster property. It is meaningful insofar as the plane-wave S-matrix elements satisfy Eq. (III-7), which in turn is true if the interactions described by the S matrix have a short range. Conversely, of course, the assumption that Eq. (III-18) is the correct expansion of the S matrix with respect to four-momentum conserving δ-functions implies Eq. (III-7), which allows the interpretation that the interactions are of short range. We prefer the first point of view, that the cluster decomposition of the S matrix derives from simple physicsl considerations.

(The second point of view has been taken by workers in the analytic S-matrix theory. They include the cluster decomposition as a postulate.)[17-18]

D. A Diagrammatic Representation of the Cluster Decomposition

We comment briefly on the possibility of a diagrammatic representation of the cluster decomposition of the S matrix. We wish to represent graphically Eq. (III-15), relating the S_{mn} to the A_{mn}. To this end we associate with each cluster amplitude A_{mn} a diagram like the one shown in Fig. 1. There are n lines which enter from below, which are labelled with the initial momentum variables, and m lines leaving from above, which are labelled with the final momentum variables. The circles represent the interactions which the A_{mn} describe. Since A_{11} is the three-momentum δ-function and thus describes no interactions, it can be represented by a straight line.

We represent the S_{mn} by a sum of diagrams, each term in the sum given by one or more diagrams of the type in Fig. 1, each corresponding to a cluster amplitude $A_{m'n'}$, with $1 \leq m' \leq m$, $1 \leq n' \leq n$.

For example, S_{54} has contributions from four terms, diagrammatically given in Fig. 2, (a) through (d). Fig. 2(a) represents A_{54}, (b) represents the terms of the form $A_{32}A_{22}$ which differ only in permutations of the momentum variables, (c) represents terms of the form $A_{43}A_{11}$, and (d), terms of the form $A_{32}A_{11}A_{11}$. It should be noted that numerical coefficients are suppressed in these figures. For a large space-time translation of one initial and one final-state particle, only diagrams (c) and (d) will contribute to the S-matrix element. Similarly, if two particles in the initial state and two particles in the final state are given a large space-time translation (together), only diagrams (b) and (d) will contribute to the S-matrix element. The diagram (a) does not contribute to either matrix element and hence must represent the amplitude for all four initial particles interacting among themselves to produce the desired final state. The diagrams (b) through (d) represent amplitudes in which some of the particles do not interact with some other particles.

These remarks are, of course, quite general. The cluster amplitude A_{mn} represents the amplitude for n initial particles all interacting among themselves and producing m final particles. The contributions to the S-matrix element arising from the case when some of the particles do not interact with others come from products of two or more cluster amplitudes. These considerations are, in fact, one of the justifications that Olive uses in introducing the cluster decomposition as a postulate in the analytic S-matrix theory.[18] They would lead to the following expansion of the S_{mn}:

$$S_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}''; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}') = A_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}''; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}')$$

$$+ \sum_{\text{part}''} \sum_{\text{part}'} c(m_i m_j \cdots; n_k n_\ell \cdots; \cdots)$$

$$\times A_{m_i n_k}(\underset{\sim}{p}''_{i_1}, \underset{\sim}{p}''_{i_2}, \cdots; \underset{\sim}{p}'_{k_1}, \underset{\sim}{p}'_{k_2}, \cdots) A_{m_j n_\ell}(\underset{\sim}{p}''_{j_1}, \cdots; \underset{\sim}{p}'_{\ell_1}, \underset{\sim}{p}'_{\ell_2}, \cdots) \cdots , \qquad \text{(III-15')}$$

with the same notation as above and an arbitrary numerical coefficient $c(m_i m_j \cdots; n_k n_\ell \cdots; \cdots)$. Thus these considerations by themselves do not lead to the cluster decomposition of the S matrix, whereas the first cluster property, with its result, Eq. (III-7), gives unambiguously the expansion, Eq. (III-15).

In a perturbation theory with a diagrammatic representation, for example, the Feynman perturbation theory, each A_{mn} is defined by an infinite set of connected graphs, each graph having m outgoing lines and n incoming lines and an arbitrarily complicated structure. In the Feynman perturbation theory, there is associated with each connected graph an overall four-momentum conserving δ-function, and no other four-momentum δ-functions. Thus the identification of the A_{mn} with the sum of connected graphs is unambiguous. It may be, however, that in terms of amplitudes so defined by a single four-momentum δ-function, the plane-wave S-matrix elements are not given by Eq. (III-15). It turns out that the Feynman perturbation theory gives plane-wave S-matrix elements and cluster amplitudes which satisfy Eq. (III-15), but it is a trivial matter to write down a "perturbation theory" which does not, as will be seen in part H below.

We can summarize this discussion by stating that a diagrammatic representation of the cluster decomposition of the S matrix is possible and that the cluster amplitude is given in perturbation theories by the sum of connected graphs with the appropriate number of incoming and outgoing lines.

E. Lorentz Invariance and the First Cluster Property

First, we wish to point out that we have not depended on the invariance of the S matrix with respect to proper homogeneous Lorentz transformations in our discussion of the cluster properties. The invariance under the homogeneous group is, in fact, irrelevant to the formulation of cluster properties, and our results would be equally valid in a non-relativistic theory. However, if we do deal with a relativistic theory we must naturally convince ourselves that the cluster properties which we assume are consistent with relativistic invariance.

It should be noted that we do depend on the translational invariance of the S matrix. This property is equivalent to the requirement that the plane-wave S-matrix elements contain δ-functions which imply conservation of overall four-momentum. This alone, of course, does not imply the cluster decomposition, or, equivalently, the first cluster property.

To make explicit the relativistic invariance of the cluster decomposition, we introduce the invariant cluster amplitude G_{mn}:

$$A_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') = \delta_4\left(\sum_{i=1}^{n} p_i' - \sum_{j=1}^{m} p_j"\right)$$

$$\left\{\prod_{i=1}^{n} [2\omega(\underset{\sim}{p}_i')]^{-1/2}\right\}\left\{\prod_{j=1}^{m} [2\omega(\underset{\sim}{p}_j")]^{-1/2}\right\} G_{mn}(p_1", \cdots, p_m"; p_1', \cdots, p_n') . \qquad \text{(III-19)}$$

In Eq. (III-19), the overall four-momentum-conserving δ-function is exhibited, as are certain "relativistic" factors. On the right-hand side, the $p_j{}''_4$ are given by $\omega(\underset{\sim}{p}_j{}'')$ and the $p_i{}'_4$ by $\omega(\underset{\sim}{p}_i{}')$. Thus the G_{mn} are defined on the same 3(m+n)-4 dimensional manifold that the A_{mn} (and S_{mn}) are. Let us denote this manifold by $\mathcal{M}_{mn}$; the Lorentz transformations map $\mathcal{M}_{mn}$ onto itself.

In light of the covariance properties of the S_{mn}, their relation to the A_{mn}, and the definition above, the G_{mn} satisfy

$$G_{mn}(Mp_1{}'', \cdots, Mp_m{}'';Mp_1{}', \cdots, Mp_n{}') = G_{mn}(p_1{}'', \cdots, p_m{}'';p_1{}', \cdots, p_n{}') , \quad \text{(III-20)}$$

where M is a four-by-four matrix representing the action of a proper homogeneous Lorentz transformation on a four-vector. Thus the G_{mn} are Lorentz scalars.

In terms of the distributions G_{mn}, the limit, Eq. (III-17), is expressed as

$$\lim_{|z|\to\infty} e^{iz\cdot\Delta} G_{mn}(p_1{}'', \cdots, p_m{}'';p_1{}', \cdots, p_n{}') = 0 \quad \text{(III-21)}$$

where, in $\Delta = \Sigma' p_i{}' - \Sigma'' p_j{}''$, Σ' denotes a summation over any subset of the primed variables, and Σ'' denotes a summation over any subset of the double-primed variables, such that Δ contain at least one variable, but not all of them. The condition expressed by Eq. (III-21) is obviously Lorentz invariant.

For completeness, we express the cluster decomposition of the S matrix in terms of the G_{mn}:

$$S = \mathcal{T}\left(\exp\left[\sum_{m,n\geq 2}^{\infty} (m!n!)^{-1} \int d^4p_1'' \cdots d^4p_m'' p_1' \cdots d^4p_n'\right.\right.$$

$$\left\{\prod_{i=1}^{m} [\delta^{(+)}(p_i'';m)(2\omega(\underset{\sim}{p}_i''))^{1/2} a^{\dagger}(\underset{\sim}{p}_i'')]\right\}$$

$$\left\{\prod_{j=1}^{n} [\delta^{(+)}(p_j';m)(2\omega(\underset{\sim}{p}_j'))^{1/2} a(\underset{\sim}{p}_j')]\right\}$$

$$\left.\left.\delta_4\left(\sum_{i=1}^{m} p_i'' - \sum_{j=1}^{n} p_j'\right) G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')\right]\right) , \quad \text{(III-22)}$$

where $\delta^{(+)}(p;m) = \delta(p\cdot p - m^2)$ if p is forward time-like but vanishes otherwise. In Eq. (III-22), the $\delta^{(+)}$ functions and four-momentum-conserving δ-function restrict the integration to the mainfold $\mathcal{M}_{mn}$.

F. Unitarity and the First Cluster Property

That unitarity and the first cluster property are distinct properties of the S matrix should be clear, because unitarity is concerned with the normalization of state vectors, which has nothing at all to do with the interactions. However, the first cluster property is a property of the interactions described by a unitary S matrix so that these properties must be compatible. This compatibility is expressed by the fact that the unitarity relations, Eq. (II-8a), when given solely in terms of the cluster amplitudes, contain only "connected" terms. That is, in a given unitarity relation expressed in terms of the A_{mn}, there are no non-vanishing terms which contain a factor of a δ-function corresponding to the conservation of a subset of initial and final four-momenta. In particular, the three-momentum δ-functions on the right-hand side of Eq. (II-8a) do not appear.

We give some examples of this last remark:

$$A_{22}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'';\underset{\sim}{p}_1',\underset{\sim}{p}_2') + A^*_{22}(\underset{\sim}{p}_1',\underset{\sim}{p}_2';\underset{\sim}{p}_1'',\underset{\sim}{p}_2'') + \frac{1}{2!}\int d^3\underset{\sim}{p}_1 d^3\underset{\sim}{p}_2 \times A_{22}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'';\underset{\sim}{p}_1,\underset{\sim}{p}_2)A^*_{22}(\underset{\sim}{p}_1',\underset{\sim}{p}_2';\underset{\sim}{p}_1,\underset{\sim}{p}_2) = 0 \qquad \text{(III-23a)}$$

$$\begin{aligned} & A_{33}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'',\underset{\sim}{p}_3'';\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{p}_3') + A^*_{33}(\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{p}_3';\underset{\sim}{p}_1'',\underset{\sim}{p}_2'',\underset{\sim}{p}_3'') \\ & + \frac{1}{4}\sum_{\begin{pmatrix}1&2&3\\ \alpha&\beta&\gamma\end{pmatrix}}\ \sum_{\begin{pmatrix}1&2&3\\ \pi&\rho&\sigma\end{pmatrix}} \int d^3\underset{\sim}{p} A_{22}(\underset{\sim}{p}_\alpha'',\underset{\sim}{p}_\beta'';\underset{\sim}{p}_\sigma',\underset{\sim}{p})A_{22}(\underset{\sim}{p}_\pi',\underset{\sim}{p}_\rho';\underset{\sim}{p}_\gamma'',\underset{\sim}{p}) \\ & + \frac{1}{4}\sum_{\begin{pmatrix}1&2&3\\ \pi&\rho&\sigma\end{pmatrix}} \int d^3\underset{\sim}{p}_1 d^3\underset{\sim}{p}_2 A_{33}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'',\underset{\sim}{p}_3'';\underset{\sim}{p}_1,\underset{\sim}{p}_2,\underset{\sim}{p}_\sigma')A^*_{22}(\underset{\sim}{p}_\pi',\underset{\sim}{p}_\rho';\underset{\sim}{p}_1,\underset{\sim}{p}_2) \\ & + \frac{1}{4}\sum_{\begin{pmatrix}1&2&3\\ \alpha&\beta&\gamma\end{pmatrix}} \int d^3\underset{\sim}{p}_1 d^3\underset{\sim}{p}_2 A_{22}(\underset{\sim}{p}_\alpha'',\underset{\sim}{p}_\beta'';\underset{\sim}{p}_1,\underset{\sim}{p}_2)A^*_{33}(\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{p}_3';\underset{\sim}{p}_1,\underset{\sim}{p}_2,\underset{\sim}{p}_\gamma'') \\ & + \frac{1}{3!}\int d^3\underset{\sim}{p}_1 d^3\underset{\sim}{p}_2 d^3\underset{\sim}{p}_3 A_{33}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'',\underset{\sim}{p}_3'';\underset{\sim}{p}_1,\underset{\sim}{p}_2,\underset{\sim}{p}_3)A^*_{33}(\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{p}_3';\underset{\sim}{p}_1,\underset{\sim}{p}_2,\underset{\sim}{p}_3) \\ & + \frac{1}{2}\int d^3\underset{\sim}{p}_1 d^3\underset{\sim}{p}_2 A_{32}(\underset{\sim}{p}_1'',\underset{\sim}{p}_2'',\underset{\sim}{p}_3'';\underset{\sim}{p}_1,\underset{\sim}{p}_2)A^*_{32}(\underset{\sim}{p}_1',\underset{\sim}{p}_2',\underset{\sim}{p}_3';\underset{\sim}{p}_1,\underset{\sim}{p}_2) = 0 \ . \end{aligned} \qquad \text{(III-23b)}$$

These relations hold when the total four-momentum in each is below the threshold for particle production.

The increase in complexity beyond the three-particle case is considerable. The unitarity relations may be expressed diagrammatically with more ease. Olive gives examples of such diagrams.[18] Integration over an "intermediate" momentum variable is represented by a straight line between the two circles (representing cluster amplitudes) sharing that variable. The complex conjugation of the cluster amplitude can be represented by a circle with, for example, a minus sign, with the cluster amplitude itself being given by a circle with a plus sign. External momenta are given by straight lines going to and from the circles, as before. (For completeness, there should be associated with the diagrammatical representation of the unitarity relations a prescription for determining the numerical coefficients.)

We now indicate how the unitarity relations can be expressed entirely in "connected" terms by using the functional formulation of Appendix B. The totality of unitarity relations involving both "connected" and "disconnected" parts is summarized by Eq. (II-16a):

$$F(\alpha^{\dagger}(\underset{\sim}{p});\alpha(\underset{\sim}{p}))F^{*}(\alpha^{*}(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})) + \left\langle F(\alpha^{\dagger}(\underset{\sim}{p});\alpha(\underset{\sim}{p})),F^{*}(\alpha^{*}(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p}))\right\rangle = 1 \quad , \qquad \text{(III-24)}$$

using the functional product bracket introduced in Appendix B. Now, it can be shown that for any two connected functionals A and B, and real variables s and t:

$$\{e^{sA},e^{tB}\} = e^{sA+tB}\left[\exp\left(\left\langle e^{sA},e^{tB}\right\rangle_c\right) - 1\right]$$

where $\{\ \}_c$ indicates the connected part of the functional product bracket, which has a well-defined meaning. This is just the generating function for the expansion of the functional product bracket $\{A^m,B^n\}$ into connected functionals. Hence, for $F = \exp A$, the left-hand side of Eq. (III-24) is identically:

$$\exp\left(A\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} + A^*\{\alpha^*(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})\} + \left\{F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\}, F^*\{\alpha^*(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})\}\right\}_c\right).$$

Therefore, Eq. (III-24) can be expressed entirely by means of an equation of its connected parts:

$$A\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} + A^*\{\alpha^*(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})\}$$

$$+ \sum_{m,n\geq 1} (m!n!)^{-1}\left\{A^m\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\}, A^{*n}\{\alpha^*(\underset{\sim}{p});\alpha^{\dagger *}(\underset{\sim}{p})\}\right\}_c = 0 . \qquad \text{(III-25)}$$

This functional expression summarizes the totality of "connected" unitarity relations. For example, the terms $A + A^* + \{A,A^*\}_c$ contain the expressions on the left-hand sides of the Eqs. (III-23).

Conversely, if one requires that the expansion of the scattering functional into connected parts be determined by the condition that the unitarity relations be expressed completely in terms of connected functionals alone, then one obtains $F = \exp A$. In this sense, one could say that the cluster decomposition follows from the connectedness of the unitarity relations. Of course, both follow from the physical condition expressed by Eq. (III-7), so that the last remark does not imply a logical dependence of the first cluster property on unitarity.

We will later give examples of operators which satisfy Lorentz invariance and unitarity but not the first cluster property, which further illustrates the obvious independence of these properties.

G. Feynman Perturbation Theory

We now give three examples of S operators which satisfy the requirements of the first cluster property, i.e., they admit the cluster decomposition. The first will be Feynman perturbation theory, by which we mean the power series expansion of operators of the form:

$$S = T\{\exp(i\int d^4x\, \mathcal{L}(x))\} \quad , \tag{III-28}$$

where T indicates the familiar time-ordered product:

$$T\left(\mathcal{L}(x_1)\cdots\mathcal{L}(x_n)\right) \equiv \sum_{\begin{pmatrix}1 & \cdots & n\\ \alpha_1 & \cdots & \alpha_n\end{pmatrix}} \theta(t_{\alpha_1}-t_{\alpha_2})\cdots\theta(t_{\alpha_{n-1}}-t_{\alpha_n}) \times\mathcal{L}(x_{\alpha_1})\cdots\mathcal{L}(x_{\alpha_n}) \quad , \tag{III-29}$$

and the operators $\mathcal{L}(x)$ satisfy casual commutation relations in order that the T product have a Lorentz-invariant meaning:

$$[\mathcal{L}(x),\mathcal{L}(x')] = 0 \qquad \text{if} \quad x - x' \text{ is spacelike} \quad . \tag{III-30}$$

We wish to show that whenever Eq. (III-28) defines an S matrix, this S matrix has the cluster decomposition.

If $\mathcal{L}(x)$ is constructed by taking products of free-field operators at the point x, then the expansion of Eq. (III-28) can be represented by a well-known diagrammatical method, the method of the Feynman-Dyson graphs. These graphs represent the expansion, as given by Wick's theorem, of the time-ordered operator, Eq. (III-28), into sums of normal-ordered products. In terms of these graphs, the cluster decomposition of the S matrix is obvious: each cluster amplitude A_{mn} is determined by summing the contributions from all connected graphs with n incoming and m outgoing lines, because a connected graph has a δ-function expressing conservation of overall four-momentum and none expressing conservation of a subset of four-momenta. It must be shown, however, that the combinatorial aspects of the cluster decomposition are satisfied, i.e., that Eq. (III-15) is satisfied.

We solve this problem not by considering the Feynman-Dyson graphs, but by showing that the scattering functional given by Eq. (III-28) is the exponential of a connected functional. In this way, those details of the structure of the graphs other than their connectedness can be ignored. We do assume that there is associated with each connected graph an overall four-momentum δ-function, so that it will be sufficient to show that the S operator of Eq. (III-28) satisfies the spatial version of the first cluster property.

In perturbation theory $\mathcal{L}(x)$ is assumed to be a sum of products of field operators, (or their derivatives), taken at the point x . (The products are normal-ordered in order to avoid a certain type of divergence.) Because of this form of $\mathcal{L}(x)$, we can infer that the functional

$$L\{\alpha^\dagger;\alpha\} \equiv e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}\int d^4x\,\mathcal{L}(x)e^{\alpha\cdot a^\dagger}|vac\rangle$$

be a connected functional in the sense that all the kernels $L_{mn}(\underset{\sim}{p}_1'', \cdots; \underset{\sim}{p}_1', \cdots)$ defining the functional contain only one four-momentum conserving δ-function. In particular, the kernels defining

$$L(\alpha^{\dagger};\alpha:t) \equiv e^{-\alpha\cdot\alpha^{\dagger}} \langle vac|e^{\alpha^{\dagger}\cdot a}\mathcal{L}(t)e^{\alpha\cdot a^{\dagger}}|vac\rangle \qquad \text{(III-31)}$$

with

$$\mathcal{L}(t) = \int d^3\underset{\sim}{x}\,\mathcal{L}(x) \qquad \text{(III-32)}$$

contain one and only one three momentum conserving δ-function. In the following discussion we thus assume, without further specifying the nature of $\mathcal{L}(x)$, that the functional $L\{\alpha^{\dagger};\alpha\}$ is a connected functional. We also assume, for simplicity, that including the details of a renormalization program will not affect our result.

