

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Constrained Model Predictive Control, State Estimation and Coordination

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requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Aerospace Engineering)

by

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To my family.

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ABSTRACT OF THE DISSERTATION

Constrained Model Predictive Control, State Estimation and Coordination

by

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In this dissertation, we study the interaction between the control performance and the quality of the state estimation in a constrained Model Predictive Control (MPC) framework for systems with stochastic disturbances. This consists of three parts: (i) the development of a constrained MPC formulation that adapts to the quality of the state estimation via constraints; (ii) the application of such a control law in a multi-vehicle formation coordinated control problem in which each vehicle operates subject to a no-collision constraint posed by others' imperfect prediction computed from finite bit-rate, communicated data; (iii) the design of the predictors and the communication resource assignment problem that satisfy the performance requirement from Part (ii).

Model Predictive Control (MPC) is of interest because it is one of the few control design methods which preserves standard design variables and yet handles constraints. MPC is normally posed as a full-state feedback control and is implemented in a certainty-equivalence fashion with best estimates of the states being used in place of the exact state. However, if the state constraints were handled in the same certainty-equivalence fashion, the resulting control law could drive the real state to violate the constraints frequently. Part (i) focuses on exploring the inclusion of state estimates into the constraints. It does this by applying constrained MPC to a system with stochastic disturbances. The stochastic nature of the problem requires re-posing the constraints in a probabilistic form. Using a gaussian assumption, the original problem is approximated by a standard deterministic constrained MPC problem for the conditional mean process of the state (the prediction). The state estimates' conditional covariances appear

in tightening the constraints as measuring the necessary standoff from the bound on the real state. ‘Closed-loop covariance’ is introduced to reduce the infeasibility and the conservativeness caused by using long-horizon, open-loop prediction covariances. The resulting control law is applied to a telecommunications network traffic control problem as an example. The idea of posing and transforming a probabilistic MPC problem works well, but not limited to, linear systems.

In Part (ii), we consider applying constrained MPC as a local control law in a coordinated control problem of a group of distributed autonomous systems. Interactions between the systems are captured via constraints. First, we inspect the application of constrained MPC to a completely deterministic case. Formation stability theorems are derived for the subsystems and conditions on the local constraint set are derived in order to guarantee local stability or convergence to a target state. If these conditions are met for all subsystems, then this stability is inherited by the overall system. For the case when each subsystem suffers from disturbances in the dynamics, own self-measurement noises, and quantization errors on neighbors’ information due to the finite-bit-rate channels, the constrained MPC strategy developed in Part (i) is appropriate to apply. Disturbance attenuation, or “string stability”, is studied in this framework and it is shown that inactivity of the MPC constraints implies stability. This then provides a connection between control objective, communications resource assignment and performance. A one-dimensional vehicle example is computed to crystallize ideas. The application of this part is not restricted to linear systems.

In Part (iii), we discuss the local predictor design and bandwidth assignment problem in a coordinated vehicle formation context. The MPC controller used in Part (ii) relates the formation control performance and the information quality in the way that large standoff implies conservative performance. If the communication channels used to exchange local information are noiseless, but have only finite bit-rate, the bits assigned to each variable in the information package will change the prediction error covariance, and hence the control performance, via the quantization errors which can be regarded as measurement noises. In this part, we aim at deriving the minimal communication resource and the corresponding bit-rate assignment strategy the corresponding stable state predictors that is used to formulate the MPC constraints.

We first develop an LMI (Linear Matrix Inequality) formulation for cross-estimator design in a simple two-vehicle scenario with non-standard information: one vehicle does not have access to the other's exact control value applied at each sampling time, but to its known, pre-computed, coupling linear feedback control law. Then a similar LMI problem is formulated for the bandwidth assignment problem that minimizes the total number of bits by adjusting the prediction gain matrices and the number of bits assigned to each variable. This LMI formulation takes care of the constraint on steady state prediction error covariance imposed by the formation performance requirement, the constraint on the limited total bandwidth, and the constraint on the predictors being stable. Some linear approximation is used to include the bandwidth assignment variables in the LMI formulation. The solution of the resulting LMIs guarantees the feasibility of the bandwidth assignment scheme and stable predictors, but not optimality. An example of a three-vehicle formation is also provided. The LMI formulation here is restricted to linear systems.

Chapter 1

Introduction

1.1 Main Ideas

In this dissertation, we study the constrained Model Predictive Control (MPC) formulation for systems with stochastic disturbances and its interaction with the state estimation. The main ideas can be illustrated with Figure 1.1. When constrained MPC is used to provide control for a system with disturbances, the constraints can only be posed on the state estimates/predictions. The direct use of the constraints on the real states as per the constraints on the predicted states may not be appropriate since the corresponding control law can only guarantee the state prediction satisfying the constraints while the unpredicted disturbance may drive the real state to trespass the boundary. Thus, a properly formulated bound on the state prediction needs to be formulated so that to include a necessary stand-off term representing the quality of the prediction, usually measured by the prediction covariance. When such a constrained MPC strategy is applied, the prediction covariance is tied to the control performance via the constraints. Hence, a control performance requirement should be translated into an upper bound on the prediction covariance as a guide to the predictor design. If the prediction is derived based on the data communicated via a finite-capacity channel, the bound on the prediction covariance can also be regarded as a requirement on the necessary communication resource. These are the issues explored in this thesis.

MPC is an increasingly significant and popular control approach because of its possible application to nonlinear multi-variable processes and its ability to handle

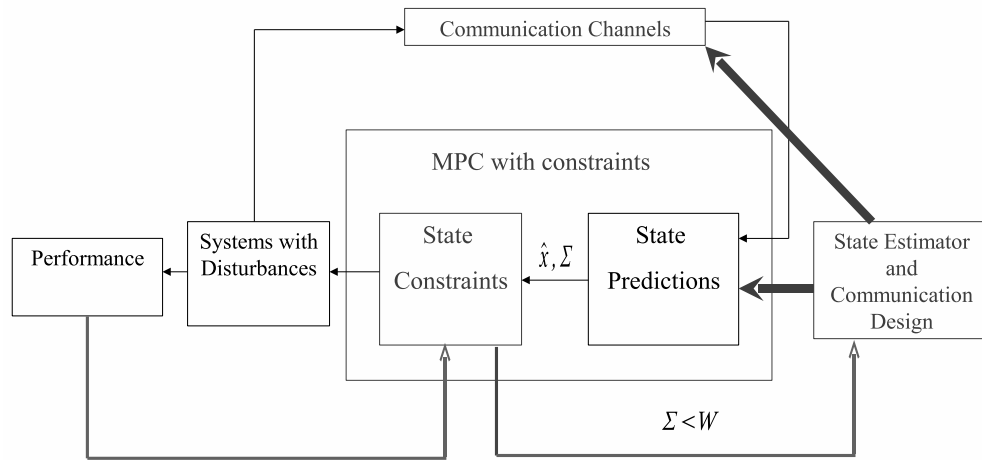


Figure 1.1: The interaction between the constrained Model Predictive Controller and the state estimation.

constraints on inputs, states and outputs. It uses open-loop constrained optimization of finite-horizon control criteria in a receding horizon approach. A model is used to predict the future behavior of the system up to the horizon, starting from its current state, and a constrained optimization based on the prediction yields an optimal open-loop control sequence over the complete horizon. Only the first element in this sequence is applied to the plant. New measurements available at the next sample time permit the calculation of an updated initial state value, and the optimization is then re-solved. The introduction of each output measurement, via the mechanism of state estimate update, results in the overall method yielding a closed-loop control.

