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**ANTI-CLASSIFICATION RESULTS
FOR SMOOTH DYNAMICAL SYSTEMS.
PRELIMINARY REPORT.**

MATTHEW FOREMAN AND ANTON GORODETSKI

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Author's Note The main results in this paper, Theorems 1 and 2, were made obsolete by joint results of the authors in March 2022. The improved results showed the unclassifiability of diffeomorphisms in every dimension. The results in this paper were proved in the fall of 2019 and winter of 2020. The authors post this manuscript on arXiv because the proofs in this manuscript illustrate fundamentally different techniques and the manuscript contains expository information that may be useful to researchers not familiar with Borel reducibility.

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1. INTRODUCTION

In his ICM talk in Stockholm in 1962, Stephen Smale proposed a program of studying the topological conjugacy classes of diffeomorphisms of a given manifold, [43]. The program was motivated by the related problem on qualitative behaviour of solutions of differential equations on a given manifold. The roots of the problems came from Poincare [40] and Birkhoff [4] who studied diffeomorphisms as cross sections of smooth flows—differentiable \mathbb{R} -actions. Smale pointed out that the topological conjugacy class of a diffeomorphism preserved all of its *qualitative behavior*. This includes properties such as having attractors, repelling points, being minimal, or transitive, or topologically mixing, or uniquely ergodic, and so forth. Quoting [42]:

“... the same phenomena and problems of the qualitative theory of ordinary differential equations are present in their simplest form in the diffeomorphism problem. Having first found theorems in the diffeomorphism case, it is usually a secondary task to translate the results back into the differential equations framework.”

The problem is stated precisely as follows. Fix a compact manifold M and $f, g \in \text{Diff}(M)$. Put $f \sim_{top} g$ if and only if there is a homeomorphism $h : M \rightarrow M$ such that $f \circ h = h \circ g$. This equivalence relation is referred to as *topological conjugacy*. Smale suggested classifying the equivalence classes by “numerical and algebraic invariants.”

An obvious problem with this definition is that the equivalence relation does not arise as the orbit equivalence relation of a group action. On the surface then, a better notion might be that f is equivalent to g if and only if they are conjugate by a *diffeomorphism* h . However Smale rejected this as being too fine an equivalence relation, in part because “the eigenvalues of the derivative at a fixed point are differentiable conjugacy invariants”, and suggested to consider topological conjugacy as the most relevant equivalence relation for smooth dynamical systems.

Author’s Update In the winter of 2022, P. Kunde showed that the equivalence relation E_0 is continuously reducible to the equivalence relation of *conjugacy by diffeomorphisms* on diffeomorphisms of the unit circle with Liouvillean rotation numbers. Thus even in one dimension, that equivalence relation cannot be understood by complete numerical invariants. (See below [why \$E_0\$ is relevant.](#))

1.1. Anti-classification: Main results. The main result of this paper is that *Smale’s Program* is hopeless in the rigorous sense that there is no *Borel* classification of topological conjugacy, even in the very

regular case of C^∞ -diffeomorphisms. More precisely we show the following¹:

Theorem 1. *Let M be a manifold of dimension at least 5. Consider the space $\text{Diff}(M)$ of C^∞ -diffeomorphisms of M with the C^∞ -topology. Then $\sim_{\text{top}} \subseteq \text{Diff}(M) \times \text{Diff}(M)$ is not a Borel set in the product topology.*

For dimension 2 we show that complete numerical invariants don't exist².

Theorem 2. *Let M a manifold of dimension at least two. Then there is no Borel function f from $\text{Diff}(M)$ to Y , where Y is a Polish Space such that $S \sim_{\text{top}} T$ if and only if $f(S) = f(T)$. In particular there is no Borel function $f : \text{Diff}(M) \rightarrow \mathbb{R}^n$ that gives complete numerical invariants for the equivalence relation \sim_{top} on $\text{Diff}(M)$.*

It remains to explain why these results show that classification is impossible.

In Section 2 we prove Theorem 2. In Section 3 we prove Theorem 1. In Section 4 we formulate some open questions and conjectures related to our results.

1.2. Impossibility Results and Generalized Church's Thesis.

Many of the most famous results in mathematics are *impossibility results*: You can't square the circle using a ruler and compass, you can't solve the general quintic polynomial using radicals and so forth. What these results have in common is a precise description of what it would mean to solve the problem.

In the 20th century, impossibility results took a different turn. The tools of Gödel and Turing made it possible to show results of the form:

It is impossible to solve the problem P using purely finitary methods.

But this demands an explanation of what the definition of *finitary methods* means. The rigorous interpretation of a problem being solvable with finitary methods is that it can be solved by a *recursive algorithm*: there is a Turing machine that can check each instance and correctly give the answer.³

¹After this text was written, we realized that using a different construction one can show that Theorem 1 in fact holds for any closed manifold (even for the circle). We intend to provide that stronger result in the final version of the paper.

²In Theorem 2 one can replace topological conjugacy by a smooth (C^r for any given $r = 0, 1, \dots, \infty$) change of coordinates.

³More formally, there is a Turing machine with a finite set of instructions that, on input of the appropriate data always halts in finite time and outputs the correct

Some famous examples of impossibility results include:

Novikov:: The word problem is not recursive: Given a finite presentation of a group it is impossible to tell whether an arbitrary given word is trivial in the group.

Davis, Matiyevich, Putnam, Robinson:: Hilbert's 10th problem is undecidable: it is impossible to tell whether an arbitrary given diophantine polynomial has a solution with strictly finitary methods.

The meta-assertion that all notions of finitary computation are equivalent is referred to as *Church's Thesis*.

A Generalized Church's Thesis. Theorems 1 and 2 are from a genre of even stronger impossibility results. They can be interpreted as saying:

It is impossible to solve a problem P using inherently countable techniques.

For this to make sense the problem has to be posed in the context of a Polish space— a topological space X whose topology is compatible with a complete separable metric.

The notion of a Polish space captures the idea that a countable amount of information can be used to completely specify the problem P. The countable dense set D and a compatible complete metric $d(x, y)$ on $D \times D$ constitutes a countable amount of information. Since D is dense every point of X is a limit of elements of a fixed countable set.

Having fixed D and d , basic open intervals, consisting of rational neighborhoods of this dense set are then viewed as countable information. Having basic open intervals, countable Boolean operations (union, intersection and complement) then exhaust what one can do inherently countably. The result is the collection of Borel subsets of X .

As summarized above, a Church's Thesis can be viewed as stating that a problem P is recursive if and only if it can be solved using inherently finite techniques in any sense of *inherently finite*. The generalized Church's thesis we are suggesting is that a problem can be solved with *inherently countable techniques* just in case it can be viewed as *Borel in a Polish Space*.

Is this a reasonable definition of inherently countable techniques? It may be too liberal—perhaps the metric on the Polish Space itself uses the uncountable axiom of choice to construct. But we will be showing

answer. There are no time or space limitations, such as *polynomial time* on the computation.

things are *not Borel* so our results are even stronger if the notion is too liberal.

But is it too restrictive? Are there methods that should be considered inherently countable that are not Borel? Any construction that involves taking countable limits preserves the property of being Borel. Anything that can reasonably be called a *construction* that takes countably many steps is Borel.

Moreover Borel sets are preserved under pre-images of Borel functions. Hence any computation that results in some sort of complete computational invariant for an equivalence relation implies that that equivalence relation is Borel. As a non-trivial example, the Spectral Theorem for Normal Operators gives complete invariants for the equivalence relation of Unitary Equivalence, showing that equivalence relation of Unitary Equivalence of Normal Operators is a Borel equivalence relation.

To clarify the idea, we give example of some invariants commonly attached to a diffeomorphism.

We will denote by $\text{Fix}(f)$ the set of fixed points of the map $f : M \rightarrow M$, i.e.

$$\text{Fix}(f) = \{x \in M \mid f(x) = x\},$$

and by $\text{Per}_n(f)$ the set of periodic points of period n , i.e.

$$\text{Per}_n(f) = \{x \in M \mid f^n(x) = x\}.$$

Proposition 3. *Let M be a compact manifold. For any $r = 0, 1, 2, \dots, \infty$ the following statements hold:*

- (1) *The set $\mathcal{U}_0 = \{f \in \text{Diff}^r(M) \mid \text{Fix}(f) = \emptyset\}$ is open;*
- (2) *For each $m \in \mathbb{N}$ the set $\mathcal{U}_m = \{f \in \text{Diff}^r(M) \mid \#\text{Fix}(f) = m\}$ is Borel;*
- (3) *For each $m, n \in \mathbb{N}$ the set $\mathcal{U}_m^n = \{f \in \text{Diff}^r(M) \mid \#\text{Per}_n(f) = m\}$ is Borel;*
- (4) *For any sequence of integers $A = \{a_n\}$, $a_n \geq 0$, the set*

$$\mathcal{U}_A = \{f \in \text{Diff}^r(M) \mid \#\text{Per}_n(f) = a_n, n = 1, 2, \dots\}$$

is Borel;

- (5) *The set*

$$\{(A, f) \mid \#\text{Per}_n(f) = a_n, n = 1, 2, \dots, \text{ where } A = \{a_n\}\}$$

in the space $\mathbb{N}^{\mathbb{N}} \times \text{Diff}^r(M)$ is Borel;

(6) The function $G : \text{Diff}^r(M) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ defined by

$$G(f) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \text{Per}_n(f)$$

(the exponential rate of growth of the number of periodic points) is Borel. In particular, the set of Artin-Mazur diffeomorphisms (defined by $\{f : G(f) < \infty\}$) is Borel.

Remark 4. Certainly, one could add many other Borel sets and Borel functions that can be expressed in terms of the number of periodic points to the statement of Proposition 3. We specifically mentioned the rate of growth of the number of periodic points, since it is the function that was heavily studied in the theory of dynamical systems (e.g. see [1, 24, 25, 26, 27]).

Proof of Proposition 3. It is clear that the set \mathcal{U}_0 is open.

To see that \mathcal{U}_1 is Borel, let us choose a countable base of topology on M , $\{V_k\}_{k \in \mathbb{N}}$, such that

$$\text{diam } V_1 \geq \text{diam } V_2 \geq \text{diam } V_3 \geq \dots, \quad \text{and} \quad \text{diam } V_k \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

The set

$$\mathcal{V}_k = \{f \in \text{Diff}^r(M) \mid f \text{ has no fixed points outside of } V_k\}$$

is open, and hence the set

$$\mathcal{U}_1 = \left(\bigcap_{K \geq 1} \bigcup_{k=K}^{\infty} \mathcal{V}_k \right) \setminus \mathcal{U}_0$$

is Borel.

To see that \mathcal{U}_2 is Borel, notice that each of the sets

$$\mathcal{V}_{k_1, k_2} = \{f \in \text{Diff}^r(M) \mid f \text{ has no fixed points outside of } V_{k_1} \cup V_{k_2}\}.$$

is open, and hence

$$\mathcal{U}_2 = \left(\bigcap_{K \geq 1} \bigcup_{k_1, k_2 \geq K} \mathcal{V}_{k_1, k_2} \right) \setminus (\mathcal{U}_0 \cup \mathcal{U}_1)$$

is Borel. Similarly one can show that \mathcal{U}_m is Borel for each $m \in \mathbb{N}$.

Since for each given $n \geq 2$ the map $f \mapsto f^n$ is continuous, the set $\{f^n \text{ has no fixed points outside of } V_k\}$ is open, and the arguments similar to those above prove that the sets \mathcal{U}_m^n are Borel. Also, one has $\mathcal{U}_A = \bigcap_n \mathcal{U}_{a_n}^n$, hence \mathcal{U}_A is Borel.

To show that the set

$$\{(A, f) \mid \#\text{Per}_n(f) = a_n, \quad n = 1, 2, \dots, \text{ where } A = \{a_n\}\}$$

is Borel, notice that it can be represented as $\bigcup_{n \in \mathbb{N}} X_n$, where

$$X_n = \bigcup_{m \in \mathbb{N}} \{(A, f) \mid n\text{-th element of } A \text{ is } m, \text{ and } f \in U_m^n\}$$

is a Borel set. Finally,

$$\{f \in \text{Diff}^r(M) \mid G(f) < C\} = \bigcup_{k \in \mathbb{N}} \bigcup_{M \in \mathbb{N}} \bigcap_{n \geq M} \bigcup_{m < \exp(nC - \frac{n}{k})} U_m^n,$$

hence $G : \text{Diff}^r(M) \rightarrow \mathbb{R} \cup \{\pm\infty\}$ is a Borel function. \dashv

In contrast, Theorem 1 says that the equivalence relation of topological conjugacy is *not* Borel for manifolds of dimension 5 or greater, and Theorem 2 claims that there is no way to provide complete numerical invariants that would be Borel functions of a diffeomorphism and provided classification up to a topological conjugacy for manifolds of dimension 2 or greater. Thus it precludes any reasonable complete invariants for the qualitative behavior of these diffeomorphisms.

Certainly, if we restrict ourselves to some smaller class of diffeomorphisms, or consider phase space of small dimension (say, the circle), then it is reasonable to expect that a classification up to topological conjugacy can exist. In Section 1.3 below we give some examples of such results.

1.3. Examples. To put our results into context, here we provide some examples of successful classification of some classes of smooth dynamical systems, as well as some known results on absence of such classification (mostly regarding topological and measure preserving dynamics). We would like to emphasize that both lists are far from being even remotely complete, and are provided as an illustration rather than a survey of all relevant results. Another list of problems will appear in the forthcoming paper by Foreman, *The complexity of the Structure and Classification of Dynamical Systems*.

Examples of successful classifications in smooth dynamics:

- Real analytic diffeomorphisms of an interval: Any two homeomorphisms of an interval $[0, 1]$ that do not have any fixed points except 0 and 1 are topologically conjugate, as can be shown using a method of fundamental domain. Using this fact, one can show that there exist complete numerical invariants for a topological conjugacy on the space of real analytic diffeomorphisms of the unit interval.

- Classification of real analytic diffeomorphisms of the circle up to a topological conjugacy: The rotation number is an invariant of (orientation preserving) topological conjugacy. Due to Denjoy's Theorem [9], any C^2 diffeomorphism with irrational rotation number is topologically conjugate to a rotation of the circle. For real analytic diffeomorphisms with rational rotation number the number, type (stable/unstable/semistable), and relative order of the periodic points form a complete invariant of topological conjugacy. See [28] for more details on rotation numbers and Poincaré's classification.

- Franks–Newhouse Theorem on Anosov (uniformly hyperbolic) diffeomorphisms: If f is an Anosov diffeomorphism of co-dimension one (i.e. either stable or unstable direction has dimension one) on a compact Riemannian manifold M , then f is topologically conjugate to a hyperbolic toral automorphism, see [19] and [34], and also [22] for an alternative proof.

- Anosov maps on tori: Any Anosov diffeomorphism of a torus is topologically conjugate to a linear automorphism of a torus, and two linear automorphisms of a torus are topologically conjugate if and only if the corresponding matrices are conjugate by elements of $GL_2(\mathbb{Z})$, see [18], and also [28, Section 18.6]. The linear automorphism of the torus associated with an Anosov Diffeomorphism can be computed with a Borel function. This provides an explicit classification of Anosov maps of a torus.

- Structurally stable diffeomorphisms: Fix a compact manifold M . A C^r diffeomorphism $f : M \rightarrow M$ is *structurally stable* if there is a C^1 -open set O containing f such that every $g \in O$ is topologically conjugate to f . The collection of structurally stable diffeomorphisms is open and there can only be countably many \sim_{top} equivalence classes of structurally stable diffeomorphisms. It follows immediately that they are classifiable in the sense that there is a (locally constant) function R from the C^r -structurally stable diffeomorphisms to the natural numbers so that two structurally stable diffeomorphisms f, g are conjugate if and only if $R(f) = R(g)$. Extending this function so that it takes value -1 on the complement of the open set of structurally stable diffeomorphisms gives a Borel classification of the structurally stable diffeomorphisms by complete numerical invariants.

However, it is not a classification of all diffeomorphisms that are topologically conjugate to a structurally stable diffeomorphism. This is an open problem, as of this writing.

- An important case of structurally stable diffeomorphisms that in lower dimensions can be classified by very explicit invariants is the collection of Morse-Smale diffeomorphisms. To define Morse-Smale diffeomorphisms, we need the following definition:

Definition A point x is a wandering point of the map f if it has a neighborhood $U = U(x)$ such that $f^n(U) \cap U = \emptyset$ for all $n \geq 1$. The non-wandering set of f is the set of all points that are not wandering.

Definition 5. A diffeomorphism f is Morse-Smale if

- (1) f has only finite number of periodic orbits and these form the non-wandering set of f ,
- (2) all of the periodic orbits are hyperbolic, and
- (3) the stable and unstable manifolds of any two periodic points intersect transversally.

Remark 6. A well-understood method of constructing some of the Morse-Smale diffeomorphisms uses gradient flows. Let $\{T_t : t \in \mathbb{R}\}$ be a flow on M such that

- (1) There is a function $L : M \rightarrow \mathbb{R}$ such that L has a finite number of critical points x_1, \dots, x_n , all of the critical points are non-degenerate and are such that for $i \neq j$, $L(x_i) \neq L(x_j)$.
- (2) T_t is the gradient flow for L .

Then the time-1 shift $f = T_1$ for this flow is a Morse-Smale diffeomorphism.

Because Morse-Smale flows are structurally stable, the previous discussion shows that they have an abstract classification by numerical invariants. However, more is known: there are classifications that give conceptual information about their properties.

Morse-Smale flows on surfaces were classified in [41]. A different set of invariants was suggested in [39]. A complete topological classification of Morse-Smale diffeomorphisms on compact orientable surfaces was obtained in [8], see also [3] and [21]. Topological classification of Morse-Smale diffeomorphisms on 3-manifolds was given in [7]; a complete topological invariant in this case is shown to be an equivalence class of a “scheme”, which contains an information on a periodic data and a topology of embedding of two-dimensional invariant manifolds of the saddle periodic points of a diffeomorphism into the ambient manifold. For a detailed historical background of the topological classification of Morse-Smale systems see Section 1.4 in [7].

In Remark 14 we discuss how to give a family of examples of unclassifiable diffeomorphisms that live on the boundary of the Morse-Smale diffeomorphisms.

Examples of anti-classification:

An early dramatic success of anticlassification results was the theorem due independently to Kaufman and Solovay (building on work of Debs, St. Raymond, Kechris and Louveau) who showed in 1983-84, that the collection of closed *sets of uniqueness* for trigonometric functions is not a Borel set. (A set $E \subseteq [0, 1]$ is a *set of uniqueness* if whenever $\sum c_n e^{2\pi i n x}$ on $[0, 1] \setminus E$ the series $\sum c_n e^{2\pi i n x}$ is identically 0.) Hence determining whether the complement of a given closed set determines the values of a trigonometric series (a problem posed by Kronecker, and studied by Cantor and many others) is simply not possible using anything resembling even a countable transfinite computation.

The first applications of the Borel/non-Borel distinction in dynamical systems were the results in [5] and [6] which showed the the collections of minimal topologically distal transformations and, respectively, generalized discrete spectrum transformations are not Borel.⁴ It has been used more recently in [15], [16] and [17] to show that ergodic diffeomorphisms of compact manifolds are not classifiable.

1.4. Borel sets, analytic sets and reductions. We now give some basic definitions. Fix a Polish space (X, τ) . A set is *Borel* if and only if it is in the smallest σ -algebra containing the open set τ . If X, Y are Polish spaces and $f : X \rightarrow Y$ is continuous, then for all Borel sets B the f -image of B , $f[B]$ is called an *analytic* set.

