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INTERACTING-STRING PICTURE OF THE NEVEU-SCHWARZ-RAMOND MODEL*

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August 29, 1973

ABSTRACT

The interacting-string picture is extended to the Neveu-Schwarz and Ramond models. The picture is manifestly dual, even for the Ramond model, and amplitudes with an arbitrary number of external fermions can be constructed. In all channels the only singularities are poles corresponding to the expected particles. There exist both $q\bar{q}$ and zero-quark mesons, which differ in the parity of the states with odd Neveu-Schwarz g -parity. Calculations of $f\bar{f}$ scattering amplitudes are performed. The results are B-functions multiplied by simple kinematical factors, and it is easily verified that the amplitudes possess all the necessary properties.

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1. INTRODUCTION

In a previous publication [1], hereafter referred to as I, we have extended the string formalism of Goddard, Goldstone, Rebbi and Thorn [2] to allow for scattering by the joining and separation of strings. Our aim here is to extend the picture to the Neveu-Schwarz and Ramond models. We shall show that the duality properties of the latter model are just as evident as those of the orbital or Neveu-Schwarz models. Amplitudes for scattering of n fermions present no difficulty, and their only singularities, in all channels, are factorizable poles corresponding to the particles of the model. As we work within the physical Hilbert space, all amplitudes are manifestly ghost-free. We have summarized the results in a previous communication [3].

Let us begin by reminding the reader why n -fermion amplitudes presented difficulties in the previous approach. The Neveu-Schwarz and Ramond models both work with anti-commuting operators b_n^μ , but n assumes half-integral values in the Neveu-Schwarz model and integral values in the Ramond model. As long as the main channel remains mesonic or fermionic, one need operate with one type of b -operator only, and all manipulations can be carried through. It is therefore easy to treat processes with no fermions or with only one fermion line. Other processes present difficulty, since the relationship between the two types of b 's is not evident. Thorn [4] was able to prove at considerable effort that amplitudes with one fermion line factorized as expected in the meson channels. Olive and Scherk [5] have used the projected propagator to show that one could sum the amplitude for $f\bar{f}$ scattering over physical states, and that the result was Lorentz

invariant. They did not calculate the amplitude explicitly or analyze the singularities in the t -channel.

The string formalism operates with amplitudes $b(\rho)$, where ρ is the co-ordinate of a point on the string, instead of with the Fourier components b_n . Meson and fermion strings are characterized by the same dynamical variables; the difference between them is characterized by the boundary conditions at the end of the string, as was originally pointed out by Gervais and Sakita [6]. Processes of $f\bar{f}$ annihilation are essentially no more complicated than those of absorption of mesons by other mesons or by fermions.

While n -fermion amplitudes in our model are dual, they are not self-dual. The singularities in every channel are factorizable poles with positive residue, but the spectra of particles in all meson channels are not identical. For $f\bar{f}$ scattering, the s and t channels will be different. The meson channels are most easily characterized by adding quark lines to duality diagrams. Fermions have a single quark line, mesons either two or none. In models with internal symmetry, the quark lines carry quantum numbers in the usual way. We then find that the parity of zero-quark mesons with odd Neveu-Schwarz g -parity is opposite to that of the corresponding $q\bar{q}$ mesons. The parity of the mesons with odd Neveu-Schwarz g -parity is defined only to within an overall sign; this is true even in a model with zero-mass fermions. We arbitrarily define the $q\bar{q}$ "vacuum" to be pseudo-scalar. The zero-quark vacuum will then be scalar.

Schwarz [7] and Thorn [8] have pointed out that fermionic physical-particle creation operators can be constructed only if the fermion "vacuum" has $m^2 = 0$. In fact, the model possesses chiral

symmetry, and all excited fermion states are parity doubled. We emphasize, however, that the parity doubling is not of the trivial type which produces ghosts. The meson states with even NS g -parity are not necessarily parity doubled.

A model of a free string with spin variables has already been constructed by Iwasaki and Kikkawa [9]. Their model appears to be that of an anti-ferromagnetic string consisting of an infinite number of "partons" with spin $\frac{1}{2}$ [10]. Our method will be to extend the Iwasaki-Kikkawa model to interacting strings.

The Neveu-Schwarz string model without longitudinal degrees of freedom is covariant only if d , the number of dimensions, is equal to 10. If we wish to construct a model in fewer dimensions we must start with a ten-dimensional model and then restrict the initial states. By doing so we take into account the longitudinal degrees of freedom which are necessary for Lorentz invariance if $d < 10$. The crossed-loop amplitudes of a model with $d < 10$ possess unphysical Pomeron singularities, and we shall not attempt to treat such amplitudes here.

In sec. 2 we shall introduce the dynamical variables and shall show how the boundary conditions are different for mesons and fermions. We shall also examine the zero-mode operators for fermionic strings. In our non-covariant treatment the zero-mode operators will be Pauli σ -matrices as opposed to the Dirac γ -matrices of Ramond, and the eigenstates of σ^2 will be the helicity states of the fermion.

The functional integral for the scattering amplitude will be set up in sec. 3. In the purely orbital model the functional integrand was similar to that for free strings, but the integral was taken over all possible motions of strings which joined and separated at given times. In our present model such a procedure would not give a Lorentz-

invariant result, and we must add a factor $G(\rho)$ at each point where strings join or separate. The factor is linear in the b -variables, and it is analogous to a similar factor in the vertex of the Neveu-Schwarz model as treated by conventional methods. In sec. 4 we shall show that the functional integral may be evaluated in terms of Neumann functions.

We discuss the duality structure of the model in sec. 5 and, in the final two sections, we perform specific calculations. The pion scattering amplitude will be calculated in a general Lorentz frame, and we shall see how our non-covariant methods lead to the known Lorentz-invariant result, provided $d = 10$. We shall also give the calculation of the $f\bar{f}$ scattering amplitude in order to show that the present scheme can deal with amplitudes which have hitherto eluded treatment in the older formalisms.

2. SPIN VARIABLES AND FERMIONIC ZERO MODES

The independent variables σ , τ , η , ζ of our system, as well as the displacement variables $x^i(\sigma, \tau)$ of the string, are defined the same way as in I. In addition there are two anti-commuting variables $S_1^i(\sigma, \tau)$, $S_2^i(\sigma, \tau)$ defined at each point on the string. The S 's may be expanded in a Fourier series:

$$S_1^i(\sigma, \tau) = e^{i\phi(2\alpha)^{-\frac{1}{2}}} \sum b_k e^{ik\eta + k\zeta}, \quad (2.1a)$$

$$S_2^i(\sigma, \tau) = \pm e^{i\phi(2\alpha)^{-\frac{1}{2}}} \sum b_k e^{-ik\eta + k\zeta}, \quad (2.1b)$$

$$b_k^i = e^{-i\phi(2\alpha)^{\frac{1}{2}}} (2\pi)^{-1} \int_0^\pi d\eta \left\{ S_1^i(\sigma, \tau) e^{-ik\eta} \right. \\ \left. \pm S_2^i(\sigma, \tau) e^{ik\eta} \right\} e^{-k\zeta}. \quad (2.2)$$

The phase factors $e^{i\phi}$ will be specified later (eqs. (2.4)). The variables b_k satisfy anti-commutation relations:

$$(b_k^i, b_{k'}^j) = \delta_{ij} \delta_{k, -k'}.$$

For mesons the mode number k takes half-integral values, whereas for fermions it takes integral values.

The sign in (2.1b) and (2.2) depends on the type of string. In diagrams such as fig. 1, strings with the arrows at both ends going forwards correspond to fermions, those with the arrows at both ends going backwards to anti-fermions, whereas strings with the arrows going in opposite directions correspond to mesons. In sec. 5 we shall identify those mesons with the top arrow directed forward as $q\bar{q}$ mesons, those with the top arrow directed backwards as zero-quark mesons. We use the plus sign in (2.1b) for anti-fermions and $q\bar{q}$ mesons, the minus sign for fermions and zero-quark mesons. Equations (2.1) then lead to the following boundary conditions at the ends of the string:

$$\text{Forward arrow: } S_1 = -S_2 \quad \frac{\partial S_1}{\partial \sigma} = \frac{\partial S_2}{\partial \sigma}, \quad (2.3a)$$

$$\text{Backward arrow: } S_1 = S_2 \quad \frac{\partial S_1}{\partial \sigma} = -\frac{\partial S_2}{\partial \sigma}. \quad (2.3b)$$

The difference between the boundary conditions for mesons and fermions has been pointed out by Gervais and Sakita [6].

Let us now fix the phase factors in (2.1). We remind the reader that, according to the conventions of I, η runs from 0 to π as we go from the bottom to the top of an incoming string or from the top to the bottom of an outgoing string. Also, α is defined to be negative for outgoing strings. We shall take

$$(2\alpha)^{\frac{1}{2}} = i(2|\alpha|)^{\frac{1}{2}} \quad (\text{outgoing strings}). \quad (2.4a)$$

For outgoing mesons the phase factor in (2.4) just cancels the phase factor due to our definition of η . For outgoing fermions, however, (2.4a) gives an extra uncanceled phase factor i . We therefore add an extra phase factor $-i$ for incoming anti-fermions or outgoing fermions. In other words, we set

$$e^{i\phi} = 1 \quad \text{mesons, incoming fermions, outgoing anti-fermions.}$$

$$e^{i\phi} = -1 \quad \text{outgoing fermions, incoming anti-fermions.} \quad (2.4b)$$

The phase factors (2.4a) and (2.4b) combine to give a factor $-i$ for incoming anti-fermions and i for outgoing anti-fermions; such a phase factor is of course permissible.

The variables S have conformal weight $\frac{1}{2}$, i.e., under a conformal transformation they behave as follows:

$$\tilde{S}_1 = \left(\frac{\partial \tilde{\rho}}{\partial \rho} \right)^{-\frac{1}{2}} S_1, \quad S_2 = \left\{ \left(\frac{\partial \tilde{\rho}}{\partial \rho} \right)^* \right\}^{-\frac{1}{2}} S_2. \quad (2.5)$$

As in I, we shall be particularly interested in the transformation of ρ onto a half-plane. We then observe from (2.3) and (2.5) that the relative sign between $\tilde{S}_1(z)$ and $\tilde{S}_2(z)$ at the boundary changes each time the variable z passes through a value Z_r corresponding to $\tau = \pm \infty$ on a fermion or anti-fermion line.

Fermion strings possess a zero mode. In his original formulation of the model, Ramond took the b_0^μ 's to be Dirac matrices, which satisfied the required anti-commutation relations. We shall define generalized Pauli matrices σ^i ($1 \leq i \leq d-2$), which satisfy the relations

$$\sigma^i \sigma^j + \sigma^j \sigma^i = 2 \delta^{ij}. \quad (2.6)$$

Such matrices must have a minimum dimension of $2^{(d-2)/2}$. We then set

$$b_0^i = 2^{-\frac{1}{2}} \sigma^i. \quad (2.7)$$

We may group the σ^i 's into pairs and construct a set of $\frac{1}{2}(d-2)$ commuting σ^2 matrices $-i\sigma^1\sigma^2, -i\sigma^3\sigma^4$ etc. The eigenstates of the σ^2 's will be the basic states of the fermion sector, the fermion vacuum being $2^{(d-2)/2}$ -fold degenerate.

