ABSTRACT INTERPOLATION AND OPERATOR-VALUED KERNELS

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ABSTRACT

The complex interpolation method is used to prove a sharp inequality of Hausdorff–Young type for integral operators with operator-valued kernels.

1. *Introduction*

Let $\mathscr{H}$ be a complex Hilbert space, and let M be a σ-finite measure space. Denote by $L^2(M; \mathscr{H})$ the space of equivalence classes of square-integrable, measurable, $\mathscr{H}$-valued functions on M; then $L^2(M; \mathscr{H})$ is also a complex Hilbert space. Denote by $\mathscr{B}(\mathscr{H})$ the space of bounded, linear operators on $\mathscr{H}$.

In this paper, we consider $\mathscr{B}(\mathscr{H})$-valued functions on $M \times M$, and we show that if such a function k belongs to certain mixed-norm spaces, then we can associate to k a bounded operator K on $L^2(M; \mathscr{H})$, by letting

$$Kf(x) = \int_M k(x, y) \cdot f(y)\, dy, \tag{1}$$

for all f in $L^2(M; \mathscr{H})$, and almost all x in M. In Theorem 1, below, we give a sharp estimate for the norm of K in a certain space of operators on $L^2(M; \mathscr{H})$, in terms of the norm of k in an appropriate mixed-norm space. A similar estimate, for scalar-valued kernels, was proved in [**4**; Theorem 1], and used to estimate the norm of the L^p Fourier transform on certain groups. A variant of Theorem 1 of the present paper will be applied in a future publication.

Theorem 1 is stated and proved in Section 3, and the results in the complex interpolation method that are needed in the proof of Theorem 1 are summarized in Section 2.

2. *The complex interpolation method*

The standard reference concerning this method is Calderón's paper [**2**]. We begin with the basic definitions.

An *interpolation pair* (B^0, B^1) is a pair of complex Banach spaces, both continuously imbedded in a complex, topological vector space V. The subspace $B^0 + B^1$ of V becomes a Banach space relative to the norm

$$\|x\|_{B^0+B^1} = \inf\{\|y\|_{B^0} + \|z\|_{B^1} : x = y+z,\, y \in B^0,\, z \in B^1\}.$$

Given an interpolation pair (B^0, B^1), let $\mathscr{F} = \mathscr{F}(B^0, B^1)$ be the space of all functions f of the complex variable $\zeta = s+it$, defined in the strip where $0 \leqslant s \leqslant 1$, with values in $B^0 + B^1$, continuous and bounded relative to the norm in $B^0 + B^1$, analytic in the strip where $0 < s < 1$, and such that $f(it) \in B^0$, is B^0-continuous, and tends to 0 as $|t| \to \infty$, and $f(1+it) \in B^1$, is B^1-continuous, and tends to 0 as $|t| \to \infty$. Then $\mathscr{F}$ is a Banach space with the norm

$$\|f\|_{\mathscr{F}} = \max\{\sup_t \|f(it)\|_{B^0},\ \sup_t \|f(1+it)\|_{B^1}\}.$$

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Given a real number s, with $0 < s < 1$, let B_s be the subspace of all x in B^0+B^1 such that $x = f(s)$ for some f in $\mathscr{F}$; this space is also denoted by $[B^0, B^1]_s$, and is called a *complex interpolation space*. Then B_s becomes a Banach space relative to the norm

$$\|x\|_{B_s} = \inf\{\|f\|_{\mathscr{F}} : f(s) = x\}.$$

It follows immediately from these definitions [**2**; p. 115], that if (A^0, A^1) and (B^0, B^1) are interpolation pairs, and $L : A^0+A^1 \to B^0+B^1$ is a linear mapping such that the restriction of L to A^0 is a bounded operator from A^0 to B^0 with norm M_0, and the restriction of L to A^1 is a bounded operator from A^1 to B^1 with norm M_1, then, when $0 < s < 1$, the restriction of L to A_s is a bounded operator from A_s to B_s with norm at most ${M_0}^{1-s} {M_1}^s$.