The receding horizon approach behind MPC relies on state estimation even though most analyses of the stability, feasibility and performance of these schemes treat the controller as a full-state-feedback strategy (Mayne, Rawlings, Rao & Sokaert 2000). From MPC's early linear unconstrained variants such as Generalized Predictive Control

and its connection to LQG (Clarke, Mohtadi & Tuffs 1987, Bitmead, Gevers & Wertz 1990, Maciejowski 2002, Camacho & Bordons 2005, Goodwin, Seron & Doná 2004), the formulation of MPC has included state estimation either inherently or explicitly in the construction of the predictor using observer polynomials or via the Kalman Filter. The core issue is that, when stochastic disturbances and/or model uncertainty come into the picture, the state estimates are at variance with the actual state. In this case, the conventional predictive controller cannot necessarily guarantee the fulfillment of state or output constraints in the real system, even though they might be satisfied for the predicted system.

Therefore, the first main issue of this dissertation is dedicated to the study of posing proper constraints on the predicted states. The stochastic nature of the problem requires re-posing the constraints in a probabilistic form, meaning that the constraints on the real state can be violated but only at a very low rate. Then by the distribution of the state prediction errors, this stochastic MPC problem can be re-posed as a deterministic one on the predicted states. This transformation becomes simple when the system is linear with gaussian noises and the constraints on the states are linear. The modified constraint posed on the predicted state consists of the bound on the true state and a necessary stand-off term determined by the prediction error covariance. Furthermore, the ‘Closed-loop covariance’ will be introduced to reduce the infeasibility and the conservativeness caused by using long-horizon, open-loop prediction covariances. The resulting control law is applied to a telecommunication network traffic control problem as an example.

The second issue of this dissertation is to study the application of the constrained MPC approach to the multi-system coordination problem, which is an effective way of dealing with large-scale systems (Šiljak 1990), such as unmanned aerial vehicles (Stipanović, Inalhan, Teo & Tomlin 2004), spacecraft formation (Beard, Lawton & Hadaegh 2001), and power systems (Camponogara, Jia, Krogh & Talukdar 2002).

The main idea of this part is that coordinated control has a natural specification via imposed constraints between neighbors’ actions, typically a no-collision requirement should be imposed. As in vehicle control problems, recent work of (Gerdes & Rossetter 2001) indicates that common coordination tasks such as lane-keeping and

distance-keeping are well captured by potential functions which may be interpreted as the action of constraints on the controlled driving dynamics. An-MPC like constrained control is a natural choice for developing local feedback control laws using communicated state information from the neighboring vehicles (Camponogara, Jia, Krogh & Talukdar 2002, Dunbar & Murray 2004, Richards & How 2004a). We consider two problems.

- (i) When there are no disturbances, the global formation maintenance can be guaranteed by the local stability implied by the feasibility and optimality of the local constrained MPC ((Mayne, Rawlings, Rao & Scokaert 2000)). The current focus on the decentralized MPC application to coordinated control is to generate the local MPC problems by partitioning a centralized MPC (Camponogara, Jia, Krogh & Talukdar 2002, Keviczky, Borrelli & Balas 2004, Dunbar & Murray 2004). The coordinated vehicles considered in this dissertation have decoupled dynamics and decoupled local criteria and are coupled by the no-collision constraints only. To keep the reconfigurability of the formation, we do not consider a centralized MPC problem.
- (ii) When there are disturbances, especially those associated with the communicated neighbors' information, the constrained MPC formulation developed in the first part can be used. In particular, with the prediction error covariances of the neighbors appearing in the constraints, this formulation ties the control performance and the quality of the communication links between the vehicles. A simple 1-D vehicle coordination example is studied to illustrate this idea.

The third part of this dissertation is to study the information architecture design problem for the vehicle formations. The information architecture deals with the design of the communication links between vehicles, the bandwidth assignment of each link within available resources, and the design of local estimators. Communication has been shown to be necessary for vehicle formation coordination (Eren, Belhumeur, Anderson & Morse 2002, Eren, Belhumeur & Morse 2002, Fax & Murray 2004, Moreau 2005, Muhammad & Egerstedt 2004). In our framework, each vehicle uses a Kalman predictor for its own state prediction and needs estimators for its neighbors' state prediction based

on the communicated data to formulate the no-collision constraints. We call this state estimator for one's neighbor the *cross-estimator*.

Different from each vehicle's self Kalman filter, the cross-estimator design needs to consider how the neighbors share their control information. We consider two design problems: (i) vehicles apply pre-determined local coupled linear feedback control law, the known control laws are provided to the vehicles but not the exact control values; (ii) vehicles apply the local constrained MPC developed in part two, the current state and the control sequence with one-step delay are communicated among the vehicles to compute a proper state predictions, starting from the current time up to the control horizon. For linear systems, both problems can be solved via Linear Matrix Inequality (LMI) tools.

In the second design problem, the finite capacity of each channel has the effect of adding quantization errors to the transmitted data, which is also the measurement for the cross-estimators. Hence, the bit-rate assignment of each channel's finite capacity to the variables will directly change the cross-estimator error covariances that appear in the MPC constraints, which ties the covariances and the control performance. Therefore, the cross-estimator design and the bandwidth assignment are synthesized into one LMI formulation that minimizes the total communication resources subject to an over-bounding condition on the covariance posed by the formation control requirement, in addition to the stable cross-estimator requirement.

1.2 Relevant History and Literature Survey

This section provides a review of the relevant literature on constrained MPC, multi-vehicle coordination and information structure design.

1.2.1 Constrained MPC

Model Predictive Control was first developed in practical use in process engineering back to 1970's. Various algorithms have been developed using the transfer function model, such as Model Predictive Heuristic Control, Dynamic Matrix Control etc, and have successful applications in the process control area (Qin & Badgwell 2003,

Maciejowski 2002, Camacho & Bordons 2005). Later development includes Generalized Predictive Control (Clarke, Mohtadi & Tuffs 1987, Bitmead, Gevers & Wertz 1990) and the use of state space models (Maciejowski 2002, Goodwin, Seron & Doná 2004, Camacho & Bordons 2005).

MPC applies a receding horizon strategy that at each sampling time solves an open-loop optimization problem from the current state measurement and derives a control sequence. Only the first element of the control sequence is applied to the plant. At the next sampling time, a new state measurement is available and the open-loop optimization is then re-posed and re-solved making MPC an overall closed-loop approach to reject the uncertainties of the system. It works for both linear and nonlinear systems (Allgöwer & Zheng 2000) with the possible inclusion of constraints. Though this approach often requires on-line optimization, the development of new computational hardware and fast algorithms (Bemporad, Morari, Dua & Pistikopoulos 2002) makes this less difficult for application in systems with fast dynamics.

