Title
The algebraic Brauer-Manin obstruction on Chatelet surfaces, degree 4 del Pezzo surfaces, and Enriques surfaces

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Publication Date
2010

Peer reviewed|Thesis/dissertation

The algebraic Brauer-Manin obstruction on Châtelet surfaces, degree 4 del Pezzo surfaces, and Enriques surfaces

by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2010

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Abstract

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We construct infinitely many Châtelet surfaces, degree 4 del Pezzo surfaces, and Enriques surfaces that are everywhere locally solvable, but have no global rational points. The lack of rational points on these surfaces is explained by an algebraic Brauer-Manin obstruction. The Enriques surfaces arise as quotients of certain K3 surfaces that are ramified double covers of a degree 4 del Pezzo surface with no rational points. We also construct an algebraic family of Châtelet surfaces over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ such that exactly 1 $\mathbb{Q}$-fiber has no $\mathbb{Q}$-points. This example is in stark contrast to the philosophy "geometry controls arithmetic".

To Tony,
the best mathematical older sibling I could ever ask for.

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Acknowledgments

It is an honor to be able to thank my advisor, Bjorn Poonen, for the immense amounts of guidance and support he has provided during my graduate studies. He has been integral in this thesis and in my growth as a mathematician; his mathematical insight, patience, and kindness are continually inspiring.

I also would like to give a special thanks to Anthony Várilly-Alvarado. His tireless mentoring, teaching, and constant friendship have helped me enormously and I would not be where I am today without him.

I have been extremely fortunate to have had numerous excellent mentors during my graduate career. I thank Kristin Lauter, Diane MacLagan, and Ronald van Luijk for the opportunities and support that they have provided me. I thank Matt Baker, Jon Hanke, Martin Olsson, Bernd Sturmfels, and Ravi Vakil for the many pieces of advice they have given me at various points during my graduate career.

I thank Kiril Datchev and Daniel Erman for being incredible friends and being there whenever I needed them.

I thank the members of Unbounded Representation and the Noetherian Ring for fostering a friendly and supportive community at Berkeley. It has been a pleasure and an honor to work with them.

I thank my parents and my sisters for their constant love and encouragement and I thank Cole for always being by my side, through everything.

Chapter 1

Introduction

Throughout this thesis, we are interested in determining whether $X(k) \neq \emptyset$ for a given variety X over a global field k. Since $X(k) \hookrightarrow X(\mathbb{A}_k)$, it is necessary that $X(\mathbb{A}_k) \neq \emptyset$. If the converse holds for all varieties in some set $\mathcal{S}$, then we say that $\mathcal{S}$ satisfies the Hasse principle. There are a smattering of classes of varieties that satisfy the Hasse principle, for example genus 0 curves, quadric hypersurfaces in projective space, and cubic hypersurfaces in at least 9-dimensional projective space over $\mathbb{Q}$ [Sko01, Thm 5.1.1]. However, as one may have gathered from this list, it is expected that the varieties that satisfy the Hasse principle are scarce.

If a class of varieties does not satisfy the Hasse principle, then we would like to understand the obstructions that account for this failure. The guiding philosophy of this research is summarized by the statement "Geometry controls arithmetic". That is to say, the geometric properties of a class of varieties should determine whether this class satisfies the Hasse principle, and if not, the complexity of the obstructions that account for its failure.

The simplest type of obstruction is the algebraic Brauer-Manin obstruction. It is conjectured that for curves and rational surfaces the algebraic Brauer-Manin obstruction explains all failures of the Hasse principle [CT03, Poo06]. In Chapters 4 and 5, we construct many examples of two kinds of rational surfaces, namely Châtelet surfaces and del Pezzo surfaces of degree 4, that have an algebraic Brauer-Manin obstruction to the Hasse principle.

One expects that this obstruction is no longer sufficient to explain all failures of the Hasse principle in the case of Enriques surfaces. However, there are no Enriques surfaces that are known to have no rational points and no algebraic Brauer-Manin obstruction. In Chapter 6 we construct Enriques surfaces with an étale-Brauer obstruction from del Pezzo surfaces of degree 4 and explain how this obstruction actually arises from an algebraic Brauer element.

In Chapter 7, we show that the "geometry controls arithmetic" philosophy does not apply to all arithmetic properties, in particular the property of a variety having a $\mathbb{Q}$-point. We construct an algebraic family of Châtelet surfaces over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ such that exactly 1 fiber has no rational point.

1.1 Notation

Throughout, k will denote a field. If additional hypotheses on k are needed, they will be stated at the beginning of the chapter or section. Let k^{sep} denote a fixed separable closure of k, let $\overline{k}$ denote a fixed algebraic closure, and let G_k denote the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. For a scheme X over k, let $X^{\mathrm{sep}} := X \times_{\operatorname{Spec} k} (\operatorname{Spec} k^{\mathrm{sep}})$ and let $\overline{X} := X \times_{\operatorname{Spec} k} \left(\operatorname{Spec} \overline{k}\right)$. If k is a global field, let $\mathbb{A}_k$ denote the adèle ring of k, let $\mathcal{O}_k$ denote the ring of integers, and, for any finite set of places S, let $\mathcal{O}_{k,S}$ denote the ring of S-integers. For a place v of a global field k, let k_v denote the completion, let $\mathcal{O}_v$ denote the ring of integers of k_v, and let $\mathbb{F}_v$ denote the residue field.

By k-variety we mean a separated scheme of finite type over k and by nice k-variety we mean a smooth projective geometrically integral k-variety. We will sometimes omit mention of k if it is obvious from context. By curve and surface we mean a smooth projective variety of dimension 1 or 2 respectively (unless otherwise stated). For a scheme X over k, let $\mathrm{Pic}(X)$ denote the Picard group and, if X is a smooth k-variety, let K_X denote the canonical divisor. For an integral scheme, let $\mathbf{k}(X)$ denote the function field, and for a point $x \in X$, let $\kappa(x)$ denote the residue field.

For any separable quadratic extension L/k and element $b \in k^{\times}$, let $(L/k, b)_2$ denote the rank 4 cyclic k-algebra

$$L \oplus L \cdot y, \text{ where } y^2 = b \text{ and } \ell y = y\sigma(\ell) \text{ for all } \ell \in L,$$

where σ is the nontrivial element in $\mathrm{Gal}(L/k)$. Recall that $(L/k, b)_2$ is trivial in $\operatorname{Br} k$ if and only if $b \in \mathrm{N}_{L/k}(L)$. If the characteristic of k is different than 2, then there is some $a \in k^{\times}$ such that $L = k(\sqrt{a})$. In this case we also denote $(L/k, b)_2$ by $(a, b)_2$ and this algebra is isomorphic to the quaternion algebra

$$k \oplus k \cdot i \oplus k \cdot j \oplus k \cdot ij, \quad \text{where } i^2 = a, j^2 = b, \text{ and } ij = -ji.$$

Chapter 2

Surfaces

In this thesis we focus on four particular kinds of surfaces: Châtelet surfaces, degree 4 del Pezzo surfaces, Enriques surfaces, and K3 surfaces. In order to give a sense of how the surfaces fit together, we give a brief overview of the classification of surfaces. We also review a construction of smooth proper models of certain conic bundles.

2.1 Overview of the classification of surfaces

The definitions and theorems in this chapter can be found in various sources. See, for example, [Bea96, IS96] in the characteristic 0 case, and [Mum69, BM77, BM76] in the positive characteristic case.

Definition 2.1.1. *For a regular projective variety X, let ϕ_{nK} denote the rational map associated to the complete linear system $|nK_X|$. The* Kodaira dimension *of X*

$$\kappa(X) = \begin{cases} -\infty & \text{if } |nK_X| = \emptyset \text{ for all } n \in \mathbb{Z}_{>0}, \\ \max_{n\in\mathbb{Z}_{>0}} \dim \phi_{nK}(X) & \text{otherwise.} \end{cases}$$

The Kodaira dimension is a coarse measure of the geometric complexity of a variety. For instance, curves with Kodaira dimension $-\infty, 0$, or 1 correspond to curves with genus $0, 1$ or ≥ 2 respectively. For surfaces, we have the following correspondence.

Theorem 2.1.2 (Classification of surfaces)**.** *Let X be a k-*minimal surface*, that is to say, X has the property that any birational morphism $X \to Y$ where Y is regular is an isomorphism. Then exactly one of the following holds:*

1. *There exists an integral curve C on X such that $K_X \cdot C < 0$. In this case X is a geometrically ruled surface or a geometrically rational surface and $\kappa(X) = -\infty$.*

2. *The canonical divisor K_X is numerically equivalent to 0, or, equivalently, $12K_X$ is linearly equivalent to 0. Either condition implies that $\kappa(X) = 0$.*

3. *$\kappa(X) = 1$ and X is either geometrically elliptic or geometrically quasi-elliptic. This means that there exists a morphism $\overline{X} \to C$ where the generic fiber is a smooth genus 1 curve (elliptic), or a singular rational curve with arithmetic genus 1 (quasi-elliptic). The latter case only occurs in characteristic 2 or 3.*

4. *X is a surface of general type and $\kappa(X) = 2$.*

The surfaces in classes (1) and (2) have been studied and classified further. The geometric genus, $g = \mathrm{H}^0(X, K_X)$, of a geometrically ruled surface agrees with the geometric genus of the curve C that gives a ruling of $\overline{X}$; that is to say, C is the unique curve such that there is a smooth morphism $\overline{X} \to C$ with the property that the generic fiber is rational. Geometrically ruled surfaces of genus 0 are geometrically rational and thus fall under the following classification theorem of Iskovskikh.