The simplest way of generating an analytic set is to start with a Borel subset $B \subseteq X \times Y$ and taking its projection to the Y -axis. Indeed all analytic subsets of Y can be built this way.⁵ The claim that all projections of Borel sets is Borel is a famous mistake of Lebesgue that was rectified by examples of Suslin.

Results such as Theorems 1 and 2 are shown using *reductions*, a technique due to Friedman and Stanley and developed independently by Harrington, Kechris and Louveau.

The idea starts with the cliché that:

⁴For a compact separable metric space X , a homeomorphism of X is *distal* if and only if for all $x \neq y$ there is an $\epsilon > 0$ for all n , $d(T^n x, T^n y) > \epsilon$. The definition dates to Hilbert.

⁵For this and other information about Borel and analytic sets see [12], [31] or [29].

To solve A you reduce it to a problem B which you already know how to solve.

Taking the contrapositive:

To show that solving B is impossible, you start with a *known* impossible problem A and reduce it to B .

Formally:

Definition 7. *Let X and Y be Polish spaces, $A \subseteq X$, $B \subseteq Y$. Then A is Borel reducible to B if and only if there is a Borel function $f : X \rightarrow Y$ such that for $x \in X$:*

$$x \in A \text{ if and only if } f(x) \in B.$$

Thus if A is not Borel, B cannot be either, since the inverse image of a Borel set by a Borel function is not Borel. The function f is a *Borel reduction*.

Define $A \preceq_{\mathcal{B}} B$ if A is Borel reducible to B . Then $\preceq_{\mathcal{B}}$ is transitive since one can compose Borel reductions. Defining the equivalence relation $A \sim_{\mathcal{B}} B$ if $A \preceq_{\mathcal{B}} B$ and $B \preceq_{\mathcal{B}} A$ we see that $\preceq_{\mathcal{B}}$ induces a partial ordering of the $\sim_{\mathcal{B}}$ equivalence classes.

The heuristic above interprets $A \preceq_{\mathcal{B}} B$ as saying that B is at least as complicated as A (with respect to countably feasible computations) and $A \sim_{\mathcal{B}} B$ as saying that they have the same complexity. Among analytic sets, there is a $\preceq_{\mathcal{B}}$ -maximal equivalence class, the *complete* analytic sets. Since the Borel sets are $\preceq_{\mathcal{B}}$ -downwards closed, and there are non-Borel analytic sets, any complete analytic set cannot be Borel.

Two dimensional reductions and equivalence relations

Let E and F be equivalence relations on Polish spaces X and Y respectively. We define $E \preceq_{\mathcal{B}} F$ if and only if there is a Borel function $f : X \rightarrow Y$ such that for all $x_1, x_2 \in X$,

$$x_1 E x_2 \text{ if and only if } f(x_1) F f(x_2).$$

We note that this differs from the definition above because it relates pairs of elements of X and Y that are connected by a unary function—hence a *two dimensional reduction*.

For the notion of reduction to be useful, one must have complicated examples. Indeed [11] describes a whole collection of benchmark equivalence relations that can be used to calibrate the complexity of a given equivalence relation.

All uncountable Polish Spaces are Borel Isomorphic. It is a classical fact that all uncountable Polish spaces are Borel isomorphic.

(See, for example [12].) Hence the particular spaces being used for reductions are chosen so the the mathematical tools appropriate for the context are available. The strongest classifications are Borel reductions to the identity equivalence relation on a Polish space. Because every Polish space can be injected into $[0, 1]$ (or the irrational numbers in $[0, 1]$), we call these *numerical invariants*. In particular to show that there is *no* reduction to the identity relation on any Polish space it suffices to show there is no reduction to the identity equivalence relation on $[0, 1]$.

1.5. The space of countable graphs and Theorem 1. One prominent example consists of the space $Graphs \subseteq \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ consisting of those $f : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ such that for all (n, m) , $f(n, m) = f(m, n)$. If $G = \langle \mathbb{N}, E \rangle$ is a graph with vertices \mathbb{N} and edges E , then we can associate the function $f_G : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ by putting $f_G(m, n) = 1$ just in case m and n are connected by an edge E . This process can be reversed: any symmetric $f : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ defines a graph G_f by putting an edge between m and n if $f(m, n) = 1$. Hence we can identify graphs with vertex set \mathbb{N} with elements of the subspace $Graphs \subseteq \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$. We will say that the function f_G *codes* the graph G .

Since the space $\{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ can be viewed as a countably infinite product of the two point space $\{0, 1\}$, it can be endowed with the product topology. With this topology, $\{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is homeomorphic to the Cantor set. The collection $Graphs \subseteq \{0, 1\}^{\mathbb{N} \times \mathbb{N}}$ is a closed subset, and hence the induced topology makes it a Polish space.

Let S_∞ be the space of (arbitrary) permutations of \mathbb{N} . Then S_∞ acts on $Graphs$ by setting, for $\phi \in S_\infty$ and $f_G \in Graphs$,

$$(\phi f_G)(m, n) = 1 \text{ if and only if } f_G(\phi^{-1}m, \phi^{-1}n) = 1. \quad (1)$$

Then ϕf_G is a graph with vertex set \mathbb{N} .

It is routine to check that for graphs G, H with vertices \mathbb{N} , G and H are isomorphic just in case f_G and f_H are in the same S_∞ orbit.⁶

Here are two standard results:

Theorem. *Consider the isomorphism relation \sim_{iso} of $Graphs$. Then:*

- (1) *As a subset of the Polish space $Graphs \times Graphs$, the relation \sim_{iso} is complete analytic.*
- (2) *If X is a Polish space and $E \subset X \times X$ is the orbit equivalence relation arising from an S_∞ -action then $E \preceq_{\mathcal{B}} \sim_{iso}$.*

⁶In the jargon of Hjorth's influential book, *Isomorphism of Countable Graphs* is an example of an equivalence relation "classifiable by countable models." Indeed the graphs themselves are the models.

We can now restate the main theorem of this paper:

Theorem 8. *Let M be a manifold of dimension at least 5. Then the isomorphism relation on Graphs is Borel reducible to the equivalence relation \sim_{top} on $Diff(M)$.*

Indeed, Theorem 1 follows immediately from this result.

1.6. The relation E_0 and Theorem 2. In this section we discuss assigning complete numerical invariants to an equivalence relation. Given any two uncountable Polish spaces X, Y there is a Borel bijection between X and Y . Hence all uncountable Polish spaces are Borel isomorphic to the irrational numbers in $[0, 1]$. For this reason the phrase *complete numerical invariant* is used to refer to invariants x belonging to any fixed Polish space X .

The first anti-classification result in dynamical systems The Kolmogorov-Sinai entropy is a function that computes a positive real number in a Borel way. Ornstein [38] showed that for Bernoulli shifts, the entropy is a complete invariant: two Bernoulli shifts that have the same entropy are isomorphic by a measure preserving transformation. After Ornstein’s classification there was optimism that the methods could be extended to larger collections.

The natural first step was to \mathcal{K} -automorphisms, those ergodic measure preserving transformations where “the present is asymptotically independent of the past.” Ornstein and Shields produced an uncountable family of non-isomorphic \mathcal{K} -automorphisms of any given entropy—thus ruling out entropy as a potential complete invariant.

Feldman ([10]) gave the first general anti-classification result that showed that there is *no* numerical invariant that completely classifies the \mathcal{K} -automorphisms up to measure isomorphism. We describe his result in contemporary language.

The benchmark equivalence relation he considers is called E_0 . It is an equivalence relation on the Cantor set. Viewing the Cantor set as $\{0, 1\}^{\mathbb{N}}$, two elements f, g are E_0 equivalent if and only if there is an N for all $n \geq N, f(n) = g(n)$.⁷

The significance of E_0 is captured by the following theorem:

Theorem. *(Harrington, Kechris, Louveau) Let E be an analytic equivalence relation on a Polish space X . Then exactly one of the following holds:*

⁷In general we use the notation X^Y for the space of functions from Y to X , so $\{0, 1\}^{\mathbb{N}}$ consists of all functions from \mathbb{N} to $\{0, 1\}$; all countable sequences of 0’s and 1’s.

E is smooth: *There is a Borel function $f : X \rightarrow Y$ where Y is a Polish space that reduces E to the identity equivalence relation on Y .*

E_0 reduced to E : *The equivalence relation E_0 is reducible to E .*

The first case means that f attaches complete numerical invariants (elements of Y) to E . The theorem then says that either there is a Borel function that computes complete numerical invariants for E or E_0 is reducible to E .

Feldman's theorem can be restated as saying:

Theorem. *There is a continuous map $f : 2^{\mathbb{N}} \rightarrow \{\mathcal{K}\text{-automorphisms}\}$ that reduces E_0 to measure isomorphism.*

For the reader's benefit we sketch the argument for the incompatibility of the two alternatives described in the Harrington, Kechris, Louveau theorem. This is the portion of their theorem we use.

Proposition 9. *Suppose that E is an equivalence relation on a Polish space X and E_0 is Borel reducible to E . Then there is no Borel function $g : X \rightarrow Y$ with Y a Polish space such that for all $x_1, x_2 \in X$,*

$$x_1 E x_2 \text{ if and only if } g(x_1) = g(x_2).$$

⊢ By the earlier remarks after the definition of two dimensional reductions, we can assume that $Y = [0, 1]$.

If $f : \{0, 1\}^{\mathbb{N}} \rightarrow X$ is the reduction of E_0 to E then $g \circ f : \{0, 1\}^{\mathbb{N}} \rightarrow [0, 1]$ and for all $\vec{x}, \vec{y} \in \{0, 1\}^{\mathbb{N}}$, we have \vec{x} eventually agrees with \vec{y} if and only if $(g \circ f)(\vec{x}) = (g \circ f)(\vec{y})$. So without loss of generality we can take $E = E_0$ and f is a reduction to the identity equivalence relation on $[0, 1]$.

Since f is a Borel map, for each subinterval I of $[0, 1]$, $f^{-1}(I)$ has the property of Baire. Fix an n and look at decimal expansions of elements of $[0, 1]$ of length n . These partition $[0, 1]$ into 10^n many sets. For each $\vec{a} = a_0 a_1 \dots a_n$ with $0 \leq a_i \leq 9$, let $A_{\vec{a}}$ be the set of x in $[0, 1]$ such that the first n digits of x are $a_0 a_1 \dots a_n$. By the Baire Category Theorem, there must be some \vec{a} such that $f^{-1}(A_{\vec{a}})$ is not first category.⁸

A standard result proved using the Banach-Mazur game [2] says that there must be a basic open interval $[s]$ in the product topology on $\{0, 1\}^{\mathbb{N}}$ given by a finite sequence of 0's and 1's such that $f^{-1}(A_{\vec{a}})$ is residual in $[s]$. Since equivalence by E_0 just depends on tails of elements $\vec{x} \in \{0, 1\}^{\mathbb{N}}$, $f^{-1}(A_{\vec{a}})$ must be residual in $\{0, 1\}^{\mathbb{N}}$.

⁸In Descriptive Set Theory the terms "meager" and "comeager" are usually used, while in Dynamical Systems their synonyms, "first category" and "residual," are mostly preferred.

Hence, again by the Baire Category theorem, \vec{a} must be unique. Denote the sequence of length n by \vec{a}_n . Since $f^{-1}(A_{\vec{a}_n})$ and $f^{-1}(A_{\vec{a}_{n+1}})$ are both residual we must have \vec{a}_{n+1} extending \vec{a}_n .

This defines a collection $\langle \vec{a}_n : n \in \mathbb{N} \rangle$ of finite subsequences of $\{0, 1, \dots, 9\}$ that extend each other. If $a \in [0, 1]$ has decimal expansion given by $\langle \vec{a}_n : n \in \mathbb{N} \rangle$ then, again by the Baire Category Theorem, $f^{-1}(a)$ is residual.

Let $B = f^{-1}(a) \subseteq \{0, 1\}^{\mathbb{N}}$. Since B is residual, it contains a perfect set—in particular it contains x, y that disagree infinitely often. But then $x \not\equiv_0 y$ and $f(x) = f(y)$, a contradiction. \dashv

We can now state the content of Theorem 2 more precisely:

Theorem 10. *Let M be a manifold of dimension at least two and \sim_{top} the relation of topological equivalence on $\text{Diff}(M)$. Then there is a Borel reduction of E_0 to Diff/\sim_{top} .*

2. TWO DIMENSIONAL DIFFEOMORPHISMS

Here we prove Theorem 10. By Proposition 9 that will imply Theorem 2.

Let

$$\Sigma = \{0, 1\}^{\mathbb{N}} = \{ \bar{\omega} = \omega_1 \omega_2 \dots, \omega_i \dots \in \{0, 1\} \}.$$

The standard topology makes Σ homeomorphic to the Cantor set. This topology is induced by the metric:

$$d(\bar{\omega}, \bar{\omega}') = \begin{cases} 0, & \text{if } \bar{\omega} = \bar{\omega}'; \\ 2^{-k}, & \text{if } \omega_i = \omega'_i \text{ for } i < k, \text{ and } \omega_k \neq \omega'_k. \end{cases}$$

Recall the definition of E_0 :

$$\bar{\omega} \sim_{E_0} \bar{\omega}' \quad \text{if and only if} \quad \omega_i = \omega'_i \quad \text{for all sufficiently large } i \in \mathbb{N}.$$

In other words, $\bar{\omega} \sim_{E_0} \bar{\omega}'$ iff $\bar{\omega}$ and $\bar{\omega}'$ belong to the inverse orbit of the same point under the topological Bernoulli shift.

Theorem 11. *There exists a continuous map $\mathbf{F} : \Sigma \rightarrow \text{Diff}^{\infty}(\mathbb{R}^2)$ such that $\mathbf{F}(\bar{\omega})$ is topologically conjugate to $\mathbf{F}(\bar{\omega}')$ if and only if $\bar{\omega} \sim_{E_0} \bar{\omega}'$.*

Since the support of each diffeomorphism in the range of F is compact, the proof of Theorem 11 for $\text{Diff}^{\infty}(\mathbb{R}^2)$ works for $\text{Diff}^r(M)$ for any $r = 0, 1, \dots, \infty$, and any two dimensional surface M . It is easily adapted to any manifold of dimension at least two as discussed in Corollary 13.

Before giving a formal proof, let us provide an informal description of the construction.

First we will choose two points $a, b \in \mathbb{R}^2$, and two sequences of disjoint closed discs $D(c_n^a, r_n)$ and $D(z_n^a, r_n)$ converging to the point a and corresponding discs $D(c_n^b, r_n)$ and $D(z_n^b, r_n)$ converging to the point b . These are shown in figure 1.

For each $\bar{\omega} \in \Sigma$, the map $\mathbf{F}(\bar{\omega})$ will be supported on the union of these discs, and the restrictions of $\mathbf{F}(\bar{\omega})$ to any two of those discs are going to be different (i.e. not topologically conjugate). The restrictions $\mathbf{F}(\bar{\omega})|D(c_n^a, r_n)$ and $\mathbf{F}(\bar{\omega})|D(z_n^a, r_n)$ will tend to the identity much faster than c_n^a or z_n^a converge to a and the analogous fact holds for the discs corresponding to b . This ensures smoothness of the map at the points a and b . The restrictions $\mathbf{F}(\bar{\omega})$ to the discs $D(c_n^a, r_n)$ and $D(c_n^b, r_n)$ will not depend on ω . They are needed to make sure that if $\mathbf{F}(\bar{\omega})$ and $\mathbf{F}(\bar{\omega}')$ are topologically conjugate, then the conjugacy fixes the points a and b .

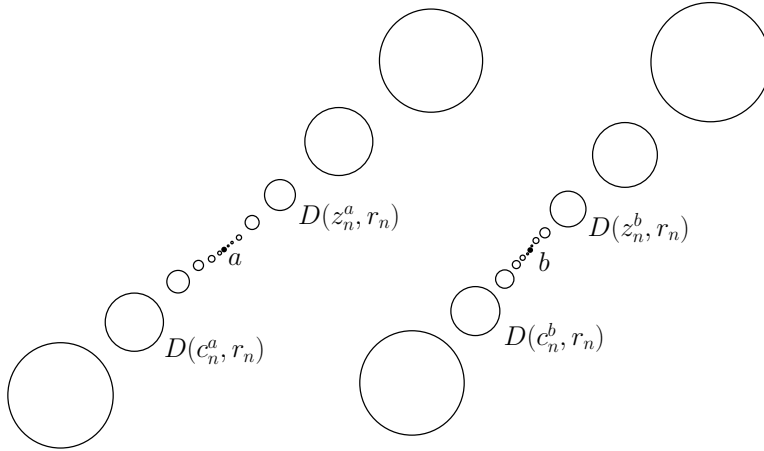


FIGURE 1. The construction of $\mathbf{F}(\bar{\omega})$.

Let us now explain how we choose the restrictions of $\mathbf{F}(\bar{\omega})$ to $D(z_n^a, r_n)$ and $D(z_n^b, r_n)$. First fix pairwise different maps f_n, g_n on discs of radius r_n . For any sequence $\bar{\omega} = \omega_1 \omega_2 \dots$ with $\omega_i \in \{0, 1\}$:

- (1) if $\omega_n = 0$ we let $\mathbf{F}(\bar{\omega})|D(z_n^a, r_n) = f_n$ and $\mathbf{F}(\bar{\omega})|D(z_n^b, r_n) = g_n$,
- (2) if $\omega_n = 1$, then we will swap them: $\mathbf{F}(\bar{\omega})|D(z_n^a, r_n) = g_n$ and $\mathbf{F}(\bar{\omega})|D(z_n^b, r_n) = f_n$.

We will show that $\bar{\omega} \sim_{E_0} \bar{\omega}'$, if and only if $\mathbf{F}(\bar{\omega})$ is topologically conjugate to $\mathbf{F}(\bar{\omega}')$.

Let us now provide the formal proof.

Proof of Theorem 11. For each $\bar{\omega} \in \Sigma$ we need to define a diffeomorphism $\mathbf{F}(\bar{\omega})$ of \mathbb{R}^2 . We denote $\mathbf{F}(\bar{\omega})$ by $f_{\bar{\omega}}$.

The diffeomorphism $f_{\bar{\omega}}$ is the identity on the complement of the union

$$D(c_n^a, r_n) \cup D(c_n^b, r_n) \cup D(z_n^a, r_n) \cup D(z_n^b, r_n).$$

It remains to define $f_{\bar{\omega}}$ on the discs themselves.

Fix two points $a, b \in \mathbb{R}^2$ such that $\text{dist}(a, b) = \frac{1}{2}$. Choose four sequences $\{c_n^a\}$, $\{c_n^b\}$, $\{z_n^a\}$, and $\{z_n^b\}$, $n \geq 2$, of points on the torus such that for each $n \geq 2$ we have

$$\text{dist}(a, c_n^a) = \text{dist}(b, c_n^b) = \frac{1}{n^4},$$

and

$$\text{dist}(a, z_n^a) = \text{dist}(b, z_n^b) = \frac{1}{2n^4}.$$

Set $r_n = \frac{1}{n^{10}}$, $n \geq 2$.

For any $c \in \mathbb{R}^2$ and $r > 0$ denote by $D(c, r)$ a closed disc centered at c of radius r .

Fix a C^∞ function $\varphi : [0, 1] \rightarrow [0, 1]$ in such a way that the following properties hold:

- 1) φ is non-increasing;
- 2) $\varphi|_{[0, 1/4]} \equiv 1$;
- 3) $\varphi|_{[3/4, 1]} \equiv 0$.