The advantages of MPC lie in the on-line reconfigurability and stability guaranteed by the feasibility and optimality of the open-loop optimization problem via posing the terminal constraint (Mayne, Rawlings, Rao & Sckaert 2000). We are interested in applying this stability argument to the multi-system coordination problem in a way that the stability of the distributed MPC implies global convergence.

Also, when applying constrained MPC to systems with stochastic disturbances, we start with formulating a probabilistic state constraint which, under certain assumption of the disturbance distribution, can be easily transformed into deterministic constraints on the state predictions. Probabilistically constrained MPC has been studied. For instance, (Schwarm & Nickolaou 1998), (Li, Wendt & Wozny 2000) and (Li, Wendt & Wozny 2002) consider the input-output model of a linear system with normally distributed disturbances. An MPC strategy with probabilistic constraints on the output is posed and then is transformed into a deterministic nonlinear programming problem. Only open-loop prediction is used and the computational efficiency is also considered. In (Schwarm & Nickolaou 1998), the authors appealed to the constraint softening approaches (Zafirou & Chiou 1993, Maciejowski 2002) to resolve the infeasibility problem.

1.2.2 Multi-vehicle Coordination and Information Structure Design

Multi-subsystem coordination is an alternative approach to the globalized control of large-scale, complex systems (Šiljak 1990). As an important branch, multi-vehicle coordination has drawn plenty of attention from different areas of engineering, such as robotics (Balch & Arkin 1998, Desai, Ostrowski & Kumar 2001), unmanned aerial vehicles (Stipanović, Inalhan, Teo & Tomlin 2004) or spacecraft (Beard, Lawton & Hadaegh 2001), and automated highway systems (Varaiya 1993, Swaroop & Hedrick 1999), just to mention a few.

The main theme is the derivation of decentralized control laws based on decentralized information. This leads to two design problems: the local feedback control law and the information architecture.

The local control design of interest here is the decentralized MPC scheme. One may expect the local MPCs to retain the desirable properties of a global MPC problem, such as feasibility and stability of the entire system. Many existing works follow this idea (Camponogara, Jia, Krogh & Talukdar 2002, Jia & Krogh 2001, Dunbar & Murray 2004, Keviczky, Borrelli & Balas 2004). In (Camponogara, Jia, Krogh & Talukdar 2002), the authors provided a set of conditions under which the local MPC solutions together converge to the global one and hence the global optimality can be achieved. These conditions ensure the convergence of the locally shared variables and require multi-step communication and control calculation at each sampling time before making the final control decision. In practical cases, the local communication and control calculation must be done in a short period of time and usually are limited to once per sample time. Then the locally shared variables may be different at different subsystems. In (Jia & Krogh 2001) and (Dunbar & Murray 2004), this difference is limited via certain constraints in the local MPC formulation and the stability of the global system can be deduced, though the global optimality cannot be guaranteed.

Our interest mainly lies in applying constrained MPC to coordinated systems with stochastic disturbances and the no-collision constraint is modified to accommodate the prediction errors. There are existing works considering robust (min-max) decentralized MPC with bounded disturbances (Richards & How 2004a, Richards & How 2004b, Jia & Krogh 2002, Jia & Krogh 2005). In (Richards & How 2004a), the

dependence on the centralized MPC is only through an initial feasible solution. The local no-collision constraints are modified according to the bounded disturbances and the feasibility is guaranteed by the initial feasible solution. This idea has been applied to cooperating UAVs (Richards & How 2004*b*). Discussion of an optimal partitioning strategy of the MPC problem has been studied in (Motee & Sayyar-Rodsari 2003).

The information architecture deals with the design of the communication links between vehicles, the bandwidth assignment of each link within available resources, and the design of local estimators. In decentralized MPC strategies, neighbors always share their future state information to form the local MPCs. The usual way is to use the state prediction from the last optimization and then special MPC constraints are used to guarantee the overall stability or feasibility, (Camponogara, Jia, Krogh & Talukdar 2002, Dunbar & Murray 2004, Richards & How 2004*a*). In (Fax & Murray 2004) and (Moreau 2005), the formation problems are defined via the relative distances between the vehicles and graph theory tools are used to establish the communication links so that the stability of the entire formation is achieved. In these results, the communication links are assumed to provide perfect data transmission which needs an infinite communication resource.

In this dissertation, we consider coordinated vehicles with process noises and capacity-limited communication channels. In this scenario, the latter two parts of the information architecture design become essential, as the state prediction of one's neighbors must be provided by cross-estimators and the local no-collision constraint works as a bridge linking the control performance and the prediction quality, which can be managed via adjusting the bit-rate assigned to each variable.

The cross-estimator design needs to share control information among the neighbors, such as known control laws or shared control values with errors. This type of design is also known as estimator design with unknown inputs which has been considered occasionally, (Meditch & Hostetter 1971, Johnson 1975, Akhenak, Chadli, Maquin & Ragot 2004). Typical solution starts with assuming the estimator structure and a model for the unknown inputs. The estimation problem for linear systems with measurement noise on both state and control variables is also known as errors-in-variables. As in (Guidorzi, Diversi & Soverini 2002), the authors discussed the optimal filtering problem

in this framework.

The effect of finite communication resource over control systems has been considered, the existing result focuses on minimum permissible communication resource assignment problem for a single system to achieve guaranteed performance, such as stability or stabilizability, (Wong & Brockett 1997, Nair & Evans 2003, Tatikonda & Mitter 2004).

1.3 Outline

Chapter 2 discusses the constrained MPC formulation for a system with process disturbances and measurement noises. A thorough analysis of incorporating state estimate into MPC constraints is provided. We first propose a general stochastic MPC problem with state constraints in a probabilistic form. For linear systems with gaussian noises, the distribution of the future state prediction error is completely determined by the covariances which are independent of the current control. Hence the probabilistically constrained MPC problem can be easily transformed into a deterministic one on the state prediction with stand-off terms defined by the covariances. However, the prediction error covariance is typically increasing along the horizon and this leads to overly tightened constraints that result in very conservative control law or even infeasible problems. We propose to replace large covariance terms by a smaller ‘closed-loop’ covariances along the horizon. The use of the closed-loop covariance is in a sense dual to the receding-horizon control idea and leads to the improved treatment of feasibility. This control strategy is then applied to an ATM (Asynchronous Transfer Mode) telecommunication network congestion control problem. The idea of setting stand-off terms in the constraint is not limited to linear systems.

Chapter 3 provides a formulation of coordinated control of autonomous systems via MPC with constraints. First, we examine the applicability of constrained MPC as the distributed control law. Coordinated vehicle formation is our example where the vehicles have decoupled dynamics, decoupled trajectory following tasks, and are coupled via the no-collision requirement. In the deterministic case, we briefly derive conditions on the local MPC problems so that each vehicle can converge to their desired positions

without any collision. We are more interested in the case with disturbances, such as process noises and measurement noises. The MPC controller proposed in Chapter 2 is a natural choice for this scenario and the state prediction quality is tied to the formation control performance via the MPC constraints. String instability, as a performance measure, may occur since active constraints in one vehicle's MPC problem involving others' state predictions will introduce additional noise into the vehicle closed-loop dynamics. We study a 1-D problem with disturbances to show the relation between the control performance and the communication between the vehicles.