To use this machinery, we need to know more about the interpolation spaces in specific instances. Given a σ-finite measure space M, we follow the usual convention of regarding two complex-valued, measurable functions on M as being equivalent if they agree almost everywhere, and we let V be the space of all such equivalence classes of measurable functions, with the topology of convergence in measure. Call a subspace X of V a *Banach lattice* on M if it is a Banach space relative to a norm $\|\cdot\|_X$ such that, if $f \in X$, $g \in V$ and $|g| \leqslant |f|$ almost everywhere, then $g \in X$, and $\|g\|_X \leqslant \|f\|_X$. Say that X has the *dominated convergence property* if the conditions

$$f \in X, \quad f_n \in V, \quad |f_n| \leqslant |f| \quad \text{and} \quad f_n \to 0 \text{ almost everywhere}$$

imply that $\|f_n\|_X \to 0$. Now let X_0 and X_1 be two Banach lattices on M, and let $0 < s < 1$; then let X be the subspace of all elements f of V for which there exist a positive number λ and non-negative elements g of X_0 and h of X_1 such that $\|g\|_{X_0} = \|h\|_{X_1} = 1$, and

$$|f| \leqslant \lambda g^{1-s} h^s. \tag{2}$$

Let $\|f\|_X = \inf\{\lambda : \text{relation (2) holds}\}$. Then X is a Banach lattice on M relative to the norm $\|\cdot\|_X$ [**2**; p. 123]; the space X is also denoted by ${X_0}^{1-s}{X_1}^s$. Given any Banach lattice X on M, and any complex Banach space B, form the Banach space $X(B)$ of B-valued measurable functions f such that $\|f(\cdot)\|_B \in X$, and define $\|f\|_{X(B)}$ to be $\|\,\|f(\cdot)\|_B\,\|_X$. The key result concerning complex interpolation of these spaces is the following theorem of Calderón [**2**; p. 125].

THEOREM 0. *Let (B^0, B^1) be an interpolation pair, and let X_0 and X_1 be Banach lattices on a σ-finite measure space M. Let $0 < s < 1$, let $B = [B^0, B^1]_s$, and let $X = {X_0}^{1-s}{X_1}^s$. Suppose that at least one of the spaces X_0 and X_1 has the dominated convergence property. Then*

$$[X_0(B^0), X_1(B^1)]_s = X(B), \tag{3}$$

with equality of norms.

Next, we consider subspaces of the space $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. Let $C_\infty(\mathscr{H}) = \mathscr{B}(\mathscr{H})$; if $1 \leqslant p < \infty$, let

$$C_p(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : \operatorname{tr}[(T^* T)^{p/2}] < \infty\}.$$

Evidently, $C_\infty(\mathscr{H})$ is a Banach space with the norm $\|\cdot\|_\infty = \|\cdot\|_{B(\mathscr{H})}$; so is $C_p(\mathscr{H})$, when $1 \leqslant p < \infty$, with the norm

$$\|T\|_p = (\operatorname{tr}[(T^* T)^{p/2}])^{1/p}.$$

Given indices p_0 and p_1, in the interval $[1, \infty]$, and s, in the interval $(0, 1)$, define p_s by

$$\frac{1}{p_s} = \frac{1-s}{p_0} + \frac{s}{p_1}.$$

It is shown in [3; p. 44] that

$$[C_{p_0}(\mathscr{H}), C_{p_1}(\mathscr{H})]_s = C_{p_s}(\mathscr{H}),$$

with equality of norms.

3. *Integral operators with operator-valued kernels*

In this section, we show that, if a measurable $\mathscr{B}(\mathscr{H})$-valued function k on $M \times M$ belongs to a certain mixed-norm space $X_p(C_{p'})$, with $1 \leqslant p \leqslant 2$, then the operator K, defined by formula (1) above, belongs to $C_{p'}(L^2(M; \mathscr{H}))$, and, in fact, $\|K\|_{p'} \leqslant \|k\|_{X_p(C_{p'})}$; here p' denotes the index conjugate to p. We follow [1] in defining mixed-norm spaces.

If $1 \leqslant r,\ s < \infty$, let $L^{r, s}(M \times M)$ be the space of all equivalence classes of complex-valued, measurable functions k on $M \times M$ such that

$$\|k\|_{r, s} = \left\{ \int_M \left[\int_M |k(x, y)|^r \, dx \right]^{s/r} dy \right\}^{1/s} < \infty; \tag{4}$$

make the obvious modifications when one or both of r and s are infinite. Then $L^{r, s}(M \times M)$, with the norm $\| \cdot \|_{r, s}$, is a Banach lattice on $M \times M$, whenever $1 \leqslant r, s \leqslant \infty$; this lattice has the dominated convergence property if r and s are both finite. Now consider functions on $M \times M$ with values in $\mathscr{B}(\mathscr{H})$, and denote the norm in $L^{r, s}(M \times M)(C_p(\mathscr{H}))$ by $\| \cdot \|_{p, r, s}$; that is,

$$\|k\|_{p, r, s} = \left[\int_M \left\{ \int_M (\|k(x, y)\|_p)^r \, dx \right\}^{s/r} dy \right]^{1/s}, \tag{5}$$

if r and s are finite, and the appropriate modifications are made if some indices are infinite. Observe that, in both formulae (4) and (5), the indices are subscripted to the norm in the order in which the norm is computed; in formula (5), for instance, on first computes the norm of $k(x, y)$ in $C_p(\mathscr{H})$, for fixed x and y, then the L^r-norm with respect to x, and, finally, the L^s-norm with respect to y.