Theorem 2.1.3 ([Isk79]). *Let X/k be a geometrically rational surface. Then X is k-birational to a del Pezzo surface or a conic bundle.*

We will give a construction of a particular type of conic bundle in §2.2. For more information on rational varieties, see [MT86].

In order to further classify surfaces of Kodaira dimension 0, we need to examine additional numerical invariants, namely Betti numbers and the topological Euler characteristic.

Definition 2.1.4. *The i^{th} Betti number is denoted b_i. If $\operatorname{char} k = 0$ then b_i is the dimension of the $\mathbb{C}$-vector space given by singular cohomology $\mathrm{H}^1(X^{\mathrm{an}}, \mathbb{C})$. If $\operatorname{char} k = p > 0$, then b_i is the dimension of the $\mathbb{Q}_\ell$-vector space given by étale cohomology $\mathrm{H}^i_{\mathrm{et}}(X, \mathbb{Q}_\ell)$ where ℓ is a prime different from p. In either case, the topological Euler characteristic is $\chi_{\mathrm{top}}(X) = \sum_i (-1)^i b_i$.*

Theorem 2.1.5. *Let X be a nice surface of Kodaira dimension 0. Then X satisfies exactly one of the following.*

1. *$b_1 = 0, b_2 = 22, \chi_{\mathrm{top}}(X) = 24$ and $\chi(\mathcal{O}_X) = 2$. In this case $K_X \sim 0$ and X is called a K3 surface.*

2. *$b_1 = 0, b_2 = 10, \chi_{\mathrm{top}}(X) = 12$ and $\chi(\mathcal{O}_X) = 1$. This implies that $2K_X \sim 0$, and if the characteristic of k is different from 2, $K_X \nsim 0$. In this case, we call X an Enriques surface.*

3. *$b_1 = 4, b_2 = 6, \chi_{\mathrm{top}}(X) = 0$ and $\chi(\mathcal{O}_X) = 0$. In this case X is an abelian surface.*

4. *$b_1 = 2, b_2 = 2, \chi_{\mathrm{top}}(X) = 0$ and $\chi(\mathcal{O}_X) = 0$. In this case X is a bielliptic surface.*

For further information on these surfaces, there is a wealth of references to choose from, such as [BHPVdV04, CD89, IS96, Mum08]. In §2.3, we review some constructions and properties of Enriques surfaces.

2.2 Conic bundles

Let $\mathcal{E}$ be a rank 3 vector sheaf on a k-variety B. A conic bundle C over B is the zero locus in $\mathbb{P}\mathcal{E}$ of a nowhere vanishing zero section $s \in \Gamma(\mathbb{P}\mathcal{E}, \mathrm{Sym}^2(\mathcal{E}))$. A diagonal conic bundle is a conic bundle where $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$ and $s = s_1 + s_2 + s_3, s_i \in \Gamma(\mathbb{P}\mathcal{E}, \mathcal{L}_i^{\otimes 2})$.

In this thesis we focus on one particular class of conic bundles: Châtelet surfaces. Over a field of characteristic different from 2, Châtelet surfaces are defined to be smooth projective models of an affine hypersurface

$$y^2 - az^2 = P(x),$$

where $a \in k^\times$ and $P(x)$ is a separable polynomial of degree 3 or 4. We can construct these models in the following way.

Let $a \in k^\times$, and let $P(x) \in k[x]$ be a separable polynomial of degree 3 or 4. Consider the diagonal conic bundle X given by $B = \mathbb{P}^1, \mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(2), s_1 = 1, s_2 = -a, s_3 = -w^4 P(x/w)$. This smooth conic bundle contains the affine hypersurface $y^2 - az^2 = P(x) \subset \mathbb{A}^3$ as an open subscheme and so we say that X is the Châtelet surface given by

$$y^2 - az^2 = P(x).$$

Note that since $P(x)$ is not identically zero, X is an integral surface.

This construction fails in characteristic 2 due to the inseparability of $y^2 - az^2$. Thus to define Châtelet surfaces in characteristic 2, we replace $y^2 - az^2$ with its Artin-Schreier analogue $y^2 + yz + az^2$. We can construct these models as a conic bundle where $\mathcal{E} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$ and $s = s_1 - s_2$ where s_1 is a global section of $\mathrm{Sym}^2(\mathcal{O} \oplus \mathcal{O})$ and s_2 is a global section of $\mathcal{O}(2)^{\otimes 2} = \mathcal{O}(4)$. Take $a \in k$ and $P(x)$ a separable polynomial over k of degree 3 or 4. If $s_1 = y^2 + yz + az^2$ and $s_2 = w^4 P(x/w)$, then X contains the affine variety defined by $y^2 + yz + az^2 = P(x)$ as an open subset.

Lemma 2.2.1. *X is smooth over k.*

Proof. Let π be the morphism $X \to \mathbb{P}^1$. Since $\mathbb{P}^1$ is smooth over k, X is smooth over k at all points where π is smooth. Therefore it remains to check smoothness at the singular points in the fibers where $w^4 P(x/w) = 0$. Except when $P(x)$ has degree 3, all of these singular points are contained in the subscheme isomorphic to $y^2 + yz + az^2 = P(x) \subset \mathbb{A}^3$. Therefore, we can use the Jacobian criterion to show smoothness. If $P(x)$ is degree 3, then we can make a change of variables on $\mathbb{P}^1$ to interchange 0 and ∞ and then use the Jacobian criterion. □

We say that X is the Châtelet surface given by

$$y^2 + yz + az^2 = P(x).$$

As above, since $P(x)$ is not identically zero, X is integral.

2.3 Enriques surfaces

In this section, we assume that the characteristic of k is different from 2. Over fields of characteristic 2, non-classical Enriques surfaces exist. For more details on these non-classical surfaces see [CD89].

We are interested in the study of Enriques surfaces; however, as the following theorem shows, the study of Enriques surfaces is intimately related to the study of certain K3 surfaces. Recall that an invertible sheaf $\mathcal{L}$ of order 2 on a variety X corresponds to an étale double cover $f\colon Y \to X$ such that $f^*\mathcal{L} \cong \mathcal{O}_Y$.

Theorem 2.3.1. *Let X be an Enriques surface and $f\colon Y \to X$ the étale double cover corresponding to $K_X \in (\operatorname{Pic} X)[2]$. Then Y is a K3 surface.*

Conversely, the quotient of a K3 surface by a fixed-point free involution is an Enriques surface.

Proof. The first statement is Proposition 1.3.1 of [CD89]. For the converse, let Y be a K3 surface, let X be the quotient of Y by the fixed-point free involution, and let $\pi\colon Y \to X$ be the natural map. Since $\pi^*K_X = K_Y = \mathcal{O}_Y$, K_X is numerically trivial and $\kappa(X) = 0$. We also have that $\chi_{\text{top}}(X) = \frac{1}{2}\chi_{\text{top}}(Y) = 12$. Thus, by Theorem 2.1.5, we have completed the proof. □

Example 2.3.2. *Let Y be the variety*

$$V\left(\langle P_i(s,t,u) - Q_i(x,y,z) : i = 0,1,2\rangle\right) \subset \mathbb{P}^5,$$

where $\deg P_i = \deg Q_i = 2$ for all i and the polynomials P_i, Q_i are generic. Then one can show that Y is a K3 surface. Let ι denote the involution on $\mathbb{P}^5$

$$\iota\colon (s:t:u:x:y:z) \mapsto (-s:-t:-u:x:y:z).$$

Due to our genericity assumption, $\iota|_Y$ has no fixed points so $X := Y/\iota$ is an Enriques surface.

While the set of complete intersections of 3 quadrics is not dense in the moduli space of K3 surfaces, a theorem proved independently by Cossec and Verra shows that, from the point of view of Enriques surfaces, it suffices to consider these examples.

Theorem 2.3.3 ([Cos83, Ver83])**.** *Let Y be a K3 surface over an algebraically closed field with a fixed point free involution ι_Y. Then there exists a regular map of degree 1, $\psi\colon Y \to \mathbb{P}^5$ such that $\psi(Y)$ is the complete intersection of 3 quadrics given in Example 2.3.2. In addition, $\psi \circ \iota_Y = \iota$, with ι as in the example above, and $\psi(Y)$ does not intersect the fixed locus of ι.*

Chapter 3

Rational points

In this chapter, we review some possible obstructions to the Hasse principle, namely the Brauer-Manin obstruction (§3.3.1), the descent obstruction (§3.3.2), and the étale-Brauer obstruction (§3.3.3). In order to define these obstructions, we first review some facts about the Brauer group (§3.1) and torsors (§3.2). Most of the information in this section can be found in [Sko01]; we give an alternate source when needed.

3.1 The Brauer group

The (cohomological) Brauer group of a scheme X, denoted $\operatorname{Br} X$, is the étale cohomology group $\mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$. Note that if $X = \operatorname{Spec} k$ then this agrees with the usual notion of $\operatorname{Br} k$, namely $\mathrm{H}^2(G_k, k^{\text{sep}\,\times})$. If X is a regular quasi-projective variety over a field, then the Brauer group can be identified with the group of Azumaya algebras up to equivalence; we explain this further in §3.1.1.