Chapter 4 deals with the cross-estimator design in vehicle formations. The design problem is set in a two-vehicle scenario with coupling linear feedback control law, i.e. the feedback control law involves both vehicles' states. The coupling control law makes this a non-standard estimator design problem, since the standard optimal estimator design requires perfect knowledge of the control value. We assume that the cross-estimator takes the same structure as the Kalman filter. An LMI formulation in terms of the prediction error covariance and the cross-estimator gain is derived so that stable cross-estimators are guaranteed, i.e. the error covariance is bounded. Approximation is introduced to capture the structural constraints on the estimator gain matrix and the solution provides a feasible estimator but not necessarily the optimal one.

Chapter 5 deals with the cross-estimator design and the bandwidth assignment problem for a vehicle formation applying the constrained MPC control law proposed in Chapter 3. A similar LMI problem for minimizing the total communication resource by choosing the cross-estimator gain matrices and the bandwidth assignment strategy can be formulated. The vehicles' MPC controllers need proper prediction of their neighbors' states. Due to the requirement of synchronization, the communicated data package sent by each vehicle consists of its current state and the control sequence computed at the previous sampling time. A model is used to update the control sequence to compute the predicted state with additional white noises modelling the difference. The only error in the transmitted data is the quantization error due to the finite bit-rate associated with each variable. Therefore, by treating the transmitted data as measurements, the measurement noise becomes the quantization error and its covariance is solely determined by the available bit-rate assigned to the variable. Hence, by making the measurement

noise covariance a variable in the LMI formulation, we are able to solve the bandwidth assignment problem. The LMI bears the following constraints: (1) the prediction error covariance should be finite; (2) the total sum of the bit-rates is bounded and to be minimized; (3) formation performance poses a bound on the prediction error covariance matrix. Due the requirement of linearity, the inverse of the measurement noise covariance is replaced by a matrix linear in the bit-rates. The resulting LMI provides a feasible solution of the cross-estimator gain and the bandwidth assignment strategy rather than the optimal solution.

1.4 Contributions

The contributions of this dissertation can be summarized as the follows:

1. Incorporating state estimation into model predictive control via constraints (Chapter 2).
 - Formulation of the MPC constraints on the predicted states with stand-off terms determined by the prediction error covariance and probability of violation.
 - Development of the use of the closed-loop covariance in the constraints to improve feasibility and performance.
2. Applying the constrained MPC strategy to multi-system coordination (Chapter 3).
 - Conditions on the local MPC problem that guarantee the local and the global convergence are derived.
 - String instability in the coordinated systems with persistent disturbances and a sufficient condition to avoid it is identified.
 - The connection between the control performance and the communication requirement via the constraints is identified.
3. Local estimator design and communication resource assignment (Chapter 4, 5).
 - Designs and developed via LMI for local state estimators for systems with known, coupling linear feedback control law via LMI.

- Designs via LMI the feasible local state predictor for the local MPC controller and the corresponding bandwidth assignment strategy that satisfy the formation performance requirement.

Chapter 2

Incorporating State Estimation into Model Predictive Control and its Application to Network Traffic Control

2.1 Introduction

Model Predictive Control (MPC) is an increasingly significant and popular control approach because of its use of a possibly nonlinear multivariable process model and its ability to handle constraints on inputs, states and outputs. It uses open-loop constrained optimization of finite-horizon control criteria in a receding horizon approach. A model is used to predict the future behavior of the system up to the horizon, starting from its current state, and a constrained optimization based on the prediction yields an optimal open-loop control sequence over the complete horizon. Only the first element in this sequence is applied to the plant. New measurements available at the next sample time permit the calculation of an updated initial state value, and the optimization is then re-solved. The introduction of each output measurement, via the mechanism of state update, results in the overall method yielding a closed-loop control.

The receding horizon approach behind MPC relies on state estimation even

though most analyses of the stability, feasibility and performance of these schemes treat the controller as a full-state-feedback strategy (Mayne, Rawlings, Rao & Scokaert 2000). From MPC's early linear unconstrained variants such as Generalized Predictive Control and its connection to LQG (Clarke, Mohtadi & Tuffs 1987, Bitmead, Gevers & Wertz 1990), the formulation of MPC has included state estimation either inherently or explicitly in the construction of the predictor using observer polynomials or via the Kalman Filter. However, to our knowledge, there has been no satisfactory treatment of the inclusion of state estimation into the formulation of the full constrained problem other than our own (Yan & Bitmead 2002) and some very interesting recent work of van Hessen and Bosgra (van Hessen & Bosgra 2002), which we build on here. This is in spite of the industrial impetus for MPC being its application in process control, where the rejection of stochastic disturbances is the central control objective. The formulations cited above are certainty equivalence controllers, in which the optimal state estimate is used as a replacement for the exact state in the control law. The natural sequencing of the receding horizon control calculation would see the one-step-ahead state prediction used as the initial state in the computation of the MPC control sequence.

The core issue is that, when stochastic disturbances and/or model uncertainty come into the picture, the state estimates are at variance with the actual state. In this case, the conventional predictive controller cannot necessarily guarantee the fulfillment of state or output constraints in the real system, even though they might be satisfied for the predicted system. There are two sources of error involved: the error between the state estimate used and the actual state at the initial time of the optimization interval, and the future state errors introduced with the evolution of the system along its prediction horizon.

Our approach to accommodating the state estimate error into constrained MPC will involve three steps.

Step i: Replace the deterministic state and output constraints by probabilistic constraints.

Step ii: Use an approximate distribution for the state estimate error to convert these probabilistic constraints into deterministic constraints on the conditional mean of

the state.

Step iii: Solve this standard deterministic constrained MPC problem for the future control sequence.

The first step weakens the hard constraints to a probabilistic form, at least for unbounded distributions. But this is necessary if indeed the disturbance process admits such signals. This modification has also been proposed by (Schwarm & Nickolaou 1998, Li, Wendt & Wozny 2000, Li, Wendt & Wozny 2002) and potentially leads to a rather intractable stochastic programming problem. The second step is the crux of our approach, where the state estimates and their covariances are introduced together to yield a non-certainty-equivalence MPC controller. The third step is computationally standard in MPC.

The novelties of our approach to incorporating state estimates into MPC are:

- introduction of the probabilistic constraints, building on some ideas of (Yan & Bitmead 2002, van Hessen & Bosgra 2002),
- inclusion of the state estimate’s conditional mean and conditional covariance into the deterministic MPC formulation,
- development of the concepts of *closed-loop covariances*, in which the estimate covariances are moderated along the prediction horizon to reflect only minimal-delay values.

The net outcome of the approach is that the MPC with state estimates remains an MPC problem using the prediction of the state in place of the actual state. The *quality* of the state estimate, typically as measured by its covariance, is used to tighten the constraints in this modified deterministic MPC problem. This is a naturally appealing idea of compromising or de-tuning performance in order to augment robustness to stochastic or other disturbances. The concept of closed-loop covariances is an intriguing dual to the receding horizon idea of MPC, and mitigates against infeasibility being caused by covariance growth with long horizons, as will be discussed.