We define

$$L(\alpha^{\dagger};\alpha:t_1 \cdots t_n) \equiv e^{-\alpha\cdot\alpha^{\dagger}}\langle vac|e^{\alpha^{\dagger}\cdot a}\mathcal{L}(t_1) \cdots \mathcal{L}(t_n)e^{\alpha\cdot a^{\dagger}}|vac\rangle \qquad \text{(III-33)}$$

so that

$$F\{\alpha^{\dagger};\alpha\} = \sum_{n=0}^{\infty} i^n(n!)^{-1}\int dt_1 \cdots dt_n T\left\{L(\alpha^{\dagger};\alpha:t_1 \cdots t_n)\right\} \quad . \qquad \text{(III-34)}$$

Now using the multiplication rule, Eq. (B-21), we note that

$$L(\alpha^{\dagger};\alpha:t_1t_2) = L(\alpha^{\dagger};\alpha:t_1)L(\alpha^{\dagger};\alpha:t_2) + \left\{L(\alpha^{\dagger};\alpha:t_1),L(\alpha^{\dagger};\alpha:t_2)\right\} \quad .$$

Since a functional product bracket of connected functionals is also connected, this is an expansion into connected functionals:

$$L(\alpha^{\dagger};\alpha:t_1t_2) = L_c(\alpha^{\dagger};\alpha:t_1)L_c(\alpha^{\dagger};\alpha:t_2) + L_c(\alpha^{\dagger};\alpha:t_1t_2) \quad .$$

Proceeding in this way, we obtain the hierarchy:

$$L(\alpha^\dagger;\alpha:t_1 \cdots t_n) = L_c(\alpha^\dagger;\alpha:t_1 \cdots t_n)$$

$$+ \sum_{\text{part.}} L_c(\alpha^\dagger;\alpha:t_{i_1} t_{i_2} \cdots) L_c(\alpha^\dagger \alpha:t_{j_1} t_{j_2} \cdots) \cdots \tag{III-35}$$

where the L_c are connected functionals in the sense that

$$\lim_{|x| \to \infty} L_c(\alpha^\dagger + \beta^\dagger e^{-ix\cdot p};\alpha + \beta e^{ix\cdot p}:t_1 \cdots t_n)$$

$$= L_c(\alpha^\dagger;\alpha;t_1 \cdots t_n) + L_c(\beta^\dagger;\beta:t_1 \cdots t_n) \quad . \tag{III-36}$$

That is, the kernels defining L_c contain one and only one three-momentum conserving δ-function. The sum over partitions $\sum_{\text{part.}}$ has been defined before; the order of the variables in each L_c on the right-hand side must be the same as that on the left-hand side. Then it is trivially true that

$$T\{L(\alpha^\dagger;\alpha:t_1 \cdots t_n)\} = T\{L_c(\alpha^\dagger;\alpha:t_1 \cdots t_n)\}$$

$$+ \sum_{\text{part.}} T\{L_c(\alpha^\dagger;\alpha:t_{i_1} t_{i_2} \cdots)\} T\{L_c(\alpha^\dagger;\alpha:t_{j1} t_{j2} \cdots)\} \cdots \tag{III-37}$$

The scattering functional given by Eq. (III-34) is therefore

$$F\{\alpha^\dagger;\alpha\} = \exp\left(A\{\alpha^\dagger;\alpha\}\right) \tag{III-38a}$$

$$A\{\alpha^\dagger;\alpha\} = \sum_{n=1}^{\infty} i^n (n!)^{-1} \int dt_1 \cdots dt_n T\{L_c(\alpha^\dagger;\alpha:t_1 \cdots t_n)\} \ , \tag{III-38b}$$

using the result, Eq. (B-27). Thus the scattering functional satisfies the spatial version of the first cluster property. In the diagrammatic representation, $A\{\alpha^\dagger;\alpha\}$ is given by the sum of all graphs which have one overall three-momentum conserving δ-function and no others. But these graphs can be unambiguously identified with the connected graphs, each of which having one four-momentum δ-function and no others. Thus the operator defined by Eqs. (III-28) through (III-30) satisfies the first cluster property.

It should be mentioned that the Feynman-Dyson graphs represent amplitudes $\mathcal{G}_{mn}(p_1'', \cdots, p_m''; p_1', \cdots p_n')$ defined on a $4(m+n) - 4$ dimensional manifold, which reduce to what we have called the invariant cluster amplitude $G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots p_n')$ when the four-momenta are on the mass shell, i.e., $p_i''_4 = \omega(\underset{\sim}{p}_i'')$, $p_j'_4 = \omega(\underset{\sim}{p}_j')$. The G_{mn} given by the Feynman perturbation theory thus have a well-defined continuation off the manifold $\mathcal{M}_{mn}$. One example of this continuation is crossing symmetry:

$$\mathcal{G}_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = \mathcal{G}_{m+1,n-1}(p_1'', \cdots, p_m'', - p_n'; p_1', \cdots, p_{n-1}')$$

$$= \mathcal{G}_{m-1,n+1}(p_1'', \cdots, p_{m-1}''; - p_m'', p_1', \cdots, p_n') \quad . \qquad \text{(III-39)}$$

This relationship is, in general, meaningless for the invariant cluster amplitudes we have defined unless there is given a prescription for continuing the G_{mn}'s from the manifold $\mathcal{M}_{mn}$ to the manifolds $\mathcal{M}_{m+1,n-1}$ and $\mathcal{M}_{m-1,n+1}$. When it is defined, crossing symmetry is related to the first cluster property in that it is the connected amplitudes which possess the symmetry.

H. The τ-Functions

The second example we give of a theory which satisfies the first cluster property is the so-called LSZ formalism.[10] It is shown in Appendix C that the scattering functional is given by

$$F\{\alpha^\dagger(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} = \sum_{n=0}^{\infty} (n!)^{-1}\int d^4x_1 \cdots d^4x_n \varphi(x_1) \cdots \varphi(x_n)K_{x_1} \cdots K_{x_n}\tau(x_1, \cdots, x_n) \tag{III-40}$$

where

$$\varphi(x) = (2\pi)^{-3/2}\int d^3\underset{\sim}{p}[2\omega(\underset{\sim}{p})]^{-1/2}\{\alpha(\underset{\sim}{p})e^{-ip\cdot x} + \alpha^\dagger(\underset{\sim}{p})e^{ip\cdot x}\} \quad . \tag{III-41}$$

From Zimmermann's work it will be shown that there is a connected functional A, which satisfies Eq. (III-11), such that $F = \exp A$.[11]

We first express the scattering functional in terms of the Fourier transforms of the τ-functions:

$$\tau(x_1, \cdots, x_n) \equiv (2\pi)^{-3n/2}\int d^4q_1 \cdots d^4q_n \exp\left[-i\left(\sum_{j=1}^{n} q_j\cdot x_j\right)\right]\tau(q_1, \cdots, q_n) \quad . \tag{III-42}$$

By suitable manipulations, employing the $\delta^{(+)}$ function defined as in Eq. (III-22), Eq. (III-41) becomes

$$\varphi(x) = (2\pi)^{-3/2}\int d^4p e^{-ip\cdot x}[\delta^{(+)}(p;m)(2\omega(\underset{\sim}{p}))^{1/2}\alpha(\underset{\sim}{p})$$

$$+ \delta^{(+)}(-p;m)(2\omega(\underset{\sim}{p}))^{1/2}\alpha^{\dagger}(-\underset{\sim}{p})] . \qquad \text{(III-43)}$$

Then from the last two equations comes the result

$$F\{\alpha^{\dagger}(\underset{\sim}{p});\alpha(\underset{\sim}{p})\} = \sum_{n=0} (n!)^{-1}\int d^4p_1 \cdots d^4p_n$$

$$\times \prod_{i=1}^{n} [(\delta^{(+)}(-p_i;m)\alpha(-\underset{\sim}{p}_i) + \delta^{(+)}(p_i;m)\alpha^{\dagger}(\underset{\sim}{p}_i))(2\omega(\underset{\sim}{p}_i))^{1/2}]$$

$$\times \left\{\left(\prod_{j=1}^{n} [2\pi i(-p_j^{\,2} + m^2)]\right)\tau(p_1, \cdots, p_n)\right\} . \qquad \text{(III-44)}$$

Zimmermann showed that the momentum-space τ-functions could be expanded in the following way.

$$\tau(p_1, \cdots, p_n) = \delta_4(p_1 + \cdots + p_n)\tilde{\eta}(p_1, \cdots, p_n)$$

$$+ \sum_{\text{part.}} \delta_4(p_{i_1} + p_{i_2} + \cdots)\tilde{\eta}(p_{i_1}, p_{i_2}, \cdots)\delta_4(p_{j_1} + p_{j_2} + \cdots)$$

$$\times \tilde{\eta}(p_{j_1}, p_{j_2}, \cdots) \cdots \qquad \text{(III-45)}$$

where the sum over partitions $\sum_{\text{part.}}$ has been defined before and where the $\tilde{\eta}$ functions do not have any more four-momentum δ-functions. This is just the cluster expansion mentioned in Appendix B. The corresponding property for the space-time τ-functions is

$$\lim_{|a|\to\infty} \tau(x_1, \cdots, x_m, y_1 + a, \cdots, y_n + a) = \tau(x_1, \cdots, x_m)\tau(y_1, \cdots, y_n) \;. \tag{III-46}$$

Equation (III-44) shows that F may be interpreted as the generating functional for

$$\left\{\left(\prod_{j=1}^{n} [2\pi i(-p_j^{\,2} + m^2)]\right) \tau(p_1, \cdots, p_n)\right\} \;,$$

defined on the "function"

$$(2\omega(\underset{\sim}{p}))^{1/2}(\delta^{(+)}(-p;m)\alpha(-\underset{\sim}{p}) + \delta^{(+)}(p;m)\alpha^\dagger(\underset{\sim}{p})) \;.$$

Then, from Eqs. (B-25) and (III-45), ln F is in the same way the generating functional for

$$\delta_4(p_1 + \cdots + p_n)\left\{\left(\prod_{j=1}^{n} [2\pi i(-p_j^{\,2} + m^2)]\right) \tilde{\eta}(p_1, \cdots, p_n)\right\}$$

defined on the same function. Hence

$$F\{\alpha^\dagger;\alpha\} = \exp\Bigg[\sum_{n=1}^{\infty} (n!)^{-1}\int d^4p_1 \cdots d^4p_n \delta_4(p_1 + \cdots + p_n)$$

$$\times \left\{\prod_{i=1}^{n} [((\delta^{(+)}(-p_i,m)\alpha(-\underset{\sim}{p}_i) + \delta^{(+)}(p_i;m)\alpha^\dagger(\underset{\sim}{p}_i))(2\omega(\underset{\sim}{p}_i))^{1/2}]\right\}$$

$$\times \left\{\left(\prod_{j=1}^{n} [2\pi i(-p_j^{\,2} + m^2)]\right) \tilde{\eta}(p_1, \cdots, p_n)\right\}\Bigg] \qquad \text{(III-47)}$$

Thus $F\{\alpha^\dagger;\alpha\} = \exp A\{\alpha^\dagger;\alpha\}$ with

$$A\{\alpha^\dagger;\alpha\} = \sum_{m,n=2}^{\infty} (m!n!)^{-1}\int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}_m{}'' d_3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}_n{}' \alpha^\dagger(\underset{\sim}{p}_1{}'') \cdots \alpha^\dagger(\underset{\sim}{p}_m{}'')$$

$$\times \alpha(\underset{\sim}{p}_1{}') \cdots \alpha(\underset{\sim}{p}_n{}') \left\{\prod_{i=1}^{m} [2\pi i(2\omega(\underset{\sim}{p}_i{}''))^{-1/2}(-p_i{}''^2 + m^2)]\right\}$$

$$\times \left\{\prod_{j=1}^{n} [2\pi i(2\omega(\underset{\sim}{p}_j{}'))^{-1/2}(-p_j{}^2 + m^2)]\right\}$$

$$\times \delta_4(p_1{}'' + \cdots + p_m{}'' - p_1{}' \cdots - p_n{}')\tilde{\eta}(p_1{}'', \cdots, p_m{}'', -p_1{}', \cdots, -p_n{}') \qquad \text{(III-48)}$$

Since the $\tilde{\eta}$ do not have any further four-momentum-conserving δ-functions, it is obvious that the A given by Eq. (III-48) satisfies Eq. (III-12).

The invariant cluster amplitude is thus given by

$$G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = (2\pi i)^{m+n} \left\{ \prod_{i=1}^{m} (-p_i''^2 + m^2) \right\}$$

$$\times \left\{ \prod_{j=1}^{n} (-p_j'^2 + m^2) \right\} \tilde{\eta}(p_1'', \cdots, p_m'', -p_1', \cdots, -p_n') \quad . \qquad \text{(III-49)}$$

It should be emphasized again here that G_{mn} is defined only on the manifold $\mathcal{M}_{mn}$, so that on the right-hand side of the above equation, $p_i'{}_4 = \omega(\underset{\sim}{p}_i')$ and $p_j''{}_4 = \omega(\underset{\sim}{p}_j'')$. (The function $\tilde{\eta}$ is singular at these points, but the singularities do not occur in the G_{mn} because of the factors $-p_i''^2 + m^2$ and $-p_j'^2 + m^2$.)

I. The Connected Phase Matrix

As a final example of an operator satisfying the requirements of the first cluster property we give

$$S = \exp(i\eta) \quad , \qquad \text{(III-50)}$$

with the phase operator η defined by

$$= \sum_{m,n=2}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n' a^\dagger(\underset{\sim}{p}_1'') \cdots a^\dagger(\underset{\sim}{p}_m'')$$

$$\times a(\underset{\sim}{p}_1') \cdots a(\underset{\sim}{p}_n') \left\{ \prod_{i=1}^{m} [2\omega(\underset{\sim}{p}_i')]^{-1/2} \right\} \left\{ \prod_{j=1}^{n} [2\omega(\underset{\sim}{p}_j')]^{-1/2} \right\}$$

$$\delta_4\left(\sum_{i=1}^{m} p_1'' - \sum_{j=1}^{n} p_j' \right) \tilde{h}_{m+n}(-p_1'', \cdots, -p_m'', p_1', \cdots, p_n') \quad ,$$

$$\text{(III-51)}$$

the distributions $\tilde{h}_n$ having no further four-momentum conserving δ-functions. Lehmann, Symanzik and Zimmermann introduced this representation to discuss causality properties of the S matrix in field theories.[24] They specified that the $\tilde{h}_n$ have no four-dimensional δ-functions in order that "all observable quantities like cross-sections and their generalization for many-particle processes would be finite." We shall show that this prescription gives exp(iη) the correct cluster decomposition structure.

There is no simple relationship between the cluster amplitudes and the $\tilde{h}_n$. We bypass this difficulty by using the functional formalism: as in the previous examples, we show that the scattering functional F is the exponential of a connected functional. To this end we define

$$E_n\{\alpha^\dagger;\alpha\} = e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}\eta^n e^{\alpha\cdot a^\dagger}|vac\rangle \quad . \qquad \text{(III-52)}$$

Thus

$$F\{\alpha^\dagger;\alpha\} = \sum_{n=0}^{\infty} i^n(n!)^{-1}E_n\{\alpha^\dagger;\alpha\} \quad .$$

It is shown in Appendix D that the E_n are related to their connected parts $E_n^{\ c}$ by the formal power series

$$\sum_{n=0}^{\infty}(it)^n(n!)^{-1}E_n\{\alpha^\dagger;\alpha\} = \exp\left(\sum_{n=1}^{\infty}(it)^n(n!)^{-1}E_n^{\ c}\{\alpha^\dagger;\alpha\}\right) \qquad \text{(III-53)}$$

where the functionals $E_n^{\ c}\{\alpha^\dagger;\alpha\}$ satisfy

$$\lim_{|z|\to\infty} E_n^{\ c}\{\alpha^\dagger + \beta^\dagger e^{iz\cdot p};\alpha + \beta e^{-iz\cdot p}\} = E_n^{\ c}\{\alpha^\dagger;\alpha\} + E_n^{\ c}\{\beta^\dagger;\beta\} \quad . \qquad \text{(III-54)}$$

Hence

$$F\{\alpha^\dagger;\alpha\} = \exp\left(\sum_{n=1}^{\infty} i^n(n!)^{-1}E_n^{\,c}\{\alpha^\dagger;\alpha\}\right) \qquad \text{(III-55)}$$

and because of Eq. (III-54), the scattering functional satisfies Eq. (III-10).

In this example, the first cluster property is introduced rather artificially. In the examples of the Feynman perturbation theory and the τ-functions this property follows from more fundamental assumptions. In the former the dynamical principles from which it is derived assure the cluster decomposition. In the latter, the cluster decomposition follows from locality and the asymptotic condition.

J. A Counter-Example

We now give an example of a unitary and Lorentz-invariant operator which does not satisfy the first cluster property. Let

$$S' = \exp(i H) \qquad \text{(III-56)}$$

where

$$H = \int d^3\underset{\sim}{p}_1{}'' d^3\underset{\sim}{p}_2{}'' d^3\underset{\sim}{p}_1{}' d^3\underset{\sim}{p}_2{}' (2\omega(\underset{\sim}{p}_1{}'')2\omega(\underset{\sim}{p}_2{}'')2\omega(\underset{\sim}{p}_1{}')2\omega(\underset{\sim}{p}_2{}'))^{-1/2}$$

$$\delta_4(p_1{}'' + p_2{}'' - p_1{}' - p_2{}')h(p_1{}'', p_2{}''; p_1{}', p_2{}')$$

$$a^\dagger(\underset{\sim}{p}_1{}'')a^\dagger(p_2{}'')|vac\rangle\langle vac|a(\underset{\sim}{p}_1{}')a(\underset{\sim}{p}_2{}') \quad , \qquad \text{(III-57)}$$

and $h^*(p_1{}'', p_2{}''; p_1{}', p_2{}') = h(p_1{}', p_2{}'; p_1{}'', p_2{}'')$. If $\tilde{h}$ is a Lorentz-invariant function of the four-momentum variables, then S' is unitary and Lorentz invariant.

S' has the action of the identity on all the n-particle subspaces for $n \neq 2$ and a non-trivial action on the two-particle subspace. Thus S' describes the interactions of two particles with each other, and only if there are no other particles present. The mere presence of other particles, no matter how far away, drastically affects the two-particle interaction: it turns it off. This is clearly a violation of the first cluster property. The latter requires that h vanish identically and thus that S' be the identity operator.

This violation is also seen in the functional formulation. The scattering functional given by S' is

$$F'\{\alpha^{\dagger};\alpha\} = e^{-\alpha\cdot\alpha^{\dagger}}\langle vac|e^{\alpha^{\dagger}\cdot a}S'e^{\alpha\cdot a^{\dagger}}|vac\rangle = 1 + e^{-\alpha\cdot\alpha^{\dagger}}D\{\alpha^{\dagger};\alpha\} \quad ,$$

where, in

$$D\{\alpha^{\dagger};\alpha\} = \int d^3\underset{\sim}{p}_1{}''d^3\underset{\sim}{p}_2{}''d^3\underset{\sim}{p}_1{}'d^3\underset{\sim}{p}_2{}'\alpha^{\dagger}(\underset{\sim}{p}_1{}'')\alpha^{\dagger}(\underset{\sim}{p}_2{}'')\alpha(\underset{\sim}{p}_1{}')\alpha(\underset{\sim}{p}_2{}')$$

$$\times\, \delta_4(p_1{}'' + p_2{}'' - p_1{}' - p_2{}')d(p_1{}'',p_2{}'';p_1{}',p_2{}') \quad ,$$

the function d is determined by h and contains no δ-functions. Then

$$\lim_{|z|\to\infty} F'\{\alpha^{\dagger} + \beta^{\dagger}e^{-iz\cdot p};\ \alpha + \beta e^{iz\cdot p}\} = 1 + e^{-\beta\cdot\beta^{\dagger}}(F'\{\alpha^{\dagger};\alpha\} - 1) + e^{-\alpha\cdot\alpha^{\dagger}}(F'\{\beta^{\dagger};\beta\} - 1) \quad .$$

which manifestly differs from the correct expression, Eq. (III-10). To conform to that equation, we must have $F' = 1$, which corresponds to $S = I$.

