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Osculating spaces of minimal surfaces in Euclidean space

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Eric Carmody

Dissertation Committee:
Professor Richard Schoen, Chair
Associate Professor Li-Sheng Tseng
Professor Jeff Viaclovsky

2023

DEDICATION

To my God,

For the Lord gives wisdom;
from his mouth come knowledge and understanding;

&

To Keilyn,

my wife and my support in all things.

Knowledge puffs up,
but love builds up.
Love never falls,
but knowledge will pass away.

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I give thanks to my God for all the people that have been instrumental in my research and the completion of this dissertation. It is by his grace that he has granted me the gift and the ability to study mathematics and author this dissertation. May all glory be to my Lord and Savior Jesus Christ.

VITA

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ABSTRACT OF THE DISSERTATION

Osculating spaces of minimal surfaces in Euclidean space

By

Eric Carmody

Doctor of Philosophy in Mathematics

University of California, Irvine, 2023

Professor Richard Schoen, Chair

In this paper we introduce the notion of the osculating space of a minimal surface as a vector bundle formed from the span of the z-derivatives of a conformal parametrization of the surface. We show that a minimal surface being J-holomorphic is equivalent to the dot product being fully degenerate on the osculating space. We then show that any minimal surface lying fully in $\mathbb{R}^n$ with osculating space of dimension half the ambient space must either be J-holomorphic or has a complex structure that is not compatible with the Euclidean metric. Considering the dot product as a bilinear form on the osculating space, we prove some strong restrictions on the dimensionality of the null space of this bilinear form. We show that the osculating space behaves nicely with respect to decomposable minimal surfaces and that we can always decompose a minimal surface into a J-holomorphic minimal surface and a minimal surface for which the dot product is non-degenerate on the osculating space. Finally, we prove that complete finite total curvature stable surfaces of genus one in $\mathbb{R}^5$ are holomorphic under the condition that a certain cover of the surface is stable and the normal bundle is not topologically trivial.

Chapter 1

Introduction

1.1 Bernstein's problem

The relevant starting point for our study of minimal surfaces is with Bernstein's problem regarding minimal graphs. The question in general can be stated as

Question (Bernstein's problem). If the graph of $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is minimal in $\mathbb{R}^n$, is f necessarily linear?

Bernstein [2] answered this question in the affirmative in 1915 for the case $n = 3$, referred to now as the classical Bernstein theorem.

Theorem (Classical Bernstein theorem)**.** *If the graph of a global function $z = f(x, y)$ defines a minimal surface in $\mathbb{R}^3$, then the surface is in fact a plane.*

However, the general question remained unanswered.

Several decades later in 1962, Fleming [7] provided an alternate proof of the classical Bernstein theorem by showing that if it were false that would imply the existence of a non-trivial minimal cone in $\mathbb{R}^3$, which was a known contradiction.

Inspired by Fleming, De Giorgi [6] improved and extended the argument in 1965. He proved that Bernstein's problem being false in dimension n implied the existence of non-trivial minimal cones in $\mathbb{R}^{n-1}$, one dimension lower. Thus, Bernstein's problem in $n = 4$ was answered in the affirmative, and the path forward to solving it in higher dimensions became proving the non-existence of these non-trivial minimal cones.

Almgren [1] showed in 1966 that non-trivial minimal cones did not exist in $n = 4$, which was followed by Simons [14] in 1968 with a proof that non-trivial minimal cones did not exist in dimension $n \leq 7$. This together with the observation by De Giorgi meant that Bernstein's theorem was thus proven for $n \leq 8$.

In that same paper, Simons conjectured that certain cones in $\mathbb{R}^{2m}$ for $m \geq 4$ were in fact minimal. The following year, Bombieri, De Giorgi, and Giusti [3] proved that these cones were indeed minimal, meaning that this method used to prove Bernstein's theorem for $n \leq 8$ could not be used to prove it in any higher dimension. However, in the same paper they then proceeded to prove a negative answer to the Bernstein problem for $n \geq 9$.

Thus, the question of Bernstein's problem was fully answered, and the result was Bernstein's theorem.

Theorem (Bernstein's theorem)**.** *If $n \leq 8$ and the graph of $f : \mathbb{R}^{n-1} \to \mathbb{R}$ is a minimal surface in $\mathbb{R}^n$, then f is a linear function, i.e. a degree 1 polynomial.*

Furthermore, the bound of $n \leq 8$ is strict.

1.2 Extensions of Bernstein's theorem

With Bernstein's problem answered, we can consider other extensions of the classical Bernstein theorem. In Bernstein's problem we are always considering the codimension one case, but we may also consider the case of a two-dimensional surface in higher codimension. We might formulate the question as follows:

Question (Extension I of Bernstein's problem). If $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$ is a global function and the graph of f defines a 2-dimensional minimal surface in $\mathbb{R}^n$, is the surface necessarily a plane?

For $n = 3$ this is just the classical Bernstein theorem, which is promising, however the immediate answer to this question is negative for all $n \geq 4$. For example, the graph of the map

$$f(x, y) = \left(x^2 - y^2, 2xy\right)$$

is a minimal surface in $\mathbb{R}^4$ which is clearly not a plane, and this example is by no means unique.

This is simply the wrong question to ask as an extension of Bernstein's problem. One consideration is that minimality is a much stronger condition in codimension 1 than in higher codimension. Minimal graphs in codimension 1, as considered in Bernstein's problem, are in fact stable in addition to minimal. However, this is not the case in higher codimension, so it is quite natural as an extension of Bernstein's problem to expand our hypotheses to include stability.

Question (Extension II of Bernstein's problem). If $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$ is a global function and the graph of f defines a 2-dimensional stable minimal surface in $\mathbb{R}^n$, is the surface necessarily a plane?

Unfortunately, the answer to this question is also negative for $n \geq 4$. Holomorphic curves serve as an important class of counterexamples to this particular question. If $F : \mathbb{C} \to \mathbb{C}^m$

is holomorphic, then the theory of calibrations introduced by Harvey and Lawson [9] easily shows that the map realized as a two-dimensional real surface in $\mathbb{R}^{2m}$ is stable. Since the surface comes from a holomorphic map it is Kähler, and Wirtinger's inequality shows that the symplectic form is a calibration on the surface. The standard result from calibrated geometry is that calibrated submanifolds minimize area in their homology class, which implies the surface is in fact stable.

In fact, the counterexample given above to address Extension I of Bernstein's Problem is a holomorphic curve. If we take the holomorphic map $F(z) = (z, z^2)$ and view it as a two-dimensional parametrized surface in $\mathbb{R}^4$ we get

$$(x, y) \mapsto \left(x, y, x^2 - y^2, 2xy\right),$$

which is the same as graph of f given above. So then the same counterexample for Extension I also serves as a counterexample for Extension II. Since the graph of f is a holomorphic curve, it is stable as shown by the calibration argument, but it is clearly not a plane.

So we have once again arrived at the wrong question to ask as an extension of Bernstein's problem. However, this counterexample motivates another modification which leads us to the correct question to properly extend Bernstein's problem to higher codimension.

Question (Extension III of Bernstein's problem). If $f : \mathbb{R}^2 \to \mathbb{R}^{n-2}$ is a global function and the graph of f defines a 2-dimensional stable minimal surface in $\mathbb{R}^n$, is the surface necessarily a holomorphic curve?

As we showed above, we cannot hope for stability to imply the surface is a plane, but our important class of stable graphs coming from holomorphic curves may mean stability implies the surface is a holomorphic curve. There are currently no known classes of examples of stable graphs that are not holomorphic curves, which means we are unable to trivially disprove this conjecture.

Before we continue, we must formalize the conclusion of this question, as it is not quite clear what it means for a surface to be a holomorphic curve. To illustrate the dilemma, consider the holomorphic curve given by $F(z) = (z, z^2, z^3)$, which when taken as a real map looks like

$$(x, y) \mapsto \left(x, y, x^2 - y^2, 2xy, x^3 - 3xy^2, 3x^2y - y^3\right).$$

From our discussion so far, we know that this is the graph of a two-dimensional surface in $\mathbb{R}^6$ which is stable.

Now consider the following modified surface

$$(x, y) \mapsto \left(x, y, x^2 - y^2, x^3 - 3xy^2, 2xy, 3x^2y - y^3\right),$$

where all we have done is swapped the fourth and fifth component of the map. Clearly this surface is isomorphic to the original and so it is also stable. However, there is no holomorphic curve $F(z) = (F_1(z), F_2(z), F_3(z))$ which realizes this two-dimensional surface in $\mathbb{R}^6$.

Now with the problem illuminated, let us introduce some definitions so that we can formalize what it means for a surface to be a holomorphic curve.

Definition. A *linear complex structure* is a linear transformation $J : \mathbb{R}^n \to \mathbb{R}^n$ such that $J^2 = -\text{id}$, where id is the identity map on $\mathbb{R}^n$.

Definition. If g is an inner product on $\mathbb{R}^n$, then we say a linear complex structure J is *compatible with* g if $g(Ju, Jv) = g(u, v)$ for all $u, v \in \mathbb{R}^n$.

Definition. Let J be a linear complex structure compatible with the Euclidean metric on $\mathbb{R}^n$ and $\iota : S \to \mathbb{R}^n$ an immersion of a Riemann surface S. The surface S is *J-holomorphic* if J restricts to a map $J : d\iota(TS) \to d\iota(TS)$.

Remark. The push forward of the tangent space T_pS at a point $p \in S$ is some subspace of the ambient tangent space $T_{\iota(p)}\mathbb{R}^n \cong \mathbb{R}^n$, where we always choose the canonical representation of this tangent space as $\mathbb{R}^n$. A linear complex structure J is a linear transformation on $\mathbb{R}^n$, so we can restrict it to this pushforward of the tangent space of S at this particular point. The claim of the definition then is that J preserves this pushforward of the tangent space at every point.

So then the correct formalization of what it means for a surface to be a holomorphic curve is that it is J-holomorphic, i.e. there exists a linear complex structure J that is compatible with the Euclidean metric such that the surface is J-holomorphic. This is the correct formalization because there is some orthogonal coordinate change that transforms J into the standard complex structure and thus transforms the surface into a form which is the direct realization of some holomorphic curve $F : \mathbb{C} \to \mathbb{C}^m$. Now we can properly state our question extending Bernstein's problem as a conjecture.

1.3 Holomorphic curves and Micallef's theorem

Conjecture. *Let S be a two-dimensional stable minimal surface in $\mathbb{R}^n$ which is an entire graph. Then S is J-holomorphic.*

For $n = 3$, this is just a weaker version of Bernstein's problem as a plane has an obvious complex structure, so it is clearly true.

For the $n = 4$ case, the conjecture was proven by Micallef [11] in 1984. In particular, Micallef proved a stronger result for which this conjecture in $\mathbb{R}^4$ is a corollary.

Theorem (Micallef's theorem)**.** *Let $F : M^2 \to \mathbb{R}^4$ be an isometric stable minimal immersion of a complete oriented parabolic surface into Euclidean 4-space. Then F is holomorphic with respect to some orthogonal complex structure on $\mathbb{R}^4$.*

The above theorem appears as Theorem I in Micallef's paper. The author references a corollary of Theorem 5.1 in [13] due to Osserman which states that an entire minimal graph is conformally equivalent to the complex plane, and so in particular is parabolic. Thus, this theorem directly proves the conjecture in the $n = 4$ case.

It is worth discussing some of the details of Micallef's proof of this theorem in order to see what obstructs this theorem from being extended to dimensions $n > 4$.

The most obvious use of the $n = 4$ condition is that since the normal bundle is rank two, we may define a complex structure on a local frame. If e_3, e_4 are a local orthonormal frame for the normal bundle, then we can define a complex structure that can be thought of simply as rotation by 90 degrees. That is the complex structure maps e_3 to e_4 and e_4 to $-e_3$. The reason why this is so helpful is that if the surface were indeed holomorphic, the complex structure would preserve the normal bundle and would either be equivalent to the complex structure that we have defined on this local frame or equivalent to its inverse which corresponds to rotation in the opposite direction. Thus, we can now make calculations with this local complex structure knowing that they must correspond with the actual holomorphic structure of the surface if indeed the conclusion of the theorem holds. If the normal bundle were rank four, we could similarly define a local complex structure on it, but even if the surface is holomorphic, there is no guarantee that the local complex structure that we have defined corresponds to the actual complex structure from the surface being holomorphic.

Another benefit of this local complex structure on the normal bundle is that it constrains it quite a bit. Micallef defines V to be the $(1, 0)$ part of the normal bundle, i.e. the span of $e_3 - ie_4$ with respect to the local frame. The normal derivative maps sections of V to sections of V, and using a theorem proved earlier about constructing orthogonal complex structures, Micallef is able to reduce the problem down to finding sections s of V such that $(\partial s)^T = 0$. That is, if at every point on the surface there exists a local nonzero section s of V such that $(\partial s)^T = 0$, then the surface is indeed holomorphic. This is only true for surfaces in $\mathbb{R}^4$, as it relies on the restrictions given by having a rank two normal bundle.

The complex stability inequality takes the form

$$\int \left\|(\partial s)^T\right\|^2 \leq \int \left\|(\overline{\partial} s)^N\right\|^2.$$

So there is a motivation to find holomorphic sections of the normal bundle and by stability these sections must satisfy $(\partial s)^T = 0$.

Although there are many details from the proof omitted here, Micallef finishes the proof using a version of the stability inequality to show that the second fundamental form must have either vanishing $(1,0)$ or $(0,1)$ part. Since we can swap these parts arbitrarily by changing the definition of the complex structure, we can suppose that the $(0,1)$ part vanishes, and this is enough the prove that in fact every local non-zero section s of V satisfies $(\partial s)^T = 0$. By the discussion above, this is enough to show the surface is indeed holomorphic with respect to some orthogonal complex structure on $\mathbb{R}^4$.

So indeed it is clear that the proof of this theorem has many obstructions preventing it from being extended in any obvious way for $n > 4$. Indeed, for the $n > 4$ case this conjecture still remains open. In the same paper, Micallef proved a result similar to this conjecture for all n, but with the graphical condition replaced with the condition that the surface has finite total curvature and genus zero. The proof in this result relies on Grothendieck line bundle splitting over the compactified surface $\mathbb{CP}^1$. It may be that some eventual proof of the conjecture in the $n > 4$ case more closely resembles this line bundle splitting argument than it does the proof of Micallef's theorem in the $n = 4$ case.

1.4 Osculating spaces in the study of minimal surfaces

The titular topic of this paper, namely osculating spaces, has rarely been used in the study of minimal surfaces. There is a paper of Calabi [4] from 1967 which uses a very similar idea to these osculating spaces in the analysis of minimal 2-spheres immersed in Euclidean spheres.

In the paper, Calabi considers a map $F : \Sigma \to \mathbb{R}^n$ such that $F \cdot F = r^2$ for some fixed $r \in \mathbb{R}$. Thus, the image of the surface lies in a sphere $rS^{n-1} \subset \mathbb{R}^n$. Then, taking z as a local isothermal parameter of Σ, Calabi establishes that if the immersion of the surface in the sphere is minimal then $F_{z\bar{z}}$ must be proportional to the position vector F. More concretely,

$$F_{z\bar{z}} = -\frac{|F_z|^2}{r^2}F.$$

Calabi then considers the wedge product of the first k derivatives of the immersion

$$T_k F = \partial F \wedge \partial^2 F \wedge \cdots \wedge \partial^k F,$$

and its conjugate. These sections of $\Lambda^k(\mathbb{C}^n)$ for $1 \leq k \leq n$ are essentially the same as the osculating spaces as defined in this paper with a few key differences. Calabi notably does not use the term osculating spaces in his paper.

Calabi uses some calculations involving the derivatives and norms of these sections $T_k F$ in order to prove a bound on the area of the immersed S^2. The convergence of the integrals is dependent on the compactness of the surface, which does not in general carry over to our setting of surfaces immersed in Euclidean space.

For these reasons, we cannot meaningfully extend Calabi's results for minimal spheres to minimal surfaces in $\mathbb{R}^n$.