An alternative natural approach to the inclusion of a state estimate into MPC would be to take the worst case strategy with the assumption of the disturbances taking values in a compact set. Such a design presents numerical issues as well as an inherent

conservativeness because of the use of open-loop control in the optimization. However, this is necessary if one demands guarantee of respect of constraints. This paper focuses on applying a naturally posed probabilistically constrained MPC scheme to a system with potentially but not necessarily unbounded stochastic disturbances.

Stochastic optimal control problems have been considered for some time (Kushner 1971, Åström 1970, Fleming & Rishel 1975) and generically involve solution via dynamic programming in which the entire conditional distribution function evolves with time. Attempts to pose and solve probabilistically constrained optimal control problems have been made under the banner of MPC. The main tack is to find a more simply soluble approximation of the original problem. For instance, (Li, Wendt & Wozny 2000) and (Li, Wendt & Wozny 2002) consider the input-output model of a linear system with normally distributed disturbances. An MPC strategy with probabilistic constraints on the output is posed and then is transformed into a deterministic nonlinear programming problem. Only open-loop prediction is used and the computational efficiency is also considered. In (Schwarm & Nickolaou 1998), the authors appealed to the constraint softening approaches (Zafirou & Chiou 1993) to resolve the infeasibility problem.

In (Kouvaritakis, Rossiter & Schuurmans 2000), the authors used a special form of control composed of a nominal value and a linear feedback, in the probabilistically constrained MPC problem. The probability of a set of linear constraints on the state is replaced by an ellipsoidal approximation, yielding a conic optimization problem. A number of other recent articles, see e.g. (Bemporad 1998, van Hessem & Bosgra 2002, Fukushima & Bitmead 2003, Fukushima & Bitmead 2005), also treat MPC as a perturbation of a nominal stabilizing feedback and accommodate disturbances through tightening of the constraints. This is also an outcome of our approach, although here the focus will also be on maintaining an accessible MPC reformulation. Further, in Section 2.5, an example from telecommunications network traffic control will be developed, which illustrates the implementation and properties of the approach.

As will be shown in Section 2.3.1, under a certain gaussian assumption or approximation, the probabilistic constraints provide a convenient and natural point of entry for the estimate covariance. This is done by imposing stricter constraints reflecting the estimate quality. By including this extra information, the stochastic optimal control

problem is converted to a soluble deterministic optimization problem in standard MPC form, in which the initial state is replaced by its one-step-ahead conditional-mean estimate and the constraints are modified to accommodate the state estimate's conditional covariance. This is equivalent to the original stochastic problem and is a non-certainty-equivalence controller, because of its dependence on the covariance.

The state prediction covariance normally is an increasing function of prediction horizon and so one might expect that constraint modification reflecting such a covariance would lead to a conservative control law or even infeasibility as the constraints tighten along the horizon, because the closed-loop property of the system is not taken into account. However, we know that future measurements will become available and, accordingly, in Section 2.4.2 we use the one-step-ahead covariance throughout the horizon to modify the constraints. We denominate this the *closed-loop covariance*. The resultant approximation problem can be easily solved by quadratic programming (QP) routines while the probabilistic feasibility is ensured.

2.2 Model Predictive Control with Probabilistic Constraints

The archetypal deterministic MPC problem is as follows.

$$\begin{aligned} \min_u J(N, x_n, u_n^{n+N-1}) &= x_{n+N}^T P_N x_{n+N} + \sum_{i=0}^{N-1} (x_{n+i}^T Q_i x_{n+i} + u_{n+i}^T R_i u_{n+i}), \\ \text{subject to : } x_{n+i+1} &= f(x_{n+i}, u_{n+i}) \\ x_{n+i} &\in \mathbb{X}_i, \quad (i = 1 \dots N), \\ u_{n+i} &\in \mathbb{U}_i, \quad (i = 0, \dots, N-1). \end{aligned}$$

The minimization commences at time n from initial state value x_n and yields the N -step solution sequence for time n ,

$$u_n^{n+N-1} = \{u_n, u_{n+1}, \dots, u_{n+N-1}\}.$$

Clearly output equations and constraints can be accommodated in this formulation, since the output depends explicitly on the state and input.

Now consider the following discrete-time stochastic system.

$$x_{n+1} = f(x_n, u_n, \omega_n),$$

$$y_n = g(x_n, u_n, \nu_n),$$

where the first equation is the recursion of state variable x_n with process noise term ω_n and the second equation is that of the measurement y_n perturbed by noise ν_n . Denote by \mathcal{Y}_n the set of the collected input and output measurements $\{(u_i, y_i) : i = 1, \dots, n\}$ and by $E_{\mathcal{Y}_n}(\cdot)$ and $P_{\mathcal{Y}_n}(\cdot)$ the conditional expectation and conditional distribution with respect to it.

The probabilistically constrained MPC problem we consider is:

MPC1:

$$\min_u J(N, x_n, u_n^{n+N-1}) = E_{\mathcal{Y}_n} \left[x_{n+N}^T P_N x_{n+N} + \sum_{i=0}^{N-1} (x_{n+i}^T Q_i x_{n+i} + u_{n+i}^T R_i u_{n+i}) \right],$$

subject to :

$$x_{n+i+1} = f(x_{n+i}, u_{n+i}, \omega_{n+i}),$$

$$y_{n+i} = g(x_{n+i}, u_{n+i}, \nu_{n+i}),$$

$$P_{\mathcal{Y}_n}(x_{n+i} \in \mathbb{X}_i) \geq p_i, (i = 1 \dots N),$$

$$u_{n+i} \in \mathbb{U}_i, i = (0 \dots N - 1).$$

Remarks:

1. The criterion is the conditional expectation of a quadratic cost function of x and u conditioned on \mathcal{Y}_n . By minimizing this as part of a receding horizon strategy, we obtain a feedback control law.
2. The constraints on state variables are posed in terms of probabilities, i.e. the constraints may be violated but only at a specific (presumably very low, $0 \leq (1 - p_i) \ll 1$) rates. This is because of the inclusion of stochastic noises in our model. Especially when the noises are unbounded, such as gaussian distributed random noises, the chance of violating the constraints always exists.
3. If the disturbances belong to a compact set, then min-max or worst case methods might be applied. Two types of min-max MPC have been proposed: open-loop

and closed-loop. The open-loop strategy, as shown in (Lee & Yu 1997), inevitably leads to an overly-conservative control scheme. The closed-loop strategies (Lee & Yu 1997, Scokaert & Mayne 1998, Bemporad 1998) can reduce the conservativeness but are computationally demanding.

