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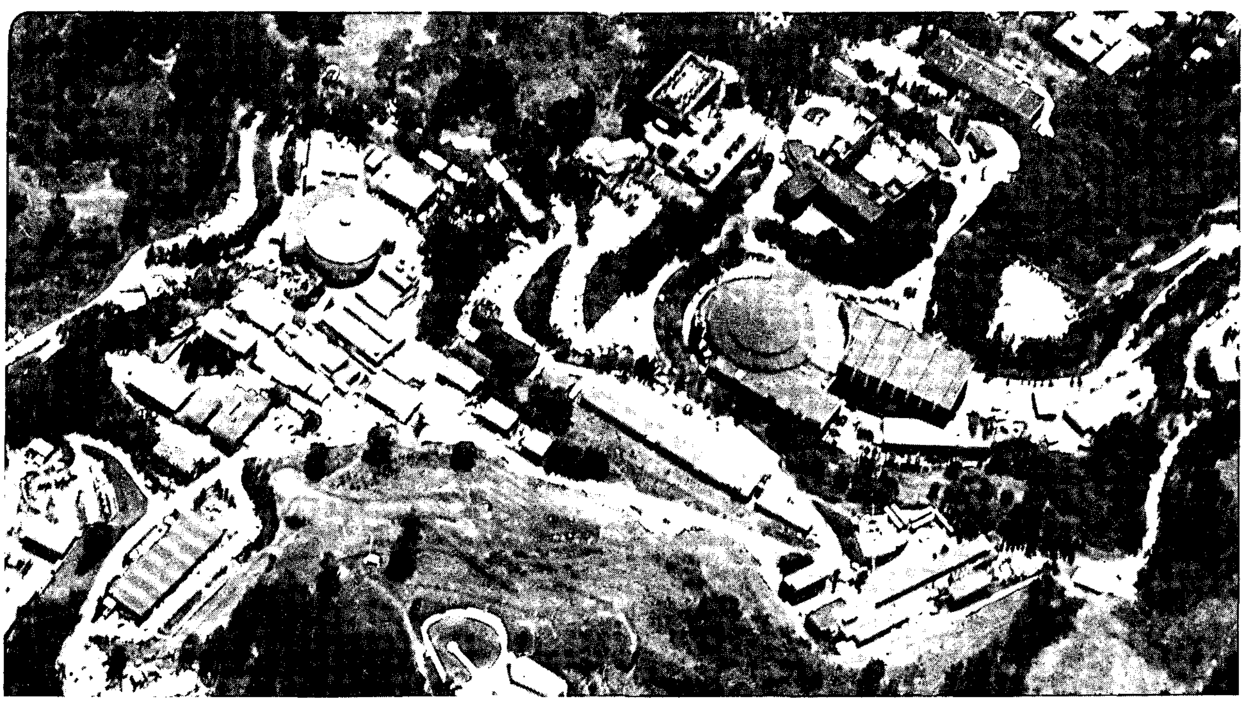

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Derivation and Solution of the Discrete Pressure Equations for the Incompressible Navier-Stokes Equations[1]

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DERIVATION AND SOLUTION OF THE DISCRETE PRESSURE EQUATIONS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

CHRISTOPHER R. ANDERSON*

Abstract. In this paper we derive the equations for the pressure which must be solved in order to advance several commonly used finite difference approximations to the Navier-Stokes equations on rectangular domains. We consider the equations for both staggered and non-staggered grids. We present solution techniques for these equations which require on the order of O(N log N) operations where N is the number of points in the computational domain.

1. Introduction. The equations describing the motion of a two dimensional incompressible fluid of constant density consist of the momentum equation,

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla \mathrm{P} + \nu \Delta \vec{u} \tag{1.1}$$

and the equation expressing incompressibility,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.2}$$

Here $\vec{u} = (u, v)$ is the velocity field, P is the pressure and ν is the coefficient of viscosity. We assume that the domain in which the equations are solved, Ω, is a rectangular region and the following initial and boundary conditions are specified,

$$\vec{u}(x, y, 0) = \vec{u}_0(x, y) \qquad (x, y) \quad \in \Omega \tag{1.3}$$

$$\vec{u}(x, y, t) = \vec{u}_b(x, y) \qquad (x, y) \quad \in \partial\Omega. \tag{1.4}$$

One fundamental difficulty in advancing approximations to these equations in time is that there is no explicit equation for the pressure. From a continuous viewpoint the pressure in (1.1) is determined by incompressibility constraint (1.2), but the translation of this fact into the discrete equations is somewhat tricky. Our goal here is to derive discrete equations for the pressure which will guarantee that discrete approximations to (1.1)-(1.2) are completely satisfied at every timestep in an explicit time marching scheme. The main idea is that if the non-linear terms are treated explicitly, then the determination of the pressure becomes a problem in linear algebra. In the completely discrete formulation there is no need to worry about the equations for the pressure on the boundary - they are determined by the choice of discretization of (1.1)-(1.2) and some basic principles of matrix operations. The concept of obtaining equations for the pressure by focussing completely on the discrete equations is implicit in the work of Chorin [3] and has been advocated more recently by Strikwerda [10], Gresho and Sani [6] and Maday, Patera and Ronquist [8].

We are interested in the equations for the pressure because of a desire to construct "fast" methods for their solutions, i.e. methods which take on the order of O(N log N) operations where N is the number of unknowns in the computational domain. A fast solution procedure is highly desirable because the solution of the pressure equations must be carried

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out at each time step in the advancement of approximations to (1.1)-(1.2). The equations for the pressure are quite similar to those of the discrete Laplacian and it originally appeared that one could easily incorporate existing fast Laplace solvers to the design effective techniques for their solution. However, this task proved much more difficult than expected. It was realized that the primary difficulty comes about because the algebraic nature of the equations can change greatly depending on the type of grid used (staggered or unstaggered) and on the approximations to the divergence operator used. It became evident that in order to design an effective iterative or direct technique it was necessary to understand in detail the structure of these equations. In this paper we will derive the equations for three popular choices of grids and discretizations and discuss the algebraic nature of the resulting equations. In the first formulation the equations can be solved by an application of an existing fast solver. The equations in the other two cases cannot be solved by applications of standard Dirichlet and Neumann problem fast solvers. We therefore present an iterative method based on preconditioned conjugate gradients [4] and a direct method based on the method of matrix partitioning [7].

The focus of this paper is to work with the discrete approximations of (1.1)-(1.2) to obtain equations for the pressure, rather than finding continuous equations for the pressure (including boundary conditions) and then discretizing them. The key problem in working with the continuous pressure equations is the choice of boundary conditions. There has been much work on the formulation of good continuous boundary conditions, some of which leads to reasonable methods, but we avoid using these boundary conditions and the continuous approach in general for two reasons. Firstly, if one decides on the discrete equations for the divergence condition and the momentum equations, then the discrete pressure equations are determined automatically - there is no choice. The boundary equations one gets are not inconsistent with continuous boundary conditions obtained by taking the normal component of the momentum equation. In fact, as Gresho and Sani point out [6], the boundary equations associated with the three problems we consider here can be seen to arise from some approximation to Neumann derivative type boundary conditions. However, in the equations for the pressure on a non-staggered grid there is a decoupling of the equations onto subgrids. While it is true that when all of the equations are taken together, the discretization appears to be an approximation to a Laplace operator with a Neumann boundary condition, the equations on the subgrids do not necessarily have this nice resemblance. Second, if one does follow the continuous approach then there are typically several different discrete implementations which seem reasonable. If you happen to be lucky you will chose the implementation which guarantees the discrete divergence conditions are satisfied exactly, if you are not lucky, then the discrete divergence condition will not be satisfied. This problem is most simply illustrated by the set of equations which result from a staggered grid. As will be seen in section three the pressure equations are that of a discrete Laplacian with Neumann boundary conditions. The incorporation of Neumann boundary conditions is, however, done using first order approximations to the boundary normal derivative, rather than the customary second order approximations.

We wish to emphasize that our goal is to focus on the discrete equations for the pressure in finite difference approximations to the Navier-Stokes equations and appropriate solutions procedures for them. There are other approaches to the problem of finding the pressure and advancing finite difference solutions of the Navier-Stokes equations in time. For example, if one is willing to relax the requirement that the divergence free condition be satisfied exactly, then this freedom can be used effectively to design numerical methods [10]. Alternatively the advancement of an approximation to the Navier-Stokes equations can proceed without the explicit determination (and solution) of equations for the pressure [2]. The idea is to project the solution at every time-step onto a divergence free basis - i.e. a set of functions which all satisfy the divergence condition exactly. We are also not discussing approximations based on finite elements. One paper of interest is that of Maday, Patera and Ronquist [8] in which ideas and solution procedures similar to those given here are presented for the finite element formulation of the Stokes equations. (For more on the finite element approach see [5]).

In the first section we show formally how to derive the equations for the pressure. In the second and third sections we explicitly derive the pressure equations for discretizations on staggered and non-staggered grids. In the fourth section we discuss fast solution procedures for these equations and present the results of some computational experiments. We follow this section with our conclusions and an appendix which gives the pressure equations and orthogonality conditions associated with a non-staggered grid.

2. Construction of the Equations (I). In this section we construct the equations which determine the pressure when an explicit time stepping scheme is used to advance an approximation to the Navier-Stokes equations. The same approach can be use to construct equations for other time discretization methods.

Assume the rectangular domain Ω is covered by a uniform grid of mesh width h and that a choice of the approximations of the individual terms in (1.1) and (1.2) has been made. Let ∇_x^h and ∇_y^h be the difference approximations to the derivatives used in the calculation of the gradient of the pressure, i.e. $\nabla \mathrm{P} \approx (\nabla_x^h \mathrm{P}, \nabla_y^h \mathrm{P})$ and let Δ^h be a difference approximation to the Laplacian. We denote by D_x^h and D_y^h the approximations to the derivatives used in the approximation of the divergence operator, so that $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx \mathrm{D}_x^h u + \mathrm{D}_y^h v$. Consider the following explicit scheme to advance an approximation to (1.1) - (1.2) in time,

$$\frac{\vec{u}^{n+1} - \vec{u}^n}{\delta t} + \vec{\mathrm{N}}(\vec{u}^n) = -(\nabla_x^h, \nabla_y^h)\mathrm{P}^{n+1} + \nu \Delta^h \vec{u}^n \tag{2.1}$$

$$\mathrm{D}_x^h u^{n+1} + \mathrm{D}_y^h v^{n+1} = 0. \tag{2.2}$$

Here $\vec{\mathrm{N}}(\vec{u}^n)$ is an approximation to the non-linear terms in the momentum equation. The problem is now to solve (2.1)-(2.2) to obtain $\vec{u}^{n+1}$. The method which is typically employed is to determine the pressure P^{n+1} first and then use (2.1) to find $\vec{u}^{n+1}$.