For any $c \in \mathbb{R}^2$, $r > 0$, and $q > 0$ denote by $\psi(c, r, q)$ the diffeomorphism

$$\psi(c, r, q) : D(c, r) \rightarrow D(c, r)$$

defined by setting

$$\psi(c, r, q)(c + \rho e^{i\theta}) = c + \rho e^{i(\theta + q\varphi(\frac{\rho}{r}))}.$$

Fix a strictly increasing function $m : \mathbb{N} \rightarrow \mathbb{N}$ such that $m(n)$ takes only odd values, and grows sufficiently fast (for example $m(n) > e^n$ for all $n \in \mathbb{N}$). Set $q_n = \frac{1}{m(n)}$.

For $n \geq 2$ define $f_{\bar{\omega}}$ on the discs $D(c_n^a, r_n)$ and $D(c_n^b, r_n)$ by setting

$$\begin{aligned} f_{\bar{\omega}}|_{D(c_n^a, r_n)} &= \psi(c_n^a, r_n, \frac{1}{8}q_n)(x) \\ f_{\bar{\omega}}|_{D(c_n^b, r_n)} &= \psi(c_n^b, r_n, \frac{1}{16}q_n)(x). \end{aligned}$$

Then on the discs $D(c_n^a, r_n/4)$ and $D(c_n^b, r_n/4)$ $f_{\bar{\omega}}$ is a rotation by $\frac{1}{8}q_n(x)$ and $\frac{1}{16}q_n(x)$ respectively. Hence they have periodic points of different orders. It follows easily that $f_{\bar{\omega}}|_{D(c_n^a, r_n)}$ and $f_{\bar{\omega}}|_{D(c_n^b, r_n)}$ are not

conjugate by homeomorphisms, nor are the restrictions of $f_{\bar{\omega}}$ to discs of different radii conjugate.

On the discs having z_n^a and z_n^b as centers and $n \geq 2$ we define

$$f_{\bar{\omega}}|D(z_n^a, r_n) = \begin{cases} \psi(z_n^a, r_n, q_n) & \text{if } \omega_{n-1} = 0 \\ \psi(z_n^a, r_n, q_n/2) & \text{if } \omega_{n-1} = 1 \end{cases}$$

and

$$f_{\bar{\omega}}|D(z_n^b, r_n) = \begin{cases} \psi(z_n^b, r_n, q_n/2) & \text{if } \omega_{n-1} = 0 \\ \psi(z_n^b, r_n, q_n) & \text{if } \omega_{n-1} = 1. \end{cases}$$

The reason for these definitions is that under the map $f_{\bar{\omega}}$

- points within radius $r_n/4$ of c_n^a have period $8m_n$
- points within radius $r_n/4$ of c_n^b have period $16m_n$
- if $\omega_{n-1} = 0$, then points within radius $r_n/4$ of z_n^a have period m_n and points within radius $r_n/4$ of z_n^b have period $2m_n$
- if $\omega_{n-1} = 1$, the periodicities of the points within radius $r_n/4$ of z_n^a and z_n^b are reversed.

Thus the restriction of $f_{\bar{\omega}}$ to the z_n^a discs are not conjugate to the restriction of $f_{\bar{\omega}}$ to the z_n^b discs, nor to restrictions of $f_{\bar{\omega}}$ to discs of different radii or discs centered at the c_n^a 's or c_n^b 's.

Everywhere but at the points a and b the map $f_{\bar{\omega}}$ is C^∞ by definition. It is not hard to see that the points a and b are flat fixed points of $f_{\bar{\omega}}$ (since q_n is exponentially small). Therefore $f_{\bar{\omega}}$ is a C^∞ diffeomorphism of \mathbb{R}^2 .

Let us show that

- I. If $\bar{\omega} \sim_{E_0} \bar{\omega}'$, then $f_{\bar{\omega}}$ is topologically conjugated to $f_{\bar{\omega}'}$.
- II. If $f_{\bar{\omega}}$ is topologically conjugated to $f_{\bar{\omega}'}$, then $\bar{\omega} \sim_{E_0} \bar{\omega}'$.

Let us start with the first statement. If $\bar{\omega} \sim_{E_0} \bar{\omega}'$, then for some $k \in \mathbb{N}$, $\omega_n = \omega'_n$ for all $n \geq k$. The following statement is immediate.

Lemma 12. *There is a homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that*

- a) $h|D = Id$ for all discs of radius r_n centered at c_n^a or c_n^b , $n \geq 2$;
- b) $h|D$ is an isometry of discs D of radius r_n centered at z_n^a or z_n^b , all $n \geq 2$;

c) for each $n \leq k$ one has

$$h|D(z_n^a, r_n), \text{ and } h|D(z_n^b, r_n) \text{ are the identity maps if } \omega_n = \omega'_n$$

and

$$h[D(z_n^a, r_n)] = D(z_n^b, r_n), \text{ and } h[D(z_n^b, r_n)] = D(z_n^a, r_n) \text{ if } \omega_n \neq \omega'_n;$$

ω'_n ;

d) for $n > k$, $h|D(z_n^a, r_n)$ and $h|D(z_n^b, r_n)$ are the identity maps.

Now it is easy to see that

$$h \circ f_{\bar{\omega}} = f_{\bar{\omega}'} \circ h.$$

Thus if $\bar{\omega} \sim_{E_0} \bar{\omega}'$, then $f_{\bar{\omega}}$ is topologically conjugated to $f_{\bar{\omega}'}$.

For the converse, let us assume now that $f_{\bar{\omega}}$ is topologically conjugated to $f_{\bar{\omega}'}$, i.e. there exists a homeomorphism $h : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$h \circ f_{\bar{\omega}} = f_{\bar{\omega}'} \circ h.$$

The homeomorphism h must send the set of periodic points of order p for $f_{\bar{\omega}}$ to the set of periodic points of order p for $f_{\bar{\omega}'}$. Hence h sends the compliment of the union of the discs $D(z_n^a, \frac{3}{4}r_n)$, $D(z_n^b, \frac{3}{4}r_n)$, $D(c_n^a, \frac{3}{4}r_n)$ and $D(c_n^b, \frac{3}{4}r_n)$ as well as the collection of points $\{z_n\}, \{c_n\}$ to itself. In particular this implies that h must map the discs $D(z_n^a, \frac{3}{4}r_n)$, $D(z_n^b, \frac{3}{4}r_n)$, $D(c_n^a, \frac{3}{4}r_n)$, $D(c_n^b, \frac{3}{4}r_n)$ to each other one by one.

Since h preserves the period of points $D(c_n^a, \frac{3}{4}r_n)$ must be sent to $D(c_n^a, \frac{3}{4}r_n)$ and similarly for the $D(c_n^b, \frac{3}{4}r_n)$'s. Because the c_n^a 's converge to a and the c_n^b 's converge to b , h is the identity on $\{a, b\}$.

Finally h if $\bar{\omega}_n = \bar{\omega}'_n$ then h must send the discs $D(z_n^a, \frac{3}{4}r_n)$ and $D(z_n^b, \frac{3}{4}r_n)$ to themselves and transpose them if $\bar{\omega}_n \neq \bar{\omega}'_n$.

By continuity, for all but finite number of $n \geq 2$, we must have

$$h(D(z_n^a, \frac{3}{4}r_n)) = D(z_n^a, \frac{3}{4}r_n), \quad \text{and} \quad h(D(z_n^b, \frac{3}{4}r_n)) = D(z_n^b, \frac{3}{4}r_n).$$

Hence, $\bar{\omega} \sim_{E_0} \bar{\omega}'$. The proof of Theorem 11 is complete. \dashv

Corollary 13. *Let M be a C^∞ manifold of dimension at least 2. Then there is a continuous map $\mathbf{F} : \Sigma \rightarrow \text{Diff}^\infty(M)$ such that $\mathbf{F}(\bar{\omega})$ is topologically conjugate to $\mathbf{F}(\bar{\omega}')$ if and only if $\bar{\omega} \sim_{E_0} \bar{\omega}'$.*

\vdash It suffices to show this for a bounded open ball of \mathbb{R}^n for $n \geq 2$. It contains an open subset diffeomorphic to a product of two-dimensional disc and a torus of dimension $n - 2$. The map \mathbf{F} can be constructed in such a way that $\mathbf{F}(\bar{\omega})$ is identity away from that open set, and on that open set it acts as a product of a map on the disc that was constructed above, in the proof of Theorem 11, and identity map on the $(n - 2)$ -dimensional torus. \dashv

Remark 14. *The analog of Theorem 2 holds even if we restrict ourselves to the boundary of the set structurally stable, or even stronger, of the set of Morse-Smale diffeomorphisms.*

One can show that exact construction of \mathbf{F} described above can be carried out on the boundary of the set of Morse-Smale diffeomorphisms,

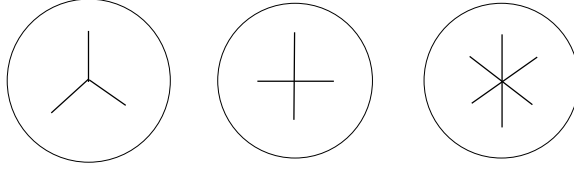


FIGURE 2. Non-homeomorphic sets of critical points inside of the discs.

but it becomes more transparent in case we make a slight modification in the construction. Namely, rather than the construction above define the map $F(\bar{\omega})$ in the discs $D(c_n^a, r_n)$, $D(c_n^b, r_n)$, $D(z_n^a, r_n)$, and $D(z_n^b, r_n)$ as time one shifts of C^∞ -gradient vector fields that vanish outside of a corresponding disc, decay exponentially fast as n goes to ∞ , and have pairwise non-homeomorphic sets of critical points (e.g. as in Fig. 2). The argument for \mathbf{F} being a reduction remains substantially the same.

With this change the diffeomorphisms in the range of the reduction belong to the boundary of the set of Morse-Smale diffeomorphisms. Indeed, the set of functions L satisfying the first condition of the construction of Morse-Smale diffeomorphisms given in Remark 6 forms an open and dense subset in the space of C^∞ functions, and the time one shift along a gradient vector field generated by such a function L must be a Morse-Smale diffeomorphism. Therefore, the time one shift along a gradient vector field (in particular, the diffeomorphisms in the image of \mathbf{F} constructed in this way) must be on the boundary of the set of Morse-Smale diffeomorphisms.

This remark is related to the open question 2 in Section 4.

3. FIVE DIMENSIONAL DIFFEOMORPHISMS

Here we prove Theorem 8, and hence Theorem 1.

3.1. Statement of the result. Recall that the space *Graphs* is given by the collection of symmetric maps $\mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ endowed with a topology that makes it homeomorphic to a Cantor set.

Theorem 15. *There exists a continuous map*

$$\mathbf{R} : \text{Graphs} \rightarrow \text{Diff}^\infty(\mathbb{R}^5)$$

such that $\mathbf{R}(E_1)$ and $\mathbf{R}(E_2)$ are topologically conjugate if and only if the graphs G_{E_1} and G_{E_2} are isomorphic.

Remark 16. *In Theorem 15, $\text{Diff}^\infty(\mathbb{R}^5)$ can be replaced by $\text{Diff}^\infty(M)$ for any smooth manifold of dimension at least 5. Indeed, in our construction any diffeomorphism in the range of \mathbf{R} is an identity outside*

of a finite ball, and therefore one can replace \mathbb{R}^5 by a single chart of a manifold M . This is explained further in Remark 19.

3.2. Construction of the map \mathbf{R} . The idea is to embed a complete countable graph into a bounded domain in \mathbb{R}^5 and to associate a given countable graph with a diffeomorphism that is supported in a union of small domains around the edges of the embedded graph. Each such domain is either a basin of attraction of a segment of fixed points (for the edges of the graph) or a repelling domain (for those edges of the complete graph that are missing in the given one).

Let us start by placing the vertices of the complete countable graph into \mathbb{R}^5 . In fact, we will place them all into a three dimensional subspace, and the extra two dimensions will be used to construct a topological conjugacy between the diffeomorphisms defined by isomorphic graphs. For this purpose we adopt the following notation:

We denote the subset of \mathbb{R}^5 given by $\{(x_1, x_2, x_3, 0, 0) : x_i \in \mathbb{R}\}$ by \mathbb{E}_3 . Its orthogonal complement, $\{(0, 0, 0, x_4, x_5) : x_i \in \mathbb{R}\}$ is denoted \mathbb{E}_2 . Note that $\mathbb{E}_3 \perp \mathbb{E}_2$ and they span \mathbb{R}^5 .

Proposition 17. *In \mathbb{E}_3 there exists a bounded sequence $\{x_n\}_{n \in \mathbb{N}}$, a point $r \notin \{x_n\}_{n \in \mathbb{N}}$ and a sequence of positive numbers $\{\eta_n\}$ such that*

- 1) $x_n \neq x_m$ if $n \neq m$;
- 2) $\text{dist}(x_n, r) < 2^{-n-1}$, so the sequence $\{x_n\}_n$ converges to r ,
- 3) No three points from the set $\{r, \{x_n\}_{n \in \mathbb{N}}\}$ are on the same line;
- 4) No four points from the set $\{r, \{x_n\}_{n \in \mathbb{N}}\}$ are on the same two dimensional plane.
- 5) The sequence of directions in the tangent space $T_r \mathbb{R}^5$ given by vectors $\{x_n - r\}$ converges to some direction that does not belong to this sequence.
- 6) Let $I_{m,n}$ be the closed interval with the end points x_m and x_n , and \mathbf{I}_n be the closed interval with the end points x_n and r . For any $n \neq k, m$, and any point $y \in I_{k,m}$ we have $\angle([r, y], \mathbf{I}_n) > \eta_n$.

⊢ Fix a point $r \in \mathbb{E}_3$ and a ray with end point r . Let x_1 be any point not on the ray. If $\{x_1, x_2, \dots, x_n\}$ are constructed, let us choose $x_{n+1} \in \mathbb{E}_3$ in such a way that for $\{x_1, \dots, x_{n+1}\}$ the properties 1)-4) are satisfied, and the angle between the initial ray and an interval $[r, x_{n+1}]$

is positive but smaller than 2^{-n-1} . A sequence $\{x_1, x_2, \dots\}$ constructed in this way satisfies all the required properties. \dashv

Notice that the properties in Proposition 17 imply also the following:

7) We have

$$I_{m,n} \cap I_{k,l} = \emptyset \quad \text{if } \{k, l\} \cap \{m, n\} = \emptyset,$$

and $I_{m,n} \cap I_{k,l}$ consists of at most one point (that must be an end point of each interval) if $\{m, n\} \neq \{k, l\}$,

8) The interval \mathbf{I}_n is disjoint from $I_{m,k}$ if $n \neq m, k$.

From now on we fix a sequence of vertices $\{x_n\}$ provided by Proposition 17. A picture of the lines $I_{m,n}$ and \mathbf{I}_n is given in figure 3.

Definition 18. Let $\mathbf{I} = \bigcup_n \mathbf{I}_n$.

Notice that \mathbf{I} is a compact set, and that

$$\begin{aligned} \overline{\bigcup_{m,n} I_{m,n}} &= \mathbf{I} \cup (\bigcup_{m,n} I_{m,n}), \\ \mathbf{I} \cap (\bigcup_{m,n} I_{m,n}) &= \{x_n\}_{n \in \mathbb{N}}. \end{aligned}$$

Remark 19. The points $\{x_n\}_n$ represent the vertices of a countable graph and the interval $I_{m,n}$ connects x_n to x_m . The intervals $I_{m,n}$ represent the edges of the countable complete graph on $\{x_n\}_n$. All the diffeomorphisms, homotopies, and conjugacies that we will consider are supported inside a ball of radius 1 around the point r . This justifies Remark 16 above.

Fix m, n and set

$$\alpha_{m,n} = \frac{1}{1000} \inf_{\{k,l\} \cap \{n,m\} = \emptyset} \min\{\angle(I_{m,k}, I_{m,n}), \angle(I_{n,k}, I_{m,n}), \text{dist}(I_{m,n}, I_{k,l})\}.$$

Lemma 20. For a sequence $\{x_n\}$ as above, for all m, n , $\alpha_{m,n} > 0$.

† If not there would be a sequence of k 's and l 's going to infinity such that either

a.) for one of m or n , $\angle(I_{m,k}, I_{m,n}) \rightarrow_k 0$ or $\angle(I_{n,k}, I_{m,n}) \rightarrow_k 0$ or

b.) $\text{dist}(I_{m,n}, I_{k,l}) \rightarrow_{k,l} 0$

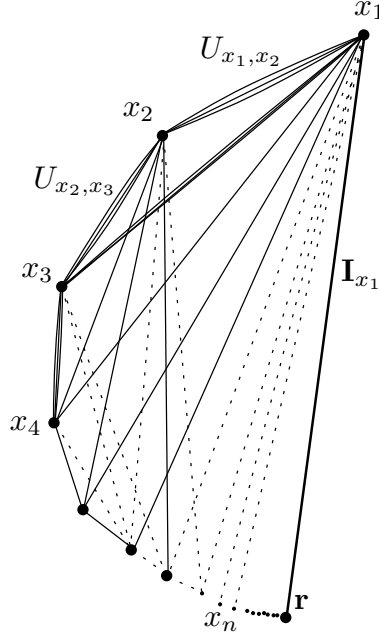


FIGURE 3. The vertices of an embedded graph.

In both cases, because the x_m and x_n are fixed and x_k, x_l converge to r , the minimum of the sequences in a.) or b.) is achieved by some k, l in the sequence. \dashv

Let us denote by ψ the function $\psi : [-1, 1] \rightarrow \mathbb{R}$ given by

$$\psi(x) = \frac{1 - x^2}{2}. \quad (2)$$

Notice that ψ is an upside down parabola symmetric about $x = 0$ with

- (1) $0 < \psi(x) \leq \frac{1}{2}$ if $x \in (-1, 1)$, and $\psi(-1) = \psi(1) = 0$;
- (2) ψ is convex;
- (3) $\psi'(-1) = 1, \psi'(1) = -1$.

Definition 21. *We now give the definitions used to define the supports of our diffeomorphisms.*

Central Axes: *Given $x, y \in \mathbb{R}^5$, $x \neq y$, denote by $\gamma_{x,y}$ the line connecting x and y given by*

$$\gamma_{x,y}(t) = \frac{x + y}{2} + t \frac{y - x}{2}.$$

Note that $\gamma_{x,y}(-1) = x$ and $\gamma_{x,y}(1) = y$.

Normal Hyperplanes: For $t \in \mathbb{R}$, let $\Pi_{x,y}(t)$ the affine 4-dimensional subspace that contains the point $\gamma_{x,y}(t)$ and is orthogonal to the line $\gamma_{x,y}$.

Atomic Domains (“cigars”): Denote by $U_{x,y,\varepsilon} \subset \mathbb{R}^5$ an open set such that $\Pi_{x,y}(t) \cap U_{x,y,\varepsilon} = \emptyset$ if $|t| \geq 1$, and $\Pi_{x,y}(t) \cap U_{x,y,\varepsilon}$ is an open ball in $\Pi_{x,y}(t)$ of radius $\varepsilon\psi(t)$ centered at $\gamma_{x,y}(t)$ if $|t| < 1$.

To simplify the notation, let us also denote

$$\gamma_{m,n} = \gamma_{x_m,x_n}, \quad \Pi_{m,n}(t) = \Pi_{x_m,x_n}(t), \quad \text{and} \quad U_{m,n,\varepsilon} = U_{x_m,x_n,\varepsilon}.$$

We will occasionally informally refer to $U_{m,n,\varepsilon}$ as the ε -cigar between x_m and x_n .

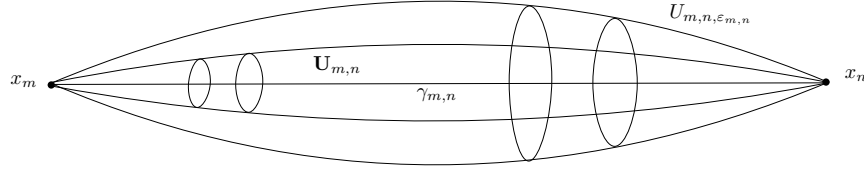


FIGURE 4. “Cigars” $U_{x_m,x_n,\varepsilon_{m,n}}$ and $U_{m,n}$.

