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SUPERFICIAL FIBRES OF GENERIC PROJECTIONS

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Abstract We consider a general fibre of given length in a generic projection of a variety. Under the assumption that the fibre is of local embedding dimension 2 or less, an assumption which can be checked in many cases, we prove that the fibre is reduced and its image on the projected variety is an ordinary multiple point.

Keywords: multisection planes; generic projections; multiple points; punctual schemes

2010 *Mathematics subject classification:* 14N05

The study of generic linear projections of a smooth projective variety $X \subset \mathbb{P}^N$, especially their singularities, was a favourite topic of classical projective geometers for generations (see e.g. [9]), and these projections remain among the main sources of finite, nonflat morphisms in Algebraic Geometry. Modern interest in this area seems to begin with Mather [5], whose work was reinterpreted in algebro-geometric language by Alzati–Ottaviani [1]. One may formulate a folklore ‘generic projection conjecture’ to the effect that the projection of X from a generic linear subspace $\Lambda \in \mathbb{P}^N$ has only the ‘expected’ singularities. After numerous partial results including [2, 7], the conjecture was recently proven for projections *from a point* by Gruson and Peskine [4] (which the reader may consult for further introductory comments, references, as well as examples and applications).

In [8], we gave various extensions of the Gruson–Peskine theorem, including one about projections from *lines*. More generally, we showed inter alia that the locus of fibres of given length k of a generic projection from a centre Λ of any dimension λ is smooth of the expected dimension, just as long as the fibres have local embedding dimension 2 or less. The latter hypothesis is of course automatic for $\lambda = 0, 1$, but as we shall see below (cf. Lemma 1 and Example 3), it also holds in a fair range of cases beyond that.

One question which was left open in [8] was as to the nature of the general fibre of given length k and embedding dimension ≤ 2 : was it in fact reduced, i.e. a collection of distinct points; even more optimistically, are those in general position relative to the $(\lambda + 1)$ -plane they are constrained to lie on? The purpose of this paper is to settle these questions. We will settle the first question affirmatively, and show further that the resulting k -fold point on the projected image is ‘ordinary’ or ‘transverse’ (see below for definitions).

The second question will be settled negatively by a counterexample (see Example 6). Such counterexamples seem rare though, and the possibility remains that they could be effectively characterized in terms of low-degree defining equations, though we do not have a specific conjecture – much less, result, to offer.

We begin with some notation. Let

$$X \subset \mathbb{P}^m$$

be an irreducible closed subvariety of codimension c . Let

$$\Lambda = \mathbb{P}^\lambda \subset \mathbb{P}^m, \quad \lambda < c$$

be a generic linear space. Let

$$X_k^\Lambda \subset \mathbb{P}^{m-\lambda-1}$$

be the (locally closed) locus of $(\lambda + 1)$ -planes L disjoint from the singular locus of X , containing Λ and meeting X in a scheme of length k or more. Now consider the following condition on $L \in X_k^\Lambda$.

(EDIM2) The local embedding dimension of $Z = L \cap X$ is 2 or less everywhere.

Under this condition, it was proven in [8] that X_k^Λ is smooth of the expected codimension in $\mathbb{P}^{m-\lambda-1}$, viz. $k(c - \lambda - 1)$, at L . The tangent space to X_k^Λ at a point L can be identified with $H^0(N^s)$ where N^s is a certain subsheaf of the normal bundle $N_{L/\mathbb{P}^m}$ which differs from it only at points of $Z = L \cap X$, and at those points consists grosso modo of the vectors normal to L and tangent to X .

Before stating the new result, we make a remark about the EDIM2 condition.

Lemma 1. *Notations as above, condition (EDIM2) holds for all $L \in X_k^\Lambda$ provided either*

- (i) $\min(\lambda + 1, m - c) \leq 2$; or
- (ii) $m < 4c - 3\lambda + 6$.

