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Abstract

A refined version of Hartle's perturbative method of solving Einstein's equations for rotating massive objects is applied for the investigation of the general relativistic Kepler frequency of a rotating neutron star. From the *precise* determination of rotating limiting mass neutron star models for seventeen representative neutron matter equations of state we find Kepler frequencies which are systematically larger by ≈10-15% than those obtained from the empirical formula of Haensel and Zdunik. The latter results however can be reproduced from Hartle's method as well by accounting for a small mass uncertainty of the limiting star model of $\lesssim 2\%$. In comparison with an exact numerical solution, Hartle's method is relatively easy to implement and should prove to be a practical tool for testing the compatibility of nuclear equations of state on pulsar periods.

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It is well known that neutron star masses impose a constraint on theories of nuclear matter. Similarly fast rotation can impose constraints, perhaps more detailed, although a pulsar sufficiently fast to do so has not been found to date. This situation could easily change in the near future in view of the rapid pace of discovery of millisecond pulsars in globular clusters [1]. In view of this we report on our refinement of Hartle's perturbative solution of Einstein's equations for massive, rotating objects. The method is much easier to implement than an exact solution of Einstein's equations for a rotating star, and makes it a practical tool for *studying the implications of fast rotation on the equation of state of neutron star matter.*

Hartle's method, because it is perturbative with respect to deviations from spherical symmetry caused by rotation, is referred to as applying to "slow" rotation [2, 3]. To the best of our knowledge it has remained an unanswered question to date up to which rotational star frequencies, Ω, it is applicable. Light on this topic will be shed in a subsequent investigation [4], where we validate the method down to periods of $P \approx 0.5$ msec, i.e. values which are much shorter than the smallest yet observed periods (like P(PSR 1937+21)=1.6 msec, see [5]), and which are unlikely to be achieved by *neutron stars* [6]. Reservations concerning the applicability of Hartle's method have been expressed in the literature (cf. [7, 8]). These have their origin in the rotation-induced mass increase obtained for neutron stars which rotate at that limiting frequency beyond which the (Maclaurin) secular instability sets in, which is too small compared to calculations based on the exact general relativistic equations [8, 9]. In Ref. [10] however it has been found that this is no longer a problem once self-consistency, inherent in the determination of star models rotating at their respective Kepler frequencies, is imposed.

Another basic problem concerns the empirically established formula for the Kepler frequency, Ω_K. We recall that for rotation at frequencies beyond the Kepler value, mass shedding at the equator sets in which makes the star unstable. Therefore Ω_K sets an absolute upper bound on the rotational frequency. For every nuclear equation of state, which is the essential input quantity for the construction of models of neutron stars, there are uniquely determined values of Ω_K for each star in the sequence up to the limiting mass value. The Kepler frequency of the limiting-mass star obeys the so-called empirical formula which has been extracted from *exact* numerical studies of rapidly rotating neutron star models, performed for a sample of different neutron matter equations of state [11]. It relates both mass and radius of a non-rotating, spherically symmetric neutron star model of limiting mass, denoted respectively by M_s and R_s, with Ω_K. (To date there exist only heuristic arguments which motivate such a dependence (cf. [12]). In [4] we perform an analysis which motivates this dependence quantitatively.) The empirical formula is given by [11]

$$\Omega_K \approx C \cdot \sqrt{[M_s/M_\odot] \, / \, [R_s/10\,\mathrm{km}]^3}\,. \tag{1}$$

The quantity C in Eq. (1) denotes a constant. Values of $C_{\rm FIP} = 7200\ \mathrm{s}^{-1}$ [11] and $C_{\rm HZ} = 7700\ \mathrm{s}^{-1}$ [13] have been extracted for it.

As we shall see, the refined self-consistent Hartle method leads to Kepler frequencies which show a similar dependence on spherical mass and radius as the exact method. In a subsequent work we will show how this dependence can be analytically motivated, however with C depending weakly on the equation of state [4]. In our investigation we construct self-consistent models of rotating neutron stars from Hartle's method, based on a collection of a total of seventeen representative equations of state.