Let us set

$$\varepsilon_{m,n} = \min(0.001\alpha_{m,n}, 2^{-n-m}), \quad (3)$$

and denote $U_{m,n} = U_{x_m,x_n,\frac{1}{2}\varepsilon_{m,n}}$. See figure 4 for illustration.

Remark 22. Later in the proof we will also consider cigars of smaller sizes $\varepsilon'_{m,n}$ and $\varepsilon''_{m,n}$, so that $\varepsilon''_{m,n} \ll \varepsilon'_{m,n} \ll \varepsilon_{m,n}$, and therefore $U_{m,n,\varepsilon''_{m,n}} \subset U_{m,n,\varepsilon'_{m,n}} \subset U_{m,n,\varepsilon_{m,n}}$. We will also use the notation $U'_{m,n} = U_{x_m,x_n,\frac{1}{2}\varepsilon'_{m,n}}$ and $U''_{m,n} = U_{x_m,x_n,\frac{1}{2}\varepsilon''_{m,n}}$.

The fact that $\alpha_{m,n} > 0$ guarantees that the following statement holds:

Proposition 23. For each $\{m, n\}$, $\{k, l\}$:

- (1) $U_{m,n,\varepsilon_{m,n}} \cap U_{k,l,\varepsilon_{k,l}} = \emptyset$ if $\{m, n\} \neq \{k, l\}$;
- (2) If $m = k$, $n \neq l$, then

$$\overline{U_{m,n,\varepsilon_{m,n}}} \cap \overline{U_{k,l,\varepsilon_{k,l}}} = x_m;$$

- (3) If the edges $I_{m,n}$ and $I_{k,l}$ are disjoint, then the sets $\overline{U_{m,n,\varepsilon_{m,n}}}$ and $\overline{U_{k,l,\varepsilon_{k,l}}}$ are also disjoint.

The dynamics on the atomic domains $\mathbf{U}_{m,n}$. We start by defining maps $f_{m,n}^+$ and $f_{m,n}^-$, that serve as basic building blocks for the final diffeomorphisms. They each send $\mathbf{U}_{m,n}$ to itself and are the identity on the complement of $\mathbf{U}_{m,n}$. The map $f_{m,n}^+$ smoothly moves the elements of $\mathbf{U}_{m,n}$ towards the centerline $\gamma_{m,n}$ while preserving $\Pi_{m,n}$, and $f_{m,n}^-$ moves the elements $\mathbf{U}_{m,n}$ towards the outer boundary (away from the centerline) while preserving $\Pi_{m,n}$.

In more detail:

- (1) Let $\lambda_{m,n} : \mathbb{R}^5 \rightarrow \mathbb{R}$ be C^∞ function such that $\lambda > 0$ in $\mathbf{U}_{m,n}$, $\lambda = 0$ in $\mathbb{R}^5 \setminus \mathbf{U}_{m,n}$, and $\|\lambda_{m,n}\|_\infty \leq 1$.
- (2) Let $w_{m,n}$ be the vector field orthogonal to $\gamma_{m,n}$, directed towards $\gamma_{m,n}$, and such that $\|w_{m,n}(x)\| = \text{dist}(x, \gamma_{m,n})$.
- (3) Let $\Theta \in C^\infty(\mathbb{R}^5)$ be a smooth function such that
 - (a) $\Theta(x) \in [0, 1]$ for all $x \in \mathbb{R}^5$;
 - (b) $\Theta(x) = 0$ if and only if $x \in \mathbf{I}$;
 - (c) Every point of the set \mathbf{I} is a flat zero point of Θ .

We can now define f^+ and f^- .

Definition 24. Let $v_{m,n}$ be the vector field on $\overline{\mathbf{U}_{m,n}}$ given by

$$v_{m,n} = \Theta \cdot \lambda_{m,n} \cdot w_{m,n}. \quad (4)$$

Let $f_{m,n}^+$ be time-1 shift associated with the vector field $v_{m,n}$, and $f_{m,n}^-$ be time-1 shift associated with the vector field $(-v_{m,n})$.

Notice that while the properties above do not define the choice of $\lambda_{m,n}$ and Θ completely, dynamics of $f_{m,n}^+$ (or $f_{m,n}^-$) does not depend on this choice (up to a continuous change of coordinates), as the following statement shows.

Proposition 25. Fix two pairs (m, n) and (m^*, n^*) and let g be either $f_{m,n}^+$ or $f_{m,n}^-$ and h be f_{m^*,n^*}^+ or f_{m^*,n^*}^- . Then g is topologically conjugate to h if and only either

- $g = f_{m,n}^+$ and $h = f_{m^*,n^*}^+$ or
- $g = f_{m,n}^-$ and $h = f_{m^*,n^*}^-$.

† Suppose first that $g = f_{m,n}^+$ and $h = f_{m^*,n^*}^+$. The case $g = f_{m,n}^-$ and $h = f_{m^*,n^*}^-$ is exactly analogous. Note that it suffices to show the result for $m^* = 0$ and $n^* = 1$.

There is an affine map ϕ taking the cigar $\overline{\mathbf{U}_{m,n}}$ to $\overline{\mathbf{U}_{0,1}}$. The map ϕ conjugates g to a diffeomorphism g^* of $\overline{\mathbf{U}_{0,1}}$ that only has fixed points on the boundary and on the line $\gamma_{0,1}$. Moreover g^* moves points directly towards the centerline through the subspaces $\Pi_{x,y}(t)$. To finish showing

that g is conjugate to h it suffices to show that g^* is conjugate to h by a map ψ .

To show that g^* and h are conjugate we use the method of *fundamental domains*. Let V be the boundary of $U_{0,1,\frac{\epsilon_{0,1}}{4}}$. If $p \in V$ then p lies in a hyperplane $\Pi_{x,y}(t)$ and $g^*(p)$ lies in the same $\Pi_{x,y}(t)$ and on the line segment between p the centerline of $U_{0,1}$. For $j \in \mathbb{Z}$ let $\psi((g^*)^j(p)) = h^j(p)$. For each point q belonging to the half open line segment between p and $g^*(p)$ let $\psi((g^*)^j(q)) = h^j(q)$. We claim that ψ is a homeomorphism of $\bar{U}_{0,1}$ that conjugates g^* to g .

Note that for each q not on the centerline, $(g^*)^j(q)$ converges monotonically to the centerline and j goes to ∞ and to the boundary as j goes to $-\infty$. For all $q \in U_{0,1}$ there is a unique $p \in V$ on the line segment connecting the centerline to the boundary that passes through q . There is a unique $k \in \mathbb{Z}$ such that $(g^*)^k(q)$ lies in the half open line segment between p and $g^*(p)$. The analogous facts hold for h as well.

Hence ψ is well-defined on all of $U_{0,1}$ and since both g^* and h converge monotonically along the line segments to the centerline ψ is a conjugacy.

Now suppose that $g = f_{m,n}^+$ and $h = f_{m^*,n^*}^-$ and ϕ is a homeomorphism from $\bar{U}_{m,n}$ to \bar{U}_{m^*,n^*} such that $\phi \circ g = h \circ \phi$.

Since ϕ must take the fixed points of g to the fixed points of h , ϕ takes $\gamma_{m,n}$ to γ_{m^*,n^*} . Let $x \in U_{m,n}$ and consider the sequence $\langle g^n(x) : n \in \mathbb{N} \rangle$. By compactness this sequence has a limit point x^* in $\bar{U}_{m,n}$. Since g is continuous $g(x^*) = x^*$. Since g takes points towards $\gamma_{m,n}$ along a line orthogonal to $\gamma_{m,n}$, x^* must be on $\gamma_{m,n}$ and be the limit of $\langle g^n(x) : n \in \mathbb{N} \rangle$.

It follows that $\{h^n(\phi(x)) : n \in \mathbb{N}\}$ converges to a point on γ_{m^*,n^*} . However no point not already on γ_{m^*,n^*} has a point on γ_{m^*,n^*} as a limit point of the orbit of f_{m^*,n^*}^- . This is a contradiction. \dashv

The definition of the reduction \mathbf{R} . Recall that the space *Graphs* consists of symmetric functions $E : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ that encode a graph G_E with vertices \mathbb{N} and n is connected to m if and only $E(m, n) = 1$.

Now Theorem 15 can be reformulated more explicitly:

Proposition 26. *Consider the map*

$$\mathbf{R} : \text{Graphs} \rightarrow \text{Diff}^\infty(\mathbb{R}^5)$$

be defined so that $\mathbf{R}(E)$ is a diffeomorphism supported on $\cup_{m,n} \mathbf{U}_{m,n}$, and

$$\mathbf{R}(E)|_{\mathbf{U}_{m,n}} = \begin{cases} f_{m,n}^+, & \text{if } E(m,n) = 1; \\ f_{m,n}^-, & \text{if } E(m,n) = 0. \end{cases} \quad (5)$$

Then $\mathbf{R}(E_1)$ and $\mathbf{R}(E_2)$ are topologically conjugate if and only if E_1 and E_2 are isomorphic graphs.

Remark 27. Notice that the map $\mathbf{R}(E)$ defined above is C^∞ diffeomorphism. Indeed, it is clear that it is smooth at the points outside the set of accumulation points of the collection of the sets $\mathbf{U}_{m,n}$ (which is exactly \mathbf{I}). Property 3c of the function Θ implies that the map $\mathbf{R}(E)$ has a flat fixed point at every point of \mathbf{I} .

Moreover the map R is a continuous map with respect to the topologies on Graphs and $\text{Diff}^\infty(\mathbb{R}^5)$

Now we need to prove Proposition 26.

3.3. The Easy Part. Here we prove that if $\mathbf{R}(E_1)$ and $\mathbf{R}(E_2)$ are topologically conjugate, then E_1 and E_2 are isomorphic.

Suppose $\mathbf{R}(E_1) = F_1$ and $\mathbf{R}(E_2) = F_2$ are topologically conjugate, i.e. there exists a homeomorphism $H : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ such that $F_2 \circ H = H \circ F_1$. In this case $H(\text{Fix}(F_1)) = \text{Fix}(F_2)$, and $H(\mathbb{R}^5 \setminus \text{Fix}(F_1)) = \mathbb{R}^5 \setminus \text{Fix}(F_2)$, where $\text{Fix}(F_i)$ is the set of fixed points of the map F_i . Moreover, since H is a homeomorphism, it must send every connected component of $\mathbb{R}^5 \setminus \text{Fix}(F_1)$ into a connected component of $\mathbb{R}^5 \setminus \text{Fix}(F_2)$. Hence H sends $\mathbf{U}_{m,n}$ to $\mathbf{U}_{k,l}$ for some $k, l \in \mathbb{N}$, and $\partial\mathbf{U}_{m,n}$ to $\partial\mathbf{U}_{k,l}$. Moreover, the set of points that belong to the boundary of more than one of the domains $\{\mathbf{U}_{m,n}\}$ must be invariant under H , but this is exactly the sequence $\{x_n\}_{n \in \mathbb{N}}$. Therefore H restricted to $\{x_n\}_{n \in \mathbb{N}}$ permutes these points and thus induces a permutation $\varphi(H) = \varphi : \mathbb{N} \rightarrow \mathbb{N}$.

Let us show that φ must be an isomorphism between the graphs E_1 and E_2 . Indeed, if $\varphi(n) = k$ and $\varphi(m) = l$, then $H(\mathbf{U}_{m,n}) = \mathbf{U}_{k,l}$, so $H(\overline{\mathbf{U}_{m,n}}) = \overline{\mathbf{U}_{k,l}}$, and $H|_{\overline{\mathbf{U}_{m,n}}}$ conjugates $F_1|_{\overline{\mathbf{U}_{m,n}}}$ and $F_2|_{\overline{\mathbf{U}_{k,l}}}$.

By Proposition 25, if $F_1|_{\overline{\mathbf{U}_{m,n}}} = f_{m,n}^+$ then $F_2|_{\overline{\mathbf{U}_{k,l}}} = f_{k,l}^+$ and similarly if $F_1|_{\overline{\mathbf{U}_{m,n}}} = f_{m,n}^-$. It follows that m and n are connected in E_1 just in case k and l are connected in E_2 . Hence ϕ is an isomorphism of E_1 with E_2 . \dashv

3.4. Construction of conjugacy. Suppose now that the graphs G_{E_1} and G_{E_2} are isomorphic, i.e. there exists a permutation $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$E_1(m, n) = 1 \quad \Leftrightarrow \quad E_2(\varphi(m), \varphi(n)) = 1.$$

Let us show that diffeomorphisms $F_1 = \mathbf{R}(E_1)$ and $F_2 = \mathbf{R}(E_2)$ are topologically conjugate. First we will present the permutation $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ as an infinite composition of transpositions (Lemma 30 below). Since ϕ is an isomorphism between the graphs E_1 and E_2 , this presentation can be viewed as giving a sequence of intermediate graphs $E^{(i)}$. For each transposition ϕ_i , we construct a conjugacy between transformations corresponding to $E^{(i)}$ and $E^{(i+1)}$. This is done in a manner that the infinite sequence of intermediate conjugacies converges to a conjugacy between $\mathbf{R}(E_1)$ and $\mathbf{R}(E_2)$.

Remark 28. *There is a temptation to start with a construction of a homotopy of the graph, and then use general results to extend it to a homotopy of an ambient space. Unfortunately some simple, but surprising, examples show that such an extension does not have to exist in general, e.g. see Example 1.14 from [32]. Therefore, we give an explicit construction.*

Decomposing Permutations. In order to construct such a conjugacy $H : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ we will use a representation of the permutation $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ as an infinite converging composition of transpositions

$$\varphi = \dots \circ \dots \circ \varphi_2 \circ \varphi_1. \quad (6)$$

Each transposition ϕ_i will correspond to a homeomorphism h_i and, setting

$$H_n = h_n \circ h_{n-1} \circ \dots \circ h_2 \circ h_1$$

we will have $H = \lim_n H_n$.

In order for that infinite composition of h_i 's to converge to a homeomorphism we need to choose a representation of a permutation $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ to have some special properties. This is what we do now.

Recall that for functions $\varphi, \langle \Phi_n : n \in \mathbb{N} \rangle$ mapping from \mathbb{N} to \mathbb{N} ,

$$\varphi = \lim_{n \rightarrow \infty} \Phi_n$$

if and only if for all m there is an N for all $n > N$, $\Phi_n(m) = \Phi_N(m)$ and $\varphi(m) = \Phi_N(m)$. If each Φ_n is a permutation then φ is a permutation if and only if $\lim_{n \rightarrow \infty} \Phi_n^{-1}$ also converges (in which case the limit is φ^{-1}).

Definition 29. *Suppose that $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ is a permutation of the natural numbers, and $\varphi = \lim_{n \rightarrow \infty} \Phi_n$, where $\Phi_n = \varphi_n \circ \varphi_{n-1} \circ \varphi_{n-2} \cdots \circ \varphi_1$ and each φ_i is a transposition. For each $m \in \mathbb{N}$ define the contamination set $Cont_m$ as*

$$Cont_m = \{m\} \cup \{n \in \mathbb{N} \mid \exists k \in \mathbb{N}, \text{ and } \{n_1, \dots, n_{k+1}\}, \text{ with } n_1 = m, n_{k+1} = n, \text{ and } i_1 < i_2 < \dots < i_k, \text{ such that } \varphi_{i_j} \text{ permutes } n_j \text{ and } n_{j+1}\}.$$

Lemma 30. *Let $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ be a permutation of the natural numbers. Then there is a sequence of transpositions $\langle \varphi_n : n \in \mathbb{N} \rangle$, such that the following holds:*

1) *If $\Phi_n = \varphi_n \circ \varphi_{n-1} \circ \varphi_{n-2} \cdots \circ \varphi_1$, then*

$$\lim_{n \rightarrow \infty} \Phi_n = \varphi. \quad (7)$$

2) *Any given transposition is encountered at most once,*

3) *For each m , there are at most two i 's with $\phi_i(m) \neq m$.*

4) *For each $m \in \mathbb{N}$ the set $Cont_m$ is contains at most four elements;*

5) *Each k belongs to only finite number of the sets $\{Cont_m\}_{m \in \mathbb{N}}$.*

⊢ Every permutation f of the natural numbers induces a countable partition

$\mathbb{N} = \bigcup_k A_k$ such that:

(1) If A_k is infinite, then for each $m \in A_k$, $A_k = \{\varphi^n(m) : m \in \mathbb{Z}\}$

(2) If A_k is finite then ϕ induces a single cycle on A_k .

By dovetailing, one can treat each A_k separately. We start by assuming that we have a single A_k and it is infinite. Since we are ignoring the other elements of the partition we can assume that every element of \mathbb{N} is moved by ϕ , hence $A_k = \mathbb{N}$. Write \mathbb{N} as the double ended sequence

$$\mathbb{N} = \langle \dots m_{-n}, m_{-n+1}, \dots m_{-2}, m_{-1}, m_0, m_1, \dots m_n, m_{n+1}, \dots \rangle,$$

where $m_{\pm n} = \varphi^{\pm n}(m_0)$, $n \in \mathbb{N}$.

We construct the φ_n 's in the following way:

$$\begin{aligned} \varphi_1 &= (-1, 1), \varphi_2 = (0, 1), \varphi_3 = (-2, 2), \varphi_4 = (-1, 2), \dots, \\ \varphi_{2k-1} &= (-k, k), \varphi_{2k} = (-k+1, k), \dots \end{aligned}$$

It is easy to see that the sequence of transpositions $\langle \varphi_n : n \in \mathbb{N} \rangle$ constructed in this way satisfies the required properties.

A very similar construction works for finite cycles. ⊣

Notation: Denote pair of natural numbers transposed by ϕ_i by n_i and m_i .

Let us formulate several definitions that we will use later in the text:

Definition 31. *Given a permutation of the natural numbers $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ with a chosen representation $\varphi = \dots \circ \varphi_n \circ \varphi_{n-1} \circ \dots \circ \varphi_1$, and a pair $\{n, m\}$, $n, m \in \mathbb{N}$, $n \neq m$, define*

$$Cont(n, m) = \{(n', m') \mid n' \in Cont_n, m' \in Cont_m\}.$$

Remark 32. *We will be constructing a sequence of homeomorphisms such that each homeomorphism will swap a pair of vertices. In order to control the orbits that initiate in a small neighborhood of a vertex, we need to take into account that some points could move to a neighborhood of an image (so that second vertex becomes “contaminated”), while some points could stay in a small neighborhood of the initial vertex. If one of the homeomorphisms from our sequence moves one of the previously “contaminated” vertices, some points from its neighborhood can be moved to a neighborhood of another vertex, which is therefore also becomes “contaminated”. Definition 29 defines the maximal set of vertices $Cont_n$ that can become “contaminated” in this process, starting with the vertex n . Definition 31 serves a similar purpose, but for edges instead of vertices.*

Definition 33. *Let us say that $n \in \mathbb{N}$ is frozen at stage i , if $\varphi_j(n) = n$ for any $j \geq i$.*

Definition 34. *Define*

$$J(n) = \min\{j \in \mathbb{N} \mid \forall m \in Cont_n, m \text{ is frozen at stage } j\}$$

and

$$T(n) = \max\{n_i, m_i \mid i \leq J(n)\},$$

where n_i, m_i are the elements of \mathbb{N} that are swapped by φ_i .