The first step in deriving the pressure equation consists of rewriting the incompressibilty constraint (2.2). The boundary values of the velocity are known so in equation (2.2) we may put the components of the equations which involve known quantities onto the right hand side. We obtain a set of equations of the form

$$\tilde{\mathrm{D}}_x^h u^{n+1} + \tilde{\mathrm{D}}_y^h v^{n+1} = \tilde{\mathrm{F}}. \tag{2.3}$$

The difference operators $\tilde{\mathrm{D}}_x^h$ and $\tilde{\mathrm{D}}_y^h$ are the terms in the divergence approximation which contain only unknown values of the velocity. For example, if we have a grid and velocities as in Figure 1, then using centered difference approximations to the incompressiblity constraint at the node $(i, 1)$ would give

$$\tilde{\mathrm{D}}_x^h = \frac{u_{i+1,1} - u_{i-1,1}}{2h} \quad \text{and} \quad \tilde{\mathrm{D}}_y^h = \frac{v_{i,2}}{2h} \tag{2.4}$$

We shall refer to the operator represented by $(\tilde{\mathrm{D}}_x^h, \tilde{\mathrm{D}}_y^h)$ as the *reduced* divergence operator. The known values on the boundary are already incorporated into the right hand side of (2.1), so there is no need to consider reduced versions of the finite difference operators involved in the discretization of the momentum equations. (The situation would be different, however, if we had a different time stepping scheme.)

A convenient way to determine the equations for the pressure is to write the equations (2.1) and (2.3) in the following block matrix form,

$$\begin{pmatrix} I & 0 & \nabla_x^h \\ 0 & I & \nabla_y^h \\ \tilde{D}_x^h & \tilde{D}_y^h & 0 \end{pmatrix} \begin{pmatrix} u^{n+1} \\ v^{n+1} \\ P^{n+1} \end{pmatrix} = \begin{pmatrix} u^n - \delta t(N_1(\overline{u^n}) + \nu\Delta u^n) \\ v^n - \delta t(N_2(\overline{v^n}) + \nu\Delta v^n) \\ \tilde{F} \end{pmatrix} \tag{2.5}$$

The equations for the pressure are constructed by finding the Schur compliment of the (3,3) entry of the matrix on the left hand side of (2.5). (Forming the Schur compliment is the same as performing block Gaussian elimination.) This results in

$$\begin{aligned} &-(\tilde{D}_x^h\nabla_x^h \;+\; \tilde{D}_y^h\nabla_y^h)P^{n+1} = \\ &\tilde{F} \;-\; \tilde{D}_x^h(u^n - \delta t(N_1(\overline{u^n}) + \nu\Delta^h u^n)) - \tilde{D}_y^h(v^n - \delta t(N_2(\overline{v^n}) + \nu\Delta^h v^n)) \end{aligned} \tag{2.6}$$

These are the equations which determine the pressure P^{n+1}.

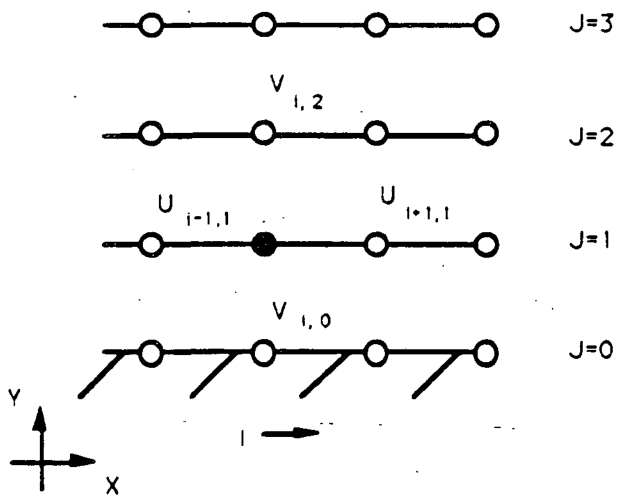

Figure 1

The important point to realize is that in deriving the equations for the pressure no assumptions about the values of the pressure or its normal derivative on the boundary have been made. The particular combination of the gradient and divergence approximations used are all that is needed. Once these approximations are specified, there is sufficient information to obtain the pressure equations. It is very tempting to conclude that the equation which determines the pressure is the discrete Laplacian, since $(\tilde{D}_x^h\nabla_x^h + \tilde{D}_y^h\nabla_y^h)$ appears to approximate div(grad) $= \Delta$, but this is precisely the conclusion one should be very careful of making. The reason for caution is that the operators $\tilde{D}_x^h$ and $\tilde{D}_y^h$ are not complete approximations to their differential counterparts at points near the boundaries - they are just pieces of the approximations. So, depending on the discretization of the gradient and divergence operators and the grid used (either staggered or non-staggered) the operators in (2.6) may or may not combine to form the standard Laplacian at interior points. Furthermore if they do combine to form a "nice" Laplacian in the interior, then because reduced stencils occur in the discrete divergence operators at the boundaries, the pressure equations at the boundaries may not look like the discrete version of any typical boundary conditions for the discrete Laplacian. For some particular discretizations (one of which we shall discuss later) these operators do form a discrete Laplacian with recognizable boundary conditions, but in other discretizations, such as that used with a non-staggered grid they do not.

The scheme for advancing the solution of the Navier-Stokes equations in time (2.1)-(2.2) can be viewed as a projection method similar to the one introduced by Chorin [3]. This is understood by writing the scheme in a fractional step fashion. First, the momentum equation is advanced without regard to the pressure,

$$\frac{\tilde{u}^{n+1} - \vec{u}^{n}}{\delta t} + \vec{\mathrm{N}}(\vec{u}^{n}) = \nu \Delta^{h} \tilde{u}^{n+1}. \tag{2.7}$$

This is followed by a step where the pressure gradient is taken into account and the incompressiblity constraint is enforced.

$$\frac{\vec{u}^{n+1} - \tilde{u}^{n+1}}{\delta t} = -(\nabla_x^h, \nabla_y^h)\mathrm{P}^{n+1}$$

$$(\mathrm{D}_x^h, \mathrm{D}_y^h) \cdot \vec{u}^{n+1} = 0. \tag{2.8}$$

This last system of equations (2.8), corresponding to the fractional step using the pressure, can be viewed as projection of the velocity field $\tilde{u}^{n+1}$ onto a divergence free vector field. These equations can be rewritten as

$$\tilde{u}^{n+1} = \vec{u}^{n+1} + \delta t(\nabla_x^h, \nabla_y^h)\mathrm{P}^{n+1}$$

$$(\mathrm{D}_x^h, \mathrm{D}_y^h) \cdot \vec{u}^{n+1} = 0$$

i.e. $\tilde{u}^{n+1}$ is expressed as a divergence free vector field and the gradient of a scaler. The solution of (2.8) can be obtained by finding the pressure first and then constructing $\vec{u}^{n+1}$. The equations one uses to find the pressure are identical to (2.6) (just eliminate $\tilde{u}^{n+1}$ in (2.7)-(2.8)).

The generality of the above presentation obscures some important facts about the system of equations for the pressure. The first of which is that the equations which determine the pressure (2.6) are usually singular. This is a manifestation of the fact that the pressure in the fluid is only determined up to a constant. Secondly, if the divergence and gradient operators are adjoints, then one can show that the equations which determine the pressure are semi-definite. We have

$$(u, v, \mathrm{P}) \begin{pmatrix} \mathrm{I} & 0 & \nabla_x^h \\ 0 & \mathrm{I} & \nabla_y^h \\ \mathrm{D}_x^h & \mathrm{D}_y^h & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ \mathrm{P} \end{pmatrix} =$$

$$(u, u) + (u, \nabla_x^h \mathrm{P}) + (v, v) + (v, \nabla_y^h \mathrm{P}) + (\mathrm{D}_x^h u, \mathrm{P}) + (\mathrm{D}_y^h, \mathrm{P})$$

$$= (u, u) + (v, v) \geq 0 \tag{2.9}$$

for all (u, v, P) if $(u, \nabla_x^h \mathrm{P}) = -(\mathrm{D}_x^h u, \mathrm{P})$ and $(v, \nabla_y^h \mathrm{P}) = -(\mathrm{D}_y^h v, \mathrm{P})$. (This occurs, for example, when one uses a staggered grid and central differencing.) Thus, the solution of the pressure equations using iterative methods based on relaxation are appropriate as long as care is taken concerning the null spaces of the equations. Unfortunately, other than examining the specific equations one gets for each discretization, the author knows of no general procedure for determining what the null spaces are. As two of the examples in the third part of this paper will show, this understanding of the null spaces and the orthogonality conditions which must be satisfied by the right hand side of (2.6) can become somewhat complicated.

One can formally derive equations for the pressure if other time discretizations are used. For example, it is common to make the evaluation of the discrete Laplacian implicit (for stability reasons). In this instance the equations in block form would be

$$(2.10)\qquad \begin{pmatrix} I-\delta t\nu\Delta^h & 0 & \nabla_x^h \\ 0 & I-\delta t\nu\Delta^h & \nabla_y^h \\ \tilde{D}_x^h & \tilde{D}_y^h & 0 \end{pmatrix} \begin{pmatrix} u^{n+1} \\ v^{n+1} \\ P^{n+1} \end{pmatrix} = \begin{pmatrix} u^n - \delta t N_1(\vec{u}^n) \\ v^n - \delta t N_2(\vec{v}^n) \\ \tilde{F} \end{pmatrix}$$

The general structure of the equations for the pressure becomes one of the form

$$-(\tilde{D}_x^h(I+\delta t\nu\Delta^h)\nabla_x^h + \tilde{D}_y^h(I+\delta t\nu\Delta^h)\nabla_y^h)P^{n+1} = \tilde{F}.$$

While this system of equations has its own peculiarities, if $\nu\delta t$ is small enough then the equations can be seen as a perturbation of those for the pressure equations when an explicit scheme is used. Thus, our concentration on the explicit time-stepping scheme is not without merit, for the inverse of the pressure equations in this case may form a good approximate inverse to be used in the iterative solution of the pressure equations corresponding to the semi-implicit time stepping scheme.

3. Construction of the Equations (II). In the equations for the pressure (2.6) we find that they are composed of finite difference operators. One can form the matrices which correspond to these operators and multiply them all out to get the equations for the pressure, but this is cumbersome. An easier way to construct the equations is to go through the procedure of determining the Schur compliment as indicated by (2.6), but when a matrix multiply is called for, don't think of it as multiplying matrices, but think of it as applying a particular finite difference operator. The procedure is best illustrated with a concrete example, and we therefore consider an implementation of the Navier-Stokes equations using a staggered grid.

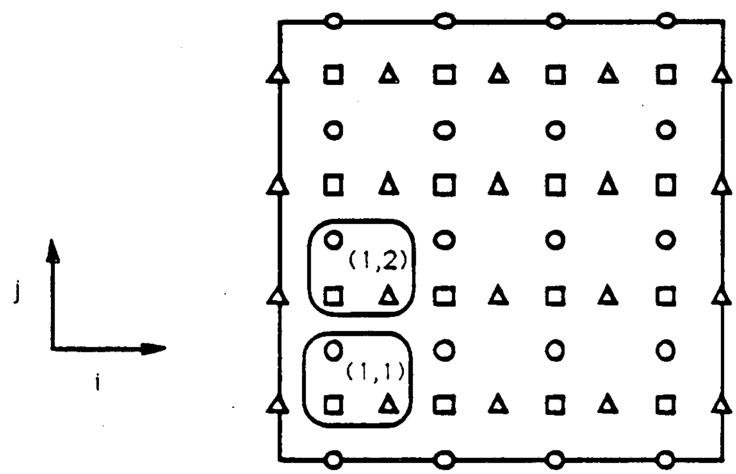

Figure 2

In figure 2 we present the staggered grid used. The physical boundary is denoted by the solid lines around the perimeter. On this grid the pressure is tabulated at the square points, the u velocity at the triangle points, and the v velocity at the circle points. Note that this grid is a replication of a triplet of points consisting of one square, one triangle, and one circle. We number the unknowns as triplets so that all the variables in the region labelled $(1,1)$ have subscripts $(1,1)$. This is an unambiguous labelling since distinct variables are labelled at distinct types of grid points. This is more convenient for our presentation than the usual labelling of the points which uses half-indices. Let h be the spacing between successive pairs of similar points, and assume there are M sets of points in the x direction and N sets of points in the y direction.