The example given here is a representative of a class of operators, exponentials of a finite sum of dyads in which the unitarity and Lorentz-invariance are easily expressed, which violate the first cluster property. Imposing the first cluster property on each operator reduces it to the identity.

This concludes our discussion of this property and the cluster decomposition of the S matrix.

IV. THE SECOND CLUSTER PROPERTY

We have shown in the previous section that there is a well-defined meaning for the "connected parts" of the plane-wave S-matrix elements. Accepting this consequence of the first cluster property, we proceed to discuss more precisely the second with the aid of the cluster amplitudes, the A_{mn} . Because the second cluster property depends decisively on the effects of wave-packet spreading, as we have indicated in the introduction, we first investigate the effect of this phenomenon on S-matrix elements in which a single initial or final-state particle is given an arbitrarily large time-like displacement. (It is obvious that such matrix elements are of the same order in the displacement parameter as the matrix elements for the one-particle transfer processes.) We then obtain the transition probability amplitude for a one-particle transfer process, from which we can obtain a limiting equation for the cluster amplitudes. The totality of these equations represent the consequences of the second cluster property for the structure of the S matrix.

A. Wave-Packet Spreading

In this section, we will deal with time-like translations only, determined by a four-vector z , which we parametrize by the real number τ and the three-vector $\underset{\sim}{v}$, with $\underset{\sim}{v}\cdot\underset{\sim}{v} < 1$:

$$z = (1 - \underset{\sim}{v}^2)^{-1/2}\tau(\underset{\sim}{v},1) \equiv (\tau/m)p_0;p_0 = \frac{m}{\sqrt{1 - v^2}}(\underset{\sim}{v},1) \quad .$$

Thus $z\cdot z = \tau^2 > 0$ and $p_0{}^2 = m^2$. From what has been said already in Section I and II, it is clear that S-matrix elements of the form

$$\langle vac|A\{\phi_f\}S\, A^\dagger\{\phi_{i1}\}U(I,(\tau/m)p_o)A^\dagger\{\phi_{i2}\}|vac\rangle$$

give the transition probability amplitude between a final state, represented by the momentum-space wave function ϕ_f and the initial state represented by $\phi_{i1}\phi'_{i2}$. Whatever ϕ_{i2} is, there is defined, at least crudely, a region in the coordinate space of the particles and in the time, far away from which the time-dependent coordinate-space wave function is vanishingly small. The corresponding region for the function ϕ'_{i2} is displaced with respect to that for ϕ_{i2} by the amount $(\tau/m)p_o$. Again, for sufficiently large τ, either positive or negative, the amount of overlap of ϕ'_{i2} with ϕ_{i1} can be made arbitrarily small. Of course, the first cluster property requires that

$$\lim_{\tau\to\pm\infty} \langle vac|A\{\phi_f\}S\, A^\dagger\{\phi_{i1}\}U(I,(\tau/m)p_o)A^\dagger\{\phi_{i2}\}|vac\rangle = 0 \quad .$$

To determine the leading terms of the matrix element in inverse powers of $|\tau|$, it is helpful to exhibit the τ dependence:

$$\langle vac|A\{\phi_f\}S\, A^\dagger\{\phi_{i1}\}U(I,(\tau/m)p_o)A^\dagger\{\phi_{i2}\}|vac\rangle = \int d^3\underset{\sim}{p} f(\underset{\sim}{p}) \exp\,[i(\tau/m)p_o\cdot p] \quad ,$$

where

$$f(\underset{\sim}{p}) = \int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}_m{}'' d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}_n{}' \phi_f{}^*(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}_m{}'')$$

$$\times\, S_{m,n+1}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}''; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}', \underset{\sim}{p})\phi_{i1}(\underset{\sim}{p}_1{}' \cdots \underset{\sim}{p}_n{}')\phi_{i2}(\underset{\sim}{p}) \quad .$$

We may assume $f(\underset{\sim}{p})$ to be a square-integrable, continuous function of $\underset{\sim}{p}$. In this form, the matrix element has the appearance of a time-dependent coordinate-space wave function. The asymptotic behavior of such wave functions is well known. The asymptotic behavior gives the wave-packet spreading effect, the $\tau^{-3/2}$ dependence for large τ. This result is obtained by evaluating the integral in the asymptotic limit by the method of stationary phase. The necessary results derived from this method are reviewed in Appendix A. Applying these results to the S-matrix element above, we obtain

$$\lim_{\tau\to\pm\infty} |\tau|^{3/2} e^{-i\tau m} \langle vac|A\{\phi_f\} S\, A^\dagger\{\phi_{i1}\} U(I,(\tau/m)p_o) A^\dagger\{\phi_{i2}\}|vac\rangle$$

$$= (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m] \exp(\pm 3\pi i/4) \phi_{i2}(\underset{\sim}{p}_o) \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n'$$

$$\times \phi_f^*(\underset{\sim}{p}_1'' \cdots \underset{\sim}{p}_m'') \phi_i(\underset{\sim}{p}_1' \cdots \underset{\sim}{p}_n') S_{m,n+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n', \underset{\sim}{p}_o) \; . \quad \text{(IV-1)}$$

Following the argument of Section III, Part A, we see that Eq. (IV-1) implies the following limiting equation, in the sense of tempered distributions, among the plane-wave S-matrix elements

$$\lim_{\tau\to\pm\infty} |\tau|^{3/2} \exp[i(\tau/m)p_o\cdot(p - p_o)] S_{m,n+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n', \underset{\sim}{p})$$

$$= (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m] \exp(\pm 3\pi i/4) \delta_3(\underset{\sim}{p} - \underset{\sim}{p}_o)$$

$$\times S_{m,n+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n', \underset{\sim}{p}_o) \quad \text{(IV-2)}$$

and a similar equation for translations of a single particle in the final state. Clearly the same limits must hold for the cluster amplitudes. Thus

$$\lim_{\tau \to \pm\infty} |\tau|^{3/2} \exp [i(\tau/m)p_o \cdot (p - p_o)] A_{m,n+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n', \underset{\sim}{p})$$

$$= (2\pi m)^{3/2} [\omega(\underset{\sim}{p}_o)/m] \exp (\pm 3\pi i/4) A_{m,n+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n', \underset{\sim}{p}_o)$$

$$\times \delta_3(\underset{\sim}{p} - \underset{\sim}{p}_o) \quad , \qquad \text{(IV-3a)}$$

$$\lim_{\tau \to \pm\infty} |\tau|^{3/2} \exp [-i(\tau/m)p_o \cdot (p - p_o)] A_{m+1,n}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m'', \underset{\sim}{p}; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')$$

$$= (2\pi m)^{3/2} [\omega(\underset{\sim}{p}_o)/m] \exp (\mp 3\pi i/4) A_{m+1,n}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m'', \underset{\sim}{p}_o; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')$$

$$\times \delta_3(\underset{\sim}{p} - \underset{\sim}{p}_o) \quad . \qquad \text{(IV-3b)}$$

These equations should be compared with Eq. (III-17). Here we see just how fast the $A_{mn}e^{iz\cdot\Delta}$ vanish with $|z|$ for large $|z|$, for these particular choices of Δ and z. In general, if more than one momentum variable occurs in Δ, then we would expect additional factors of $\tau^{-3/2}$ to determine the rate at which $A_{mn}e^{i(\tau/m)p_o\cdot\Delta}$ vanishes for large τ. Thus an S-matrix element describing an interaction involving two initial-state particles which are displaced by the four-vector $(\tau/m)p_o$ is of order $\tau^{-3/2}$ compared with the same matrix element, but in which only one initial-state particle is so displaced, (assuming there are four or more initial-state particles). The latter matrix element is, in turn, of order $\tau^{-3/2}$ compared with the same matrix element, but in which no particles are displaced.

One "intermediate-state" particle, in the sense used in the Introduction, similarly displaced will give a factor of $\tau^{-3/2}$ and two or more such particles displaced will give a factor of τ^{-3}, as was mentioned before. Thus, with the second cluster property and these initial and final-state single-particle wave-packet spreading effects, we are determining all the terms of $A_{mn}e^{i(\tau/m)p_o\cdot\Delta}$ of order $\tau^{-3/2}$, no matter what subsets of initial and final momentum variables we include in Δ.

B. Requirements of the Second Cluster Property on the Cluster Amplitudes

We proceed as indicated in Section I, with the notation of Section III, Part A. For the initial state

$$|\psi_i\rangle = A^{\dagger}\{\emptyset_{i1}\}U(I,(\tau/m)p_o)A^{\dagger}\{\emptyset_{i2}\}|vac\rangle \qquad \text{(IV-4a)}$$

and the final state

$$|\psi_f\rangle = A^{\dagger}\{\emptyset_{f1}\}U(I,(\tau/m)p_o)A^{\dagger}\{\emptyset_{f2}\}|vac\rangle \quad , \qquad \text{(IV-4b)}$$

we consider the leading terms in inverse powers of τ, for large τ, of $\langle\psi_f|S|\psi_i\rangle$, having first subtracted out as suggested by the first cluster property,

$$\langle\emptyset_{f1}|S|\emptyset_{i1}\rangle\langle\emptyset_{f2}|S|\emptyset_{i2}\rangle = \langle vac|A\{\emptyset_{f1}\}S\,A^{\dagger}\{\emptyset_{i1}\}|vac\rangle\,\langle vac|A\{\emptyset_{f2}\}S\,A^{\dagger}\{\emptyset_{i2}\}|vac\rangle \;.$$

For definiteness, we take τ to be positive.

Let us now calculate the contribution of the one-particle transfer processes. (We offer Figure 3 as a "visual aid" for the following discussion.) We define a state vector, assuming $n_1 \geqslant 2$:

$$S\, A^{\dagger}\{\phi_{i1}\}|vac\rangle \equiv \sum_{N=2}^{\infty} A^{\dagger}\{\phi_{i1}^{(N)}\}|vac\rangle \quad ,$$

$$A^{\dagger}\{\phi_{i1}^{(N)}\}|vac\rangle = (N!)^{-1/2}\int d^3\underset{\sim}{k}_1 \cdots d^3\underset{\sim}{k}_N (N!n_1!)^{-1/2}\int d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_{n_1}'$$

$$\times\, S_{Nn_1}(\underset{\sim}{k}_1, \cdots, \underset{\sim}{k}_N; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')\phi_{i1}(\underset{\sim}{p}_1' \cdots \underset{\sim}{p}_{n_1}')$$

$$\times\, a^{\dagger}(\underset{\sim}{k}_1) \cdots a^{\dagger}(\underset{\sim}{k}_N)|vac\rangle \quad . \qquad \text{(IV-5)}$$

Then we also define the state vector

$$|I\rangle = \sum_{N=2}^{\infty} A^{\dagger}\{\phi_{i1}^{(N)}\}U(I,(\tau/m)p_o)A^{\dagger}\{\phi_{i2}\}|vac\rangle \quad . \qquad \text{(IV-6)}$$

In constructing the vector $|I\rangle$ we have taken into account all of the interactions of the particles in the first system among themselves. We have not taken into account any of the interactions of the particles of the second system among themselves or of interactions between the systems. Now we wish to take into account the interactions of the particles of the second system with themselves and with one particle from the first system. To do this, we let S act only on these particles in the vector $|I\rangle$. That is, S has the action of the identity on all but one of the particles in the first system, because the interactions of those particles among themselves are already taken care of in $|I\rangle$.

Let us designate by $|II\rangle$ the new vector obtained in this way from $|I\rangle$. Explicitly, taking into account a factor for the possible choices of the particle from the first system, we obtain

$$|II\rangle = \sum_{N=2}^{\infty} [(N-1)!]^{-1} \int d^3\underset{\sim}{k}_1 \cdots d^3\underset{\sim}{k}_N (n_1! n_2!)^{-1/2}$$

$$\times \int d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_{n_1}' S_{Nn_1}(\underset{\sim}{k}_1, \cdots, \underset{\sim}{k}_N; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}') \phi_{i1}(\underset{\sim}{p}_1' \cdots \underset{\sim}{p}_{n_1}')$$

$$\times \int d^3\underset{\sim}{q}_1' \cdots d^3\underset{\sim}{q}_{n_2}' \phi_{i2}(\underset{\sim}{q}_1' \cdots \underset{\sim}{q}_{n_2}') \exp\left[i(\tau/m)p_o \cdot \sum_{i=1}^{n_2} q_i'\right]$$

$$\times\ a^\dagger(\underset{\sim}{k}_1) \cdots a^\dagger(\underset{\sim}{k}_{N-1}) S\, a^\dagger(\underset{\sim}{k}_N) a^\dagger(\underset{\sim}{q}_1') \cdots a^\dagger(\underset{\sim}{q}_{n_2}')|vac\rangle \quad . \tag{IV-7}$$

The contributions to $\langle\psi_f|S|\psi_i\rangle$ arising from one-particle transfer processes are to be found in $\langle\psi_f|II\rangle$, because of the way in which we have constructed the vector $|II\rangle$. With our particular choice of $|\psi_i\rangle$ and $|\psi_f\rangle$, the second cluster property may be expressed as follows: The leading terms of $\langle\psi_f|S|\psi_i\rangle - \langle\phi_{f1}|S|\phi_{i1}\rangle\langle\phi_{f2}|S|\phi_{i2}\rangle$ in inverse powers of τ as τ tends to infinity are obtained from $\langle\psi_f|II\rangle$.

The identification of the leading terms (in inverse powers of τ) in $\langle\psi_f|II\rangle$ is straightforward. We use the asymptotic limit already familiar to us in the discussion of the wave-packet spreading effects. The result is

$$\lim_{\tau\to+\infty} \tau^{3/2} e^{im\tau} \langle \psi_f | II \rangle$$

$$= (2\pi m)^{3/2} [\omega(\underset{\sim}{p}_o)/m] \exp(-3\pi i/4)(m_1! m_2! n_1! n_2!)^{-1/2}$$

$$\times \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_{m_1}'' d^3\underset{\sim}{q}_1'' \cdots d^3\underset{\sim}{q}_{m_2}'' \phi_{f1}^*(\underset{\sim}{p}_1'' \cdots \underset{\sim}{p}_{m_1}'')\phi_{f2}^*(\underset{\sim}{q}_1'' \cdots \underset{\sim}{q}_{m_2}'')$$

$$\times \int d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_{n_1}' d^3\underset{\sim}{q}_1' \cdots d^3\underset{\sim}{q}_{n_2}' \phi_{i1}(\underset{\sim}{p}_1' \cdots \underset{\sim}{p}_{n_1}')\phi_{i2}(\underset{\sim}{q}_1' \cdots \underset{\sim}{q}_{n_2}')$$

$$\times \; S_{m_2,n_2+1}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}', \underset{\sim}{p}_o)$$

$$\times \; S_{m_1+1,n_1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{p}_o; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}') \quad . \tag{IV-8}$$

Equation (IV-8) thus represents the leading terms in $\langle \psi_f | S | \psi_i \rangle$ due to one-particle transfer processes. It should be noted that they are causal, by construction. We have not included anti-causal contributions, namely those in which the later interactions influence the earlier ones, because, by the argument in the Introduction, they vanish faster than $\tau^{-3/2}$.

We must emphasize that there are other terms in $\langle \psi_f | S | \psi_i \rangle$ of order $\tau^{-3/2}$, namely, those due to wave-packet spreading effects. Thus there are contributions to the overall S-matrix element of order $\tau^{-3/2}$ arising from the following situations:

(1) A particle in the first, or "earlier," cluster does not interact with other particles in that cluster, but instead interacts with particles in the second, or "later," cluster.

(2) Likewise, a particle in the second cluster does not interact with particles in that cluster, but with particles in the first cluster.

(3) A particle from the first cluster is observed with respect to the "translated" final states although it did not interact with the particles in the second cluster.

(4) Similarly, a particle from the second cluster is observed with respect to the "untranslated" final states although it did not interact with the particles in the first cluster.

The important thing to notice about these four types of contributions is that they are "disconnected," that is, they do not involve any interaction between the first and second clusters. Thus the contributions listed above can be ignored if we consider only the <u>connected</u> part of $\langle \psi_f | S | \psi_i \rangle$, i.e., those contributions which involve interactions between particles which are originally in different clusters. The mathematical statement of the second cluster property, then, is the following: In the limit that τ goes to infinity, the connected part of $\langle \psi_f | S | \psi_i \rangle$ is dominated by the connected part of $\langle \psi_f | II \rangle$, the causal one-particle transfer contribution

$$\lim_{\tau \to +\infty} \tau^{3/2} [\langle \psi_f | S | \psi_i \rangle_{\text{connected}} - \langle \psi_f | II \rangle_{\text{connected}}] = 0 \quad . \qquad \text{(IV-9)}$$

This limit implies the following limiting equation, in the sense of tempered distributions, among the cluster amplitudes

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(\sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j'' + p_o\right)\right]$$

$$\times A_{m_1+m_2,n_1+n_2}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}', \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}')$$

$$= (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m] \exp(-3\pi i/4) A_{m_2,n_2+1}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}', \underset{\sim}{p}_o)$$

$$A_{m_1+1,n_1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{p}_o; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}') \quad , \qquad \text{(IV-10)}$$

where $m_1 \geqslant 1$, $m_2 \geqslant 2$, $n_1 \geqslant 2$, $n_2 \geqslant 1$. This limit involving tempered distributions follows from Eq. (IV-9), because of arguments similar to those used in Section III, Part B. This limit, we claim, must hold in any reasonable theory giving a unitary, Lorentz-invariant S matrix with the cluster decomposition. It obviously requires a structure in the S-matrix elements not given by these other properties.

The simplest example of Eq. (IV-10) is for the case of three particles in the initial state and three in the final state. If two of the outgoing particles and one of the incoming particles are given the same forward time-like translation, we obtain the result

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp[i(\tau/m)p_o\cdot(p_o + q_1' - q_1'' - q_2'')] A_{33}(\underset{\sim}{p}_1'', \underset{\sim}{q}_1'', \underset{\sim}{q}_2''; \underset{\sim}{p}_1', \underset{\sim}{p}_2', \underset{\sim}{q}_1')$$

$$= (2\pi m)^{3/2} [\omega(\underset{\sim}{p}_o)/m] \exp(-3\pi i/4) A_{22}(\underset{\sim}{q}_1'', \underset{\sim}{q}_2''; \underset{\sim}{q}_1', \underset{\sim}{p}_o) A_{22}(\underset{\sim}{p}_1'', \underset{\sim}{p}_o; \underset{\sim}{p}_1', \underset{\sim}{p}_2') \ . \qquad \text{(IV-11)}$$

This simple case also illustrates the causal nature of Eq. (IV-10). Suppose one outgoing and two incoming particles are given the same forward time-like translation. It is clear physically that the connected part of the relevant S-matrix element must vanish faster than $\tau^{-3/2}$, as was discussed in the Introduction. Correspondingly, Eq. (IV-10) gives

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp [i(\tau/m)p_o\cdot(p_o + q_1' + q_2' - q_1")]$$

$$A_{33}(\underset{\sim}{p}_1",\underset{\sim}{p}_2",\underset{\sim}{q}_1";\underset{\sim}{p}_1',\underset{\sim}{q}_1',\underset{\sim}{q}_2') = (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m] \exp (-3\pi i/4)$$

$$A_{13}(\underset{\sim}{q}_1";\underset{\sim}{q}_1',\underset{\sim}{q}_2',\underset{\sim}{p}_o)A_{31}(\underset{\sim}{p}_1",\underset{\sim}{p}_2",\underset{\sim}{p}_o;\underset{\sim}{p}_1') \quad . \qquad \text{(IV-12)}$$

But A_{13} and A_{31} vanish identically, because of the impossibility of conserving overall four-momentum, so that

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp [i(\tau/m)p_o\cdot(q_1' + q_2' - q_1")]A_{33}(\underset{\sim}{p}_1",\underset{\sim}{p}_2",\underset{\sim}{q}_1";\underset{\sim}{p}_1',\underset{\sim}{q}_1',\underset{\sim}{q}_2') = 0 \qquad \text{(IV-13)}$$

as required.