Let d_i be the distance between x_{m_i} and x_{n_i} (where n_i and m_i are the pair swapped by ϕ_i). By property 2 of Proposition 17 and the triangle inequality, $d_i \leq 2^{-n_i-1} + 2^{-m_i-1}$. Property 3 of the sequence of transpositions $\{\varphi_i\}$ provided by Lemma 30 together with the choice of the sequence of vertices $\{x_n\} \subset \mathbb{R}^5$ given by Proposition 17 imply the following:

Lemma 35.

$$\sum_{i=1}^{\infty} d_i < \infty$$

⊢ We have

$$d_i = \text{dist}(x_{n_i}, x_{m_i}) \leq 2^{-n_i-1} + 2^{-m_i-1} \leq 2^{-\min(n_i, m_i)}.$$

By property 3 of Lemma 30), $(n_1, m_1), (n_2, m_2), \dots$ each natural number appears at most 2 times. This implies that

$$\sum_{i=1}^{\infty} d_i \leq \sum_{k=1}^{\infty} 2 \cdot 2^{-k} = 2.$$

⊔

Interpolating graphs. Suppose we are given symmetric maps $E_1 : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ and $E_2 : \mathbb{N} \times \mathbb{N} \rightarrow \{0, 1\}$ and a permutation $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ giving an isomorphism between the corresponding graphs. Let us consider the sequence of graphs coded by $E^{(1)}, E^{(2)}, \dots, E^{(i)}, \dots$ such that $E^{(1)} = E_1$, and for $i \geq 1$ and any $m, n \in \mathbb{N}$ we have

$$E^{(i+1)}(m, n) = E^{(i)}(\varphi_i^{-1}(m), \varphi_i^{-1}(n)) = E^{(1)}((\varphi_i \dots \varphi_1)^{-1}(m), (\varphi_i \dots \varphi_1)^{-1}(n)).$$

Since $\varphi_i \circ \dots \circ \varphi_1(n) = \varphi(n)$ for all sufficiently large i , for given $m, n \in \mathbb{N}$ and large i we also have

$$\begin{aligned} E_2(\varphi(m), \varphi(n)) &= E_1(m, n) = E^{(1)}(m, n) = \\ &E^{(i+1)}(\varphi_i \circ \dots \circ \varphi_1(m), \varphi_i \circ \dots \circ \varphi_1(n)) = E^{(i+1)}(\varphi(m), \varphi(n)). \end{aligned}$$

Hence the sequence of graphs $\{E^{(i)}\}$ converges to E_2 .

Recall R is the map defined in Proposition 26. Denote $G_i = \mathbf{R}(E^{(i)}) \in \text{Diff}^\infty(\mathbb{R}^5)$. Then $G_1 = F_1 = \mathbf{R}(E_1)$, and G_i converge to $F_2 = \mathbf{R}(E_2)$ in C^0 -topology⁹. We will complete the proof of Theorem 15 by proving the following statement:

Proposition 36. *There exists a sequence of homeomorphisms $\{h_i\}_{i \in \mathbb{N}}$, $h_i : \mathbb{R}^5 \rightarrow \mathbb{R}^5$, $i = 1, 2, \dots$, such that:*

1) *for each $i = 1, 2, \dots$ we have $h_i \circ G_i = G_{i+1} \circ h_i$, i.e. the h_i is a topological conjugacy between diffeomorphisms G_{i+1} and G_i ;*

2) *one has $\sum_{i \in \mathbb{N}} \text{dist}_{C^0}(\text{id}, h_i) < \infty$ and*

3) *for any $x \in \mathbb{R}^5$, there exists $N = N(x) \in \mathbb{N}$ such that for any $n \geq N$ one has*

$$h_n \circ \dots \circ h_1(x) = h_N \circ \dots \circ h_1(x).$$

Notice that Proposition 36 implies that the maps F_1 and F_2 are topologically conjugate. Indeed, the property 2) implies that the sequence of compositions $h_n \circ \dots \circ h_1$ converges as $n \rightarrow \infty$ to a continuous map $H : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ in the space $C^0(\mathbb{R}^5, \mathbb{R}^5)$. The property 3) implies that H is one-to-one, and because it is a continuous bijection whose support is contained in a compact subset of \mathbb{R}^5 , it must be a homeomorphism. Finally, the property 1) implies that

$$h_n \circ \dots \circ h_1 \circ G_1 = G_{n+1} \circ h_n \circ \dots \circ h_1,$$

and, since G_{n+1} converges to F_2 as $n \rightarrow \infty$, we get $H \circ F_1 = F_2 \circ H$, i.e. the homeomorphism H conjugates the maps F_1 and F_2 . This concludes the proof of Theorem 15 modulo Proposition 36.

⁹In fact, in our construction a sequence $\{G_i\}$ converges to F_2 in C^∞ -topology; we are not going to use it.

Now we need to prove Proposition 36. To motivate the construction we list in advance some properties of the h_i 's.

- (1) Each h_i is equal to ϕ_i on the set of vertices $\{x_n\}_n$.
- (2) Each h_i takes cigars to cigars.
- (3) Every $x \in \mathbb{R}^5$ is only moved by finitely many h_i .

3.4.1. Initial choices and building blocks: definition of $\{B_{Q_i}\}$, $\{V_{k,i}\}$, and $\{W_i\}$. We start by choosing the neighborhoods that will contain the supports of the homeomorphisms h_i .

Choice of neighborhoods of the vertices. Choose a sequence of balls $\{B_{x_n}\}$, such that B_{x_n} is centered at x_n , and radius of B_{x_n} is smaller than

$$\rho_n = \frac{1}{1000} \min_{k,m \neq n} (\text{dist}(x_n, x_m), \text{dist}(x_n, I_{m,k})). \quad (8)$$

Choice of vectors $\{w_i\}$ in \mathbb{E}_2 . Let us recall that we are given a sequence of transpositions $\varphi_1, \varphi_2, \dots$, such that if $\Phi_n = \varphi_n \circ \dots \circ \varphi_2 \circ \varphi_1$ then $\varphi = \lim_{N \rightarrow \infty} \Phi_N$, and φ_i is the transposition of m_i and n_i . Let us assign to each φ_i a positive number $0 < q_i < \frac{1}{2}$. We can do so in such a way that for any $i \in \mathbb{N}$ we have $0 < q_{i+1} < q_i$. We also require $q_i \rightarrow 0$ as $i \rightarrow \infty$.

Set $w_i = (\cos \pi q_i, \sin \pi q_i)$. Notice that $w_i \rightarrow (1, 0) \in \mathbb{R}^2 \cong \mathbb{E}_2$ as $i \rightarrow \infty$. Recall that we consider the whole space \mathbb{R}^5 as $\mathbb{R}^5 = \mathbb{E}_3 \oplus \mathbb{E}_2$, with $\mathbb{E}_3 \times \{(0, 0)\}$ being the hyperspace where all the points $\{x_n\}$ belong, and $\{(0, 0, 0)\} \times \mathbb{E}_2$ being the space where the vectors w_i belong. Note that the $(0, 0, 0, w_i)$ are orthogonal to every element of \mathbb{E}_3 .

Definition 37. Denote by $\beta_j > 0$ the minimum among all the angles between w_j and any other vector w_i , $i \neq j$. Denote by K_j the cone in \mathbb{E}_2 with central axis the line containing the origin and w_j and having angle $\frac{1}{3}\beta_j$.

We now define the sets that will contain all points that will be moved by h_i . In order to do so we start by defining some ‘‘building blocks’’ that will be used for each h_i , and locate them in \mathbb{R}^5 in such a way that the only intersections are for obvious reasons (e.g. two of them contain the same vertex or the same edge of an embedded graph). This is where we need to use the additional dimensions in $\mathbb{R}^5 = \mathbb{E}_3 \oplus \mathbb{E}_2$, and our choice of the vectors $\{w_i\} \subset \mathbb{E}_2$.

Defining $\{Q_i\}$ and $\{B_{Q_i}\}$. Recall that d_i is the distance between x_{m_i} and x_{n_i} where ϕ_i transposes m_i and n_i . Denote by Q_i^u the union of

three line segments,

$$Q_i^u = I_{x_{n_i}, x_{n_i} + w_i d_i} \cup I_{x_{n_i} + w_i d_i, x_{m_i} + w_i d_i} \cup I_{x_{m_i} + w_i d_i, x_{m_i}},$$

and Q_i^d the union of the corresponding line segments:

$$Q_i^d = I_{x_{m_i}, x_{m_i} - w_i d_i} \cup I_{x_{m_i} - w_i d_i, x_{n_i} - w_i d_i} \cup I_{x_{n_i} - w_i d_i, x_{n_i}}.$$

and let $Q_i = Q_i^u \cup Q_i^d$.

Thus Q_i is a rectangle of width d_i and height $2w_i d_i$. Informally Q_i^u and Q_i^d are the upper and lower portions of the rectangle.

Let B_{Q_i} be δ_i -neighborhood of Q_i , where $\{\delta_i\}$ is a fast decaying sequence of small positive numbers such that for $i \neq j$:

- if $\{m_i, n_i\} \cap \{m_j, n_j\} = \emptyset$, then $B_{Q_i} \cap B_{Q_j} = \emptyset$.
- otherwise $B_{Q_i} \cap B_{Q_j}$ is a subset of either $B_{x_{n_i}}$ or $B_{x_{m_i}}$.

In other words, B_{Q_i} is a small neighborhood of Q_i such that intersection of two such neighborhoods must be inside of the small neighborhoods of the vertices that were specified above.

Defining the open sets Δ . Let points $x, y \in \mathbb{R}^5$, $x \neq y$, and a vector $w \in \mathbb{R}^5$ that is not parallel to $(y - x)$ be given. Let $L(x, y, w)$ be a three dimensional plane that contains x and is orthogonal to the plane that contains the points x, y , and $y + w$, see Figure 5.

Definition 38. Let $v = v(x, y, w)$ be an affine vector field in \mathbb{R}^5 defined by the following properties:

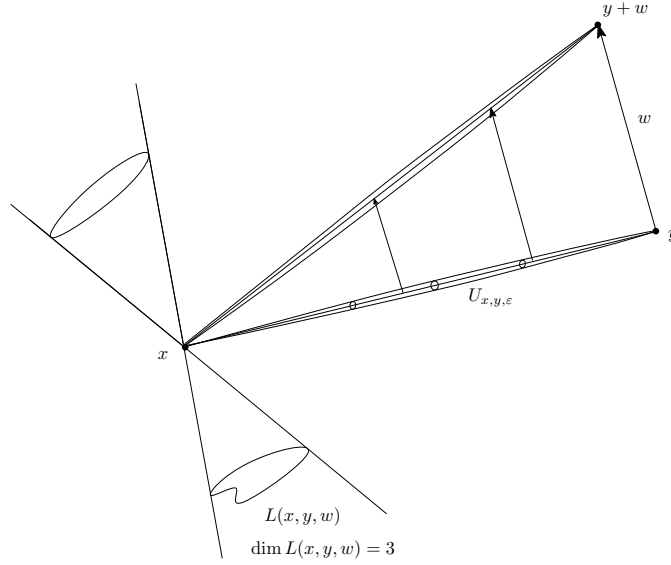
- (1) $v|_L = 0$;
- (2) $v(y) = w$;
- (3) $\frac{\partial}{\partial v}(v) = 0$ (in other words, non-singular orbits of v are straight lines, and restricted to each of these lines, v is constant).

Remark 39. The vector field $v(x, y, w)$ can be represented as a product of a constant vector field \bar{w} such that $\bar{w}(z) = w$ for all $z \in \mathbb{R}^5$ and a C^∞ function $\psi : \mathbb{R}^5 \rightarrow \mathbb{R}$, $\psi = \psi_{x, y, w}$. We will use this notation later.

The vector field v with these properties for given x, y, w is uniquely defined. Denote by $\{g_v^t\}$ the flow defined by the vector field v .

Let $U_{x, y, \varepsilon}$ be the cigar defined in definition 21. Set $\Delta = \Delta(x, y, \varepsilon, w)$ as

$$\Delta = \cup_{t=0}^1 g_v^t(U_{x, y, \varepsilon}).$$

FIGURE 5. The construction of $\Delta(x, y, w)$.

Remark 40. *The set $\Delta = \Delta(x, y, \varepsilon, w)$ is defined to be the union of images of $U_{x,y,\varepsilon}$ when one fixes one end point (in this case x), and “moves” another end point along the vector w . In this and other similar definitions (such as the definition of $V(r, x_{n_i}, x_{m_i}, \eta_i, d_i w_i)$ below) the first argument represents the end point that is fixed.*

Defining the open sets $V_{k,i}$ and W_i . We now define the sets that will contain the points moved by h_i and are not on cigars connected to the point r .

Given points $x, y, z \in \mathbb{E}_3$, $\xi > 0$, and a vector $w \in \mathbb{E}_2$, denote

$$V(z, x, y, \xi, w) = \Delta(z, x, \xi, w) \cup \Delta(z, x + w, \xi, y - x) \cup \Delta(z, y, \xi, w) \cup \Delta(z, y, \xi, -w) \cup \Delta(z, y - w, \xi, x - y) \cup \Delta(z, x, \xi, -w).$$

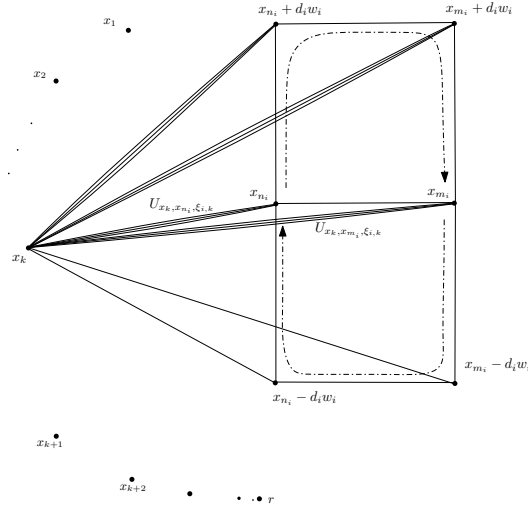
Also, given points $x, y \in \mathbb{E}_3$, $\xi > 0$, and vector $w \in \mathbb{E}_2$, using definition 38 we set $\tilde{w} = v(\frac{x+y}{2}, x, w)$, $\tilde{u} = v(\frac{x+y}{2}, x + w, y - x)$, and denote

$$W(x, y, \xi, w) = \left(\cup_{t=-1}^1 g_{\tilde{w}}^t(U_{x,y,\xi}) \right) \cup \left(\cup_{t=0}^1 g_{\tilde{u}}^t(U_{x+w,y-w,\xi}) \right)$$

We will denote

$$V_{k,i} = V(x_k, x_{n_i}, x_{m_i}, \xi_{i,k}, d_i w_i), \quad W_i = W(x_{n_i}, x_{m_i}, \xi_i, d_i w_i),$$

where $\xi_{i,k}$, ξ_i are small constants to be chosen later, in Section 3.4.4.


 FIGURE 6. The construction of $V(x_k, x_{n_i}, x_{m_i}, \xi_{i,k}, d_i w_i)$.

3.4.2. Preliminary step: shrinking the cigars. In the construction of the map $\mathbf{R} : \text{Graph} \rightarrow \text{Diff}^\infty(\mathbb{R}^5)$ in Proposition 26 the choice of the cigars $U_{m,n,\varepsilon_{m,n}}$ was uniform, independent of the graph E . The homeomorphisms h_i are built in such a manner that every point in \mathbb{R}^5 is only moved by finitely many h_i 's. The transformation h_i must swap m_i and n_i and thus move every cigar having either as an endpoint. To avoid moving unnecessary points (and points already considered and in the appropriate places) each h_i is a composition of maps. The first of the maps contracts the relevant cigars diameters. Then the cigars are swapped and finally re-expanded. The contraction and expansion preserves the conjugacy classes of the relevant maps.

The map used to shrink the cigars will be denoted S . For any family of positive small constants $\{\varepsilon'_{m,n}\}$, $\varepsilon'_{m,n} < \varepsilon_{m,n}$, we can choose S to be a homeomorphism of \mathbb{R}^5 supported on $\overline{\cup_{m \neq n} U_{m,n,\varepsilon_{m,n}}}$ such that $S|_{U_{m,n}}$ is a contraction towards $\gamma_{m,n}$, and maps $\mathbf{U}_{m,n}$ to $\mathbf{U}'_{m,n}$. The size of $\varepsilon'_{m,n}$ is determined by the transpositions ϕ_i built in the decomposition of ϕ in Lemma 30.

3.4.3. Distortion property of homeomorphisms h_i . It is crucial to be able to make sure that in our construction a point from $U_{x_k, x_{n_i}}$ that was close to x_k and was mapped by h_i to $U_{x_k, x_{m_i}}$ will remain close to x_k . In order to give a precise statement and justify it let us introduce some extra notation.

Suppose $U_{x,y,\varepsilon}$ is a **cigar**. Let us introduce a function on $U_{x,y,\varepsilon}$ that would show whether a given point $z \in U_{x,y,\varepsilon}$ is closer to x or to y .

Namely, let us recall that an interval $[x, y]$ is parameterized by $t \in [-1, 1]$,

$$\gamma_{x,y}(t) = \frac{x+y}{2} + t \frac{y-x}{2},$$

so $\gamma_{x,y}(-1) = x$ and $\gamma_{x,y}(1) = y$. Also, $U_{x,y,\varepsilon}$ was defined as

$$U_{x,y,\varepsilon} = \bigcup_{-1 \leq t \leq 1} \{z \in \Pi_{x,y}(t) \mid |z - \gamma_{x,y}(t)| < \varepsilon \psi(t)\}$$

where ψ is given in equation 2.

Let us define

$$Pos : U_{x,y,\varepsilon} \rightarrow [-1, 1], \quad Pos(z) = t \text{ if } z \in U_{x,y,\varepsilon} \cap \Pi_{x,y}(t). \quad (9)$$

Then if $Pos(z)$ is close to -1 , then z is close to x , and if $Pos(z)$ is close to 1 , that z is close to y . The function Pos depends on the cigar $U_{x,y,\varepsilon}$, and which of the vertices of the ‘‘cigar’’ (x or y in our case) is chosen as first ($Pos(x) = -1$), and which one as a second ($Pos(y) = 1$).

Remark 41. *The choice of ρ_n in equation (8) implies that for any $x \in B_{x_n} \cap U_{x_n, x_m, \varepsilon_{m,n}}$ we have $Pos(x) \in [-1, -1 + \frac{1}{1000})$, where $Pos(x)$ is the relative position of x in $U_{x_n, x_m, \varepsilon_{m,n}}$.*

Proposition 42. *Suppose points $x, y \in \mathbb{R}^5$, and a vector $w \in \mathbb{R}^5, w \neq 0$, w is not collinear to $y - x$, are given. For any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that the following holds. If $\varepsilon \leq \varepsilon_0$, $z_1 \in U_{x,y,\varepsilon}$, $z_2 \in U_{x,y+w,\varepsilon}$, and the vector $z_2 - z_1$ is collinear with w , then*

$$|Pos_2(z_2) - Pos_1(z_1)| < \delta$$

where Pos_1 is the position in $U_{x,y,\varepsilon}$ and Pos_2 is the position in $U_{x,y+w,\varepsilon}$.