Proof. (i) is trivial. As for (ii), assuming $m - c \geq 3, \lambda > 1$, note that if Z has embedding dimension 3 or more at p , then the embedded tangent space $\hat{T}_p X$ meets L in at least a 3 (projective)-dimensional space, hence meets Λ , a hyperplane in L , in at least a 2-dimensional space. The space of pairs (p, U) , where U is a 2-dimensional subspace of $\hat{T}_p X$, is $(4(m - c) - 6)$ -dimensional, and it is $3(m - \lambda)$ conditions for U to be contained in Λ . Since Λ is general with respect to X , the Bertini theorem on transversality of a general translate yields the result. $\square$

As a matter of terminology, a collection of subspaces A_i of a vector space B is said to be *transverse* if

$$\text{codim}\left(\bigcap A_i, B\right) = \sum \text{codim}(A_i, B).$$

A point y on a subvariety Y in a smooth variety P is said to be a *transverse k -fold point* if locally at y , Y is a union of k smooth branches $Y_1, \dots, Y_k$ and $T_y Y_i, i = 1, \dots, k$ is a transverse collection of subspaces of $T_y P$.

Theorem 2. *Notations as above, assume*

- (a) L is general in X_k^Λ ;
- (b) $c > \lambda + 1$;
- (c) condition (EDIM2) holds.

Then

- (i) Z is reduced;
- (ii) each point of Z is general on X ;
- (iii) Z projects to a transverse k -fold point on the projection $\pi_\Lambda(X)$.

Note that in (ii), it is not claimed that Z is a generic k -tuple on X , only each point in itself is generic. Thus given any subvariety $Y \subsetneq X$, Z may be assumed disjoint from Y by taking Λ general enough.

Example 3. For $m - c = 3$, i.e. X a 3-fold, and $\lambda = 2$, Lemma 1 shows that condition (EDIM2) is weaker than $c > \lambda + 1 = 3$. Therefore the theorem is applicable to the generic projection of any smooth 3-fold $X \subset \mathbb{P}^m$, $m \geq 7$, to $\mathbb{P}^{m-3}$. In particular, for $m = 7$, the projected image of X in $\mathbb{P}^4$ has finitely many transverse 4-fold points, each with four linearly independent tangent hyperplanes.

In fact, for $\lambda = 2$, i.e. projection from a plane, and X of codimension $c = 4$, the theorem applies for X of dimension up to 11, i.e. for projection from $\mathbb{P}^m$ to $\mathbb{P}^{m-3}$ for $m \leq 15$. $\square$

The proof of the theorem is a continuation of the argument used to prove Theorem 5.1 in [8]. A key role is played again by the vanishing lemma [8, Lemma 5.8], for subsheaves of the normal bundle N_L whose sections move with Λ (which, we recall, is a generic λ -plane). Originally, the lemma was applied to the secant sheaf N^s . Here, we note that it can be applied as well to suitable subsheaves of N^s , and exploit the consequences. Incidentally, we neglected to mention in [8] that the vanishing lemma implies a regularity result of the secant sheaves, and this omission will be rectified below.

Because of the key role it plays in this paper, we proceed to state without proof a special case, sufficient for our purposes, of the lemma in question. This concerns the following data:

- Λ , a $\mathbb{P}^\lambda$ in $\mathbb{P}^m$ that is generic with respect to the subvariety X as above (in particular, $X \cap \Lambda = \emptyset$),
- L , an arbitrary $\mathbb{P}^{\lambda+1}$ containing Λ ,
- $M \subset N_{L/\mathbb{P}^m}$, a coherent subsheaf of the normal bundle, such that the support of $N_{L/\mathbb{P}^m}/M$ is finite and disjoint from Λ , and having the property that $H^0(M)$ ‘moves infinitesimally with Λ ’, in the sense that given any $v_1 \in H^0(N_{\Lambda/\mathbb{P}^m})$, there exists $v_2 \in H^0(M)$ having the same image as v_1 in $H^0(M \otimes \mathcal{O}_\Lambda)$.

Vanishing lemma (See [8, Lemma 5.8]). *Assumptions as above, we have $H^1(M(t)) = 0$ for $t \geq -1$.*

The proof is given in [8] and will not be reproduced here.

Proof of theorem. We fix a sufficiently general L as in the theorem. Then by [8, Theorem 5.1], we may assume that $Z = L \cap X$ has length exactly k . Pick any $p \in Z$.