In contrast to Calabi's work, we define osculating spaces as vector bundles over the surface rather than simply as sections of $\Lambda^j(\mathbb{C}^n)$. We would like to define the jth osculating space as the span of the first j derivatives pointwise over the surface. This turns out to be a fine definition almost everywhere on the surface, but there are potentially isolated points where the dimension of the span is less than j. To remedy this, we prove a result that shows the j-dimensional spaces around these isolated points have a notion of a limit so that we can assign a j-dimensional vector space at these points in a smoothly varying way. Thus, we prove the following theorem, where $\tilde{E}_j$ denotes the pointwise span of the first j derivatives.

Theorem. *If $\tilde{E}_j$ has dimension j for all $z \in D$ except for some isolated points, then there exists a vector bundle E_j of rank j such that $E_j(z) = \tilde{E}_j(z)$ at all points in D besides those isolated points.*

It is then these E_j that we define as the jth osculating space of the minimal surface and top-dimensional osculating space $E = E_k$ we will also call the osculating space of the surface.

Following this definition of the osculating spaces proper, we then derive several important properties of osculating spaces, including the following propositions.

Proposition. *Let $F : D \to \mathbb{R}^n$ be a harmonically parametrized surface with osculating spaces $E_1, \ldots, E_k$. The top dimensional osculating space $E = E_k$ is a constant vector subbundle of $\mathbb{C}^n$, i.e. $E_k(z)$ is the same k-dimensional subspace of $\mathbb{C}^n$ for all $z \in D$.*

Proposition. *Let z and w be two isothermal parameters for a neighborhood of a minimal surface in $\mathbb{R}^n$. Then the osculating spaces as defined using each of these two coordinates are the same.*

The latter proposition shows that despite the osculating spaces being computed in coordinates, they are independent of our choice of coordinates and so are geometrical in nature. This is very important because otherwise we could not hope that they would tell us anything meaningful about the surface.

Following the derivation of these properties of osculating spaces, we then explore several areas relating to minimal surfaces in which osculating spaces provide some illumination.

The first is regarding holomorphic curves, and we show that osculating spaces provide an equivalent condition to determine whether a minimal surface is a holomorphic curve. We define a vector subspace of $\mathbb{C}^n$ to be null if the dot product is fully degenerate on that subspace. Then, as specified in the following proposition and theorem, the osculating space being null is equivalent to the surface being a holomorphic curve. Note that the converse is much more verbose to state formally because there are surfaces which are morally holomorphic curves but technically do not fit the definition.

Proposition. *The osculating space E is null for any J-holomorphic surface.*

Theorem. *Every minimal surface with an osculating space E that is null must lie inside an even-dimensional affine subspace $\mathbb{R}^{2k} \subset \mathbb{R}^n$ and there exists a linear complex structure J compatible with the Euclidean metric on $\mathbb{R}^{2k}$ for which the surface is J-holomorphic.*

A second area is regarding the dot product as a bilinear form on the osculating space. We examine the null space of this bilinear form and what properties it can have in relation to the osculating space itself. This gives us important information about which subspaces of $\mathbb{C}^n$ are able to be an osculating space of some minimal surface. One powerful result we are able to prove is the following theorem.

Theorem. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k that has a null space E^0 of dimension l. Then $l \neq k-1$ and $l \neq k-2$.*

The third area is decomposable minimal surfaces, defined as a minimal surface immersed in $\mathbb{R}^n$ that can be written as the direct sum of two minimal surfaces immersed in $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$. We show that the dimension of the osculating space behaves nicely with respect to decomposition.

Proposition. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a decomposable minimal surface where we represent the decomposition as $F = (F^1, F^2)$ with $F^1 : D \to \mathbb{R}^m$*

and $F^2 : D \to \mathbb{R}^{n-m}$*. As usual we let* k *be the dimension of the osculating space of* F*, and we let* k_1 *and* k_2 *be the dimensions of the osculating spaces of* F^1 *and* F^2 *respectively.*

Then we have that

$$k_1, k_2 \le k \le k_1 + k_2.$$

We also prove that every minimal surface in $\mathbb{R}^n$ can be decomposed into a holomorphic surface and a minimal surface with an osculating space on which the dot product is non-degenerate.

Proposition. *Let* $F : D \to \mathbb{R}^n$ *be a local conformal parametrization of a minimal surface with osculating space* E *of dimension* k *and null space* E^0 *of dimension* l*. Then the surface is decomposable with* $F = (H, L)$ *where* $H : D \to \mathbb{R}^{2l}$ *is* J*-holomorphic with respect to some linear complex structure on* $\mathbb{R}^{2l}$ *and* $L : D \to \mathbb{R}^{n-2l}$ *is minimal.*

Chapter 2

Setting

Let $\iota : S \to \mathbb{R}^n$ be an immersion of a Riemann surface into Euclidean n-space. Let z be a local parameter for S around a point p, such that $\varphi : U \to \mathbb{C}$ is the coordinate function taking some neighborhood U of p to the value of the parameter z at that point, with $\varphi(p) = 0$. Then let $F : D \subset \mathbb{C} \to \mathbb{R}^n$ be the map that takes the value of the local parameter z to the location of the associated point in S within $\mathbb{R}^n$, i.e. $F = \iota \circ \varphi^{-1}$. Let us compute some geometric properties of S in terms of the parametrization F.

2.1 Metric

First, let us compute the induced metric g on $\mathbb{C}$ in terms of F. Now the metric on $\mathbb{C}$ is just the pullback of the Euclidean metric δ on $\mathbb{R}^n$ by F, so $g(V, W) = dF(V) \cdot dF(W)$, where $V, W \in T_0\mathbb{C}$ and $\cdot$ is the Euclidean dot product in $\mathbb{R}^n$.

Consider then $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ which are a basis for the tangent space of $\mathbb{C} = \mathbb{R}^2$ at every point with the real coordinates $z = x + iy$. If we define a curve $\gamma : (-\epsilon, \epsilon) \to D$ around a point

(x, y) by $\gamma(t) = (x + t, y)$, then

$$dF\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial t}\bigg|_{t=0} (F \circ \gamma) = \frac{\partial F}{\partial x}(x, y). \tag{2.1}$$

We will use F_x to denote $\frac{\partial F}{\partial x}$ and by a similar argument we have that $dF\left(\frac{\partial}{\partial y}\right) = F_y$.

So then we have that

$$g = (F_x \cdot F_x)dx^2 + 2(F_x \cdot F_y)dxdy + (F_y \cdot F_y)dy^2.$$

However, by definition of z being a local parameter we have that

$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right) \text{ and}$$
$$g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0.$$

This tells us that $F_x \cdot F_x = F_y \cdot F_y$ and $F_x \cdot F_y = 0$. If we define $\lambda = F_x \cdot F_x$ as the conformal parameter, then we get that

$$g = \lambda\left(dx^2 + dy^2\right). \tag{2.2}$$

Now so far we have been working with the real tangent spaces of $\mathbb{C}$ and $\mathbb{R}^n$, but we can also extend this to the complexified tangent spaces. Since we have $z = x+iy$ and so $dz = dx+idy$, we define

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \text{ and}$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$

so that

$$dz\left(\frac{\partial}{\partial z}\right) = d\bar{z}\left(\frac{\partial}{\partial \bar{z}}\right) = 1 \text{ and}$$
$$dz\left(\frac{\partial}{\partial \bar{z}}\right) = d\bar{z}\left(\frac{\partial}{\partial z}\right) = 0.$$

Then we get that

$$dF\left(\frac{\partial}{\partial z}\right) = \frac{1}{2}\left(F_x - iF_y\right) = F_z$$

where F_z is the partial derivative of F with respect to z, and we also get a similar result for $\bar{z}$. So we also have

$$F_z \cdot F_z = F_{\bar{z}} \cdot F_{\bar{z}} = 0 \text{ and}$$
$$|F_z|^2 = F_z \cdot F_{\bar{z}} = \frac{\lambda}{2}.$$

Then in terms of these complex forms we have

$$g = \lambda dz d\bar{z}. \tag{2.3}$$

2.2 Second fundamental form

Next let us compute the second fundamental form of S as a submanifold of $\mathbb{R}^n$ in terms of F, but first we must understand the connection on $\mathbb{R}^n$ in terms of F. Let $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ be the standard basis for the tangent space on $\mathbb{R}^n$ and let $V = V^1\frac{\partial}{\partial x^1} + \ldots + V^n\frac{\partial}{\partial x^n}$ be a vector field on the tangent space of S immersed in $\mathbb{R}^n$ where each $V^i : D \subset \mathbb{C} \to \mathbb{R}$ is a real function in terms of (x, y). Then, since the coordinate partials are constant with respect to the connection on $\mathbb{R}^n$, we have that

$$\nabla_{F_x} V = \sum_{i=1}^{n} \left(\nabla_{F_x} V^i\right) \frac{\partial}{\partial x^i},$$

so we only need to care about computing the derivative of smooth functions. As in Equation 2.1, we have that

$$\nabla_{F_x} V^i = dV^i \left(\frac{\partial}{\partial x}\right) = \frac{\partial V^i}{\partial x},$$

and a similar property holds for the covariant derivative with respect to F_y. Thus, we get that

$$\begin{aligned} \nabla_{F_x} F_x &= F_{xx}, \\ \nabla_{F_y} F_x &= F_{xy}, \\ \nabla_{F_x} F_y &= F_{yx}, \text{ and} \\ \nabla_{F_y} F_y &= F_{yy}, \end{aligned}$$

where F_{xx} denotes the second partial derivative of F with respect to x, and similarly for the others. Then, if we let indices i, j take values corresponding to x, y, we get that the second fundamental form is

$$h_{ij} = \left(\nabla_{F_i} F_j\right)^{\perp} = (F_{ij})^{\perp}, \tag{2.4}$$

where the superscript $\perp$ is denoting projection to the normal bundle of S in $\mathbb{R}^n$.

We can again think about this in terms of the complexified tangent spaces. If we let α, β take values corresponding to $z, \bar{z}$, then we can compute in the same way that $h_{\alpha\beta} = (F_{\alpha\beta})^{\perp}$. Now we have equality of mixed partials so $F_{z\bar{z}} = F_{\bar{z}z}$ and since $F_z \cdot F_z = F_{\bar{z}} \cdot F_{\bar{z}} = 0$, by

Leibniz rule we also get

$$F_{zz} \cdot F_z = 0,$$
$$F_{z\bar{z}} \cdot F_z = 0,$$
$$F_{z\bar{z}} \cdot F_{\bar{z}} = 0, \text{ and}$$
$$F_{\bar{z}\bar{z}} \cdot F_{\bar{z}} = 0.$$

In particular,

$$h_{zz} = (F_{zz})^{\perp} = F_{zz} - \frac{F_{zz} \cdot F_{\bar{z}}}{|F_z|^2} F_z,$$
$$h_{z\bar{z}} = (F_{z\bar{z}})^{\perp} = F_{z\bar{z}},$$
$$h_{\bar{z}\bar{z}} = (F_{\bar{z}\bar{z}})^{\perp} = F_{\bar{z}\bar{z}} - \frac{F_{\bar{z}\bar{z}} \cdot F_z}{|F_z|^2} F_{\bar{z}},$$

and since $F_{z\bar{z}} = F_{\bar{z}z}$ we of course also get $h_{z\bar{z}} = h_{\bar{z}z}$, which is expected as the second fundamental form should be symmetrical.

2.3 Curvature

Next, we can compute the curvature tensor of S and other related quantities based on what we have determined about the second fundamental form. Moving forward we will only be working with respect to complex coordinates.

For a 2-dimensional surface we know that the Riemann curvature tensor only has one independent component, and we can write

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta}\gamma),$$

where K is the Gaussian curvature of the surface. Using Equation 2.3 we get that $R_{z\bar{z}z\bar{z}} = -\frac{\lambda^2}{4}K$, and so if we can compute that term of the curvature tensor in some other way then we can solve for the Gaussian curvature, which completely determines the curvature tensor.

We have that

$$
\begin{aligned}
R_{z\bar{z}z\bar{z}} &= g_{z\bar{z}} R_{z\bar{z}}{}^{\bar{z}}{}_{\bar{z}} \\
&= \frac{\lambda}{2} d\bar{z} \left(R\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}} \right) \\
&= \frac{\lambda}{2} \frac{2}{\lambda} g \left(R\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} \right) \frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z} \right) \\
&= g\left(h_{zz}, h_{\bar{z}\bar{z}}\right) - g\left(h_{z\bar{z}}, h_{z\bar{z}}\right) \\
&= |\left(F_{zz}\right)^{\perp}|^2 - |F_{z\bar{z}}|^2 ,
\end{aligned}
$$

where the fourth equality is by the Gauss equation. Thus, we get that

$$
K = \frac{4}{\lambda^2} \left(|F_{z\bar{z}}|^2 - |\left(F_{zz}\right)^{\perp}|^2 \right). \tag{2.5}
$$

Next, we can compute the mean curvature vector H as half of the g-trace of the second fundamental form h, so

$$
\begin{aligned}
H &= \frac{1}{2} g^{\alpha\beta} h_{\alpha\beta} \\
&= \frac{1}{2} \left(g^{z\bar{z}} h_{z\bar{z}} + g^{\bar{z}z} h_{\bar{z}z} \right) \\
&= \frac{2}{\lambda} F_{z\bar{z}}.
\end{aligned}
$$

Thus, S is minimal if and only if $F_{z\bar{z}} = 0$, which means each of the component functions of F is harmonic.

2.4 Minimal surfaces

As we've seen, the condition that S is minimal is equivalent to $F_{z\bar{z}} = 0$. By equality of mixed partials we have that any mixed partial derivative of F containing z and $\bar{z}$ will vanish. We define

$$G_j = \frac{\partial^j F}{\partial z^j}$$

and note that since F is a real-valued function we get

$$\frac{\partial^j F}{\partial \bar{z}^j} = \overline{\frac{\partial^j F}{\partial z^j}} = \overline{G_j}$$

Thus, all non-zero derivatives of F can be represented in terms of G_j. Furthermore, since

$$\frac{\partial G_j}{\partial \bar{z}} = 0$$

due to equality of mixed partials as mentioned above, G_j is holomorphic for all $j \geq 1$.

Proposition 2.1. *Let $\phi : D \to \mathbb{C}^n$ be a holomorphic map on some open disk $D \subset \mathbb{C}^n$ that satisfies $\phi \cdot \phi = 0$ i.e. ϕ is an isotropic vector field. Then there exists a local minimal surface with conformal parametrization $F : D \to \mathbb{R}^n$ such that $F_z = \phi$.*

Proof. Let us write

$$\phi(z) = (\phi^1(z), \ldots, \phi^n(z))$$

so that ϕ^j represents the component functions of ϕ as a vector field. Let $c \in \mathbb{R}^n$ be any constant vector. We define

$$F = c + \left(2\text{Re} \int \phi^1(z)dz, \ldots, 2\text{Re} \int \phi^n(z)dz\right).$$

Notice these integrals are well-defined since D is simply-connected.

It is straightforward to verify that indeed $F_z = \phi$. Therefore, F is the conformal parametrization of some surface since ϕ is isotropic and that surface is minimal because ϕ is holomorphic. □

Remark. This proposition tells us that we can talk interchangeably of a local minimal surface and of a holomorphic isotropic vector field which we can presumptively call F_z.

Chapter 3

Osculating spaces

3.1 Definition of osculating spaces

Our primary interest in this paper is minimal surfaces, and indeed we will use the properties of osculating spaces in application to minimal surfaces. However, the construction of osculating spaces and their properties does not rely on the fact that F is a conformal parametrization, only that it is harmonic. Thus, we will define osculating spaces for harmonically parametrized surfaces for the sake of generality, with the understanding that we are interested in the specific case of surfaces that are simultaneously harmonically and conformally parametrized, i.e. minimal surfaces.

Definition 3.1. A parametrization $F : D \to \mathbb{R}^n$ of a surface with $D \subset \mathbb{C}$ an open disk is *harmonic* if $F_{z\bar{z}} = 0$.

Suppose we have such a locally defined harmonic surface $F : D \to \mathbb{R}^n$ such that $D \subset \mathbb{C}$ is an open disk. Then, as stated above, we may define $G_j : D \to \mathbb{C}^n$ to be the jth derivative of F with respect to z, and due to F being harmonic we get that each of these G_j are holomorphic.