C. Functional Formulation

We wish to summarize the totality of Eqs. (IV-3) and (IV-10) as a condition on the scattering functional F. Of course, we must first obtain the condition on the functional A. By straightforward expansion of

$$A\{\alpha^\dagger(\underset{\sim}{p}) + \beta^\dagger(\underset{\sim}{p}) \exp [-i(\tau/m)p_o\cdot p];\alpha(\underset{\sim}{p}) + \beta(\underset{\sim}{p})\exp[i(\tau/m)p_o\cdot p]\}$$

using the two equations cited above and the assumption that they give all the terms of order $\tau^{-3/2}$, on arrives at

$$\lim_{\tau\to+\infty} \tau^{3/2} \Big\{ A\{\alpha^\dagger + \beta^\dagger \exp[-i(\tau/m)p_o\cdot p];\alpha + \beta \exp[i(\tau/m)p_o\cdot p]\}$$

$$- A\{\alpha^\dagger;\alpha\} - A\{\beta^\dagger;\beta\} - \int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\beta(\underset{\sim}{q}) \frac{\delta A\{\alpha^\dagger;\alpha\}}{\delta\alpha(\underset{\sim}{q})}$$

$$- \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\beta^\dagger(\underset{\sim}{q}) \frac{\delta A\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})} - \int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\alpha^\dagger(\underset{\sim}{q})$$

$$\times \frac{\delta A\{\beta^\dagger;\beta\}}{\delta\beta^\dagger(\underset{\sim}{q})} - \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\alpha(\underset{\sim}{q}) \frac{\delta A\{\beta^\dagger;\beta\}}{\delta\beta(\underset{\sim}{q})}$$

$$- \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q] \frac{\delta A\{\beta^\dagger;\beta\}}{\delta\beta(\underset{\sim}{q})} \times \frac{\delta A\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})} \Big\} = 0 \quad . \qquad \text{(IV-14)}$$

One recognizes the first three terms from Eq. (III-12). The next four terms represent, completely, the effects of the wave-packet spreading of a single final or initial particle, given by Eq. (IV-3). The last term summarizes the totality of connected, causal, one-particle transfer processes.

With the relationship $F = \exp A$, and again by a straightforward procedure, one obtains

$$
\lim_{\tau\to+\infty} \tau^{3/2} \Bigg\{ F\{\alpha^\dagger + \beta^\dagger \exp[-i(\tau/m)p_o\cdot p];\alpha + \beta \exp[i(\tau/m)p_o\cdot p]\}
$$

$$
- F\{\alpha^\dagger;\alpha\}F\{\beta^\dagger;\beta\} - \Bigg[\int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\beta(\underset{\sim}{q}) \frac{\delta}{\delta\alpha(\underset{\sim}{q})}
$$

$$
+ \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\beta^\dagger(\underset{\sim}{q}) \frac{\delta}{\delta\alpha^\dagger(\underset{\sim}{q})} + \int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\alpha^\dagger(\underset{\sim}{q}) \frac{\delta}{\delta\beta^\dagger(\underset{\sim}{q})}
$$

$$
+ \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\alpha(\underset{\sim}{q}) \frac{\delta}{\delta\beta(\underset{\sim}{q})}\Bigg] F\{\alpha^\dagger;\alpha\}F\{\beta^\dagger;\beta\}
$$

$$
- \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q] \frac{\delta F\{\beta^\dagger;\beta\}}{\delta\beta(\underset{\sim}{q})} \times \frac{\delta F\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})} \Bigg\} = 0 \quad . \qquad \text{(IV-15)}
$$

As with the previous equation, it is easy to pick out the first cluster property, the wave-packet spreading effects, and the causal one-particle transfer contribution. These equations are completely equivalent and faithfully represent the totality of Eqs. (III-7), (IV-3) and (IV-10).

Causality can be viewed in the functional formalism in the following way. It should be noticed that these functional equations are invariant under the simultaneous interchange of both the functions $\alpha^\dagger(\underset{\sim}{p})$ with $\beta^\dagger(\underset{\sim}{p}) \exp[-i(\tau/m)p_o\cdot p]$ and the functions $\alpha(\underset{\sim}{p})$ with $\beta(\underset{\sim}{p}) \exp[i(\tau/m)p_o\cdot p]$, except for the last term, the causal one-particle transfer term. Thus a sort of symmetry in the scattering functional is broken by the requirement that the S matrix have macroscopic causality. Performing the interchange of the functions is seen to effectively change the sign of τ, so that what Eq. (IV-15) summarizes is again the influencing of later events by earlier events but not vice versa. This "formulation of the principle of causality in differential form" should be compared with that of Bogoliubov and Shirkov.[25]

D. The Plane-Wave S-Matrix Elements

The limiting equation for the plane-wave S-matrix elements corresponding to Eq. (IV-10) is obtained in a straightforward way by functionally differentiating Eq. (IV-15): m_1 times with respect to $\alpha^\dagger(\underset{\sim}{p})$, m_2 times with respect to $\beta^\dagger(\underset{\sim}{p})$, n_1 times with respect to $\alpha(\underset{\sim}{p})$, and n_2 times with respect to $\beta(\underset{\sim}{p})$. Here we assume $n_1, n_2, m_1, m_2 > 1$. The result is

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(p_o + \sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j''\right)\right]$$

$$\times \Bigg\{ S_{m_1+m_2,n_1+n_2}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}', \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}')$$

$$- S_{m_2 n_2}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}') S_{m_1 n_1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')$$

$$- \sum_{i=1}^{n_2} S_{m_2,n_2-1}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, [\underset{\sim}{q}_i'] \cdots, \underset{\sim}{q}_{n_2}')$$

$$\times S_{m_1,n_1+1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}''; \underset{\sim}{q}_i', \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')$$

$$- \sum_{j=1}^{m_1} S_{m_2+1,n_2}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}'', \underset{\sim}{p}_j''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}')$$

$$\times S_{m_1-1,n_1}(\underset{\sim}{p}_1'', \cdots [\underset{\sim}{p}_j''] \cdots, \underset{\sim}{p}_{m_1}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')$$

$$+ \sum_{i=1}^{n_2} \sum_{j=1}^{m_1} \delta_3(\underset{\sim}{q}_i' - \underset{\sim}{p}_j'') S_{m_2,n_2-1}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots [\underset{\sim}{q}_i'] \cdots, \underset{\sim}{q}_{n_2}')$$

$$\times S_{m_1-1,n_1}(\underset{\sim}{p}_1'', \cdots [\underset{\sim}{p}_j'] \cdots, \underset{\sim}{p}_{m_1}''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}')$$

$$- \int d^3k\, S_{m_2,n_2+1}(\underset{\sim}{q}_1'', \cdots, \underset{\sim}{q}_{m_2}''; \underset{\sim}{q}_1', \cdots, \underset{\sim}{q}_{n_2}', \underset{\sim}{k}) S_{m_1+1,n_1}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_{m_1}'', \underset{\sim}{k}; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_{n_1}') \Bigg\}$$

$$= 0 \qquad \text{(IV-16)}$$

The bracketing of a momentum variable in certain of the S_{mn} above is to indicate that the variable is not present in the argument of that S_{mn}, but is someplace else in the same term. The first two terms in Eq. (IV-16) of course represent the first cluster property. The last term includes the causal one-particle transfer contributions. To intrepret the wave-packet spreading effects, we refer to the list enumerated in Part B of this section, between Eqs. (IV-8) and (IV-9). The third and fourth terms in the above equation represent contributions of types (2) and (4), respectively. The fifth term represents the situation in which a particle initially in the second cluster does not interact at all, but is observed with respect to final states appropriate to the particles emerging from the interactions of the first cluster. Identical terms are contained in both the third and fourth terms, in their disconnected parts, so that the fifth term assures that this contribution is counted correctly. The wave-packet spreading phenomena of types (1) and (3) are contained in the disconnected parts of the last term.

E. The Possibility of a Diagrammatic Representation

At this point, it may be hoped, and, indeed, Eq. (IV-10) suggests, that there is an immediate extension of the diagrammatic technique introduced in Section III, Part D. It might be thought that, after all contributions from physically realizable one-particle intermediate states are subtracted from a cluster amplitude, the remainder tends to zero faster than $\tau^{-3/2}$ under any forward time-like translation of a non-trivial subset of initial and final particles. Thus, if one would define a "reduced" amplitude R_{mn} by

$$A_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \equiv R_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')$$

$$+ \sum_{\begin{pmatrix}1 & \cdots & m \\ \alpha_1 & \cdots & \alpha_m\end{pmatrix}} \sum_{\begin{pmatrix}1 & \cdots & n \\ \beta_1 & \cdots & \beta_n\end{pmatrix}} \sum_{r=2}^{m-1} \sum_{s=2}^{n-1} [r!(s-1)!(m-r)!(n-s+1)!]^{-1}$$

$$\times \int d^3\underset{\sim}{k} A_{rs}(\underset{\sim}{p}''_{\alpha_1}, \cdots, \underset{\sim}{p}''_{\alpha_r}; \underset{\sim}{p}'_{\beta_1}, \cdots, \underset{\sim}{p}'_{\beta_{s-1}}, \underset{\sim}{k})$$

$$\times A_{m-r+1,n-s+1}(\underset{\sim}{k}, \underset{\sim}{p}''_{\alpha_{r+1}}, \cdots, \underset{\sim}{p}''_{\alpha_m}; \underset{\sim}{p}'_{\beta_s}, \cdots, \underset{\sim}{p}'_{\beta_n}) \qquad \text{(IV-17)}$$

one would hope that

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp [i(\tau/m)p_o\cdot(\Sigma_i' p_i' - \Sigma_j'' p_j")] R_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') = 0 , \qquad \text{(IV-18)}$$

where Σ' and $\Sigma"$ denote sums over subsets of the initial and final particles, respectively. However, such is obviously not the case because the decomposition, Eq. (IV-17), is not causally consistent. Anti-causal terms, as well as causal, occur in the sum on the right-hand side, so that if the causal nature of the A_{mn} is to be maintained, the R_{mn} must cancel the anti-causal terms. Since the anti-causal terms, which were <u>not</u> present in Eq. (IV-10), can also vanish as slowly as $\tau^{-3/2}$, Eq. (IV-18) is false. Thus diagrams based on the decomposition, Eq. (IV-17), are not of much significance.

Of course, it may be possible to introduce a factor inside the integrand on the right-hand side of Eq. (IV-17) which would tend to eliminate the unwanted terms in the appropriate limit. For example, the causal nature of the distribution $\lim_{\epsilon\to 0+} (\omega + i\epsilon)^{-1}$ is well known:

$$\lim_{\tau\to+\infty} \exp(i\omega\tau) \lim_{\varepsilon\to 0+} (\omega + i\varepsilon)^{-1} = -2\pi i\delta(\omega)$$

$$\lim_{\tau\to\infty} \exp(i\omega\tau) \lim_{\varepsilon\to 0+} (\omega + i\varepsilon)^{-1} = 0 \quad . \tag{IV-19}$$

However, to use this causal function, it is necessary to continue the A_{mn} off the manifold on which they are physically defined. Without further assumptions beyond unitarity and Lorentz invariance, it is not known how this can be done. There are, of course, theories which allow for this and in those theories the second cluster property has an obvious diagrammatic representation. We are thinking, for example, of the Feynman perturbation theory, which will be discussed in more detail later.

F. Lorentz Invariance

That the second cluster property is consistent with the Lorentz invariance requirement on the S matrix is clearly seen by expressing it in terms of the invariant cluster amplitudes, the G_{mn} . From Eqs. (III-19) and (IV-10) one obtains on the manifold $\mathcal{M}_{m_1+m_2,n_1+n_2}$

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(\sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j''\right)\right]$$
$$\times\Big\{G_{m_1+m_2,n_1+n_2}(p_1'', \cdots, p_{m_1}'', q_1'', \cdots, q_{m_2}''; p_1', \cdots, p_{n_1}', q_1', \cdots, q_{n_2}')$$
$$- \delta^{(+)}(\sum_j q_j'' - \sum_i q_i'; m) G_{m_2,n_2+1}(q_1'', \cdots, q_m''; q_1', \cdots, q_{n_2}', \sum_j q_j'' - \sum_i q_i')$$
$$\times G_{m_1+1,n_1}(\sum_j q_j'' - \sum_i q_i', p_1'', \cdots, p_{m_1}''; p_1', \cdots, p_{n_1}')\Big\} = 0$$

(IV-20)

The $\delta^{(+)}$-function assures that the distributions G_{m_2,n_2+1} and G_{m_1+1,n_1} need be defined only on the manifolds $\mathcal{M}_{m_2,n_2+1}$ and $\mathcal{M}_{m_1+1,n_1}$, respectively. Making explicit the asymptotic limit in the second term, we have

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(p_o + \sum_{i=1}^{n_2} q_i' - \sum_{j=1}^{m_2} q_j''\right)\right]$$

$$\times G_{m_1+m_2,n_1+n_2}(p_1'', \cdots, p''_{m_1}, q_1'', \cdots, q''_{m_2}; p_1', \cdots, p'_{n_1}, q_1', \cdots, q'_{n_2})$$

$$= \tau^{3/2}(2m)^{1/2} \exp(-3\pi i/4)\delta_4\left(\sum_{j=1}^{m_2} q_j'' - \sum_{i=1}^{n_2} q_i' - p_o\right)$$

$$\times G_{m_2,n_2+1}(q_1'', \cdots, q''_{m_2}; q_1', \cdots, q''_{n_2}, p_o)$$

$$\times G_{m_1+1,n_1}(p_o, p_1'', \cdots, p''_{m_1}; p_1', \cdots, p'_{n_1}) \quad . \qquad \text{(IV-21)}$$

These expressions are manifestly Lorentz invariant.

G. Unitarity

Again, as with the first cluster property, it should be obvious that unitarity and the second cluster property are distinct properties of the S matrix. We will illustrate this remark later with an example of a unitary operator which does not have the second cluster property.

It should be noticed however, for example in the functional formalism, that the one-particle transfer term in Eq. (IV-14) bears a certain resemblance to the contributions to the unitarity relations with one-particle intermediate states, i.e. there is a similarity between

$$\int d^3\underset{\sim}{q} \exp\left[-i(\tau/m)p_o\cdot q\right] \frac{\delta A\{\beta^\dagger;\beta\}}{\delta\beta(\underset{\sim}{q})} \times \frac{\delta A\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})}$$

and

$$\int d^3\underset{\sim}{q} \; \frac{\delta A\{\alpha^\dagger;\alpha\}}{\delta\alpha(\underset{\sim}{q})} \times \frac{\delta A^*\{\alpha^*;\alpha^{\dagger *}\}}{\delta\alpha^\dagger(\underset{\sim}{q})} \quad .$$

It might be thought, on the basis of this similarity, that the one-particle transfer contributions in the A_{mn} are implied by the one-particle intermediate state contributions to the unitarity relations, and can be derived from them. This is false, as we show for a specific case.

Consider the cluster amplitude A_{33} . The unitarity relation for A_{33} is given by Eq. (III-23b); the relevant one-particle term is on the second line of that equation. The requirements of the second cluster property for A_{33} are shown by Eq. (IV-12). We proceed as follows: Define a new amplitude $\hat{A}_{mn}$ such that $\hat{A}_{22} = A_{22}$, $\hat{A}_{23} = A_{23}$, $\hat{A}_{32} = A_{32}$,

$$A_{33}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'',\underset{\sim}{p}_3{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}',\underset{\sim}{p}_3{}') \equiv \hat{A}_{33}(\underset{\sim}{p}_1{}'',\underset{\sim}{p}_2{}'',\underset{\sim}{p}_3{}'';\underset{\sim}{p}_1{}',\underset{\sim}{p}_2{}',\underset{\sim}{p}_3{}')$$
$$+ \frac{1}{8} \sum_{\begin{pmatrix}1&2&3\\ \alpha_1&\alpha_2&\alpha_3\end{pmatrix}} \sum_{\begin{pmatrix}1&2&3\\ \beta_1&\beta_2&\beta_3\end{pmatrix}} \int d^3\underset{\sim}{k}\; \hat{A}_{22}(\underset{\sim}{p}''_{\alpha_1},\underset{\sim}{p}''_{\alpha_2};\underset{\sim}{p}'_{\beta_1},\underset{\sim}{k}) A_{22}(\underset{\sim}{p}''_{\alpha_3},\underset{\sim}{k};\underset{\sim}{p}'_{\beta_2},\underset{\sim}{p}'_{\beta_3}) \quad . \tag{IV-22}$$

Substituting these definitions into Eq. (III-23b), one finds that the unitarity relation in terms of the $\hat{A}_{mn}$ no longer has a one-particle term. Thus, if one infers the existence of the one-particle transfer term in A_{33} from the one-particle term in the unitarity relation, then one must infer that A_{33} has no such term. Then the one-particle transfers must come from the terms $\sum\int d^3k\ \hat{\tilde{A}}_{22}\hat{A}_{22}$, which have both causal and anti-causal contributions, violating Eq. (IV-12). Therefore, as we claimed, the suggesting of the previous paragraph is false.

We see from this example the more general fact, that unitarity and the second cluster property are distinct because the former contains nothing of causality while the latter is definitely causal in nature. Of course, unitarity along with some other condition which implies causality, for example certain analyticity properties of the G_{mn}, may be sufficient to insure the requirements of the second cluster property.

H. Feynman Perturbation Theory

We now give two examples of operators which satisfy the requirements of the second cluster property. The first will be the Feynman perturbation theory, discussed previously in Section III, Part G. There it was argued that the sum of all connected graphs obtained by the Wick decomposition of Eq. (III-28) with m outgoing and n incoming lines, labelled by four-momenta $p_1'', \cdots, p_m''$ and $p_1', \cdots, p_n'$ respectively, represents a Lorentz-invariant distribution $\mathcal{G}_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$, defined on a $4(m+n) - 4$ dimensional manifold, which, when put on the mass shell, becomes a possible candidate for the invariant cluster amplitude G_{mn}:

$$G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$$

$$= \mathcal{G}_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')\Big|_{p''_{i4}=\omega(\underset{\sim}{p}_i''); p'_{j4}=\omega(\underset{\sim}{p}_j')} \quad . \tag{IV-23}$$

We now show that the G_{mn} so defined satisfy Eq. (IV-21), and hence the second cluster property. We rely on the graphical approach for this demonstration.