We say that an element $\mathcal{A} \in \operatorname{Br} X$ is algebraic if

$$\mathcal{A} \in \ker(\operatorname{Br} X \to \operatorname{Br} X^{\text{sep}}),$$

and transcendental otherwise. The subgroup of algebraic elements is denoted $\operatorname{Br}_1 X$. The Hochschild-Serre spectral sequence relates the algebraic part of the Brauer group to Galois cohomology by the following long exact sequence:

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} X^{\text{sep}})^{G_k} \to \operatorname{Br} k \to \operatorname{Br}_1 X \xrightarrow{r} \mathrm{H}^1(G_k, \operatorname{Pic} X^{\text{sep}}) \to \mathrm{H}^3(k, \mathbb{G}_m).$$

If k is a global field, then $\mathrm{H}^3(k, \mathbb{G}_m) = 0$ [NSW08, 8.3.11(iv), 8.3.17] so

$$\frac{\operatorname{Br}_1 X}{\operatorname{Br}_0 X} \xrightarrow{\sim} \mathrm{H}^1(G_k, \operatorname{Pic} X^{\text{sep}}), \text{ where } \operatorname{Br}_0 X := \operatorname{im} \operatorname{Br} k.$$

This map is unfortunately very difficult to explicitly compute and even harder to explicitly invert. Nevertheless, this isomorphism is usually the main tool used when studying the algebraic part of the Brauer group.

If X is a regular noetherian integral scheme, the purity theorem gives a method of testing whether an element of $\operatorname{Br}\mathbf{k}(X)$ extends to an element of $\operatorname{Br} X$.

Theorem 3.1.1 (Gabber, [Fuj02]). *Let X be a regular noetherian integral scheme. Then the following sequence is exact*

$$0 \to \operatorname{Br} X \to \operatorname{Br}\mathbf{k}(X) \to \bigoplus_{Y \text{ codim } 1} \mathrm{H}^1(\kappa(Y), \mathbb{Q}/\mathbb{Z})',$$

where $\mathrm{H}^1(\kappa(Y), \mathbb{Q}/\mathbb{Z})'$ means that we ignore the p-primary part if $\kappa(Y)$ is imperfect of characteristic p.

In particular, this theorem implies that if $Z \subseteq X$ has codimension greater than or equal to 2 then $\operatorname{Br} X \cong \operatorname{Br} X \setminus Z$.

3.1.1 Azumaya algebras

The Brauer group of a field k can be defined as the group of central simple algebras over k up to equivalence. The notion of a central simple algebra over a field is generalized to an arbitrary base scheme by Azumaya algebras.

Definition 3.1.2. *An Azumaya algebra $\mathcal{A}$ is an $\mathcal{O}_X$-algebra $\mathcal{A}$ that is coherent as an $\mathcal{O}_X$-module and that the fiber $\mathcal{A} \otimes_{\mathcal{O}_X} \kappa(x)$ is a central simple algebra over $\kappa(x)$ for all points $x \in X$.*

We say that two Azumaya algebras $\mathcal{A}, \mathcal{A}'$ are similar ($\mathcal{A} \sim \mathcal{A}'$) if there exist locally free coherent $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{E}'$ such that

$$\mathcal{A} \otimes_{\mathcal{O}_X} \mathbf{End}_{\mathcal{O}_X}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} \mathbf{End}_{\mathcal{O}_X}(\mathcal{E}')$$

Definition 3.1.3. *The Azumaya Brauer group of X, denoted $\operatorname{Br}_{\mathrm{Az}} X$, is the set of Azumaya algebras up to similarity. It is a group under tensor product.*

Grothendieck showed that the Azumaya Brauer group always injects into the cohomological Brauer group [Gro68a, Eqn. 2.1]. With additional assumptions on X, we have the following stronger relationship.

Theorem 3.1.4 (Gabber, de Jong [dJ]). *If X has an ample invertible sheaf, then the natural map $\operatorname{Br}_{\mathrm{Az}} X \hookrightarrow \operatorname{Br} X$ induces an isomorphism*

$$\operatorname{Br}_{\mathrm{Az}} X \cong (\operatorname{Br} X)_{\mathrm{tors}}.$$

Together Theorems 3.1.1 and 3.1.4 imply that if X is a quasi-projective variety over a field, then

$$\operatorname{Br}_{\mathrm{Az}} X \cong (\operatorname{Br} X)_{\mathrm{tors}} = \operatorname{Br} X.$$

3.2 Torsors

Let G be an fppf group scheme over a scheme S.

Definition 3.2.1. *A (right) S-torsor under G is an fppf S-scheme X equipped with a right G-action such that the morphism*

$$\begin{array}{rcl} X \times_S G & \to & X \times_S X \\ (x, g) & \mapsto & (x, xg) \end{array}$$

is an isomorphism.

In this thesis, we are concerned with the case where G is a linear algebraic group over k. In this case, since G is affine, isomorphism classes of S-torsors under G are classified by the pointed set defined by Cech cohomology $\check{\mathrm{H}}^1_{\mathrm{fppf}}(S, G)$[Sko01, §2.2].

Lemma 3.2.2. [Sko01, Lemma 2.2.3] *Let X be a right S-torsor under G and F be an affine S-scheme equipped with a left action of G. Then the quotient of $X \times_S F$ by the action of G given by $(x, f) \mapsto (xg^{-1}, gf)$ exists as an affine S-scheme Y; in particular, there exists a morphism of S-schemes $X \times_S F \to Y$ whose geometric fibers are orbits of G.*

Definition 3.2.3. *The scheme Y is called the contracted product of X and F with respect to G and it is denoted $X \times_S^G F$ or $X \times^G F$. It is also referred to as the twist of F by X and denoted F^X.*

3.3 Obstructions to the Hasse principle

3.3.1 The Brauer-Manin obstruction

For any scheme T and point $x \in X(T)$, the functoriality of the Brauer group induces a map

$$\operatorname{Br} X \to \operatorname{Br} T, \text{ denoted } \mathcal{A} \mapsto \operatorname{ev}_{\mathcal{A}}(x).$$

For a fixed element $\mathcal{A} \in \operatorname{Br} X$, using the above induced map and class field theory, we obtain

$$\begin{tikzcd} & X(k) \arrow[r] \arrow[d, "\operatorname{ev}_{\mathcal{A}}"] & X(\mathbb{A}_k) \arrow[d, "\operatorname{ev}_{\mathcal{A}}"] \arrow[dr, "\varphi_{\mathcal{A}}"] & & \\ 0 \arrow[r] & \operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r, "\sum \operatorname{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0 \end{tikzcd} \tag{3.3.1}$$

Remark 3.3.1. A priori, $\operatorname{ev}_{\mathcal{A}}(X(\mathbb{A}_k)) \subseteq \prod_v \operatorname{Br} k_v$. However, one can show that for any $\{P_v\} \in X(\mathbb{A}_k), \operatorname{ev}_{\mathcal{A}}(P_v) = 0$ for almost all v, so the image actually lies in the direct sum.

The diagram shows that $X(k) \subseteq \varphi_{\mathcal{A}}^{-1}(0)$ for any $\mathcal{A} \in \operatorname{Br} X$. In particular

$$X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}_1} := \bigcap_{\mathcal{A} \in \mathrm{Br}_1 X} \varphi_{\mathcal{A}}^{-1}(0), \quad \text{and } X(k) \subseteq X(\mathbb{A}_k)^{\mathrm{Br}} := \bigcap_{\mathcal{A} \in \mathrm{Br}\, X} \varphi_{\mathcal{A}}^{-1}(0).$$

We say there is a Brauer-Manin obstruction to the Hasse principle if

$$X(\mathbb{A}_k) \neq \emptyset \text{ and } X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset.$$

The obstruction is termed algebraic if, in addition, $X(\mathbb{A}_k)^{\mathrm{Br}_1} = \emptyset$.

The continuity of $\mathrm{ev}_{\mathcal{A}}$ is often needed when computing Brauer-Manin obstructions. While this result is well-known, it is difficult to find in the literature so we give a proof for the reader's convenience.

Lemma 3.3.2. *Let k_v be a local field and let V be a smooth projective scheme over k_v. For any $[\mathcal{B}] \in \operatorname{Br} V$,*

$$\mathrm{ev}_{\mathcal{B}} : V(k_v) \to \operatorname{Br} k_v$$

is continuous for the discrete topology on $\operatorname{Br} k_v$.

Proof. To prove continuity, it suffices to show that $\mathrm{ev}_{\mathcal{B}}^{-1}(\mathcal{A})$ is open for any $\mathcal{A} \in \operatorname{Br} k_v$. By replacing $[\mathcal{B}]$ with $[\mathcal{B}] - [\mathcal{A}]$, we reduce to showing that $\mathrm{ev}_{\mathcal{B}}^{-1}(0)$ is open. Since V is a smooth projective scheme over a field, $\mathrm{Br}_{\mathrm{Az}} V = \operatorname{Br} V$, so without loss of generality we can assume that $\mathcal{B}$ is an Azumaya algebra.