Now, for all $z \in D$ we define $\tilde{E}_j(z) = \text{span}\{G_1(z), \ldots, G_j(z)\}$. This is not quite the formal definition for the osculating space, but it is morally the definition for the osculating space. The name for the osculating space is borrowed from the notion of the osculating plane of a curve which is spanned by the tangent and normal vector to the curve. Our intuition is that the jth osculating space is the span of the first j derivatives of F with respect to z, but we will define it slightly different in the end to get better behavior at certain points that are in some sense singular.

In order to formally define osculating spaces, we must first explore the properties of this vector space $\tilde{E}_j(z)$ and how varying j and z may affect it.

Proposition 3.2. *Let A_j be the set of points $z \in D$ for which the dimension of $\tilde{E}_j(z)$ is less than j i.e.*

$$A_j = \{z \in D : dim\ \tilde{E}_j(z) < j\}.$$

Then either $A_j = D$ or A_j consists only of isolated points in D.

Proof. Fix j and consider the wedge product $G_1 \wedge \cdots \wedge G_j$ as a section of the vector bundle $\bigwedge^j (\mathbb{C}^n)$ over D. We can write this product with respect to the standard basis of the wedge product as follows

$$G_1 \wedge \cdots \wedge G_j = \sum_{1 \leq i_1 \leq \ldots \leq i_j \leq n} a_{i_1,\ldots,i_j} e_{i_1} \wedge \cdots \wedge e_{i_j},$$

where the coefficients are holomorphic functions $a_{i_1,\ldots,i_j} : D \to \mathbb{C}$ since each of the factors $G_1, \ldots, G_j$ are holomorphic. Then the set of points A_j where the dimension of $\tilde{E}_j(z)$ is less than j is precisely the intersection of the zero sets of each of the $a_{i_1,\ldots,i_j}$ functions, proving the claim. □

Remark. This proposition tells us that for a given j, our pseudo-osculating space $\tilde{E}_j(z)$ is either precisely of dimension j for almost every point in D or it's never dimension j.

Intuitively, it should also be the case that if it isn't of dimension j, then for any larger indices it should also not have a dimension matching the index. In order to prove this we first need a lemma.

Lemma 3.3. *The function dim $\tilde{E}_j : D \to \mathbb{Z}$ is lower semicontinuous.*

Proof. Let $z_0 \in D$ be an arbitrary point and suppose that dim $\tilde{E}_j(z_0) = m \leq j$. Then let $G_{i_1}(z_0), \ldots, G_{i_m}(z_0)$ be m linearly independent vectors that span $\tilde{E}_j(z_0)$. Then $G_{i_1} \wedge \cdots \wedge G_{i_m}$ is holomorphic and nonzero at z_0 and thus it must also be nonzero in a neighborhood of z_0. Therefore in that neighborhood of z_0, the dimension of $\tilde{E}_j(z)$ is greater than or equal to m, proving the claim. □

Proposition 3.4. *There exists an index k such that dim $\tilde{E}_j(z) \leq k$ for all j and $z \in D$ and dim $\tilde{E}_j(z) = j$ for all $j \leq k$ and all $z \in D$ except for some isolated points.*

Proof. Let A_j be as defined in Proposition 3.2 and let us choose k such that $k+1$ is the smallest index for which $A_{k+1} = D$ (this is guaranteed to exist because of Proposition 3.2 together with the fact that the dimension cannot be more than n).

Since $k+1$ is the smallest index where $A_{k+1} = D$, $\tilde{E}_j(z)$ must have dimension j for all $j \leq k$ and all $z \in D$ except for some isolated points. It remains only to show that dim $\tilde{E}_j(z) \leq k$ for all j and all $z \in D$. Consider the open set $U = D \setminus A_k$ which is just D minus the isolated points where $\tilde{E}_k(z)$ has dimension less than k.

For every $z \in U$ we have that $G_1(z), \ldots, G_k(z)$ are linearly independent and $G_{k+1}(z)$ lies in their span. So we can write $G_{k+1} = f_1 G_1 + \ldots + f_k G_k$ where each $f_j : D \to \mathbb{C}$ is meromorphic on D and holomorphic on U. This follows from writing the components of $G_1, \ldots, G_k$ as columns of a matrix with G_{k+1} as an augmented column and row-reducing over the field of meromorphic functions on D.

Notice that if we take the z-derivative of that expression for G_{k+1} we get

$$\begin{aligned} G_{k+2} &= \partial_z f_1 G_1 + (f_1 + \partial_z f_2) G_2 + \ldots + (f_{k-1} + \partial_z f_k) G_k + f_k G_{k+1} \\ &= (f_1 f_k + \partial_z f_1) G_1 + (f_1 + f_2 f_k + \partial_z f_2) G_2 + \ldots \\ &\quad + \left(f_{k-1} + f_k^2 + \partial_z f_k\right) G_k. \end{aligned}$$

Thus, G_{k+2} is also a linear combination of $G_1, \ldots, G_k$ over meromorphic functions that are holomorphic on U. Continuining this process by taking more z-derivatives gives us an expression for G_j as a linear combination of $G_1, \ldots, G_k$ over meromorphic functions that are holomorphic on U for any $j > k$.

Thus, the dimension of $\tilde{E}_j(z)$ is at most k for all j and all $z \in U$. At each of the isolated points in A_k we may apply Lemma 3.3 which shows the dimension at those points must also be at most k, completing the proof. □

We would like for the osculating space to be a vector subbundle of $\mathbb{C}^n$ over D. However, if the fibers are to be the vector spaces $\tilde{E}_j(z)$, then there are potentially isolated points of D where the dimension is lower than j. Our hope is that the j-dimensional spaces have a well-defined limit at those isolated points so that we can in fact define such a vector bundle.

Theorem 3.5. *If $\tilde{E}_j$ has dimension j for all $z \in D$ except for some isolated points, then there exists a vector bundle E_j of rank j such that $E_j(z) = \tilde{E}_j(z)$ at all points in D besides those isolated points.*

Proof. The idea of the proof is to look at the image of these vector spaces in the Grassmannian with a clever choice of coordinates and show the limit exists as we approach one of these isolated points.

Without loss of generality suppose we have an isolated point at $z = 0$, i.e. the span of

$G_1(0), \ldots, G_j(0)$ has dimension less than j. Recall that coordinate charts of the Grassmannian $G(n, j)$ come from choosing j vectors from your subspace, arranging them as rows of a $j \times n$ matrix, and then rearranging the columns such that the first j columns are linearly independent. Each choice of rearranging the columns gives us a different coordinate chart in the Grassmannian, and after we've rearranged the columns we row reduce the matrix and the rightmost $j \times (n-j)$ block of the matrix gives us the coordinates of the subspace in that chart.

Now take $G_1, \ldots, G_j$ as the rows of a $j \times n$ matrix such that the elements of the matrix are from the ring of holomorphic functions on D. If we were to evaluate all these functions at $z = 0$, then we would get a matrix with complex elements with some rank $r < j$ which is equal to the dimension of the vector space $\tilde{E}_j(0)$. We can rearrange the columns of the matrix with holomorphic elements such that the first r are linearly independent after evaluating at $z = 0$. We can then proceed to row reduce just the first r columns, leaving the leftmost $j \times r$ block with entries of 1 along the main diagonal and 0 everywhere else. Since the first r columns were linearly independent at $z = 0$, the remaining entries in the rightmost $j \times (n-r)$ block of the matrix are all meromorphic functions that are holomorphic at $z = 0$.

Since we started with the assumption that the matrix evaluated at $z = 0$ had rank r, all the entries in the lower right $(j - r) \times (n - r)$ block must be zero when evaluated at $z = 0$, otherwise there would be more than r pivots. For each of these $j - r$ rows, we can divide by z to the power of the largest order zero common to all the meromorphic functions within that row. If all the functions in that row are equivalently zero, then $\tilde{E}_j(z)$ has dimension less than j at all points in D, contradicting our assumption. After dividing each of those rows by some power of z, we get a matrix whose entries are all still meromorphic functions that are holomorphic at $z = 0$ and each of the rows of that $(j - r) \times (n - r)$ block are not all zero at $z = 0$.

Now we can repeat this entire process with this $(j - r) \times (n - r)$ matrix. It must have some nonzero rank s when evaluated at $z = 0$ since we ensured it was not all zero. Therefore, we can rearrange the columns so that the first s of the $(j-r) \times (n-r)$ block are linearly independent when evaluated at $z = 0$, making sure that we rearrange the columns of the original matrix too. Then, we can row reduce those s columns within the larger matrix, and for the same reasoning as above, all the remaining entries of the matrix are meromorphic functions that are holomorphic at $z = 0$. That will leave a $(j-r-s) \times (n-r-s)$ block that we can divide out by the highest powers of z in each row, and continue the process. That lower right block cannot be identically zero at any step because that would mean the matrix has rank less than j for all $z \in D$ which implies the dimension of $\tilde{E}_j(z)$ is less than j for all $z \in D$, contradicting our assumption. Thus, the lower block will always have rank at least one and so this process is guaranteed to terminate in at most j steps.

In the end we are left with some rearrangement of the original columns of the matrix such that after row-reducing the left $j \times j$ block is the identity matrix and the right $j \times (n-j)$ block has entries which are meromorphic functions that are holomorphic at $z = 0$. Thus, in this coordinate chart on the Grassmannian there is some neighborhood of $z = 0$ which is well-defined.

Thus, the limit of the nearby vector spaces at the isolated points are well-defined and so we can form the vector bundle E_j over D. □

This proposition tells us that although there are potentially singularities at some isolated points in D where the dimension of $\tilde{E}_j$ is less than j, all those singularities are in some sense removable, and there is a natural way of assigning a j-dimensional subspace of $\mathbb{C}^n$ to those points. This allows us to finally define the osculating spaces as vector bundles.

Definition 3.6. Let $F : D \to \mathbb{R}^n$ be a harmonically parametrized surface on a disk $D \subset \mathbb{C}$. We define the *jth osculating space* of the surface F to be the vector bundle E_j as in Theorem 3.5 for all $1 \leq j \leq k$ with k as in Proposition 3.4. We also write $E = E_k$ for the top-dimensional osculating space.

3.2 Properties of osculating spaces

Now that we have formally defined osculating spaces for harmonic (and minimal) surfaces, we would like to prove some properties that they have and build some helpful intuition for them.

So far our understanding is that every harmonic (minimal) surface $F : D \to \mathbb{R}^n$ has a nested sequence of vector bundles over D,

$$E_1 \subsetneq \ldots \subsetneq E_k = E,$$

up to some number k that depends on the surface. Each osculating space E_j has rank j and its fibers above each $z \in D$ besides isolated points are the subspaces $\tilde{E}_j(z) = \operatorname{span}\{G_1(z), \ldots, G_j(z)\} \subset \mathbb{C}^n$. Taking the span of more than k derivatives will give us a subspace that is at most k-dimensional.

Since

$$E_1 = \operatorname{span}\{G_1\} = \operatorname{span}\{F_z\},$$

the complexified tangent space is

$$TM \otimes \mathbb{C} = E_1 \oplus \overline{E_1}.$$

Now to expand our intuition, first notice that at all points in D except for some isolated points, the fibers of $E = E_k$ are

$$E_k(z) = \text{span}\{G_1(z), \ldots, G_k(z)\} = \text{span}\{G_1(z), \ldots, G_k(z), \ldots\},$$

where we use $E_k(z)$ as shorthand for the fiber of E_k at z. That is, the fibers are the span of the first k z-derivatives of F which is equal to the span of all z-derivatives of F because of Proposition 3.4 that tells us the span of derivatives of F has dimension at most k. Because E is the span of all z-derivatives of F (away from isolated points), sections of E are closed under taking z-derivatives and $\bar{z}$-derivatives. Intuitively this suggests that the subspaces $E(z) \subset \mathbb{C}^n$ are the same subspace at every point $z \in D$.

Proposition 3.7. *Let $F : D \to \mathbb{R}^n$ be a harmonically parametrized surface with osculating spaces $E_1, \ldots, E_k$. The top dimensional osculating space $E = E_k$ is a constant vector subbundle of $\mathbb{C}^n$, i.e. $E_k(z)$ is the same k-dimensional subspace of $\mathbb{C}^n$ for all $z \in D$.*

Proof. Let A_k be the set of isolated points in D where $\tilde{E}_k(z)$ has dimension less than k as in Proposition 3.2. Equivalently, A_k is the set of points where $E_k(z) \neq \tilde{E}_k(z)$.

Without loss of generality, let $0 \in D \backslash A_k$, i.e. $E_k(0)$ is equal to the span of $G_1(0), \ldots, G_k(0)$. Then, since A_k consists only of isolated points, let $U \subset D \backslash A_k$ be an open disk containing 0.

Now suppose for some $z_0 \in U$ we have that $E_k(z_0) \neq E_k(0)$, so there is a vector $v \in E_k(z_0)$ such that $v \notin E_k(0)$. Let $v = v_1 G_1(z_0) + \ldots + v_k G_k(z_0)$ for $v_1, \ldots, v_k \in \mathbb{C}$. Now we define $s = \frac{z}{z_0}(v_1 G_1 + \ldots + v_k G_k)$ which is a holomorphic section of E.

Since we can always write G_{k+1} as a holomorphic linear combination of $G_1, \ldots, G_k$ on U, all derivatives of s remain local sections of E over U. Since s is a holomorphic section of E, it is complex analytic and so

$$s(z) = \sum_{j=0}^{\infty} \partial_z^j s(0) \frac{z^j}{j!}$$

for all $z \in U$. Since all derivatives of s are sections of E, $\partial_z^j s(0) \in E_k(0)$ for all j.

Therefore, each partial sum in the series above lies in $E_k(0)$ and the partial sums converge to $s(z)$. Then, since $E_k(0)$ is a subspace of $\mathbb{C}^n$ and therefore closed in $\mathbb{C}^n$, $s(z) \in E_k(0)$ for all $z \in U$. In particular, $v = s(z_0) \in E_k(0)$, a contradiction. So we conclude that $E_k(z)$ is the same vector subspace of $\mathbb{C}^n$ for all $z \in U$.

From the above we get that $E_k(z)$ is constant for all $z \in D \backslash A_k$. Then, at each of the isolated points in A_k, we use the limit construction of E_k at those points to get that E_k is in fact constant for all of D. □

The next piece of helpful intuition is that the dimension of the osculating space gives us some information about the dimension of the surface. By analogy, for a planar curve, all derivatives of the parametrization of the curve are vectors that lie in the plane that contains the curve. The same idea is true for harmonic surfaces except that we have the z-derivative and the $\bar{z}$-derivative to consider. If the harmonic surface lies in a hyperplane, then the osculating space E and its conjugate $\overline{E}$ must also lie in the same hyperplane.

Before we can prove that, we first need a lemma which tells us that a complex subspace closed under the complex conjugate must be the complexification of some real subspace.

Lemma 3.8. *Let W be a linear subspace of $\mathbb{C}^n$. If for all $v \in W$ we have that $\overline{v} \in W$, then W is the complexification of some real vector subspace $V \subset \mathbb{R}^n$.*

Proof. Define V to be the real vector space consisting of all real vectors contained in W and define $V^{\mathbb{C}}$ to be the complexification of V.

Any vector in $V^{\mathbb{C}}$ is of the form $u + iv$ for $u, v \in V$, so $u + iv \in W$.

For any $w \in W$ we know that $\overline{w} \in W$, so Re $w = \frac{1}{2}(w + \overline{w}) \in W$ and Im $w = \frac{1}{2i}(w - \overline{w}) \in W$. Since Re w and Im w are real vectors they must also be in V, and thus $w =$ Re $w + i$Im $w \in$

$V^{\mathbb{C}}$.

So in fact $W = V^{\mathbb{C}}$. □

Proposition 3.9. *Let $F : D \to \mathbb{R}^n$ be a harmonically parametrized surface and fix some real vector $a \in \mathbb{R}^n$. The surface F lies in the affine hyperplane $a \cdot F = b$ for some $b \in \mathbb{R}$ if and only if $a \cdot v = 0$ for all $v \in E + \overline{E}$.*

Remark. Note that due to Proposition 3.7 we are able to simultaneously think of E as both a vector bundle and a vector space. Thus, when we write $E + \overline{E}$, we mean this in the standard linear algebra sense of the sum of subspaces, rather than as the sum of vector subbundles. Notice also that we do not write this as a direct sum, as in general $E \cap \overline{E}$ may be non-trivial.