In order to obtain a physical interpretation of our basic states, we investigate how they transform under Lorentz transformations. We find from ref. [9] that the Lorentz operators M^{i-} and M^{i+} , in so far as they affect the modes in question, are

$$M^{i-} = \frac{1P^j}{2P^+} (\sigma^j \sigma^i - \sigma^i \sigma^j), \quad M^{i+} = 0. \quad (2.8)$$

The operators (2.8) are just the operators which transform the helicity states of zero-mass fermions under the transformations in question. We may therefore identify our basic states with the helicity states of the fermions. The phase factors in the helicity states are defined according to the convention of Cohen-Tannoudji, Morel and Navelet [11].

Since the relation between the physical-particle creation operators of our present theory and the covariant operators of the conventional theory is known [7], we can obtain the relation between our σ 's and Ramond's γ 's. In this way we again find that the eigenstates of the matrices σ^2 are the helicity states of the fermions.

In order to treat all modes uniformly, we should like to consider the zero-mode operators as creation or annihilation operators.

We therefore regard positive-helicity fermion states as "empty" and negative-helicity states as "full"; we could equally well have made the opposite choice. The operators $(2)^{-\frac{1}{2}}(b_0^1 + ib_0^2)$, $(2)^{-\frac{1}{2}}(b_0^3 + ib_0^4) \dots$ will then be annihilation operators, while their conjugate complexes will be creation operators. For anti-fermions we regard the negative-helicity states as empty, so that the operators $(2)^{-\frac{1}{2}}(b_0^1 - ib_0^2), \dots$ are the annihilation operators.

3. FUNCTIONAL INTEGRAL AND LORENTZ INVARIANCE

We make the ansatz that the transition amplitude is given by the following functional integral:

$$A = \int dZ_2 \dots dZ_{N-2} H(Z_1, \dots, Z_N) \quad (3.1a)$$

$$H = V(Z_1, \dots, Z_N) \prod_{r=1}^N \prod_{n,1} \int dp_{n,r}^i \int \prod dx^i(\sigma, \tau) dS_1(\sigma, \tau) dS_2(\sigma, \tau) \\ \times \Psi_r(p_{n,r}^i, b_{k,r}^i) \left\{ \exp \left[\sum_{r,1} \alpha_r \int_0^\pi d\eta_r p_r^i(\eta_r) x^i(\sigma, \tau_1) \right. \right. \\ \left. \left. + \int d\sigma d\tau \mathcal{L}(\sigma, \tau) - \tau_1 \sum P_1^- - \tau_f \sum P_f^- \right] \right\} \prod_\alpha G_1(\rho_\alpha + \delta\rho)^\Gamma(Z_1, \dots, Z_N) \quad (3.1b)$$

$$\Gamma(Z_1, \dots, Z_N) = \prod_\alpha \left(\frac{\partial \rho}{\partial Z} \right)^{3/2}_{\rho_\alpha + \delta\rho} \prod_{\alpha < \beta} (z_\alpha - z_\beta)^{-1} \prod_{r < s} (z_r - z_s)^{-1} \prod_{r < s} (z_r - z_s)^2 \\ \times \prod_{\alpha, r} (z_\alpha - z_r) \prod_r (z_r - z_{r+1}) \exp \left\{ \frac{1}{2\pi^2} \sum_r \int_0^\pi d\eta_r \int_0^\pi d\eta'_r N(\sigma, \tau_1; \sigma', \tau'_1) \right\} \quad (3.1c)$$

$$\mathcal{L}(\sigma, \tau) = \frac{1}{2\pi} \left\{ -\frac{1}{2} \left(\frac{\partial x^1}{\partial \sigma} \right)^2 - \frac{1}{2} \left(\frac{\partial x^1}{\partial \tau} \right)^2 - S_1 \left(\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \sigma} \right) S_1 \right. \\ \left. - S_2 \left(\frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \sigma} \right) S_2 \right\}, \quad (3.1d)$$

$$G_1(\rho) = S_1^1(\rho) \left(\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \sigma} \right) x^1(\rho). \quad (3.1e)$$

Apart from the last two factors, the expression on the right of (3.1b) is precisely the analogue of the corresponding expression for the orbital model given in I. The anti-commuting Neveu-Schwarz variables appear in a slightly different form from the commuting variables. A treatment of functional integration over anti-commuting variables has been given by Candlin [12], who showed that the wave-function ψ corresponding to the bra $(\alpha \beta)$ of a particular mode k is*

$$\psi(b_k) = \alpha + \beta b_k. \quad (3.2a)$$

The wave-function of the ket will involve the creation operator b_{-k} in the same way. In addition, both bra and ket require a normalization factor

$$1 - \frac{1}{2} b_{-k} b_k. \quad (3.2b)$$

For the zero mode we use the annihilation and creation operators defined in the previous section.

* We have made a slight change in the formalism of ref. [12], quoted in ref. [3], where the wave-function of a ket was a function of the annihilation operator. In our reformulation the wave-function of an empty state is unity. We now require a minus sign instead of a plus sign in (3.2b).

The factor $G(\rho_\alpha + \delta\rho)$ in (3.1b) has no analogue in the orbital model, and is inserted for Lorentz invariance. The variable ρ_α refers to a point at which two strings join or separate; as G is singular at this point, we have changed ρ by a small quantity $\delta\rho$ which is allowed to tend to zero at the end of the calculation. We shall now show that the action integral acquires a contribution at the point $\rho = \rho_\alpha$ under a Lorentz transformation, and that the last two factors of (3.1b) provide a term which cancels this contribution.

We pointed out in I that we could obtain a superficial proof of Lorentz invariance (ignoring questions of factor ordering) by subjecting the variables of the functional integral to a Lorentz transformation. For the Lorentz transformation involving the generator M^{1-} , it is necessary also to make a conformal transformation to restore the condition $x^+ = \tau$. The initial and final states then change according to the transformations given by G.G.R.T. and, since the S matrix is not altered by the transformations, we have proved Lorentz invariance.

With our present model we have to supplement the conformal transformation by a super-gauge transformation [6], [9] to restore the condition $S^+ = 0$. After the Lorentz transformation $1 - i \in M^{1-}$, S^+ will be given by the equation:

$$S^+ = \epsilon S^1.$$

We therefore make the super-gauge transformation

$$S_1^\mu \rightarrow S_1^\mu - \epsilon S_1^1 \left(\frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \sigma} \right) x^\mu, \quad (3.3a)$$

$$S_2^\mu \rightarrow S_2^\mu - \epsilon S_2^1 \left(\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \sigma} \right) x^\mu, \quad (3.3b)$$

$$x^\mu \rightarrow x^\mu + 2\epsilon S_1^1 S_1^\mu + 2\epsilon S_2^1 S_2^\mu, \quad (3.3c)$$

where the superscript μ runs from 1 to d . Under the transformation, the Lagrangian changes by a total derivative, apart from terms involving factors

$$S_1^\mu \left(\frac{\partial}{\partial \tau} - i \frac{\partial}{\partial \sigma} \right) x^\mu \quad \text{or} \quad S_2^\mu \left(\frac{\partial}{\partial \tau} + i \frac{\partial}{\partial \sigma} \right) x^\mu,$$

which vanish according to the I.K. gauge condition. The surface terms are

$$\delta \left\{ \int d\sigma d\tau \mathcal{L} \right\} = \frac{\epsilon}{\pi} \left\{ \int d\rho S_1^j S_1^1 \frac{\partial x^j}{\partial \sigma} + \int d\rho^* S_2^j S_2^1 \frac{\partial x^j}{\partial \sigma} \right\}. \quad (3.4)$$

In deriving (3.4) we have again used the gauge condition to eliminate terms involving S^- .

The contributions to (3.4) from $\tau = \pm \infty$ provide terms in the Lorentz generators, while those from the ends of the string vanish owing to the boundary conditions, but contributions from the points where strings join or separate do not vanish. The singular nature of the boundary is most easily handled by making the local co-ordinate transformation:

$$(\rho - \rho_\alpha) = -(z - z_\alpha)^2 \equiv -(\xi + i\eta - \xi_\alpha)^2.$$

By (2.5), the variable S_1 becomes

$$\bar{S}_1^1 = (-2(z - z_\alpha))^{1/2} S_1^1$$

and, from (2.3), we may set $\bar{S}_2^1 = \bar{S}_1^1$. On integrating (3.4) over the line $\eta = \delta$ and passing to the limit $\delta = 0$, we find:

$$\delta \left\{ \int d\sigma d\tau \mathcal{L} \right\} = \frac{\epsilon}{\pi} \int dx \frac{1}{(\xi - \xi_\alpha)^2 + \delta^2} \bar{S}_1^j \bar{S}_1^i \left\{ \delta \frac{\partial x^j}{\partial \xi} + (\xi - \xi_\alpha) \frac{\partial x^j}{\partial \eta} \right\} \\ \approx \epsilon \bar{S}_1^j(\xi_\alpha) \bar{S}_1^i(\xi_\alpha) \frac{\partial x^j(\xi_\alpha)}{\partial \xi} . \quad (3.5)$$

Now let us recalculate (3.5) with an extra factor $G(\rho_\alpha)$.

We make use of the formulas, valid when $z \approx z'$:

$$\bar{S}_1^i(z) \bar{S}_1^j(z') = -\frac{1}{2} (z - z')^{-1} \delta^{ij} + (1 - \delta^{ij}) O(1) + O(z - z'), \quad (3.6a)$$

$$\frac{\partial x^i(z)}{\partial \xi} \frac{\partial x^j(z')}{\partial \xi} = 2(z - z')^{-2} \delta^{ij} + O(1) . \quad (3.6b)$$

Equations (3.6) may easily be derived by expanding S and x in terms of annihilation and creation operators and using the commutation or anti-commutation relations. They hold when S_1 , S_2 and x are operators or factors within Feynman functional integrals. Inserting (3.6) in (3.5), we find:

$$\delta \left\{ \int d\sigma d\tau \mathcal{L} \right\} \bar{G}_1(\xi_\alpha + \delta\xi) = \frac{\epsilon}{8\xi} \left\{ \frac{1}{2} \bar{S}_1^j(\xi_\alpha) \frac{\partial x^j(\xi_\alpha)}{\partial \xi} \frac{\partial x^i(\xi_\alpha)}{\partial \xi} \right. \\ \left. - \frac{1}{2} \bar{S}_1^i(\xi_\alpha) \left(\frac{\partial x^j(\xi_\alpha)}{\partial \xi} \right)^2 + \frac{2}{\xi} \bar{S}_1^j(\xi_\alpha) S_1^i(\xi_\alpha) S_1^j(\xi_\alpha + \delta\xi) \right\} \\ = \frac{\epsilon}{8\xi} \left\{ \frac{1}{2} \bar{G}_1(\xi_\alpha) \frac{\partial x^i(\xi_\alpha)}{\partial \xi} + 2\pi \bar{S}_1^i(\xi_\alpha) \mathcal{L}(\xi_\alpha) \right\} . \quad (3.7)$$

We may now transform back to the variable ρ ; as the right-hand side has conformal weight $5/2$ while the left-hand side has conformal

weight $3/2$, we must add an extra factor $\partial\rho/\partial z$ ($= -2\delta\xi$) to the right-hand side. Therefore:

$$\delta \left\{ \int d\sigma d\tau \right\} G_1(\rho_\alpha + \delta\rho) = \epsilon \left\{ -G_1(\rho_\alpha + \delta\rho) \frac{\partial x^i(\rho_\alpha + \delta\rho)}{\partial \tau} \right. \\ \left. + 4\pi S_1^i(\rho_\alpha + \delta\rho) \mathcal{L}(\rho_\alpha + \delta\rho) \right\} . \quad (3.8)$$

We notice that the term (3.8) is almost cancelled by the change of G itself. In fact, under the Lorentz transformation in question:

$$G_1(\rho) \rightarrow G_1(\tilde{\rho}) - \frac{3\epsilon}{2} G_1(\rho) \frac{\partial x^i}{\partial \tau} + \epsilon G_1(\rho) \frac{\partial x^i}{\partial \tau} - 4\pi \epsilon S_1^i(\rho) \mathcal{L}(\rho) , \quad (3.9)$$

provided that the point ρ is at the end of the string. The last two terms of (3.9) are equal and opposite to the right-hand side of (3.8).