In the typical discretization of the equations of motion for this grid, one centers the approximation to the u component of the momentum equation about the triangle points, the v component of the momentum equation about the circle points, and the divergence condition at the square points. For example, an approximation to the u momentum equation at the point (i,j) in the interior would be,

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\delta t}+\mathrm{A}_{i,j}^{n}=-\frac{\mathrm{P}_{i+1,j}^{n+1}-\mathrm{P}_{i,j}^{n+1}}{h}+\nu\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}\right)$$

Here we have used $\mathrm{A}_{i,j}^{n}$ to denote an approximation to the non-linear advection term based on the velocities at time n. For specific examples of some choices of $\mathrm{A}_{i,j}^{n}$ see [9]. The equation for the v component of the momentum is similar. We abbreviate the description of the two momentum equations by

$$u_{i,j}^{n+1}=u_{i,j}^{n}-\delta t\left(\frac{\mathrm{P}_{i+1,j}^{n+1}-\mathrm{P}_{i,j}^{n+1}}{h}\right)+\delta t\,\mathrm{F}_{i,j}^{n} \tag{3.1}$$

$$v_{i,j}^{n+1}=v_{i,j}^{n}-\delta t\left(\frac{\mathrm{P}_{i,j+1}^{n+1}-\mathrm{P}_{i,j}^{n+1}}{h}\right)+\delta t\,\mathrm{G}_{i,j}^{n} \tag{3.2}$$

The approximation to the incompressiblity condition is centered at the square points. If we use second order centered approximations to the derivatives, then the equations for the interior square points are of the form

$$\frac{u_{i,j}^{n+1}-u_{i-1,j}^{n+1}}{h}+\frac{v_{i,j}^{n+1}-v_{i,j-1}^{n+1}}{h}=0 \tag{3.3}$$

For the square points nearest the boundary, the difference approximation is not the complete stencil as that above. For the bottom edge we would have the equation

$$\frac{u_{i,1}^{n+1}-u_{i-1,1}^{n+1}}{h}+\frac{v_{i,1}^{n+1}}{h}=\frac{v_{i,0}^{b}}{h}\quad\text{for}\quad i=2,3,\ldots,M-1 \tag{3.4}$$

and for the left and right bottom corner points,

$$\frac{u_{1,1}^{n+1}}{h}+\frac{v_{1,1}^{n+1}}{h}=\frac{u_{0,1}^{b}}{h}+\frac{v_{1,0}^{b}}{h} \tag{3.5}$$

$$\frac{-u_{M,1}^{n+1}}{h}+\frac{v_{M,1}^{n+1}}{h}=\frac{-u_{M-1,1}^{b}}{h}+\frac{v_{M,0}^{b}}{h}. \tag{3.6}$$

We have used $u_{i,j}^{b}$ and $v_{i,j}^{b}$ to denote the known boundary values of the u and v velocity. For the other sides and corner points the equations are similar to (3.5) - (3.6). The presence of boundary data has the effect of causing reduced stencils to be used near boundary points as well as giving rise to discrete equations which are not homogeneous - i.e. the boundary data enters in as a forcing function.

The key to deriving equations for the pressure is to apply the reduced divergence operator (defined by the left hand sides of equations (3.3)-(3.6)) to the momentum equations

(3.1)-(3.2). In essence, forming the Schur compliment in equation (2.5) is the same as taking the discrete divergence of the momentum equation, but one does not use the complete discrete divergence operator - only that part defined by the reduced divergence operator.

In deriving equations for the pressure we express (3.1)-(3.2) as

$$\frac{P^{n+1}_{i+1,j} - P^{n+1}_{i,j}}{h} = \frac{u^n_{i,j}}{\delta t} - \frac{u^{n+1}_{i,j}}{\delta t} - F^n_{i,j} \tag{3.7}$$

$$\frac{P^{n+1}_{i,j+1} - P^{n+1}_{i,j}}{h} = \frac{v^n_{i,j}}{\delta t} - \frac{v^{n+1}_{i,j}}{\delta t} - G^n_{i,j}. \tag{3.8}$$

We now apply the reduced divergence operators defined by equations similar to one of (3.3)-(3.6) at each point of the square grid. Depending on the form of the reduced divergence operator we get different equations. For points which are in the interior, the reduced divergence operator is given by (3.4) and this results in a pressure equation of the form

$$\left(\frac{P^{n+1}_{i+1,j} - 2P^{n+1}_{i,j} + P^{n+1}_{i-1,j}}{h^2} + \frac{P^{n+1}_{i,j+1} - 2P^{n+1}_{i,j} + P^{n+1}_{i,j-1}}{h^2}\right) = R^n_{i,j} \tag{3.9}$$

where $R^n_{i,j}$ is the reduced discrete divergence operator applied to the terms on the right hand side of (3.7) which don't involve $u^{n+1}_{i,j}$ and $v^{n+1}_{i,j}$. For edge and corner points, we apply the operator defined by (3.5) and (3.6) respectively and we get equations of the form

$$\left(\frac{P^{n+1}_{i+1,1} - 2P^{n+1}_{i,1} + P^{n+1}_{i-1,1}}{h^2} + \frac{P^{n+1}_{i,2} - P^{n+1}_{i,1}}{h^2}\right) = R^n_{i,1} - \frac{1}{\delta t}\frac{v^b_{i,0}}{h} \tag{3.10}$$

for $i = 2, 2, \ldots, M-1$ and

$$\left(\frac{P^{n+1}_{2,1} - P^{n+1}_{1,1}}{h^2} + \frac{P^{n+1}_{1,2} - P^{n+1}_{1,1}}{h^2}\right) = R^n_{1,1} - \frac{1}{\delta t}\left(\frac{u^b_{0,1}}{h} + \frac{v^b_{1,0}}{h}\right) \tag{3.11}$$

for the bottom lower left corner. The equations for the other edges and corners is similar. As before, the terms $R^n_{i,j}$ result from the reduced discrete divergence operator applied to the right hand side of (3.7) which don't involve $u^{n+1}_{i,j}$ and $v^{n+1}_{i,j}$. The boundary values have entered the equation because the application of the reduced divergence operator to $u^{n+1}_{i,j}$ and $v^{n+1}_{i,j}$ does not yield zero but the values on the right hand side of (3.5)-(3.6).

By inspection, the equations which determine the pressure are those of a discrete Laplacian centered at the square points with Neumann boundary conditions applied at the edges. If one solves these equations for the pressure P^{n+1} then defines u^{n+1} and v^{n+1} using the (3.1) and (3.2) then it is easy to verify that u^{n+1} and v^{n+1} satisfy the discrete incompressiblity constraint as well.

There is a non-trivial null-space corresponding to these equations - namely the pressure field which is identically constant. It can be proven using the discrete maximum principle that this is the only null vector. Therefore, in order that there be a solution of the equations determining the pressure we must have that the right hand side is orthogonal to this vector. The terms which are denoted by $R_{i,j}$ are obtained by applying the reduced discrete divergence operator to a vector consisting of known terms in the momentum equation, and it can be verified that

$$\sum_{i,j} R^n_{i,j} = 0$$

i.e. this component of the right hand side is orthogonal to the constant vector. Thus, the only contribution to the inner product of the constant vector with the right hand side is the boundary velocity data, and we find that

$$-\sum_{i=1,M} v^b_{i,0} - \sum_{j=1,N} u^b_{0,j} + \sum_{i=1,M} v^b_{i,N} + \sum_{j=1,N} u^b_{M,j} = 0$$

must be satisfied. If we multiply this expression by h we see that this is a midpoint rule approximation to the condition that the integral of the normal velocity around the boundary vanishes.

Although the equations which the pressure satisfy consist of a discrete Laplace operator with Neumann boundary conditions, these are not the standard second order equations one uses for Neumann problems. Specifically, let us consider a point along the bottom boundary. The typical method for incorporating Neumann boundary conditions consists of eliminating the point of the difference stencil outside the domain by using a centered approximation to the derivative at the boundary. When this is carried out (and the equation multiplied by 1/2 to make the system symmetric) the difference equation which results is of the form

$$\frac{\frac{1}{2}\,\mathrm{P}^{n+1}_{i+1,1} - 2\mathrm{P}^{n+1}_{i,1} + \frac{1}{2}\,\mathrm{P}^{n+1}_{i-1,1}}{h^2} + \frac{\frac{1}{2}\mathrm{P}^{n+1}_{i,2}}{h^2}.$$

On the other hand, the difference equation (3.10) is of the form

$$\frac{\mathrm{P}^{n+1}_{i+1,1} - 3\mathrm{P}^{n+1}_{i,1} + \mathrm{P}^{n+1}_{i-1,1}}{h^2} + \frac{\mathrm{P}^{n+1}_{i,2}}{h^2}.$$

This would be the result if a first order one-sided difference approximation were used to eliminate the point outside the domain. The fact that the boundary conditions are different illuminates why there is difficulty if one tries to determine the pressure from a continuous approach. If one started with the idea of using a Laplace operator with Neumann boundary conditions then one must chose between two (and possibly many more) ways of implementing this boundary condition. As can be seen in this example, there are two reasonable choices, one of which insures that the discrete divergence condition is satisfied, and the other which does not.

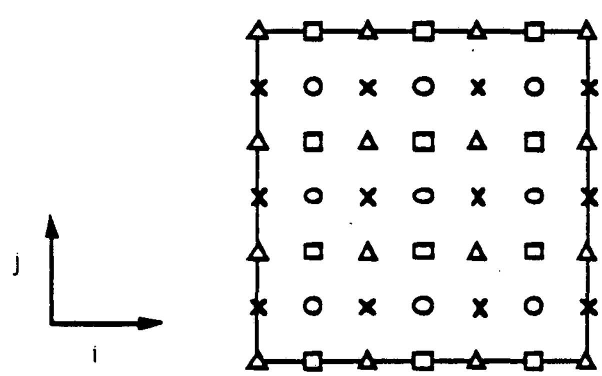

Figure 3

4. Equations For A Non-Staggered Grid. In this section we derive the equations for the pressure on a non-staggered grid. In Figure 3 we exhibit the non-staggered grid. We assume that the grid has an even number of panels in each direction (M in the x direction and N in the y direction). The numbering of the points is the same as that in the first quadrant of the Cartesian plane - the $(0,0)$ node being the lower left corner, the $(1,0)$ node being the one just to the right of it, etc.. In anticipation of the equations for the pressure, we have labelled the points by one of four symbols - a circle, a triangle, a square or a cross.