4. The constraints on the controls u_n^{n+N-1} in *MPC1* are deterministic. Since they are free variables in the optimization, there is no need to put probabilistic constraints on them.
5. This is a difficult constrained stochastic optimization problem (Fleming & Rishel 1975). The solution would nominally involve the determination of the entire conditional distribution of the state and not just its first few moments. This, in turn, would require evolution of this distribution from its initial value and the distributions of the noise processes.
6. Our approach will be to take an alternative system based on a gaussian assumption or approximation of the state estimate error distribution. This will then permit a simple recasting of the probabilistic MPC problem. We shall require some fairly strong (but often acceptable) assumptions about the underlying system, notably that the future state covariance is independent of the current control. This is a property trivially satisfied for linear systems.

2.3 Incorporating State Estimates into MPC

The task of this section is to seek a tractable deterministic approximation of the problem *MPC1* of Section 2.2. For simplicity, we commence with a constrained linear system to develop the approach and later highlight the role played by linearity in our technique.

Consider a k -dimensional linear system with mutually independent gaussian initial condition $x_0 \sim N(\hat{x}_0, \Sigma_0)$, gaussian process noise $\omega_n \sim N(0, \Gamma_n)$ and gaussian measurement noise $\nu_n \sim N(0, \Lambda_n)$. Take the constraints in *MPC1* as linear, giving

MPC2:

$$\min_u J(N, x_n, u_n^{n+N-1}) = E_{\mathcal{Y}_n} \left[x_{n+N}^T P_N x_{n+N} + \sum_{i=0}^{N-1} (x_{n+i}^T Q_i x_{n+i} + u_{n+i}^T R_i u_{n+i}) \right], \quad (2.1)$$

$$\text{subject to : } x_{n+i+1} = Ax_{n+i} + Bu_{n+i} + G\omega_{n+i}, \quad (2.2)$$

$$y_{n+i} = Cx_{n+i} + D\nu_{n+i}, \quad (2.3)$$

$$P_{\mathcal{Y}_n}(x_{n+i} \leq \beta_i) \geq p_i, \quad (i = 1 \dots N), \quad (2.4)$$

$$u_{n+i} \leq \mu_i, \quad (i = 0, \dots, N-1). \quad (2.5)$$

2.3.1 Replacing probabilistic constraints by deterministic constraints

The problem at hand is that for the system (2.2) at time n we want to solve the constrained stochastic optimal control problem *MPC2*. Ignoring for the moment the constraints, we have a linear system with gaussian disturbances ω_n and ν_n . If the initial state distribution were gaussian and control were linear, then the state would be also gaussian, as would its conditional distribution function conditioned on \mathcal{Y}_n . We could then propagate the conditional distribution $P_{\mathcal{Y}_n}(x_{n+i})$ using the finite-dimensional Kalman filter.

$$\hat{x}_{n|n} = \hat{x}_{n|n-1} + \Sigma_{n|n-1} C^T (C \Sigma_{n|n-1} C^T + \Lambda_n)^{-1} (y_n - C \hat{x}_{n|n-1}), \quad (2.6)$$

$$\hat{x}_{n+1|n} = A \hat{x}_{n|n} + B u_n, \quad (2.7)$$

$$\Sigma_{n|n} = \Sigma_{n|n-1} - \Sigma_{n|n-1} C^T (C \Sigma_{n|n-1} C^T + \Lambda_n)^{-1} C \Sigma_{n|n-1}, \quad (2.8)$$

$$\Sigma_{n+1|n} = A \Sigma_{n|n} A^T + G \Gamma_n G^T. \quad (2.9)$$

The Kalman filter and its predictor variants (see (Anderson & Moore 1979)) compute the conditional means, $\hat{x}_{n+i|n}$, and conditional variances $\Sigma_{n+i|n}$, which parametrize completely the gaussian conditional distribution functions.

We remark here, that since all the past control inputs $\{u_{n-i} : i = 0, 1, \dots, n\}$ are known, assuming zero mean gaussian initial state estimate error, $\tilde{x}_{0|0} = x_0 - \hat{x}_{0|0}$, it follows that, by linearity, the filtered and predicted state estimate's errors, $\tilde{x}_{n+i|n} =$

$x_{n+i} - \hat{x}_{n+i|n}$, are gaussian with means zero and covariances $\Sigma_{n+i|n}$, even though the feedback control might be nonlinear due to the constraints. [This is discernible from the Kalman filter error equations. We also note that the distribution of the state and of its estimate need not be gaussian for this to hold.] Further, the conditional state estimate error covariance is independent from the control. Thus,

$$\tilde{x}_{n+i|n} \sim N(0, \Sigma_{n+i|n}).$$

These conditional predicted means, prediction errors and covariances are computed from initial conditions $\hat{x}_{n|n}$, $\tilde{x}_{n|n}$ and $\Sigma_{n|n}$ using the open-loop predictor:

$$\hat{x}_{n+i|n} = A\hat{x}_{n+i-1|n} + Bu_{n+i-1}, \quad (2.10)$$

$$\tilde{x}_{n+i|n} = A\tilde{x}_{n+i-1|n} + G\omega_{n+i-1}, \quad (2.11)$$

$$\Sigma_{n+i|n} = A\Sigma_{n+i-1|n}A^T + G\Gamma_nG^T, \quad (i = 1, \dots, N). \quad (2.12)$$

We use this to replace the probabilistic constraints of *MPC2* by deterministic constraints on the conditional means. We draw attention to the additional property that these conditional means are optimized in the solution of the MPC control problem.

We first consider a scalar state system in modifying *MPC2* into a deterministic form. This is as derived in (Yan & Bitmead 2002). The multivariable case is a geometric extension using inscribed conic sets, see (van Hessem & Bosgra 2002) and some later comments.

The gaussian distribution of the prediction errors makes it possible to modify the constraints (2.4), $P_{y_n}(x_{n+i} \leq \beta_i) \geq p_i$, into a deterministic form. Since $\tilde{x}_{n+i|n} \sim N(0, \Sigma_{n+i|n})$, the derived variable

$$\xi_{n+i} = \Sigma_{n+i|n}^{-\frac{1}{2}} \tilde{x}_{n+i|n}$$

has standard normal distribution $N(0, 1)$. Then (2.4),

$$P_{y_n}(x_{n+i} \leq \beta_i) \geq p_i$$

is equivalent to

$$P_{y_n}(\xi_{n+i} \leq \check{\beta}_i) \geq p_i, \quad \text{where, } \check{\beta}_i = \Sigma_{n+i|n}^{-\frac{1}{2}}(\beta_i - \hat{x}_{n+i|n}).$$

Denote by β_i^* the solution of $\Phi(\beta_i^*) = p_i$, where $\Phi(\cdot)$ is the standard normal distribution function. Then (2.4) can be recast as:

$$P_{\mathcal{Y}_n}(x_{n+i} \leq \beta_i) \geq p_i \Leftrightarrow \hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+i|n}^{\frac{1}{2}} \beta_i^*. \quad (2.13)$$

The probabilistic constraints may thus be replaced by deterministic ones on the conditional mean process of the state.

The derivation has assumed that the state process is scalar, in order that the linear inequality constraint on x_{n+i} translates into a linear constraint on the $N(0,1)$ variable ξ_{n+i} . For the vector state case, it is necessary to convert the set of linear constraints into an inscribed ellipsoidal constraint on the standard normal variables. This process is developed in more detail and in a similar context in (van Hessen & Bosgra 2002). The end result is the conversion of the linear probabilistic constraints on x_{n+i} into tighter deterministic linear constraints on $\hat{x}_{n+i|n}$.