Fix a representative $\mathcal{B}$ of the element $[\mathcal{B}] \in \mathrm{Br}_{\mathrm{Az}} V$. Let n^2 denote the rank of $\mathcal{B}$ and let $f_{\mathcal{B}} \colon Y_{\mathcal{B}} \to V$ be the PGL_n-torsor associated to $\mathcal{B}$. Then we observe that the set $\mathrm{ev}_{\mathcal{B}}^{-1}(0)$ is equal to $f_{\mathcal{B}}(Y_{\mathcal{B}}(k_v)) \subset V(k_v)$. This set is open by the implicit function theorem [Igu00, Thm. 2.2.1]. □

3.3.2 The descent obstruction

Let G be a linear algebraic group over k. An element $x \in X(k)$ induces a map

$$\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, G) \to \check{\mathrm{H}}^1_{\mathrm{fppf}}(k, G), \text{ denoted } \tau \mapsto \tau(x).$$

For any X-torsor under G, $f \colon Y \to X$, let τ_Y denote the associated element of $\check{\mathrm{H}}^1_{\mathrm{fppf}}(X, G)$. Then we have the following partition of rational points

$$X(k) = \coprod_{\sigma \in \check{\mathrm{H}}^1_{\mathrm{fppf}}(k,G)} \{x \in X(k) : \tau_Y(x) = \sigma\}.$$

Using the twisted torsor construction as explained in 3.2, one can show that [Sko01, §2.2, p. 22]

$$\{x \in X(k) : \tau_Y(x) = \sigma\} = f^{\sigma}(Y^{\sigma}(k)).$$

Thus, we can rewrite the partition as follows.

$$X(k) = \coprod_{[\sigma]\in\check{\mathrm{H}}^1_{\mathrm{fppf}}(k,G)} f^\sigma\left(Y^\sigma\left(k\right)\right)$$

Therefore $X(k)$ is contained in the descent set

$$X(\mathbb{A}_k)^{\mathrm{descent}} := \bigcap \bigcup_{[\sigma]\in\check{\mathrm{H}}^1_{\mathrm{fppf}}(k,G)} f^\sigma\left(Y^\sigma\left(\mathbb{A}_k\right)\right),$$

where the intersection is taken over all linear algebraic groups G and all torsors $f\colon Y \to X$ under G. We say there is a descent obstruction to the Hasse principle if

$$X(\mathbb{A}_k) \neq \emptyset \text{ and } X(\mathbb{A}_k)^{\mathrm{descent}} = \emptyset.$$

We note that if we instead take the intersection over only *connected* linear algebraic groups, the obstruction we obtain is no stronger than the Brauer-Manin obstruction, at least in the case of geometrically integral varieties over a number field [Har02, Thm. 2(2)]. Since an arbitrary linear algebraic group is the extension of a finite étale group by a connected linear algebraic group, this leads us to a study of the étale-Brauer obstruction.

3.3.3 The étale-Brauer obstruction

The following definitions can be found in [Poo08]. Let G be a finite étale group over k. Using the same partition of rational points as described in §3.3.2, we see that $X(k)$ is contained in the étale-Brauer set

$$X(\mathbb{A}_k)^{\mathrm{et,Br}} := \bigcap \bigcup_{[\sigma]\in\mathrm{H}^1_{\mathrm{fppf}}(k,G)} f^\sigma\left(Y^\sigma\left(\mathbb{A}_k\right)^{\mathrm{Br}}\right),$$

where the intersection is taken over all torsors $f\colon Y \to X$ under finite étale groups G. We say there is an étale-Brauer obstruction to the Hasse principle if

$$X(\mathbb{A}_k) \neq \emptyset \text{ and } X(\mathbb{A}_k)^{\mathrm{et,Br}} = \emptyset.$$

Harari's result [Har02, Thm. 2(2)] (mentioned above in §3.3.2) leads us to ask if the étale-Brauer obstruction is at least as strong as the descent obstruction. Demarche has answered this question in the affirmative, for nice varieties over a number field [Dem09]. Under the same assumptions, Skorobogatov showed that the reverse inclusion also holds [Sko09, Cor. 1.2], thereby showing that

$$X(\mathbb{A}_k)^{\mathrm{et,Br}} = X(\mathbb{A}_k)^{\mathrm{descent}}$$

for any nice variety X over a number field k.

Chapter 4

Failure of the Hasse principle for characteristic 2 Châtelet surfaces

Recently Poonen showed that, for any global field k of characteristic different from 2, there exists a Châtelet surface over k that violates the Hasse principle [Poo09, Prop 5.1 and §11]. We extend Poonen's result to characteristic 2 by proving the following.

Theorem 4.0.3. *Let k be a global field of characteristic 2. There exist infinitely many Châtelet surfaces over k with a Brauer-Manin obstruction to the Hasse principle.*

The proof of Theorem 4.0.3 is constructive. The difficulty in the proof lies in finding suitable equations so that the Brauer set is easy to compute and empty.

Interestingly, even though the obstruction for the Châtelet surface is Brauer-Manin, Poonen showed that the existence of such a surface can be used to show that the étale-Brauer obstruction is insufficient to explain all failures of the Hasse principle [Poo08]. While Poonen's paper assumes that the characteristic is different from 2, all of the arguments go through in characteristic 2 by using Theorem 4.0.3 instead of [Poo09, Prop. 5.1 and §11] and replacing any polynomial of the form $by^2 + az^2$ by its Artin-Schreier analogue, $by^2 + byz + az^2$.

4.1 Proof of Theorem 4.0.3

Let k be a global field of characteristic 2. Let κ denote its constant field and let n denote the order of $\kappa^\times$. Choose $\gamma \in \kappa$ such that $T^2 + T + \gamma$ is irreducible in $\kappa[T]$. Fix a prime $\mathfrak{p}$ of k of odd degree and let $S = \{\mathfrak{p}\}$. By class field theory and the Chebotarev density theorem [Neu99, Thm 13.4, p. 545], we can find elements $a, b \in \mathcal{O}_{k,S}$ that generate prime ideals of even and odd degree, respectively, such that $a \equiv \gamma \pmod{b^2\mathcal{O}_{k,S}}$. These conditions imply that $v_\mathfrak{p}(a)$ is even and negative and that $v_\mathfrak{p}(b)$ is odd and negative.

Define

$$\begin{aligned} f(x) &= a^{-4n}bx^2 + x + ab^{-1}, \\ g(x) &= a^{-8n}b^2x^2 + a^{-4n}bx + a^{1-4n} + \gamma. \end{aligned}$$

Note that $g(x) = a^{-4n}bf(x) + \gamma$. Let X be the Châtelet surface given by

$$y^2 + yz + \gamma z^2 = f(x)g(x). \tag{$*$}$$

Lemma 4.1.1 shows that $X(\mathbb{A}_k) \neq \emptyset$, and Lemma 4.1.2 shows that $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. Together, these show that X has a Brauer-Manin obstruction to the Hasse principle.

Lemma 4.1.1. *The Châtelet surface X has a k_v-point for every place v.*

Proof. Suppose that $v = v_a$. Since a generates a prime of even degree, the left-hand side of $(*)$ factors in $k_v[y, z]$. Therefore, there is a solution over k_v.

Now suppose that $v \neq v_a$. Since $y^2 + yz + az^2$ is a norm form for an unramified extension of k_v for all v, in order to prove the existence of a k_v-point, it suffices to find an $x \in k_v$ such that the valuation of the right-hand side of $(*)$ is even. Suppose further that $v \neq v_{\mathfrak{p}}, v_b$. Choose x such that $v(x) = -1$. Then the right-hand side of $(*)$ has valuation -4 so there exists a k_v-point.

Suppose that $v = v_{\mathfrak{p}}$. Let π be a uniformizer for v and take $x = \pi a^2/b$. Then

$$f(x) = b^{-1}a^{4-4n}\pi^2 + a^2b^{-1}\pi + ab^{-1}.$$

Since a has negative even valuation and $n \geq 1$, we have $v(f(x)) = v(a^2b^{-1}\pi)$ which is even. Now let us consider

$$g(x) = a^{4-8n}\pi^2 + a^{2-4n}\pi + a^{1-4n} + \gamma.$$

By the same conditions mentioned above, all terms except for γ have positive valuation. Therefore $v(g(x)) = 0$.

Finally suppose that $v = v_b$. Take $x = \frac{1}{b} + 1$. Then

$$f(x) = \frac{1}{b}\left(a^{-4n} + a + 1 + b + a^{-4n}b^2\right).$$

Note that by the conditions imposed on a, $(a^{-4n} + a + 1 + b + a^{-4n}b^2) \equiv \gamma + b \pmod{b^2\mathcal{O}_{k,S}}$. Thus $v(f(x)) = -1$. Now consider

$$g(x) = a^{-8n} + a^{-8n}b^2 + a^{-4n} + a^{-4n}b + a^{1-4n} + \gamma$$

modulo $b^2\mathcal{O}_{k,S}$. By the conditions imposed on a, we have

$$g(x) \equiv 1 + 1 + b + \gamma + \gamma \equiv b \pmod{b^2\mathcal{O}_{k,S}}.$$

Thus $v(g(x)) = 1$, so $v\left(f(x)g(x)\right)$ is even. $\square$

Let $L = k[T]/(T^2+T+\gamma)$ and let $\mathcal{A}$ denote the class of the cyclic algebra $(L/k, f(x))_2$ in $\operatorname{Br} k(X)$. Using the defining equation of the surface, we can show that $(L/k, g(x))_2$ is also a representative for $\mathcal{A}$. Since $g(x)+a^{-4n}bf(x)$ is a v-adic unit, $g(x)$ and $f(x)$ have no common zeroes. Since $\mathcal{A}$ is the class of a cyclic algebra of order 2, the algebra $(L/k, f(x)/x^2)_2$ is another representative for $\mathcal{A}$. Note that for any point P of X, there exists an open neighborhood U containing P such that one of $f(x)$, $g(x)$, $f(x)/x^2$ is a nowhere vanishing regular function on U. Therefore, $\mathcal{A}$ is an element of $\operatorname{Br} X$.

Lemma 4.1.2. *Let $P_v \in X(k_v)$. Then*

$$\operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = \begin{cases} 1/2 & \text{if } v = v_b, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $X(\mathbb{A}_k)^{\operatorname{Br}} = \emptyset$.

Proof. The surface X contains an open affine subset that can be identified with

$$V(y^2 + yz + az^2 - P(x)) \subseteq \mathbb{A}^3.$$

Let X_0 denote this open subset. Since $\operatorname{ev}_{\mathcal{A}}$ is continuous by Lemma 3.3.2 and inv_v is an isomorphism onto its image, it suffices to prove that inv_v takes the desired value on the v-adically dense subset $X_0(k_v) \subset X(k_v)$.