The values of all the variables u, v, and P are associated with the same grid points. The discretization of both components of the momentum equation is centered about each point, so for example, at the point (i,j) the equations for the u component would have the form,

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\delta t}+A_{i,j}^{n}=-\frac{P_{i+1,j}^{n+1}-P_{i-1,j}^{n+1}}{2h}+\nu\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{h^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{h^2}\right)$$

As in the last section we lump the discretization of the Laplacian and the non-linear terms into one forcing function and write the momentum equations as

$$u_{i,j}^{n+1}=u_{i,j}^{n}-\delta t\left(\frac{P_{i+1,j}^{n+1}-P_{i-1,j}^{n+1}}{2h}+F_{i,j}^{n}\right) \tag{4.1}$$

$$v_{i,j}^{n+1}=v_{i,j}^{n}-\delta t\left(\frac{P_{i,j+1}^{n+1}-P_{i,j-1}^{n+1}}{2h}+G_{i,j}^{n}\right). \tag{4.2}$$

These approximations are made for $i=1$ to $i=M-1$ and $j=1$ to $j=N-1$ - all points in the interior.

At the boundary points the velocity is specified but the pressure is not. In order to find an equation for the pressure at boundary points it is necessary to approximate the divergence condition at the boundary as well. For the interior points, the incompressibility constraint is approximated using centered differences

$$\frac{u_{i+1,j}^{n+1}-u_{i-1,j}^{n+1}}{2h}+\frac{v_{i,j+1}^{n+1}-v_{i,j-1}^{n+1}}{2h}=0. \tag{4.3}$$

On the boundary one sided derivatives are used, and we shall discuss the equations for the pressure which result from two different choices. One choice is to use first order accurate approximations and the other to use second order accurate approximations.

Consider now the first order accurate approximation. At the point $(i,0)$ along the bottom, we use a discretization of the form

$$\frac{u_{i+1,0}^{b}-u_{i-1,0}^{b}}{2h}+\frac{v_{i,1}^{n+1}-v_{i,0}^{b}}{h}=0. \tag{4.4}$$

Here we have denoted the known boundary velocity values by a super-script b. Similar approximations hold for the other edges and corners. To define the reduced divergence operator defined by this approximation, we write the incompressibilty equation with the unknown values of u^{n+1} and v^{n+1} on the left hand side and the boundary data terms on the right hand side. For (4.4) the equation becomes

$$\frac{v_{i,1}^{n+1}}{h}=-\frac{u_{i+1,0}^{b}-u_{i-1,0}^{b}}{2h}+\frac{v_{i,0}^{b}}{h}. \tag{4.5}$$

Just as in the previous section, the procedure to determine the pressure equations consists of applying the reduced divergence operator to (4.1)- (4.2) and rearranging terms. The result of this procedure is a set of four *uncoupled* systems of equations. Each set of equations corresponds to a subgrid which consists of points labelled with one of the symbols in Figure 3. To describe the equations for each of these subgrids we distinguish between boundary points and interior points. By boundary points we shall mean the points of the subgrid which are on the physical boundary, or, in the case that the subgrid does not extend to the physical boundary, then those points which are closest to the boundary.

The equation for the pressure at the interior points of each of the four subgrids is the same, namely

$$(4.6)\qquad (\frac{P^{n+1}_{i+2,j} - 2P^{n+1}_{i,j} + P^{n+1}_{i-2,j}}{4h^2} + \frac{P^{n+1}_{i,j+2} - 2P^{n+1}_{i,j} + P^{n+1}_{i,j-2}}{4h^2}) = R^n_{i,j}$$

The right hand side of this equation $R^n_{i,j}$ is given by

$$(4.7)\qquad R^n_{i,j} = \frac{F^{n+1}_{i+1,j} - F^{n+1}_{i-1,j}}{2h} + \frac{G^{n+1}_{i,j+1} - G^{n+1}_{i,j-1}}{2h}$$

The equation (4.6) is just the standard five point Laplacian, but with a mesh spacing of $2h$ and not h.

The equations for the boundary points differ depending on the particular grid. On the grid denoted by circles, the application of the reduced divergence operator along the bottom boundary points yields

$$(4.8)\qquad (\frac{P^{n+1}_{i+2,1} - 2P^{n+1}_{i,1} + P^{n+1}_{i-2,1}}{4h^2} + \frac{P^{n+1}_{i,3} - P^{n+1}_{i,1}}{4h^2}) = R^n_{i,1} - \frac{v^b_{i,0}}{2h}$$

Similar equations hold for the other three edges. At the lower left corner the equation becomes

$$(4.9)\qquad (\frac{P^{n+1}_{3,1} - P^{n+1}_{1,1}}{4h^2} + \frac{P^{n+1}_{1,3} - P^{n+1}_{1,1}}{4h^2}) = R^n_{1,1} - \frac{u^b_{0,1}}{2h} - \frac{v^b_{1,0}}{2h}$$

The other corners are analogous.

For the square points, along the left and right edges $i = 2$ and $i = M - 1$, the equations are of the form (4.8). For the top and bottom edges, there is only coupling between the boundary point and one interior point. Along the bottom, for example, the equations are

$$(4.10)\qquad \frac{P^{n+1}_{i,2} - P^{n+1}_{i,0}}{2h^2} = R^n_{i,0} + (\frac{u^b_{i+1,0} - u^b_{i-1,0}}{2h}) - \frac{v^b_{i,0}}{h}$$

In order to make the complete system of equations symmetric, this last equation should be multiplied by $\frac{1}{2}$. When we refer to (4.10) later on, we will assume that we are working with the equation after this scaling. For the cross points, the boundary equations are a rotated version of the square points. On the top and bottom, we have equations of the form (4.8) (with the i and j indices suitably interchanged) while on the left and right we have equations of the form (4.10). Finally, for the boundary triangle points, all of the boundary equations are of the form (4.10). We note that for the corner points of the triangle subgrid we do not get an equation for the pressure. This does not cause any problems because in our formation of the pressure gradients in the momentum equations, these corner points are not needed.

When the equations associated with each subgrid are lumped together and the complete system is expressed in block matrix form, then the structure of the equations is given by

$$(4.11)\qquad \begin{pmatrix} A_{1,1} & 0 & 0 & 0 \\ 0 & A_{2,2} & 0 & 0 \\ 0 & 0 & A_{3,3} & 0 \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix}.$$

For future reference we shall assume that the equations associated with $A_{1,1}$, $A_{2,2}$, $A_{3,3}$ and $A_{4,4}$ are the triangle, box, cross, and circle points respectively. Each set of diagonal blocks $A_{i,i}$ is a discretization of the Laplacian (with mesh width $2h$) in the interior, but only the boundary equations for the circle subgrid ($A_{4,4}$) resemble an approximation to the Laplacian with Neumann boundary conditions. For each of the four sets of equations there is a non-trivial null space consisting of the constant vector. It can be shown (using the discrete maximum principle) that this null vector is unique. The null space for the whole system is therefore four dimensional. Associated with the existence of this null space are four orthogonality conditions which must be satisfied by the right hand side of these equations. As in the staggered grid case the terms on the right hand side of the equations corresponding to $R^n_{i,j}$ are orthogonal to each of these null vectors. The orthogonality condition therefore involves only the boundary data and we get the following separate conditions.

For the circle points we have the condition

$$\sum_{i=1,3,\ldots}^{M-1} (-v^b_{i,0} + v^b_{i,N}) + \sum_{j=1,3,\ldots}^{N-1} (-u^b_{0,j} + u^b_{M,j}) = 0,$$

for the square points,

$$\sum_{i=1,3,\ldots}^{M-1} (-v^b_{i,0} + v^b_{i,N}) - \frac{u^b_{0,0}}{2} - \sum_{j=2,4,\ldots}^{N-2} u^b_{0,j} - \frac{u^b_{0,N}}{2} + \frac{u^b_{M,0}}{2} + \sum_{j=2,4,\ldots}^{N-2} u^b_{M,j} + \frac{u^b_{M,N}}{2} = 0,$$

for the cross points,

$$-\frac{v^b_{0,0}}{2} - \sum_{i=2,4,\ldots}^{M-2} v^b_{i,0} - \frac{v^b_{M,0}}{2} + \frac{v^b_{0,N}}{2} + \sum_{i=0,2,\ldots}^{M-2} v^b_{i,N} + \frac{v^b_{M,N}}{2} + \sum_{j=1,3,\ldots}^{N-1} (-u^b_{0,j} + u^b_{M,j}) = 0.$$

and for the triangle points

$$\sum_{i=2,4,\ldots}^{M-2} (-v^b_{i,0} + v^b_{i,N}) + \sum_{j=2,4,\ldots}^{N-2} (-u^b_{0,j} + u^b_{M,j}) -$$

$$\frac{1}{2}(u^b_{1,0} - u^b_{M-1,0} + u^b_{1,N} - u^b_{M-1,N} + v^b_{0,1} - v^b_{0,N-1} + v^b_{M,1} - v^b_{M,N-1}) = 0$$

If the boundary velocities satisfy a discrete divergence condition in the corners, i.e. if in the lower left corner

$$\frac{v^b_{0,1} - v^b_{0,0}}{h} + \frac{u^b_{1,0} - u^b_{0,0}}{h} = 0$$

and related approximations hold for the other corners, then the 'extra' terms in this last condition can be incorporated into the sum. The result is the following condition,

$$-\frac{u^b_{0,0}}{2} - \sum_{j=2,4,\ldots}^{N-2} u^b_{0,j} - \frac{u^b_{0,N}}{2} + \frac{u^b_{M,0}}{2} + \sum_{j=2,4,\ldots}^{N-2} u^b_{M,j} + \frac{u^b_{M,N}}{2}$$

$$-\frac{v^b_{0,0}}{2} - \sum_{i=2,4,\ldots}^{M-2} v^b_{i,0} - \frac{v^b_{M,0}}{2} + \frac{v^b_{0,N}}{2} + \sum_{i=0,2,\ldots}^{M-2} v^b_{i,N} + \frac{v^b_{M,N}}{2} = 0$$

(The requirement that the divergence in the corners vanish is actually too stringent - it can be replaced by the requirement that the sum of the discrete divergence in the four corners be zero.) As is the case with the staggered grid, when these conditions are multiplied by $2h$, the terms in the conditions can be see to be either a midpoint or trapezoidal rule approximation to the integral of the normal velocity around the boundary.

In view of the structure of these equations and the existence of four separate orthogonality conditions, it is clear why there has been difficulty in solving these equations. Firstly, if one attempts to find the pressure by solving a five point Laplacian with h spacing instead of $2h$ spacing then it unlikely this will work. The reason is that the h spaced Laplacian is coupling unknowns which are not coupled in the algebraic system which defines the pressure. If the procedure works at all, it is because the solution of the equations on the h and $2h$ grids are presumably close to some continuous solution, and hence are close to each other. Another difficulty which arises is the fact that there is a distinct null space associated with each grid. On each of these grids, the pressure is defined up to a constant. If one selects different constants for the different grids, then the complete pressure field will appear to have oscillations on the order of the mesh spacing. Although this appears to be a disaster, this may not have an adverse effect on the evolution in the momentum equation. Here the difficulty does not appear because the gradients in the momentum equation are formed with pressure values which share the same constant. If one uses the pressure from different grids to calculate some other quantity, then the differing constants will certainly have an adverse effect.