We also have to consider the norms that arise when the order of integration is reversed. Given any function k on $M \times M$, define a function k^* by letting $k^*(x, y) = k(y, x)$ for all x and y in M; given any Banach space B of functions on $M \times M$, let

$$B^* = \{k : k^* \in B\}, \quad \text{and let} \quad \|k\|_{B^*} = \|k^*\|_B. \tag{6}$$

Given any index p, with $1 \leqslant p \leqslant \infty$, define a Banach lattice X_p by letting

$$X_p = [L^{p, p'}(M \times M)]^{\frac{1}{2}} [L^{p, p'}(M \times M)^*]^{\frac{1}{2}};$$

again $p' = p/(p-1)$. Finally, we use the abbreviation $X_p(C_{p'})$ for the space $X_p(C_{p'}(\mathscr{H}))$, which consists of all measurable, $C_{p'}(\mathscr{H})$-valued functions k on $M \times M$ such that

$\|k(\cdot,\cdot)\|_{C_{p'}(\mathscr{H})}$ belongs to X_p; the norm on this space is given by

$$\|k\|_{X_p(C_{p'})} = \| \, \|k(\cdot,\cdot)\|_{C_{p'}(\mathscr{H})}\|_{X_p}.$$

We can now state and prove our main result.

THEOREM 1. *Let $\mathscr{H}$ be a complex Hilbert space, and let M be a σ-finite measure space. Let $1 \leqslant p \leqslant 2$, and let $k \in X_p(C_{p'})$. Then the integral operator K, with kernel k, belongs to $C_{p'}(L^2(M;\mathscr{H}))$, and*

$$\|K\|_{p'} \leqslant \|k\|_{X_p(C_{p'})}. \tag{7}$$

Proof. For brevity, we use the notation E_p for $X_p(C_{p'}(\mathscr{H}))$. First, let $p = 2$. The lattice X_2 is just $L^{2,2}(M \times M)$, that is $L^2(M \times M)$, and the space E_2 is just $L^2(M \times M; C_2(\mathscr{H}))$. Now it is fairly well known that the map $k \mapsto K$ given by formula (1) is an isometry of $L^2(M \times M; C_2(\mathscr{H}))$ onto $C_2(L^2(M;\mathscr{H}))$. Thus inequality (7) holds, with equality, when $p = 2$.

Now let $p = 1$. The proof in this case is a modification of a standard argument; we include it for completeness. Let $k \in E_1$, and let $f \in L^2(M;\mathscr{H})$. Fix a number λ such that $\lambda > \|k\|_{E_1}$; by the definition of the space E_1, there exist non-negative, measurable, functions g and h on $M \times M$ such that $\|g\|_{1,\infty} = \|h^*\|_{1,\infty} = 1$, and

$$\|k(x,y)\|_{C_\infty(\mathscr{H})} \leqslant \lambda g(x,y)^{\frac{1}{2}} h(x,y)^{\frac{1}{2}},$$

for all x and y in M. Let

$$l(x) = \int_M \|k(x,y)\cdot f(y)\|_{\mathscr{H}}\, dy.$$

Since $C_\infty(\mathscr{H}) = \mathscr{B}(\mathscr{H})$,

$$\begin{aligned} l(x) &\leqslant \int_M \|k(x,y)\|_{C_\infty(\mathscr{H})}\|f(y)\|_{\mathscr{H}}\, dy \\ &\leqslant \int_M \lambda h(x,y)^{\frac{1}{2}} g(x,y)^{\frac{1}{2}} \|f(y)\|_{\mathscr{H}}\, dy \\ &\leqslant \lambda \left[\int_M h(x,y)\, dy\right]^{\frac{1}{2}} \left[\int_M g(x,y)\, \|f(y)\|_{\mathscr{H}}^2\, dy\right]^{\frac{1}{2}} \end{aligned}$$