Since L/k is an unramified extension for all places v, evaluating the invariant map reduces to computing the parity of the valuation of $f(x)$ or $g(x)$.

Suppose that $v \neq v_a, v_b, v_{\mathfrak{p}}$. If $v(x_0) < 0$, then by the strong triangle inequality, $v(f(x_0)) = v(x_0^2)$. Now suppose that $v(x_0) \geq 0$. Then both $f(x_0)$ and $g(x_0)$ are v-adic integers, but since $g(x) - a^{-4n}bf(x) = \gamma$ either $f(x_0)$ or $g(x_0)$ is a v-adic unit. Thus, for all $P_v \in X_0(k_v)$, $\operatorname{inv}_v(\mathcal{A}(P_v)) = 0$.

Suppose that $v = v_a$. Since a generates a prime of even degree, $T^2 + T + \gamma$ splits in k_a. Therefore, $(L/k, h)$ is trivial for any $h \in k_a(V)^\times$ and so $\operatorname{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in X_0(k_v)$.

Suppose that $v = v_{\mathfrak{p}}$. We will use the representative $(L/k, g(x))$ of $\mathcal{A}$. If $v(x_0) < v(a^{4n}b^{-1})$ then the quadratic term of $g(x_0)$ has even valuation and dominates the other terms. If $v(x_0) > v(a^{4n}b^{-1})$ then the constant term of $g(x_0)$ has even valuation and dominates the other terms. Now assume that $x_0 = a^{4n}b^{-1}u$, where u is v-adic unit. Then we have

$$g(x_0) = u^2 + u + \gamma + a^{1-4n}.$$

Since γ was chosen such that $T^2+T+\gamma$ is irreducible in $\mathbb{F}[T]$ and $\mathfrak{p}$ is a prime of odd degree, $T^2+T+\gamma$ is irreducible in $\mathbb{F}_{\mathfrak{p}}[T]$. Thus, for any v-adic unit u, $u^2+u+\gamma \not\equiv 0 \pmod{\mathfrak{p}}$. Since $a \equiv 0 \mod \mathfrak{p}$, this shows $g(x_0)$ is a v-adic unit. Hence $\operatorname{inv}_v(\mathcal{A}(P_v)) = 0$ for all $P_v \in X_0(k_v)$.

Finally suppose that $v = v_b$. We will use the representative $(L/k, f(x))$ of $\mathcal{A}$. If $v(x_0) < -1$ then the quadratic term has odd valuation and dominates the other terms in $f(x_0)$. If

$v(x_0) > -1$ then the constant term has odd valuation and dominates the other terms in $f(x_0)$. Now assume $x_0 = b^{-1}u$ where u is any v-adic unit. Then we have

$$f(x_0) = \frac{1}{b}\left(a^{-4n}u^2 + u + a\right).$$

It suffices to show that $a^{-4n}u^2 + u + a \not\equiv 0 \pmod{b\mathcal{O}_{k,S}}$. Since $a \equiv \gamma \pmod{b\mathcal{O}_{k,S}}$, we have

$$a^{-4n}u^2 + u + a \equiv \overline{u}^2 + \overline{u} + \gamma.$$

Using the same argument as in the previous case, we see that $a^{-4n}u^2+u+a \not\equiv 0 \pmod{b\mathcal{O}_{k,s}}$ and thus $v(g(x_0)) = -1$. Therefore $\mathrm{inv}_v(\mathcal{A}(P_v)) = \frac{1}{2}$ for all $P_v \in X_0(k_v)$. $\square$

Chapter 5

Failure of the Hasse principle for degree 4 del Pezzo surfaces

In 1975, Birch and Swinnerton-Dyer produced two examples of del Pezzo surfaces of degree 4 over $\mathbb{Q}$ that have a Brauer-Manin obstruction to the Hasse principle [BSD75, Thm. 3]. We generalize their result to the following.

Theorem 5.0.3. *Let k be any global field of characteristic not 2. There exist infinitely many degree 4 del Pezzo surfaces with Brauer-Manin obstructions to the Hasse principle.*

The proof of Theorem 5.0.3 is constructive. For ease of exposition, we first give the proof in the case that k is a number field, and then we describe the changes that must be made in the case of positive characteristic in §5.1.1.

5.1 Proof of Theorem 5.0.3

Let k be a number field. By class field theory and the Chebotarev density theorem [Neu99, Thm 13.4, p. 545], there exists a totally positive element $a \in k$ such that a generates a prime ideal, -1 is a square in k_a, the residue field of k_a has at least 3 elements, and all primes lying over $2, 3$, and 5 split completely in $k(\sqrt{a})/k$. Let $b \in \mathcal{O}_k$ be an a-adic unit that is a nonsquare in k_a. Let $c \in \mathcal{O}_k$ be an element such that $b^2c + b \equiv 1 \pmod{a}$. Note that these conditions imply that c and $b^2c^2 - 1$ are a-adic integers.

Let X be the intersection of the following two quadrics in $\mathbb{P}^4$.

$$\begin{aligned} st &= x^2 - ay^2 \\ \left(s - b^2ct\right)\left(cs + (1 - b^2c^2)t\right) &= x^2 - az^2 \end{aligned}$$

Since $1 - b^2c^2$ is nonzero, X is smooth and thus is a del Pezzo surface of degree 4. In Lemma 5.1.1, we show that X has everywhere local points and in Lemma 5.1.2, we show that X has a Brauer-Manin obstruction to the existence of rational points. Together these show that X fails to satisfy the Hasse principle.

Lemma 5.1.1. *X has a k_v-point for every place v.*

Proof. Let v be archimedean or a place lying over $2, 3$ or 5. By our assumption on a, $\sqrt{a} \in k_v$. Thus $(0 : 0 : \sqrt{a} : 1 : 1)$ is a k_v-point of X.

Now suppose that v is a place dividing c that lies over a prime which is at least 7. The point $(-a : 1 : 0 : 0 : 1)$ is a non-singular point of $\mathbb{F}_v$ and thus lifts to a k_v-point.

Now suppose that v is a finite place dividing $b(b^2c^2 - 1)$ that does not divide $30ac$. Let C_1, C_2 be the arithmetic genus 1 curves over k obtained by intersecting X with $V(t-y)$ and $V(t-z)$ respectively. If C_1 has bad reduction at a place dividing b then $2ac-2$ has positive valuation at that place. If C_2 has bad reduction at a place dividing b then $8ac - 2$ has positive valuation at that place. Thus for v dividing b under the above assumptions, then at least one of C_1, C_2 has good reduction at v. A similar argument shows that for $v | b^2c^2 - 1$ and $v \nmid 30ac$, at least one of C_1, C_2 has good reduction at v. Thus, by Hensel's Lemma and the Weil conjectures, either C_1 or C_2 has a k_v-point. Therefore, X has a k_v-point.

Now let v be a finite place lying over a prime greater than 5 and that $v \nmid acb(b^2c^2 - 1)$. Then v is a place of good reduction, and so the Weil conjectures imply that, since $\#\mathbb{F}_v \geq 7$, $\#X(\mathbb{F}_v) \neq \emptyset$. Thus, X has at least one smooth $\mathbb{F}_v$-point, which, by Hensel's lemma, lifts to a k_v-point.

Lastly, suppose that $v = a$. The equations modulo a are

$$\begin{aligned} x^2 &\equiv st && \pmod{a} \\ 0 &\equiv cs^2 - 2b^2c^2st - b^2c(1-b^2c^2)t^2 && \pmod{a} \\ &\equiv c\left(s - \tfrac{2b^2c+\sqrt{4b^2}}{2}t\right)\left(s - \tfrac{2b^2c-\sqrt{4b^2}}{2}t\right) && \pmod{a} \\ &\equiv c\left(s - t(b^2c+b)\right)\left(s - t(b^2c-b)\right) && \pmod{a} \end{aligned}$$

Since $b^2c + b \equiv 1 \pmod{a}$, $(1 : 1 : 1 : 0 : 0)$ is a smooth $\mathbb{F}_a$-point of X and thus lifts to a k_a-point of X. $\square$

Let $\mathcal{A}$ denote the class of the quaternion algebra $\left(a, \frac{s-b^2ct}{s}\right)_2$ in $\operatorname{Br} k(X)$. Using the defining equations of the surface, we conclude that

$$\left(a, \frac{cs + (1-b^2c^2)t}{s}\right)_2, \left(a, \frac{cs + (1-b^2c^2)t}{t}\right)_2, \text{ and } \left(a, \frac{s - b^2ct}{t}\right)_2$$

are also representatives for $\mathcal{A}$. Since for any point P of $X \cap (D_+(s) \cup D_+(t))$ there exists a neighborhood U containing P such that one of $\frac{s-b^2ct}{s}, \frac{s-b^2ct}{t}, \frac{cs+(1-b^2c^2)t}{s}, \frac{cs+(1-b^2c^2)t}{t}$ is a nowhere vanishing regular function, $\mathcal{A}$ is an element of $\operatorname{Br}(X \cap (D_+(s) \cup D_+(t)))$. The subscheme $X \cap V(s,t)$ is codimension 2 in X so, by the Purity theorem, $\mathcal{A}$ extends to an element of $\operatorname{Br} X$. We will show that $\mathcal{A}$ gives a Brauer-Manin obstruction to the Hasse principle.