The idea of Hartle's treatment of rotating, fully relativistic massive objects is to develop a solution of Einstein's field equations that is based on a perturbation of the line element from that of a static star. The perturbed form is [2, 3],

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}\,(d\phi - \omega\, dt)^2 + e^{2\mu}\, d\theta^2 + e^{2\lambda}dr^2 + O\left(\Omega^3\right)\,. \tag{2}$$

The metric functions in Eq. (2), ν, ψ, μ, and λ depend on the radial coordinate as well as on the star's frequency Ω, which is not shown for the purpose of brevity. These functions are obtained in terms of monopole and quadrupole components which are themselves the solutions of a set of coupled differential equations which follow from Einstein's field equations [2, 8]. We refer to this system of equations as Hartle's stellar structure equations.

Table 1: Equations of state of this work

Label	Equation of state	Description[†]	Ref.
1	G_{300}	$R, H, K{=}300$	[14]
2	HV	$R, H, K{=}285$	[15, 16]
3	G^{DCM2}_{B180}	$R, Q, K{=}265, B^{1/4} = 180$	[17]
4	G^{DCM2}_{265}	$R, H, K{=}265$	[17]
5	G^{π}_{300}	$R, H, \pi, K{=}300$	[14]
6	G^{π}_{200}	$R, H, \pi, K{=}200$	[18]
7	$\Lambda^{00}_{Bonn} + HV$	$R, H, K{=}186$	[10]
8	G^{DCM1}_{225}	$R, H, K{=}225$	[17]
9	G^{DCM1}_{B180}	$R, Q, K{=}225, B^{1/4} = 180$	[17]
10	HFV	$R, H, \Delta, K{=}376$	[16]
11	$\Lambda^{00}_{HEA} + HFV$	$R, H, \Delta, K{=}115$	[10]
12	BJ(I)	$NR, H, \Delta,$	[19]
13	WFF(UV_{14}+TNI)	$NR, NP, K{=}261$	[20]
14	FP(V_{14}+TNI)	$NR, N, K{=}240$	[21]
15	WFF(UV_{14}+UVII)	$NR, NP, K{=}202$	[20]
16	WFF(AV_{14}+UVII)	$NR, NP, K{=}209$	[20]
17	Pan(C)	$NR, H, \Delta, K{=}60$	[22]

[†] The following abbreviations are used: R = relativistic; NR = non-relativistic; N = pure neutron; NP = n, p, leptons; π = pion condensation; H = composed of n, p, hyperons, leptons; Δ = Δ_{1232}-resonance; Q = quark hybrid composition; K = incompressibility in MeV; $B^{1/4}$ = bag constant in MeV.

The ingredient missing from earlier applications of Hartle's method is the transcendental equation for the general relativistic Kepler frequency. It is given by [7]

$$\Omega_K \;=\; e^{\nu-\psi}\; V(\Omega_K) \;+\; \omega(\Omega_K)\,, \tag{3}$$

$$V(\Omega_K) \;\equiv\; \frac{\omega'}{2\psi'}\, e^{\psi-\nu} \;+\; \sqrt{\frac{\nu'}{\psi'} + \Big(\frac{\omega'}{2\psi'}\, e^{\psi-\nu}\Big)^2}\,. \tag{4}$$

Equations (3) and (4) are to be evaluated at the star's equator. The quantity V defined in Eq. (4) denotes the orbital velocity measured by an observer with zero angular momentum in the ϕ-direction. Primes refer to derivatives with respect to the radial coordinate. The quantity ω occuring in Eqs. (2) - (4) denotes the frequency of the local inertial frames (dragging effect). An essential feature is that V and ω (like the metric functions ν and ψ) depend on Ω_K. The latter quantity is uniquely related to the mass of the rotating neutron star model, $M_{\rm rot}$. Once a neutron matter equation of state has been specified, Hartle's stellar structure equations can be solved in combination with Eqs. (3) and (4) to obtain the properties of a neutron star model rotating at $\Omega = \Omega_{\rm K}$. We stress once again that in this Letter we are only interested in non-rotating as well as rotating neutron star models at their limiting gravitational masses. The construction of non-rotating neutron star models is relatively easy to accomplish by solving the Oppenheimer-Volkoff equations. In contrast, the construction of models of rotating neutron stars of limiting mass is more involved since a self-consistency problem inherent in the determination of the Kepler frequency of Eq. (3) is encountered. In short, the problem is to find that value of the star's central energy density, ϵ_c, for which Hartle's stellar structure equations lead for $M_{\rm rot}(\Omega)$ to its limiting value (i.e. $[\partial M_{\rm rot}/\partial \epsilon_c] = 0$) for $\Omega = \Omega_{\rm K}$ which *simultaneously* satisfies Eqs. (3) and (4). Surprisingly this has not been recognized in applications of Hartle's method. (A detailed description for finding the Kepler frequency is outlined in Refs. [10, 23].)