Proof of Proposition 42. Projection of a point in $U_{x,y,\varepsilon}$ to the 2-dimensional plane that contains points x, y , and $y + w$ does not change the value of the function Pos . Therefore, without loss of generality we can assume that both z_1 and z_2 are the points on that plane. It is enough to take

$$\varepsilon < \frac{1}{4} \min(\text{dist}(x, y), \text{dist}(x, y + w)) \min(|\tan \alpha|, |\tan \beta|) \delta,$$

where α is an angle between an interval $[x, y]$ and the vector w , and β is an angle between an interval $[x, y + w]$ and the vector w . Let $z'_1 \in [x, y]$ be a point of the intersection of the interval $[x, y]$ with the line containing z_1 and z_2 , and $z'_2 \in [x, y]$ be a point of the intersection of the interval $[x, y + w]$ with that line. Then we have

$$|Pos(z_1) - Pos(z'_1)| \leq \frac{2}{\text{dist}(x, y)} \frac{\varepsilon}{|\tan \alpha|} \leq \frac{\delta}{2}, \quad \text{and}$$

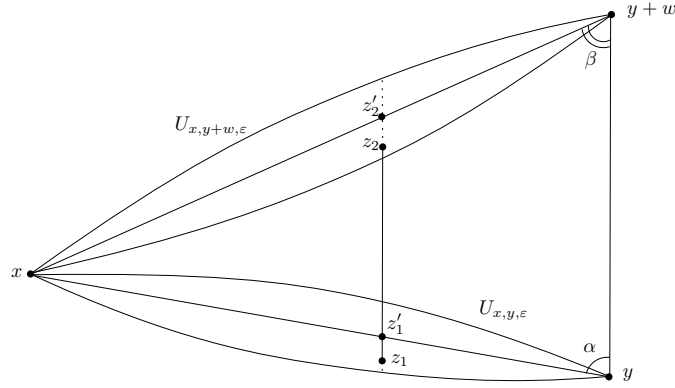


FIGURE 7. Proof of Proposition 42.

$$|Pos(z_2) - Pos(z'_2)| \leq \frac{2}{\text{dist}(x, y + w)} \frac{\epsilon}{|\tan \beta|} \leq \frac{\delta}{2},$$

and since $Pos(z'_1) = Pos(z'_2)$, we have

$$|Pos(z_2) - Pos(z_1)| \leq |Pos(z_1) - Pos(z'_1)| + |Pos(z_2) - Pos(z'_2)| \leq \delta.$$

□

Similarly to Proposition 42, the following statement holds:

Proposition 43. *Suppose points $x, y \in \mathbb{R}^5$, and a vector $w \in \mathbb{R}^5, w \neq 0$, w is not collinear to $y - x$, are given. For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that the following holds. If $\epsilon \leq \epsilon_0$, $z_1 \in U_{x,y,\epsilon}$, $z_2 \in U_{x+w,y-w,\epsilon}$, and the vector $z_2 - z_1$ is collinear with w , then*

$$|Pos(z_2) - Pos(z_1)| < \delta$$

The proof is very similar to the proof of Proposition 42, so we will skip it.

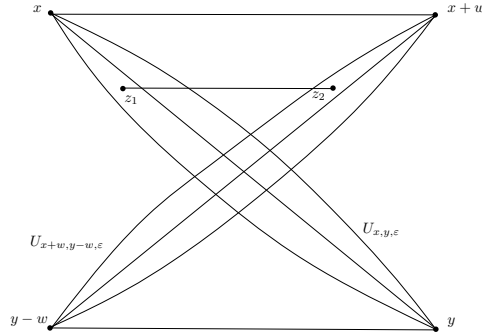


FIGURE 8. Proposition 43.

Choice of the sizes of cigars. As remarked in section 3.4.2 the initial sizes of cigars can be chosen as small as needed. There are several conditions that we want the choice to satisfy. As one of them, we want to make sure that after we constructed a homeomorphism h_i that sends one cigar to another, any point from the first cigar that is mapped by h_i to the second cigar does not change its position within a ‘‘cigar’’ very much.

Let us start by noticing that Proposition 42 immediately implies the following statement:

Proposition 44. *Given points $x, y, t \in \mathbb{E}_3$ that do not belong to the same line, and a vector $w \in \mathbb{E}_2$, for any $\delta > 0$ there exists $\varepsilon > 0$ such that the following holds. For any points*

$$z_1 \in U_{t,x,\varepsilon}, \quad z_2 \in U_{t,x+w,\varepsilon}, \quad z_3 \in U_{t,y+w,\varepsilon}, \quad z_4 \in U_{t,y,\varepsilon},$$

such that

$$(z_2 - z_1) \parallel w, \quad (z_3 - z_2) \parallel (y - x), \quad (z_4 - z_3) \parallel (-w),$$

we have

$$|Pos(z_4) - Pos(z_1)| < \delta,$$

where $Pos(z_4)$ is a position of z_4 in $U_{t,y,\varepsilon}$, and $Pos(z_1)$ is a position of z_1 in $U_{t,x,\varepsilon}$, as defined in (9).

Similarly, Proposition 43 implies that following statement:

Proposition 45. *Given distinct points $x, y \in \mathbb{E}_3$ and a vector $w \in \mathbb{E}_2$, for any $\delta > 0$ there exists $\varepsilon > 0$ such that the following holds. For any points*

$$z_1 \in U_{x,y,\varepsilon}, \quad z_2 \in U_{x+w,y-w,\varepsilon}, \quad z_3 \in U_{y+w,x-w,\varepsilon}, \quad z_4 \in U_{y,x,\varepsilon},$$

such that

$$(z_2 - z_1) \parallel w, \quad (z_3 - z_2) \parallel (y - x), \quad (z_4 - z_3) \parallel (-w),$$

we have

$$|Pos(z_4) - Pos(z_1)| < \delta,$$

where $Pos(z_4)$ is a position of z_4 in $U_{y,x,\varepsilon}$, and $Pos(z_1)$ is a position of z_1 in $U_{x,y,\varepsilon}$, as defined in (9).

Take distinct n, m and k in \mathbb{N} . From item 3 of Lemma 30, there are at most four values $i \in \mathbb{N}$ such that the transposition φ_i moves either n or m (or both). Let us recall that $\rho_n > 0$ and $\rho_m > 0$ were defined in equation (8). Choose $\varepsilon_{n,m} > 0$ sufficiently small that if for some $i \in \mathbb{N}$ the transposition φ_i moves n and does not move m (or moves m and does not move n), then Proposition 44 is applicable to the points

$x = x_n, y = x_k, t = x_m$ (or to the points $x = x_m, y = x_k, t = x_n$) with $\delta = 0.001 \min(\rho_n, \rho_m)$, and if φ_i transposes n and m , Proposition 45 is applicable with the same $\delta > 0$.

This choice of the sizes of the cigars will be used in the proof of to Corollary 54.

3.4.4. Absence of intersections. We would like to be able to control the sets of points that are being moved by homeomorphisms h_i in order to justify the property 3) from Proposition 36. This can be done via the following statements:

Proposition 46. *There exists a collection of positive numbers $\xi_{i,k}, \xi_i, i \in \mathbb{N}, k \in \mathbb{N}$, such that the following holds. Recall the notation:*

$$V_{k,i} = V(x_k, x_{n_i}, x_{m_i}, \xi_{i,k}, d_i w_i), \quad W_i = W(x_{n_i}, x_{m_i}, \xi_i, d_i w_i),$$

then for $i \neq j$:

- (1) $V_{k,i} \cap V_{l,j} \neq \emptyset$ if and only if one of the pairs $\{n_i, k\}, \{m_i, k\}$ coincides with one of the pairs $\{n_j, l\}, \{m_j, l\}$,
- (2) $W_i \cap W_j = \emptyset$ if $i \neq j$,
- (3) $W_i \cap V_{k,j} \neq \emptyset$ if and only if $\{n_i, m_i\} = \{n_j, k\}$ or $\{n_i, m_i\} = \{m_j, k\}$.

Proof of Proposition 46. Let us denote by $\pi_2 : \mathbb{R}^5 \rightarrow \mathbb{E}_2$ the projection to the two-dimensional subspace, and by $\pi_3 : \mathbb{R}^5 \rightarrow \mathbb{E}_3$ the projection to the three dimensional subspace. We first consider the case $k \neq l$. For given $i \in \mathbb{N}$, consider the plane $\mathcal{P}_{i,k}$ that contains x_{n_i}, x_{m_i} , and x_k . The family of planes $\mathcal{P}_{i,k}$ converges to the plane that contains points x_{n_i}, x_{m_i} , and r as $k \rightarrow \infty$. Denote the limit by $\mathcal{P}_{i,\infty}$. It follows that for each k the infimum of the angles between $\mathcal{P}_{i,k}$ and any $\mathcal{P}_{i,k'}, k' \neq k$, is positive. This implies that if $\{\xi_{i,k}\}_{k \in \mathbb{N}}$ are sufficiently small, then for any $k \neq k'$ we have $\pi_3(V_{k,i}) \cap \pi_3(V_{k',i}) = \emptyset$, hence $V_{k,i} \cap V_{k',i} = \emptyset$.

If $k = l$ and none of the pairs coincide, then $\{n_i, m_i\} \cap \{n_j, m_j\} = \emptyset$. For $\xi_{i,k}, \xi_{j,l}$ small enough, we have $V_{k,i} \cap V_{l,j} = \emptyset$. Indeed, consider $U_{x_k, x_{n_i}, \varepsilon_{n_i, k}} \cup B_{x_{n_i}}$. There exists $\delta > 0$ such that if $\xi_{i,k}$ is sufficiently small, then for any $t \in [-\delta, \delta]$ we have

$$g_v^t(U_{x_k, x_{n_i}, \xi_{i,k}}) \subset U_{x_k, x_{n_i}, \varepsilon_{n_i, k}} \cup B_k,$$

where the vector field $v = v(x_k, x_{n_i}, d_i w_i)$ is given by definition 38, and for any $t \in [-1, -\delta] \cup [\delta, 1]$ we have that $\pi_2(g_v^t(U_{x_k, x_{n_i}, \xi_{i,k}}))$ is in the cone K_i (the cone in \mathbb{E}_2 of size $\frac{\beta_i}{3}$ around w_i , given in definition 37 above). Similarly, we will have for any $t \in [-\delta, \delta]$

$$g_v^t(U_{x_k, x_{m_i}, \xi_{i,k}}) \subset U_{x_k, x_{m_i}, \varepsilon_{m_i, k}} \cup B_{x_{m_i}},$$

where $v = v(x_k, x_{m_i}, d_i w_i)$, and for any $\tau \in [-1, -\delta] \cup [\delta, 1]$ we will have that $\pi_2(g_v^\tau(U_{x_k, x_{m_i}, \xi_{i,k}}))$ is in the cone K_i . Finally, by choosing $\xi_{i,k}$ sufficiently small, we can guarantee that for any $s \in [0, 1]$ the projection $\pi_2(g_v^s(U_{x_k, x_{n_i} + d_i w_i, \xi_{i,k}}))$, where $v = v(x_k, x_{n_i} + d_i w_i, x_{m_i} - x_{n_i})$, is in the cone K_i . Since the cones K_i and K_j are disjoint if $i \neq j$, this implies that if $\{n_i, m_i\} \cap \{n_j, m_j\} = \emptyset$, then $V_{k,i} \cap V_{l,j} = \emptyset$ for any $k \in \mathbb{N} \setminus \{n_i, m_i\}$ and any $l \in \mathbb{N} \setminus \{n_j, m_j\}$. This proves the first item.

Similar arguments show that if $\{\xi_n\}$ are positive and sufficiently small and go to zero fast enough, then $W_i \cap W_j = \emptyset$ if $i \neq j$. Indeed, if $\xi_i > 0$ are small, then projections of W_i and W_j to \mathbb{E}_3 must be disjoint. Hence the second item holds.

What is left is to show that $W_i \cap V_{k,j} = \emptyset$ if $\{n_i, m_i\} \neq \{n_j, k\}$ and $\{n_i, m_i\} \neq \{m_j, k\}$. Notice that if ξ_i is small, then $\pi_3(W_i) \subseteq U_{x_{n_i}, x_{m_i}, \varepsilon_{n_i, m_i}}$. In particular, for any $\{n_i, m_i\} \neq \{n_j, k\}$ and $\{n_i, m_i\} \neq \{m_j, k\}$ we have

$$\begin{aligned} \left(\bigcup_{t=-1}^1 g_{v(x_k, x_{n_j}, d_j w_j)}^t(U_{x_k, x_{n_j}, \varepsilon_{k, n_j}}) \right) \cap W_i &= \emptyset, \\ &\text{and} \\ \left(\bigcup_{t=-1}^1 g_{v(x_k, x_{m_j}, d_j w_j)}^t(U_{x_k, x_{m_j}, \varepsilon_{k, m_j}}) \right) \cap W_i &= \emptyset. \end{aligned}$$

Now, if ξ_i is sufficiently small, for any point $p \in W_i \setminus U_{x_{n_i}, x_{m_i}, \varepsilon_{n_i, m_i}}$ the projection $\pi_2(p)$ is in the cone K_i . Therefore, for any $j \neq i$ we have

$$\left(\bigcup_{t=0}^1 g_{v(x_k, x_{n_j} + d_j w_j, x_{m_j} - x_{n_j})}^t(U_{x_k, x_{n_j} + d_j w_j, \xi_j}) \right) \cap W_i = \emptyset,$$

and

$$\left(\bigcup_{t=0}^1 g_{v(x_k, x_{m_j} - d_j w_j, x_{n_j} - x_{m_j})}^t(U_{x_k, x_{m_j} - d_j w_j, \xi_j}) \right) \cap W_i = \emptyset.$$

This implies that $V_{k,j} \cap W_i = \emptyset$.

Finally, since the angles between planes containing the points x_{n_i} , x_{m_i} , and x_k and the plane containing points x_{n_i} , x_{m_i} , and $x_{n_i} + d_i w_i$ are non-zero, if ξ_i and $\xi_{i,k}$ are sufficiently small, we also have

$$V(x_k, x_{n_i}, x_{m_i}, \xi_{i,k}, d_i w_i) \cap W_i = \emptyset.$$

Proof of Proposition 46 is complete. \dashv

We also would like to choose the sizes $\{\xi_{i,k}\}$ and $\{\xi_i\}$ so small that after we construct the homeomorphisms $\{h_i\}$, a map h_i will not change a relative position Pos of a point too much. Formally, it can be done by the following statement (that follows immediately from Propositions 44 and 45):

Proposition 47. *There exists a collection of positive numbers $\xi_{i,k}$, ξ_i , $i \in \mathbb{N}$, $k \in \mathbb{N}$, such that additionally to the conditions stated in Proposition 46 the following holds:*

For any points

$z_1 \in U_{x_k, x_{n_i}, \xi_{i,k}}$, $z_2 \in U_{x_k, x_{n_i} + d_i w_i, \xi_{i,k}}$, $z_3 \in U_{x_k, x_{m_i} + d_i w_i, \xi_{i,k}}$, $z_4 \in U_{x_k, x_{m_i}, \xi_{i,k}}$,
such that

$$(z_2 - z_1) \parallel w_i, \quad (z_3 - z_2) \parallel (x_{m_i} - x_{n_i}), \quad (z_4 - z_3) \parallel (-w_i),$$

we have

$$|Pos(z_4) - Pos(z_1)| < \frac{1}{500 \cdot 2^i},$$

where $Pos(z_4)$ is a position of z_4 in $U_{x_k, x_{m_i}, \xi_{i,k}}$, and $Pos(z_1)$ is a position of z_1 in $U_{x_k, x_{n_i}, \xi_{i,k}}$, as defined in (9).

Also, for any points

$z_1 \in U_{x_{n_i}, x_{m_i}, \xi_i}$, $z_2 \in U_{x_{n_i} + d_i w_i, x_{m_i} - d_i w_i, \xi_i}$, $z_3 \in U_{x_{n_i} + d_i w_i, x_{m_i} - d_i w_i, \xi_i}$, $z_4 \in U_{x_{m_i}, x_{n_i}, \xi_i}$,
such that

$$(z_2 - z_1) \parallel w_i, \quad (z_3 - z_2) \parallel (x_{m_i} - x_{n_i}), \quad (z_4 - z_3) \parallel (-w_i),$$

we have

$$|Pos(z_4) - Pos(z_1)| < \frac{1}{500 \cdot 2^i},$$

where $Pos(z_4)$ is a position of z_4 in $U_{x_{m_i}, x_{n_i}, \xi_i}$, and $Pos(z_1)$ is a position of z_1 in $U_{x_{n_i}, x_{m_i}, \xi_i}$, as defined in (9).

Now we would like to choose the sizes of the initial cigars so small that after we construct a homeomorphism h_i , their images will still be contained in the cigars of sizes dictated by Propositions 46 and 47. Let us start with the following simple statement:

Proposition 48. *Suppose a collection of positive numbers $\xi_{i,k}$, ξ_i , $i \in \mathbb{N}$, $k \in \mathbb{N}$, satisfies the conditions stated in Propositions 46 and 47. Then there exists a collection of positive numbers $\xi'_{i,k}$, ξ'_i , $i \in \mathbb{N}$, $k \in \mathbb{N}$, such that the following inclusions hold for all $i \in \mathbb{N}$ and $k \neq n_i, m_i$:*

$$\begin{aligned} g_{v(x_k, x_{n_i}, d_j w_j)}^1(U_{x_k, x_{n_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{n_i} + d_i w_i, \frac{1}{2} \xi_{i,k}}, \\ g_{v(x_k, x_{n_i} + d_i w_i, x_{m_i} - x_{n_i})}^1 \circ g_{v(x_k, x_{n_i}, d_i w_i)}^1(U_{x_k, x_{n_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{m_i} + d_i w_i, \frac{1}{2} \xi_{i,k}}, \\ g_{v(x_k, x_{n_i} + d_i w_i, -d_i w_i)}^1 \circ g_{v(x_k, x_{n_i} + d_i w_i, x_{m_i} - x_{n_i})}^1 \circ g_{v(x_k, x_{n_i}, d_i w_i)}^1(U_{x_k, x_{n_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{m_i}, \frac{1}{2} \xi_{i,k}}, \\ g_{v(x_k, x_{m_i}, -d_j w_j)}^1(U_{x_k, x_{m_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{m_i} - d_i w_i, \frac{1}{2} \xi_{i,k}}, \\ g_{v(x_k, x_{m_i} - d_i w_i, x_{n_i} - x_{m_i})}^1 \circ g_{v(x_k, x_{m_i}, -d_i w_i)}^1(U_{x_k, x_{m_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{n_i} - d_i w_i, \frac{1}{2} \xi_{i,k}}, \\ g_{v(x_k, x_{n_i} - d_i w_i, d_i w_i)}^1 \circ g_{v(x_k, x_{m_i} - d_i w_i, x_{n_i} - x_{m_i})}^1 \circ g_{v(x_k, x_{m_i}, -d_i w_i)}^1(U_{x_k, x_{m_i}, \xi'_{i,k}}) &\subset U_{x_k, x_{n_i}, \frac{1}{2} \xi_{i,k}}, \end{aligned}$$

$$\begin{aligned}
& g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{n_i},d_j w_j)}^1(U_{x_{n_i},x_{m_i},\xi'_i}) \subset U_{x_{n_i}+d_i w_i,x_{m_i}-d_i w_i,\frac{1}{2}\xi_i}, \\
& g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{n_i}+d_i w_i,x_{m_i}-x_{n_i})}^1 \circ g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{n_i},d_j w_j)}^1(U_{x_{n_i},x_{m_i},\xi'_i}) \subset U_{x_{m_i}+d_i w_i,x_{n_i}-d_i w_i,\frac{1}{2}\xi_i}, \\
& g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{m_i}+d_i w_i,-d_i w_i)}^1 \circ g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{n_i}+d_i w_i,x_{m_i}-x_{n_i})}^1 \circ \\
& \quad g_{v(\frac{1}{2}(x_{n_i}+x_{m_i}),x_{n_i},d_j w_j)}^1(U_{x_{n_i},x_{m_i},\xi'_i}) \subset U_{x_{m_i},x_{n_i},\frac{1}{2}\xi_i}.
\end{aligned}$$

Notice that the same pair of indices of vertices can serve as a pair (n_i, k) for some $i \in \mathbb{N}$, and at the same time, as a pair (n_j, m_j) for some $j \neq i$, which gives several restrictions on the size of an initial ‘‘cigar’’ with vertices x_n and x_m . So we will set

$$\varepsilon'_{m,n} = \min\{\xi'_{i,k}, \xi'_i \mid \{m, n\} = \{n_i, k\} \text{ or } \{m, n\} = \{m_i, k\} \text{ or } \{m, n\} = \{n_i, m_i\}\}$$

Because any n is only moved by finitely many $\{\varphi_i\}$ the minimum above is taken over a finite set. Now we can apply the arguments from Section 3.4.2 and start with the initial cigars of sizes $\{\varepsilon'_{m,n}\}$.