Before proceeding further, we should like to remark that the cancellation encountered here is similar to the cancellation which occurs when a gauge operator is commuted through a vertex in the conventional formulation of the Neveu-Schwarz model. The right-hand side of (3.8) has a relatively simple form because the two factors on the left-hand side involve identical fermion operators. All contributions come from the commutators or anti-commutators between the operators, from which we derived (3.6). These contributions are themselves cancelled by the change in G .

The second term in (3.9) is not cancelled by (3.8). It is simply equal to $(\partial\tilde{\rho}/\partial\rho)^{-3/2}$, and occurs because G has conformal weight $3/2$. The first factor in Γ , given by (3.1c), has been inserted

to cancel this contribution. However, we saw in I that the conformal transformation from ρ to z is not unique, but that the z 's may be subjected to an arbitrary projective transformation without changing the shape of fig. 1. The factor $(\partial\rho/\partial z)^{3/2}$ will change under a projective transformation, and the remaining factors of (3.1c) have been added to restore projective invariance. In the first product, z_α and z_β range over all points where strings join or separate. The product $\prod_1$ is over all Z 's where at least one of Z_r and Z_s is on an external meson lines, while in $\prod_2$ one of Z_r and Z_s is on an incoming fermion line and the other on an incoming anti-fermion line. (Throughout this paper, the term incoming fermion line will be taken to include outgoing anti-fermion line and vice versa.) The coefficients m_r^2 in the exponent of the last factor are equal to the squares of the masses of the relevant ground states ($-\frac{1}{2}$ for mesons, 0 for fermions). As was pointed out in I, the coefficients must have these values if the functional integral is to be finite, since the exponent of (6.2) gives rise to a similar term with the opposite sign, and the Neumann function is infinite. It is fortunate that we were able to obtain a projectively invariance function with the correct values for m^2 .

Our proof of Lorentz invariance ignores questions of factor ordering. We should construct a proof along the lines of that given in I. We have not done so but, in secs. 6 and 7, we shall perform explicit calculations of scattering amplitudes and shall find that they are indeed covariant for $d = 10$.

To define the right-hand side of (3.1b) unambiguously it is necessary to specify the order of the anti-commuting factors. We order

all factors ψ and G from left to right as we go around fig. 1 in a counter-clockwise direction, starting from the bottom right-hand corner. This rule fulfills the requirement that the wave-function for a bra be the conjugate complex of that for the corresponding ket. The z 's in the factors of Γ (eq. (3.1d)) and in the factors $(Z_r - Z_s)^{-2P_r P_s}$ are ordered in the same way.

4. EVALUATION IN TERMS OF NEUMANN FUNCTIONS

In this section we shall evaluate the functional integral over the anti-commuting variables in terms of Neumann functions, so as to complement the integration over the commuting variables which has been performed in I. Gervais and Sakita [6] have extended their calculation of functional integrals to anti-commuting variables. We shall summarize some of their work and shall extend it to cover the requirements of the present problem.

The functional integral (3.1) involves factors b_k from the wave-functions, as well as factors $S(\rho_\alpha)$ which occur in the variable G . We multiply all such factors by "c-numbers" which anti-commute with every variable $S(\rho)$ and with one another. Next we bring the variables into the exponent by using the equation

$$B = e^B - 1 \quad (4.1)$$

where B is any quantity whose square is zero; in particular, where it is the product of two anti-commuting c-numbers. The terms in the exponent of the functional integral which involve anti-commuting factors are thus:

$$\begin{aligned} & \frac{1}{2\pi} \int d\sigma d\tau \left\{ -S_1^1 \left(\frac{\vec{\partial}}{\partial\tau} + i \frac{\vec{\partial}}{\partial\sigma} \right) S_1^1 - S_2^1 \left(\frac{\vec{\partial}}{\partial\tau} - i \frac{\vec{\partial}}{\partial\sigma} \right) S_2^1 \right\} \\ & + \sum_a \int d\sigma d\tau S_a^1(\sigma, \tau) f_a^1(\sigma, \tau) - \frac{1}{2} \sum_{\mathbf{k}, r} b_{\mathbf{k}, r}^1(\tau_1) b_{\mathbf{k}, r}^1(\tau_1) . \end{aligned} \quad (4.2)$$

The function f is a known anti-commuting c-number. The third term of (4.2) arises from the normalization factors (3.2b) of the wave-functions, and it is to be summed over all incoming and outgoing strings.

We define Neumann functions for the anti-commuting fields by the equations:

$$\left(\frac{\partial}{\partial\tau} + i \frac{\partial}{\partial\sigma} \right) K_{1b}(\rho, \rho') = 2\pi \delta_{1b} \delta^2(\rho - \rho'), \quad (4.3a)$$

$$\left(\frac{\partial}{\partial\tau} - i \frac{\partial}{\partial\sigma} \right) K_{2b}(\rho, \rho') = 2\pi \delta_{2b} \delta^2(\rho - \rho'), \quad (4.3b)$$

where the subscripts correspond to the subscripts on S . At the ends of the string, K must satisfy boundary conditions corresponding to (2.3):

$$\rho \text{ at backward arrow: } K_{1b}(\rho, \rho') = K_{2b}(\rho, \rho'), \quad (4.4a)$$

$$\rho \text{ at forward arrow: } K_{1b}(\rho, \rho') = -K_{2b}(\rho, \rho'), \quad (4.4b)$$

$$\rho' \text{ at backward arrow: } K_{a1}(\rho, \rho') = K_{a2}(\rho, \rho'), \quad (4.4c)$$

$$\rho' \text{ at forward arrow: } K_{a1}(\rho, \rho') = -K_{a2}(\rho, \rho'). \quad (4.4d)$$

The second term of (4.2) can be eliminated in the usual way by making the change of variables:

$$S_a^1(\rho) \rightarrow S_a^1(\rho) + \frac{1}{2} \int d\sigma' d\tau' K_{ab}(\rho, \rho') f_b^1(\rho'). \quad (4.5)$$

The expression (4.2) then becomes

$$\begin{aligned} & \frac{1}{2\pi} \int d\sigma d\tau \left\{ -S_1^1 \left(\frac{\vec{\partial}}{\partial\tau} + i \frac{\vec{\partial}}{\partial\sigma} \right) S_1^1 - S_2^1 \left(\frac{\vec{\partial}}{\partial\tau} - i \frac{\vec{\partial}}{\partial\sigma} \right) S_2^1 \right\} \\ & + \frac{1}{2} \sum_{\mathbf{k}, r} b_{\mathbf{k}, r}^1(\tau_1) b_{\mathbf{k}, r}^1(\tau_1) \\ & - \frac{1}{4} \int d\sigma d\tau d\sigma' d\tau' f_a^1(\rho) K_{ab}(\rho, \rho') f_b^1(\rho') \\ & - \sum_{k=-\infty}^{\infty} \left\{ \sum_r (\text{inc.}) - \sum_r (\text{outg.}) \right\} \left(\frac{\alpha_r}{2} \right)^{\frac{1}{2}} b_{\mathbf{k}, r}^1(\tau_1) \int d\sigma d\sigma' d\tau' \\ & \quad \times \left\{ e^{ik\sigma} K_{1b}(\sigma, \tau_1; \rho') \pm e^{-ik\sigma} K_{2b}(\sigma, \tau_1; \rho') \right\} f_b^1(\rho') \\ & + \left(\sum_{k>0} - \sum_{k<0} \right) \sum_r \left(\frac{\alpha_r}{2} \right)^{\frac{1}{2}} b_{\mathbf{k}, r}^1(\tau_1) \int d\sigma d\sigma' d\tau' \\ & \quad \times \left\{ e^{ik\sigma} K_{1b}(\sigma, \tau_1; \rho') \pm e^{-ik\sigma} K_{2b}(\sigma, \tau_1; \rho') \right\} f_b^1(\rho'). \end{aligned} \quad (4.6)$$

The fourth term of (4.6) is the surface term at $\tau = \pm\infty$ from the substitution (4.5), while the fifth term results from the substitution (4.5) in the last term of (4.2). In evaluating both of these terms we have used (2.1) and (2.2), and the $\pm$ sign corresponds to the $\pm$ sign in the latter equation.

In order to carry through the usual manipulations it is necessary to show that the last two terms in (4.6) cancel. Let us begin with the non-zero modes. The terms involving annihilation operators b_k for incoming states and creation operators b_{-k} for outgoing states do cancel. The others do not, but they are negligible in the limit $\tau \rightarrow \pm \infty$ owing to the exponential decrease of the Fourier component of K with τ . We saw in I that only annihilation operators acting on kets or creation operators acting on bras need be kept under such circumstances. The unwanted terms thus disappear for the non-zero modes.

The zero modes require special treatment and, in fact, we shall have to impose further boundary conditions on the K 's. Let us provisionally attach superscripts i, j , corresponding to the $(d-2)$ dimensions, to the K 's. The formulas (4.5) and (4.6) are modified appropriately. We set $K^{ij} = 0$ unless i and j are two dimensions which have been paired in our treatment of the zero mode, and we examine one such pair of dimensions x and y . It will in fact be more convenient to use the helicity combinations:

$$K^{\pm\pm} = \frac{1}{2} \left\{ K^{xx} \pm i K^{xy} \pm i K^{yx} \pm (\pm i) K^{yy} \right\}. \quad (4.7)$$

The summation over the zero-mode b 's in (4.5) is rewritten in the form

$$b_0^x K^x + b_0^y K^y = b_0^+ K^- + b_0^- K^+ \quad (4.8a)$$

where

$$b_0^\pm = 2^{-\frac{1}{2}} (b_0^x \pm i b_0^y). \quad (4.8b)$$

The annihilation operators, as defined at the end of sec. 2, are included in the sum $\sum_{k>0}$, the creation operators in the sum $\sum_{k<0}$. As with the non-zero modes, the unwanted terms in (4.6) fail automatically to cancel for creation operators acting on incoming states or for annihilation operators acting on outgoing states. We enforce the vanishing of such terms by setting the new boundary conditions:

$$K^{\pm j}(\rho, \rho') = 0 \quad \tau = \tau_1 \quad (4.9)$$

where the sign is $+$ for an incoming fermion string and $-$ for an incoming anti-fermion string.