We consider the set of all connected graphs contributing to the plane-wave S-matrix element for n incoming and m outgoing particles with momenta $\underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n'$ and energies $\omega(\underset{\sim}{p}_1'), \cdots, \omega(\underset{\sim}{p}_n')$, and momenta $\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''$ and energies $\omega(\underset{\sim}{p}_1''), \cdots, \omega(\underset{\sim}{p}_m'')$, respectively. Aside from kinematical factors and an overall four-momentum conserving δ-function, the sum of the contributions from the connected graphs is the amplitude $G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$. From this set of graphs, a subset of the following type can be selected without ambiguity: those graphs for which r outgoing and s incoming lines are disjoint from the remaining $m-r$ outgoing and $n-s$ incoming lines except for a single internal line. Calling the sum of contributions from this subset $G^{(1)}_{mn:rs}$, such a selection uniquely defines a decomposition of G_{mn} for a given r and s:

$$G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = G^{(1)}_{mn:rs}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') + G^{(2)}_{mn:rs}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') \quad . \tag{IV-24}$$

For convenience, in the subset of graphs giving $G^{(1)}_{mn:rs}$, let the four-momentum of the internal line be directed toward the part of the graph containing the r outgoing and s incoming lines. This assumption is no restriction because of the crossing symmetry of the Feynman amplitudes. For any given subgraph of $n-s$ lines going to $m-r+1$ lines, in this subset, there will be a sum of graphs which correspond to all possible subgraphs connecting $s+1$ lines to r lines. Summing over the subgraphs of n-s lines going to m-r+1 lines gives back the contribution from the entire subset. Then, from the Feynman rules

$$G^{(1)}_{mn:rs}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = N_{nm;rs} \sum_{\begin{pmatrix}1 & \cdots & m\\ \alpha_1 & \cdots & \alpha_m\end{pmatrix}} \sum_{\begin{pmatrix}1 & \cdots & n\\ \beta_1 & \cdots & \beta_n\end{pmatrix}}$$

$$\times \lim_{\epsilon\to 0+} \left[\left(\sum_{i=1}^{r} p''_{\alpha_i} - \sum_{j=1}^{s} p'_{\beta_j} \right)^2 - m^2 + i\epsilon \right]^{-1}$$

$$\times \mathcal{G}_{r,s+1}\left(p''_{\alpha_1}, \cdots, p''_{\alpha_r}; p'_{\beta_1}, \cdots, p'_{\beta_s}, \sum_i p''_{\alpha_i} - \sum_j p'_{\beta_j}\right)$$

$$\times \mathcal{G}_{m-r+1,n-s}\left(p''_{\alpha_{r+1}}, \cdots, p''_{\alpha_m}, \sum_i p''_{\alpha_i} - \sum_j p'_{\beta_j}; p'_{\beta_{s+1}}, \cdots, p'_{\beta_n}\right), \tag{IV-25}$$

where $p''_{i4} = \omega(\underset{\sim}{p}_i')$, $p'_{j4} = \omega(\underset{\sim}{p}_j')$, and $N_{nm:rs}$ is a numerical factor to be determined below.

One contribution to $N_{nm:rs}$ is $i(2\pi)^{-4}$ associated with the Feynman denominator, but one factor of $(2\pi)^{-3/2}$ is needed for the conventional external-line normalization of both $\mathcal{G}_{r,s+1}$ and $\mathcal{G}_{m-r+1,n-s}$, so there remains $i(2\pi)^{-1}$. The need for another factor comes from the fact that the sum over permutations of all outgoing momentum variables among themselves and all incoming momentum variables among themselves includes too much. Any two terms in the sum which are the same except for a permutation of the r momentum variables among themselves, the s momentum variables among themselves, etc., are in fact identical. (For simplicity, we assume that $r \neq m - r$ and $s \neq n - s$.) The same graphs are summed in each, although perhaps in some different order. Hence, a factor of $[r!s!(m-r)!(n-s)!]^{-1}$ must be introduced to keep the counting correct. Therefore

$$N^{-1}_{nm:rs} = -2\pi i \; r!s!(m-r)!(n-s)! \quad . \tag{IV-26}$$

(If both $r = m - r$ and $s = n - s$, there should be an additional factor of two in this expression.)

It is clear from Eq. (IV-25) that the properties of the Feynman denominator will determine whether or not the G_{mn} satisfy Eq. (IV-21). It is shown in Appendix A that

$$\lim_{\tau\to+\infty} \lim_{\varepsilon\to 0+} \tau^{3/2}[p^2 - m^2 + i\varepsilon]^{-1} \exp[i(\tau/m)p_o\cdot(p_o - p)]$$
$$= -2\pi i(2m)^{-1}(2\pi m)^{3/2} \exp(-3\pi i/4)\delta_4(p - p_o) \quad . \tag{IV-27}$$

Because $G^{(2)}_{mn:rs}$ has at least two internal lines joining the subgraphs of interest, there are at least two corresponding Feynman denominators. Thus, for a particular set of r outgoing and s incoming lines:

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(\sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)\right]$$
$$G^{(2)}_{mn:rs}(p_1'', \cdots, p_m''; p_1', \cdots, p_n') = 0 \quad . \tag{IV-28}$$

Let us consider a similar expression for $G^{(1)}_{mn:rs}$. For some of the terms in the sum over the outgoing and incoming momentum variables in $G^{(1)}_{mn:rs}$, the momentum variables in the Feynman denominator will correspond to the momentum variables in the argument of the exponential representing the time-like translation. These terms will vanish as $\tau^{-3/2}$. In other terms, there will be the "wrong" variables in the denominator, and they will vanish faster than $\tau^{-3/2}$. Thus

$$\lim_{\tau \to \infty} \tau^{3/2} \exp\left[i(\tau/m)p_o \cdot \left(p_o + \sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)\right]$$

$$\times G^{(1)}_{mn:rs}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$$

$$= r!s!(m-r)!(n-s)!N_{nm:rs} \mathcal{G}_{r,s+1}(p_1'', \cdots, p_r''; p_1', \cdots, p_s', p_o)$$

$$\times \mathcal{G}_{m-r+1,n-s}(p''_{r+1}, \cdots, p_m'', p_o; p'_{s+1}, \cdots, p_n')(-2\pi i)(2\pi m)^{3/2}(2m)^{-1}\exp(-3\pi i/4)$$

$$\times \delta_4\left(p_o + \sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right) \Big| p''_{i4} = \omega(\underset{\sim}{p}_i''); p'_{j4} = \omega(\underset{\sim}{p}_j') \quad . \qquad \text{(IV-29)}$$

Since $p_{o4} = \omega(\underset{\sim}{p}_o)$, all the momenta on the right-hand side are on the mass shell. Therefore, with consideration of Eqs. (IV-24), (IV-26) and (IV-28), Eq. (IV-29) shows that the G_{mn} given by Feynman perturbation theory satisfy the second cluster property.

The fact that Feynman perturbation theory is thus consistent with the crude notion of causality embodied in the second cluster property is in no way surprising, of course. Causality, in the sense we have been using it, is determined by the time-ordering operation, so that the crude notion of "later" events occuring after, and being influenced by, "earlier" events is assured. Because Eq. (IV-21) refers to the limit that τ goes to infinity, it says nothing about the much stronger property, microscopic causality, which, in the Feynman perturbation theory is imposed by both the locality conditions and the time-ordering.

I. The τ-Functions

As was explained in Section III, Part G, Zimmermann showed that the functions $\tau(p_1, \cdots, p_n)$, the Fourier transforms of the τ-functions in coordinate space, have a unique expansion into certain functions $\tilde{\eta}(p_1, \cdots, p_n)$ which are free of vacuum singularities, i.e., δ-functions conserving a subset of four-momenta. Because of this property, the invariant cluster amplitudes in the LSZ formalism are:

$$G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$$

$$= (2\pi i)^{m+n} \left\{ \left[\prod_{i=1}^{m} (-p_i''^2 + m^2) \right] \left[\prod_{j=1}^{n} (-p_j'^2 + m^2) \right] \right.$$

$$\left. \times \tilde{\eta}(p_1'', \cdots, p_m'', -p_1', \cdots, -p_n') \Big|_{p_{i4}'' = \omega(p_i''); p_{j4}' = \omega(p_j')} \right\} \quad . \qquad \text{(III-49)}$$

We now show from Zimmermann's work that the G_{mn} so defined satisfy Eq. (IV-21), and hence the second cluster property.

Using the basic assumptions of the LSZ formalism, Zimmermann found a function $\sigma(p_1, \cdots, p_n)$ which has neither vacuum nor one-particle singularities. The latter are mass-shell δ-functions, $\delta(p^2 - m^2)$, and principal part denominators, $P(p^2 - m^2)^{-1}$, corresponding to one-particle intermediate states. This function is

$$\sigma(p_1, \cdots, p_n) = \left\{ \prod_{\mu=1}^{M} (q_\mu{}^2 - m^2) \right\} \tilde{\eta}(p_1, \cdots, p_n) \quad , \qquad \text{(IV-30)}$$

defined on the $4(n-1)$-dimensional manifold spanned by the four-momentum variables constrained by the condition $\sum_{i=1}^{n} p_i = 0$. Here the q_μ are the set of all partial sums of the momentum variables. The one-particle singularities can then be exhibited as follows:

$$\tilde{\eta}(p_1, \cdots, p_n) = \left[\prod_{\mu=1}^{M} \lim_{\varepsilon\to 0+} (q_\mu^2 - m^2 + i\varepsilon)^{-1}\right] \sigma(p_1, \cdots, p_n) \quad . \tag{IV-31}$$

When one of the q_μ is put on the mass shell, $\tilde{\eta}$ is of course not defined, but the residue $\lim_{q_\mu^2\to m^2} [(q_\mu^2 - m^2)\tilde{\eta}(p_1, \cdots, p_n)]$ has a well-defined meaning. Zimmermann showed that

$$[(p^2 - m^2)\tilde{\eta}(p_1, \cdots, p_n)]\Big|_{p^2=m^2}$$

$$= -2\pi i[(p^2 - m^2)\tilde{\eta}(p_{i_1}, \cdots, p'_{i_r}, - p)]\Big|_{p^2=m^2}$$

$$\times [(p^2 - m^2)\tilde{\eta}(p_{i_{r+1}}, \cdots, p_{i_n}, p)]\Big|_{p^2=m^2} \quad . \tag{IV-32}$$

if $p = \sum_{\nu=1}^{r} p_{i_\nu}$ is a particular subset of four-momenta.

One can then combine Eqs. (III-49), (IV-30) and (IV-31) to obtain

$$\lim_{\tau\to+\infty} \tau^{3/2} \exp\left[i(\tau/m)p_o\cdot\left(p_o + \sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)\right] G_{mn}(p_1'', \cdots, p_m''; p_1', \cdots, p_n')$$

$$= (2\pi i)^{m+n}\left[\prod_{i=1}^{m}(-p_i''^2 + m^2)\right]\left[\prod_{j=1}^{n}(-p_j'^2 + m^2)\right]\left[\left(\sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)^2 - m^2\right]$$

$$\times \tilde{\eta}(p_1'', \cdots, p_m'', - p_1', \cdots, - p_n') \lim_{\tau\to+\infty} \lim_{\varepsilon\to 0+} \tag{IV-33}$$

$$\times \tau^{3/2}\left[\left(\sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)^2 - m^2 + i\varepsilon\right]^{-1}$$

$$\times \exp\left[i(\tau/m)p_o\cdot\left(p_o + \sum_{j=1}^{s} p_j' - \sum_{i=1}^{r} p_i''\right)\right]\Big|_{p''_{i4}=\omega(\underset{\sim}{p}_i''); p'_{j4}=\omega(\underset{\sim}{p}_j')} \quad .$$

Applying the limit, Eq. (IV-27), and using Eq. (IV-32), which is allowable since p_o is on the mass shell, one arrives at Eq. (IV-21). Hence the S matrix given by the τ-functions satisfies the requirements of the second cluster property.

It should be emphasized that Zimmermann's results are not consequences of perturbation theory, but follow from the assumptions of the LSZ formalism: locality, Lorentz invariance, asymptotic conditions, etc. Zimmermann was also able to give a diagrammatic representation of this theory to show explicitly the singularities due to the one-particle intermediate states; that is, he showed that the functions $\tilde{\eta}$ could be expanded uniquely with respect to such singularities. That such a diagrammatic representation is possible here is due to the fact that the theory gives the functions involved a well-defined meaning off the mass shell.

J. A Counter-Example

We now inquire what conditions the second cluster property imposes on the Hermitian phase matrix of LSZ II. The invariant cluster amplitudes are given by infinite sums of integrals involving products of the distributions $\tilde{h}_n$. For example

$$G_{22}(p_1'',p_2'';p_1',p_2') = i\tilde{h}_4(-p_1'',-p_2'',p_1',p_2') + \cdots ,$$

$$G_{33}(p_1'',p_2'',p_3'';p_1',p_2',p_3') = i\tilde{h}_6(-p_1'',-p_2'',-p_3'',p_1',p_2',p_3')$$
$$- \frac{1}{8} \sum_{\begin{pmatrix}1&2&3\\ \alpha_1&\alpha_2&\alpha_2\end{pmatrix}} \sum_{\begin{pmatrix}1&2&3\\ \beta_1&\beta_2&\beta_3\end{pmatrix}} \int \frac{d^3p}{2\omega(\underset{\sim}{p})} \delta_4(p''_{\alpha_1} + p''_{\alpha_2} - p'_{\beta_1} - p)\tilde{h}_4(-p''_{\alpha_1},-p''_{\alpha_2},p'_{\beta_1},p)$$
$$\times \tilde{h}_4(-p, -p''_{\alpha_3},p'_{\beta_2},p'_{\beta_3}) + \cdots \qquad ((IV\text{-}34b)$$

This structure for the G_{mn} is reminiscent of the reduction of the unitarity relations attempted in Part G. There we attempted to remove from the unitarity relation the term with a one-particle intermediate state. It is clear that representing the G_{mn} in terms of the $\tilde{h}_n$ is equivalent to defining new amplitudes from the G_{mn}, so that successively the terms with one-particle, two-particle, etc. intermediate states are removed from the unitarity relations for such amplitudes. It is also clear that this reduction is not causal, in the sense of the second cluster property. That is to say, the causal one-particle transfer contributions are not isolated by this reduction. Thus, for example, in Eq. (IV-34b) they reside partly in $\tilde{h}_6$, partly in $\delta^{(+)}\tilde{h}_4\tilde{h}_4$, and so on. The second cluster property requires that, e.g.

$$\lim_{\tau\to\infty} \tau^{3/2} \exp\,[i(\tau/m)p_o\cdot(p_1' - p_1'' - p_2'')]\{\tilde{h}_6(p_1'', -p_2'', -p_3'', p_1', p_2', p_3')$$
$$-i/2\;\delta^{(+)}(p_1'' + p_2'' - p_1'; m)\tilde{h}_4(-p_1'', -p_2'', p_1', p_1'' + p_2'' - p_1')$$
$$\tilde{h}_4(-p_1'' - p_2'' + p_1', -p_3'', p_2', p_3') + \cdots\} = 0 \;. \qquad \text{(IV-35)}$$

These are clearly new conditions on the $\tilde{h}_n$, beyond the conditions given in LSZ II. Because of the difficulty of relating the cluster amplitudes to the $\tilde{h}_n$, we prefer to express these conditions functionally in terms of the functionals $E_n^{\,c}$ introduced in Part I, Section III:

$$E_n^c\{\alpha^\dagger;\alpha\} = \text{connected part of } e^{-\alpha\cdot\alpha^\dagger}\langle \text{vac}|e^{\alpha^\dagger\cdot a}a_n^n e^{\alpha\cdot a^\dagger}|\text{vac}\rangle \quad , \qquad \text{(IV-36)}$$

$$\lim_{\tau\to\infty} \tau^{3/2}\Big\{E_n^c\{\alpha^\dagger + \beta^\dagger \exp[-i(\tau/m)p_o\cdot p];\ \alpha + \beta \exp[i(\tau/m)p_o\cdot p]\}$$

$$- E_n^c\{\alpha^\dagger;\alpha\} - E_n^c\{\beta^\dagger;\beta\}$$

$$- \int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\beta(\underset{\sim}{q}) \frac{\delta E_n^c\{\alpha^\dagger;\alpha\}}{\delta\alpha(\underset{\sim}{q})}$$

$$- \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\beta^\dagger(\underset{\sim}{q}) \frac{\delta E_n^c\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})}$$

$$- \int d^3\underset{\sim}{q} \exp[i(\tau/m)p_o\cdot q]\alpha^\dagger(\underset{\sim}{q}) \frac{\delta E_n^c\{\beta^\dagger\beta\}}{\delta\beta^\dagger(\underset{\sim}{q})} - \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q]\alpha(\underset{\sim}{q}) \frac{\delta E_n^c\{\alpha^\dagger\alpha\}}{\delta\beta(\underset{\sim}{q})}$$

$$- \sum_{s=1}^{n-1} \binom{n}{s} \int d^3\underset{\sim}{q} \exp[-i(\tau/m)p_o\cdot q] \frac{\delta E_s^c\{\beta^\dagger;\beta\}}{\delta\beta(\underset{\sim}{q})} \frac{\delta E_{n-s}^c\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{q})}\Big\} = 0 \quad . \qquad \text{(IV-37)}$$

Equation (IV-35) immediately suggests an operator which is unitary, Lorentz invariant and which satisfies the cluster decomposition, but does not satisfy the requirements of the second cluster property. This is

$$S'' = \exp(i\eta_{22}) \quad , \tag{IV-38}$$

$$\eta_{22} = \frac{1}{4}\int d^3\underset{\sim}{p}_1'' d^3\underset{\sim}{p}_2'' d^3\underset{\sim}{p}_1' d^3\underset{\sim}{p}_2' a^\dagger(\underset{\sim}{p}_1'')a^\dagger(\underset{\sim}{p}_2'')a(\underset{\sim}{p}_1')(\underset{\sim}{p}_2')$$

$$\times [2\omega(\underset{\sim}{p}_1'')2\omega(\underset{\sim}{p}_2'')2\omega(\underset{\sim}{p}_1')2\omega(\underset{\sim}{p}_2')]^{-1/2}\delta_4(p_1'' + p_2'' - p_1' - p_2')$$

$$\times \tilde{h}_4(-p_1'', -p_2'', p_1', p_2') \quad , \tag{IV-39}$$

which h_4 satisfies all the properties listed in Appendix D. In fact, the operator

$$S^* = \exp\left(i \sum_{m=0}^{M} \sum_{n=0}^{N} \eta_{mn}\right) \quad ,$$

for any finite M and N greater than zero, violates the second cluster property. Such operators do not give correctly the causal properties of the one-particle transfer processes. The existence of such operators proves conclusively the independence of the second cluster property from unitarity, Lorentz invariance and the first cluster property.

V. CONCLUSION

We wanted to formulate very general conditions on the S matrix; conditions which arise from the short-range nature of interactions and, therefore, conditions which must be satisfied in any realistic theory. This we did in Eqs. (III-7) and (IV-16), which are limiting equations, in the sense appropriate to tempered distributions, involving plane-wave S-matrix elements. We point to these equations and the structure they imply for the S matrix as the results of our study.

Of course, no completely consistent, non-trivial example of an S matrix in closed form has yet been given. Nevertheless, it seemed reasonable to see if these conditions were satisfied in present-day theories of elementary-particle interactions. We checked, in particular, the Feynman perturbation theory and the LSZ formalism and found that they satisfy these conditions.

This is certainly not very surprising, of course. Without going deeply into the matter, we could say that the cluster properties depend on the "locality," i.e., microcausality, of the field operators in both theories. It would then be interesting to see how these properties could be formulated in nonlocal field theories.

We should note that it is important, for macroscopic causality, that the operators of interest to us in both of these theories be time-ordered in the conventional sense. If the time-ordering in these operators were different, the causality properties would be wrong. (This is seen, for instance, in the counter-example given in Section IV, in which there is no time-ordering at all.)

As to what further cluster properties should be formulated, perhaps these two theories could be investigated for indications. It is hardly necessary to point out that we have made extremely weak assumptions. Certainly, stronger assumptions could be made about the short-range nature of the interactions. (As an example, we could specify how fast an S-matrix element vanishes when a subset of particles are given a large space-like translation.)

In closing, we should like to express our conviction that it is a meaningful task to find further cluster properties which any realistic theory of elementary-particle interactions must satisfy.

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APPENDICES

A. Some Asymptotic Limits

In this appendix, we wish to state some well-known asymptotic limits and to discuss their applicability to the present study.