Now recall the ‘secant subsheaf’ $N^s \subset \text{Hom}(\mathcal{I}_L, \mathcal{O}_L)$ defined in [8], which fits in an exact diagram

$$\begin{array}{ccccccc}
 0 & & & & 0 & & \\
 \downarrow & & & & \downarrow & & \\
 N^s & \rightarrow & \text{Hom}(\mathcal{I}_Z/\mathcal{I}_X, \mathcal{O}_Z) & \rightarrow & 0 & & \\
 \downarrow & & \downarrow & & & & \\
 \text{Hom}(\mathcal{I}_L, \mathcal{O}_L) & \rightarrow & \text{Hom}(\mathcal{I}_L \cdot \mathcal{O}_Z) & \rightarrow & 0 & & (1) \\
 \downarrow & & \downarrow & & & & \\
 \text{Hom}(\mathcal{I}_L \cap \mathcal{I}_X, \mathcal{O}_Z) & = & \text{Hom}(\mathcal{I}_L \cap \mathcal{I}_X, \mathcal{O}_Z) & & & &
 \end{array}$$

(note $\mathcal{I}_Z/\mathcal{I}_X = (\mathcal{I}_L + \mathcal{I}_X)/\mathcal{I}_X = \mathcal{I}_L/(\mathcal{I}_L \cap \mathcal{I}_X)$). Of course, $\text{Hom}(\mathcal{I}_Z/\mathcal{I}_X, \mathcal{O}_Z)$ is just a skyscraper sheaf with stalk equal to the tangent space at Z to the punctual Hilbert scheme of X , that is $X^{[k]}$ where $k = l(Z)$. Now we have a support decomposition

$$\text{Hom}(\mathcal{I}_Z/\mathcal{I}_X, \mathcal{O}_Z) = \bigoplus T_p, \quad T_p = \text{Hom}(\mathcal{I}_Z/\mathcal{I}_X, \mathcal{O}_Z)_p$$

which corresponds to a local analytic decomposition

$$X^{[k]} = \prod X^{[k(p)]}, \quad k(p) := l_p(Z).$$

via the identification of Zariski tangent spaces

$$T_{Z_p} X^{[k(p)]} = H^0(T_p). \quad (2)$$

It was observed in [8] – and of course is well known – that the assumption of embedding dimension 2 or less implies, via Fogarty’s theorem on the smoothness of the Hilbert scheme of a smooth surface [3], that $X^{[k]}(p)$ is smooth at Z_p and its general point corresponds to a reduced scheme.

Now I claim that the surjection

$$N^s \rightarrow T_p$$

(cf. (1)) induces a surjection on H^0 :

$$H^0(N^s) \rightarrow H^0(T_p). \quad (3)$$

If this holds, then the natural (local) map from the secant scheme X_k^Λ to $X^{[r(p)]}$ is smooth, hence surjective, near Z . Therefore Z_p is general on $X^{[r(p)]}$, hence reduced. Since Z_p is supported at p , it follows that $Z_p = p$ is a single reduced general point of X , this proving parts (i) and (ii) of the theorem.

To prove surjectivity of (3), note that its cokernel is anyhow of the form $H^0(A)$ for some quotient A of the skyscraper sheaf T_p , such that the natural map $H^0(N^s) \rightarrow H^0(A)$ vanishes. Let $N_0^s \subset N^s$ be the corresponding subsheaf of N^s under the surjection $N^s \rightarrow T_p$. Thus we have an exact sequence

$$0 \rightarrow N_0^s \rightarrow N^s \rightarrow A \rightarrow 0. \quad (4)$$

Then by the generality of L , N_0^s satisfies the hypotheses of our vanishing lemma, and hence $H^1(N_0^s) = 0$. But then the exact sequence (4) yields surjectivity of $H^0(N^s) \rightarrow H^0(A)$, which is a contradiction unless $A = 0$.

Note, parenthetically, that there is no reason to think sections to N^s must go to zero in T_p ; thus we cannot apply the vanishing lemma directly to the kernel of the map $N^s \rightarrow T_p$.