3.4.5. Coating the needles. Here we choose a sequence of cigars $U_{r,x_n,\tilde{\theta}_n}$ around the ‘‘needles’’ of \mathbf{I} . After that we will make the sizes of those cigars even smaller, so we reserve the notation $\{\theta_n\}$ for the preliminary sizes of those neighborhoods, and will use $\{\tilde{\theta}_n\}$ for the adjusted sizes.

Lemma 49. *There exists a sequence $\{\tilde{\theta}_n\}_{n \in \mathbb{N}}$ of positive numbers such the following hold:*

- 1) For any $m \neq n$ one has $U_{r,x_n,\tilde{\theta}_n} \cap U_{r,x_m,\tilde{\theta}_m} = \emptyset$
- 2) For any distinct m, n, k that are all different we have

$$U_{r,x_n,\tilde{\theta}_n} \cap U_{x_m,x_k,\varepsilon_{m,k}} = \emptyset$$

- 3) If $\{n_i, m_i\} \cap \{n_j, m_j\} = \emptyset$, then

$$V(r, x_{n_i}, x_{m_i}, \tilde{\theta}_{n_i}, d_i w_i) \cap V(r, x_{n_j}, x_{m_j}, \tilde{\theta}_{n_j}, d_j w_j) = \emptyset,$$

and for any $k \in \mathbb{N}$

$$V(r, x_{n_i}, x_{m_i}, \tilde{\theta}_{n_i}, d_i w_i) \cap V(x_k, x_{n_j}, x_{m_j}, \xi_{j,k}, d_j w_j) = \emptyset$$

- 4) For any $i, j \in \mathbb{N}$ (including the case when $j = i$) we have

$$W_j \cap V(r, x_{n_i}, x_{m_i}, \tilde{\theta}_{n_i}, d_i w_i) = \emptyset$$

Proof of Lemma 49. Let us recall that the vertices $\{x_n\} \subset \mathbb{R}^5$ were chosen in such a way that properties listed in Proposition 17 are satisfied. In particular, due to property 6) from Proposition 17, it is enough to choose some $\theta'_n < \eta_n$ for each $n \in \mathbb{N}$ to make sure that conditions 1) and 2) from Lemma 49 are satisfied for any $\tilde{\theta}_n \leq \theta'_n$.

The arguments to show that for sufficiently small $\tilde{\theta}_n$ the properties 3) and 4) hold are parallel to the arguments used in the proof of Lemma 46, so we will omit them here. \dashv

Let us now explain how we are going to choose the actual sizes $\{\theta_n\}$ of the neighborhoods U_{r,x_n,θ_n} . Recall from definition 34:

$$J(n) = \min\{j \in \mathbb{N} \mid \text{for all } m \in \text{Cont}_n, m \text{ is frozen at stage } j\}$$

and

$$T(n) = \max\{n_i, m_i \mid i \leq J(n)\},$$

where n_i, m_i are the elements of \mathbb{N} that are swapped by φ_i .

For each $n \in \mathbb{N}$ choose small $\theta_n > 0$ such that $\theta_n \leq \tilde{\theta}_n$, and $U_{x_n,r,\theta_n} \cap U_{x_n,x_m,\varepsilon_{n,m}} = \emptyset$ for all $m \leq T(n)$.

Remark 50. *Suppose $U_{x_n,r,\theta_n} \cap U_{x_n,x_k,\varepsilon_{n,k}} \neq \emptyset$. Then:*

- (1) $k \geq T(n)$.
- (2) *If h_i moves x_k , then for any $j \geq i$ the map h_j does not move x_n (in other words, n is frozen at stage i).*

Let us define the sets $Base_i \subset \mathbb{R}^5$ as

$$Base_i = \left(\bigcup_{n \in \mathbb{N}} B_{x_n} \right) \cup \left(\bigcup_{n \neq m} U_{x_n,x_m,\varepsilon'_{n,m}} \right) \cup \left(\bigcup_{n \text{ is not frozen at stage } i} U_{x_n,r,\theta_n} \right) \quad (10)$$

3.4.6. Construction of homeomorphisms $\{h_i\}$. Here we present the construction of the homeomorphisms $\{h_i\}$. Fix an i for the rest of this subsection.

We will need the following simple statement:

Lemma 51. *Suppose $U \subset \mathbb{R}^d$ is bounded open set, and $\mathfrak{K} \subset \bar{U}$ is compact. Suppose $\psi : \mathfrak{K} \rightarrow \mathbb{R}$ is a continuous function such that $\psi(x) = 0$ for any point $x \in \mathfrak{K} \cap \partial U$. Then there exists a continuous function $\Psi : \bar{U} \rightarrow \mathbb{R}$ such that $\Psi|_{\mathfrak{K}} = \psi$, and $\Psi|_{\partial U} = 0$.*

We will construct h_i as a composition of six maps,

$$h_i = R_i \circ \hat{h}_i \circ h_i^\uparrow \circ h_i^\rightarrow \circ h_i^\uparrow \circ S_i.$$

The homeomorphism S_i will shrink the sizes of the cigars connecting the x_k 's and r to the points x_{m_i} and x_{n_i} and the cigar connecting x_{m_i} to x_{n_i} . This is done to make them small enough that moving them along the paths determined by the superscripted h_i 's keeps their intersections with the other cigars very small. The final homeomorphism R_i will increase the sizes back to match the sizes of the target cigars.

Let us describe each of them separately.

Definition of S_i .

The homeomorphism S_i will be supported on

$$\left(\bigcup_{k \neq n_i, m_i} \left(U_{x_k, x_{n_i}, \varepsilon_{k, n_i}} \cup U_{x_k, x_{m_i}, \varepsilon_{k, m_i}} \right) \right) \cup U_{x_{n_i}, x_{m_i}, \varepsilon_{n_i, m_i}} \quad (11)$$

All sets in this union are pairwise disjoint, so one can define S_i on each of them separately. For example, construct S_i on $U_{x_k, x_{n_i}, \varepsilon_{k, n_i}}$. This will depend on a small positive parameter δ which in turn depends on k and i . The size of δ will be specified later. Let S_i restricted to $\mathbf{U}_{x_k, x_{n_i}} = U_{x_k, x_{n_i}, \frac{1}{2}\varepsilon_{k, n_i}}$ be a contraction toward $\gamma_{x_k, x_{n_i}}$ (the line that connects x_k and x_{n_i}) with coefficient $\frac{2\delta}{\varepsilon_{k, n_i}}$, so that $S_i(\mathbf{U}_{x_k, x_{n_i}}) = U_{x_k, x_{n_i}, \delta}$. Also, we will require that $S|_{\partial U_{x_k, x_{n_i}, \varepsilon_{k, n_i}}} = id$, and for any ray \mathbf{r} that originates at a point of $\gamma_{x_k, x_{n_i}}$ and is orthogonal to it, S_i is defined on $\mathbf{r} \cap \left(U_{x_k, x_{n_i}, \varepsilon_{k, n_i}} \setminus \mathbf{U}_{x_k, x_{n_i}} \right)$ as an affine map that sends it to $\mathbf{r} \cap \left(U_{x_k, x_{n_i}, \varepsilon_{k, n_i}} \setminus U_{x_k, x_{n_i}, \delta} \right)$. Notice that these properties define S_i on $U_{x_k, x_{n_i}, \varepsilon_{k, n_i}}$ uniquely.

Definition of h_i^\uparrow .

Let us apply Lemma 51 to an open set U given by the union

$$U = B_{Q_i} \cup W_i \cup \left(\bigcup_{k \neq n_i, m_i} V_{k, i} \right) \cup V(r, x_{n_i}, x_{m_i}, \theta_i, d_i w_i) \quad (12)$$

Define

$$\mathfrak{K} = \overline{\bigcup_{k \neq n_i, m_i} \left(\Delta(x_k, x_{n_i}, \frac{1}{2}\xi_{k, i}, \pm d_i w_i) \cup \Delta(x_k, x_{m_i}, \frac{1}{2}\xi_{k, i}, \pm d_i w_i) \right)} \cup \overline{W(x_{n_i}, x_{m_i}, \frac{1}{2}\xi_i, d_i w_i)}$$

Notice that $\mathfrak{K} \subset \bar{U}$, and $\mathfrak{K} \cap \partial U = \{x_k\}_{k \neq n_i, m_i} \cup \{r\}$. Now define $\psi : \mathfrak{K} \rightarrow \mathbb{R}$ in the following way:

- If $x \in \Delta(x_k, x_{n_i}, \frac{1}{2}\xi_{k, i}, d_i w_i) \cup \Delta(x_k, x_{n_i}, \frac{1}{2}\xi_{k, i}, -d_i w_i)$, then ψ is given by Remark 39 applied to $v = v(x_k, x_{n_i}, d_i w_i)$.
- If $x \in \Delta(x_k, x_{m_i}, \frac{1}{2}\xi_{k, i}, -d_i w_i) \cup \Delta(x_k, x_{m_i}, \frac{1}{2}\xi_{k, i}, d_i w_i)$, then ψ is given by Remark 39 applied to $v = v(x_k, x_{m_i}, -d_i w_i)$.
- If $x \in \overline{W(x_{n_i}, x_{m_i}, \frac{1}{2}\xi_i, d_i w_i)}$, then ψ is given by Remark 39 applied to $v = v\left(\frac{x_{n_i} + x_{m_i}}{2}, x_{n_i}, d_i w_i\right)$.

By continuity, ψ is defined on the whole set \mathfrak{K} , including the needles $U_{r, x_{m_i}, \tilde{\theta}_{m_i}}$ and $U_{r, x_{n_i}, \tilde{\theta}_{n_i}}$. Since $\mathfrak{K} \cap \partial U = \{x_k\}_{k \neq n_i, m_i} \cup \{r\}$, it is clear that $\psi|_{\mathfrak{K} \cap \partial U} = 0$. Moreover using the techniques involving the function

Θ (Item 3) used in Definition 24, we can assume that every element of \mathbf{I} is a flat fixed point of ψ .

So we can apply Lemma 51 to construct a continuous function $\Psi : \mathbb{R}^5 \rightarrow \mathbb{R}$, such that $\Psi|_{\mathfrak{K}} = \psi$ and $\Psi|_{\mathbb{R}^5 \setminus U} = 0$.

Define h_i^\uparrow to be a time-one shift along the vector field $\Psi \cdot \bar{w}_i$, where \bar{w}_i is a constant vector field in the direction of w_i .

Definition of h_i^\rightarrow .

We will use a similar strategy. First, define U in the same way as in (12). Now define

$$\mathfrak{K} = \overline{\bigcup_{k \neq n_i, m_i} \Delta(x_k, x_{n_i} + w_i, \frac{1}{2}\xi_{k,i}, x_{m_i} - x_{n_i}) \cup \bigcup_{k \neq n_i, m_i} \Delta(x_k, x_{m_i} - w_i, \frac{1}{2}\xi_{k,i}, x_{n_i} - x_{m_i})} \cup \overline{W(x_{n_i}, x_{m_i}, \frac{1}{2}\xi_i, d_i w_i)}$$

Notice that $\mathfrak{K} \subset \bar{U}$, and

$$\mathfrak{K} \cap \partial U = \{x_k\}_{k \neq n_i, m_i} \cup \{r\}$$

Let us define a function $\psi : \mathfrak{K} \rightarrow \mathbb{R}$ in the following way:

- If $x \in \Delta(x_k, x_{n_i} + w_i, \frac{1}{2}\xi_{k,i}, x_{m_i} - x_{n_i})$, then ψ is given by Remark 39 applied to $v = v(x_k, x_{n_i} + d_i w_i, x_{m_i} - x_{n_i})$.
- If $x \in \Delta(x_k, x_{m_i} - w_i, \frac{1}{2}\xi_{k,i}, x_{n_i} - x_{m_i})$, then ψ is given by Remark 39 applied to $v = v(x_k, x_{n_i} - d_i w_i, x_{n_i} - x_{m_i})$.
- If $x \in \overline{W(x_{n_i}, x_{m_i}, \frac{1}{2}\xi_i, d_i w_i)}$, then ψ is given by Remark 39 applied to $v = v(\frac{x_{n_i} + x_{m_i}}{2}, x_{n_i} + d_i w_i, x_{m_i} - x_{n_i})$.

Exactly in the same way as in definition of h_i^\uparrow , by continuity ψ is defined on the whole set \mathfrak{K} . Since $\mathfrak{K} \cap \partial U = \{x_k\}_{k \neq n_i, m_i} \cup \{r\}$, it is clear that $\psi|_{\mathfrak{K} \cap \partial U} = 0$. So we can apply Lemma 51 to construct a continuous function $\Psi : \mathbb{R}^5 \rightarrow \mathbb{R}$, such that $\Psi|_{\mathfrak{K}} = \psi$ and $\Psi|_{\mathbb{R}^5 \setminus U} = 0$.

Define h_i^\rightarrow to be a time-one shift along the vector field $\Psi \cdot v$, where v is a constant vector field in the direction of $x_{m_i} - x_{n_i}$.

Definition of \hat{h}_i .

To define \hat{h}_i , we will need the following statement:

Lemma 52. *Let $L : \mathbb{R}^5 \rightarrow \mathbb{R}^5$ be an affine map that sends an interval $I = [x, y]$ into the interval $I' = [x', y']$. Suppose open sets $U_{x_1, x_2, \varepsilon}$ and $U_{x'_1, x'_2, \varepsilon'}$ for some $\varepsilon, \varepsilon' > 0$ are two cigars given via definition 21. Fix a small $\mu > 0$. Then for any sufficiently small $\delta, \delta' > 0$ the following holds.*

Let Ξ (respectively, Ξ') be a real valued C^∞ function that it is positive in $U_{x,y,\varepsilon}$ (respectively, in $U_{x',y',\varepsilon'}$) and vanishes outside of it. Let w (respectively, w') be a vector field orthogonal to the line $\gamma_{x,y}$ containing points x and y (respectively, to the line $\gamma_{x',y'}$), directed towards this line, and such that $\|w(z)\| = \text{dist}(z, \gamma_{x,y})$ (respectively, $\|w'(z)\| = \text{dist}(z, \gamma_{x',y'})$). Define f (respectively f') to be the time-1 shift along the flow defined by the vector field $\Xi \cdot w$ (respectively, $\Xi' \cdot w'$).

Define a map $S^* : U_{x,y,\varepsilon} \rightarrow U_{x,y,\delta}$ as a contraction toward the line $\gamma_{x,y}$ with coefficient $\frac{\delta}{\varepsilon} < 1$. Similarly, define a map $R^* : U_{x',y',\delta'} \rightarrow U_{x',y',\varepsilon'}$ as an inverse of a contraction toward the line $\gamma_{x',y'}$ with coefficient $\frac{\delta'}{\varepsilon'} < 1$. There exists a homeomorphism

$$\hat{h} : \overline{U_{x',y',\varepsilon'}} \rightarrow \overline{U_{x',y',\delta'}},$$

such that

- (1) $\hat{h}|_{\partial U_{x',y',\varepsilon}} = \text{id}$,
- (2) $\hat{h}(L(U_{x,y,\delta})) = U_{x',y',\delta'}$,
- (3) if $\tilde{h} = R^* \circ \hat{h} \circ L \circ S^*$, then $f' \circ \tilde{h} = \tilde{h} \circ f$,
- (4) $\text{dist}_{C^0}(\text{id}, \hat{h}) < \mu$.

Notice that $S^* = S^*(x, y, \varepsilon, \delta)$ is a function of x, y, ε and δ , and similarly $R^* = R^*(x', y', \varepsilon', \delta')$ is a function of x', y', ε' and δ' .

Remark 53. In Lemma 52 we do not assume that $\overline{U_{x,y,\varepsilon}}$ and $\overline{U_{x',y',\varepsilon'}}$ are disjoint. For example, it is possible that $x = x'$.

Proof of Lemma 52. For given small δ, δ' there exists an affine invertible orientation preserving map L' such that the line $\gamma_{x',y'}$ is a line of fixed points for L' , $L' \circ L(U_{x,y,\delta}) = U_{x',y',\delta'}$, and the composition $L' \circ L$ sends subspaces orthogonal to $\gamma_{x,y}$ to subspaces orthogonal to $\gamma_{x',y'}$. The C^0 -norm of restriction $L'|_{L(U_{x,y,\delta})}$ tends to zero as $\delta, \delta' \rightarrow 0$. Moreover, the restriction $L'|_{L(U_{x,y,\delta})}$ can be extended to a homeomorphism $H : \overline{U_{x',y',\varepsilon'}} \rightarrow \overline{U_{x',y',\delta'}}$ such that $H|_{L(U_{x,y,\delta})} = L'|_{L(U_{x,y,\delta})}$, and $\|H\|_{C^0} \rightarrow 0$ as $\delta, \delta' \rightarrow 0$. The composition $L' \circ L \circ S^*$ is an affine map that sends the vector field w to a vector field \tilde{w} in $U_{x',y',\delta'}$ directed towards the line $\gamma_{x',y'}$. The push-forward of the vector field w' under the map $(R^*)^{-1}$ (let us denote it by \tilde{w}') is also a vector field in $U_{x',y',\delta'}$ directed towards the line $\gamma_{x',y'}$. The standard fundamental domain arguments show that the time-1 shift along the flows defined by \tilde{w} and \tilde{w}' are conjugate, and C^0 -norm of the conjugacy is not greater than δ' . Let us denote that conjugacy by h^* . Then $\hat{h} = h^* \circ L'$ satisfies all the conditions of Lemma 52 if δ, δ' are sufficiently small. \dashv

Finally, there exists a collection of sufficiently small numbers $\delta_{k,i} > 0$ and $\delta_i > 0$ such that setting S_i to be the S constructed with δ_i and $\delta_{k,i}$, the restriction of $h_i^\uparrow \circ h_i^\rightarrow \circ h_i^\uparrow \circ S_{\delta_{k,i}}$ to $U_{x_k, x_{n_i}, \varepsilon_{k,i}}$ is an affine map. An application of Lemma 52 to these choices finishes the construction of \hat{h}_i .

Definition of R_i .

The constructions of R_i is very similar to the construction of S_i described above, except that it is an expansion rather than a contraction.

