Weights for Computing Vertex Normals from Facet Normals

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Abstract

I propose a new equation to estimate the normal at a vertex of a polygonal approximation to a smooth surface, as a weighted sum of the normals to the facets surrounding the vertex. It accounts for the difference in size of these facets by assigning larger weights for smaller facets. When tested on random cubic polynomial surfaces, it is superior to other popular weighting methods.

Introduction

When a surface is approximated by polygonal facets, often only the vertex positions are known, but associated vertex normals are required for smooth shading. Sometimes the model is just a polyhedron, and there is no single correct normal at a vertex. However, in other cases, there is an underlying smooth surface approximated by the facets, whose true normals are to be estimated from the facet geometry. Examples are range data, or measured vertices on a plaster character model. A vertex normal is usually taken as a weighted sum of the normals of facets sharing that vertex. Gouraud [1] suggested equal weights, and Thürmer and Wüthrich [2] propose weighting by the facet angles at the vertex. Here I propose a new set of weights, which help to handle the cases when the facets surrounding a vertex differ greatly in size. The weights are appropriate to, and tested for, the situation when the smooth surface is locally represented near the vertex by a height field approximated by a taylor series polynomial.

The problem of facets of different sizes is illustrated in a 2D analogue in figure 1, where a circle is approximated by an inscribed polygon *DEFG*... with unequal sides. The normal to the circle at *E* passes through the center *C* of the circle. An unweighted average of the normals to segments *DE* and *EF* will lie on the angle bisector of angle *DEF*, which does not pass through *C*. The correct weights are inversely proportional to the lengths of the segments. This can be shown by solving simultaneous linear equations, as done in the 3D case below.

The new weights

The weights I propose in 3D give the correct normals for a polyhedron inscribed in a sphere. Suppose Q is a vertex of the polyhedron, and its adjacent vertices are V_0, V_1, ..., V_{n-1}. The center C of the sphere can be found as the intersection of the perpendicular bisector planes of the edges QV_i. For simplicity in the derivation, translate the coordinates so that Q is at the origin. Then the midpoint of the edge between $Q = (0, 0, 0)$ and $V_i = (x_i, y_i, z_i)$ is $(1/2)V_i$, and the vector V_i is normal to any plane perpendicular to this edge, so the equation of the perpendicular bisector plane is

$$x_i x + y_i y + z_i z = |V_i|^2 / 2 .$$

We first consider the case $n = 3$, where we get 3 linear equations in 3 unknowns:

$$x_0 x + y_0 y + z_0 z = |V_0|^2 / 2$$
$$x_1 x + y_1 y + z_1 z = |V_1|^2 / 2$$
$$x_2 x + y_2 y + z_2 z = |V_2|^2 / 2\,.$$

Rearranging and grouping the terms of the Cramer's rule solution [3] of this system, one gets

$$C = (x, y, z) = (|V_2|^2 V_0 \times V_1 + |V_0|^2 V_1 \times V_2 + |V_1|^2 V_2 \times V_0) / (2D)$$

where D is the determinant of the coefficient matrix, and is negative if the vertices V_0, V_1, and V_2 are in counterclockwise order, when viewed from Q. Divide this equation by the negative quantity $|V_0|^2 |V_1|^2 |V_2|^2 / (2D)$, to get a vector pointing in the outward normal direction, that is, a positive multiple c of the normal vector:

$$\frac{V_0 \times V_1}{|V_0|^2 |V_1|^2} + \frac{V_1 \times V_2}{|V_1|^2 |V_2|^2} + \frac{V_2 \times V_0}{|V_2|^2 |V_0|^2} = cN\,. \qquad (1)$$

In the degenerate case when Q, V_0, V_1, and V_2 are all in the same plane P, D is zero, but since Q, V_0, V_1, and V_2 then all lie on the circle where the plane P intersects the sphere, one can show that $|V_2|^2 V_0 \times V_1 + |V_0|^2 V_1 \times V_2 + |V_1|^2 V_2 \times V_0$ is also zero, so equation (1) holds with $c = 0$. If V_0, V_1, and V_2 are in clockwise order when viewed from Q, c is negative.

To express this equation in terms of the facet normals, let N_i be the normal to the i^{th} facet, between V_i and V_{i+1}, with all indices taken mod n (which, for now, is 3), and let α_i be the angle between V_i and V_{i+1}. Then

$$V_i \times V_{i+1} = N_i |V_i| |V_{i+1}| \sin\alpha_i$$

so

$$\sum_{i=0}^{2} \frac{V_i \times V_{i+1}}{|V_i|^2 |V_{i+1}|^2} = \sum_{i=0}^{2} \frac{N_i \sin\alpha_i}{|V_i| |V_{i+1}|} = cN\,. \qquad (2)$$

Therefore the proposed weight for normal N_i is $\sin\alpha_i/(|V_i||V_{i+1}|)$. The $\sin\alpha_i$ factor is analogous to the α_i factor in [2] and the reciprocals of the edge lengths are related to the reciprocal segment lengths discussed in the 2D case above. In practice, the left hand sum in equation (2) is easier to compute, since it involves neither square roots nor trigonometry.