The equations (4.3) and the boundary conditions (4.4) and (4.9) are just sufficient to determine the K 's. As in I, we make a conformal transformation of fig. 1 onto the half-plane. The K 's, like the S 's, have conformal weight $\frac{1}{2}$ so that, under a transformation from ρ to $\tilde{\rho}$,

$$\tilde{K}_{ab}(\rho, \rho') = \left\{ \left(\frac{\partial \tilde{\rho}}{\partial \rho} \right)^{(*)} \left(\frac{\partial \tilde{\rho}'}{\partial \rho'} \right)^{(*)} \right\}^{-\frac{1}{2}} K_{ab}(\rho, \rho') \quad (4.10)$$

the conjugate complex being used when the subscript has the value 2.

Equation (4.10), together with the boundary conditions (4.4), imply that the z -plane $\bar{K}$'s satisfy the boundary condition

$$\bar{K}_{1b}(z, z') = \pm \bar{K}_{2b}(z, z'), \quad z \text{ real} \quad (4.11a)$$

$$\bar{K}_{a1}(z, z') = \pm \bar{K}_{a2}(z, z'), \quad z' \text{ real} \quad (4.11b)$$

where the sign is $+$ when $Z_1 < z < \infty$ or $Z_1 < z' < \infty$ (in the notation of I), but changes each time z (4.11a) or z' (4.11b) passes a value Z_r corresponding to an external fermion or anti-fermion line. The following functions K now satisfy all our conditions:

$$\bar{K}^{+-}(z, z') = -\bar{K}^{-+}(z', z) = \frac{1}{z - z'} \left\{ \prod_r' (z - Z_r)^{\frac{1}{2}} (z' - Z_r)^{-\frac{1}{2}} \right\} \\ \times \left\{ \prod_r'' (z - Z_r)^{-\frac{1}{2}} (z' - Z_r)^{\frac{1}{2}} \right\}, \quad (4.12a)$$

$$\bar{K}^{++}(z, z') = \bar{K}^{--}(z, z') = 0, \quad (4.12b)$$

the product $\prod_r'$ being over all Z_r corresponding to incoming fermion lines and $\prod_r''$ over all Z_r corresponding to incoming anti-fermion lines. One can obtain the other K 's from the equations

$$\bar{K}_{2b}(z, z') = \bar{K}_{1b}(z^*, z'), \quad \bar{K}_{a2}(z, z') = \bar{K}_{a1}(z, z'^*). \quad (4.12c)$$

The boundary conditions (4.11) and (4.9) determine the factors in the square roots of (4.12c) uniquely; (4.11) alone would not be sufficient.

The ρ -plane functions $K_{ab}(\rho, \rho')$ can now be found from Eq. (4.10). If the variable z approaches Z_r , the conversion factor $(\partial\rho/\partial z)^{-\frac{1}{2}}$ contains a factor $(z - Z_r)^{\frac{1}{2}}$ which cancels the factor $(z - Z_r)^{\frac{1}{2}}$ in (4.12a) if it is in the denominator, and leads to a zero result if it is in the numerator. We notice in particular that either K^{+-} or K^{-+} will vanish if τ or τ' approaches $\pm\infty$. Furthermore, if τ or τ' both approach $\pm\infty$ on incoming fermion lines or both approach $\pm\infty$ on incoming anti-fermion lines, K^{+-} and K^{-+} both vanish.

The first two terms of (4.6) are independent of the external states and may be absorbed in the volume element, so that all dependence on the states is in the last term. We now use (4.1) to take this term out of the exponent, and we can state a simple prescription for performing the functional integral over the anti-commuting variables.

The right-hand side of (3.1b) contains factors of $S_1(\rho)$ and $S_2(\rho)$, both from the wave-functions corresponding to full fermion states and from the factors G . Pairs of such factors are contracted in all possible ways, with appropriate sign changes. For each contraction we add a factor $\frac{1}{2} K_{ab}(\rho, \rho')$. Pairs with ρ and ρ' interchanged are not counted separately.

Terms involving $x^1(\rho)$ in the exponent of (3.16) are identical to those in the corresponding expression of I. In addition, (3.1b) involves factors of $\partial x^1(\rho)/\partial\tau$ which occur in the definition of G . (The derivatives $\partial x^1(\rho)/\partial\sigma$ vanish because of the boundary conditions. To evaluate the functional integral we write

$$\partial x^1(\rho)/\partial\tau = \epsilon^{-1} \{ \exp [\epsilon \partial x^1(\rho)/\partial\tau] - 1 \} \quad (|\epsilon| \ll 1)$$

and use the methods of Hsue, Sakita and Virasoro and of Gervais and Sakita [13]. We can easily calculate the effect of the extra factors on the general S-matrix element, but for simplicity we shall only quote the prescription to be used when the external particles have no orbital excitations. The factors $\partial x^1(\rho)/\partial\tau$ are contracted in pairs, with some left uncontracted. This is done in all possible ways. For each contracted pair we add a factor

$$(d-2) \frac{\partial^2}{\partial\tau \partial\tau'} N(\rho, \rho') \quad (4.13a)$$

while for each uncontracted factor we add a factor

$$\frac{\partial}{\partial\tau} \sum_r P_r^1 N(\rho, \rho_r) \quad (4.13b)$$

where ρ_r is a point at $\tau = \pm\infty$ on the r th string.

We are left with the question of the volume element. As in I, the easiest way of answering this question is to take the case where some of the strings are infinitely short, and then to proceed to the general case using Lorentz invariance. If all or all but two of the external particles are mesons, we can repeat the procedure used in I. For external fermions we must modify the procedure somewhat, since the reasoning is only applicable if the character (mesonic or fermionic) of the string with finite length does not change at the interaction point. The calculations are given in appendix A. In all cases we confirm that after the functional integration has been performed, the volume element is identical to that used in I. The Veneziano integrand now has an additional factor Γ , defined by (3.1c), and, in fact, the calculation of the volume element is the real justification of the formula for Γ .

As with the purely orbital model, the duality properties of the amplitude are manifest. If we analytically continue the momentum variables in such a way that one of the α 's changes sign, the corresponding string will pass from the left to the right of fig. 1 or vice versa. The amplitude will remain analytic through the transition. With the present model we do have to be careful about phase factors; in appendix B we shall indicate how such factors appear and how they cancel one another out.

The functional integral (3.2) determines the finite-time transition amplitudes as well as the S matrix. As in I, we now have a complete finite-time quantum-mechanical system. The S matrix is Lorentz-invariant, factorizable and dual, and the problem of ghosts does not appear.

5. DUALITY STRUCTURE

Our model possesses two types of mesons, depending on whether the arrow at the top goes forwards or backwards. We shall distinguish between them by applying dotted quark lines as in fig. 1. A quark line is applied to an arrow going forwards at the top of a string or to an arrow going backwards at the bottom. The distinction between $q\bar{q}$ mesons and zero-quark mesons is made even in a theory without internal degrees of freedom but, if we do have internal symmetry, the quark lines carry quantum numbers in the usual way.

It is not difficult to see that the coupling of a $q\bar{q}$ meson to a system of fermions of given helicity is equal in magnitude to the coupling of the corresponding zero-quark meson. If fig. 1 is turned upside down, the intermediate meson changes from a $q\bar{q}$ meson to a zero-quark meson or vice versa. By conformally mapping the new diagram onto the lower half-plane, with the points Z_i corresponding to the same external lines as before, we conclude that the new Neumann functions are obtained from the old by simply changing i to $-i$. The anti-commuting factors in (3.2b) will have to be re-ordered. Both these changes can affect only the sign of the amplitude and not its magnitude.

Since Neveu-Schwarz g -parity is subject to odd-even conservation, the parity of the odd- g -parity states is undetermined by an overall sign. This is so even in our present model with massless ground-state fermions; the number of b -excitations plus the number of particles cannot change from even to odd in an interaction. By examining the simplest meson-fermion vertex, we shall now show that the parity of a zero-quark meson with odd g -parity is opposite to the

parity of the corresponding $q\bar{q}$ meson.

In figs. 2, all three particles are unexcited (except possibly in the zero mode). Since the functional integral (3.2b) has one factor $G(\rho)$, the amplitude will be non-zero only if one of the anti-fermions is in the "full" helicity state. The two non-zero helicity amplitudes are therefore $\langle + | T | - \rangle$ and $\langle - | T | + \rangle$; they will be equal in magnitude but will have opposite signs if the parity of the meson is negative. When going from fig. 2(a) to fig. 2(b), the change $1 \rightarrow -1$ in the Neumann function cannot reverse the sign of one of the amplitudes without also reversing the sign of the other.* The amplitude $\langle + | T | - \rangle$, in which the incoming anti-fermion has a zero-mode excitation, will undergo an additional sign change due to the re-ordering of its wave-function with the factor $G(\rho)$, while the amplitude $\langle - | T | + \rangle$ is not affected by factor re-ordering. The relative sign of the two amplitudes is therefore different in fig. 2(a) from what it is in fig. 2(b). In other words, the parities of the mesons in the two diagrams are opposite.

It is not difficult to calculate the vertex functions explicitly and to confirm the result just stated.

In appendix C we shall provide a further check on our result by examining the transition amplitude from a general excited meson to a system of n ground-state fermions. We shall compare the parities of $q\bar{q}$ and zero-quark mesons and shall find that they are the same

* The phase factor of the amplitude is the product of the phase factors of the wave-functions and of the Neumann functions K . The amplitudes are either both real or both imaginary, and the wave-functions for the full $\bar{f}$ state are both imaginary. It follows that the Neumann functions are both real or both imaginary.

or different according to whether the g -parity of the meson is even or odd.

We arbitrarily define the parity of the $q\bar{q}$ vacuum to be negative, since we should like to identify this particle with the pion. Independently of any experimental evidence, we could base our definition of the parity on the eventual hope of obtaining a theory where the mass of the fermion vacuum is not zero. It is known that the Ramond model then requires a pseudoscalar pion.

The zero-quark meson vacuum is scalar. Within the framework of our present model, such a meson is necessary for duality. The fermions should be regarded as quarks rather than baryons, and duality diagrams with zero-quark intermediate states are unavoidable. In a realistic model the baryons would be three-quark fermions, and duality diagrams with zero-quark intermediate states need not occur. The scalar meson vacuum and its excited states might well be absent. At present it is not known how to construct a model of baryons. The three-string model suggested by Gross and Kaku appears to possess all the correct physical properties, but the detailed development of the model seems to require techniques for handling interacting strings non-perturbatively.

In a system with fermions it is not possible to identify Neveu-Schwarz g -parity with physical g -parity, since we require complete exchange degeneracy, i.e., each isotopic spin state must be exchange degenerate. Complete exchange degeneracy is an immediate consequence of the requirement that the u -channel in $f\bar{f}$ scattering should have no resonances at all (in the Born approximation). By identifying the two g -parities, Neveu and Schwarz required the isotopic spin to

alternate between 0 and 1 on their exchange-degenerate trajectories. The lightest $q\bar{q}$ mesons in our model (with $SU(2)$ symmetry) would be $\pi(\eta(m^2 = -\frac{1}{2})), \quad \omega(m^2 = 0), \quad \rho'\omega'(m^2 = \frac{1}{2})$. We can equally well incorporate $SU(3)$ in the model since the motivation of Neveu and Schwarz for not doing so, namely their desire to identify the two g -parities, is now lost. Conservation of NS g -parity will presumably be violated in a more realistic model.