Lemma 5.1.2. *Let $P_v \in X(k_v)$. Then*

$$\operatorname{inv}_v\left(\operatorname{ev}_{\mathcal{A}}(P_v)\right) = \begin{cases} 1/2 & \text{if } v = v_a, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $X(\mathbb{A}_k)^{\mathcal{A}} = \emptyset$.

Proof. By Lemma 3.3.2, it suffices to show the above formula for all $P_v \in (X \cap D_+(u,v))(k_v)$. Suppose that v is archimedean. Since a is totally positive, it is a square in k_v. Thus, $\mathcal{A}$ is trivial for all points P_v and $\operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 0$.

Now suppose that v is a finite prime different from a. Since $k_v(\sqrt{a})/k_v$ is an unramified extension, it suffices to show that one of $\frac{s-b^2ct}{s}, \frac{s-b^2ct}{t}, \frac{cs+(1-b^2c^2)t}{s}, \frac{cs+(1-b^2c^2)t}{t}$ has even valuation. If $v(s) < v(t)$, then $v\left(\frac{s-b^2ct}{s}\right) = 0$. Now assume that $v(s) \geq v(t)$. Then $\frac{s-b^2ct}{t}$ and $\frac{cs+(1-b^2c^2)t}{t}$ are both v-adically integral. Since $\frac{cs+(1-b^2c^2)t}{t} - c\frac{s-b^2ct}{t} = 1$, by the strong triangle equality, one of these two values is a v-adic unit. Hence $\operatorname{inv}_v(\operatorname{inv}_{\mathcal{A}}(P_v)) = 0$.

Lastly, suppose that $v = v_a$. Since $k_a(\sqrt{a})/k_a$ is a ramified extension, it suffices to show that the reduction modulo a of $\frac{s-b^2ct}{s}$ is not a square in $\mathbb{F}_a$. Let $(s : t : x : y : z) \in X(k_a)$. After scaling we may assume that $s, t, x, y, z \in \mathcal{O}_a$ and that at least one coordinate is an a-adic unit. We note that the s, t and x-coordinates must be a-adic units so, after scaling, we may assume $x = 1$. Modulo a, we have the following system of equations,

$$st = 1, \quad cs^2 - 2b^2c^2st + b^2c(b^2c^2 - 1)t^2 = 0.$$

By eliminating t from the second equation, we see that $s^2 \equiv b^2c \pm b$. Thus

$$\frac{s - b^2ct}{s} = \frac{s^2 - b^2cst}{s^2} = \frac{b^2c \pm b - b^2c}{s^2} = \frac{\pm b}{s^2},$$

which, by our assumption on b, is never a square. Therefore, $\operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 1/2$. □

5.1.1 Global function field case

Let k be a global field of characteristic different from 2. Fix a place ∞ of k, and let $\mathcal{O}_k$ be the ring of elements integral at all places except ∞. Let $a \in \mathcal{O}_k$ be such that a generates a prime ideal, -1 is a square in k_a, and ∞ and all primes with residue field consisting of less than 6 elements split completely in $k(\sqrt{a})/k$. Let b and c satisfy the same assumptions as above. Then the same statements hold, with slight modifications to the proofs as explained below.

Lemma 5.1.2 goes through exactly as stated. Consider Lemma 5.1.1. The first case (v archimedean or a place lying over 2, 3, or 5) is replaced with the case where v is ∞ or such that $\#\mathbb{F}_v < 6$. Then the rest of the proof goes through unless $\operatorname{char} k = 3$ or 5. If $\operatorname{char} k \mid 15$, then we need a new argument for the places dividing $b(b^2c^2-1)$ that do not divide $2ac$. Instead of considering the curves obtained by intersecting X with $V(t-y)$ and $V(t-z)$, now let C_1, C_2 be the arithmetic genus 1 curves obtained by intersecting X with $V(t+y+z)$ and $V(t+z)$. By a similar argument, we see that at most one of C_1, C_2 has bad reduction at v, so by the Weil Conjectures and Hensel's Lemma, at least one of C_1, C_2 has a k_v-point. Therefore X has a k_v-point.

Chapter 6

Failure of the Hasse principle for Enriques surfaces

In 2007, Cunnane showed that Enriques surfaces need not satisfy the Hasse principle by constructing a family of surfaces such that the K3 double covers and all of their twists had no adelic points [Cun07]. He shows that whenever a K3 double cover and all of its twists have no adelic points the lack of rational points on the Enriques surface is caused by an algebraic Brauer-Manin obstruction.

We construct a family of Enriques surfaces with no rational points and such that each K3 double cover has local points everywhere. We show that these counterexamples to the Hasse principle are explained by an algebraic Brauer-Manin obstruction.

6.1 Main result

Let d be a nonzero squarefree integer and define Y_d to be the K3 surface in $\mathbb{P}^5_{\mathbb{Q}}$ cut out by the following quadrics

$$\begin{aligned} st &= x^2 - 5y^2 \\ (s+t)(s+2t) &= x^2 - 5z^2 \\ du^2 &= 10x^2 - 3y^2 + z^2. \end{aligned}$$

Let ι be the involution $(s:t:u:x:y:z) \mapsto (-s:-t:-u:x:y:z)$. Since $\iota|_{Y_d}$ has no fixed points, the quotient $X_d := Y_d/\iota$ is an Enriques surface. We let f_d denote the natural map $Y_d \to X_d$. Let $\mathcal{P} = \{2,3,5,7,11,13,17,19,23,41,383,\infty\}$ and let $\Sigma = \{d \in \mathbb{Z} \mid d \in \mathbb{Z}_v^{\times 2}$ for all $v \in \mathcal{P}$ and $p \nmid d$ for all $p < 98\}$.

Theorem 6.1.1. *Assume that $d \in \Sigma$. Then $Y_d(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ and $X_d(\mathbb{Q}) = \emptyset$.*

In §6.2 we show that X_d has local points everywhere and in §6.3 we show that X_d has no $\mathbb{Q}$-points. Together these complete the proof of Theorem 6.1.1. In §6.4 we show that this lack of rational points is explained by an algebraic Brauer-Manin obstruction.

6.2 Local solvability

Proposition 6.2.1. *$Y_d(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$ for all $d \in \Sigma$.*

Proof. Since Y_d is projective, it is equivalent to prove that $Y_d(\mathbb{Q}_v)$ is nonempty for all completions $\mathbb{Q}_v$ of $\mathbb{Q}$.

If v is a finite prime of good reduction and $v \geq 22$, then by the Weil conjectures and Hensel's lemma Y_d has a $\mathbb{Q}_v$-point.

So it remains to consider the archimedean place, the finite primes less than 22 and the primes of bad reduction of Y_d, which are $2, 3, 5, 7, 41, 383$ and the primes dividing d.

Now assume that v is a finite prime which divides d. Consider the smooth genus 5 curve $Z = V(u) \cap Y_d$. The curve Z has good reduction at all primes dividing d. Since all primes dividing d are at least 98, the Weil conjectures and Hensel's lemma show that $Z(\mathbb{Q}_v) \neq \emptyset$ and thus $Y_d(\mathbb{Q}_v) \neq \emptyset$.

One can check that the only places remaining are those in $\mathcal{P}$. If $v \in \mathcal{P}$ then, by assumption, d is in $\mathbb{Q}_v^{\times 2}$. Also, if $v \neq 2$ then one of $-1, 5, -5$ is in $\mathbb{Q}_v^{\times 2}$. Therefore, one of the following is a $\mathbb{Q}_v$-point

$$(1 : 1 : 3\sqrt{1/d} : 1 : 0 : \sqrt{-1}), \quad (10 : -10 : 6\sqrt{5/d} : 5 : 5 : \sqrt{5}), \quad (5 : 0 : \sqrt{-5/d} : 0 : 0 : \sqrt{-5}).$$

If $v = 2$ then $(-25 : 5 : \sqrt{-135/d} : 0 : 5 : 2\sqrt{-15})$ is a $\mathbb{Q}_v$-point. This completes the proof. □

6.3 Absence of rational points

By [Sko01, Form. 2.12], we have

$$X_d(\mathbb{Q}) = \bigcup_{\sigma \in \mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z})} f_d^\sigma\left(Y_d^\sigma(\mathbb{Q})\right).$$

Any element $\sigma \in \mathrm{H}^1(\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$ can be represented by a squarefree integer e. The associated twist, which we will denote $f_{d,e} \colon Y_{d,e} \to X_d$, is given by the vanishing of the following quadrics in $\mathbb{P}^5$

$$est = x^2 - 5y^2 \tag{6.3.1}$$
$$e(s+t)(s+2t) = x^2 - 5z^2 \tag{6.3.2}$$
$$edu^2 = 10x^2 - 3y^2 + z^2. \tag{6.3.3}$$

We will prove that if $5 \nmid d$ then for all squarefree integers e, $Y_{d,e}$ has no rational points and therefore obtain

Proposition 6.3.1. *If $5 \nmid d$, then $X_d(\mathbb{Q}) = \emptyset$.*

For any nonzero squarefree integer e, define S_e to be the vanishing locus of

$$\begin{aligned} est &= x^2 - 5y^2 \\ e(s+t)(s+2t) &= x^2 - 5z^2 \end{aligned}$$

in $\mathbb{P}^4$. The surface S_e is a del Pezzo surface of degree 4 and there is an obvious projection morphism $\pi_{d,e}\colon Y_{d,e} \to S_e$. Therefore if $S_e(\mathbb{Q}) = \emptyset$, then $Y_{d,e}(\mathbb{Q}) = \emptyset$ for all d.