2.3.2 New deterministic MPC problem

Since the criterion function (2.1) is a conditional expectation, we are able to change it into a deterministic function of $\hat{x}_{n+i|n}$, u_{n+i} and $\Sigma_{n+i|n}$:

$$\begin{aligned} J &= E_{\mathcal{Y}_n} [tr(x_{n+N}^T P_N x_{n+N})] + \sum_{i=0}^{N-1} \left\{ E_{\mathcal{Y}_n} [tr(x_{n+i}^T Q_i x_{n+i})] + u_{n+i}^T R_i u_{n+i} \right\} \\ &= tr [E_{\mathcal{Y}_n}(x_{n+N} x_{n+N}^T) P_N] + \sum_{i=0}^{N-1} \left\{ tr [E_{\mathcal{Y}_n}(x_{n+i} x_{n+i}^T) Q_i] + u_{n+i}^T R_i u_{n+i} \right\} \\ &= tr \left[(\Sigma_{n+N|n} + \hat{x}_{n+N|n} \hat{x}_{n+N|n}^T) P_N \right] + \sum_{i=0}^{N-1} \left\{ tr [(\Sigma_{n+i|n} \right. \\ &\quad \left. + \hat{x}_{n+i|n} \hat{x}_{n+i|n}^T) Q_i] + u_{n+i}^T R_i u_{n+i} \right\} \\ &= tr(\Sigma_{n+N|n} P_N) + \sum_{i=0}^{N-1} tr(\Sigma_{n+i|n} Q_i) + \hat{x}_{n+N|n}^T P_N \hat{x}_{n+N|n} + \\ &\quad \sum_{i=0}^{N-1} \left(\hat{x}_{n+i|n}^T Q_i \hat{x}_{n+i|n} + u_{n+i}^T R_i u_{n+i} \right). \end{aligned} \quad (2.14)$$

From (2.12) the state prediction covariance $\Sigma_{n+i|n}$ will not be changed by the controls. Therefore, minimizing (2.14) is equivalent to minimizing

$$J'(N, \hat{x}_{n|n}, u_n^{n+N-1}) = \hat{x}_{n+N|n}^T P_N \hat{x}_{n+N|n} + \sum_{i=0}^{N-1} \left(\hat{x}_{n+i|n}^T Q_i \hat{x}_{n+i|n} + u_{n+i}^T R_i u_{n+i} \right).$$

Hence we have a new receding horizon control problem to be solved at each time instant.

MPC3:

$$\min_u J'(N, \hat{x}_{n|n}, u_n^{n+N-1}) = \hat{x}_{n+N|n}^T P_N \hat{x}_{n+N|n} + \sum_{i=0}^{N-1} \left(\hat{x}_{n+i|n}^T Q_i \hat{x}_{n+i|n} + u_{n+i}^T R_i u_{n+i} \right)$$

subject to: $\hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+i|n}^{\frac{1}{2}} \beta_i^*$, ($i = 1, \dots, N$), (2.15)

$u_{n+i} \leq \mu_i$, ($i = 0, \dots, N - 1$).

The overall control strategy is the receding horizon implementation of this. As the state estimate covariance is used to modify the constraints, the control u is not certainty equivalence. Figure 2.1 depicts two conditional probability density functions

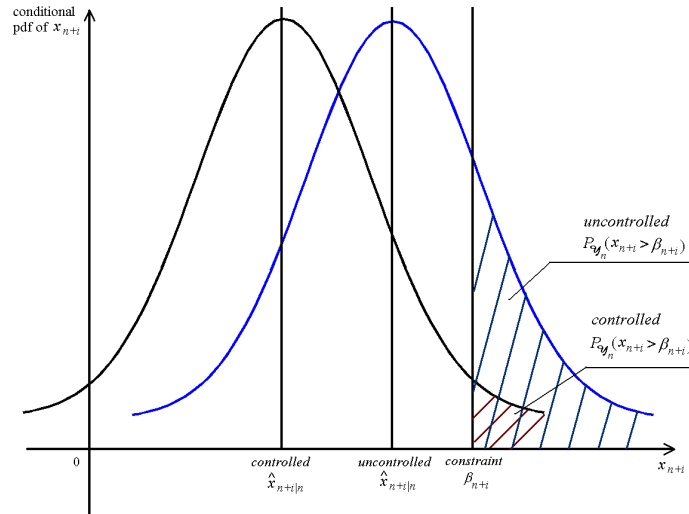


Figure 2.1: Controlling the conditional mean process to achieve the probabilistic constraint.

of the scalar state x_{n+i} . Their means are two differing values of $\hat{x}_{n+i|n}$ and they share a common variance. The probabilistic constraint represents the requirement that the shaded area under the conditional density curve to the right of β_{n+i} in Figure 2.1 should be small enough. The new MPC task is to control the conditional mean values of state x

to a proper level so that the probabilistic constraint can be achieved. The transformations and calculations earlier are designed to achieve this via (2.13).

Despite the depiction in Figure 2.1, it is not true that x is gaussian. Because of the presence of the constraints, the control u is a nonlinear function of x or \hat{x} , as shown in (Maciejowski 2002). Hence, the closed-loop system has nonlinear dynamics of state propagation and the gaussian property cannot be preserved in such a nonlinear process. However, the prediction error remains gaussian, which is the underpinning of our method.

As mentioned earlier, *MPC3* is derived for a scalar system. In this case, we may state the following theorem,

Theorem 1. *For a linear scalar system in the form of (2.2) with measurement (2.3), if the initial estimation error and the random disturbances are independent zero-mean gaussian variables, then the minimizing solution of MPC3 solves MPC2.*

This is evident from the derivation of *MPC3*, since each step in the transform is exactly equivalent. Hence, the solution of the new deterministic problem *MPC3* is the optimal solution of the original stochastic optimization problem *MPC2*.

The transformation from *MPC2* to *MPC3* could be explained as that we tighten the constraints posed on the model according to the estimate quality (measured by the covariance of the estimation error) to ensure the satisfaction by that on the plant. The covariance of the open-loop prediction that evolves as in (2.12) has the effect of modifying the constraints. Typically, this covariance is increasing with i and in (2.13) this leads to successively more stringent restrictions being applied on the control. For large horizon N , feasibility could be lost. Even if there exists a feasible solution for the problem, that solution might be very conservative. The next section will be devoted to proposing a remedy for this difficulty using closed-loop covariances instead of the open-loop ones.

If the system is multi-dimensional, then (2.4) represents a joint probability keeping the state in a polyhedron in state space. To evaluate the probability of vector constraints requires the calculation of multiple integrals of multi-dimensional normal density function; this is computationally demanding. It is convenient to blend the estimate covariance into the constraints. We propose to do the same ellipsoidal approximation as derived in (van Hessen & Bosgra 2002). Replacing the probability of the polyhedron

on the left hand side of (2.4) by that of the maximal inscribed ellipsoid, provides an overbound of (2.4) which is suitable to apply with the gaussian assumption. However, the solution turns out to be conservative. Another option is to pose the constraints on the states in ellipsoidal form, if that makes sense, in (2.4) from the beginning.