Proof. Suppose $a \cdot F = b$, then taking any number of z or $\bar{z}$ derivatives gives $a \cdot G_j = a \cdot \overline{G_j} = 0$ for all j. Then, for any $v \in E + \overline{E}$, there exists some point z_0 at which we can write v as a linear combination $v = a_1 G_1(z_0) + \ldots + a_k G_k(z_0) + b_1 \overline{G_1}(z_0) + \ldots + b_k \overline{G_k}(z_0)$. Thus, we get that $a \cdot v = 0$.

Now suppose that $a \cdot v = 0$ for all $v \in E + \overline{E}$. Then in particular $a \cdot F_z = 0$ and $a \cdot F_{\bar{z}} = 0$ for all $z \in D$. So $a \cdot F : D \to \mathbb{R}$ is a harmonic function with zero z-derivative and zero $\bar{z}$-derivative, thus it must be a constant function. So $a \cdot F = b$ for some $b \in \mathbb{R}$. □

Corollary 3.10. *If $E + \overline{E}$ is an m-dimensional subspace of $\mathbb{C}^n$, then the surface lies in an m-dimensional affine subspace of $\mathbb{R}^n$.*

Proof. By Lemma 3.8 we have that $E + \overline{E}$ is the complexification of some real m-dimensional vector space V. Then $V^{\perp}$ is an $(n - m)$-dimensional subspace consisting of all vectors a such that $a \cdot v = 0$ for all $v \in V$. Thus by Proposition 3.9 we have that the surface lies in the intersection of $n - m$ non-degenerate hyperplanes. The resulting intersection is precisely an m-dimensional affine subspace of $\mathbb{R}^n$. □

Corollary 3.11. *Let k be the dimension of E. Then the surface lies in an affine subspace of $\mathbb{R}^n$ of dimension $2k - \dim E \cap \overline{E}$.*

Proof. This follows immediately from $\dim E + \overline{E} = \dim E + \dim \overline{E} - \dim E \cap \overline{E}$ and $\dim E = \dim \overline{E}$. ☐

Corollary 3.12. *If $k < n/2$, then the surface cannot lie fully in $\mathbb{R}^n$, i.e. there is some strict affine subspace of $\mathbb{R}^n$ in which the surface lies.*

Proof. $2k - \dim E \cap \overline{E} \leq 2k < n$. ☐

Corollary 3.13. *Let $F : D \to \mathbb{R}^n$ be a surface parametrized by harmonic polynomials of degree at most d. Then the surface lies in an affine subspace of $\mathbb{R}^n$ of dimension $2d$.*

Proof. If F consists of harmonic polynomials of degree d then $G_j = 0$ for $j > d$, thus $k \leq d$. ☐

The above proposition and subsequent corollaries are all various ways of representing the same idea that the real part of $E + \overline{E}$ is parallel to the affine subspace that the surface lies in. If the surface lies fully in $\mathbb{R}^n$ then $E + \overline{E} = \mathbb{C}^n$.

Another piece of intuition connected with the above is that since the osculating space E is the same at every point and is defined as the span of derivatives at that point, the osculating space E is a local property of the harmonic surface. Therefore, if any open subset of a harmonic surface lies in some affine subspace of $\mathbb{R}^n$, then in fact the whole surface must lie in that affine subspace.

This same idea can be expressed in terms of F_z and E. That is, E should be completely determined by the values of F_z at points in any open set.

Proposition 3.14. *Let $F : D \to \mathbb{R}^n$ be a harmonic surface with osculating space E. Then for any open set $U \subset D$ we have that $E = span\ \{F_z(p) : p \in U\}$.*

Proof. Let $U \subset D$ be any open set and let A_k be defined as in Proposition 3.2. Then we let $z_0 \in U \backslash A_k$ so that by the definition of A_k we have that $G_1(z_0), \ldots, G_k(z_0)$ span E. We now seek to show that $E^\perp = (\text{span } \{F_z(p) : p \in U\})^\perp$.

Suppose that $a \in (\text{span } \{F_z(p) : p \in U\})^\perp$, so that $a \cdot F_z = 0$ on U. If we take z-derivatives of this equation then we get that $a \cdot G_j = 0$ on U for all j. In particular we get that $a \cdot G_j(z_0) = 0$ for all j. Since the vectors $G_1(z_0), \ldots, G_k(z_0)$ span E we get that $a \cdot v = 0$ for all $v \in E$. Thus $a \in E^\perp$ and so $E^\perp \supset (\text{span } \{F_z(p) : p \in U\})^\perp$.

Next suppose that $a \in E^\perp$, so that $a \cdot v = 0$ for all $v \in E$. Then in particular we have that $a \cdot F_z(p) = 0$ for all $p \in U$, since $F_z(p) \in E$. Thus $a \in (\text{span } \{F_z(p) : p \in U\})^\perp$ and so $E^\perp \subset (\text{span } \{F_z(p) : p \in U\})^\perp$.

Thus, we have proven that $E^\perp = (\text{span } \{F_z(p) : p \in U\})^\perp$, which implies the claim since both E and span $\{F_z(p) : p \in U\}$ are vector spaces. □

The above properties of osculating spaces have all been true in general for harmonic surfaces. However, for the remainder of this paper we will be concerned for minimal surfaces and will use these properties and intuition specifically for the study of minimal surfaces. A paper of T. Milnor [12] from 1983 discusses what properties of minimal surfaces are extensions or special cases of properties of harmonically immersed surfaces. More research could be done into whether osculating spaces can be used to prove anything more about harmonic surfaces beyond what we already have done.

The last property of osculating spaces is perhaps the most important if they are to be of any significance, and that is whether they are geometrical. That is, are these osculating spaces invariant under changes of coordinates on the surface. If they are not, then they are not a geometric property of the surface but only a transient property of our particular coordinates chosen on the surface.

First, we prove a lemma regarding coordinate changes between isothermal parameters.

Lemma 3.15. *Let z be an isothermal parameter on a neighborhood of a minimal surface in $\mathbb{R}^n$ and let $F : D \to \mathbb{R}^n$ be the conformal parametrization of that neighborhood of the surface with respect to the coordinate z. If $w = f(z, \bar{z})$ is another isothermal parameter on this neighborhood of the surface then f must be holomorphic or anti-holomorphic.*

Proof. Let $\tilde{D} \subset \mathbb{C}$ be the region of the complex plane on which the coordinate w is defined corresponding to the neighborhood of the minimal surface. Then $f : D \to \tilde{D}$ maps the z-coordinates to the w-coordinates. If $\tilde{F} : \tilde{D} \to \mathbb{R}^n$ is the conformal parametrization of that neighborhood of the surface with respect to the coordinate w, then $F = \tilde{F} \circ f$.

Now taking the derivatives of F with respect to z and $\bar{z}$ gives us

$$
\begin{aligned}
F_z &= \tilde{F}_w f_z + \tilde{F}_{\bar{w}} \bar{f}_z \\
F_{\bar{z}} &= \tilde{F}_w f_{\bar{z}} + \tilde{F}_{\overline{W}} \bar{f}_{\bar{z}}.
\end{aligned}
$$

Since w is an isothermal parameter, we know that $\tilde{F}_w$ and $\tilde{F}_{\bar{w}}$ are isotropic, so

$$
\begin{aligned}
0 &= F_z \cdot F_z \\
&= \left(\tilde{F}_w f_z + \tilde{F}_{\bar{w}} \bar{f}_z\right) \cdot \left(\tilde{F}_w f_z + \tilde{F}_{\bar{w}} \bar{f}_z\right) \\
&= 2 f_z \bar{f}_z |\tilde{F}_w|^2.
\end{aligned}
$$

As the surface is regular, we cannot have $\tilde{F}_w = 0$ so one of f_z or $\bar{f}_z$ must be 0. These options correspond to f being anti-holomorphic or holomorphic respectively, completing the proof. □

Proposition 3.16. *Let z and w be two isothermal parameters for a neighborhood of a minimal surface in $\mathbb{R}^n$. Then the osculating spaces as defined using each of these two coordinates are the same.*

Proof. As in the lemma, we will let $F : D \to \mathbb{R}^n$ and $\tilde{F} : \tilde{D} \to \mathbb{R}^n$ represent the conformal parametrizations of the surface with respect to the z and w coordinate respectively. We will use E_j and $\tilde{E}_j$ to represent the osculating spaces with respect to the z and w coordinate respectively (even though we have used $\tilde{E}_j$ to mean something different in previous propositions).

From Lemma 3.15 we know that w must be either a holomorphic or anti-holomorphic transformation of z. Suppose then that $w = f(z)$ where f is holomorphic. Then we have that $\partial_z = f_z \partial_w$. Thus we have that

$$F_z = f_z \tilde{F}_w$$

and

$$F_{zz} = f_{zz} \tilde{F}_w + f_z^2 \tilde{F}_{ww}.$$

It is not particularly illustrative to compute further, but these seem to suggest that the jth derivative of F with respect to z is some linear combination of the first j derivatives of $\tilde{F}$ with respect to w.

In general then suppose that we have

$$G_j = a_1(z)\tilde{G}_1 + \ldots + a_j(z)\tilde{G}_j,$$

where $G_j = \partial_z^j F$ and $\tilde{G}_j = \partial_w^j \tilde{F}$. As above if $\partial_z = f_z \partial_w$ then we have that

$$\begin{aligned} G_{j+1} &= a_1'(z)\tilde{G}_1 + (a_1(z)f(z) + a_2'(z))\,\tilde{G}_2 + \dots \\ &\quad + \left(a_{j-1}(z)f(z) + a_j'(z)\right)\tilde{G}_j + a_j(z)f(z)\tilde{G}_{j+1}. \end{aligned}$$

So then by induction it is clear that G_j is a linear combination of $\tilde{G}_1, \dots, \tilde{G}_j$ for all j.

Furthermore, notice that $a_j(z) = f_z^j$ for all j. The linear transformation for the change of basis from $\tilde{G}_1, \dots, \tilde{G}_j$ to $G_1, \dots, G_j$ is triangular and the diagonal is precisely $f_z, f_z^2, \dots, f_z^j$. Thus, $E_j = \tilde{E}_j$ because f_z should be nowhere zero as the change of coordinate function should not be singular in its domain.

So although in general the particular vectors G_j are not geometrical, since $G_j \neq \tilde{G}_j$, the osculating spaces are geometrical since $E_j = \tilde{E}_j$.

Now, at the beginning of the proof we assumed that the change of coordinates $w = f(z)$ was holomorphic, but in general it could also be anti-holomorphic. In that case we take $\overline{w} = f(z)$ is holomorphic and by the same argument as above we get that the osculating spaces E_j are the complex conjugates of $\tilde{E}_j$. So geometrically there is no preference between E_j or $\overline{E_j}$ and any change in the isothermal parameter will generate new osculating spaces that agree with the original osculating spaces or are their complex conjugates. □

3.3 Graphical minimal surfaces

Upon considering minimal surfaces that are graphical over some 2-plane, we get a restriction on the dimension of the osculating space. Namely the osculating space must have dimension less than n.

Proposition 3.17. *Let S be a minimal surface that is graphical over some 2-plane in $\mathbb{R}^n$*

and has osculating space E of dimension k. Then $k < n$.

Proof. Without loss of generality suppose that the surface is graphical over the x_1x_2-plane in $\mathbb{R}^n$. Then, due to a theorem of Osserman [13], there exists a global isothermal parameter $u = u_1 + iu_2$ on the surface such that the parametrization of the surface with respect to u satisfies $x_1 = u_1$ and $x_2 = au_1 + bu_2$ with $a, b \in \mathbb{R}$ and $b > 0$. We will only use that the equations for x_1 and x_2 are linear in u_1 and u_2.

Let $F : D \to \mathbb{R}^n$ be the local conformal parametrization corresponding to the parameter u above. Then, the first two components of F are linear in u_1 and u_2 as discussed above. Therefore, the first two components of the first derivative F_u are constants and the first two components of any higher derivatives in u must be zero.

Therefore, $G_2, \ldots, G_k$ span at most an $n-2$-dimensional subspace of $\mathbb{C}^n$. Thus, since E is spanned almost everywhere by $G_1, \ldots, G_k$, it can be at most $n-1$-dimensional. So indeed $k < n$. □

Chapter 4

Holomorphic curves

We repeat the following definitions from the introduction as we will now be making use of them again.

The motivation behind these definitions is that they generalize the notion of a holomorphic curve $f : \mathbb{C} \to \mathbb{C}^m = \mathbb{R}^{2m}$ as a real two-dimensional surface. The issue is that we can take some real orthogonal transformation of a holomorphic curve thought of as a real surface such that the resulting map is not easily identified as a holomorphic curve.

The intuition then is that these definitions are able to capture when the surface is a holomorphic curve under some orthogonal transformation.

Definition 4.1. A *linear complex structure* is a linear transformation $J : \mathbb{R}^n \to \mathbb{R}^n$ such that $J^2 = -\mathrm{id}$, where id is the identity map on $\mathbb{R}^n$.

Definition 4.2. If g is an inner product on $\mathbb{R}^n$, then we say a linear complex structure J is *compatible with* g if $g(Ju, Jv) = g(u, v)$ for all $u, v \in \mathbb{R}^n$.

Definition 4.3. Let J be a linear complex structure compatible with the Euclidean metric on $\mathbb{R}^n$ and $\iota : S \to \mathbb{R}^n$ an immersion of a Riemann surface S. The surface S is *J-holomorphic* if J restricts to a map $J : d\iota(TS) \to d\iota(TS)$.

Remark. The push forward of the tangent space T_pS at a point $p \in S$ is some subspace of the ambient tangent space $T_{\iota(p)}\mathbb{R}^n \cong \mathbb{R}^n$, where we always choose the canonical representation of this tangent space as $\mathbb{R}^n$. A linear complex structure J is a linear transformation on $\mathbb{R}^n$, so we can restrict it to this pushforward of the tangent space of S at this particular point. The claim of the definition then is that J preserves this pushforward of the tangent space at every point.

Proposition 4.4. *Let J be a linear complex structure compatible with the Euclidean metric and $\iota : S \to \mathbb{R}^n$ a J-holomorphic surface with $F : D \to \mathbb{R}^n$ a local conformal parametrization. Then $JF_z = iF_z$ and $JF_{\bar{z}} = -iF_{\bar{z}}$ for all $z \in D$, or vice-versa up to a choice in local parameters.*

Proof. From the definition of J-holomorphic we know at any point $z \in D$ that J restricts to a real linear map from the real 2-dimensional tangent space of the surface to itself. By extension to complex scalars, this means J acts as a linear transformation on the 2-dimensional complex space spanned by F_z and $F_{\bar{z}}$. Then, with respect to these two vectors as a basis we can think of J as a complex matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

So, for example, we have that $JF_z = aF_z + cF_{\bar{z}}$. We can take the complex conjugate of this equation, and since J is a real transformation we get $JF_{\bar{z}} = \bar{c}F_z + \bar{a}F_{\bar{z}}$, which tells us $b = \bar{c}$ and $d = \bar{a}$. Notice also that the compatibility property of J extends to complex scalars so we have that

$$\begin{aligned} 0 &= F_z \cdot F_z \\ &= JF_z \cdot JF_z \\ &= (aF_z + cF_{\bar{z}}) \cdot (aF_z + cF_{\bar{z}}) \\ &= 2acF_z \cdot F_{\bar{z}}. \end{aligned}$$

The surface is regular so we know that F_z cannot have magnitude zero and so this tells us that either $a = 0$ or $c = 0$. The final property of J that we have yet to take advantage of is that $J^2 = -\mathrm{id}$. If $a = 0$ then

$$\begin{aligned} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} &= J^2 \\ &= \begin{pmatrix} 0 & \bar{c} \\ c & 0 \end{pmatrix}^2 \\ &= \begin{pmatrix} c\bar{c} & 0 \\ 0 & c\bar{c} \end{pmatrix}, \end{aligned}$$

which is evidently impossible since $c\bar{c} \geq 0$.