The spectrum of mesons in our model agrees in a rough qualitative way with that observed. One obtains a better spectrum by assuming that the interaction raises the square of the masses by about $(1 \text{ BeV})^2$; the ω and $\rho'\omega'$ would be separated by repulsion of energy levels. However, it is obviously premature to attempt any sort of a quantitative comparison between predicted and observed mass spectra.

6. PION-PION SCATTERING

As an illustration of our methods we shall first find the amplitude for pion-pion scattering. We shall work in a general Lorentz frame and shall show that the known, Lorentz-invariant result does follow from our non-covariant approach if $d = 10$. For the reader who is not interested in the recalculation of known results, we remark that sec. 7 will be independent of the present section.

The momenta of the four pions will be denoted by $P_1, \dots, P_4$, the ρ -coordinates of their strings at $\tau = \pm\infty$ by $\rho_1, \dots, \rho_4$, and the widths of the strings by $\alpha_1, \dots, \alpha_4$, where $\sum \alpha_r = 0$. We shall denote the ρ -coordinates of the points where the strings join and separate by ρ_α, ρ_β . The Veneziano integrand will be given by the function

$$I = \tilde{I}(\rho) \Gamma(\rho) \left\{ \frac{1}{2} K(\rho_\alpha - \rho_\beta) \right\} \left\{ \left[\frac{\partial}{\partial \tau_\alpha} \sum_{r=1}^4 P_r^{-1} N(\rho_\alpha, \rho_r) \right] \times \left[\frac{\partial}{\partial \tau_\beta} \sum_{r=1}^4 P_r^{-1} N(\rho_\beta, \rho_r) \right] + (d-2) \frac{\partial^2}{\partial \tau_\alpha \partial \tau_\beta} N(\rho_\alpha, \rho_\beta) \right\}. \quad (6.1)$$

$\tilde{I}$ is the ordinary Veneziano integrand which we would obtain from (3.1b) by ignoring the anti-commuting variables and the function Γ , defined by (3.2c). The last two factors of (6.1) result from treating the two factors $G(\rho_\alpha)$ and $G(\rho_\beta)$ according to the prescription given at the end of sec. 4; since our mesons are unexcited there are no further anti-commuting factors from the wave functions. The factor $S_1(\rho_\alpha)$ and $S_2(\rho_\alpha)$ in the G 's give the factor $-\frac{1}{2} K(\rho_\alpha - \rho_\beta)$, while the factors $\partial x^1(\rho_\alpha)/\partial \tau$ and $\partial x^1(\rho_\beta)/\partial \tau$ lead to the last factor of (6.1).

Let us make the conformal transformation

$$\rho = \alpha_1 \ln(z-1) + \alpha_2 \ln(z-Z) + \alpha_3 \ln z, \quad (6.2)$$

so that the points $\rho_1, \dots, \rho_4$ map onto 1, Z , 0 and ∞ respectively. The points ρ_α and ρ_β will map onto z_α and z_β .

Then

$$\begin{aligned} \tilde{I}(\rho) &= (1-Z)^{-2P_1 P_2} Z^{-2P_2 P_3} \\ &= (1-Z)^{-s-1} Z^{-t-1} \end{aligned} \quad (6.3a)$$

where

$$s = (P_1 + P_2)^2, \quad t = (P_2 + P_3)^2 \quad (6.3b)$$

$$\Gamma = \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{3/2} \left(\frac{\partial \rho_\beta}{\partial \tau_\beta} \right)^{3/2} (z_\alpha - z_\beta)^{-1} (1 - Z)^{-1} Z^{-1} (1 - z_\alpha)(z_\alpha - Z)z_\alpha \\ \times (1 - z_\beta)(Z - z_\beta)(-z_\beta)(1 - Z)Z, \quad (6.3c)$$

$$K(\rho_\alpha - \rho_\beta) = \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1/2} (z_\alpha - z_\beta)^{-1}, \quad (6.3d)$$

$$N(\rho_\alpha, \rho_r) = 2 \ln |z_\alpha - z_r|, \quad (6.3e)$$

$$N(\rho_\beta, \rho_r) = 2 \ln |z_\beta - z_r|. \quad (6.3f)$$

In (6.3d) we have added the two factors of $(\partial \rho / \partial z)^{-1/2}$ to convert the function $\bar{K}(z_\alpha, z_\beta)$ to $K(\rho_\alpha, \rho_\beta)$. We note that the square-root factors of (4.12) are absent, as there are no external fermions.

From (6.3), we may write

$$\frac{\partial}{\partial \tau_\alpha} N(\rho_\alpha, \rho_r) = \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1} (z_\alpha - z_r)^{-1}, \quad (6.4a)$$

$$\frac{\partial}{\partial \tau_\beta} N(\rho_\beta, \rho_r) = \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1} (z_\beta - z_r)^{-1}, \quad (6.4b)$$

and it follows from (6.1), (6.3c), (6.3d) and (6.4) that the factors of $(\partial \rho_\alpha / \partial z_\alpha)$ and $(\partial \rho_\beta / \partial z_\beta)$ cancel. Such a cancellation always occurs and, in fact, the explicit factors of $(\partial z / \partial z)^{3/2}$ in Γ were introduced to cancel similar hidden factors in G which would violate Lorentz invariance. We may now write

$$\Gamma = 2(1 - Z)^{-s-1} Z^{-t-1} (z_\alpha - z_\beta)^{-2} (1 - z_\alpha)(z_\alpha - Z)z_\alpha(1 - z_\beta)(Z - z_\beta)(-z_\beta) \\ \times \left\{ \left[P_1^1 (z_\alpha - 1)^{-1} + P_2^1 (z_\alpha - Z)^{-1} + P_3^1 z_\alpha^{-1} \right] \right. \\ \times \left[P_1^1 (z_\beta - 1)^{-1} + P_2^1 (z_\beta - Z)^{-1} + P_3^1 z_\beta^{-1} \right] \\ \left. + \frac{1}{2} (d - 2)(z_\alpha - z_\beta)^{-2} \right\}. \quad (6.5)$$

We remind the reader that the superscripts i run from 1 to $d - 2$ only; our calculations are not manifestly covariant.

The expression (6.5) is most easily evaluated in a frame where two of the strings are infinitely short. If α_2 and α_3 are small, we may write:

$$z_\alpha \approx Z, \quad z_\beta \approx 0,$$

so that

$$\Gamma = -2(1 - Z)^{-s} Z^{-t-1} P_2^1 P_3^1. \quad (6.6)$$

We may replace the product $P_2^1 P_3^1$ by $-P_2 P_3$ ($= -P_2^\mu P_{3\mu}$), since, in our present frame, $P_2^+ = P_3^+ = 0$. On multiplying (6.6) by the volume element $dZ Z^{-1}(1 - Z)^{-1}$ and integrating, we obtain the usual result; in fact, our calculation resembles the usual calculation fairly closely. As our aim is to show that we obtain a Lorentz-invariant amplitude when working in a general frame, we have to perform a slightly longer calculation.

The points z_α and z_β are given by the equation

$$\frac{\partial \rho}{\partial z} = 0,$$

i.e.,

$$\frac{\alpha_1}{z-1} + \frac{\alpha_2}{z-Z} + \frac{\alpha_3}{z} = 0. \quad (6.7)$$

It is convenient to introduce the parameters

$$a = -\alpha_1/\alpha_4, \quad c = \alpha_3/\alpha_4,$$

so that z_α and z_β are given by the equation

$$z^2 + z(a(1-Z) + cZ - 1) - cZ = 0,$$

from which we find:

$$z_\alpha z_\beta = -cZ, \quad z_\alpha + z_\beta = (1-a)(1-Z) + (1-c)Z$$

$$z_\alpha - z_\beta = \{CZ^2 + 2BZ(1-Z) + A(1-Z)^2\}^{\frac{1}{2}}, \quad (6.8a)$$

$$\equiv \{D(Z)\}^{\frac{1}{2}},$$

$$\text{where} \quad A = (1-a)^2, \quad B = 1+c-a+ac, \quad C = (1+c)^2. \quad (6.8b)$$

To evaluate the kinematical quantities $P_r^1 P_s^1$, we first note that:

$$P_r^{12} - 2P_r^- P_r^+ = \frac{1}{2},$$

so that

$$P_r^- = \frac{1}{2} (P_r^+)^{-1} (P_r^{12} - \frac{1}{2}).$$

Also

$$P_r^1 P_s^1 - P_r^- P_s^+ - P_r^+ P_s^- = -P_r P_s,$$

i.e.,

$$2P_r^1 P_s^1 = -(P_r + P_s)^2 - 1 + \frac{\alpha_s}{\alpha_r} (P_r^{12} - \frac{1}{2}) + \frac{\alpha_r}{\alpha_s} (P_s^{12} - \frac{1}{2}). \quad (6.9a)$$

Hence

$$2P_1^1 P_2^1 = -s-1 + (P_1^{12} - \frac{1}{2})(1+c-a)a^{-1} + (P_2^{12} - \frac{1}{2})a(1+c-a)^{-1}, \quad (6.9b)$$

$$2P_2^1 P_3^1 = -t-1 - (P_2^{12} - \frac{1}{2})c(1+c-a)^{-1} - (P_3^{12} - \frac{1}{2})(1+c-a)c^{-1}, \quad (6.9c)$$

$$2P_1^1 P_3^1 = s+t+1 - (P_1^{12} - \frac{1}{2})c a^{-1} - (P_3^{12} - \frac{1}{2})a c^{-1}. \quad (6.9d)$$

If (6.8) and (6.9) are inserted into (6.5), we find that terms involving P_r^{12} do cancel, as is necessary for Lorentz invariance.