In Fig. 1 we plot our results for $\Omega_{\rm K}$, calculated from Eq. (3), as a function of $\sqrt{[M_{\rm s}/M_\odot]/[R_{\rm s}/10\,{\rm km}]^3}$ for the seventeen neutron matter equations of state (labeled "1" through "17") of Table 1. This collection contains equations of state derived from relativistic nuclear field theory ("1" through "11") as well as from non-relativistic Schroedinger theory based on potential models for the nucleon-nucleon interaction ("12" through "17"). It turned out that a very *precise* determination of the limiting mass is required since $\Omega_{\rm K}$ changes very rapidly as a function of $M_{\rm rot}$ (related to rapid changes in radius) at the mass limit [7], which can be represented by

$$\frac{\Delta\Omega_{\rm K}}{\Omega_{\rm K}} \approx \pm 10 \, \frac{\Delta M_{\rm rot}}{M_{\rm rot}} \, . \tag{5}$$

This sensitive dependence of $\Omega_{\rm K}$ on limiting mass is grahically depicted by means of the vertical bars in Fig. 1 for all equations of state of our collection. Each of these bars shows the change of $\Omega_{\rm K}$ with respect to relative rotational mass changes $\Delta M_{\rm rot}/M_{\rm rot} \lesssim 1\%$. Typically the associated error in determining $\Omega_{\rm K}$ is, according to Eq. (5), $\approx \pm 10\%$. We stress that the precise determination of mass (and radius) in the framework of the *exact* method is restricted by the fact that a compromise between numerical accuracy and radial grid spacing is to be made. This implies errors in mass and radius of respectively 1% and 5% [7]. In other words the determination of the mass limit to better than 1% has not been performed for the exact method.