Then if $c = 0$,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = J^2$$
$$= \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix}^2$$
$$= \begin{pmatrix} a^2 & 0 \\ 0 & \overline{a}^2 \end{pmatrix},$$

which gives us $a^2 = \overline{a}^2 = -1$. Thus, we have that either $a = i$ or $a = -i$.

The solution $a = i$ gives our claimed result, that $JF_z = iF_z$. The solution $a = -i$ would give $JF_z = -iF_z$. When choosing local parameters, we could always make a choice to swap x and y, which would also swap z and $\bar{z}$, meaning that we can always make $JF_z = iF_z$ by the right choice of z. Thus, we have our desired result. □

A surface being J-holomorphic for some linear complex structure J on $\mathbb{R}^n$ is quite a strong property to have. As discussed in the introduction, a problem of particular interest is to show under what additional assumptions a stable minimal surface in $\mathbb{R}^n$ is J-holomorphic. This came about as a natural extension of Bernstein's theorem and from holomorphic curves being a well-known class of stable 2-dimensional surfaces. In particular, every J-holomorphic surface is stable, since it is a calibrated surface, and stability itself is already a strong condition on the surface. Although stability certainly implies minimality, we can also directly show that any J-holomorphic surface is minimal without reference to stability.

Proposition 4.5. *Every J-holomorphic surface is minimal.*

Proof. Recall that a surface is minimal if and only if the conformal parametrization is harmonic, i.e. $F_{z\bar{z}} = 0$, at every point. From Proposition 4.4 we have that $JF_z = iF_z$ and

$JF_{\bar{z}} = -iF_{\bar{z}}$. Recall that J is constant, so if we take the $\bar{z}$ derivative of the first equality and the z derivative of the second equality then we get that

$$iF_{z\bar{z}} = JF_{z\bar{z}} = JF_{\bar{z}z} = -iF_{\bar{z}z}.$$

But now equality of mixed partials gives us that $F_{z\bar{z}} = 0$. ☐

4.1 Null osculating space

Definition 4.6. A linear subspace $V \subset \mathbb{C}^n$ is *null* if for all $u, v \in V$ we have that $u \cdot v = 0$.

The osculating space of a minimal surface gives us a convenient way to identify whether the surface is J-holomorphic. Namely, the osculating space being null is equivalent to the surface being J-holomorphic, with a caveat discussed below. We prove both directions of the equivalence in two parts over the following proposition and theorem.

Proposition 4.7. *The osculating space E is null for any J-holomorphic surface.*

Remark. Note that by Proposition 4.5 it now makes sense to talk about the osculating space of a J-holomorphic surface since we know it must be minimal.

Proof. Again from Proposition 4.4 we have that $JF_z = iF_z$. Taking the z derivative of this equality arbitrarily many times gives us that $JG_j = iG_j$ for all j. Since E is spanned by $G_1, \ldots, G_k$ we get that $Jv = iv$ for all $v \in E$. Then, using the compatibility of J with the Euclidean metric, we get that

$$\begin{aligned} u \cdot v &= Ju \cdot Jv \\ &= iu \cdot iv \\ &= -u \cdot v \end{aligned}$$

for all $u, v \in E$, which implies $u \cdot v = 0$. □

Now, the caveat in stating the converse of Proposition 4.7 comes from the problem that there are no linear complex structures in $\mathbb{R}^n$ when n is odd. This is because every operator on an odd-dimensional real vector space has an eigenvalue, which would mean $-v = J^2 v = \lambda^2 v$ for some non-zero vector v, which is a contradiction since there is no real λ for which $\lambda^2 = -1$.

The reason this poses a problem is that we can take any holormorphic curve, or even a flat plane, and write it as an immersion in some odd-dimensional space. For example, the xy-plane

$$(x, y) \mapsto (x, y, 0)$$

in $\mathbb{R}^3$ should clearly be J-holomorphic, but by a technicality there is no linear complex structure in $\mathbb{R}^3$. Thus, the formal way that we must phrase this idea is that there exists an even-dimensional affine subspace $\mathbb{R}^{2k} \subset \mathbb{R}^n$ which contains the surface. Then, without loss of generality, we can act as if the surface was defined as an immersion into $\mathbb{R}^{2k}$ in the first place, and that it is J-holomorphic with respect to some linear complex structure J on $\mathbb{R}^{2k}$.

Theorem 4.8. *Every minimal surface with an osculating space E that is null must lie inside an even-dimensional affine subspace $\mathbb{R}^{2k} \subset \mathbb{R}^n$ and there exists a linear complex structure J compatible with the Euclidean metric on $\mathbb{R}^{2k}$ for which the surface is J-holomorphic.*

Proof. Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of the surface and take k to be the dimension of E.

By Lemma 3.8 we know that $E \cap \overline{E}$ must be the complexification of a real vector space. However, since E is null, the only real vector it can contain is the zero vector, and so the intersection $E \cap \overline{E}$ must actually be trivial. Thus, the sum of the two spaces

$$E + \overline{E} = E \oplus \overline{E}$$

is in fact a direct sum and therefore is $2k$-dimensional. So by Corollary 3.10 we have that the minimal surface lies in a $2k$-dimensional affine subspace of $\mathbb{R}^n$. Then without loss of generality we may assume that F lies fully in $\mathbb{R}^n$ and $n = 2k$.

There exists a unique complex linear map $J : \mathbb{C}^n \to \mathbb{C}^n$ that has eigenvalue i on E and eigenvalue $-i$ on $\overline{E}$. Clearly then $J^2 = -\mathrm{id}$, and so it only remains to show that J is also a real linear map $J : \mathbb{R}^n \to \mathbb{R}^n$ and that J is compatible with the Euclidean metric to conclude the proof.

First, we show that J is actually real. Let $e_1, \dots, e_k$ be a basis for E so that $\overline{e_1}, \dots, \overline{e_k}$ is a basis for $\overline{E}$. Then, since $E \oplus \overline{E} = \mathbb{C}^n$, we have that $e_1, \dots, e_k, \overline{e_1}, \dots, \overline{e_k}$ is a basis for $\mathbb{C}^n$. If we define

$$\begin{aligned} u_j &= \frac{1}{2}\left(e_j + \overline{e_j}\right) \\ v_j &= \frac{1}{2i}\left(e_j - \overline{e_j}\right) \end{aligned}$$

for $1 \le j \le k$, then $u_1, \dots, u_k, v_1, \dots, v_k$ also span $\mathbb{C}^n$. Furthermore, since each u_j and v_j are in fact real vectors, they span $\mathbb{R}^n$ over real scalars as well. Then, by our definition of how J acts on E and $\overline{E}$, we get that

$$\begin{aligned} Ju_j &= \frac{1}{2}\left(Je_j + J\overline{e_j}\right) \\ &= \frac{1}{2}\left(ie_j - i\overline{e_j}\right) \\ &= -\frac{1}{2i}\left(e_j - \overline{e_j}\right) \\ &= -v_j, \end{aligned}$$

and by a similar calculation we get that $Jv_j = u_j$, for all $1 \le j \le k$. Thus, since J maps a spanning set of real vectors to real vectors, J is in fact a real linear map.

Next, we show that J is compatible with the Euclidean metric. We need to show that $Ju \cdot Jv = u \cdot v$ for all $u, v \in \mathbb{R}^n$, but we will show it for all $u, v \in \mathbb{C}^n$. Note that we can uniquely decompose any vector $u \in \mathbb{C}^n$ as $u = u^{(1,0)} + u^{(0,1)}$, where $u^{(1,0)} \in E$ and $u^{(0,1)} \in \overline{E}$, since we know that $E \oplus \overline{E} = \mathbb{C}^n$. Recalling the fact that E (and thus $\overline{E}$) is null, we compute

$$\begin{aligned} u \cdot v &= (u^{(1,0)} + u^{(0,1)}) \cdot (v^{(1,0)} + v^{(0,1)}) \\ &= u^{(1,0)} \cdot v^{(1,0)} + u^{(0,1)} \cdot v^{(1,0)} + u^{(1,0)} \cdot v^{(0,1)} + u^{(0,1)} \cdot v^{(0,1)} \\ &= u^{(0,1)} \cdot v^{(1,0)} + u^{(1,0)} \cdot v^{(0,1)} \end{aligned}$$

and

$$\begin{aligned} Ju \cdot Jv &= J(u^{(1,0)} + u^{(0,1)}) \cdot J(v^{(1,0)} + v^{(0,1)}) \\ &= (iu^{(1,0)} - iu^{(0,1)}) \cdot (iv^{(1,0)} - iv^{(0,1)}) \\ &= -u^{(1,0)} \cdot v^{(1,0)} + u^{(0,1)} \cdot v^{(1,0)} + u^{(1,0)} \cdot v^{(0,1)} - u^{(0,1)} \cdot v^{(0,1)} \\ &= u^{(0,1)} \cdot v^{(1,0)} + u^{(1,0)} \cdot v^{(0,1)}. \end{aligned}$$

Thus, $u \cdot v = Ju \cdot Jv$ for all $u, v \in \mathbb{C}^n$ and so J is in fact compatible with the Euclidean metric, completing the proof. $\square$

Chapter 5

Constructing minimal surfaces

We would like to be able to classify what properties of osculating spaces are or are not possible for minimal surfaces to have.

For example, we already know from Corollary 3.12 that if the surface lies fully in $\mathbb{R}^n$, then $n/2 \leq k \leq n$, where k is the dimension of the osculating space E. However, we could ask whether there actually exists such a minimal surface with osculating space of every dimension between $n/2$ and n. Furthermore, although any subspace of a real vector space is isometric to any other of the same dimension, the same is not true for subspaces of a complex vector space with respect to the bilinear dot product on $\mathbb{C}^n$. We may also seek to classify how the bilinear dot product can and cannot act on the osculating space.

With this goal in mind, we introduce a sharp criterion for a surface to be minimal based on the osculating space.

Proposition 5.1 (Criteria for constructing minimal surfaces)**.** *Let $D \subset \mathbb{C}$ be some neighborhood in the complex plane, $\varphi : D \to \mathbb{C}^n$ be a holomorphic vector field, and $G_j = \partial_z^{j-1}\varphi$ so that $\varphi = G_1$. Then the following are equivalent* *(i)* $\varphi(z) \cdot \varphi(z) = 0$ *for all* $z \in D$,

(ii)

$$\sum_{i+j=l} \frac{G_i(z)}{(i-1)!} \cdot \frac{G_j(z)}{(j-1)!} = 0$$

for all $l \geq 2$ *and all* $z \in D$,

(iii) there exists $z_0 \in D$ *such that*

$$\sum_{i+j=l} \frac{G_i(z_0)}{(i-1)!} \cdot \frac{G_j(z_0)}{(j-1)!} = 0$$

for all $l \geq 2$.

Proof.

(i) $\implies$ (ii): For any $z_0 \in D$ we can write

$$\varphi(w) = \sum_{j=1}^{\infty} G_j(z_0) \frac{(w-z_0)^{(j-1)}}{(j-1)!},$$

since φ is holomorphic and thus complex analytic. Then we get that

$$\begin{aligned} 0 &= \varphi(w) \cdot \varphi(w) \\ &= \sum_{i=1}^{\infty} G_i(z_0) \frac{(w-z_0)^{(i-1)}}{(i-1)!} \cdot \sum_{j=1}^{\infty} G_j(z_0) \frac{(w-z_0)^{(j-1)}}{(j-1)!} \\ &= \sum_{l=2}^{\infty} \left(\sum_{i+j=l} \frac{G_i(z_0)}{(i-1)!} \cdot \frac{G_j(z_0)}{(j-1)!} \right) z^{l-2}. \end{aligned}$$

Thus, we have that each coefficient

$$\sum_{i+j=l} \frac{G_i(z_0)}{(i-1)!} \cdot \frac{G_j(z_0)}{(j-1)!} = 0,$$

for all $l \geq 2$ and all $z_0 \in D$.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (i): As before, we can expand φ at $w = z_0$ and we get

$$\begin{aligned}\varphi(w) \cdot \varphi(w) &= \sum_{i=1}^{\infty} G_i(z_0) \frac{(w - z_0)^{(i-1)}}{(i-1)!} \cdot \sum_{j=1}^{\infty} G_j(z_0) \frac{(w - z_0)^{(j-1)}}{(j-1)!} \\ &= \sum_{l=2}^{\infty} \left(\sum_{i+j=l} \frac{G_i(z_0)}{(i-1)!} \cdot \frac{G_j(z_0)}{(j-1)!} \right) (w - z_0)^{l-2} \\ &= 0,\end{aligned}$$

which holds for all w in a neighborhood of z_0 where $\varphi(w)$ is equal to its Taylor expansion. But since $\varphi(z) \cdot \varphi(z)$ is a holomorphic function on D and equal to zero on this neighborhood of z_0, it must be zero for all $z \in D$. □

Remark. If we think of a (potentially infinite) matrix with elements

$$a_{ij} = \frac{G_i(z_0)}{(i-1)!} \cdot \frac{G_j(z_0)}{(j-1)!}$$

then the criterion just becomes that the skew-diagonals of this matrix must sum to zero. If F_z consists of only holomorphic polynomials then it only has finitely many non-zero derivatives and so this will in fact be a finite matrix whose skew-diagonals sum to zero.

One way of constructing a minimal surface using this criterion is as follows: if we find $e_1, \ldots, e_m \in \mathbb{C}^n$ and we consider $e_j = 0$ for all $j > m$ such that

$$\sum_{i+j=l} e_i \cdot e_j = 0$$

for all $l \geq 2$, then we can define

$$\varphi = \sum_{j=1}^{m} e_j z^{j-1}.$$

Then, Proposition 2.1 tells us that there exists some local conformal parametrization of a minimal surface $F : D \to \mathbb{R}^n$, where in this case D can be the entire complex plane, such that $F_z = \varphi$.

Remark. Note that if we consider the criterion for $l = 2$ we get that $G_1 \cdot G_1 = 0$ and for $l = 3$ we get that $G_1 \cdot G_2 = 0$.

5.1 Completion of a minimal surface

The following two propositions take their inspiration from Taylor series and the idea that you can specify the first several derivatives of a function and construct many functions which have derivatives of exactly those values at a certain point. In attempting to do the same thing for minimal surfaces, we can select a collection of vectors in $\mathbb{C}^n$ and ask whether there exists a minimal surface with z-derivatives of exactly those vectors at a point.

The first issue we can see is that by Proposition 5.1 the vectors in $\mathbb{C}^n$ that we select must satisfy certain necessary conditions for any minimal surface to have those vectors as derivatives.

The point of the next two propositions is that as long as the vectors satisfy those necessary conditions, we have complete freedom to choose some number of them, with the number limited in certain ways by the dimension n of the space.

Proposition 5.2. *Let $e_1, \ldots, e_m \in \mathbb{C}^n$ be linearly independent and satisfy*

$$\sum_{i+j=l} e_i \cdot e_j = 0$$

for $2 \leq l \leq m+1$. Then, if $m \leq n/2$, there exists $v \in \mathbb{C}^n$ such that taking $e_{m+1} = v$ and $e_j = 0$ for all $j > m+1$ satisfies

$$\sum_{i+j=l} e_i \cdot e_j = 0$$

for all $l \geq 2$, and consequently

$$F_z = \sum_{j=1}^{m+1} e_j z^{j-1}$$

defines a minimal surface.

Proof. In order for such a $v \in \mathbb{C}^n$ to exist, it must satisfy m linear relations of the form

$$e_j \cdot v = b_j,$$

where $b_j \in \mathbb{C}$ is a value depending on the dot products of $e_1, \ldots, e_m$, and it must satisfy $v \cdot v = 0$. We get this precisely from examining the form of the criterion for $l > m+1$, since we are given that the criterion holds for $2 \leq l \leq m+1$ already. Since $e_1, \ldots, e_m$ are linearly independent, the intersection of these m hyperplanes in $\mathbb{C}^n$ is some $n-m$ dimensional affine subspace U of $\mathbb{C}^n$. If U intersects the $v \cdot v = 0$ hypersurface within $\mathbb{C}^n$, then any vector in that intersection will be a valid solution, so instead suppose that U does not intersect that hypersurface.