The expression is still not manifestly Lorentz invariant, as the α 's depend on the Lorentz frame. We obtain the result:

$$\begin{aligned} A(s,t) &= \int_0^1 dZ(1-Z)^{-1} Z^{-1} I \\ &= \int_0^1 dZ(1-Z)^{-s-1} Z^{-t-1} \left\{ \left[(t-1)CZ + tB(1-Z) + sBZ \right. \right. \\ &\quad \left. \left. + (s-1)A(1-Z) + 2CZ + 2A(1-Z) \right] \right. \\ &\quad \left. + (d-2)(1-Z)Zac(1-a+c)D^{-1}(Z) \right\} D^{-1}(Z). \end{aligned} \quad (6.10)$$

The integral can be evaluated with the aid of the formulas:

$$\begin{aligned}
 s \int dZ Z^{-t-1} (1-Z)^{-s-1} f(Z) &= (s+t) \int dZ (1-Z) Z^{-t-1} (1-Z)^{-s-1} f(Z) \\
 &\quad - \int dZ Z^{-t} (1-Z)^{-s} f'(Z), \\
 t \int dZ Z^{-t-1} (1-Z)^{-s-1} f(Z) &= (s+t) \int dZ Z Z^{-t-1} (1-Z)^{-s-1} f(Z) \\
 &\quad + \int dZ Z^{-t} (1-Z)^{-s} f'(Z),
 \end{aligned} \tag{6.11}$$

which may easily be derived by integration by parts. We thus find

$$\begin{aligned}
 &\int_0^1 dZ (1-Z)^{-t-1} Z^{-s-1} \left\{ (t-1)CZ + tB(1-Z) + sBZ + (s-1)A(1-Z) \right\} D^{-1}(Z) \\
 &= (s+t+1)B(-s, -t) - 2 \int dZ (1-Z)^{-s} Z^{-t} \\
 &\quad \times \left\{ (1-Z)^{-1} Z^{-1} + \left[(C-B)Z + (B-A)(1-Z) \right]^2 D^{-2}(Z) \right\}.
 \end{aligned} \tag{6.12}$$

The integral in (6.12) cancels against the remaining terms of (6.10), provided $d-2=8$. We are left with the result

$$A(s, t) = (s+t+1)B(-s, -t), \tag{6.13}$$

which agrees with the usual Neveu-Schwarz pion-pion amplitude. We emphasize that we obtain a Lorentz-invariant result only if $d=10$. As long as we are neglecting crossed loops we can always restrict ourselves to four dimensions, but we must do so after completing the calculation. If we were to restrict ourselves to four dimensions from the beginning, we would lose the longitudinal mode which is necessary for Lorentz invariance if $d \neq 10$.

7. FERMION ANTI-FERMION SCATTERING

For our calculation of the $f\bar{f}$ scattering amplitude, we shall take a Lorentz frame where two of the strings have infinitesimal length. The momenta P^0 , P^x , P^y , and P^z for the four particles, numbered in clockwise order from the bottom left of fig. 1, will be as follows:

$$\begin{array}{llll}
 P_1: & P_0 & 0 & -P_0 \\
 P_2: & (P_0^2 + P_1^2)^{\frac{1}{2}} & P_1 \sin \phi & P_0 \\
 P_3: & -(P_0^2 + P_1^2)^{\frac{1}{2}} & -P_1 \sin \phi & P_0 \\
 P_4: & -P_0 & 0 & -P_0
 \end{array} \tag{7.1}$$

The second and third strings will have infinitesimal length if $P_2^z \approx -P_2^0$, $P_3^z \approx -P_3^0$. We therefore take the (complex) angle ϕ to be given by the formula:

$$\cos \phi = (-(P_0^2 + P_1^2)^{\frac{1}{2}} + P_0 \delta) P_1^{-1}, \quad \sin \phi = -i(P_0 - (P_0^2 + P_1^2)^{\frac{1}{2}} \delta) P_1^{-1}, \tag{7.2}$$

where δ is small. If α_r is the length of the r th string, it follows from (7.1) and (7.2) that:

$$\frac{\alpha_2}{\alpha_1} = \frac{P_2^+}{P_1^+} = \delta, \quad \frac{\alpha_3}{\alpha_1} = -\delta. \tag{7.3}$$

In terms of $s (= (P_1 + P_2)^2)$ and $t (= (P_2 + P_3)^2)$, the momenta P_0 and P_1 will be given by the equation

* We restrict ourselves to four dimensions as we have explained in the introduction. In our present Lorentz frame the dimension number does not enter the calculation.

$$p_0 = \frac{1}{2}(-t)^{\frac{1}{2}}, \quad p_1 = [s(s+t)/t]^{\frac{1}{2}}. \quad (7.4)$$

As we shall be working with the Neumann functions (4.12), it will be convenient to use the positive and negative helicity combinations of P_r . Let us define

$$P_r^{(\pm)} = (2)^{-\frac{1}{2}}(P_r^x \pm i P_r^y),$$

which must be distinguished from $P^\pm = (2)^{-\frac{1}{2}}(P^0 \pm P^z)$. Then

$$\begin{aligned} P_1^{(+)} &= -P_1^{(-)} = i(-t)^{\frac{1}{2}}/(2\sqrt{2}) & P_2^{(+)} &= -i(-t)^{\frac{1}{2}}/\sqrt{2} \\ P_2^{(-)} &= i\delta(2s+t)(-t)^{-\frac{1}{2}}/2\sqrt{2} & P_3^{(+)} &= -i\delta(2s+t)(-t)^{-\frac{1}{2}}/(2\sqrt{2}) \\ P_3^{(-)} &= i(-t)^{\frac{1}{2}}/\sqrt{2} & P_4^{(+)} &= -P_4^{(-)} = i(-t)^{\frac{1}{2}}/(2\sqrt{2}). \end{aligned} \quad (7.5)$$

With zero-mass particles a Lorentz transformation is effected by multiplying each helicity state by a numerical factor. Since our momenta are complex the factors for transforming to the center-of-mass system will not be pure phase factors. They are as follows:

$$\text{Particles 1 and 4: } e^{\mp i\pi\lambda/2} \left\{ -\left(\frac{s}{s+t}\right)^{\frac{1}{2}} \right\}^\lambda, \quad (7.6a)$$

$$\text{Particles 2 and 3: } e^{\mp i\pi\lambda/2} \left\{ \frac{s}{t} \left[s(s+t) \right]^{\frac{1}{2}} \right\}^\lambda, \quad (7.6b)$$

where the $-$ sign in the exponent is for particles 1 and 2, the plus sign for particles 3 and 4.

Now let us write down formulas for the Neumann functions K for the process in question. We take the configuration represented by

fig. 1, so that particles 1 and 4 are incoming and outgoing anti-fermions, particles 2 and 3 incoming and outgoing fermions. As usual we shall put $Z_1 = 1$, $Z_2 = Z$, $Z_3 = 0$, $Z_4 = \infty$, so that the conformal transformation from ρ to z will be:

$$\begin{aligned} \rho &= \alpha_1 \ln(z-1) + \alpha_2 \ln(z-Z) + \alpha_3 \ln z \\ &= \alpha(\ln(z-1) + \delta \ln(z-Z) - \delta \ln z), \quad (\alpha = \alpha_1) \end{aligned} \quad (7.7)$$

from (7.3). The points z_α and z_β where the strings join and separate will be given by the equation

$$\frac{\partial \rho}{\partial z} = 0,$$

i.e.,

$$\frac{1}{z-1} + \frac{\delta}{z-Z} - \frac{\delta}{z} = 0.$$

To second order in δ , the solutions are:

$$z_\alpha = Z + \delta(1-Z)(1-\delta(1-Z)Z^{-1}), \quad (7.8a)$$

$$z_\beta = -\delta(1+\delta(2-Z^{-1})). \quad (7.8b)$$

According to (4.12), the z -plane Neumann function $\bar{K}_{||}$ will be:

$$\bar{K}_{||}^{+-}(z, z') = -\bar{K}_{||}^{+}(z', z) = \frac{1}{z-z'} \left\{ \frac{(z-Z)(z'-1)z'}{(z'-Z)(z-1)z} \right\}^{\frac{1}{2}}. \quad (7.9)$$

To convert to the ρ -plane functions K , we must multiply by $(\partial \rho / \partial z)^{-\frac{1}{2}}(\partial \rho' / \partial z')^{-\frac{1}{2}}$. We shall not calculate the derivatives $(\partial \rho_\alpha / \partial z_\alpha)^{-\frac{1}{2}}$ and $(\partial \rho_\beta / \partial z_\beta)^{-\frac{1}{2}}$, as they cancel against similar factors

in Γ , eq. (3.1c). We therefore define

$$\tilde{K}(\rho_\alpha, \rho) = \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{\frac{1}{2}} K(\rho_\alpha, \rho),$$

$$\tilde{K}(\rho_\beta, \rho) = \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{\frac{1}{2}} K(\rho_\beta, \rho)$$

$$\tilde{\tilde{K}}(\rho_\alpha, \rho_\beta) = \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \frac{\partial \rho_\beta}{\partial z_\beta} \right)^{\frac{1}{2}} K(\rho_\alpha, \rho_\beta).$$

(7.10)

As an illustration of the calculation, let us determine $\tilde{K}^{+-}(\rho_\alpha, \rho_1)$ (ρ_r is a point at $\tau = \infty$ on the r th string). We have to substitute $z' = z_\alpha$, $z = 1 + \epsilon$ in (7.9), and multiply by $(\partial \rho_1 / \partial z_1)^{-\frac{1}{2}}$. As the right-hand side of (7.9) contains a factor $(z - 1)^{\frac{1}{2}}$ in the denominator, we cannot set $z = 1$ immediately. However, for $z = 1 + \epsilon$, eq. (7.7) gives:

$$\frac{\partial \rho}{\partial z} = \frac{\alpha}{z - 1},$$

so that the factors of ϵ cancel. We thus find

$$\tilde{K}^{+-}(\rho_\alpha, \rho_1) = -1(\alpha\delta)^{-\frac{1}{2}} (Z/(1 - Z))^{\frac{1}{2}} + O(\delta^{\frac{1}{2}}). \quad (7.11)$$

When calculating the function $\tilde{K}^{+-}(\rho_\alpha, \rho_1)$, eq. (7.9) gives a factor $(z - 1)^{\frac{1}{2}}$ in the numerator, and the derivative $(\partial \rho / \partial z)^{-\frac{1}{2}}$ gives a further such factor, so that the Neumann function is zero.

The Neumann functions for other relevant values of the arguments can be calculated in the same way, and we quote the results. Terms of order δ compared to the main term are ignored except in the functions

$\tilde{K}(\rho_\alpha, \rho_3)$, $\tilde{K}(\rho_\beta, \rho_4)$ and $\tilde{\tilde{K}}(\rho_\alpha, \rho_\beta)$, where terms of order δ^2 are neglected. At least one of the functions K^{+-} or K^{-+} is always zero for the arguments of interest, and we only write down the function which fails to vanish.

$$K^{+-}(\rho_1, \rho_2) = K^{-+}(\rho_4, \rho_3) = 1 \alpha^{-1} \delta^{-\frac{1}{2}} Z^{\frac{1}{2}},$$

$$K^{-+}(\rho_2, \rho_3) = -(\alpha\delta)^{-1} (1 - Z)^{\frac{1}{2}},$$

$$K^{+-}(\rho_1, \rho_4) = -\alpha^{-1} (1 - Z)^{\frac{1}{2}},$$

$$\tilde{K}^{+-}(\rho_\alpha, \rho_1) = -1(\alpha\delta)^{-\frac{1}{2}} Z^{\frac{1}{2}} (1 - Z)^{-\frac{1}{2}},$$

$$\tilde{K}^{+-}(\rho_\alpha, \rho_2) = \alpha^{-\frac{1}{2}} \delta^{-1} (1 - Z)^{-\frac{1}{2}},$$

$$\tilde{K}^{-+}(\rho_\alpha, \rho_3) = -\alpha^{-\frac{1}{2}} \delta^{-1} (1 - \frac{1}{2} \delta),$$

$$\tilde{K}^{+-}(\rho_\alpha, \rho_4) = \tilde{K}^{-+}(\rho_\alpha, \rho_4) = 0,$$

$$\tilde{K}^{-+}(\rho_\beta, \rho_1) = \tilde{K}^{+-}(\rho_\beta, \rho_1) = 0$$

$$\tilde{K}^{+-}(\rho_\beta, \rho_2) = -\alpha^{\frac{1}{2}} \delta^{-1} (1 - Z)^{\frac{1}{2}}$$

$$\tilde{K}^{-+}(\rho_\beta, \rho_3) = \alpha^{-\frac{1}{2}} \delta^{-1}$$

$$\tilde{K}^{-+}(\rho_\beta, \rho_4) = 1(\alpha\delta)^{-\frac{1}{2}} Z^{\frac{1}{2}} (1 + \delta(Z^{-1} - \frac{3}{2}))$$

$$\tilde{\tilde{K}}^{+-}(\rho_\alpha, \rho_\beta) = \delta^{-1} (1 - 2\delta).$$

(7.12)