One can show by induction that also, for any n,

$$\sum_{i=0}^{n-1} \frac{V_i \times V_{i+1}}{|V_i|^2 |V_{i+1}|^2} = cN. \tag{3}$$

The induction starts at $n = 3$ with equation (2). To derive equation (3) for $n + 1$ just add equation (3) for n and equation (2) for the three vertices V_0, V_{n-1}, and V_n. The last term in the sum in equation (3), involving $V_{n-1} \times V_0$, will cancel the first term in equation (2), involving $V_0 \times V_{n-1}$, giving equation (3) for $n + 1$ (with some new multiple c of N).

In degenerate cases, similar to those discussed in [2], c will be zero, and no reasonable normal can be chosen. In cases like those in the tests below, when (*a*) the smooth surface is defined locally by a height function $z = f(x, y)$, (*b*) the vertices V_i are in monotonically increasing counterclockwise order around Q when viewed from above, and (*c*) the angles from V_i to V_{i+1} are all acute, every term in equation (3) will have a positive z component, so the sum will be a positive multiple of the upward surface normal. In other cases, the resulting normal may need to be flipped.

Discussion and results

The derivation above only applies to polyhedra inscribed in spheres, and the normal estimate is incorrect even for ellipsoids. Nevertheless, I believe it is superior to other estimates, because it handles cases where edges adjacent to the vertex Q have very different lengths.

To test this belief, for each n between 3 and 9, I constructed 1,000,000 surfaces of the form $z = f(x, y) = Ax^2 + Bxy + Cy^2 + Dx^3 + Ex^2y + Fxy^2 + Gy^3$, with A, B, C, D, E, F, and G all uniformly distributed pseudo random numbers in the interval [-0.1, 0.1]. These are representative of the third order behavior of smooth surfaces, translated to place the vertex Q, where the normal is to be estimated, at the origin, and rotated to make the true normal at Q point along the positive z axis. I then generated the vertices V_0, V_1, ..., V_{n-1} in cylindrical coordinates (r, θ, z) by choosing the θ's randomly and uniformly in the interval $[0, 2\pi]$, resorting them in increasing order, choosing for each θ an r randomly and uniformly in the interval [0, 1], converting (r, θ) to cartesian coordinates (x, y), and then setting $z = f(x, y)$. I rejected cases where consecutive θ's differed by more than π, because they violate condition (c) above.

For each accepted sequence V_0, V_1,..., V_{n-1}, I computed several average normals, weighting N_i by: (*a*) the area of triangle QV_iV_{i+1}, (*b*) one (*i. e*. an unweighted average), (*c*) the angle α_i as proposed in [2], (*d*) $\sin\alpha_i/(|V_i||V_{i+1}|)$ as proposed here, (*e*) $1./(|V_i||V_{i+1}|)$, and (*f*) $1./\mathrm{sqrt}(|V_i||V_{i+1}|)$. I then measured the error angle between these estimated normals and the correct normal, which is the positive z axis. Table 1 below gives the root-mean-square error angle in degrees, over the 1,000,000 trials, for each n and each of the six weighting schemes.

n	(a)	(b)	(c)	(d)	(e)	(f)
3	6.468480	7.301575	10.713165	2.976864	7.355423	7.052390
4	4.324776	5.447873	8.538115	2.275241	5.453043	5.216014
5	3.671877	4.644646	7.030462	1.952892	4.959709	4.588069
6	3.308351	4.234035	5.820398	1.783925	4.890262	4.352113
7	3.088986	4.005317	4.869360	1.674386	4.958369	4.253581
8	2.933336	3.835367	4.064310	1.590934	5.007468	4.169861
9	2.809747	3.719183	3.416770	1.534402	5.115330	4.131644

Table 1. RMS errors in degrees for a cubic surface, with all coefficients in [-0.1, 0.1].

The method proposed here does seem superior, and the method proposed in [2] seems worse than the unweighted average, at least in the situation tested here, where all vertices are on the smooth surface. The method in [2] was designed to be consistent when polygons are subdivided, possibly by adding along a facet edge a new vertex which might not be on the smooth surface.

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References

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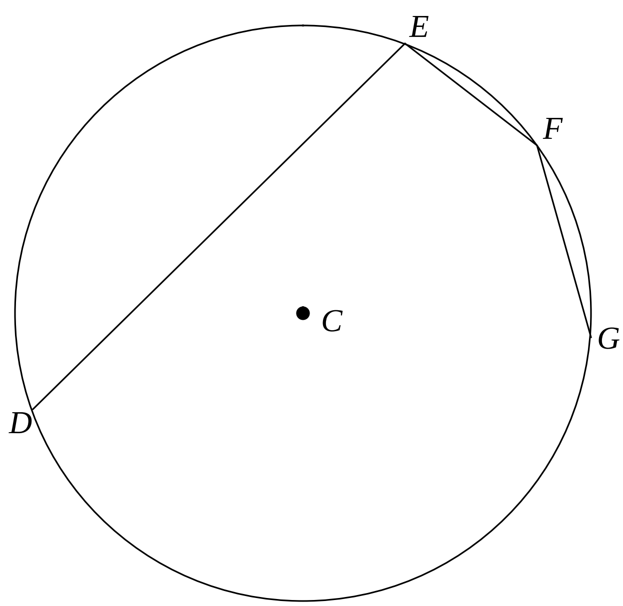

Figure 1. A polygon with unequal sides inscribed in a circle.