There exists an $n-m$ dimensional linear subspace W of $\mathbb{C}^n$ and a fixed vector $u \in \mathbb{C}^n$ such that $U = \{u + w : w \in W\}$, i.e. U is parallel to W offset by that vector u. Now let $w \in W$ be any vector and consider a set of vectors of the form $u + tw$ for $t \in \mathbb{C}$ that all lie in U. If we take the dot product with itself, we get $(w \cdot w)t^2 + 2(u \cdot w)t + (u \cdot u)$. We have assumed that this can never equal zero, since it lies in U for all t, and thus we must have that $w \cdot w = 0$, $u \cdot w = 0$, and $u \cdot u \neq 0$. However, our choice of w was arbitrary, so we get that $w \cdot w = 0$ for all $w \in W$ and $u \cdot w = 0$ for all $w \in W$. Now if we let $w_1, w_2 \in W$, then using that all vectors in W are isotropic we get

$$\begin{aligned}0 &= (w_1 + w_2) \cdot (w_1 + w_2)\\&= (w_1 \cdot w_1) + 2(w_1 \cdot w_2) + (w_2 \cdot w_2)\\&= 2(w_1 \cdot w_2).\end{aligned}$$

Thus, we conclude that in fact the dot product between any two vectors in W must be zero, i.e. W is a null subspace.

Let us define $W^\perp = \{a \in \mathbb{C}^n : \forall w \in W, a \cdot w = 0\}$. Then what we have discovered in the previous paragraph is that $W \subset W^\perp$ and $u \in W^\perp$. Since the bilinear form given by the dot product on $\mathbb{C}^n$ is non-degenerate, the dimension of $W^\perp$ must be $n - (n - m) = m$. But as W lies inside $W^\perp$ we get that $n - m \leq m$ which implies $m \geq n/2$.

In the statement of the proposition we assumed that $m \leq n/2$, so we must have that $m = n/2$. But if $m = n/2$, then W and $W^\perp$ are the same dimension which implies $W = W^\perp$ and thus $u \in W$. But that would imply $u \cdot u = 0$, contradicting our assumption that U does not intersect the $v \cdot v = 0$ hypersurface, and thus completes the proof. □

Remark. Working through the linear relations, one finds that $W^\perp$ is just the osculating space E_m spanned by $e_1, \ldots, e_m$.

Remark. Supposing then that $m \leq n/2$, this proves that the intersection of U and the $v \cdot v = 0$ hypersurface is non-empty, and from algebraic geometry we get that the intersection is some surface of dimension $n - m - 1$.

If $m = n/2$, then the intersection may lie inside $W^\perp = E_m$ and so while there is a solution v, the set $e_1, \ldots, e_m, v$ is not linearly independent.

If $m = (n-1)/2$, then the intersection is the same dimension as $W^\perp = E_m$. If the intersection lies entirely inside E_m, then it must be equal to E_m. Thus, either there is some linearly independent solution v, or the zero vector is a solution in which case $e_1, \ldots, e_m$ already define a minimal surface.

If $m < (n-1)/2$, then the dimension of the intersection is strictly larger than m, the dimension of $W^\perp = E_m$ and thus a linearly independent solution v is guaranteed.

Proposition 5.3. *Let $e_1, \ldots, e_m \in \mathbb{C}^n$ be linearly independent and satisfy*

$$\sum_{i+j=l} e_i \cdot e_j = 0$$

for $2 \leq l \leq m+1$ and let $E_m = span\{e_1, \ldots, e_m\}$. If $E_m + \overline{E_m} \neq \mathbb{C}^n$ then there exists $v \in \mathbb{C}^n$ such that taking $e_{m+1} = v$ and $e_j = 0$ for all $j > m+1$ satisfies

$$\sum_{i+j=l} e_i \cdot e_j = 0$$

for all $l \geq 2$, and consequently

$$F_z = \sum_{j=1}^{m+1} e_j z^{j-1}$$

defines a minimal surface.

Proof. Since $E_m + \overline{E_m}$ is the complexification of some real subspace of dimension less than n, without loss of generality we can take some real rotation of $\mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$ such that $e_1, \ldots, e_m \in \mathbb{C}^{n-1}$. Then we want some solution $v = (v_1, v_2)$ where $v_1 \in \mathbb{C}^{n-1}$ and $v_2 \in \mathbb{C}$.

As in Proposition 5.2, we get m linear relations of the form

$$e_j \cdot v_1 = b_j$$

where $b_j \in \mathbb{C}$ is a value depending on the dot products of $e_1, \ldots, e_m$. Since $e_1, \ldots, e_m$ are linearly independent and their span is not all of $\mathbb{C}^n$ we are guaranteed that $m \leq n-1$ and so the intersection of these m hyperplanes is non-empty in $\mathbb{C}^{n-1}$. Let v_1 be any element of this intersection.

For $v = (v_1, v_2)$ to be a solution, it suffices that v_1 satisfy the above m linear relations and for $v \cdot v = 0$. Therefore, we choose $v_2 \in \mathbb{C}$ to be a square root of $-v_1 \cdot v_1$. Thus, $v \cdot v = v_1 \cdot v_1 + v_2^2 = 0$, and so v satisfies the required conditions. □

5.2 Non-compatible complex structure

By Theorem 4.8 we know that if a minimal surface is J-holomorphic and lies fully in $\mathbb{R}^n$ then n must be even and the osculating space must be of dimension $n/2$. This naturally leads us to question whether the converse is true; if a minimal surface lies fully in $\mathbb{R}^n$ for even n and the osculating space is of dimension $n/2$, then is it necessarily J-holomorphic.

Notice that Theorem 4.8 only supposes that the osculating space is null in addition to this, so the question can be reduced to whether a minimal surface lying fully in $\mathbb{R}^n$ with osculating space of dimension $n/2$ must have a null osculating space.

Proposition 5.4. *If $F : D \to \mathbb{R}^4$ is a local conformal parametrization of a minimal surface with osculating space E of dimension 2, then there exists a linear complex structure J compatible with the Euclidean metric on $\mathbb{R}^4$ for which the surface is J-holomorphic.*

Proof. As noted above, by Theorem 4.8 it suffices to show that E is null. Since E is spanned by G_1 and G_2, this means we need only show that $G_1 \cdot G_1 = G_1 \cdot G_2 = G_2 \cdot G_2 = 0$. However, by the criteria for constructing minimal surfaces, we are guaranteed the first two are true and thus what remains is only to show that $G_2 \cdot G_2 = 0$.

Since E is two dimensional we get that $G_3(z) = f(z)G_1(z) + g(z)G_2(z)$ where f, g are meromorphic on D. Then $G_3 \cdot G_1 = f(z)G_1 \cdot G_1 + g(z)G_2 \cdot G_1 = 0$. But the criteria for constructing minimal surfaces tells us that $G_2 \cdot G_2 = -G_3 \cdot G_1 = 0$, completing the proof. $\square$

Remark. This is a nice result to have for $n = 4$. However, this unfortunately does not extend to higher dimensions. For example, in $n = 6$ take the following vectors

$$e_1 = (1, i, 0, 0, 0, 0)$$
$$e_2 = (0, 0, \sqrt{2}, i, 0, 0)$$
$$e_3 = (-1/4, i/4, 0, 0, 1/4, i/4)$$

which satisfy the criteria for constructing minimal surfaces. Thus, there exists a conformal parametrization $F : D \to \mathbb{R}^n$ of a minimal surface with $F_z = e_1 + e_2 z + e_3 z^2$. Since e_1, e_2, e_3 are linearly independent we get that the osculating space E is 3-dimensional.

By observation it is clear that no vector in E is real besides the zero vector, thus the only vector in $E \cap \overline{E}$ is the zero vector. Therefore we can conclude that $E \oplus \overline{E} = \mathbb{C}^6$ and so the surface lies fully in $\mathbb{R}^6$. However, E is not null, since $e_2 \cdot e_2 = 1$, for example.

Thus, we have constructed a minimal surface lying fully in $\mathbb{R}^6$ with osculating space of dimension 3 that is not J-holomorphic, since E is not null.

Proposition 5.5. *For all $k > 2$, there exists a minimal surface lying fully in $\mathbb{R}^{2k}$ with osculating space E of dimension k that is not J-holomorphic.*

Proof. Let $\epsilon_1, \ldots, \epsilon_{2k}$ be the standard basis for $\mathbb{C}^{2k}$. Then let

$$\begin{aligned}
e_1 &= \epsilon_1 + i\epsilon_2 \\
e_2 &= \sqrt{2}\epsilon_3 + i\epsilon_4 \\
e_3 &= -\frac{1}{4}\epsilon_1 + \frac{i}{4}\epsilon_2 + \frac{1}{4}\epsilon_5 + \frac{i}{4}\epsilon_6 \\
e_4 &= \epsilon_7 + i\epsilon_8 \\
e_5 &= \epsilon_9 + i\epsilon_{10} \\
&\vdots \\
e_k &= \epsilon_{2k-1} + i\epsilon_{2k}.
\end{aligned}$$

It is straightforward to verify that these satisfy the criteria for constructing minimal surfaces, that they are linearly independent, that their span contains no real vectors besides the zero vector, and that their span is not null. As in the remark, there exists a conformal parametrization $F : D \to \mathbb{R}^n$ of a minimal surface such that

$$F_z = \sum_{j=1}^{k} e_j z^{j-1}.$$

This constructed minimal surface lies fully in $\mathbb{R}^{2k}$ with osculating space E of dimension k that is not J-holomorphic, since E is not null. □

The above proposition shows that there are examples in $n > 4$ dimensions of minimal surfaces with osculating space of dimension $n/2$ that are nonetheless not J-holomorphic. The next two propositions show that this class of surfaces is however related to J-holomorphic surfaces by some (non-isometric) linear transformation of the ambient space.

Proposition 5.6. *Let $F : D \to \mathbb{R}^{2k}$ be a local conformal parametrization of a minimal surface lying fully in $\mathbb{R}^{2k}$ with osculating space of dimension k that is not null. Then there exists a linear complex structure J on $\mathbb{R}^{2k}$ that is not compatible with the Euclidean metric* *on $\mathbb{R}^{2k}$ such that the surface is holomorphic with respect to this J.*

Remark. The above is technically an abuse of Definition 4.3 since the given J is not compatible with the Euclidean metric. However, we still mean that J pointwise preserves the tangent plane of the surface, which does not require J to be compatible with the Euclidean metric.

Proof. Since the surface lies fully in $\mathbb{R}^{2k}$ and E has dimension k we must have that $E \cap \overline{E} = \{0\}$. Thus, there exists a unique linear operator J on $\mathbb{C}^{2k}$ that has eigenvalue i on E and eigenvalue $-i$ on $\overline{E}$.

We must show that J is actually a real operator. Let $e_1, \ldots, e_k$ be a basis for E. If we let

$$\begin{aligned} a_j &= \frac{1}{2}\left(e_j + \overline{e_j}\right) \\ b_j &= \frac{1}{2i}\left(e_j - \overline{e_j}\right) \end{aligned}$$

then $a_1, \ldots, a_k, b_1, \ldots, b_k$ is a real basis for $\mathbb{R}^{2k}$, since over complex scalars they span $E \oplus \overline{E} = \mathbb{C}^{2k}$. Then by definition of how J acts on E and $\overline{E}$ we get

$$\begin{aligned} Ja_j &= \frac{1}{2}\left(Je_j + J\overline{e_j}\right) \\ &= \frac{1}{2}\left(ie_j - i\overline{e_j}\right) \\ &= -b_j \\ Jb_j &= \frac{1}{2i}\left(Je_j - J\overline{e_j}\right) \\ &= \frac{1}{2i}\left(ie_j + i\overline{e_j}\right) \\ &= a_j. \end{aligned}$$

Then, since J sends a real basis of $\mathbb{R}^{2k}$ to real vectors, it is in fact a real operator on $\mathbb{R}^{2k}$.

Because of how we defined it, $J^2 = -I_{2k}$ and J fixes the tangent plane of the surface, so J

is a linear complex structure on $\mathbb{R}^{2k}$ and the surface is holomorphic with respect to J.

Note that J must not be compatible with the Euclidean metric since if it was then we would get that E is null, but we assumed that it was not. □

Proposition 5.7. *Let $F : D \to \mathbb{R}^{2k}$ be a local conformal parametrization of a minimal surface lying fully in $\mathbb{R}^{2k}$ with osculating space of dimension k that is not null. Then there exists a metric g on $\mathbb{R}^{2k}$ with respect to which the immersion F is still minimal and is J-holomorphic.*

Equivalently there exists a linear transformation A on $\mathbb{R}^{2k}$ such that the immersion $A \circ F : D \to \mathbb{R}^{2k}$ describes a minimal surface that is J-holomorphic.

Proof. Proposition 5.6 gives us a linear complex structure J that is not compatible with the Euclidean metric and has eigenvalue i on E and eigenvalue $-i$ on $\overline{E}$. As in the proof for Proposition 5.6, we take $a_1, \ldots, a_k, b_1, \ldots, b_k$ as a real basis for $\mathbb{R}^{2k}$. Define the matrix A^{-1} to have columns $a_1, b_1, \ldots, a_k, b_k$, which is invertible since these columns form a basis. Notice that $J_0 = AJA^{-1}$ is the standard complex structure on $\mathbb{R}^{2k}$, and so in particular it is compatible with the Euclidean metric on $\mathbb{R}^{2k}$.

Consider the immersion $A \circ F : D \to \mathbb{R}^{2k}$. The component functions of F are harmonic maps on D, so the component functions of $A \circ F$ are also harmonic maps, thus defining osculating spaces for $A \circ F$ still makes sense, although the parametrization may not be conformal. The z-derivatives of this transformed map are just $A \circ G_j$ and thus J_0 has eigenvalue i on these. Similarly, J_0 has eigenvalue $-i$ on the $\bar{z}$-derivatives of $A \circ F$. Thus, J_0 fixes the tangent planes of $A \circ F$ and so this surface is J_0-holomorphic. By Proposition 4.5 this additionally gives us that $A \circ F$ is in fact minimal.

Notice that taking the matrix $g = A^t A$ as a metric on $\mathbb{R}^{2k}$ we get that (F, g) is isometric to $(A \circ F, g_{\text{Eucl}})$, completing the proof. □

5.3 Null space of the dot product on E

The dot product on $\mathbb{C}^n$ restricts to a symmetric bilinear form on E, and so we can examine the null space of this bilinear form. Using the machinery that we have built up so far, we get some quite strong restrictions on what the null space can be and consequently what the osculating space E can look like as a subspace of $\mathbb{C}^n$.

Definition 5.8. We define E^0 to be the null space of the dot product as a bilinear form on E, i.e. $E^0 = \{v \in E : \forall u \in E, u \cdot v = 0\}$.

Proposition 5.9. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k that has a null space E^0 of dimension l. Then $l \leq n - k$ and $l \leq k$.*

Proof. Define $E^\perp = \{v \in \mathbb{C}^n : \forall u \in E, u \cdot v = 0\}$. Then, $E^\perp$ has dimension $n - k$ and $E^0 = E \cap E^\perp$, so $l \leq n - k$ and $l \leq k$. □

Corollary 5.10. *$l \leq n/2$.*

Proof. $l + l \leq (n - k) + k$. □

Remark. An alternative proof that $l \leq n/2$ is to notice that E^0 is null (every vector is isotropic) and the maximal null subspaces of $\mathbb{C}^n$ are $n/2$ dimensional.

Theorem 5.11. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k that has a null space E^0 of dimension l. Then $E_{k-l}(z) \oplus E^0 = E$ for all $z \in D$ except at isolated points.*

Proof. Let m be the smallest index at which $\dim(E_m(z) \cap E^0) \geq 1$ for all points z on some open set $U \subset D$. Then around any point $z_0 \in U$ where $G_1(z_0), \ldots, G_k(z_0)$ span E we take a local section s of $E_m(z) \cap E^0$. So we can write $s(z) = f_1(z)G_1(z) + \ldots + f_m(z)G_m(z)$ where each $f_j(z)$ is holomorphic near z_0 and $s(z) \cdot G_j(z) = 0$ near z_0 for $1 \leq j \leq k$. We must not have f_m identically zero because we initially chose m as a smallest index, thus we can write G_m as a holomorphic (aside from isolated points where f_m is zero) linear combination of $G_1, \ldots, G_{m-1}, s$. So $G_1, \ldots, G_{m-1}, s$ span E_m.