Following Goldberger, Grisaru, MacDowell and Wong [14], we now define the following helicity amplitudes, the entries in the brackets being $\langle 34|T|21 \rangle$:

$$\begin{aligned}\phi_1 &= \langle ++|T|++ \rangle & \phi_2 &= \langle --|T|++ \rangle \\ \phi_3 &= \langle +-|T|+- \rangle & \phi_4 &= \langle +-|T|+- \rangle \\ \phi_5 &= \langle ++|T|+- \rangle.\end{aligned}\quad (7.13)$$

We remind the reader that we are using a different phase convention from ref. [14], so that the signs of ϕ_2 and ϕ_4 are reversed. The amplitude ϕ_5 will be zero in our model, as the number of b-excitations changes from odd to even. The other four amplitudes are given by a Veneziano integral:

$$A_\lambda(s, t) = h_\lambda \int_0^1 dZ Z^{-1} (1-Z)^{-1} \tilde{I}(Z) \Gamma(Z) F_\lambda(Z), \quad (7.14)$$

where the subscript λ corresponds to the subscripts in (7.13). The function $\tilde{I}$ is the ordinary Veneziano integrand, obtained by performing the functional integral (3.1b) without the factors G and the anti-commuting factors in the wave function. The contributions of these factors, evaluated according to the prescription at the end of sec. 4, has been denoted by F_λ . Γ is given by (3.1c), and h_λ is the product of the Lorentz transformation factors (7.6).

The functions $\tilde{I}$ and Γ may be written down immediately:

$$\tilde{I} = (1-Z)^{-2P_1 P_2} Z^{-2P_2 P_3} = (1-Z)^{-s} Z^{-t}, \quad (7.15)$$

$$\begin{aligned}\Gamma &= \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{3/2} (z_\alpha - z_\beta)^{-1} (1-Z)^{-2} Z^{-2} (1-z_\alpha)(z_\alpha - Z) z_\alpha \\ &\quad \times (1-z_\beta)(Z-z_\beta)(-z_\beta)(1-Z)Z \\ &= \delta^2 \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{3/2} (1-Z),\end{aligned}\quad (7.16)$$

after neglecting terms of order δ^3 . The Lorentz-transformation factors will be as follows:

$$h_1 = -\frac{\delta s}{t} \quad h_2 = -1 \quad h_3 = -\frac{\delta(s+t)}{t} \quad h_4 = 1. \quad (7.17)$$

We shall begin the determination of the functions F_λ with F_3 , which is the simplest because all the helicity states are "empty." The relevant factors in (3.2b) are then:

$$G(\rho_\beta) G(\rho_\alpha) = S_1^{i(\rho_\beta)} S_1^{j(\rho_\alpha)} \frac{\partial x^{i(\rho_\beta)}}{\partial \tau} \frac{\partial x^{j(\rho_\alpha)}}{\partial \tau}. \quad (7.18)$$

Contraction of $S_1^{+}(\rho_\beta)$ with $S_1^{-}(\rho_\alpha)$ leads to a factor $-\frac{1}{2} \delta^{-1} \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-\frac{1}{2}}$ from (7.12), while contraction of other helicity combinations gives zero. The factors $\partial x^{(-)}(\rho_\beta)/\partial \tau$ and $\partial x^{(+)}(\rho_\alpha)/\partial \tau$ give the following factor, according to (4.13b):

$$\left\{ \frac{\partial}{\partial \tau_\beta} \sum_{r=1}^4 P_r^{(-)} N(\rho_\beta, \rho_r) \right\} \left\{ \frac{\partial}{\partial \tau_\alpha} \sum_{r=1}^4 P_r^{(+)} N(\rho_\alpha, \rho_r) \right\}$$

$$= t \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1} \left\{ -\frac{1}{2} (z_\beta - 1)^{-1} + z_\beta^{-1} \right\} \left\{ \frac{1}{2} (z_\alpha - 1)^{-1} - (z_\alpha - z)^{-1} \right\}. \quad (7.19)$$

We have used (7.5), together with the formula

$$\frac{\partial}{\partial z} N(z, z') = 2(z - z')^{-1}.$$

The terms z_β^{-1} and $(z_\alpha - z)^{-1}$ in the curly brackets of (7.19) are both of order δ^{-1} , and they will dominate over the other terms. We should also contract the two quantities $\partial x^{(-)}(\rho_\beta)/\partial \tau$ and $\partial x^{(+)}(\rho_\alpha)/\partial \tau$ to give a term (4.13a), but such a term will contain no factors of δ in the denominator and may be neglected. The expression (7.19) is equal to

$$\delta^{-2} t \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1} (1 - z)^{-1} \quad (7.20)$$

so that

$$F_3(z) = -\frac{1}{2} \delta^{-3} t \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-3/2} (1 - z)^{-1}. \quad (7.21a)$$

In the amplitude ϕ_1 , the helicity states of the particles 1 and 4 are "full". We shall therefore have additional factors of

$(2\alpha)^{\frac{1}{2}} S_1^{+}(\rho_1)$ and $(2\alpha)^{\frac{1}{2}} S_1^{-}(\rho_4)$, from (2.2) and (3.2a),* so that the expression to be contracted will be:

$$2\alpha S_1^{-}(\rho_4) G(\rho_\beta) G(\rho_\alpha) S_1^{+}(\rho_1).$$

The anti-commuting factors will lead to the Neumann functions:

$$\frac{\alpha}{2} \left\{ K^{-+}(\rho_\alpha, \rho_\beta) K^{+-}(\rho_1, \rho_4) - K^{-+}(\rho_\alpha, \rho_1) K^{+-}(\rho_\beta, \rho_4) \right. \\ \left. + K^{+-}(\rho_\alpha, \rho_4) K^{-+}(\rho_\beta, \rho_1) \right\}$$

$$= -\frac{1}{2} \delta^{-1} \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-\frac{1}{2}} (1 - z)^{-\frac{1}{2}}.$$

As before, the commuting factors will lead to (7.20), so that

$$F_1(z) = -\frac{1}{2} \delta^{-3} t \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-3/2} (1 - z)^{-3/2}. \quad (7.21b)$$

The calculation of $F_4(z)$ is similar, the expression to be contracted now being:

$$2i \alpha \delta^{\frac{1}{2}} S_1^{-}(\rho_4) G(\rho_\beta) S_1^{+}(\rho_3) G(\rho_\alpha).$$

The factor 1 is associated with the "full" anti-nucleon state in the bra. Contraction of the anti-commuting factors leads to the result:

* The terms involving S_1 and S_2 in (2.2) give identical contributions, so we consider only S_1 and add a factor 2.

$$\frac{1}{2} i \alpha \delta^{\frac{1}{2}} \left\{ -K^{+-}(\rho_\alpha, \rho_\beta) K^{+-}(\rho_3, \rho_4) + K^{+-}(\rho_\alpha, \rho_3) K^{+-}(\rho_\beta, \rho_4) \right. \\ \left. - K^{+-}(\rho_\alpha, \rho_4) K^{+-}(\rho_\beta, \rho_3) \right\}.$$

In this case the terms of lowest order in δ within the curly bracket cancel, and we have to include first-order corrections. The commuting factors again give the expression (7.20), so that

$$F_4(Z) = \frac{1}{2} \delta^{-2} t \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-3/2} (1-Z) Z^{-\frac{1}{2}}. \quad (7.21c)$$

Finally, for $F_2(Z)$, we have to contract the factors

$$-2 i \alpha \delta^{\frac{1}{2}} S_1^-(\rho_4) G(\rho_\beta) S_1^-(\rho_2) G(\rho_\alpha).$$

The anti-commuting factors give the Neumann functions:

$$-\frac{1}{2} i \alpha \delta^{\frac{1}{2}} \left\{ -K^{+-}(\rho_\alpha, \rho_\beta) K(\rho_2, \rho_4) + K^{+-}(\rho_\alpha, \rho_2) K^{+-}(\rho_\beta, \rho_4) \right. \\ \left. - K^{+-}(\rho_\alpha, \rho_4) K^{+-}(\rho_\beta, \rho_2) \right\}.$$

Only the second term within the curly bracket is non-zero, and the expression is equal to

$$\frac{1}{2} \delta^{-1} \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-\frac{1}{2}} Z^{\frac{1}{2}} (1-Z)^{-\frac{1}{2}}.$$

The commuting factors which occur are now $\partial x^{(-)}(\rho_\beta)/\partial \tau$ and $\partial x^{(-)}(\rho_\alpha)/\partial \tau$, so that the right-hand side of (7.19) is replaced by:

$$\left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1} z_\beta^{-1} \left\{ -\frac{1}{2} t (z_\alpha - 1)^{-1} - \frac{1}{2} \delta(2s+t)(z_\alpha - Z)^{-1} \right. \\ \left. + t z_\alpha^{-1} \right\} \\ = \delta^{-1} \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-1} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-1} Z^{-1} (1-Z)^{-1} \left\{ sZ - t(1-Z) \right\}.$$

Hence

$$F_2(Z) = \frac{1}{2} \delta^{-2} \left(\frac{\partial \rho_\alpha}{\partial z_\alpha} \right)^{-3/2} \left(\frac{\partial \rho_\beta}{\partial z_\beta} \right)^{-3/2} Z^{-\frac{1}{2}} (1-Z)^{-3/2} \left\{ sZ - t(1-Z) \right\}. \quad (7.21d)$$

It is now a straightforward matter to substitute (7.15), (7.16), (7.17) and (7.21) in (7.14). The helicity amplitudes are given by the formulas:

$$\phi_1 = s B(-s - \frac{1}{2}, -t) \quad \phi_2 = \frac{1}{2}(s-t)(s+t+1)^{-1} B(-s - \frac{1}{2}, -t - \frac{1}{2}) \\ \phi_3 = (s+t) B(-s, -t) \quad \phi_4 = t B(-s, -t - \frac{1}{2}). \quad (7.22)$$

We have omitted an irrelevant overall factor of $\frac{1}{2}$.

In an $SU(2)$ model, the states with $I = 1$ and $I = 0$ of the $q\bar{q}$ channel will have factors of $\frac{3}{2}$ and $\frac{1}{2}$, respectively. In the zero-quark channel there will be no isotopic-spin factor, but the sign of ϕ_2 will be reversed. The reversal of sign corresponds to the change of parity of the mesons with odd Neveu-Schwarz g -parity.

It is easily verified that the amplitudes (7.22) have poles corresponding only to the expected states on the leading trajectory, that all residues on the leading trajectory are positive, and that crossing is maintained.