by the Schwarz inequality. Now, for almost all x,

$$\int_M h(x,y)\, dy \leqslant \|h^*\|_{1,\infty} = 1.$$

Thus

$$\begin{aligned} \int_M l(x)^2\, dx &\leqslant \lambda^2 \int_M \left[\int_M g(x,y)\, \|f(y)\|_{\mathscr{H}}^2\, dy\right] dx \\ &= \lambda^2 \int_M \|f(y)\|_{\mathscr{H}}^2 \left[\int_M g(x,y)\, dx\right] dy, \end{aligned}$$

by Fubini's theorem. Again, for almost all y,

$$\int_M g(x, y)\, dx \leqslant \|g\|_{1,\infty} = 1,$$

so that $\|l\|_2 \leqslant \lambda \|f\|_{L^2(M,\mathscr{H})}$. In particular, $l(x) < \infty$ for almost all x, and the integral (1) defining $Kf(x)$ converges, in the norm of $\mathscr{H}$, for almost all x; moreover

$$\|Kf\|_{L^2(M,\mathscr{H})} \leqslant \|l\|_2 \leqslant \lambda \|f\|_{L^2(M,\mathscr{H})}.$$

That is, $K \in \mathscr{B}(L^2(M;\mathscr{H}))$, and $\|K\|_\infty \leqslant \lambda$. Now λ is any number greater than $\|k\|_{E_1}$. Therefore,

$$\|K\|_\infty \leqslant \|k\|_{E_1},$$

and inequality (7) holds when $p = 1$.

To complete the proof, we use the complex interpolation method, as summarized in Section 2. Whenever we write $A = B$, for Banach spaces A and B, we mean that the spaces coincide, *with equality of norms*. Now let $1 < p < 2$, and let $s = 2/p'$. Then $0 < s < 1$, and

$$\frac{1}{p} = \frac{1-s}{1} + \frac{s}{2}.$$

We claim that $E_p = [E_1, E_2]_s$. First observe that

$$\frac{1}{p'} = \frac{1-s}{\infty} + \frac{s}{2},$$

so that $C_{p'}(\mathscr{H}) = [C_\infty(\mathscr{H}), C_2(\mathscr{H})]_s$. Next, by some applications of Hölder's inequality,

$$L^{p,p'}(M \times M) = [L^{1,\infty}(M \times M)]^{1-s}[L^{2,2}(M \times M)]^s;$$

this can also be deduced from Theorem 0 and the fact, implicit in [1], that

$$L^{p,p'}(M \times M) = [L^{1,\infty}(M \times M), L^{2,2}(M \times M)]_s.$$

Finally, it is trivial that, if A, B, C and D are all Banach lattices on $M \times M$, then

$$[A^{1-s} B^s]^{\frac{1}{2}} [C^{1-s} D^s]^{\frac{1}{2}} = [A^{\frac{1}{2}} C^{\frac{1}{2}}]^{1-s}[B^{\frac{1}{2}} D^{\frac{1}{2}}]^s.$$

Therefore, $X_p = X_1{}^{1-s} X_2{}^s$. Now the lattice $X_2 = L^2(M \times M)$ has the dominated convergence property. Therefore Theorem 0 applies and yields that $E_p = [E_1, E_2]_s$, as claimed.

We also have that

$$C_{p'}(L^2(M;\mathscr{H})) = [C_\infty(L^2(M;\mathscr{H})), C_2(L^2(M;\mathscr{H}))]_s,$$

and we have shown that the map $k \mapsto K$ is bounded from E_1 to $C_\infty(L^2(M;\mathscr{H}))$, and from E_2 to $C_2(L^2(M;\mathscr{H}))$, with norm at most 1 in both cases. By interpolation, this map is also bounded from E_p to $C_{p'}(L^2(M;\mathscr{H}))$, with norm at most 1. This completes the proof of the theorem.

COROLLARY 2. *If* $1 \leqslant p \leqslant 2$, *then*

$$\|K\|_{p'} \leqslant (\|k\|_{p',p,p'})^{\frac{1}{2}}(\|k^*\|_{p',p,p'})^{\frac{1}{2}}. \tag{8}$$

Proof. There is nothing to prove if the right side of the inequality is infinite. If the right side is finite, then $k \in X_p(C_{p'})$, and

$$\|k\|_{X_p(C_{p'})} \leqslant (\|k\|_{p',\,p,\,p'})^{\frac{1}{2}}(\|k^*\|_{p',\,p,\,p'})^{\frac{1}{2}}. \tag{9}$$

Inequality (8) then follows from equalities (7) and (9). This completes the proof of the corollary.