Now taking the derivative of s we get $\partial_z s = f_1' G_1 + (f_1 + f_2') G_2 + \ldots + (f_{m-1} + f_m') G_m + f_m G_{m+1}$, and notice that in particular we get that it is a section of E_{m+1}. Notice we have f_m as the coefficient on G_{m+1}, so we can write G_{m+1} as a holomorphic linear combination of $G_1, \ldots, G_m, \partial_z s$. Since we already showed $G_1, \ldots, G_{m-1}, s$ span E_m, we have that $G_1, \ldots, G_{m-1}, s, \partial_z s$ span E_{m+1}.

We can recursively take more and more derivatives and the same argument holds, so we get $G_1, \ldots, G_{m-1}, s, \partial_z s, \ldots, \partial_z^j s$ span E_{m+j}, and in particular $G_1, \ldots, G_{m-1}, s, \partial_z s, \ldots, \partial_z^{k-m} s$ span E.

Now since $s \cdot u = 0$ for all $u \in E$, we have that $\partial_z^j s \cdot u = 0$ for all $u \in E$ and so $s, \partial_z s, \ldots, \partial_z^{k-m} s$ all lie in E^0. So we have that $l \geq k - m + 1$. If $l > k - m + 1$, then $\dim (E_{m-1} \cap E^0)$ must be greater than 1 everywhere, contradicting our choice of m as the smallest index, thus $l = k - m + 1$ and so $s, \partial_z s, \ldots, \partial_z^{k-m} s$ span E^0 except at isolated points.

Recall that $G_1, \ldots, G_{m-1}$ span E_{m-1} and $s, \partial_z s, \ldots, \partial_z^{k-m} s$ span E^0 except at isolated points on U, and if $E_{m-1} \cap E^0$ had trivial intersection only at those isolated points then there would be an open subset of U on which their intersection had dimension 1, contradicting our choice of m as the smallest index where that happens. Thus, there must be some point besides those isolated points where $E_{m-1} \cap E^0$ is trivial, and thus $E_{m-1} \oplus E^0 = E$ at that point.

But $G_1 \wedge \cdots \wedge G_{m-1} \wedge s \wedge \partial_z s \wedge \cdots \wedge \partial_z^{k-m} s$ is a holomorphic section of $\bigwedge^k (\mathbb{C}^n)$ and thus is either zero everywhere or only at isolated points. Since we know there is one point at which it's non-zero, it must only be zero at isolated points and so $E_{k-l} \oplus E^0 = E$ everywhere except at isolated points. $\square$

Theorem 5.12. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k that has a null space E^0 of dimension l. Then $l \neq k-1$ and $l \neq k-2$.*

Proof. There exists an orthogonal basis $e_1, \ldots, e_k$ for E such that

$$e_i \cdot e_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \leq k-l \\ 0, & i = j > k-l \end{cases}$$

where the number of ones and zeros on the diagonal is fixed by the dimension of the null space of the bilinear form.

Suppose that $l = k-1$, then we can write $G_1(z) = a_1(z)e_1 + \ldots + a_k(z)e_k$ where each a_j is holomorphic. Since $G_1(z) \cdot G_1(z) = 0$, we get that $a_1(z)^2 = 0$ which implies $a_1(z) = 0$ as it is holomorphic. But then each $G_j(z)$ is only in the span of $e_2, \ldots, e_k$, so E is at most $k-1$-dimensional, a contradiction.

Next, suppose that $l = k-2$, then similarly we can write $G_1(z) = a_1(z)e_1 + \ldots + a_k(z)e_k$ where each a_j is holomorphic. Again we can use $G_1(z) \cdot G_1(z) = 0$ to get that $a_1(z)^2 + a_2(z)^2 = 0$. Thus, at each z, $a_2(z)$ must either equal $ia_1(z)$ or $-ia_1(z)$. Then, at any z_0 for which $a_1(z_0) \neq 0$, by continuity there is a neighborhood around z_0 for which either $a_2(z) = ia_1(z)$ or $a_2(z) = -ia_1(z)$ for every point in that neighborhood.

Suppose then that $a_2(z) = ia_1(z)$ in that neighborhood. Then, $\partial_z^j a_2(z) = i\partial_z^j a_1(z)$ for all j and z within this neighborhood, so in particular each G_j has this specific ratio between the first and second components. Then, any vector in the span of all G_j must also have this ratio, and so in particular e_1 would not be in their span, a contradiction. Thus, $a_1(z)$ must be equivalently zero everywhere, which likewise implies $a_2(z)$ is zero everywhere. This leads us to the same conclusion as the $l = k-1$ case that suggests E is at most $k-2$-dimensional, a contradiction.

Thus, $l \neq k-1$ and $l \neq k-2$. $\square$

This theorem is a very strong restriction on what the null space of the osculating space can be. We can have that $l = k$ which means that the osculating space is null. In the case that it is not null, however, then the dimension of the null space must be at least 3 less than the dimension of the osculating space.

Notice that this gives us a very short alternate proof of Proposition 5.4 which is just that if $k = 2$ then $l = 2$.

This theorem combined with Proposition 5.9 also gives us the following proposition. Although the statement seems complicated, this complexity just comes from the difficulty of formally stating that a surface is J-holomorphic. This proposition should be understood intuitively as saying that if $l = \lfloor \frac{n}{2} \rfloor$, then the surface is J-holomorphic.

Proposition 5.13. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k that has a null space E^0 of dimension l.*

If n is even and $l = \frac{n}{2}$, then the surface is J-holomorphic with respect to some linear complex structure J on $\mathbb{R}^n$.

If n is odd and $l = \frac{n-1}{2}$, then the surface lies in an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ and is J-holomorphic with respect to some linear complex structure J on that $(n-1)$-dimensional affine space.

Proof. If $l = \frac{n}{2}$ then $k = \frac{n}{2}$ by Proposition 5.9. If $l = \frac{n-1}{2}$ then either $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$, again by Proposition 5.9, but the latter is forbidden by Theorem 5.12.

So then in either case $l = k$ which means the osculating space E is null. Thus, we can apply Theorem 4.8 to get our desired conclusion. $\square$

Chapter 6

Decomposable minimal surfaces

The following terminology of describing surfaces as *decomposable* comes from a 1967 paper by Chern and Osserman [5] on complete minimal surfaces.

Definition 6.1. A minimal surface immersed in $\mathbb{R}^n$ is *decomposable* if there is some $m < n$ for which there exists orthonormal vectors $e_1, \ldots, e_m \in \mathbb{R}^n$ such that for every local conformal parametrization $F : D \to \mathbb{R}^n$ we have that

$$\sum_{j=1}^{m} (F_z \cdot e_j)^2 = 0.$$

Note that this means we have an orthogonal decomposition of $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$ such that the projection of the surface onto each factor is a minimal surface.

Remark. It suffices that a single local conformal parametrization $F : D \to \mathbb{R}^n$ of the minimal surface satisfies the above equation, as it follows that every other local conformal parametrization must also satisfy it as well. To see this, note that $\sum_{j=1}^{m} (F_z \cdot e_j)^2$ is a holomorphic function on the surface with respect to our local choice of coordinates. Then, for any other choice of coordinates on a local neighborhood intersecting the neighborhood of our first set of coordinates, we have that the change of coordinate function is either holomorphic or anti-holomorphic on the intersection. Thus, it is equivalently zero on the intersection and so must be zero on both neighborhoods. Therefore, for any connected minimal surface, the function must be zero with respect to any isothermal parameter on any neighborhood of the surface.

Because of this, when a surface is decomposable we will commonly write $F = (F^1, F^2)$ as a decomposition of a local conformal parametrization $F : D \to \mathbb{R}^n$ into $F^1 : D \to \mathbb{R}^m$ and $F^2 : D \to \mathbb{R}^{n-m}$ as local conformal parametrizations of two minimal surfaces. We also state the following propositions in terms of the local conformal parametrizations of the surface as that suffices for discussing decomposability.

Proposition 6.2. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a decomposable minimal surface where we represent the decomposition as $F = (F^1, F^2)$ with $F^1 : D \to \mathbb{R}^m$ and $F^2 : D \to \mathbb{R}^{n-m}$. As usual we let k be the dimension of the osculating space of F, and we let k_1 and k_2 be the dimensions of the osculating spaces of F^1 and F^2 respectively.*

Then we have that

$$k_1, k_2 \leq k \leq k_1 + k_2.$$

Proof. Notice that since each F^a is just an orthogonal projection of F, each z-derivative of F^a is the same orthogonal projection of the corresponding z-derivative of F. Thus, the osculating spaces of each F^a are just the orthogonal projections of the osculating space of F. So we get that $k \geq k_1$ and $k \geq k_2$.

Furthermore, since the osculating spaces of each F^a lie in orthogonal subspaces to each other, their direct sum is precisely $k_1 + k_2$-dimensional. Each vector in the osculating space of F is precisely the sum of its projections onto the two orthogonal subspaces, so the osculating space must lie in the direct sum of the osculating spaces of each F^a. So $k \leq k_1 + k_2$. □

Remark. It may seem like in the above we might be able to show that $k = k_1 + k_2$, but this is not the case. One simple counterexample would be if $F^1 : D \to \mathbb{R}^a$ and $F^2 : D \to \mathbb{R}^b$ are both degree d minimal surfaces with osculating space of dimension d. Then, the decomposable minimal surface $F = (F^1, F^2)$ constructed from these two surfaces must also be degree d, and so can have osculating space of dimension no greater than d, whereas we might have hoped that the osculating space would be of dimension $d + d = 2d$.

However, there are examples which show that the equality $k = k_1 + k_2$ can be achieved. Let $F_z = (1, i, z, iz, \ldots, z^{d-1}, iz^{d-1})$ define a minimal surface into $\mathbb{R}^{2d}$ which is in fact holomorphic, and the osculating space has dimension d. Then we can decompose the corresponding F into F^1 and F^2 where $F^1_z = (1, i, \ldots, z^{m-1}, iz^{m-1})$ and $F^2_z = (z^m, iz^m, \ldots, z^{d-1}, iz^{d-1})$. So the osculating spaces of F^1 and F^2 have dimension m and $d - m$ respectively.

Remark. One may also hope to establish a similar result to Proposition 6.2 for the dimensions of the null spaces, i.e. if l, l_1, and l_2 are the dimensions of the null spaces of the osculating spaces of F, F^1, and F^2 respectively, then we hope to also get

$$l_1, l_2 \leq l \leq l_1 + l_2,$$

but this is not the case.

To see this, consider the example where

$$F^1_z = (z^2 - 1, iz^2 + i, 2z)$$

and

$$F^2_z = (1, i).$$

Notice that for F^1 the osculating space is 3-dimensional and so due to Theorem 5.12 the null space must be 0-dimensional, as it cannot be 3-dimensional in $\mathbb{C}^3$. For F^2 the osculating space is 1-dimensional and the null space is 1-dimensional.

Now notice that $F = (F^1, F^2)$ is a degree 3 minimal surface and so has osculating space of dimension at most 3, but due to the lower bound from Proposition 6.2 it must be precisely 3. Again due to Theorem 5.12 the null space must be 0-dimensional, as it cannot be 3-dimensional in $\mathbb{C}^5$. This contradicts the left inequality in what we hoped may be true above.

Next consider the example where

$$F^1_z = (z^2 - 1, iz^2 + i, 2z)$$

and

$$F^2_z = (iz^2 - i, -z^2 - 1, 2iz).$$

As discussed in the previous example, $l_1 = l_2 = 0$. However, the surface $F = (F^1, F^2)$ is holomorphic and has a 3-dimensional osculating space which is null, so $l = 3$. This contradicts the right inequality in what we hoped may have been true above.

6.1 Holomorphic decomposition

Proposition 6.3. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E of dimension k and null space E^0 of dimension l. Then the surface is decomposable with $F = (H, L)$ where $H : D \to \mathbb{R}^{2l}$ is J-holomorphic with respect to some linear complex structure on $\mathbb{R}^{2l}$ and $L : D \to \mathbb{R}^{n-2l}$ is minimal.*

Remark. Clearly this decomposition is only non-trivial if $0 < 2l < n$.

Proof. First, we note that $E^0 \cap \overline{E^0} = \{0\}$ since E^0 consists of only null vectors. Thus, $E^0 \oplus \overline{E^0}$ is a $2l$-dimensional complexification of some real subspace W by Lemma 3.8.

Now let $e_1, \ldots, e_l$ be an orthonormal basis for E^0. We define the following coordinate change

$$\begin{aligned} a_j &= \frac{1}{\sqrt{2}} \left(e_j + \overline{e_j} \right) \\ b_j &= \frac{1}{i\sqrt{2}} \left(e_j - \overline{e_j} \right). \end{aligned}$$

Notice that $a_1, \ldots, a_l, b_1, \ldots, b_l$ is a real orthonormal basis for W.

Let us define π_W to be the projection map in $\mathbb{R}^n$ onto W. We let

$$H = \pi_W \circ F = (F \cdot a_1)\, a_1 + (F \cdot b_1)\, b_1 + \ldots + (F \cdot a_l)\, a_l + (F \cdot b_l)\, b_l.$$

Taking the jth z-derivative of H we get

$$\begin{aligned} \partial_z^j H &= (G_j \cdot a_1)\, a_1 + (G_j \cdot b_1)\, b_1 + \ldots + (G_j \cdot a_l)\, a_l + (G_j \cdot b_l)\, b_l \\ &= \frac{1}{2} \left(G_j \cdot (e_1 + \overline{e_1}) \right) (e_1 + \overline{e_1}) - \frac{1}{2} \left(G_j \cdot (e_1 - \overline{e_1}) \right) (e_1 - \overline{e_1}) + \ldots \\ &\quad + \frac{1}{2} \left(G_j \cdot (e_l + \overline{e_l}) \right) (e_l + \overline{e_l}) - \frac{1}{2} \left(G_j \cdot (e_l - \overline{e_l}) \right) (e_l - \overline{e_l}) \\ &= (G_j \cdot \overline{e_1})\, e_1 + (G_j \cdot e_1)\, \overline{e_1} + \ldots + (G_j \cdot \overline{e_l})\, e_l + (G_j \cdot e_l)\, \overline{e_l} \\ &= (G_j \cdot \overline{e_1})\, e_1 + \ldots + (G_j \cdot \overline{e_l})\, e_l, \end{aligned}$$

where the last equality comes from the definition of E^0. Notice that this is just the projection of G_j onto E^0.

Now, since H is harmonic, it has a well-defined osculating space which is just the projection of E onto E^0. However, since E^0 lies inside E, this means the osculating space of H is just E^0 itself. Since the osculating space of H is null, H must be holomorphic and therefore also minimal.

Finally, we can let L be the projection of F onto the orthogonal complement of W. Since H and L are orthogonal components of F, it follows that H_z and L_z are orthogonal components of F_z. Thus

$$\begin{aligned}0 &= F_z \cdot F_z \\ &= H_z \cdot H_z + L_z \cdot L_z \\ &= L_z \cdot L_z,\end{aligned}$$

and so L is indeed also minimal.

So we conclude that indeed F is decomposable as $F = (H, L)$ where $H : D \to \mathbb{R}^{2l}$ is holomorphic. □

Remark. The converse of Proposition 6.3 is not true in general. That is if we have a decomposable minimal surface with a local conformal parametrization $F : D \to \mathbb{R}^n$ that can be decomposed as $F = (H, L)$ where $H : D \to \mathbb{R}^{2l}$ is holomorphic, then the null space of the osculating space of the minimal surface is not necessarily of dimension l.

For a counterexample, consider the example used in the discussion following Proposition 6.2 where

$$L_z = (z^2 - 1, iz^2 + i, 2z)$$

and

$$H_z = (1, i).$$

Clearly H is holomorphic but $F : D \to \mathbb{R}^5$ given by $F = (H, L)$ must have that the null space of its osculating space is 0-dimensional as shown in that previous discussion of this example. Therefore, although $l = 0$, there is a non-trivial decomposition of the surface into a holomorphic part.

Remark. Proposition 6.3 can also be used to give an alternate proof to Proposition 5.13 since in each case $n - 2l$ will be either 0 or 1. If $n - 2l = 0$ then $F = H$. If $n - 2l = 1$ then L must be a constant and so the surface clearly lies in an $2l$-dimensional affine subspace and when projected to that subspace the surface is just H. So in either case the surface is clearly J-holomorphic.