Lemma 6.3.2. *Let e be any squarefree integer not divisible by 2. Then S_e and S_{2e} are isomorphic. In particular,*

$$S_e(\mathbb{Q}) = \emptyset \iff S_{2e}(\mathbb{Q}) = \emptyset.$$

Proof. One can easily check that $(s : t : x : y : z) \mapsto (2t : s : x : y : z)$ gives an isomorphism from S_{2e} to S_e. $\square$

Lemma 6.3.3. *If there exists an odd prime p such that $p \equiv \pm 2 \pmod 5$ and $p \mid e$ then*

$$S_e(\mathbb{Q}_p) = \emptyset.$$

Proof. Assume $(s : t : u : x : y : z) \in Y_{d,e}(\mathbb{Q}_p)$. Since p is inert in $\mathbb{Q}(\sqrt{5})$, $v_p(est) \equiv v_p(e(s+t)(s+2t)) \equiv 0 \mod 2$. Thus s and t have different p-adic valuations. But this implies that $v_p(e(s+t)(s+2t))$ is odd, resulting in a contradiction. $\square$

Lemma 6.3.4. *If $5 \mid e$ and $5 \nmid d$, then $Y_{d,e}(\mathbb{Q}_5) = \emptyset$.*

Proof. Let $(s : t : u : x : y : z) \in Y_{d,e}(\mathbb{Z}_5)$. After scaling we may assume that at least one coordinate is in $\mathbb{Z}_5^\times$. By considering the equations 6.3.1 modulo 5 we see that $v_5(x), v_5(y), v_5(z) > 0$. Considering the equations 6.3.1 modulo 5^2 implies that $v_5(u), v_5(s), v_5(t) > 0$ which gives a contradiction. $\square$

Lemma 6.3.5. *Assume that $e = \pm \prod p_i$ where $p_i \equiv \pm 1 \mod 5$. Let $\mathcal{A} = \left[\left(5, \frac{s+t}{t}\right)\right] \in \operatorname{Br} k(S_e)$. Then $\mathcal{A} \in \operatorname{Br} S_e$ and*

$$\operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = \begin{cases} 1/2 & \text{if } v = 5, \\ 0 & \text{otherwise.} \end{cases}$$

for all $\{P_v\} \in S_e(\mathbb{A}_{\mathbb{Q}})$. Therefore, $S_e(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$.

Proof. This statement and proof is a very slight generalization of [BSD75, Thm. 3].

Let $P_v = (s : t : u : x : y : z)$. Note that

$$\mathcal{A} = \left[\left(5, \frac{s+t}{t}\right)\right] = \left[\left(5, e\frac{s+2t}{t}\right)\right] = \left[\left(5, e\frac{s+t}{s}\right)\right] = \left[\left(5, \frac{s+2t}{s}\right)\right].$$

For any point P of S_e, there exists an open set U containing P where at least one of $\frac{s+t}{t}, e\frac{s+2t}{t}, e\frac{s+t}{s}, \frac{s+2t}{s}$ is an invertible regular function, unless $P \in V(s,t)$. Since $V(s,t) \cap X$ is codimension 2 in X, $\mathcal{A}$ extends to an element of $\operatorname{Br} S_e$.

Assume $v \neq 5$. Then $\mathbb{Q}\left(\sqrt{5}\right)/\mathbb{Q}$ is unramified at v so it suffices to show that one of

$$v\left(\frac{s+t}{t}\right),\ v\left(e\frac{s+2t}{t}\right),\ v\left(e\frac{s+t}{s}\right),\ v\left(\frac{s+2t}{s}\right)$$

is even. In fact, we will show that at least one of the above quantities is 0. If $v(s) < v(t)$ then $v\left(\frac{s+2t}{s}\right) = 0$. If $v(s) > v(t)$ then $v\left(\frac{s+t}{t}\right) = 0$. Now consider the case $v(s) = v(t)$. Then $\frac{s+t}{t}$ and $\frac{s+2t}{s}$ are v-adic integers and $\frac{s}{t}$ is a v-adic unit. Since

$$\left(\frac{s+2t}{s}\right)\left(\frac{s}{t}\right) - \left(\frac{s+t}{t}\right) = 1,$$

either $\frac{s+t}{t}$ or $\frac{s+2t}{s}$ is a v-adic unit.

Now assume $v = 5$. The difference of the defining equations of S_e yields the congruence relation

$$(s-2t)(s-t) \equiv 0 \pmod 5.$$

Thus $\frac{s+t}{t} \equiv \pm 2 \pmod 5$ and hence $\operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 1/2$ for all $P_v \in S_e(\mathbb{Q}_v)$.

$\square$

Proof of Proposition 6.3.1. By [Sko01, Formula 2.12], it suffices to show that $Y_{d,e}(\mathbb{Q}) = \emptyset$ for all squarefree integers e. If $5 \mid e$, then we are done by Lemma 6.3.4. If $5 \nmid e$, we will show that $S_e(\mathbb{Q}) = \emptyset$. By Lemma 6.3.2, we may assume that $2 \nmid e$. Then either there exists an odd prime $p \equiv \pm 2 \pmod 5$ such that $p \mid e$ or $e = \pm \prod p_i$ where $p_i \equiv \pm 1 \pmod 5$ for all i. Thus by Lemmas 6.3.3 and 6.3.5 we have our result. $\square$

6.4 Algebraic Brauer set

We have an exact sequence of Galois modules

$$0 \to \langle K_X \rangle \xrightarrow{\lambda} \operatorname{Pic} \overline{X} \to \operatorname{Num} \overline{X} \to 0$$

Let $\operatorname{Br}_\lambda = r^{-1}\lambda_*\left(H^1(G_{\mathbb{Q}}, \langle K_X \rangle)\right) \subseteq \operatorname{Br}_1 X$. By [Sko01, Thm. 6.1.2], we have

$$X_d(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}_\lambda} = \bigcup_{\substack{\text{squarefree} \\ \text{integer } e}} f_{d,e}\left(Y_{d,e}(\mathbb{A}_{\mathbb{Q}})\right). \tag{6.4.1}$$

Using this we can prove

Theorem 6.4.1. *For any $d \in \Sigma, X_d(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}_1} = \emptyset$.*

Proof. By the functoriality of the Brauer group we have $\operatorname{ev}_{\mathcal{A}}(f(P)) = \operatorname{ev}_{f^*\mathcal{A}}(P)$ for any map $f\colon Y \to X$, $P \in Y(T)$, and $\mathcal{A} \in \operatorname{Br} X$. Therefore, for any subgroup $B \in \operatorname{Br} X$, we have $X(\mathbb{A}_{\mathbb{Q}})^B \cap f(Y(\mathbb{A}_{\mathbb{Q}})) = Y(\mathbb{A}_{\mathbb{Q}})^{f^*B}$. Using this in conjunction with Equation 6.4.1, we obtain

$$X_d(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}_1} = \bigcup_{\substack{\text{squarefree}\\ \text{integer } e}} f_{d,e}\left(Y_{d,e}(\mathbb{A}_{\mathbb{Q}})^{f_{d,e}^*(\operatorname{Br}_1 X_d)}\right).$$

Since $Y_{d,e}(\mathbb{A}_{\mathbb{Q}})^{(5,\frac{s+t}{t})} = \emptyset$ for all e it suffices to show that there exists an element $\mathcal{A} \in \operatorname{Br}_1 X_d$ such that $f_{d,e}^*\mathcal{A} = \left(5, \frac{s+t}{t}\right)$. The function $\frac{s+t}{t} \in \mathbf{k}(Y_{d,e})$ is fixed by σ, so it is an element of $\mathbf{k}(X_d)$. Therefore, $\left(5, \frac{s+t}{t}\right)$ is an element of $\operatorname{Br} \mathbf{k}(X_d)$. Let D be a prime divisor on X_d in the support of $\operatorname{div}\left(\frac{s+t}{t}\right)$. One can check that D must be either the push-forward of the vanishing locus of t on Y_d or the push-forward of the vanishing locus of $s+t$ on Y_d. Using the defining equations of $Y_{d,e}$, one can check that either $\left(\frac{x}{y}\right)^2 = 5$ or $\left(\frac{x}{z}\right)^2 = 5$ in $\kappa\left((f^*(D)_{\mathrm{red}})\right)$. Since $\frac{x}{y}$ and $\frac{x}{z}$ are fixed by σ, the same relation holds in $\kappa(D)$. Thus, the Purity theorem (Theorem 3.1.1) tells us that $\left(5, \frac{s+t}{t}\right)$ is an element of $\operatorname{Br} X_d$ and we have our result. $\square$

Chapter 7

A family of surfaces with exactly one pointless rational fiber

We construct a family of smooth projective geometrically integral surfaces over an open subscheme of $\mathbb{P}^1_{\mathbb{Q}}$ with the following curious arithmetic property: there is exactly one $\mathbb{Q}$-fiber with no rational points. Our proof makes explicit a non-effective construction of Poonen [Poo09, Prop. 7.2]. This example demonstrates that the guiding philosophy "geometry controls arithmetic" does not apply to all arithmetic properties, namely it doesn't apply to the existence of $\mathbb{Q}$-points. We believe that this is the first example of an algebraic family where exactly one $\mathbb{Q}$-fiber has no rational points.

Theorem 7.0.2. *Define $P_0(x) := (x^2-2)(3-x^2)$ and $P_\infty(x) := 2x^4+3x^2-1$. Let $\pi\colon X \to \mathbb{P}^1_{\mathbb{Q}}$ be the Châtelet surface bundle over $\mathbb{P}^1_{\mathbb{Q}}$ given by*

$$y^2 + z^2 = \left(6u^2 - v^2\right)^2 P_0(x) + \left(12v^2\right)^2 P_\infty(x),$$

where π is projection onto $(u : v)$. Then $\pi(X(\mathbb{Q})) = \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$.