We will be concerned first with integrals of the form

$$\mathcal{F}(z) = \int d^3\underset{\sim}{p}\; e^{-ip\cdot z} f(\underset{\sim}{p}) \quad , \qquad p_4 = \omega(\underset{\sim}{p}) \quad , \tag{A-1}$$

where $f(\underset{\sim}{p})$ is a continuous, square-integrable function of $\underset{\sim}{p}$, and where one or more components of the four-vector z tends to infinity. In general, $\mathcal{F}(z)$ tends to zero in such limits.

For example, when the four-vector z is space-like or on the light cone, the Riemann-Lebesque lemma applies and we have

$$\lim_{|z|\to\infty} \mathcal{F}(z) = 0 \quad , \qquad z \cdot z \lesssim 0 \tag{A-2}$$

Of course, it is meaningful to inquire how fast $\mathcal{F}(z)$ goes to zero as $|z|$ tends to infinity. For instance, if $f(\underset{\sim}{p})$ belongs to the class of testing functions $\mathcal{S}(P^3)$ defined in Section II, Part B, then, for z space-like, $\mathcal{F}(z)$ tends to zero faster than any power of $|z|$. In general, however, we do not take up this question for the case of space-like z.

We do need to know the asymptotic behavior of $\mathcal{F}(z)$ when z tends to infinity in a time-like direction. In this case, the integral can be evaluated in the asymptotic limit by the method of stationary phase.[26] For convenience, we parametrize z in the following way:

$$z = (\tau/m)p_o \; ; \quad \underset{\sim}{p}_o = m\underset{\sim}{v}/(1 - \underset{\sim}{v}^2)^{1/2} \quad p_{o4} = \omega(\underset{\sim}{p}_o) \quad , \tag{A-3}$$

where $\tau > 0$, $|\underset{\sim}{v}| < 1$, so that $z \cdot z = \tau^2 > 0$. Equation (A-1) is, in these terms,

$$\mathcal{F}((\tau/m)p_o) = \int d^3\underset{\sim}{p}f(\underset{\sim}{p}) \exp [-i(\tau/(1 - v^2)^{1/2})(\omega(\underset{\sim}{p}) - \underset{\sim}{p} \cdot \underset{\sim}{v})] \quad . \tag{A-4}$$

The intuitive argument is that when τ tends to infinity, the main contribution to the integral comes from the neighborhood of the point at which the phase of the integrand is stationary. In our case, this point is determined by $\nabla_{\underset{\sim}{p}}\omega(\underset{\sim}{p}) = \underset{\sim}{v}$ and is therefore $\underset{\sim}{p} = \underset{\sim}{p}_o$. Contributions from other values of the variable tend to vanish because of the rapid oscillation of the phase. Generalizing the well-known result in one dimension[26] to our three-dimensional case, we obtain

$$\lim_{\tau\to+\infty} \tau^{3/2}e^{im\tau}\mathcal{F}((\tau/m)p_o) = (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m]e^{-3\pi i/4}f(\underset{\sim}{p}_o) \quad . \tag{A-5}$$

This equation may be expressed symbolically as

$$\lim_{\tau\to\infty} \tau^{3/2} \exp [-i(\tau/m)p_o\cdot(p - p_o)] = (2\pi m)^{3/2}[\omega(\underset{\sim}{p}_o)/m]e^{-3\pi i/4}\delta_3(\underset{\sim}{p} - \underset{\sim}{p}_o) \; . \tag{A-6}$$

The last equation is a reminder that

$$\tau^{3/2} \exp [i(\tau/m)p_o\cdot(p - p_o)]$$

must be treated as a distribution in the variable $\underset{\sim}{p}$. Thus, if $f(\underset{\sim}{p})$ belongs to the space $\mathcal{S}(P^3)$ of one-particle testing functions, the function $\mathcal{F}(z)$ and the various limits indicated above are well-defined.

However, if the function $f(\underset{\sim}{p})$ is a plane-wave S-matrix element, for example $S_{m+1,n}(\underset{\sim}{p},\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}'';\underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}')$, it is not clear that the product of the two distributions, with the same variable, has any meaning. We will be able to use these limits because, in our applications, $S_{m+1,n}$ never stands alone, but always in an integral of the form

$$\int d^3\underset{\sim}{p}_1{}'' \cdots d^3\underset{\sim}{p}_m{}'' d^3\underset{\sim}{p}_1{}' \cdots d^3\underset{\sim}{p}_n{}' \phi_f^*(\underset{\sim}{p}_1{}'' \cdots \underset{\sim}{p}_m{}'')$$

$$\times S_{m+1,n}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}'', \underset{\sim}{p}; \underset{\sim}{p}'_1, \cdots, \underset{\sim}{p}_n{}')\phi_i(\underset{\sim}{p}_1{}' \cdots \underset{\sim}{p}_n{}')$$

which we assume to be a continuous, square-integrable function of $\underset{\sim}{p}$.

In the discussion of examples of the second cluster property, it is necessary to have the limiting form of the following distribution in the four-vector variable p:

$$\tau^{3/2} \exp\,[-i\tau(p_o/m)\cdot(p - p_o)] \lim_{\varepsilon\to 0+} [p^2 - m^2 + i\varepsilon]^{-1} \quad .$$

The Feynman denominator may be partially fractionated as follows:

$$[p^2 - m^2 + i\varepsilon]^{-1} = [2\omega(p)]^{-1}\{[p_4 - \omega(p) + i\varepsilon]^{-1} - [p_4 + \omega(p) - i\varepsilon]^{-1}\} \; . \tag{A-7}$$

Using the well-known limits

$$\lim_{\tau\to+\infty} e^{-i\omega\tau} \lim_{\varepsilon\to 0+} (\omega - i\varepsilon)^{-1} = 0$$

$$\lim_{\tau\to+\infty} e^{-i\omega\tau} \lim_{\varepsilon\to 0+} (\omega + i\varepsilon)^{-1} = -2\pi i\delta(\omega) \; , \tag{A-8}$$

and the three-dimensional stationary phase result, Eq. (A-6), we obtain the (symbolic) limits:

$$\lim_{\tau\to\pm\infty} \tau^{3/2} \exp\,[\pm i(\tau/m)p_o\cdot(p \pm p_o)\,\lim_{\epsilon\to 0+}\,[p^2 - m^2 + i\epsilon]^{-1}$$

$$= (-\pi i/m)(2\pi m)^{3/2} e^{-3\pi i/4}\,\delta_4(p \pm p_o)\quad. \tag{A-9}$$

We repeat for emphasis that Eqs. (A-6) and (A-9) are to be understood in the sense of distributions with respect to continuous, square-integrable testing functions.

B. Functional Formulation

The first and second cluster properties will be expressed as relationships between the distributions $S_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n')$ for all positive integers m and n. There will obviously be an infinite number of such relations. As is well known, the most economical way of summarizing the totality of relations among symmetric functions of any number of variables is by means of a generating functional.[27] Thus we define, for two arbitrary functions of three-momentum, $\alpha(\underset{\sim}{p})$ and $\alpha^\dagger(\underset{\sim}{p})$, a functional

$$\mathcal{S}_{mn}\{\alpha^\dagger;\alpha\} \equiv (m!n!)^{-1}\int d^3\underset{\sim}{p}_1", \cdots, d^3\underset{\sim}{p}_m" d^3\underset{\sim}{p}_1', \cdots, d^3\underset{\sim}{p}_n' \alpha^\dagger(\underset{\sim}{p}_1") \cdots \alpha^\dagger(\underset{\sim}{p}_m")$$

$$\times\ \alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_n') S_{mn}(\underset{\sim}{p}_1", \cdots, \underset{\sim}{p}_m"; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad , \qquad \text{(B-1a)}$$

$$\mathcal{S}\{\alpha^\dagger;\alpha\} = \sum_{m,n=0}^{\infty} \mathcal{S}_{mn}\{\alpha^\dagger;\alpha\} \quad . \qquad \text{(B-1b)}$$

We have had to introduce a functional of two functions here because the S_{mn} are symmetric in the primed and double-primed variables <u>separately</u>. If the functions $\alpha(\underset{\sim}{p})$ and $\alpha^\dagger(\underset{\sim}{p})$ belong to the space $\mathcal{S}(P^3)$ of testing functions, then the functionals have a well-defined meaning. However, this is not necessary, since we are interested only in how relations among the S_{mn} may be formally and concisely expressed.

The distributions S_{mn} may be recovered from the functional $\mathcal{S}\{\alpha^\dagger;\alpha\}$ by functional differentiation. The rules for this operation should be noted briefly here by some examples:

$$\frac{\delta\alpha(\underset{\sim}{p})}{\delta\alpha(\underset{\sim}{p}')} = \delta_3(\underset{\sim}{p} - \underset{\sim}{p}') \quad , \qquad \frac{\delta\alpha^\dagger(\underset{\sim}{p})}{\delta\alpha(\underset{\sim}{p})} = 0 \quad . \tag{B-2a}$$

Denoting $\alpha\cdot\beta = \int d^3\underset{\sim}{p}\,\alpha(\underset{\sim}{p})\beta(\underset{\sim}{p})$ for arbitrary functions $\alpha(\underset{\sim}{p})$, $\beta(\underset{\sim}{p})$, one has therefore

$$\frac{\delta\alpha\cdot\beta}{\delta\alpha(\underset{\sim}{p}')} = \beta(\underset{\sim}{p}') \qquad \frac{\delta \exp(\alpha\cdot\beta)}{\delta\alpha(\underset{\sim}{p}')} = \beta(\underset{\sim}{p}') \exp(\alpha\cdot\beta) \quad . \tag{B-2b}$$

Using Eqs. (B-2a), we thus obtain

$$S_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') = \frac{\delta^m}{\delta\alpha^\dagger(\underset{\sim}{p}_1'') \cdots \delta\alpha^\dagger(\underset{\sim}{p}_m'')} \frac{\delta^n}{\delta\alpha(\underset{\sim}{p}_1') \cdots \delta\alpha(\underset{\sim}{p}_n')} \times \mathcal{S}\{\alpha^\dagger;\alpha\}\Big|_{\alpha^\dagger=\alpha=0} \quad . \tag{B-3}$$

Equation (B-3) is also obvious if we rewrite the definition for the generating functional in the following form

$$\mathcal{S}\{\alpha^\dagger;\alpha\} = \langle \mathrm{vac}| e^{\alpha^\dagger\cdot a} S e^{\alpha\cdot a^\dagger} |\mathrm{vac}\rangle \quad , \tag{B-4}$$

where the "dot" notation has been introduced above:

$$\alpha^\dagger\cdot a = \int d^3\underset{\sim}{p}\,\alpha^\dagger(\underset{\sim}{p})a(\underset{\sim}{p}) \quad , \quad \alpha\cdot a^\dagger = \int d^3\underset{\sim}{p}\,\alpha(\underset{\sim}{p})a^\dagger(\underset{\sim}{p}) \quad . \tag{B-5}$$

The functional formalism is convenient for relating the coefficients of the ordered expansion of an operator in annihilation and creation operators to the plane-wave matrix elements of that operator. Let Q be any operator on the space $\mathcal{H}$. Then Q is uniquely specified by the coefficients Q'_{mn}:

$$Q = \sum_{m,n=0}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n' a^\dagger(\underset{\sim}{p}_1'') \cdots a^\dagger(\underset{\sim}{p}_m'')$$

$$\times a(\underset{\sim}{p}_1') \cdots a(\underset{\sim}{p}_n') Q'_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad . \qquad \text{(B-6)}$$

The generating functional for the plane-wave matrix elements of Q, defined by

$$Q_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \equiv \langle vac | a(\underset{\sim}{p}_1'') \cdots a(\underset{\sim}{p}_m'') Q a^\dagger(\underset{\sim}{p}_1') \cdots a^\dagger(\underset{\sim}{p}_n') | vac \rangle , \qquad \text{(B-7)}$$

is

$$Q\{\alpha^\dagger;\alpha\} \equiv \langle vac | e^{\alpha^\dagger \cdot a} Q e^{\alpha \cdot a^\dagger} | vac \rangle \quad . \qquad \text{(B-8)}$$

Substituting the expansion, Eq. (B-6), into the definition, Eq. (B-8), we obtain, by repeated use of the commutation relations:

$$Q\{\alpha^\dagger;\alpha\} = e^{\alpha \cdot \alpha^\dagger} \sum_{m,n=0}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1' \cdots d^3\underset{\sim}{p}_n'$$

$$\times \alpha^\dagger(\underset{\sim}{p}'') \cdots \alpha^\dagger(\underset{\sim}{p}_m'') \alpha(\underset{\sim}{p}_1') \cdots \alpha(\underset{\sim}{p}_m') Q'_{mn}(\underset{\sim}{p}_1'', \cdots, \underset{\sim}{p}_m''; \underset{\sim}{p}_1', \cdots, \underset{\sim}{p}_n') \quad . \qquad \text{(B-9)}$$

We have used here, for example, such results as

$$e^{\alpha^\dagger \cdot a} a^\dagger(\underset{\sim}{p}) e^{-\alpha^\dagger \cdot a} = a^\dagger(\underset{\sim}{p}) + \alpha^\dagger(\underset{\sim}{p}) \ , \ e^{-\alpha \cdot a^\dagger} a(\underset{\sim}{p}) e^{\alpha \cdot a^\dagger} = a(\underset{\sim}{p}) + \alpha(\underset{\sim}{p}) \quad .$$

Thus the generating functional for the coefficients Q'_{mn} is $e^{-\alpha \cdot \alpha^\dagger} Q\{\alpha^\dagger;\alpha\}$. The coefficients are obtained by functional differentiation:

$$Q'_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}''; \underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}') = \frac{\delta^m}{\delta\alpha^\dagger(\underset{\sim}{p}_1{}'') \cdots \delta\alpha^\dagger(\underset{\sim}{p}_m{}'')} \frac{\delta^n}{\delta\alpha(\underset{\sim}{p}_1{}') \cdots \delta\alpha(\underset{\sim}{p}_n{}')} \cdot$$

$$\times \Bigg\{ e^{-\alpha\cdot\alpha^\dagger} \sum_{j,k=0}^{\infty} (j!k!)^{-1} \int d^3\underset{\sim}{q}_1{}'' \cdots d^3\underset{\sim}{q}_j{}'' d^3\underset{\sim}{q}_1{}' \cdots d^3\underset{\sim}{q}_k{}' \alpha^\dagger(\underset{\sim}{q}_1{}'') \cdots \alpha^\dagger(\underset{\sim}{q}_j{}'')$$

$$\times \alpha(\underset{\sim}{q}_1{}') \cdots \alpha(\underset{\sim}{q}_k{}') Q_{jk}(\underset{\sim}{q}_1{}'', \cdots, \underset{\sim}{q}_j{}''; \underset{\sim}{q}_1{}', \cdots, \underset{\sim}{q}_k{}') \Bigg| \Bigg\}_{\alpha=\alpha^\dagger=0} \quad . \tag{B-11}$$

It is not particularly illuminating to carry out the differentiation on the right-hand side of Eq. (B-11). It is sufficient to notice that one term will be Q_{mn} and all the other terms will be products of one or more three-momentum δ-functions, each involving one primed and one double-primed variable, with one plane-wave matrix element Q_{kj} with $k < m$ and $j < n$. It is much more economical to state that given the generating functional $Q\{\alpha^\dagger;\alpha\}$ of the plane-wave matrix elements, Q_{mn}, the generating functional of the ordered-expansion coefficients Q'_{mn} is $e^{-\alpha\cdot\alpha^\dagger} Q\{\alpha^\dagger;\alpha\}$. Further, one sees from Eq. (B-6), that given the generating functional of the expansion coefficients, the operator itself is trivially recovered by making the substitutions: $\alpha^\dagger(\underset{\sim}{p}) \to a^\dagger(\underset{\sim}{p})$, $\alpha(\underset{\sim}{p}) \to a(\underset{\sim}{p})$, preserving the order given.

The last remark may be formalized in the following way. We define a linear mapping η which carries out this substitution. Let $P\{\alpha^\dagger;\alpha\}$ be any formal power-series functional of the functions $\alpha(\underset{\sim}{p})$ and $\alpha^\dagger(\underset{\sim}{p})$. The linear mapping η of the set of all such formal power-series functionals into the set of all formal power series operating on the Hilbert space is then defined by

$$\mathcal{N}(1) = 1 \quad , \tag{B-12a}$$

$$\mathcal{N}(c_1 P_1 + c_2 P_2) = c_1 \mathcal{N}(P_1) + c_2 \mathcal{N}(P_2) \quad , \tag{B-12b}$$

$$\mathcal{N}\left(\left[\prod_{r=1}^{m} \alpha^\dagger(\underset{\sim}{q}_r)\right]\left[\prod_{s=1}^{n} \alpha(\underset{\sim}{p}_s)\right]\right) = \left[\prod_{r=1}^{m} a^\dagger(\underset{\sim}{q}_r)\right]\left[\prod_{s=1}^{n} a(\underset{\sim}{p}_s)\right] \quad , \tag{B-12c}$$

where c_1 and c_2 are any two complex numbers, and P_1 and P_2 are any two power-series functionals. Thus it is seen from Eqs. (B-6) and (B-9) that for any operator Q whose action is on $\mathcal{H}$,

$$Q = \mathcal{N}(e^{-\alpha\cdot\alpha^\dagger} Q\{\alpha^\dagger;\alpha\}) = \mathcal{N}\left(e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a} Q e^{\alpha\cdot a^\dagger}|vac\rangle\right) \quad . \tag{B-13}$$

The mapping $\mathcal{N}$ defines a normal-ordered product unambiguously. If any operator Q is put into Eq. (B-13), it will come out in a normal-ordered expansion, with correct account taken of the commutation relations.

The generating functional for the ordered-expansion coefficients may also be given in terms of the vacuum expectation values of repeated commutators of the operator with annihilation and creation operators. For this purpose, we define for any two operators A and B:

$$[A, B]_0 = B \quad , \qquad [A, B]_1 = [A, B] \quad , \tag{B-14a}$$

$$[A, B]_n = [A, [A, B]_{n-1}] \quad . \tag{B-14b}$$

Then

$$e^A B e^{-A} = \sum_{n=0}^{\infty} (n!)^{-1} [A, B]_n \quad . \tag{B-15}$$

Likewise,

$$e^{A}e^{B}e^{-A} = \exp\left[\sum_{n=0}^{\infty} (n!)^{-1}[A, B]_n\right] . \qquad \text{(B-16)}$$

Using these identities, one arrives at, after some simple rearrangements:

$$Q = \mathcal{N}\left(e^{-\alpha\cdot\alpha^\dagger}\langle \text{vac}|e^{\alpha^\dagger\cdot a}Qe^{\alpha\cdot a^\dagger}|\text{vac}\rangle\right)$$

$$= \mathcal{N}\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m!n!)^{-1}\langle \text{vac}|[-\alpha\cdot a^\dagger,[\alpha^\dagger\cdot a,Q]_n]_m|\text{vac}\rangle\right) \qquad \text{(B-17)}$$

which is the desired relationship. Also

$$Q'_{mn}(\underset{\sim}{p}_1{}'', \cdots, \underset{\sim}{p}_m{}'';\underset{\sim}{p}_1{}', \cdots, \underset{\sim}{p}_n{}')$$

$$= \langle \text{vac}|[-a^\dagger(\underset{\sim}{p}_1{}'), \cdots [-a^\dagger(\underset{\sim}{p}_n{}'),[a(\underset{\sim}{p}_1{}''], \cdots [a(\underset{\sim}{p}_m{}''),Q] \cdots]] \cdots]|\text{vac}\rangle . \qquad \text{(B-18)}$$

Of course the combinatorial statements expressed above are elementary and there is no real need for the functional formalism beyond the economy in notation it offers. In the above paragraphs, we wished to emphasize the relationship between the ordered-expansion coefficients of an operator and its plane-wave matrix elements.