It remains to prove assertion (iii). To this end consider the exact sequence

$$0 \rightarrow N^s(-1) \rightarrow N_L(-1) \rightarrow \bigoplus_{p \in Z} Q_p \rightarrow 0, \quad (5)$$

where Q_p is a skyscraper $k(p)$ -module of length $c - \lambda - 1$ such that, writing

$$\mathbb{P}^m = \mathbb{P}(V^*), \quad L = \mathbb{P}(U^*), \quad U \subset V,$$

we may identify

$$H^0(Q_p) = V/(U + T_p(X)(-1)).$$

By our vanishing lemma, we have $H^1(N^s(-1)) = 0$, whence surjectivity of

$$q : H^0(N_L(-1)) = \text{Hom}(U, V/U) \rightarrow \bigoplus H^0(Q_p).$$

On the other hand, as $N_L(-1)$ is a trivial bundle on $L \simeq \mathbb{P}^{\lambda+1}$, it is easy to see from (5) that $H^i(N^s(-1)) = 0, i > 1$, and consequently

$$\chi(N^s(-1)) = h^0(N^s(-1)) = (m - \lambda - 1) - k(c - \lambda - 1). \quad (6)$$

Let W_i be the $(m - c)$ -dimensional, $(c - \lambda - 1)$ -codimensional subspace of V/U corresponding to $T_{p_i}X$. Note that each W_i projects to the tangent space of the branch of $\pi_\Lambda(X)$ corresponding to p_i . By (5), $\bigcap W_i = H^0(N^s(-1))$. Therefore (6) proves our transversality assertion. $\square$

It has been noted above that $H^i(N^s(-i)) = 0, \forall i > 1$, as follows from (5). Together with our vanishing lemma and the Castelnuovo–Mumford lemma [6, p. 99], we conclude the following corollary.

Corollary 4. *The sheaf N^s is 0-regular. In particular, N^s is globally generated.* $\square$

Remark 5. The general fibre of given length (e.g. 3) of generic projection from a line need not be in general position relative to the plane containing it. The following is an example of a smooth nondegenerate surface X in $\mathbb{P}^5$ admitting a 3-parameter family $\{D_t\}$ of trisecant lines filling a hypersurface. Then, a generic line Λ will meet finitely many of the D_t and the corresponding aligned triples $D_t \cap X$ will be fibres of projection from Λ . Our vague impression – unbacked by evidence – is that such examples are not easy to construct and may be ‘special’ in some characterizable way related to the defining equations.

Example 6. Let $F = (f_{ij})$ be a general symmetric 4×4 matrix of linear forms on $\mathbb{P}^5$. Let X be the locus $\text{rk}(F) < 3$, i.e. the zero locus of the 3×3 minors of F , which is a smooth surface cut out by cubics; let Y be the locus $\text{rk}(F) < 4$, i.e. the zero locus of $\det(F)$, which is a quartic singular along X . More invariantly, F corresponds to a quadratic form q on the trivial bundle $4\mathcal{O}_{\mathbb{P}^5}$ with values in $\mathcal{O}_{\mathbb{P}^5}(1)$, and Y and X are respectively the loci

where q drops rank by 1 or 2 at least. For $t = (t_1, \dots, t_4)$ general, let D_t be the line in $\mathbb{P}^5$ with equations $\sum_i t_i f_{ij}$, $j = 1, \dots, 4$. The lines D_t sweep out Y . I claim that a general D_t is trisecant to X . To see this, we can assume by a suitable change of notations that D_t corresponds to the 1st row (= 1st column) of F . Then the only 3×3 minor nontrivial on D_t is the one corresponds to rows and columns 2, 3, 4, which gives a single cubic equation on D_t defining the intersection $D_t \cap X$. $\square$

Open questions. In conclusion, we point out two important questions that remain open.

- (i) Let Z be a general fibre of given length in a generic projection of a smooth variety. Then is Z smoothable? (The question only arises if Z has embedding dimension > 2 .)
- (ii) Describe the singularity of X_k^Λ at a point L such that $L \cap X$ has length $> k$ and embedding dimension 2 (the case of embedding dimension ≤ 1 was settled in [8]).

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