We now turn to the equations which result from the use of a second order approximation to the divergence condition at the boundary. This approximation is one suggested by Chorin in [3]. Along the lower boundary the divergence condition is approximated by an equation of the form

$$\frac{u^b_{i+1,0} - u^b_{i-1,0}}{2h} + \frac{-\frac{1}{2}v^{n+1}_{i,2} + 2v^{n+1}_{i,1} - \frac{3}{2}v^b_{i,0}}{h} = 0. \tag{4.12}$$

instead of (4.4). Using the centered difference approximation for interior points and (4.12) for the points on the physical boundary allows us to define a reduced divergence operator. Applying this operator to the momentum equation and rearranging terms allows us to construct the equations. As above, in the equations arising from the use of first order approximations, this procedure gives rise to four sets of equations, each set corresponding to one of the subgrids labelled by the symbols in Figure 3. We now briefly describe the equations and their structure, and leave the detailed specification to the appendix.

The only difference in the pressure equations associated with the first and second order approximation to the divergence operator is at grid points on the physical boundary. For the interior points the equations are the same as those for the first order approximation (4.6). Similarly, at edge points of the individual grids which are not on the physical boundary, such as circle edge points or the top and bottom cross edge points, the equations are of the type given by (4.8)-(4.9). However, for the square points on the bottom edge, the equations obtained by applying the discrete divergence operator to the momentum equation are of the form

$$\frac{\mathrm{P}^{n+1}_{i,2} - \mathrm{P}^{n+1}_{i,0}}{h^2} - \frac{\mathrm{P}^{n+1}_{i,3} - \mathrm{P}^{n+1}_{i,1}}{4h^2} = \frac{u^b_{i+1,0} - u^b_{i-1,0}}{2h} - \frac{3}{2}\frac{v^b_{i,0}}{h} + \mathrm{R}^n_{i,0} \tag{4.13}$$

This form of equation holds for the top and bottom edge square points, the left and right edge cross points, and all the edge triangle points. What can be seen from this equation is that there is coupling of the values of the pressure on the boundary to the three pressure values perpendicular to the boundary. This coupling implies that the equations associated with the separate component grids (circles, square, cross, and triangle) are not distinct as in the first order case. The precise structure is more easily seen when the equations are written in block matrix fashion. If P_1, P_2, P_3 and P_4 represent the unknowns on the triangle, cross, square and circle subgrids respectively then the equations have the form,

$$\begin{pmatrix} A_{1,1} & B_{1,2} & B_{1,3} & 0 \\ 0 & A_{2,2} & 0 & B_{2,4} \\ 0 & 0 & A_{3,3} & B_{3,4} \\ 0 & 0 & 0 & A_{4,4} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{pmatrix} \tag{4.14}$$

If the boundary equations of the form (4.13) are multiplied by $\frac{1}{4}$ then the diagonal blocks in (4.14) are the same as those for the first order approximation. As is readily apparent from the matrix form, the equations considered as a whole are not symmetric. However, the fact that the system of equations is upper block diagonal suggests that they should be solved in the following succession.

$$A_{4,4}\, P_4 = F_4 \tag{4.15}$$

$$A_{3,3}\, P_3 = F_3 - B_{3,4}\, P_4 \tag{4.16}$$

$$A_{2,2}\, P_2 = F_2 - B_{2,4}\, P_4 \tag{4.17}$$

$$A_{1,1}\, P_1 = F_1 - B_{1,2}\, P_2 - B_{1,3}\, P_3 \tag{4.18}$$

In each of these subsystems the matrices $A_{i,i}$ are all symmetric and semi-definite. It can be verified that associated with each $A_{i,i}$ a one dimensional null space. Hence, the null space is four dimensional and the problems arising from this are the same as that for the non-staggered grid with a first order approximation to the divergence at the boundary. The null space for each $A_{i,i}$ is spanned by the vector consisting of all ones. The existence of this null vector leads to orthogonality conditions on the right hand sides of (4.15)-(4.18). These conditions are also given in the appendix. What is interesting about the orthogonality conditions is the fact that although the right hand sides of equations (4.16)-(4.18) contain values of the pressure, the conditions do not involve the pressure at all, they only involve the values of the velocity on the boundary.

5. Solution Of The Pressure Equations. In this section we describe numerical techniques which can be applied to solve the systems of equations which were described in the last two sections. Firstly, if one uses a staggered grid then the set of equations can be solved by a direct application of an existing fast solver. We used the subroutine BLKTRI from the NCAR set of routines FISHPACK. This routine was used rather than the customary 5-point Laplace solver in the package because the Neumann boundary conditions associated with the standard solver were implemented with a second order, rather than first order, approximation to normal derivatives at the boundary. If one does not have access to a fast solver which can implement the correct boundary conditions, then the techniques which we will now describe for the other sets of equations can be employed.

The solution for either of the the non-staggered grid equations (4.11) and (4.14) requires the solution of four different sub-systems of equations - one associated with each of the diagonal blocks $A_{i,i}$. Each set of equations consists of a discrete 5-point laplacian at interior points and different types of equations at the boundary points. For three of the four systems it is not apparent that their boundary equations can be cast as implementations of any standard boundary condition, so it is unlikely that an existing fast solver is available for their solution. In order to construct a fast method out of existing routines, it is natural to consider iterative methods. We chose to use the method of pre-conditioned conjugate gradients [4]. This iterative scheme is easy to implement and, as will be discussed below, is efficient computationally. After a discussion of the iterative approach we outline a procedure for constructing a direct fast solver. The construction of this fast solver requires an initial work effort of $O(N^{3/2})$ (where N is the number of points in the whole domain). After the initial construction, the application of the direct method is on the order of a fast solver for a Dirichlet problem in the domain - typically O(N log N). Since the equations for the pressure are solved at every timestep it may be worth the extra initial effort to construct the direct method. In three dimensions, there are analogous direct procedures for the equations which arise, but the work (after initialization) is on the order of $N^{4/3}$ which may be unsatisfactory. Iterative methods might therefore be the optimal choice for three dimensional calculations.

We begin by reviewing the technique of pre-conditioned conjugate gradients. If a real symmetric definite system to be solved is of the form

$$Ax = b$$

and M is a symmetric definite approximation to A whose inverse is readily computed, then preconditioned conjugate gradients is derived by applying the standard conjugate gradient method to the prepared system

$$M^{-1}Ax = M^{-1}b$$

while using the inner product defined by $< \cdot, \cdot >= (\cdot, M\cdot)$. Here $(\cdot,\cdot)$ is the standard inner product. The systems we are solving are real symmetric but only semi-definite, so some care must be taken in the implementation of the iteration. If one is using a definite preconditioner then no problems arise as long as the initial iterate is orthogonal to the null space of the equations. However, in our procedure we are using a preconditioner which is indefinite (one corresponding to a discrete Neumann problem) and so we must add an extra step. Let $\vec{c}$ denote the null vector for the system and our preconditioner. (In the systems of interest here as well as for the preconditioner, the null space is one dimensional with the null vector consisting of the constant vector.) The steps of the iteration, written using the standard inner product $(\cdot, \cdot)$ are then as follows -

$$r^0 = b - Ax^0 \qquad x^0 \text{ initial guess}$$

$$\text{For } k = 1, 2, 3, \ldots$$

$$\begin{aligned}
z^k &= M^{-1}r^k \\
z^k &= z^k - \frac{(z^k, \vec{c})}{(\vec{c}, \vec{c})} \\
\beta^k &= \frac{(z^k, r^k)}{(z^{k-1}, r^{k-1})} \qquad k \geq 1 \quad \beta^0 = 0 \\
p^k &= z^k + \beta^k p^{k-1} \\
\alpha^k &= \frac{(z^k, z^k)}{(p_k, Ap^k)} \\
x^{k+1} &= x^k + \alpha^k p^k \\
r^{k+1} &= r^k - \alpha^k Ap^k.
\end{aligned}$$

The system of equations corresponding to the circle points ($A_{4,4}$) can be solved directly using the subroutine BLKTRI mentioned above. For the remaining sets of equations we used the pre-conditioned conjugate gradient method with the inverse of $A_{4,4}$ as the preconditioner. This choice of preconditioner corresponds to using a discrete Neumann problem with a $2h$ stencil as the approximate operator M.

For a test problem we chose the domain Ω to be the unit square. The basic step in the advancement of the Navier-Stokes equations can be written as

$$\frac{\vec{u}^{n+1}}{\delta t} = F(\vec{u}^n) - (\nabla_x^h, \nabla_y^h)P^{n+1} \tag{5.1}$$

(5.2) $$(D_x^h, D_y^h) \cdot \vec{u}^{n+1} = 0$$

We therefore set $\delta t = 1$, $\vec{u}^{n+1} = 0$ on $\partial\Omega$ and

$$F(\vec{u}^n) = \frac{1}{2}\sin(\pi x)\sin(\pi y).$$

The specification of these quantities allows us to construct the right hand sides of (4.11) and (4.14) and hence determines a test problem. The ultimate goal in the determination of the pressure is to use it in equation (5.1) so that the vector field u^{n+1} can be determined which is divergence free. We therefore monitored the divergence of the velocity field which was obtained from the current pressure iterate and equation (5.1) and used this to judge the quality of the computed solution.

In our first set of experiments we solved the system (4.11) corresponding to the non-staggered grid with first order approximations to the divergence condition at the boundaries. Since the equations are uncoupled there is no need to solve them in any particular order, but we chose to solve for the circle points first (which can be done without iteration) and then used interpolations of the pressure values on this grid to construct initial guesses for the iterative solution of system of equations associated with the other grids. We used several different sized grids and found that the discrete L^2 norm the divergence of the velocity field corresponding to the circle points was always less than 1×10^{-8}. This value gives the approximate order of the accuracy on our machine of the fast solver BLKTRI which we used.

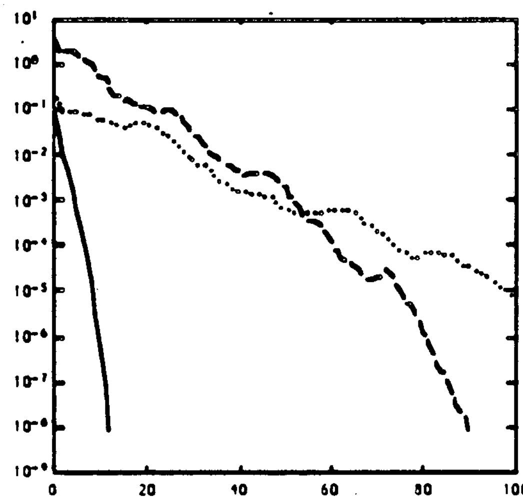

Figure 4

CG Iterations vs. L^2 Norm of Discrete Divergence

—————— : Neumann Preconditioner ($2h$) for $A_{1,1}$.

— — — : Neumann Preconditioner (h) for complete system.

.... : No Preconditioner for complete system.

As we carried out the iterative solution of the other three systems of equations we were interested in two aspects of the computation. The first was the decay of the divergence with each iteration, and the second the effect of the mesh size on the rate of decay of the divergence. In our problem, the iteration process associated with the triangle points $A_{1,1}$ had the slowest rate of decay. This is expected since it is for this problem that our preconditioner deviates most from the actual operator. We therefore report on the results for this grid only - the values for the other grids are more favorable. In Figure 4, the solid line shows the L^2 norm of the divergence of the velocity field as a function of the number of iterations for a problem which had 33 grid points in each direction. It took twelve iterations to achieve a level of accuracy comparable to that which was obtained on the circle points using the fast direct solver. In Table 1 we present the number of iterations needed to obtain a divergence of less than 1×10^{-8} as the size of the grid was changed. As we can see, there is a slight growth in the number of iterations as the mesh size is increased but nothing substantial. We were pleased with these results for we expected a much more rapid decrease in efficiency.