Note that the degenerate fibers of π do not lie over $\mathbb{P}^1(\mathbb{Q})$ so the family of smooth projective geometrically integral surfaces mentioned above contains all $\mathbb{Q}$-fibers.

The non-effectivity in [Poo09, Prop. 7.2] stems from the use of higher genus curves and Faltings' theorem. (This is described in more detail in [Poo09, §9]). We circumvent the use of higher genus curves by an appropriate choice of $P_\infty(x)$.

7.1 Background

A Châtelet surface bundle over $\mathbb{P}^1$ is a flat proper morphism $V \to \mathbb{P}^1$ such that the generic fiber is a Châtelet surface. We can construct them in the following way. Let $P, Q \in k[x, w]$ be linearly independent homogeneous polynomials of degree 4 and let $\alpha \in k^\times$. Let V be the diagonal conic bundle over $\mathbb{P}^1_{(a:b)} \times \mathbb{P}^1_{(w:x)}$ given by $\mathcal{L}_1 = \mathcal{O}, \mathcal{L}_2 = \mathcal{O}, \mathcal{L}_3 = \mathcal{O}(1,2), s_1 = 1, s_2 = -\alpha, s_3 = -(a^2P + b^2Q)$. By composing $V \to \mathbb{P}^1 \times \mathbb{P}^1$ with the projection onto the first factor, we realize V as a Châtelet surface bundle. We say that V is the Châtelet surface bundle given by

$$y^2 - \alpha z^2 = a^2P(x) + b^2Q(x),$$

where $P(x) = P(x,1)$ and $Q(x) = Q(x,1)$. We can also view a, b as relatively prime, homogeneous, degree d polynomials in u, v by pulling back by a suitable degree d map $\phi\colon \mathbb{P}^1_{(u:v)} \to \mathbb{P}^1_{(a:b)}$.

7.2 Proof of Theorem 7.0.2

By [Isk71], we know that the Châtelet surface

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violates the Hasse principle, i.e. it has $\mathbb{Q}_v$-rational points for all completions v, but no $\mathbb{Q}$-rational points. Thus, $\pi(X(\mathbb{Q})) \subseteq \mathbb{A}^1_{\mathbb{Q}}(\mathbb{Q})$. Therefore, it remains to show that $X_{(u:1)}$, the Châtelet surface defined by

$$y^2 + z^2 = (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x),$$

has a rational point for all $u \in \mathbb{Q}$.

If $P_{(u:1)} := (6u^2 - 1)^2 P_0(x) + 12^2 P_\infty(x)$ is irreducible, then by [CTSSD87a, CTSSD87b] we know that $X_{(u:1)}$ satisfies the Hasse principle. Thus it suffices to show that $P_{(u:1)}$ is irreducible and $X_{(u:1)}(\mathbb{Q}_v) \neq \emptyset$ for all $u \in \mathbb{Q}$ and all places v of $\mathbb{Q}$.

7.2.1 Irreducibility

We prove that for any $u \in \mathbb{Q}$, the polynomial $P_{(u:1)}(x)$ is irreducible in $\mathbb{Q}[x]$ by proving the slightly more general statement, that for all $t \in \mathbb{Q}$

$$P_t(x) := (2x^4 + 3x^2 - 1) + t^2(x^2 - 2)(3 - x^2) = x^4(2 - t^2) + x^2(3 + 5t^2) + (-6t^2 - 1)$$

is irreducible in $\mathbb{Q}[x]$. We will use the fact that if $a, b, c \in \mathbb{Q}$ are such that $b^2 - 4ac$ and ac are not squares in $\mathbb{Q}$ then $p(x) := ax^4 + bx^2 + c$ is irreducible in $\mathbb{Q}[x]$.

Let us first check that for all $t \in \mathbb{Q}$, $(3 + 5t^2)^2 - 4(2 - t^2)(-6t^2 - 1)$ is not a square in $\mathbb{Q}$. This is equivalent to proving that the affine curve $C\colon w^2 = t^4 + 74t^2 + 17$ has no rational points. The smooth projective model, $\overline{C} : w^2 = t^4 + 74t^2s^2 + 17s^4$ in weighted projective space $\mathbb{P}(1,1,2)$, has 2 rational points at infinity. Therefore $\overline{C}$ is isomorphic to its Jacobian. A computation in `Magma` shows that $\mathrm{Jac}(C)(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [BCP97]. Therefore, the points at infinity are the only 2 rational points of $\overline{C}$ and thus C has no rational points.

Now we will show that $(-6t^2-1)(2-t^2)$ is not a square in $\mathbb{Q}$ for any $t \in \mathbb{Q}$. As above, this is equivalent to determining whether $C' \colon w^2 = (-6t^2-1)(2-t^2)$ has a rational point. Since 6 is not a square in $\mathbb{Q}$, this is equivalent to determining whether the smooth projective model, $\overline{C'}$, has a rational point. The curve $\overline{C'}$ is a genus 1 curve so it is either isomorphic to its Jacobian or has no rational points. A computation in `Magma` shows that $\operatorname{Jac}(C')(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$ [BCP97]. Thus $\#C'(\mathbb{Q}) = 0$ or 2. If (t,w) is a rational point of C', then $(\pm t, \pm w)$ is also a rational point. Therefore, $\#C(\mathbb{Q}) = 2$ if and only if there is a point with $t=0$ or $w=0$ and one can easily check that this is not the case.

7.2.2 Local Solvability

Lemma 7.2.1. *For any point $(u:v) \in \mathbb{P}^1_{\mathbb{Q}}$, the Châtelet surface $X_{(u:v)}$ has $\mathbb{R}$-points and $\mathbb{Q}_p$-points for every prime p.*

Proof. Let $a = 6u^2 - v^2$ and let $b = 12v^2$. We will refer to $a^2P_0(x) + b^2P_\infty(x)$ both as $P_{(a:b)}$ and $P_{(u:v)}$.

$\mathbb{R}$-points: It suffices to show that given $(u:v)$ there exists an x such that

$$P_{(a:b)} = x^4(2b^2-a^2) + x^2(3b^2+5a^2) + (-6a^2-b^2)$$

is positive. If $2b^2-a^2$ is positive, then any x sufficiently large will work. So assume $2b^2-a^2$ is negative. Then $\alpha = \frac{-(3b^2+5a^2)}{2(2b^2-a^2)}$ is positive. We claim $P_{(a:b)}(\sqrt{\alpha})$ is positive.

$$\begin{aligned}
P_{(a:b)}(\sqrt{\alpha}) &= \alpha^2(2b^2-a^2) + \alpha(3b^2+5a^2) + (-6a^2-b^2) \\
&= \frac{(3b^2+5a^2)^2}{4(2b^2-a^2)} + \frac{-(3b^2+5a^2)^2}{2(2b^2-a^2)} + (-6a^2-b^2) \\
&= \frac{1}{4(2b^2-a^2)}\left(4(2b^2-a^2)(-6a^2-b^2) - (3b^2+5a^2)^2\right) \\
&= \frac{1}{4(2b^2-a^2)}\left(-17b^4 - 74a^2b^2 - a^4\right)
\end{aligned}$$

Since $2b^2-a^2$ is negative by assumption and $-17b^4-74a^2b^2-a^4$ is always negative, we have our result.

$\mathbb{Q}_p$-points:

$p \geq 5$ Without loss of generality, let a and b be relatively prime integers. Let $\overline{X}_{(a:b)}$ denote the reduction of $X_{(a:b)}$ modulo p. We claim that there exists a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$ that, by Hensel's lemma, we can lift to a $\mathbb{Q}_p$-point of $X_{(a:b)}$.

Since $P_{(a:b)}$ has degree at most 4 and is not identically zero modulo p, there is some $x \in \mathbb{F}_p$ such that $P_{(a:b)}(x)$ is nonzero. Now let y, z run over all values in $\mathbb{F}_p$. Then the polynomials $y^2, P_{(a:b)}(x) - z^2$ each take $(p+1)/2$ distinct values. By the pigeonhole principle, y^2 and $P_{(a:b)}(x) - z^2$ must agree for at least one pair $(y, z) \in \mathbb{F}_p^2$ and one can check that this pair is not $(0, 0)$. Thus, this tuple (x, y, z) gives a smooth $\mathbb{F}_p$-point of $\overline{X}_{(a:b)}$. (The proof above that the quadratic form $y^2 + z^2$ represents any element in F_p is not new. For example, it can be found in [Coh07, Prop 5.2.1].)

$p = 3$ From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_3(b/a)$ is positive. Since $\mathbb{Q}_3(\sqrt{-1})/\mathbb{Q}_3$ is an unramified extension, it suffices to show that given a, b as above, there exists an x such that $P_{(a:b)}(x)$ has even valuation. Since $v_3(b/a)$ is positive, $v_3(2b^2 - a^2) = 2v_3(a)$. Therefore, if $x = 3^{-n}$, for n sufficiently large, the valuation of $P_{(a:b)}(x)$ is $-4n + 2v_3(a)$ which is even.

$p = 2$ From the equations for a and b, one can check that for any $(u : v) \in \mathbb{P}^1_{\mathbb{Q}}$, $v_2(b/a)$ is at least 2. Let $x = 0$ and $y = a$. Then we need to find a solution to $z^2 = a^2(-7 + (b/a)^2)$. Since $v_2(b/a) > 1$, $-7 + (b/a)^2 \equiv 1^2 \mod 8$. By Hensel's lemma, we can lift this to a solution in $\mathbb{Q}_2$.

$\square$

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