Various simplifications occur when the space $\mathscr{H}$ has dimension 1. The spaces $C_q(\mathscr{H})$, and their norms, all coincide, and $X_p(C_{p'})$ is just X_p. Since $\|k\|_{q,\,r,\,s}$ is independent of the first index q, it is appropriate to suppress this index. Then Corollary 2 reads

$$\|K\|_{p'} \leqslant (\|k\|_{p,\,p'}\, \|k^*\|_{p,\,p'})^{\frac{1}{2}}, \tag{10}$$

whenever $1 \leqslant p \leqslant 2$. This inequality was proved in a different way in [**4**; Theorem 1].

We indicate briefly how inequalities (7), (8) and (10) generalize the classical Hausdorff–Young inequality. Let M be the unit circle, with measure $d\theta/2\pi$, and let g be an integrable function on the circle, with Fourier coefficients $\{\hat{g}(n)\}_{n=-\infty}^{\infty}$. Then let k_g be the function on $M \times M$ given by $k_g(x, y) = g(x-y)$ for all x and y in M, and let K_g be the integral operator associated with the kernel k_g. Now K_g is just the convolution operator $f \mapsto g * f$, and it turns out that $\|K_g\|_r = \|\hat{g}\|_r$ for all indices r. On the other hand, $\|k_g\|_{p,\,r} = \|g\|_p$ for all second indices r; also, $\|(k_g)^*\|_{p,\,r} = \|g\|_p$ for all r. In particular, $(\|k_g\|_{p,\,p'}\|(k_g)^*\|_{p,\,p'})^{\frac{1}{2}}$ is just $\|g\|_p$, and inequality (10) implies that $\|\hat{g}\|_{p'} \leqslant \|g\|_p$, whenever $1 \leqslant p \leqslant 2$. This is the Hausdorff–Young inequality for the circle; the corresponding inequality for Fourier transforms on the real line, however, is not a special case of any of the inequalities proved in this paper.

Finally, we remark that inequalities (7), (8) and (10) are sharp, and that, in general, the indices appearing in these inequalities are best possible. Indeed, if k is the characteristic function of a measurable rectangle of finite, positive measure, then equality holds in relation (10); in fact

$$\|K\|_\infty = (\|k\|_{p,\,p'}\|k^*\|_{p,\,p'})^{\frac{1}{2}}$$

in this case. It can then be shown that, if $\mathscr{H}$ and M are as in Theorem 1, then there is a non-trivial kernel k, in $X_p(C_{p'}(\mathscr{H}))$, for which equality holds in relation (8) and therefore in relation (7).

The indices appearing in relations (7), (8) and (10) are best possible in the following sense. Let q, r, s and t be indices with the property that, whenever M and $\mathscr{H}$ are as in Theorem 1, then there exists a constant D, possibly depending on M and $\mathscr{H}$, such that

$$\|K\|_t \leqslant D(\|k\|_{q,\,r,\,s}\|k^*\|_{q,\,r,\,s})^{\frac{1}{2}}, \tag{11}$$

for all $\mathscr{B}(\mathscr{H})$-valued kernels k on $M \times M$. Then

$$s = r', \qquad \text{and} \qquad t \geqslant \max(2, q, s).$$

Conversely, if the indices satisfy these conditions, then it follows from Corollary 2 that inequality (11) holds with $D = 1$; moreover, the resulting inequality is sharp. We omit the proofs of these facts.

Note added in proof. After this paper was accepted, we learned of the paper " On the membership of integral operators in classes S_p for $p \geqslant 2$ ", by G. E. Karadzhov (Problemy Matematicheskogo Analiza, No. 3: Integral'nye i Differentsial'nye Operatory. Differentsial'nye Uravneniya, pp. 28–33, 1972; English translation in

J. Soviet Math., 1 (1973), 200–204). Karadzhov deals only with scalar-valued kernels, and uses the real interpolation method to derive generalizations of our inequality (10). In particular, inequality (10) is a special case of the final inequality in Karadzhov's paper, except that, in the latter paper, an unspecified constant occurs in the right side of the inequality. This constant can be chosen to be independent of the underlying measure spaces, however, and this fact can be used as in [**4**; Lemma 5] to show that the constant can in fact be chosen to be 1. There seem to be serious difficulties in deriving our Theorem 1, even for scalar-valued kernels, by Karadzhov's method. On the other hand, the methods of the present paper can be used to generalize Karadzhov's results to the context of operator-valued kernels.

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