In Proposition 6.3 one may expect that since the null space of the osculating space of the surface has dimension l and the holomorphic part of the decomposition H is formed from projecting to this null space that L would be left over with an osculating space that has a trivial null space. However, as in the discussion following Proposition 6.2, we should not necessarily expect the null spaces of the osculating spaces of the decomposed surfaces to follow this logic.

In fact it turns out that this is true.

Proposition 6.4. *Let $F : D \to \mathbb{R}^n$ be a local conformal parametrization of a minimal surface with osculating space E that has a null space of dimension l. Let $F = (H, L)$ be the decomposition of F into holomorphic part $H : D \to \mathbb{R}^{2l}$ and the leftover minimal part $L : D \to \mathbb{R}^{n-2l}$. Then the null space of the osculating space of L is trivial.*

Proof. Suppose v is in the null space of the osculating space of L.

Now, for any $u \in E$, we may write $u = u^H + u^L$ where u^H is the orthogonal projection of u onto E^0 (which is the osculating space of H) and u^L is the orthogonal projection onto the osculating space of L as it sits inside E.

Now, H and L map to orthogonal spaces, so in particular their osculating spaces must be orthogonal to each other. Then, since v is in the osculating space of L, we know that $u^H \cdot v = 0$. Furthermore, by the definition of the null space of the osculating space of L we know that $u^L \cdot v = 0$. So $u \cdot v = 0$.

However, since this must be true for all $u \in E$, this means $v \in E^0$. But, in the decomposition, E^0 is the osculating space of H, which is orthogonal to the osculating space of L. Thus, $v = 0$, and so the null space of the osculating space of L is trivial. □

Chapter 7

Surfaces of genus one

Due to the help of Professor R. Schoen, we now show a result based on his recent paper with A. Fraser [8]. In the paper, the authors show that a complete oriented finite total curvature minimal surface in $\mathbb{R}^n$ of genus one that is covering stable is J-holomorphic. The covering stable condition is that every finite cover of the surface is stable. This is quite a strong condition on the surface, so it is desirable to try to weaken the condition in some way. For this result we will use the weakened condition of 4-covering stable, which we will define shortly.

Although we do weaken the covering stable condition, there are two extra conditions that we must assume. The first is the condition that the normal bundle is not topologically trivial, which seems like a reasonable condition to hold and we use to ensure that there is a positive bundle in the normal bundle splitting. The second condition is just that the surface is in $\mathbb{R}^5$ rather than $\mathbb{R}^n$ for any n. We need this condition because we use that the normal bundle is only rank 3 to constrain how it can split.

Due to a theorem of Chern and Osserman [5], an orientable minimal surface Σ which is complete and finite total curvature is conformally equivalent to a compact Riemann surface

M with finitely many punctures. The complexified tangent bundle and the complexified normal bundle of the surface extend across the punctures to bundles on M.

Definition 7.1. Let Σ be a complete oriented finite total curvature minimal surface in $\mathbb{R}^n$ of genus one. We will call such a surface 4-*covering stable* if the cover of Σ corresponding to the $2\mathbb{Z} \oplus 2\mathbb{Z}$ subgroup of the fundamental group of M is stable.

Remark. The term 4-covering stable is not standard and was chosen for convenience so that we may refer to this property of a surface. In particular, although there are other covers of Σ of degree 4, we only refer to this particular covering of degree 4 when we say 4-covering stable.

7.1 4-covering stable surfaces

We now prove the following lemmas as steps toward proving the final theorem below. Certain parts of this argument are very similar to the proof of Theorem IV in [11]. That theorem assumes finite total curvature and genus zero to show the surface must be J-holomorphic. The proof uses a Grothendieck line bundle splitting argument and the end of the proof is essentially the same as ours. The proof in [8] is also modeled after this proof of Theorem IV, the end goal being to show that the direct sum of E_1 and the positive bundles is the osculating space of the surface. The details in showing that this is actually true, however, all differ between the various proofs.

Lemma 7.2. *Let Σ be a complete oriented finite total curvature minimal surface in $\mathbb{R}^5$ that has a normal bundle which is not topologically trivial. Let M be the compactified Riemann surface of Σ and let N be the extension of the complexified normal bundle on Σ to M. Then we can write $N = P \oplus Z \oplus P^*$ where P is a positive line bundle and $Z = Z^*$ is a self-dual line bundle of degree zero.*

Proof. Due to a theorem of Koszul and Malgrange [10], the normal bundle N can be made into a holomorphic vector bundle that agrees with the connection, that is a section s of N is holomorphic if and only if $\partial_{\bar{z}}^{\perp} s = 0$.

Using the Krull-Schmidt theorem we get that N can be written as a direct sum of indecomposable holomorphic vector bundles on M that are unique up to rearrangement. Since N is self-dual, we get that the bundle summands of positive degree must be dual to the summands of negative degree. Since N is rank 3, the collective rank of the positive degree bundle summands must be either 0 or 1.

If it is 0, then each indecomposable bundle summand is degree zero and thus topologically trivial. Therefore, N would be topologically trivial, contradicting our assumption in the hypothesis.

If it is 1, then $N = P + Z + P^*$ where $Z = Z^*$ must be a self-dual line bundle of degree zero, which is the desired conclusion. □

Lemma 7.3. *Let Σ be a complete oriented finite total curvature stable minimal surface of genus one in $\mathbb{R}^5$ that is also 4-covering stable and has a normal bundle which is not topologically trivial. Let M be the compactified Riemann surface of Σ and let $N = P \oplus Z \oplus P^*$ be the extension of the complexified normal bundle on Σ to M. Then $(\partial s)^T = 0$ for all $s \in \Gamma(P \oplus Z)$.*

Proof. Since P has positive degree, it has a holomorphic section s. By the complex stability inequality
$$\int_{\Sigma} \|(\partial s)^T\|^2 \leq \int_{\Sigma} \|(\bar{\partial} s)^{\perp}\|^2$$
we get that $(\partial s)^T = 0$. Since P is a line bundle we therefore have that every section $s \in \Gamma(P)$ satisfies $(\partial s)^T = 0$.

Since M is a compact Riemann surface of genus one it is isomorphic to some complex torus

$\mathbb{C}/\Lambda$ for some lattice $\Lambda = \{1, \tau\}$ where $\tau \in \mathbb{C}$. Now there are four distinct self-dual degree zero line bundles over M, namely $[0], [1/2 - 0], [\tau/2 - 0], [(1/2 + \tau/2) - 0]$ where $[D]$ refers to the line bundle corresponding to the divisor D. Therefore, Z must be one of these line bundles.

Let $\pi_4 : M_4 \to M$ be the covering map of the cover corresponding to the $2\mathbb{Z} \oplus 2\mathbb{Z}$ subgroup of the fundamental group of M as mentioned in Definition 7.1. Then, $\pi_4^* Z$ is the trivial line bundle, as the pull-back maps all of the above mentioned line bundles to the trivial line bundle. Therefore, $\pi_4^* Z$ has a holomorphic section s, and by 4-covering stability we therefore have that $(\partial s)^T = 0$. So then, for any section $\tilde{s} \in \Gamma(Z)$, we get that $(\partial(\pi_4^* \tilde{s}))^T = 0$, and so $(\partial \tilde{s})^T = 0$. □

Lemma 7.4. *Let Σ be a complete oriented finite total curvature stable minimal surface of genus one in $\mathbb{R}^5$ that is also 4-covering stable and has a normal bundle which is not topologically trivial. Let M be the compactified Riemann surface of Σ and let $N = P \oplus Z \oplus P^*$ be the extension of the complexified normal bundle on Σ to M. Then $P^\perp = P \oplus Z$ where $P^\perp$ is the orthogonal complement of P in N with respect to the dot product on $\mathbb{C}^5$.*

Proof. Let $\pi_4 : M_4 \to M$ be the covering map of the cover corresponding to the $2\mathbb{Z} \oplus 2\mathbb{Z}$ subgroup of the fundamental group of M as mentioned in Definition 7.1. Then, as discussed in the proof of Lemma 7.4, we have that $\pi_4^* Z$ is the trivial line bundle and so has a holomorphic section.

Let s_1 be a holomorphic section of $\pi_4^* P$ and let s_2 be a holomorphic section of $\pi_4^* Z$. Then

$$\begin{aligned} \partial_{\bar{z}} (s_1 \cdot s_i) &= \partial_{\bar{z}}^\perp s_1 \cdot s_i + s_1 \cdot \partial_{\bar{z}}^\perp s_i \\ &= 0 \end{aligned}$$

for $i = 1, 2$. Since $s_1 \cdot s_i$ is a holomorphic function on M, it must be a constant. However, since $\pi_4^* P$ has positive degree, s_1 has a zero and therefore $s_1 \cdot s_i = 0$. Thus $P^\perp = P \oplus Z$. $\square$

Lemma 7.5. *Let Σ be a complete oriented finite total curvature stable minimal surface of genus one in $\mathbb{R}^5$ that is also 4-covering stable and has a normal bundle which is not topologically trivial. Let M be the compactified Riemann surface of Σ and let $N = P \oplus Z \oplus P^*$ be the extension of the complexified normal bundle on Σ to M. Then $E_1 \oplus P \oplus Z$ is closed under d.*

Proof. As in Lemma 7.4, let s_1 be a holomorphic section of $\pi_4^* P$ and let s_2 be a holomorphic section of $\pi_4^* Z$. We will use repeatedly below that $\partial_z s_i$ is a normal vector field by Lemma 7.3 and that $\partial_{\bar{z}} s_i$ is a tangential vector field since each s_i is holomorphic in the normal bundle.

First, we note that $dF_z = F_{zz} dz$. Then, since $F_{zz} \cdot F_z = 0$, the tangential part of F_{zz} lies in E_1. Similarly, since $F_{zz} \cdot s_i = -F_z \cdot \partial_z s_i = 0$, the normal part of F_{zz} lies in $(P \oplus Z)^\perp = P$. So d maps E_1 to $E_1 \oplus P$.

Next, we show that $(\partial_z s_i \cdot s_1) dz$ is a holomorphic differential on M. We compute

$$\begin{aligned}
\partial_{\bar{z}}(\partial_z s_i \cdot s_1) &= \partial_{\bar{z}} \partial_z s_i \cdot s_1 + \partial_z s_i \cdot \partial_{\bar{z}} s_1 \\
&= \partial_z(\partial_{\bar{z}} s_i \cdot s_1) - \partial_{\bar{z}} s_i \cdot \partial_z s_1 + \partial_z s_i \cdot \partial_{\bar{z}} s_1 \\
&= 0,
\end{aligned}$$

where in the last equality we use that each term contains a dot product between a tangential and normal vector field. Then indeed, since s_1 has a zero, we have that $(\partial_z s_i \cdot s_1) dz$ is a holomorphic differential on M with a zero. Then, by Riemann-Roch for genus 1 surfaces, this differential must be zero.

By the above, since $\partial_z s_i$ is a normal vector field and orthogonal to s_1, it must lie in $P^\perp =$

$P \oplus Z$. Then finally we compute

$$\begin{aligned} \partial_{\bar{z}} s_i \cdot F_z &= \partial_{\bar{z}}(s_i \cdot F_z) \\ &= 0. \end{aligned}$$

Since $\partial_{\bar{z}} s_i$ is a tangential vector field, this shows that it must lie in E_1. Thus d maps $P \oplus Z$ to $E_1 \oplus P \oplus Z$. □

Lemma 7.6. *Let Σ be a complete oriented finite total curvature stable minimal surface of genus one in $\mathbb{R}^5$ that is also 4-covering stable and has a normal bundle which is not topologically trivial. Then Σ lies in a 4-dimensional affine subspace of $\mathbb{R}^5$.*

Proof. Let M be the compactified Riemann surface of Σ and let $N = P \oplus Z \oplus P^*$ be the extension of the complexified normal bundle on Σ to M. From Lemma 7.5 we have that $E_1 \oplus P \oplus Z$ is closed under d and thus $E_1 \oplus P \oplus Z = M \times \Lambda$ where Λ is a constant 3-dimensional subspace of $\mathbb{C}^5$.

Let $v \in \Lambda \cap \overline{\Lambda}$, then since $v \in \overline{\Lambda}$ it is orthogonal to $E_1 \oplus P$ with respect to the Hermitian inner product. Similarly since $v \in \Lambda$, it is orthogonal to $\overline{E_1} \oplus \overline{P}$ with respect to the Hermitian inner product.

Thus, $\Lambda \cap \overline{\Lambda}$ is precisely one-dimensional and is the complexification of some real one-dimensional subspace W of $\mathbb{R}^5$. Since W is orthogonal to the tangent space $E_1 \oplus \overline{E_1}$ everywhere on the surface, it follows that the surface lies in some 4-dimensional affine subspace parallel to $W^\perp \subset \mathbb{R}^5$. □

Theorem 7.7. *Let Σ be a complete oriented finite total curvature stable minimal surface of genus one in $\mathbb{R}^5$ that is also 4-covering stable and has a normal bundle which is not topologically trivial. Then Σ lies in a 4-dimensional affine subspace of $\mathbb{R}^5$ and is J-holomorphic with respect to some linear complex structure J on that affine subspace.*

Proof. Let M be the compactified Riemann surface of Σ and let $N = P \oplus Z \oplus P^*$ be the extension of the complexified normal bundle on Σ to M. From the proof of Lemma 7.6 we have that Σ lies in some 4-dimensional affine subspace parallel to $W^\perp$.

Note that we now have that $E_1 \oplus P$ is closed under d, since $E_1 \oplus P = (E_1 \oplus P \oplus Z) \cap W^\perp$ and for a vector $w \in W$ we have that if $s \cdot w = 0$ then $ds \cdot w = 0$.

Thus, $E_1 \oplus P$ contains the osculating space of the surface Σ and is itself null, so by Theorem 4.8 we conclude that Σ is J-holomorphic with respect to some linear complex structure J on the affine $\mathbb{R}^4$ containing Σ. $\square$

Chapter 8

Conclusion

In this paper, we introduce the formal definition of osculating spaces for minimal surfaces immersed in $\mathbb{R}^n$. We proved that it is possible to define osculating spaces as vector bundles over the surface and proceeded to prove a number of important properties that govern their behavior.

Following that, we showed that osculating spaces give some important insights into various areas of study regarding minimal surfaces. These included holomorphic curves, minimal surfaces with a complex structure not compatible with the Euclidean metric, and decomposable minimal surfaces.

We finally showed that complete finite total curvature stable surfaces of genus one in $\mathbb{R}^5$ are holomorphic under some certain assumptions. This was proved using a normal bundle splitting argument in the spirit of the proof of Theorem IV from [11] which was for genus zero surfaces instead.

8.1 Directions of further study

There are several areas of research which did not yield significant progress and so are not included in this paper, yet may produce interesting results upon further study.

The first area is the extension of Bernstein's theorem to higher codimension that was discussed above. Although we do get a restriction on the dimension of the osculating space in the case of a graphical minimal surface, there seems to be few obvious next steps towards a solution, even for $n = 5$. The primary difficulty lies in finding some relation between osculating spaces and stability. As of now, stability seems to stubbornly resist any meaningful interaction with the osculating spaces of the surface, in contrast to the findings above connecting osculating spaces with the holomorphicity of the surface, for example. If some sort of connection were found, it would likely be instrumental in the advancement of this problem.

Another area is the question of how the osculating space may or may not be restricted depending on global properties of the surface. The osculating space as defined is an entirely local property, yet is the same at every point on the surface. There may be some restriction as to what the osculating space may be depending on the genus of the surface, for example. Such a relationship may be helpful in identifying the relationship between dimension and genus in the problem of whether a stable and finite total curvature minimal surface is holomorphic.

In Theorem 7.7 we must assume that the normal bundle is not topologically trivial and the 4-covering stable condition. It seems like the 4-covering stable condition may not be able to be improved, but it would be desirable to prove the result without the topologically trivial condition. Such a proof would have to handle the case where the normal bundle splits into three degree zero line bundles and only one is self-dual while the other two are dual to each other. Alternatively, it may be possible to find a counter-example of a surface that satisfies all the conditions except for the 4-covering stable or topologically trivial condition and is not J-holomorphic.

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