Notice that together with Propositions 44 and 43, the construction of homeomorphisms $\{h_i\}$ guarantees that the following holds:

Corollary 54. *The sizes of the cigars $\{\varepsilon'_{n,m}\}$ can be chosen small enough to guarantee that if $x \in U_{x_n, x_m, \varepsilon'_{n,m}}$, and $h_i(x) \in U_{x_{\bar{n}}, x_{\bar{m}}, \varepsilon'_{\bar{n}, \bar{m}}}$, then $|\text{Pos}(h_i(x)) - \text{Pos}(x)| < \frac{1}{500 \cdot 2^{i+1}}$. As a result, if an orbit of $x \in U_{x_n, x_m, \varepsilon'_{n,m}}$ is inside of $\cup_{n,m} U_{x_n, x_m, \varepsilon'_{n,m}}$, then the relative position of any point in the orbit of x differs from $\text{Pos}(x)$ by at most $\frac{1}{500}$.*

Also, let us notice that due to the construction of h_i presented above we immediately have the following estimate:

Proposition 55. *For each $i \in \mathbb{N}$ we have*

$$\text{dist}_{C^0}(\text{id}, h_i) \leq 2 \max_{k \neq n_i, m_i} (\varepsilon_{k, n_i}, \varepsilon_{k, m_i}) + 2\varepsilon_{n_i, m_i} + 6d_i$$

Combined with Lemma 35, Proposition 55 gives the following:

Corollary 56.

$$\sum_{i=1}^{\infty} \text{dist}_{C^0}(\text{id}, h_i) < \infty$$

⊢ Let us recall that we chose $\{\varepsilon_{m,n}\}$ such that $\varepsilon_{m,n} \leq 2^{-m-n}$. This implies that $\sum_{i=1}^{\infty} \max_{k \neq n_i, m_i} (\varepsilon_{k, n_i}, \varepsilon_{k, m_i}) < \infty$ and $\sum_{i=1}^{\infty} \varepsilon_{n_i, m_i} < \infty$. Combined with Lemma 35 that claims that $\sum_{i=1}^{\infty} d_i < \infty$, it completes the proof of Corollary 56. ⊣

Let us recall that the sets Base_i were defined in (10) as

$$\text{Base}_i = \left(\bigcup_{n \in \mathbb{N}} B_{x_n} \right) \cup \left(\bigcup_{n \neq m} U_{x_n, x_m, \varepsilon_{n,m}} \right) \cup \left(\bigcup_{n \text{ is not frozen at stage } i} U_{r, x_n, \theta_n} \right)$$

Lemma 57. *If $h_i(x) \neq x$, then either $h_j(h_i(x)) = h_i(x)$ for all $j > i$, or $h_i(x) \in \text{Base}_i$.*

Proof of Lemma 57. Suppose $h_i(x) \neq x$, then $x, h_i(x) \in \text{supp } h_i$, i.e.

$$h_i(x) \in \text{Base}_i \cup (B_{Q_i} \cup (\cup_{k \neq m_i, n_i} V_{k,i}) \cup W_i \cup V_{r,i}).$$

Due to our choice of B_{Q_i} , $\{V_{k,i}\}$, W_i , and $V_{r,i}$ defined in Section 3.4.1 in the case $h_i(x) \notin \text{Base}_i$ the projection $\pi_2(h_i(x))$ is in a cone K_i around w_i in \mathbb{E}_2 that is disjoint from cones around w_j , $j \neq i$, and hence $h_i(x) \notin \text{supp } h_j$ for any $j > i$. +

Define for each $n \in \mathbb{N}$ the set $S(n)$ by

$$S(n) = \{k \in \mathbb{N} \mid k > n \text{ and } \forall i \in \mathbb{N}, \text{ if } h_i(x_k) \neq x_k,$$

then for any $t \in \text{Cont}_m$ with $m \leq n$ the vertex x_t is frozen at stage $i\}$

For each $n \in \mathbb{N}$ (that will correspond to a vertex x_n) and $i \in \mathbb{N}$ (that will correspond to a homeomorphism h_i) define a "tail" $T_i(n) \subset \mathbb{N}$ in the following way:

- $T_0(n) = S(n)$ for each $n \in \mathbb{N}$;
- $T_{i+1}(n) = T_i(n)$ if $\varphi_i(n) = n$;
- $T_{i+1}(n) = T_{i+1}(m) = T_i(n) \cap T_i(m)$, if φ_i transposes n and m , $n \neq m$.

For each $n \in \mathbb{N}$ and $i \in \mathbb{N} \cup \{0\}$ define a set (that we will informally call a "comb")

$$\mathcal{B}_i(n) = U_{r, x_n, \theta_n} \cup \left(\bigcup_{k \in T_i(n)} U_{x_n, x_k, \varepsilon'_{n,m}} \right)$$

Lemma 58. *If $n_1 \neq n_2$, then for any $i, j \in \mathbb{N}$ we have $\mathcal{B}_i(n_1) \cap \mathcal{B}_j(n_2) = \emptyset$.*

Definition 59. *Let us say that a cigar $U_{x_m, x_n, \varepsilon'_{n,m}}$ is i -relaxed if it is not a part of $\mathcal{B}_i(k)$ for any k , or if it is a part of $\mathcal{B}_i(k)$ with k being frozen at stage i .*

The next three statements directly follow from the definitions:

Lemma 60. *If $n_1 \neq n_2$, then for any $i, j \in \mathbb{N}$ we have $\mathcal{B}_i(n_1) \cap \mathcal{B}_j(n_2) = \emptyset$.*

Lemma 61. *If $U_{x_m, x_n, \varepsilon'_{n,m}}$ is i -relaxed, then it is also j -relaxed for any $j > i$.*

Lemma 62. *If $U_{x_m, x_n, \varepsilon'_{n,m}}$ is i -relaxed, then $U_{h_i(x_m), h_i(x_n), \varepsilon'_{\varphi_i(m), \varphi_i(n)}}$ is also i -relaxed.*

Due to Remark 50 we get the following statement:

Lemma 63. *If n is not frozen at stage i , and $h_i(x_n) = x_n$, then $\mathcal{B}_i(n) \subseteq \text{Fix}(h_i)$.*

It will be convenient to use the following terminology:

Definition 64. *A point x_n is frozen at stage i , if n is frozen at stage i .*

A cigar $U_{x_n, x_m, \varepsilon_{n,m}}$ is frozen at stage i if both n and m are frozen at stage i .

A comb $\mathcal{B}_i(n)$ is frozen at stage i , if n is frozen at stage i .

Definition 65. *Given a point $x \in \mathbb{R}^5$ and $i \in \mathbb{N}$, denote by $O_i(x)$ an orbit of the point x starting with the homeomorphism h_i , i.e.*

$$O_i(x) = \{x, h_i(x), h_{i+1} \circ h_i(x), h_{i+2} \circ h_{i+1} \circ h_i(x), \dots\}$$

Lemma 66. *Suppose a vertex x_k is frozen at stage i (i.e. $\varphi_j(k) = k$ for all $j \geq i$). Then for any point $x \in B_{x_k}$ an orbit of the point x starting with the homeomorphism h_i (i.e. $O_i(x)$) is finite.*

Proof of Lemma 66. Let us recall that by the choice of the size of the ball B_{x_k} (see Remark 41), for any point $x \in B_{x_k} \cap U_{x_k, x_n, \varepsilon'_{m,n}}$ we have $\text{Pos}(x) < \gamma$. Denote by \mathfrak{F}_k the set

$$\mathfrak{F}_k = B_{x_k} \cup \left(\bigcup_{n \neq k} \{x \in U_{x_k, x_n, \varepsilon'_{m,n}}, \text{Pos}(x) < 2\gamma\} \right),$$

where γ is the constant from Corollary 54. Due to Lemma 57, if $x \in B_{x_k}$, and $h_i(x), h_{i+1}(x), \dots, h_{i+m}(x) \in \text{Base}_i$, then $h_{i+m}(x) \in \mathfrak{F}_k$. Also, if $x \in B_{x_k}$, and $h_j(x) \neq x$ for some $j \geq i$, then either $h_j(x) \in \mathfrak{F}_k$, or $h_l(h_j(x)) = h_j(x)$ for all $l > j$. Indeed, this follows from the assumption that x_k is frozen at stage i and Proposition 46. Therefore, we just need to show that it is impossible for a point $x \in B_{x_k}$ to have an infinite orbit inside of \mathfrak{F}_k . But if $x \in U_{x_k, x_n, \varepsilon'_{k,n}}$, then this follows from the fact that Cont_n is a finite set. \dashv

Lemma 67. *For any vertex x_n , any point $x \in B_{x_n}$, and any $i \in \mathbb{N}$, an orbit of the point x starting with the homeomorphism h_i (i.e. $O_i(x)$) is finite.*

Proof of Lemma 67. Take any $x_n \in B_{x_n}$. If n is frozen at stage i , then Lemma 66 can be applied, and hence $O_i(x)$ is finite. Suppose n is not frozen at stage i . Let $l \in \mathbb{N}$ be the smallest index $l \geq i$ such that $h_l(x_n) \neq x_n$. Denote by \mathfrak{F}_n^* the set

$$\mathfrak{F}_n^* = \mathfrak{F}_n \cup U_{x_n, r, \eta_n}.$$

There are three possibilities:

- 1) $h_l \circ \dots \circ h_i(x) \notin Base_i$;
- 2) $h_l \circ \dots \circ h_i(x) \in \mathfrak{F}_n^*$;
- 2) $h_l \circ \dots \circ h_i(x) \in \mathfrak{F}_{\varphi_l(n)}^*$.

In the first case $h_l \circ \dots \circ h_i(x)$ is a fixed point of any h_j with $j > l$ (due to Lemma 57), and hence $O_i(x)$ is finite.

In the second case, if x_n is frozen at stage $l + 1$, then the same arguments that were used in the proof of Lemma 66 show that $O_i(x)$ is finite.

In the third case, if $x_{\varphi_l(n)}$ is frozen at stage $l + 1$, then, once again, the same arguments that were used in the proof of Lemma 66 show that $O_i(x)$ is finite.

In the second case, if x_n is not frozen at stage $l + 1$, let $l' \in \mathbb{N}$ be the smallest index $l' > l$ such that $h_{l'}(x_n) \neq x_n$, and proceed with the same three cases as above.

Similarly, in the third case, if $x_{\varphi_l(n)}$ is not frozen at stage $l + 1$, let $l' \in \mathbb{N}$ be the smallest index $l' > l$ such that $h_{l'}(x_n) \neq x_n$, and proceed with the same three cases as above.

Since $Cont_n$ is finite, this process will stop after finitely many steps, hence $O_i(x)$ must be finite. \dashv

Lemma 68. *If $U_{x_m, x_n, \varepsilon'_{m,n}}$ is i -relaxed, then for any point $x \in U_{x_m, x_n, \varepsilon'_{m,n}}$, an orbit of the point x starting with the homeomorphism h_i (i.e. $O_i(x)$) is finite.*

Proof of Lemma 68. Without loss of generality we can assume that $x \in U_{x_m, x_n, \varepsilon'_{m,n}} \setminus (B_{x_m} \cup B_{x_n})$, since otherwise we can refer to Lemma 67. Let $l \geq i$ be the smallest index such that $h_l(x_n) \neq x_n$ or $h_l(x_m) \neq x_m$. Then one of the following cases must hold:

- 1) $h_l(x) \notin Base_l$;
- 2) $h_l(x) \in B_{x_m} \cup B_{x_n} \cup B_{h_l(x_n)} \cup B_{h_l(x_m)}$;
- 3) $h_l(x) \in U_{x_m, x_n, \varepsilon'_{m,n}} \setminus (B_{x_m} \cup B_{x_n})$;
- 4) $h_l(x) \in U_{h_l(x_m), h_l(x_n), \varepsilon'_{\varphi_i(m), \varphi_i(n)}} \setminus (B_{h_l(x_n)} \cup B_{h_l(x_m)})$.

In the first case the orbit $O_i(x)$ is finite by Lemma 57.

In the second case the orbit $O_i(x)$ is finite by Lemma 67.

In the third or fourth case consider the smallest index $l' > l$ such that $h_{l'}(h_l(x)) \neq h_l(x)$. Then consider the four cases similar to those

listed above, etc. Since $Cont(m, n)$ is finite, this process will stop after finitely many steps, which proves Lemma 68. \dashv

Lemma 69. *If $x \in U_{x_n, r, \eta_n}$, and x_n is frozen at stage i , then the orbit $O_i(x)$ is finite.*

Proof of Lemma 69. If x_n is frozen at stage i , $x \in U_{x_n, r, \theta_n}$, and $l \geq i$ is the smallest index such that $h_l(x) \neq x$, then x must belong to $V_{x_n, x_s, x_t, \varepsilon_{s,t}^{final}}$ for some $s, t \in \mathbb{N}$, where either s or t belongs to $T_i(n)$. Without loss of generality assume that $s \in T_i(n)$. If $h_l(x) \notin Base_i$, then $O_i(x)$ is finite by Lemma 57. If $h_l(x) \in Base_l$, then either $h_l(x) \in B_{x_n}$ (and then $O_i(x)$ is finite by Lemma 67), or $h_l(x) \in U_{x_n, x_s, \varepsilon'_{n,s}} \cup U_{x_n, x_t, \varepsilon'_{n,t}}$. Notice that both $U_{x_n, x_s, \varepsilon'_{n,s}}$ and $U_{x_n, x_t, \varepsilon'_{n,t}}$ are l -relaxed. Indeed, $U_{x_n, x_s, \varepsilon'_{n,s}}$ is l -relaxed by definition, since x_n is frozen at stage l . Let us show that $U_{x_n, x_t, \varepsilon'_{n,t}}$ is l -relaxed. If not, it must be either a part of $\mathcal{B}_l(m)$ for some $m \neq t$, or a part of $\mathcal{B}_l(t)$. If $U_{x_n, x_t, \varepsilon'_{n,t}}$ is a part of $\mathcal{B}_l(m)$ for some $m \neq t$, then x_m is frozen at stage l (otherwise h_l cannot move x_t), and then $U_{x_n, x_t, \varepsilon'_{n,t}}$ is l -relaxed by definition. And if $U_{x_n, x_t, \varepsilon'_{n,t}}$ is a part of $\mathcal{B}_l(t)$, then $n > t$, and hence $h_l(s) = s$ if t is not frozen at stage l , contradiction. Therefore, both $U_{x_n, x_s, \varepsilon'_{n,s}}$ and $U_{x_n, x_t, \varepsilon'_{n,t}}$ are l -relaxed, and by Lemma 68 the orbit $O_i(x)$ must be finite. \dashv

Lemma 70. *If $x \in \mathcal{B}_i(n)$, and then the orbit $O_i(x)$ is finite.*

Proof of Lemma 70. If n is frozen at stage i , then either x belongs to some $U_{x_n, x_m, \varepsilon'_{n,m}}$ which is i -relaxed (and then $O_i(x)$ is finite due to Lemma 68), or x belongs to U_{x_n, r, θ_n} (and $O_i(x)$ is finite by Lemma 69).

Suppose that n is not frozen at stage i . Let $l \geq i$ be the smallest index such that $h_l(x_n) \neq x_n$. Notice that for any $j < l$ we have $h_j(x) = x$ (due to Lemma 63). One of the following cases must hold:

- 1) $h_l(x) \notin Base_l$;
- 2) $x \notin \mathcal{B}_l(n)$ (which is possible since it is possible that $\mathcal{B}_l(n) \neq \mathcal{B}_i(n)$);
- 3) $x \in \mathcal{B}_l(n)$, and n is frozen at stage $l + 1$;
- 4) $x \in \mathcal{B}_l(n)$, and n is not frozen at stage $l + 1$;
- 5) $x \in \mathcal{B}_l(\varphi_l(n))$, and $\varphi_l(n)$ is frozen at stage $l + 1$;
- 6) $x \in \mathcal{B}_l(\varphi_l(n))$, and $\varphi_l(n)$ is not frozen at stage $l + 1$.

In the case 1), the orbit $O_i(x)$ is finite due to Lemma 57.

In the case 2), the orbit of $O_i(x)$ is finite due to Lemma 68.

In the case 3) or 5), the orbit of $O_i(x)$ is finite due to Lemma 68 and Lemma 69.

In the case 4) or 6), take the smallest index $l' > l$ such that $h_{l'}(x_n) \neq x_n$ (in the case 4)) or $h_{l'}(x_{\varphi_l(n)}) \neq x_{\varphi_l(n)}$ (in the case 6)), and consider the same 6 cases. Since $Cont_n$ is finite, this process will end in a finite number of steps, hence $O_i(x)$ must be finite. \dashv

Combination of Lemma 67, Lemma 68, and Lemma 70 now shows that an orbit of any point is finite, which completes the proof of Proposition 36, and hence of Theorem 15.

4. OPEN QUESTIONS AND CONJECTURES

Our results provoke a long list of farther questions. Let us briefly list some of them here. More appear in [11].

- (1) Let M be a compact manifold without boundary of dimension at least two. Is there a dense \mathcal{G}_δ subset of $\text{Diff}^\infty(M)$ such that the equivalence relation of topological conjugacy is Borel?

One candidate would be the Kupka-Smale diffeomorphisms. (Recall that a diffeomorphism is Kupka-Smale if every periodic orbit is hyperbolic, and stable and unstable manifolds of any two periodic orbits are transversal. It is known that the set of Kupka-Smale diffeomorphisms of a given manifold M forms a dense G_δ subset of $\text{Diff}^r(M)$ for each $r = 1, 2, \dots, \infty$, see [30] and [44].)

- (2) Is the equivalence relation given by existence of a topological conjugacy on the boundary of the set of Morse-Smale diffeomorphisms of a given manifold M Borel¹⁰? (Morse-Smale diffeomorphisms are defined in 5)
- (3) Is the collection of diffeomorphisms f in $\text{Diff}^\infty(M)$ that are topologically conjugate to a structurally stable diffeomorphism a Borel set?
- (4) Let X be the collection of diffeomorphisms of a manifold M . Is the equivalence relation of *conjugacy by homeomorphisms* given by a Polish group action? Reducible to an S^∞ -action?

¹⁰After this text was written, we realized that the answer to this question is “no”; we intend to provide a proof of this statement in the final version of the paper.

APPENDIX A. LOCATIONS OF NOTATIONAL DEFINITIONS

This list gives the notation and the first page it appears on.

Page 21:

$$I_{m,n}, \quad \mathbf{I}_n$$

Page 22:

$$\mathbf{I} = \bigcup_n \mathbf{I}_n, \quad \alpha_{m,n}$$

Page 23:

$$\gamma_{x,y}$$

Page 24:

$$\begin{array}{cccc} \Pi_{x,y}(t), & \gamma_{m,n}, & U_{x,y,\epsilon}, & U_{m,n,\epsilon}, \\ \text{Cigars}, & \mathbf{U}_{m,n}, & \mathbf{U}'_{m,n}, & \mathbf{U}''_{m,n} \\ \epsilon_{m,n}, & \epsilon'_{m,n}, & \epsilon''_{m,n} & \end{array}$$

Page 25:

$$f_{m,n}^+, \quad f_{m,n}^-, \quad \lambda_{m,n}, \quad w_{m,n}, \quad \Theta, \quad v_{m,n}$$

Page 28:

$$Cont_m$$

Page 29:

$$Cont(n, m)$$

Page 30:

$$\text{frozen}, \quad J(n), \quad T(n), \quad d_i$$

Page 31:

$$E_1, \quad E_2, \quad E^{(i)}, \quad h_i$$

Page 32:

$$\begin{array}{cccc} \rho_n, & w_i, & q_i, & \beta_j \\ K_j, & Q_i & B_{Q_i} & \end{array}$$

Page 33:

$$\begin{array}{cccc} Q_i^u, & Q_i^d, & \delta_i, & v = v(x, y, w), \quad g_v^t \\ \psi = \psi(x, y, \epsilon, w), & \Delta = \Delta(x, y, \epsilon, w) & & \end{array}$$

Page 35:

$$\begin{array}{ccc} V(z, x, y, \xi, w), & v_{k,i}, & W(x, y, \xi, w) \\ V_{k,i}, & W_i & \end{array}$$

Page 36:

$$Pos(z)$$

Page 42:

$$\tilde{\theta}_n$$

Page 43:

$$Base_i, \quad \Psi, \quad \mathfrak{k}$$

$$R_i, \quad \hat{h}_i, \quad h_i^\uparrow, \quad h_i^{\rightarrow}, \quad h_i^\downarrow, \quad S_i$$

Page 46:

$$\Xi, \quad \Xi'$$

Page 48:

$$T_i(n), \quad i\text{-relaxed}$$

Page 49:

$$\mathfrak{F}, \quad \mathfrak{F}^*$$

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