The generating functional for the expansion coefficients of the product of two operators P and Q , i.e.,

$$(PQ)'\{\alpha^\dagger;\alpha\} \equiv e^{-\alpha\cdot\alpha^\dagger}\langle \text{vac}|e^{\alpha^\dagger\cdot a}PQe^{\alpha\cdot a^\dagger}|\text{vac}\rangle \qquad \text{(B-19)}$$

can be obtained in terms of the corresponding functionals for P and Q , i.e.,

$$P'\{\alpha^\dagger;\alpha\} \equiv e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}Pe^{\alpha\cdot a^\dagger}|vac\rangle \quad , \tag{B-20a}$$

$$Q'\{\alpha^\dagger;\alpha\} \equiv e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}Qe^{\alpha\cdot a^\dagger}|vac\rangle \quad , \tag{B-20b}$$

A simple rearrangement of Eq. (B-19), using Eq. (B-16) yields

$$(PQ)'\{\alpha^\dagger;\alpha\} = \langle vac|e^{\alpha^\dagger\cdot a}Pe^{-\alpha^\dagger\cdot a}e^{\alpha\cdot a^\dagger}e^{\alpha^\dagger\cdot a}e^{-\alpha\cdot a^\dagger}Qe^{\alpha\cdot a^\dagger}|vac\rangle \quad .$$

Inserting on the right-hand side of this equation the plane-wave expansion of the identity operator , namely,

$$I = \sum_{n=0}^{\infty}(n!)^{-1}\int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n a^\dagger(\underset{\sim}{p}_1)\cdots a^\dagger(\underset{\sim}{p}_n)|vac\rangle\langle vac|a(\underset{\sim}{p}_1)\cdots a(\underset{\sim}{p}_n) \quad ,$$

we arrive at

$$(PQ)'\{\alpha^\dagger;\alpha\} = \sum_{n=0}^{\infty}(n!)^{-1}\int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n \frac{\delta^n}{\delta\alpha(\underset{\sim}{p}_1)\cdots\delta\alpha(\underset{\sim}{p}_n)} P'\{\alpha^\dagger;\alpha\}$$

$$\times \frac{\delta^n}{\delta\alpha^\dagger(\underset{\sim}{p}_1)\cdots\delta\alpha^\dagger(\underset{\sim}{p}_n)} Q'\{\alpha^\dagger;\alpha\} \quad . \tag{B-21}$$

For a unitary operator U, $U U^\dagger = U^\dagger U = I$, Eq. (B-21) yields

$$\sum_{n=0}^{\infty}(n!)^{-1}\int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n \frac{\delta^n U'\{\alpha^\dagger;\alpha\}}{\delta\alpha(\underset{\sim}{p}_1)\cdots\delta\alpha(\underset{\sim}{p}_n)}\left(\frac{\delta^n U'\{\alpha^*;\alpha^{\dagger *}\}}{\delta\alpha^{\dagger *}(\underset{\sim}{p}_1)\cdots\delta\alpha^{\dagger *}(\underset{\sim}{p}_n)}\right)^* = 1 \quad . \tag{B-22}$$

where $U'\{\alpha^\dagger;\alpha\} = e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}Ue^{\alpha\cdot a^\dagger}|vac\rangle$.

It will prove useful to define the functional <u>product bracket</u>, for two functionals $P'\{\alpha^\dagger;\alpha\}$ and $Q'\{\alpha^\dagger;\alpha\}$

$$\{P',Q'\} \equiv \sum_{n=1}^{\infty} (n!)^{-1} \int d^3\underset{\sim}{p}_1 \cdots d^3\underset{\sim}{p}_n \frac{\delta^n P'\{\alpha^\dagger;\alpha\}}{\delta\alpha(\underset{\sim}{p}_1)\cdots\delta\alpha(\underset{\sim}{p}_n)} \frac{\delta^n Q'\{\alpha^\dagger;\alpha\}}{\delta\alpha^\dagger(\underset{\sim}{p}_1)\cdots\delta\alpha^\dagger(\underset{\sim}{p}_n)} \quad . \tag{B-23}$$

Thus Eq. (B-21) may be rewritten as

$$(PQ')\{\alpha^\dagger;\alpha\} = P'\{\alpha^\dagger;\alpha\}Q'\{\alpha^\dagger;\alpha\} + \left\{P'\{\alpha^\dagger;\alpha\},Q'\{\alpha^\dagger;\alpha\}\right\} \quad . \tag{B-21'}$$

In connection with the product bracket, we define a <u>connected functional</u>. A functional $P'\{\alpha^\dagger;\alpha\}$ is connected if

$$\lim_{|x\cdot x|\to\infty} P'\{\alpha^\dagger + \beta^\dagger e^{ip\cdot x};\ \alpha + \beta e^{-ip\cdot x}\} = P'\{\alpha^\dagger;\alpha\} + P'\{\beta^\dagger;\beta\} \quad . \tag{B-24}$$

This means, of course, that the kernel functions defining the functional contain an overall four-momentum conserving δ-function and no δ-functions conserving a subset of four-momenta. This idea of connectedness is used in the functional expression of the first cluster property.

If P' and Q' are both connected functionals (of $\alpha(\underset{\sim}{p})$ and $\alpha^\dagger(\underset{\sim}{p})$), then the product bracket $\{P',Q'\}$ is clearly also a connected functional since each term in Eq. (B-23) is. Graphically, one can picture the kernels defining P' and Q' as vertices; each functional differentiation is a line joining these vertices. If P', Q', and R' are all connected functionals, the $\{P',Q',R'\}$ is not. Using the differentiation rules, we find that $\{P'Q',R'\} - P'\{Q',R'\} - Q'\{P',R'\}$ <u>is</u> a connected functional, which we denote by $\{P'Q',R'\}_c$. (Generally we will denote the connected part of a functional by the subscript c.)

In this functional, to use graphical language, the vertex representing P' is joined with the vertex representing R' by at least one line, as is the vertex representing Q'.

In general, if the functionals A', B', $\cdots$ P', Q', $\cdots$ are all connected functionals, the functional product bracket $\{A'B' \cdots, P'Q' \cdots\}$ can be expanded as a sum of products of connected functionals. Examples of this decomposition are given in Section III, Part F, and in Appendix D.

A well-known example of the use of the functional formalism described here will be used extensively in Section III, so we mention it here for reference. This is the so-called cluster expansion, first introduced in classical statistical mechanics by Ursell.[28] Suppose we are given two sets of functions $\tau_n(x_1, \cdots, x_n)$ and $\eta_n(x_1, \cdots, x_n)$, symmetric in the n variables, where n runs over all the positive integers, which are related in the following way:

$$\tau_1(x) = \eta_1(x) \quad , \quad \tau_2(x_1,x_2) = \eta_2(x_1,x_2) + \eta_1(x_1)\eta_1(x_2) \quad ,$$

$$\tau_3(x_1,x_2,x_3) = \eta_3(x_1,x_2,x_3) + \eta_1(x_1)\eta_2(x_2,x_3) + \eta_1(x_2)\eta_2(x_1,x_3) + \eta_1(x_3)\eta_2(x_1,x_2) + \eta_1(x_1)\eta_1(x_2)\eta_1(x_3)$$

or in general:

$$\tau_n(x_1, \cdots, x_n) = \eta_n(x_1, \cdots, x_n) + \sum_{\text{part.}} \eta_i(x_{i1},x_{i2}, \cdots)\eta_j(x_{j1},x_{j2}, \cdots)\cdots \qquad \text{(B-25)}$$

where the sum $\sum_{\text{part.}}$ is taken over all possible partitions of the indices $1, \cdots, n$ into distinct classes $i_1, i_2, \cdots; j_1, j_2, \cdots; \cdots$. Defining the generation functionals (with $\tau_0 = 1$)

$$T\{\alpha\} \equiv \sum_{n=0}^{\infty} (n!)^{-1} \int dx_1 \cdots dx_n \alpha(x_1) \cdots \alpha(x_n) \tau_n(x_1, \cdots, x_n) \qquad \text{(B-26a)}$$

and

$$E\{\alpha\} \equiv \sum_{n=1}^{\infty} (n!)^{-1} \int dx_1 \cdots dx_n \alpha(x_1) \cdots \alpha(x_n) \eta_n(x_1, \cdots, x_n) \quad , \qquad \text{(B-26b)}$$

we obtain from the totality of Eqs. (B-25), the result:[29]

$$T\{\alpha\} = \exp \left(E\{\alpha\} \right) \quad . \qquad \text{(B-27)}$$

Our preceeding discussion was limited to the case in which the Hilbert space was the one appropriate to the description of non-interacting neutral scalar bosons. These ideas can be easily generalized for a Hilbert space $\mathcal{H}'$ for an arbitrary number of different kinds of elementary particles satisfying Bose or Fermi statistics and with arbitrarily complicated internal degrees of freedom. The rule is to associate with each boson plane-wave creation operator $a_i^\dagger(\underset{\sim}{p})$, where the index i denotes the helicity state and the mass and spin of the particle as well as internal quantum numbers, a function $\alpha_i^\dagger(\underset{\sim}{p})$ and for each annihilation operator $a_i(\underset{\sim}{p})$ a function $\alpha_i(\underset{\sim}{p})$. With each fermion creation and annihilation operator $b_i^\dagger(\underset{\sim}{p})$ and $b_i(\underset{\sim}{p})$, we associate objects $\beta_i^\dagger(\underset{\sim}{p})$ and $\beta_i(\underset{\sim}{p})$ respectively, which anticommute with all fermion annihilation and creation operators and with all other other $\beta_j^\dagger(\underset{\sim}{p})$, $\beta_j(\underset{\sim}{p})$, but commute with boson operators and c-numbers. Then we generalize the "dot" notation in the following way:

$$\alpha\cdot a^\dagger \equiv \sum_i \int d^3\underset{\sim}{p}\alpha_i(\underset{\sim}{p})a_i^\dagger(\underset{\sim}{p}) + \sum_j \int d^3\underset{\sim}{p}\beta_j(\underset{\sim}{p})b_j^\dagger(\underset{\sim}{p}) \quad ,$$

$$\alpha^\dagger\cdot a \equiv \sum_i \int d^3\underset{\sim}{p}\alpha_i^\dagger(\underset{\sim}{p})a_i(\underset{\sim}{p}) + \sum_j \int d^3\underset{\sim}{p}\beta_j^\dagger(\underset{\sim}{p})b_j(\underset{\sim}{p}) \quad . \qquad \text{(B-5')}$$

Similarly we generalize the η operation as follows: the linear mapping η' of the set of all formal power-series functionals of the $\alpha_i(\underset{\sim}{p})$, $\alpha_i^\dagger(\underset{\sim}{p})$, $\beta_j(\underset{\sim}{p})$, $\beta_j^\dagger(\underset{\sim}{p})$ onto the set of all formal power series of annihilation and creation operators acting on $\mathcal{H}'$ is defined by:

$$\eta'(1) = 1 \quad , \qquad \text{(B-12a')}$$

$$\eta'\left(c_1P_1 + c_2P_2\right) = c_1\eta'\left(P_1\right) + c_2\eta'\left(P_2\right) \quad , \qquad \text{(B-12b')}$$

$$\eta'\left(\left[\prod_{r=1}^{k} \alpha_{i_r}^\dagger(\underset{\sim}{p}_r'')\right]\left[\prod_{s=1}^{\ell} \beta_{i_s}^\dagger(\underset{\sim}{q}_s'')\right]\left[\prod_{t=1}^{m} \alpha_{i_t}(\underset{\sim}{p}_t')\right]\left[\prod_{u=1}^{n} \beta_{i_u}(\underset{\sim}{q}_u')\right]\right)$$

$$= \left[\prod_{r=1}^{k} a_{i_r}^\dagger(\underset{\sim}{p}_r'')\right]\left[\prod_{s=1}^{\ell} b_{i_s}^\dagger(\underset{\sim}{q}_s'')\right]\left[\prod_{t=1}^{m} a_{i_t}(\underset{\sim}{p}_t')\right]\left[\prod_{u=1}^{n} b_{i_u}(\underset{\sim}{q}_u')\right] \quad . \qquad \text{(B-12c')}$$

Application of the commutation and anticommutation relations then gives the result, that, for any operator Q,

$$Q = \eta'\left(e^{-\alpha^\dagger\cdot\alpha}\langle vac|e^{\alpha^\dagger\cdot a}Qe^{a^\dagger\cdot\alpha}|vac\rangle\right) \quad , \qquad \text{(B-13')}$$

or equivalently,

$$Q = \mathcal{N}' \left(\sum_{m,n=0}^{\infty} (m!n!)^{-1} \langle \text{vac} | [-a^\dagger \cdot \alpha, [\alpha^\dagger \cdot a, Q]_n]_m | \text{vac} \rangle \right) \quad . \qquad \text{(B-17')}$$

In Eqs. (B-13') and (B-17') the order of the products is significant due to the presence of the fermions. Again we emphasize the motivation for this development: the totality of relationships among plane-wave matrix elements of an operator in $\mathcal{H}'$ can be concisely summarized in terms of the generating functional

$$Q\{\alpha^\dagger;\alpha\} = \langle \text{vac} | e^{\alpha^\dagger \cdot a} Q e^{a^\dagger \cdot \alpha} | \text{vac} \rangle \quad .$$

The operator itself can be recovered from the generating functional through Eq. (B-13').

C. LSZ Formalism

We review briefly in this Appendix the main features of a theory of interactions first formulated by Lehmann, Symanzik, and Zimmermann, and popularly known as the LSZ formalism.[10] In particular, we wish to consider the relationship between the S matrix and the τ-functions, defined by

$$\tau(x_1, \cdots, x_n) \equiv \langle \text{vac} | T\{\phi(x_1) \cdots \phi(x_n)\} | \text{vac} \rangle \quad \ldots \tag{C-1}$$

The conventional time-ordering symbol is defined by Eq. (III-29). The operator $\phi(x)$ is a scalar field which satisfies the condition of microcausality and the "asymptotic condition," i.e.,

$$U(M,z)\phi(x)U^{-1}(M,z) = \phi(Mx + z) \quad , \tag{C-2}$$

$$[\phi(x),\phi(y)] = 0 \quad \text{when} \quad (x - y)^2 < 0 \quad , \tag{C-3}$$

$$\lim_{x_o \to \mp\infty} \langle \Phi | \phi^\alpha(x_o) | \psi \rangle = \langle \Phi | \phi^\alpha_{\substack{in \\ out}} | \psi \rangle \quad , \tag{C-4}$$

where Φ and ψ are any two normalizable state vectors and the ϕ^α notation will be defined below. Equation (C-4), the asymptotic condition, introduces the in- and out-fields, which satisfy

$$K_x \phi_{\substack{in \\ out}}(x) \equiv (\partial_\mu \partial^\mu + m^2)\phi_{\substack{in \\ out}}(x) = 0 \quad , \tag{C-5}$$

$$[\phi_{\substack{in \\ out}}(x), \ \phi_{\substack{in \\ out}}(y)] = i\Delta(x - y) \quad . \tag{C-6}$$

Thus $\phi_{in}(x)$, $\phi_{out}(x)$ are free-field operators and can be expanded in terms of plane-wave annihilation and creation operators in the usual way:

$$\phi_{\substack{in\\out}}(\underset{\sim}{x}) = (2\pi)^{-3/2}\int d^3\underset{\sim}{p}[2\omega(\underset{\sim}{p})]^{-1/2}\,\{a^{\dagger}_{\substack{in\\out}}(\underset{\sim}{p})e^{ip\cdot x} + a_{\substack{in\\out}}(\underset{\sim}{p})e^{-ip\cdot x}\} \tag{C-7}$$

where the operators $a^{\dagger}_{in}(\underset{\sim}{p})$, $a_{in}(\underset{\sim}{p})$ satisfy the usual commutation relations, Eq. (II-1), as do $a^{\dagger}_{out}(\underset{\sim}{p})$, $a_{out}(\underset{\sim}{p})$. Furthermore, we introduce a complete, orthonormal set of solutions of the Klein-Gordon equation, $f_\alpha(x)$

$$K_x f_\alpha(x) = 0 \quad , \tag{C-8}$$

$$-i\int f_\alpha(x)\overleftrightarrow{\partial} f_\beta^{*}(x)d^3\underset{\sim}{x} \equiv -i\int\left\{f_\alpha(x)\frac{\partial f_\beta^{*}(x)}{\partial x_o} - f_\beta^{*}(x)\frac{\partial f_\alpha(x)}{\partial x_o}\right\}d^3\underset{\sim}{x}$$

$$= \delta_{\alpha\beta}; \quad \sum_\alpha f_\alpha(x)f_\alpha^{*}(x') = i\Delta^{(+)}(x-x') \quad . \tag{C-10}$$

The operators $\phi^\alpha(x)$ are defined by

$$\phi^\alpha(x_o) = -i\int f_\alpha(x)\overleftrightarrow{\partial}\phi(x)d^3\underset{\sim}{x} \quad . \tag{C-11}$$

Because the in- and out-fields satisfy the Klein-Gordon equation, ϕ^α_{in} and ϕ^α_{out} are time-independent. Furthermore, because of translational invariance, $\phi^\alpha(x_o)|vac\rangle$ is time-independent, so that

$$\phi^\alpha_{in}|vac\rangle = \phi^\alpha_{out}|vac\rangle \quad . \tag{C-12}$$

The S matrix in this formalism is the unitary operator which maps out-states into in-states

$$\phi_{in}(x) = S\phi_{out}(x)S^{-1} \quad . \tag{C-13}$$

From the asymptotic condition, Eq. (C-4), follows the so-called reduction formulae:

$$[ST\{\phi(x_1)\cdots\phi(x_n)\},\ a^{\dagger}_{in}(\underset{\sim}{k})] = i(2\pi)^{-3/2}[2\omega(\underset{\sim}{k})]^{-1/2}$$

$$\times\int d^4z e^{-ik\cdot z}K_z ST\{\phi(x_1)\cdots\phi(x_n)\phi(z)\}\quad, \tag{C-14a}$$

$$[ST\{\phi(x_1)\cdots\phi(x_n)\},a_{in}(\underset{\sim}{k})] = -i(2\pi)^{-3/2}[2\omega(\underset{\sim}{k})]^{-1/2}$$

$$\times\int d^4z e^{ik\cdot z}K_z ST\{\phi(x_1)\cdots\phi(x_n)\phi(z)\}\quad. \tag{C-14b}$$

From these reduction formulae we will obtain an expression for the scattering functional. For convenience, let us define

$$\varphi^{(+)}(x) \equiv (2\pi)^{-3/2}\int d^3\underset{\sim}{p}[2\omega(\underset{\sim}{p})]^{-1/2}\alpha(\underset{\sim}{p})e^{-ip\cdot x}\quad, \tag{C-15a}$$

$$\varphi^{(-)}(x) \equiv (2\pi)^{-3/2}\int d^3\underset{\sim}{p}[2\omega(\underset{\sim}{p})]^{-1/2}\alpha^{\dagger}(\underset{\sim}{p})e^{ip\cdot x}\quad, \tag{C-15b}$$

$$\varphi(x) \equiv \varphi^{(+)}(x) + \varphi^{(-)}(x)\quad. \tag{C-15c}$$

Then, we introduce a linear mapping $\mathcal{N}$, completely analogous to that defined in Appendix B, Eqs. (B-12), which maps $\alpha(\underset{\sim}{p})$ into $a_{in}(\underset{\sim}{p})$ and $\alpha^{\dagger}(\underset{\sim}{p})$ into $a^{\dagger}_{in}(\underset{\sim}{p})$. Thus, for example, we have

$$\phi_{in}(x) = \mathcal{N}(\varphi(x))\quad. \tag{C-16}$$

We emphasize that this mapping unambiguously defines a normal-ordering. Using the reduction formula, Eq. (C-14b), we have

$$[\alpha^{\dagger}\cdot a,\ ST\{\phi(x_1)\cdots\phi(x_n)\}] = i\int d^4x\varphi^{(-)}(x)K_x ST\{\phi(x_1)\cdots\phi(x_n)\phi(x)\} \tag{C-17}$$

employing the dot notation introduced in Eq. (B-5). In the following we take $a^\dagger(p)$ and $a(p)$ to be in operators, although for the final result we could have used out operators just as well. By iteration, it follows that

$$[\alpha^\dagger \cdot a, S]_n = i^n \int d^4x_1 \cdots d^4x_n \, \phi^{(-)}(x_1) \cdots \varphi^{(-)}(x_n)$$

$$\times \, K_{x_1} \cdots K_{x_n} ST\{\phi(x_1) \cdots \phi(x_n)\} \qquad \text{(C-18)}$$

where $[\;]_n$ is the repeated commutator introduced in Appendix B, Likewise, using Eq. (C-14a), we obtain

$$[-\alpha \cdot a^\dagger, ST\{\phi(x_1) \cdots \phi(x_n)\}] = i \int d^4x \, \varphi^{(+)}(x)$$

$$\times \, K_x ST\{\phi(x_1) \cdots \phi(x_n)\phi(x)\} \quad . \qquad \text{(C-19)}$$