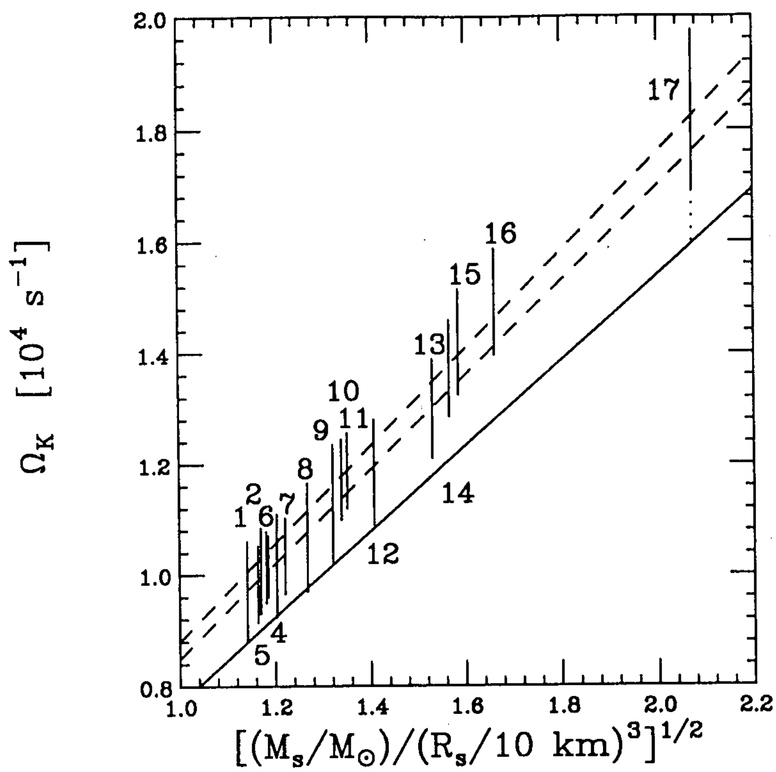

Figure 1: Comparison of the Kepler frequency Ω_K (solution of Eq. (3)) calculated for the seventeen equations of state of Table 1 (labeled "1" through "17"). The ordering at small x-values is (from left to right) "1", "2", "5", "3", "6", "4". Note that equations of state "3" and "6" are located at more or less the same x-value. Therefore only label "6" is drawn in. The vertical bars show the sensitive dependence of Ω_K on limiting rotational star mass (see text). The slope of the Haensel-Zdunik approximation (solid line), given by $C_{HZ} = 7700\ s^{-1}$, is compared with values of $8500\ s^{-1}$ and $8800\ s^{-1}$ supported by our investigation.

In contrast Hartle's method is numerically easier to treat and therefore provides an appropriate tool for a more precisely investigation of $\Omega_K(M_{rot})$.

We conclude from the above described sensitive dependence of Ω_K on limiting mass that:

1. The rotating limiting mass models must be determined to a very high degree of accuracy (1% in $\Delta M_{rot}/M_{rot}$ to get 10% in $\Delta\Omega_K/\Omega_K$). Since this degree of accuracy is a problem of the exact method, values of Ω_K derived from it [7, 11] may have underestimated Ω_K by up to 10-15%;

2. Our Ω_K values support a slope value of $8500\ s^{-1} \lesssim C_{Hartle} \lesssim 8800\ s^{-1}$ (lower and upper dashed lines in Fig. 1, respectively). In comparison with $C_{HZ} = 7700\ s^{-1}$, our C-value is 10-15% larger;

3. We believe the Haensel and Zdunik [13] results for Ω_K are underestimates for the following reason. They can be obtained from Hartle's method for the equations of state labeled "1" through "9", "12", and "14", *provided* the limiting mass values are all assumed to be underestimated by $\approx 1\%$ (see bottom ends of the vertical bars of Fig. 1). The assumed errors for the remaining equations of state are $\approx 2\%$. This is shown, for example, for equation of state "17" (Pan(C)) by means of the vertical dotted line. The lower part of it ends at the Haensel-Zdunik value (solid line). Note that an underestimate of the mass is more natural than an overestimate since one begins the calculations at a lower mass and works up to the maximum.

More details of the findings presented here will be given in a subsequent article [4] where we: (a) report the results of a sequence of stars that are rotating at their Kepler frequencies up to the limiting star of each sequence; (b) compare rotating neutron star models derived from Hartle's method with their exact counterparts (the rotation induced mass increase, for example, is very well obtained in the framework of Hartle's formalism [4]); (c) shed light on the empirical formula for Ω_K by means of an analytic investigation of Hartle's method (an expression for the slope factor C is derived which exhibits its dependence on the equation of state). In essence the applicability of Hartle's method down to rotational periods of ≈ 0.5 msec is confirmed. For this reason we feel confident as concerns the findings of this Letter.

Since Hartle's method is much simpler to implement than an exact solution of the general relativistic equations for a rotating star, we draw attention to it in this Letter as a practical tool in evaluating the implications of pulsar rotation on the equation of state (cf. [10] and reference [4]) just as the Oppenheimer-Volkoff equations are for evaluating the constraints imposed by neutron star masses. It is only a little more complicated than solving the latter problem because the equations (exact or approximate) for rotation do not simply constitute an initial value problem as for a static star.

In summary, from the investigation of the Kepler frequencies, Ω_K, of rotating neutron star models for seventeen neutron matter equations of state in the framework of Hartle's "slow" rotation formalism, we have shed light on the sensitive dependence of this quantity on the limiting neutron star mass. From the very precise determination of the limiting mass point, which turned out to be of crucial importance since even small deviations from it strongly influence Ω_K, we have demonstrated the impact of mass errors of $\approx 1\%$ (inherent in the determination of the limiting mass from the exact method) on the Kepler frequency. We arrive at a corresponding $\approx \pm 10\%$ uncertainty of Ω_K.

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