In next set of experiments we solved the system (4.14) corresponding to the non-staggered grid with a second order approximations to the divergence condition at the boundaries. In the solution of these equations it is necessary to solve them in the order given by (4.15)-(4.18). As we solved each of the systems we used interpolates of the previously determined values of the pressure to obtain the starting vector. There is not much to report except that the behavior of the iterative process was essentially identical to that of the case described just above and indicated by the solid line in Figure 4 and the results in Table 1. Since the equations we are solving are identical except for a change in the right hand sides, this is expected.

In the third set of experiments we solved the equations (4.11) as a whole without regard to the fact that they are really four decoupled systems of equations. Our goal here was to duplicate some early attempts we made at the iterative solution of the pressure equations. We chose preconditioned conjugate gradients and used a discrete Neumann problem with mesh width h as the preconditioner. In order to get the iterative method to converge at all we needed to orthogonalize the iterates with respect to the four dimensional null space associated with the problem. This aspect of the equations could not be ignored. The dotted lines in Figure 4 shows the L^2 norm of the divergence of the velocity field as a function of the number of iterates obtained using this preconditioner. The dashed line indicates the results which are obtained when the whole system is solved using conjugate gradients with no preconditioning. As can be seen from the figure, the iterative method based on a global Neumann preconditioner was actually worse than a method with no preconditioning at all! This might be expected since the Neumann preconditioner is coupling unknowns together which are completely decoupled in the system which is being solved. We did not even attempt an iterative solution of the equations (4.14) as a whole because the system is not symmetric and the method of conjugate gradients is not applicable.

Grid Size	Iterations	L^2 Norm of Divergence
17×17	7	6.71×10^{-12}
33×33	11	8.91×10^{-9}
41×41	14	2.12×10^{-9}
65×65	16	4.57×10^{-9}

Table 1

The iteration method for the three sets of equations associated with $A_{1,1}$, $A_{2,2}$ and $A_{3,3}$ we have described is easy to implement and the extensions to three dimensional calculations should not be difficult. We choose for our experiments a preconditioner which happened to be handy, and it is certainly not the best. For another preconditioner which works better see [1]. We now discuss a direct method for solving the equations.

In each of the different systems we must solve, the equations correspond to a standard five-point Laplace equation in the interior but with varied equations at the boundaries. The procedure we suggest is an implementation of the method of matrix partitioning [7]. We construct the solution by combining a solution of the interior equations - a solution which is obtained by solving a standard Dirichlet problem - and the solution of a dense system of equations for the boundary values. The derivation of the equations one must solve to carry out this procedure can be accomplished using matrix calculus, but we derive the equations in a different fashion, one which is perhaps easier to understand and which facilitates the implementation. For more concise details on this type of procedure, see [1].

To be specific we shall concentrate on the solving a set of equations which are described by

$$\frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{h^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{h^2} = f_{i,j} \tag{5.3}$$

at interior points - i.e. for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. For the top and bottom points the equations are determined by a three point stencil and have the form,

$$\frac{p_{i+1,n} - 2p_{i,n} + p_{i-1,n}}{h^2} + \frac{-p_{i,n} + p_{i,n-1}}{h^2} = f_{i,n}$$

$$\frac{p_{i+1,0} - 2p_{i,0} + p_{i-1,0}}{h^2} + \frac{-p_{i,0} + p_{i,1}}{h^2} = f_{i,1} \tag{5.4}$$

for $i = 1, \ldots, n$. At the left and right edges as well as the corners, the equations are determined by a two point stencil and have the form

$$\frac{-p_{0,j} + p_{1,j}}{h^2} = f_{1,j}$$

$$\frac{-p_{n,j} + p_{n-1,j}}{h^2} = f_{n,j} \tag{5.5}$$

for $j = 0, \ldots, n$. These equations are those associated with the cross points. We assume that the right hand side of these equations is orthogonal to the constant vector - this ensures that the system posesses a solution.

For simplicity we take the number of panels to be the same in each direction and the domain to be square with sides of length a. We set $h = a/n$. The method we describe can easily be modified to accommodate different numbers of points as well as different mesh sizes in each direction.

In the first step of the procedure we compute a solution $\bar{p}$ by solving (5.3) for the interior points and setting $\bar{p} = 0$ at the boundary points. This problem is just a standard Dirichlet problem in the domain with homogeneous boundary conditions. The next step is to add to this solution a correction of the form

$$\gamma_{i,j} = \sum_{k=1}^{n-1} a_k z_k(n-j) \sin(\frac{k\pi ih}{a}) + \sum_{k=1}^{n-1} b_k z_k(j) \sin(\frac{k\pi ih}{a}) +$$

$$\sum_{k=1}^{n-1} c_k z_k(n-i) \sin(\frac{k\pi jh}{a}) + \sum_{k=1}^{n-1} d_k z_k(i) \sin(\frac{k\pi jh}{a}) +$$

$$\delta_{0,0}\, e_1 + \delta_{0,n}\, e_2 + \delta_{n,0}\, e_3 + \delta_{n,n}\, e_4 \tag{5.6}$$

Here $\delta_{i,j}$ is the Kronicker δ function. The function $z_k(i)$ is given by

$$z_k(i) = \lambda_k^i + (\lambda_k^i - \lambda_k^{-i})\left(\frac{1}{(1-\lambda_k^{2n})} - 1\right) \tag{5.7}$$

with

$$\lambda_k = \frac{2 + 4\sin(\frac{k\pi h}{2a})^2 - \sqrt{(2 + 4\sin(\frac{k\pi h}{2a})^2)^2 - 4)}}{2} \tag{5.8}$$

Now each of the functions appearing in the sums is a function which is discrete harmonic at all interior points - it satisfies (5.3) with a zero right hand side. On boundary points it is zero except on one side where it takes on the values $\sin(\frac{k\pi jh}{a})$ or $\sin(\frac{k\pi ih}{a})$. We need four sets of such functions, one set for each side. In the representation (5.6) the coefficients a_k, b_k, c_k and d_k correspond to correction functions associated with the top, bottom, right, and left edges respectively. The functions with coefficients e_i are used to correct the corner values.

The reason for chosing this particular form for the correction is that at interior points, the equations for the correction

$$\mathrm{A}\vec{\gamma} = f - \mathrm{A}\bar{p} = \vec{r} \tag{5.9}$$

where $\vec{r}$ is the residual, are automatically satisfied for any choice of the undetermined coefficients in (5.6). Thus, we need only concern ourselves with the equations at the boundaries. We obtain the equations which determine the coefficients a_k, b_k, c_k, d_k and e_k by requiring that (5.4) and (5.5) acting on the sum $\bar{u} + \gamma$ hold at each of the boundary points. These equations form the 5 by 5 block system

$$\begin{pmatrix} \mathrm{D}_{1,1} & \mathrm{E}_{1,2} & \mathrm{E}_{1,3} & \mathrm{E}_{1,4} & \mathrm{O}_{1,5} \\ \mathrm{E}_{2,1} & \mathrm{D}_{2,2} & \mathrm{E}_{2,3} & \mathrm{E}_{2,4} & \mathrm{O}_{2,5} \\ \mathrm{E}_{3,1} & \mathrm{E}_{3,2} & \mathrm{D}_{3,3} & \mathrm{E}_{3,4} & \mathrm{O}_{3,5} \\ \mathrm{E}_{4,1} & \mathrm{E}_{4,2} & \mathrm{E}_{4,3} & \mathrm{D}_{4,4} & \mathrm{O}_{4,5} \\ \mathrm{O}_{5,1} & \mathrm{O}_{5,2} & \mathrm{O}_{5,3} & \mathrm{O}_{5,4} & \mathrm{O}_{5,5} \end{pmatrix} \begin{pmatrix} \vec{a}_k \\ \vec{b}_k \\ \vec{c}_k \\ \vec{d}_k \\ \vec{e}_k \end{pmatrix} = \begin{pmatrix} \\ \\ \vec{r} \\ \\ \\ \end{pmatrix}. \tag{5.10}$$

Here we have grouped together the equations corresponding to the top, bottom, right, left and corner points. The size of each diagonal block $\mathrm{D}_{i,i}$ is $(n-1) \times (n-1)$ where n is the number of panels on a side. The block $\mathrm{O}_{5,5}$ is of size 4×4. There are several options for solving (5.10). The simplest one is to construct this system and then, using an LU decomposition solve it. The initial cost of setting up the equations and forming the LU decomposition is $O(n^3)$, which implies a cost of $O(N^{\frac{3}{2}})$ if N is the total number of unknowns in the domain. Once the LU decomposition is constructed, the order of operations necessary to solve the system becomes O(N) and so subsequent solutions of (5.10) can be determined efficiently.

When the equations (5.10) are solved, the coefficients of the correction are determined, and the complete solution of the pressure equations is given by the sum

$$p = \gamma + \bar{p}.$$

However, instead of using (5.6) to evaluate γ in the interior and then adding it to $\bar{p}$, it is computationally more efficient to solve (5.3) once again but with boundary values given by $p = \gamma + \bar{p} = \gamma$. The cost of this latter procedure is O(N) plus the cost of a fast Dirichlet solve in the interior, while that of using (5.6) to obtain all the interior values is $O(N^{\frac{3}{2}})$. If the more efficient scheme is followed to compose p from γ and $\bar{p}$, then the net result is a method which decomposes the problem of solving (5.3)-(5.5) into a direct procedure consisting of solving a dense system of equations on the order of the number of boundary points and solving two Dirichlet problem in the interior.

There are some details concerning the formation of the LU decompostion of (5.10) which are important. To facilitate the construction of the LU decomposition we note that each of the blocks $D_{i,i}$ as well as $E_{1,2}, E_{2,1}, E_{3,4}, E_{4,3}$ become diagonal when (5.10) is multiplied by the matrix Tr of the form

$$\mathrm{Tr} = \begin{pmatrix} T_{1,1} & & & & \\ & T_{2,2} & & & \\ & & T_{3,3} & & \\ & & & T_{4,4} & \\ & & & & I \end{pmatrix} \tag{5.11}$$

where each of the diagonal blocks $T_{i,i}$ is the matrix representing the discrete sin transform of $n-1$ points, i.e. $T_{i,j} = sin(\frac{i\pi jh}{a})$. This transformation also renders the matrix (5.10) symmetric. Let H be the system of equations in (5.10) which results after the multiplication by Tr. The LU decomposition of the matrix system H must be done carefully because the matrix is singular. The singularity comes about because the complete problem we are attempting to solve is singular (the constant vector is a null vector for (5.3) -(5.5)) and the matrix system H inherits this singularity. However, with matrix analysis [1] it can be shown that the dimension of the null space for H is no larger than that of the original system, and hence the dimension of the null space is one. Furthermore, the null vector for H is given by

$$\vec{s} = \begin{pmatrix} \mathrm{Tr}(\vec{q_1}) \\ \vec{q_2} \end{pmatrix}$$

where $\vec{q_1}$ is a constant vector of size $4n-4$ and $\vec{q_4}$ is the constant vector of size 4. Since the null vector is known we can circumvent the difficulty which it imposes by solving

$$(\mathrm{H} - \vec{s}\,\vec{s}^{\,t})\,\vec{x} = \vec{b}$$

instead of solving $\mathrm{H}\vec{x} = \vec{b}$. This modified set of equations is strictly negative definite and its solution satisfies the equations

$$\mathrm{H}\vec{x} = \vec{b} \qquad \vec{x}^{\,t}\,\vec{s} = 0$$

Thus, before computing the LU decomposition we merely add to H the term $-\vec{s}\,\vec{s}^{\,t}$. This allows us to avoid the difficulties which arise because of the singularity of the system.