Combining Eqs. (C-18) and (C-19), we finally obtain

$$[-\alpha \cdot a^\dagger, [\alpha^\dagger \cdot a, S]_n]_m$$

$$= i^{m+n} \int d^4x_1 \cdots d^4x_m d^4y_1 \cdots d^4y_n \, \phi^{(+)}(x_1) \cdots \varphi^{(+)}(x_m)\varphi^{(-)}(y_1) \cdots \varphi^{(-)}(y_n)$$

$$\times \, K_{x_1} \cdots K_{x_m} K_{y_1} \cdots K_{y_n} ST\{\phi(x_1) \cdots \phi(x_m)\phi(y_1) \cdots \phi(y_n)\} \quad . \qquad \text{(C-20)}$$

This is precisely the form required to obtain the scattering functional according to Eq. (B-17):

$$F(\alpha^{\dagger};\alpha) \equiv e^{-\alpha\cdot\alpha^{\dagger}}\langle \mathrm{vac}|e^{\alpha^{\dagger}\cdot a}Se^{\alpha\cdot a^{\dagger}}|\mathrm{vac}\rangle = 1 + \sum_{m=2}^{\infty}\sum_{n=2}^{\infty}(m!n!)^{-1}$$

$$\times \langle \mathrm{vac}|[-\alpha\cdot a^{\dagger},[\alpha^{\dagger}\cdot a,S]_n]_m|\mathrm{vac}\rangle = 1 + \sum_{n=2}(n!)^{-1}i^n \; d^4x_1 \cdots d^4x_n$$

$$\times \varphi(x_1) \cdots \varphi(x_n)K_{x_1} \cdots K_{x_n}\tau(x_1, \cdots, x_n) \quad .$$

Thus, with the functional formulation we obtain in a direct way the oft-quoted result:

$$S = I + \sum_{n=2}^{\infty} (n!)^{-1}i^n\int d^4x_1 \cdots d^4x_n \eta(\varphi(x_1) \cdots \varphi(x_n))K_{x_1} \cdots K_{x_n}$$

$$\times \tau(x_1, \cdots, x_n) \quad . \qquad \text{(C-22)}$$

Since $SSS^{-1} = S$, $\eta(\varphi(x_1) \cdots \varphi(x_n))$ may be taken as the normal-ordered product of either the in-fields or the out-fields.

We now inquire whether the operator given by Eq. (C-22) satisfies the requirements for an S operator given in Section II. That this operator is unitary follows from an identity satisfied by the τ-functions.[30] Furthermore, the right-hand side of Eq. (C-22) is manifestly Lorentz invariant. Finally, because of the asymptotic condition and translational invariance, this S operator has the action of the identity on the vacuum and one-particle subspaces. Thus the operator given by Eq. (C-22) is a suitable candidate for the S operator in these respects. In fact, it is shown in Sections III and IV that this operator also satisfies the first and second cluster properties.

D. A Theorem on Connected Functionals

Before proceeding to the theorem on connected functionals which is necessary in Section III, Part I, we briefly review the properties of the hermitian phase matrix of LSZ II.[24] If the S matrix is specified by distributions $\tilde{h}_n(p_1, \cdots, p_n)$, symmetric in the n four-vector variables, in the following way

$$S = \exp(i\eta) \quad , \tag{D-1}$$

$$\eta = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} (m!n!)^{-1} \int d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_m'' d^3\underset{\sim}{p}_1'' \cdots d^3\underset{\sim}{p}_n' a^\dagger(\underset{\sim}{p}_1'') \cdots a^\dagger(\underset{\sim}{p}_m'')$$
$$\times\, a(\underset{\sim}{p}_1') \cdots a(\underset{\sim}{p}_n') \left\{\prod_{i=1}^{m} [2\omega(\underset{\sim}{p}_i'')]^{-1/2}\right\}\left\{\prod_{j=1}^{n} [2\omega(\underset{\sim}{p}_j')]^{-1/2}\right\}$$
$$\times\, \delta_4\left(\sum_{i=1}^{m} p_i'' - \sum_{j=1}^{n} p_j'\right) \tilde{h}_{m+n}(-p_1'', \cdots, -p_m'', p_1', \cdots, p_n') \tag{D-2}$$

then the general properties of the S matrix have the following representation in the $\tilde{h}_n$:

(1) Unitarity: $\tilde{h}_n(p_1, \cdots, p_n) = \tilde{h}_n^*(-p_1, \cdots, -p_n)$. (D-3a)

(2) Lorentz invariance: $\tilde{h}_n(p_1, \cdots, p_n) = \tilde{h}_n(Mp_1, \cdots, Mp_n)$. (D-3b)

(3) Invariance of the vacuum: $\tilde{h}_0 = h_1(p) = 0$. (D-3c)

(4) Invariance of one-particle states: $\tilde{h}_2(p_1,p_2) = 0$. (D-3d)

In addition to these S-matrix properties, which we have discussed in Section II, Lehmann, Symanzik and Zimmermann add: TPC invariance, which requires that the $\tilde{h}_n$ be real; the finiteness of scattering and reaction cross-sections and their generalization in multiparticle processes, which requires that the $\tilde{h}_n$ contain no four-dimensional δ-functions; and that the $\tilde{h}_n$ must be continuous functions of their invariant variables.

Using the theorem of this appendix, it is shown in Section III that the prescription that the $\tilde{h}_n$ contain no four-dimensional δ-functions gives the cluster decomposition. The theorem is this:

If $E_n^{\,c}\{\alpha^\dagger;\alpha\}$ is the connected part of the functional

$$E_n\{\alpha^\dagger;\alpha\} = e^{-\alpha\cdot\alpha^\dagger}\langle vac|e^{\alpha^\dagger\cdot a}n_n e^{\alpha\cdot a^\dagger}|vac\rangle \quad , \tag{D-4}$$

then the relationship between the two sets of functionals is given by the following equation between formal power series

$$\sum_{n=0}^{\infty} (it)^n(n!)^{-1}E_n\{\alpha^\dagger;\alpha\} = \exp\left(\sum_{n=1}^{\infty} (it)^n(n!)^{-1}E_n^{\,c}\{\alpha^\dagger;\alpha\}\right) . \tag{D-5}$$

The proof is as follows: We define generating functions for these functionals,

$$E(t) = \sum_{n=0}^{\infty} (it)^n(n!)^{-1}E_n\{\alpha^\dagger;\alpha\} \quad , \tag{D-6}$$

$$E^c(t) = \sum_{n=1}^{\infty} (it)^n(n!)^{-1}E_n^{\,c}\{\alpha^\dagger;\alpha\} \quad , \tag{D-7}$$

and introduce a new set of functionals by means of the generating function

$$D(t) = \exp E^c(t) = \sum_{n=0}^{\infty} (it)^n(n!)^{-1}D_n\{\alpha^\dagger;\alpha\} \quad . \tag{D-8}$$

Explicitly, we have

$$E_n^c\{\alpha^\dagger;\alpha\} = n! \sum_{\lambda=1}^{n} (-1)^{\lambda+1}(\lambda-1)!$$

$$\times \sum_{\substack{s_1=0 \\ \Sigma s_m=\lambda \\ \Sigma m s_m=n}}^{n} \cdots \sum_{s_n=0}^{n} \prod_{m=1}^{n} (s_m!)^{-1}[(m!)^{-1}D_m\{\alpha^\dagger;\alpha\}]^{s_m} , \qquad \text{(D-9)}$$

$$D_n\{\alpha^\dagger;\alpha\} = n! \sum_{\substack{s_1=0 \\ \Sigma m s_m=n}}^{n} \cdots \sum_{s_n=0}^{n} \prod_{m=1}^{n} (s_m!)^{-1}[(m!)^{-1}E_m^c\{\alpha^\dagger;\alpha\}]^{s_m} . \qquad \text{(D-10)}$$

It is advantageous to use here the functional product bracket introduced in Appendix B. Thus, with the definition, Eq. (D-4), we have

$$E_n\{\alpha^\dagger;\alpha\} = E_r\{\alpha^\dagger;\alpha\}E_{n-r}\{\alpha^\dagger;\alpha\} + \{E_r\{\alpha^\dagger;\alpha\} \ , \ E_{n-r}\{\alpha^\dagger;\alpha\}\} . \qquad \text{(D-11)}$$

Now, as a consequence of the rules of functional differentiation, the following results holds, for the connected functionals $\mathcal{A}_i$, X:

$$\left\{\prod_{i=1}^{m}(\alpha_i)^{s_i}, X\right\} = \left\{\prod_{i=1}^{m}(\alpha_i)^{s_i}, X\right\}_c$$

$$+ \sum_{\substack{\nu_1=0 \\ \Sigma\nu_i \neq 0}}^{s_1} \cdots \sum_{\nu_m=0}^{s_m} \prod_{i=1}^{m}\binom{s_i}{\nu_i} \prod_{j=1}^{m}(\alpha_j)^{\nu_j}\left\{\prod_{k=1}^{m}(\alpha_k)^{s_k-\nu_k}, X\right\}_c ,$$

where the subscript "c" denotes the connected part of the functional. Applying this result to the expansion of Eq. (D-10), we obtain

$$\{D_n,D_1\} = \{D_{n_1} D_1\}_c + \sum_{r=1}^{n-1}\binom{n}{r} D_r\{D_{n-r},D_1\}_c . \tag{D-13}$$

Before we can apply this result to the proof, we need the following lemma

$$D_{\nu+1} = \sum_{r=0}^{\nu}\binom{\nu}{r} D_r E^c_{\nu-r+1} . \tag{D-14}$$

This follows simply by differentiating both sides of the defining relation, Eq. (D-8), with respect to t:

$$\frac{dD(t)}{dt} = D(t)\,\frac{dE^c(t)}{dt} \tag{D-15}$$

and equating coefficients of like powers of t.

We wish to show that $E_n = D_n$ for all n. We proceed inductively. Since E_1 is connected, $E_1 = E_1^{\,c} = D_1$. Now assume $E_m = D_m$ for $m \leq n$. Then, from Eq. (D-11)

$$E_{n+1} = E_n E_1 + \{E_n, E_1\} \quad . \tag{D-16}$$

Using Eq. (D-13), one obtains

$$E_{n+1} = D_n E_1{}^c + \{D_n, E_1{}^c\}_c + \sum_{r=1}^{n-1} \binom{n}{r} D_r \{D_{n-r}, E_1{}^c\}_c \quad .$$

But

$$\{D_{n-r}, E_1{}^c\}_c = \{E_{n-r}, E_1\}_c = E^c_{n-r+1} \quad ,$$

so that

$$E_{n+1} = \sum_{r=0}^{n} \binom{n}{r} D_r E^c_{n-r+1} = D_{n+1} \quad .$$

Hence, $E_n = D_n$ for all positive integers, and Eq. (D-5) follows from the definition, Eq. (D-8).

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FIGURE CAPTIONS

Fig. 1 Diagram corresponding to the cluster amplitude A_{mn}.

Fig. 2 The four diagrams contributing to the matrix element that describes four incident and five outgoing particles.

Fig. 3 Pictorial representation of the second cluster property. It is not to be understood as a Feynman diagram, or any other diagram we discuss. It is included merely to serve as a visual aid to the discussion in Section IV, Part B.

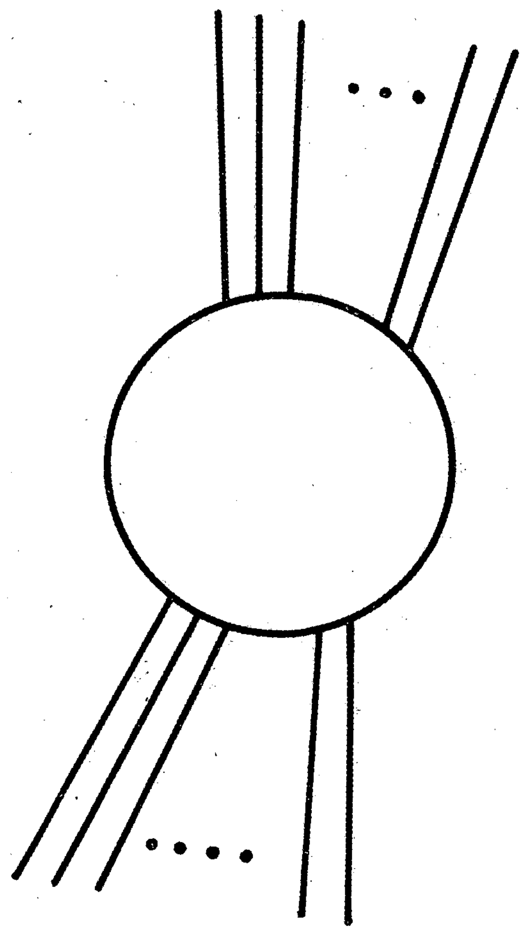

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Fig. 1

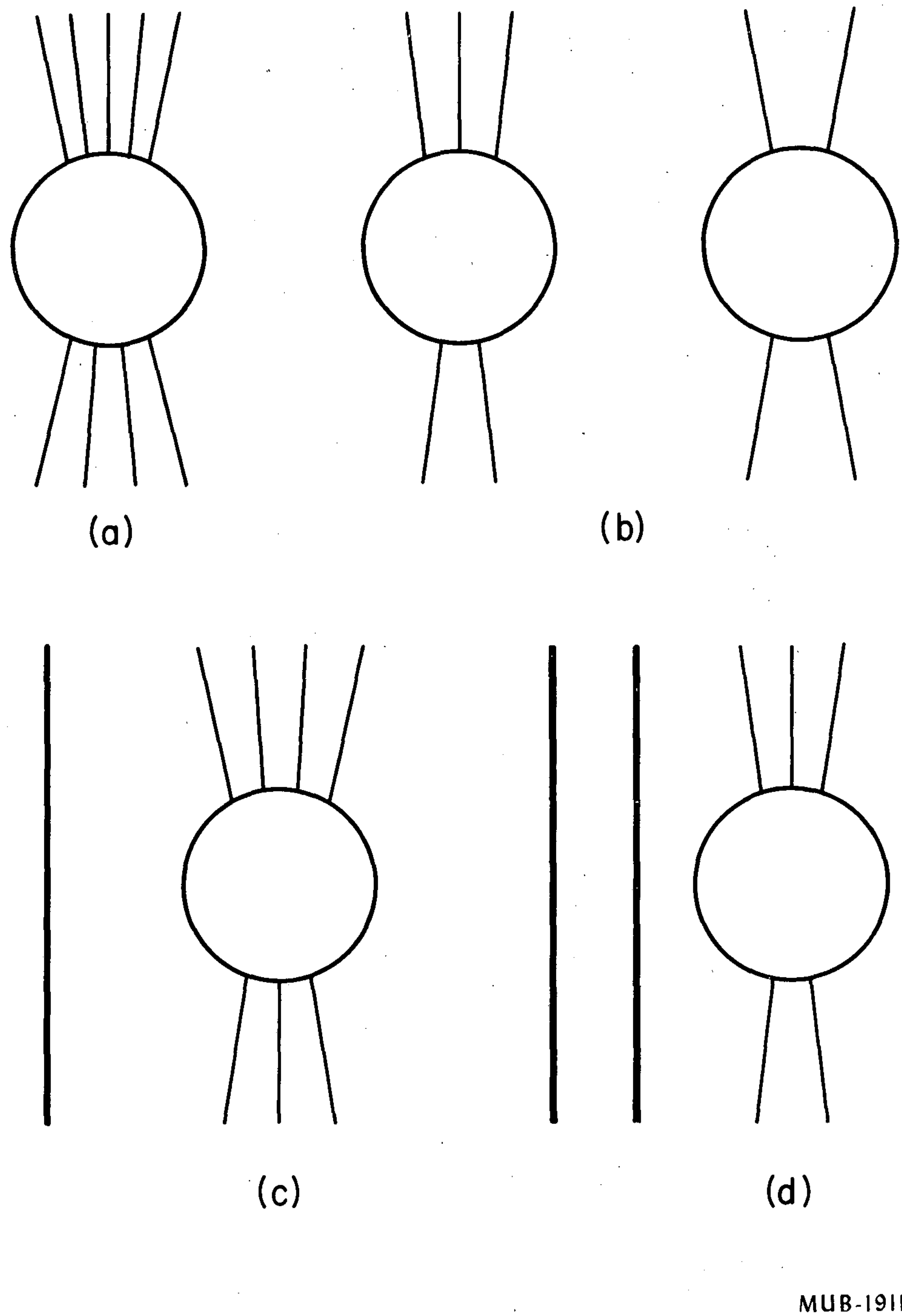

Fig. 2

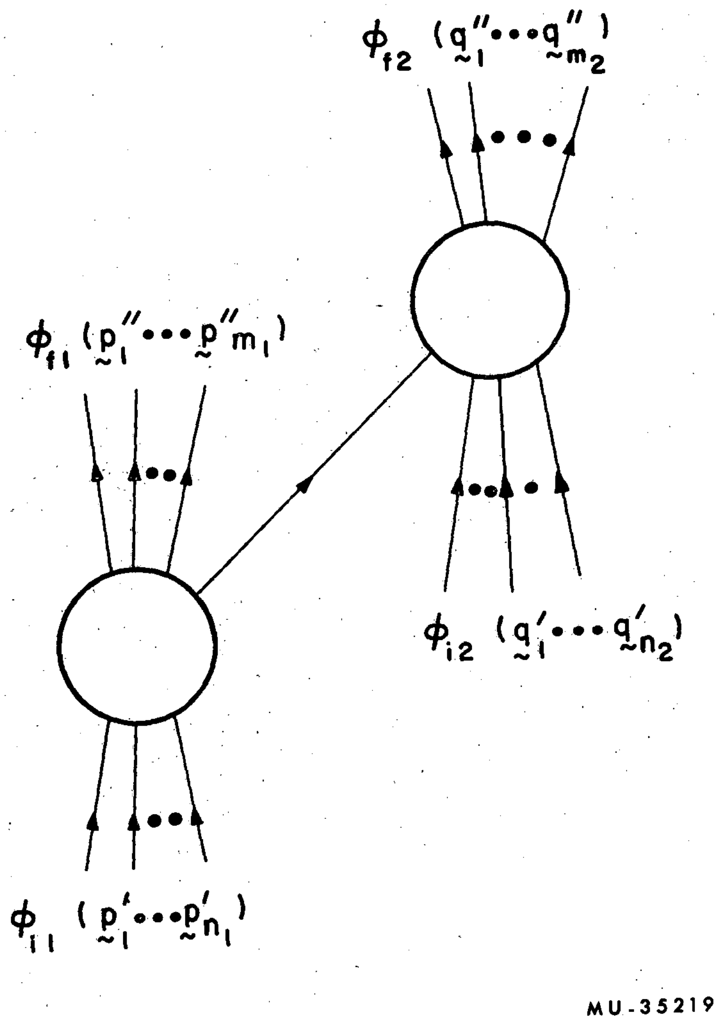
ϕ_{f2} ($q''_1 \cdots q''_{m_2}$)
ϕ_{f1} ($p''_1 \cdots p''_{m_1}$)
ϕ_{i2} ($q'_1 \cdots q'_{n_2}$)
ϕ_{i1} ($p'_1 \cdots p'_{n_1}$)
MU-35219

Fig. 3

0

0