We have chosen to display many of the details of our work, since our approach differs considerably from the usual approach. We hope that, by doing so, we have not given the reader a false impression of the length of the calculation. Once one is familiar with the method the manipulations are simple, straightforward and not too lengthy.

APPENDIX A: Volume Element with External Fermions

For simplicity we shall examine $f\bar{f}$ scattering, though the method may be extended to n-point amplitudes with hardly any change. We choose the configuration of fig. 3, where each of the strings of finite length remain fermionic throughout. We have to show that the volume element reduces to $d(\tau_\beta - \tau_\alpha)$, where τ_α and τ_β are the τ -values of the points where the strings separate and join.

The conformal transformation connecting ρ and z will be:

$$\rho = \ln(z-1) + \gamma \ln(z-Z) - (\gamma + \delta) \ln z \quad (A.1)$$

where the Z_r 's have been chosen to be 1, Z , 0, ∞ , and δ is a small quantity. The points z_α and z_β are given by the equation

$$\frac{d\rho}{dz} = \frac{1}{z-1} + \frac{\gamma}{z-Z} - \frac{\gamma+\delta}{z} = 0. \quad (A.2)$$

Solving this equation we find, to sufficient accuracy, that:

$$z_\alpha = -\delta \quad z_\beta = \gamma Z / \delta, \quad (A.3)$$

and, inserting these values in (A.1), we obtain the formula

$$\tau_\beta - \tau_\alpha = -\delta \ln Z. \quad (A.4)$$

We note that, if $\tau_\beta - \tau_\alpha$ is finite, z_α , z_β and Z are all small and that their orders of magnitude are given by:

$$|z|_\alpha \gg z_\beta \gg Z. \quad (A.5)$$

Now let us evaluate the volume element, together with the factors ΓG in (3.1b). The last factor of (3.1c) is unity and, in the limit of small Z , the penultimate factor cancels a similar factor in the volume element $\mathcal{V}$. We are left with the expression

$$G_1(\rho_\alpha + \delta\rho) G_1(\rho_\beta + \delta\rho) \left(\frac{\partial\rho}{\partial z}\right)_{\rho_\alpha+\delta\rho}^{3/2} \left(\frac{\partial\rho}{\partial z}\right)_{\rho_\beta+\delta\rho}^{3/2} \frac{z_\alpha(z_\alpha-Z)z_\beta(z_\beta-Z)}{(z_\alpha-z_\beta)Z^2} dz. \quad (A.6)$$

According to the prescription at the end of sec. 4, the factors $\partial x^1 / \partial \tau$ will give the following factors in the integrand:

$$\begin{aligned} \left\{ \frac{\partial}{\partial \tau} x^1(\rho_\alpha + \delta\rho) \right\} \left(\frac{\partial\rho}{\partial z}\right)_{\rho_\alpha+\delta\rho} &\rightarrow - \sum_r P_r^1 \frac{\partial}{\partial z} N(z_\alpha, Z_r) \\ &= - \sum_r P_r^1 \frac{1}{z_\alpha - Z_r}. \end{aligned} \quad (A.7)$$

The formula for $\{\partial x^1(\rho_\beta + \delta\rho) / \partial \tau\}$ is identical. Substituting (A.7) in (A.6) and using (A.3) and (A.4), we obtain the expression

$$k^1 k^j S_1^1(\rho_\alpha + \delta\rho) S_1^j(\rho_\beta + \delta\rho) \left(\frac{\partial\rho}{\partial z}\right)_{\rho_\alpha+\delta\rho}^{1/2} \left(\frac{\partial\rho}{\partial z}\right)_{\rho_\beta+\delta\rho}^{1/2} \frac{\gamma}{\delta^2} d(\tau_\beta - \tau_\alpha), \quad (A.8)$$

where k is the momentum carried by the intermediate meson in fig. 2. The first factors of (A.8), including the derivatives, are just the meson-fermion vertices at ρ_α and ρ_β , and the remaining factor is proportional to $d(\tau_\beta - \tau_\alpha)$ as required. We have thus verified the form of the volume element.

APPENDIX B: Phase Factors in Crossing

Here we shall outline how phase factors may appear in crossing. It will turn out that such factors always cancel one another.

Suppose that the r th string passes from the top left to the top right of fig. 1. It follows from our rules of factor ordering that its wave function has to be commuted through the factor $G(\rho_\alpha)$, ρ_α being the point of joining or separation of the string. We thereby obtain a factor of -1 for each b-excitation. This factor is cancelled by two factors of $+1$, one from (2.4a) and the other from the factor $(\partial\rho/\partial z)^{-\frac{1}{2}} \Big|_{z=Z_r}$ which occurs in the Neumann function K . The latter factor is of the form $r e^{i\eta/2}$ for a string at the top left of the diagram, but of the form $r e^{i(\eta+\pi)/2}$ for a string at the top right.

If the string in question is a fermion string there are two further phase factors independent of the number of excitations. The Neumann function arising from the factor $S_1(\rho_\alpha)$ within $G(\rho_\alpha)$ contains a factor $(Z_r - z_\alpha)^{-\frac{1}{2}}$. As the quantity $Z_r - z_\alpha$ changes from positive to negative, we acquire a factor -1 . On the other hand, the momentum P_r , which was positive time-like, now becomes negative-time-like, and one can easily show by analytic continuation that helicity states must then be defined with an extra phase factor -1 for a ket

or i for a bra. Again, therefore, the phase factors cancel.

If a meson line is moved across the bottom of the diagram, the factor-ordering problem is slightly more complicated, the effect being to give a sign change whatever the number of b-excitations in the string. The phase factor in $(\partial\rho/\partial z)^{-\frac{1}{2}}$ now cancels the phase factor in (2.4a), but there is a change of sign from re-ordering of the z 's in Γ (eq. (3.1c)) and in the factors $(Z_r - Z_s)^{-2P_r P_s}$. Once more the sign changes cancel. If a fermion line is moved from left to right across the bottom of the diagram, the phase factors from the factor $(Z_r - Z_\alpha)^{-\frac{1}{2}}$ and from the definition of the helicity states are both equal to $+1$. The amplitude thus changes sign, as it should when a fermion line is moved across the bottom of the diagram.

APPENDIX C:

Further Example of Relative Parities of $q\bar{q}$ and Zero-Quark Mesons

We are interested in the relative sign between the coupling of a $q\bar{q}$ meson to an n -fermion system and the coupling of the corresponding zero-quark meson to the same fermion system. Actually we are not interested in this relative sign itself, but in the comparison of the relative sign with a similar relative sign for the processes where the helicities of all fermions are reversed. The parities of the mesons will be the same or different according to whether the relative signs are the same or different.

As usual we examine a process such as fig. 1, where all the external particles are fermions in their ground state (except for possible zero-mode excitations).

For the left-hand side of the diagram, let

N_1 = number of positive-helicity fermions

N_2 = number of negative-helicity fermions

N_3 = number of negative-helicity anti-fermions

N_4 = number of positive-helicity anti-fermions.

Then

$$N_1 + N_2 = N_3 + N_4 \quad (C.1)$$

and, if N is the total number of particles

$$N = N_1 + N_2 + N_3 + N_4.$$

By turning fig. 1 upside down, we change the intermediate $q\bar{q}$ meson into a zero-quark meson or vice versa. As in the simpler example studied, the Neumann function for the new diagram is obtained from that for the old by the transformation $1 \leftrightarrow -1$. We therefore require to find the number of factors of 1 in the product of Neumann functions in our amplitude. This will be equal to

$$v_1 - v_2,$$

where v_1 is zero or one according to whether the amplitude is real or imaginary, and v_2 is equal to the number of factors of 1 in the wave-functions. The number v_1 will be the same for the original amplitude and the helicity reversed amplitude, so that we may ignore it, while we have seen in sec. 2 that

$$v_2 = N_4.$$

The change $1 \rightarrow -1$ in the Neumann function therefore gives rise to a factor

$$(-1)^{N_4}. \quad (C.2)$$

In addition there may be a sign change due to the re-ordering of anti-commuting factors. There are $N_2 + N_4$ such factors from the fermion wave-functions of the "full" helicity states, and $N - 1$ factors G . The order of all the factors must be reversed, so that we have to multiply the amplitude by

$$(-1)^{\frac{1}{2}(N+N_2+N_4-2)} \quad N + N_2 + N_4 \text{ even} \quad (C.3a)$$

$$(-1)^{\frac{1}{2}(N+N_2+N_4-1)} \quad N + N_2 + N_4 \text{ odd} \quad (C.3b)$$

Combining (C.2) and (C.3), we find a factor of

$$(-1)^{\frac{1}{2}(N+N_2-N_4-2)} \quad N + N_2 + N_4 \text{ even} \quad (C.4a)$$

$$(-1)^{\frac{1}{2}(N+N_2-N_4-1)} \quad N + N_2 + N_4 \text{ odd} \quad (C.4b)$$

The factor for the helicity reversed amplitude will be obtained by replacing N_2 by N_1 and N_4 by N_3 in (C.4). Multiplying the two factors, using (C.1), and remembering that N is even, we conclude that the two sign changes are the same if $N_2 + N_4$ is even but different if $N_2 + N_4$ is odd. The number of b-excitations in the intermediate meson is even or odd according as $N_2 + N_4$ is odd or even, so that the g-parity is $(-1)^{\frac{N_2+N_4}{2}}$. The relative parity of the $q\bar{q}$ and zero-quark mesons is thus the same as the g-parity, which is the result we wished to prove.

REFERENCES

- [1] S. Mandelstam, Berkeley preprint, to be published in Nuclear Physics.
- [2] P. Goddard, J. Goldstone, C. Rebbi and C. B. Thorn, Nucl. Phys. B56 (1973) 109.
- [3] S. Mandelstam, Berkeley preprint.
- [4] C. B. Thorn, Phys. Rev. D 4 (1971) 1112.
- [5] D. Olive and J. Scherk, CERN preprint.
- [6] J-L. Gervais and B. Sakita, Nucl. Phys. B34 (1971) 477; B34 (1971) 632.
- [7] J. H. Schwarz, California Institute of Technology preprint (Review article).
- [8] C. B. Thorn, private communication.
- [9] Y. Iwasaki and K. Kikkawa, Phys. Rev. D 8 (1973) 440.
- [10] Y. Aharonov, A. Casher and L. Susskind, Phys. Rev. D 2 (1971) 988.
- [11] G. Cohen-Tannoudji, A. Morel and H. Navelet, Ann. Phys. (N. Y.) 46 (1968) 239.
- [12] D. J. Candlin, Nuovo Cimento 4 (1956) 231.
- [13] C. S. Hsue, B. Sakita and M. A. Virasoro, Phys. Rev. D 2 (1970) 2857; J. L. Gervais and B. Sakita, Phys. Rev. D 4 (1971) 2291.
- [14] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell and D. Y. Wong, Phys. Rev. 120 (1960) 2250.

FIGURE CAPTIONS

- Fig. 1. String diagram for $f\bar{f}$ scattering.
- Fig. 2. String diagram for the meson-fermion vertex.
- Fig. 3. String configuration with known volume element.

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Fig. 1

(a)

(b)

Fig. 2

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$$\frac{\gamma}{\frac{\tau_a}{\tau_b}} \frac{\gamma + \delta}{1 - \delta}$$

Fig. 3

XBL738-3979

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