There are other ways to solve equations (5.10). One technique is to use the preconditioned conjugate gradient method with the inverse of the diagonal of H as the preconditioner. In the application of such an iterative scheme it appears that it would take $O(n^3)$ operations to carry out each iteration (the cost of a matrix vector multiply for H), but in reality the iteration can be carried out with the cost of a fast solve on the entire domain. The details of this are given in [1].

The direct method we have outlined can be used to solve all of the four systems associated with the determination of the pressure on a non-staggered mesh. The method can also be used to solve the pressure equations for the staggered mesh. If we ignore the start up costs, then in two dimensions, the direct scheme given here has an operation count which is on the order of the implementation of a fast solver for the domain. In three dimensions the operation count is not so favorable, and iterative methods may be more efficient.

6. Conclusions. We have presented the discrete equations for the pressure which must be solved in order to advance several commonly used finite difference approximations to the Navier-Stokes equations. The equations can be derived from a linear algebra viewpoint by writing the momentum and incompressibility conditions in block matrix form and then determining the pressure equations by constructing a Schur compliment. The derivation gives the equation for the pressure as combination of discrete divergence and gradient operators. There is no need to specify any sort of pressure boundary conditions.

The solution of the resulting pressure equations presents some challenges. Problems arise because the structure of the pressure equations can change dramatically depending on choice of the discretization or the type of grid used. We have gone through three popular choices of discretizations to illustrate this. For a staggered grid we have rederived the fact that the equations which determine the pressure correspond to a Laplace operator with Neumann boundary conditions. Associated with this system is a one-dimensional null space corresponding to the constant solution. For the non-staggered grid with a first order approximation to the divergence condition at the boundary the equations become four *uncoupled* Laplace operators with boundary conditions which, depending on the system, do and do not look like Neumann boundary conditions. Associated with each of the subsystems is a separate null space, and so there are four undetermined constants in the pressure solution. If one uses a non-staggered grid with second order approximations to the divergence conditions, then the resulting system is non-symmetric. However, it can be solved as a sequence of four subproblems each involving symmetric operators. Associated with each subsystem is a one dimensional null space, and so there are four undetermined constants in the pressure solution as well. In the case of the staggered grid, the equations can be solved by a direct application of a fast Laplace solver. For the equations on a non-staggered grid we showed that the technique of preconditioned conjugate gradients can be employed to construct fast methods for their solution. We have also outlined a direct method for the solution of these systems. The direct method combines a solution of a dense linear system (of size the number of boundary points) and two solutions of a Dirichlet problem in the interior of the domain. As our experiments indicated, the success in constructing fast iterative methods depended crucially on understanding the structure of the pressure equations. The construction of the direct method would also not have been possible without the detailed knowledge these systems. Listings of the programs used in this paper are available from the author.

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Appendix

Pressure Equations and Orthogonality Conditions For A Non-Staggered Grid

In this appendix we present the equations for the pressure associated with a non-staggered grid when a second order approximation to the divergence operator at the boundaries is used. We describe the equations with reference to the grid in Figure 3. This grid has M panels in the x direction and N panels in the y direction. The number of panels must be even in each direction. We also assume that the mesh width h in each direction is the same. The grid points are numbered in each direction from $i = 0$ to $i = N$, and $j = 0$ to $j = N$. We shall present the equations by giving them for each of the sets of points denoted by circles, squares, triangles, and crosses. This is a natural grouping which leads to the block system of equations (4.14). In writing these equations we are assuming that one is solving them in the sequence given by (4.15)-(4.18) and so some values of the pressure occur on the right hand side of the equality. When written in this way, the equations defining the operators $A_{i,i}$ for both (4.11) and (4.14) are expressed by the left hand side of the equations. We are therefore also exhibiting the equations for the non-staggered grid which occur when one uses a first order approximation to the divergence operator at the boundary. The right hand sides must still be changed for (4.11), but this is relatively straight forward. After we present the equations we shall give the orthogonality conditions which must be satisfied by the velocity boundary data in order that the equations (4.15)-(4.18) possess solutions.

The basic step in the solution of the Navier-Stokes equations takes the form

$$\frac{\vec{u}^{n+1}}{\delta t} = \mathrm{F}(\vec{u}^{n}) - (\nabla_x^h, \nabla_y^h)\mathrm{P}^{n+1}$$

$$(\mathrm{D}_x^h, \mathrm{D}_y^h) \cdot \vec{u}^{n+1} = 0. \tag{6.1}$$

It is components of these equations which combine to form the equations for the pressure alone. In order to avoid too many superscripts and subscripts, we abbreviate the problem (6.1) by

$$\vec{w} = \vec{u} - (\nabla_x^h, \nabla_y^h)\mathrm{P}$$

$$(\mathrm{D}_x^h, \mathrm{D}_y^h) \cdot \vec{w} = 0 \tag{6.2}$$

with $\vec{w} = (w, z)$ and $\vec{u} = (u, v)$. On the boundary we take $(w, z) = (w^b, z^b)$. With this notation the equations take the following form;

Equations for Circle Points

Interior, $i = 3, 5, \ldots, M-3$ $j = 3, 5, \ldots, N-3$

$$\frac{P_{i+2,j}-2P_{i,j}+P_{i-2,j}}{4h^2} + \frac{P_{i,j+2}-2P_{i,j}+P_{i,j-2}}{4h^2} = \frac{u_{i+1,j}-u_{i-1,j}}{2h} + \frac{v_{i,j+1}-v_{i,j-1}}{2h}$$

Top, $i = 3, 5, \ldots, M-3$

$$\frac{P_{i+2,N-1}-2P_{i,N-1}+P_{i-2,N-1}}{4h^2} + \frac{-P_{i,N-1}+P_{i,N-3}}{4h^2} = \frac{u_{i+1,N-1}-u_{i-1,N-1}}{2h} - \frac{v_{i,N-2}}{2h} + \frac{z^b_{i,N}}{2h}$$

Bottom, $i = 3, 5, \ldots, M-3$

$$\frac{P_{i+2,1}-2P_{i,1}+P_{i-2,1}}{4h^2} + \frac{P_{i,3}-P_{i,1}}{4h^2} = \frac{u_{i+1,1}-u_{i-1,1}}{2h} + \frac{v_{i,2}}{2h} - \frac{z^b_{i,0}}{2h}$$

Left, $j = 3, 5, \ldots, N-3$,

$$\frac{P_{3,j}-P_{1,j}}{4h^2} + \frac{P_{1,j+2}-2P_{1,j}+P_{1,j-2}}{4h^2} = \frac{u_{2,j}}{2h} + \frac{v_{1,j+1}-v_{1,j-1}}{2h} - \frac{w^b_{0,j}}{2h}$$

Right, $j = 3, 5, \ldots, N-3$,

$$\frac{-P_{M-1,j}+P_{M-3,j}}{4h^2} + \frac{P_{M-1,j+2}-2P_{M-1,j}+P_{M-1,j-2}}{4h^2} = \frac{-u_{M-2,j}}{2h} + \frac{v_{M-1,j+1}-v_{M-1,j-1}}{2h} + \frac{w^b_{M,j}}{2h}$$

Corners,

$$\frac{P_{3,N-1}-P_{1,N-1}}{4h^2} + \frac{-P_{1,N-1}+P_{1,N-3}}{4h^2} = \frac{u_{2,N-1}}{2h} - \frac{v_{1,N-2}}{2h} - \frac{w^b_{0,N-1}}{2h} + \frac{z^b_{1,N}}{2h}$$

$$\frac{-P_{M-1,N-1}+P_{M-3,N-1}}{4h^2} + \frac{-P_{M-1,N-1}+P_{M-1,N-3}}{4h^2} = -\frac{u_{M-2,N-1}}{2h} - \frac{v_{M-1,N-2}}{2h} + + \frac{w^b_{M,N-1}}{2h} \frac{z^b_{M-1,N}}{2h}$$

$$\frac{P_{3,1}-P_{1,1}}{4h^2} + \frac{P_{1,3}-P_{1,1}}{4h^2} = \frac{u_{2,1}}{2h} + \frac{v_{1,2}}{2h} - \frac{w^b_{0,1}}{2h} - \frac{z^b_{1,0}}{2h}$$

$$\frac{-P_{M-1,1}+P_{M-3,1}}{4h^2} + \frac{P_{M-1,3}-P_{M-1,1}}{4h^2} = -\frac{u_{M-2,1}}{2h} + \frac{v_{M-1,2}}{2h} + \frac{w^b_{M,1}}{2h} - \frac{z^b_{M-1,0}}{2h}$$

Equations for Cross Points

Interior, $i = 2, 4, \ldots, M-2$ $j = 3, 5, \ldots, N-3$

$$\frac{\mathrm{P}_{i+2,j} - 2\mathrm{P}_{i,j} + \mathrm{P}_{i-2,j}}{4h^2} + \frac{\mathrm{P}_{i,j+2} - 2\mathrm{P}_{i,j} + \mathrm{P}_{i,j-2}}{4h^2} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{v_{i,j+1} - v_{i,j-1}}{2h}$$

Top, $i = 2, 4, \ldots, M-2$

$$\frac{\mathrm{P}_{i+2,N-1} - 2\mathrm{P}_{i,N-1} + \mathrm{P}_{i-2,N-1}}{4h^2} + \frac{-\mathrm{P}_{i,N-1} + \mathrm{P}_{i,N-3}}{4h^2} = \frac{u_{i+1,N-1} - u_{i-1,N-1}}{2h} - \frac{v_{i,N-2}}{2h} + \frac{z^b_{i,N}}{2h}$$

Bottom, $i = 2, 4, \ldots, M-2$

$$\frac{\mathrm{P}_{i+2,1} - 2\mathrm{P}_{i,1} + \mathrm{P}_{i-2,1}}{4h^2} + \frac{\mathrm{P}_{i,3} - \mathrm{P}_{i,1}}{4h^2} = \frac{u_{i+1,1} - u_{i-1,1}}{2h} + \frac{v_{i,2}}{2h} - \frac{z^b_{i,0}}{2h}$$

Left, $j = 1, 3, \ldots, N-1$,

$$\frac{\mathrm{P}_{2,j} - \mathrm{P}_{0,j}}{4h^2} = \frac{\mathrm{P}_{3,j} - \mathrm{P}_{1,j}}{16h^2} + \frac{u_{1,j}}{2h} - \frac{u_{2,j}}{8h} - \frac{3w^b_{0,j}}{8h} + \frac{z^b_{0,j+1} - z^b_{0,j-1}}{8h}$$

Right, $j = 1, 3, \ldots, N-1$,

$$\frac{-\mathrm{P}_{M,j} + \mathrm{P}_{M-2,j}}{4h^2} = \frac{-\mathrm{P}_{M-1,j} + \mathrm{P}_{M-3,j}}{16h^2} - \frac{u_{M-1,j}}{2h} + \frac{u_{M-2,j}}{8h} + \frac{3w^b_{M,j}}{8h} + \frac{z^b_{M,j+1} - z^b_{M,j-1}}{8h}$$

Equations for Box Points

Interior, $i = 3, 5, \ldots, M-3$ $j = 2, 4, \ldots, N-2$

$$\frac{P_{i+2,j}-2P_{i,j}+P_{i-2,j}}{4h^2} + \frac{P_{i,j+2}-2P_{i,j}+P_{i,j-2}}{4h^2} = \frac{u_{i+1,j}-u_{i-1,j}}{2h} + \frac{v_{i,j+1}-v_{i,j-1}}{2h}.$$

Top, $i = 1, 3, \ldots, M-1$

$$\frac{-P_{i,N}+P_{i,N-2}}{4h^2} = \frac{-P_{i,N-1}+P_{i,N-3}}{16h^2} - \frac{v_{i,N-1}}{2h} + \frac{v_{i,N-2}}{8h} + \frac{w^b_{i+1,N}-w^b_{i-1,N}}{8h} + \frac{3z^b_{i,N}}{8h}$$

Bottom, $i = 1, 3, \ldots, M-1$

$$\frac{P_{i,2}-P_{i,0}}{4h^2} = \frac{P_{i,3}-P_{i,1}}{16h^2} + \frac{v_{i,1}}{2h} - \frac{v_{i,2}}{8h} + \frac{w^b_{i+1,0}-w^b_{i-1,0}}{8h} - \frac{3z^b_{i,0}}{8h}$$

Left, $j = 2, 4, \ldots, N-2$,

$$\frac{P_{3,j}-P_{1,j}}{4h^2} + \frac{P_{1,j+2}-2P_{1,j}+P_{1,j-2}}{4h^2} = \frac{u_{2,j}}{2h} + \frac{v_{1,j+1}-v_{1,j-1}}{2h} - \frac{w^b_{0,j}}{2h}$$

Right, $j = 2, 4, \ldots, N-2$,

$$\frac{-P_{M-1,j}+P_{M-3,j}}{4h^2} + \frac{P_{M-1,j+2}-2P_{M-1,j}+P_{M-1,j-2}}{4h^2} = \frac{-u_{M-2,j}}{2h} + \frac{v_{M-1,j+1}-v_{M-1,j-1}}{2h} + \frac{w^b_{M,j}}{2h}$$

Equations for Triangle Points

Interior, $i = 2, 4, \ldots, M-2$ $j = 2, 4, \ldots, N-2$

$$\frac{\mathrm{P}_{i+2,j} - 2\mathrm{P}_{i,j} + \mathrm{P}_{i-2,j}}{4h^2} + \frac{\mathrm{P}_{i,j+2} - 2\mathrm{P}_{i,j} + \mathrm{P}_{i,j-2}}{4h^2} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} + \frac{v_{i,j+1} - v_{i,j-1}}{2h}$$

Top, $i = 2, 4, \ldots, M-2$

$$\frac{-\mathrm{P}_{i,N} + \mathrm{P}_{i,N-2}}{4h^2} = \frac{-\mathrm{P}_{i,N-1} + \mathrm{P}_{i,N-3}}{16h^2} - \frac{v_{i,N-1}}{2h} + \frac{v_{i,N-2}}{8h} + \frac{w^b_{i+1,N} - w^b_{i-1,N}}{8h} + \frac{3z^b_{i,N}}{8h}$$

Bottom, $i = 2, 4, \ldots, M-2$

$$\frac{\mathrm{P}_{i,2} - \mathrm{P}_{i,0}}{4h^2} = \frac{\mathrm{P}_{i,3} - \mathrm{P}_{i,1}}{16h^2} + \frac{v_{i,1}}{2h} - \frac{v_{i,2}}{8h} + \frac{w^b_{i+1,0} - w^b_{i-1,0}}{8h} - \frac{3z^b i,0}{8h}$$

Left, $j = 2, 4, \ldots, N-2$,

$$\frac{\mathrm{P}_{2,j} - \mathrm{P}_{0,j}}{4h^2} = \frac{\mathrm{P}_{3,j} - \mathrm{P}_{1,j}}{16h^2} + \frac{u_{1,j}}{2h} - \frac{u_{2,j}}{8h} - \frac{3w^b_{0,j}}{8h} + \frac{z^b_{0,j+1} - z^b_{0,j-1}}{8h}$$

Right, $j = 2, 4, \ldots, N-2$,

$$\frac{-\mathrm{P}_{M,j} + \mathrm{P}_{M-2,j}}{4h^2} = \frac{-\mathrm{P}_{M-1,j} + \mathrm{P}_{M-3,j}}{16h^2} - \frac{u_{M-1,j}}{2h} + \frac{u_{M-2,j}}{8h} + \frac{3w^b M,j}{8h} + \frac{z^b_{M,j+1} - z^b_{M,j-1}}{8h}$$

Orthogonality Conditions

For each of the above systems, the operator represented by the left hand side is singular. In each case it can be shown that the null space is one dimensional and consists of the span of the vector of all ones. In order that these systems have solutions, it is therefore necessary that the right hand sides of the equations be orthogonal to this null vector. We now write down for each system the requirements on the data which will ensure that this orthogonality condition is satisfied. The derivation of these conditions consists of just summing the right hand sides of the particular equations. There is some subtlety in evaluating the summation because in the equations associated with the cross, box, and triangle points, the right hand sides involve pressure differences. One eliminates these pressure differences by utilizing the equations which they satisfy to express these differences in terms of the velocities. In writing down these orthogonality conditions we are therefore assuming that the equations for the pressure at a previous steps in the solution of (4.15) - (4.18) are satisfied exactly. As an example of this procedure, in the determination of the orthogonality condition for the cross points, one has the sum

$$\sum_{i=3,5,\ldots}^{N-1} \frac{\mathrm{P}_{3,j} - \mathrm{P}_{1,j}}{16h^2}$$

to contend with. This sum is eliminated by using the equations that are satisfied at the circle points, namely,

$$\begin{aligned}
\sum_{j=1,3,\ldots}^{N-1} \frac{\mathrm{P}_{3,j} - \mathrm{P}_{1,j}}{16h^2} = \frac{1}{4}\Bigg(& \frac{\mathrm{P}_{1,3} - \mathrm{P}_{1,1}}{4h^2} + \sum_{j=3,5,\ldots}^{N-3} \frac{\mathrm{P}_{1,j+2} - 2\mathrm{P}_{1,j} - \mathrm{P}_{1,j-2}}{4h^2} + \frac{\mathrm{P}_{1,N-3} - \mathrm{P}_{1,N-1}}{4h^2} \\
& + \frac{u_{2,N-1}}{2h} - \frac{v_{1,N-2}}{2h} - \frac{w^b_{0,N-1}}{2h} + \frac{z^b_{1,N}}{2h} \\
& + \sum_{j=3,5,\ldots}^{N-3} \frac{u_{2,j}}{2h} + \frac{v_{1,j+1} - v_{1,j-1}}{2h} - \frac{w^b_{0,j}}{2h} \\
& + \frac{u_{2,1}}{2h} + \frac{v_{1,2}}{2h} - \frac{w^b_{0,1}}{2h} - \frac{z^b_{1,0}}{2h}\Bigg)
\end{aligned}$$

In this expression all of the P terms on the right hand side drop out, and one is left with a sum of velocities. These remaining terms combine with the other terms in the condition and the result is an expression which only involves velocity boundary data.

The orthogonality condition for the circle points is given by

$$(6.3) \qquad -\sum_{j=3,5,\ldots}^{M-1} \frac{w^b_{0,j}}{2h} + \sum_{j=3,5,\ldots}^{M-1} \frac{w^b_{M,j}}{2h} - \sum_{i=3,5,\ldots}^{N-1} \frac{z^b_{i,0}}{2h} + \sum_{j=3,5,\ldots}^{N-1} \frac{z^b_{i,N}}{2h} = 0$$

When multiplied by $4h^2$ this can be seen as a midpoint rule approximation to the integral of the normal velocity around the boundary.

For the cross points, assuming the equations on the circle points have been satisfied, the orthogonality condition is

$$-\sum_{j=3,5,\ldots}^{N-1} \frac{w_{0,j}^b}{2h} + \sum_{j=3,5,\ldots}^{N-1} \frac{w_{M,j}^b}{2h} - \frac{1}{8h}(z_{0,0}^b + z_{1,0}^b) - \sum_{i=2,4,\ldots}^{M-2} \frac{z_{i,0}^b}{2h} - \frac{1}{8h}(z_{M-1,0}^b + z_{M,0}^b)$$
$$+ \frac{1}{8h}(z_{0,N}^b + z_{1,N}^b) + \sum_{i=2,4,\ldots}^{M-2} \frac{z_{i,N}^b}{2h} + \frac{1}{8h}(z_{M-1,N}^b + z_{M,N}^b) = 0$$

similarly the orthogonality condition for the box points is given by

$$-\sum_{i=1,3,\ldots}^{M-1} \frac{z_{i,0}^b}{2h} + \sum_{i=1,3,\ldots}^{M-1} \frac{z_{i,N}^b}{2h} - \frac{1}{8h}(w_{0,0}^b + w_{0,1}^b) - \sum_{j=2,4,\ldots}^{N-2} \frac{w_{0,j}^b}{2h} - \frac{1}{8h}(w_{0,N-1}^b + w_{0,N}^b)$$
$$+ \frac{1}{8h}(w_{M,0}^b + z_{M,1}^b) + \sum_{j=2,4,\ldots}^{N-2} \frac{w_{M,j}^b}{2h} + \frac{1}{8h}(w_{M,N-1}^b + w_{M,N}^b) = 0$$

When each of these expressions is multiplied by $4h^2$ then the sums can be seen either as a midpoint rule or combined midpoint and trapezoidal rule approximation to the integral of the normal velocity on an edge. In the combined approximation, the midpoint rule is used to approximate the integral over the interior of the edge and the trapezoidal rule used to approximate the integral at the ends of the edge.

Lastly we give the orthogonality conditions for the triangle points. These conditions assume that the equations associated with the circle, box and triangle points are satisfied. Furthermore, in order to get nice formulas, we assume that at the corners the velocity data is specified in such a way that if the divergence at the corners is computed using a one-sided approximation in each direction, then the sum of these four quantities is zero. With these assumptions, the condition becomes

$$- \frac{1}{8h}(w_{0,0}^b + w_{0,1}^b) - \sum_{j=2,4,\ldots}^{N-2} \frac{w_{0,j}^b}{2h} - \frac{1}{8h}(w_{0,N-1}^b + w_{0,N}^b)$$
$$+ \frac{1}{8h}(w_{M,0}^b + z_{M,1}^b) + \sum_{j=2,4,\ldots}^{N-2} \frac{w_{M,j}^b}{2h} + \frac{1}{8h}(w_{M,N-1}^b + w_{M,N}^b)$$
$$- \frac{1}{8h}(z_{0,0}^b + z_{1,0}^b) - \sum_{i=2,4,\ldots}^{M-2} \frac{z_{i,0}^b}{2h} - \frac{1}{8h}(z_{M-1,0}^b + z_{M,0}^b)$$
$$+ \frac{1}{8h}(z_{0,N}^b + z_{1,N}^b) + \sum_{i=2,4,\ldots}^{M-2} \frac{z_{i,N}^b}{2h} + \frac{1}{8h}(z_{M-1,N}^b + z_{M,N}^b) = 0$$

When each of these equations is multiplied by $4h^2$ the sums are a combined midpoint and trapezoidal rule approximation to the integral of the normal derivative around the boundary.

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