To derive our deterministic MPC problem, we have chosen a linear system framework. Since MPC is attractive because of its ability to handle nonlinearities, it behoves us to comment on the dependence on this property. Its importance lies in establishing the quality of the gaussian approximation to the state, which permits the conversion of a stochastic problem to a deterministic one. Clearly, if an overbounding gaussian variable can be found for the state then this might be admissible in constraint management. The linearity also has been used to develop adequate descriptions of the distributions, via the Kalman filter or other nonlinear estimators. This suggests that it might be possible to consider nonlinear systems operating with the Extended Kalman Filter. For our method, the central property of linear systems is that the state estimate covariance does not depend on the control signal chosen - this helps separate the constraint specification from the control solution. In a nonlinear context this would require other tricks to bound.

2.4 Closed-loop Covariances

2.4.1 Covariance properties along the horizon

Simulation results of *MPC3* in Section 2.5 show that it can be either conservative or even infeasible. The reason is that the modified constraints of (2.15) in *MPC3* are increasingly stringent. The number β_i^* is positive (since p_i should represent a large probability) and fixed by the original constraint. Computation of $\Sigma_{n+i|n}$ proceeds according to the time-update portion of the Kalman filter (2.9). For the linear, stationary system earlier, this is given by

$$\Sigma_{n+i+1|n} = A\Sigma_{n+i|n}A^T + G\Gamma_{n+i}G^T,$$

and is initialized by $\Sigma_{n+1|n}$, the one-step-ahead prediction covariance. In the asymptotically stationary case, we have

$$\Sigma_{n+i|n} \geq \Sigma_{n+i-1|n} \geq \cdots \geq \Sigma_{n+1|n}. \quad (2.16)$$

The net result of inequalities such as (2.16) is that incorporation of the state estimate covariance into MPC, as developed in Section 2.3, implies possibly increasingly demanding modification of the constraints on the states as one moves along the horizon. Eventually, unless Σ_∞ is strongly bounded, infeasibility of the MPC problem will result through the growth of $\Sigma_{n+i|n}$ for large values of i . This is problematic because it would appear to militate against long horizons in MPC, which normally are associated with improved dynamical properties.

As stated in Theorem 1, at least for the scalar case, *MPC2* and *MPC3* are equivalent. Hence the deficiency of *MPC3* argued above originates from *MPC2*, which is a pure open-loop strategy and does not capture the closed-loop property of the system. It is not obvious how to pose a new stochastic MPC problem incorporating the closed-loop behavior. A simple remedy to *MPC3* will be now proposed.

2.4.2 Closed-loop Covariance Values

Our solution to the conundrum of ever-increasingly tighter constraints along the horizon of MPC is to fix all the state estimate covariances at their one-step-ahead, or minimal-control-delay values. That is, the constraint re-formulation from (2.15),

$$\hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+i|n}^{\frac{1}{2}} \beta_i^*,$$

is replaced by its one-step-ahead variant,

$$\hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+1|n}^{\frac{1}{2}} \beta_i^*.$$

We denominate this substitution as using the *closed-loop covariance*.

The reasoning behind the closed-loop covariance lies in the recognition that a receding horizon control strategy is being used. Thus, we know that before the next step of the MPC is taken a new measurement will become available. Thus, before the constraint involving $\Sigma_{n+2|n}$ might need to be active, a new measurement at time

$n + 1$ will have been available and *MPC3* will be re-posed and re-solved. The control u_{n+1} that feeds into the plant will be computed subject to the constraint with bound $(\beta_2 - \Sigma_{n+2|n+1}^{\frac{1}{2}}\beta_2^*)$, which would be equal to $(\beta_2 - \Sigma_{n+1|n}^{\frac{1}{2}}\beta_2^*)$ in the stationary case. From Section 2.4.1, this quantity is no smaller than $(\beta_2 - \Sigma_{n+2|n}^{\frac{1}{2}}\beta_n^*)$. Repeating this argument up to time $n + N$, we conclude that replacing the bounds $(\beta_i - \Sigma_{n+i|n}^{\frac{1}{2}}\beta_i^*)$ in the constraints (2.15) of *MPC3* by $(\beta_2 - \Sigma_{n+1|n}^{\frac{1}{2}}\beta_2^*)$ will result in a new MPC problem representing the closed-loop feature of the system in the stationary case.

Therefore, we reach our final version of MPC strategy,

MPC4:

$$\min_u J'(N, \hat{x}_{n|n}, u_n^{n+N-1}) = \hat{x}_{n+N|n}^T P_N \hat{x}_{n+N|n} + \sum_{i=0}^{N-1} \left(\hat{x}_{n+i|n}^T Q_i \hat{x}_{n+i|n} + u_{n+i}^T R_i u_{n+i} \right)$$

subject to: $\hat{x}_{n+i|n} \leq \beta_i - \Sigma_{n+1|n}^{\frac{1}{2}}\beta_i^*, (i = 1, \dots, N),$

$u_{n+i} \leq \mu_i, (i = 0, \dots, N - 1).$

[Here β_i^* is the solution of $\Phi(\beta_i^*) = p_i$, where $\Phi(\cdot)$ is the standard normal distribution function and p_i is the design probability of constraint satisfaction $P_{\mathcal{Y}_n}(x_{n+i} \leq \beta_i) \geq p_i$.]

Remarks:

- First we make two simple observations:
 - (i) The feasible region defined in *MPC4* is typically bigger than that in *MPC3*.
 - (ii) The criteria in *MPC4* and *MPC3* are the same function. From (i), we can conclude that the optimized criterion functions satisfy $J'_{MPC4}{}^{opt} \leq J'_{MPC3}{}^{opt}$.
- In a sense, the use of the closed-loop covariance is dual to the receding-horizon control idea. While a sequence of N control values is computed, only the first is applied. The state update then converts this open-loop control into closed-loop and issues such as asymptotic stability are addressable. Similarly, while the prediction occurs over N time steps, only the one-step-ahead prediction covariance information plays a role in modifying the constraints. This again reflects the disguised closed-loop nature of the control and leads to the improved treatment of

feasibility.

- If the stochastic disturbances fail to be gaussian, then application of *MPC4* may result in a risky control law or a conservative one. However, the idea will still work with different transformation from *MPC2* to *MPC3* and the proper choice of the closed-loop covariance to form *MPC4*.
- *For linear systems with gaussian noises, the stability issue is not treatable since the unboundedness of a gaussian random variable may drive the state arbitrarily far from zero. For systems with bounded noises, properly posed MPC4 can provide a set of over-bounding constraints on the real system. Hence, the stability can be guaranteed as in (Mayne, Rawlings, Rao & Scokaert 2000) via imposing terminal constraints.*

The stochastic nature of the original problem has been absorbed into the probabilistic constraints, which in turn have been mapped into modified more stringent deterministic constraints on the controlled conditional mean process, reflecting the state estimate quality. The result is a deterministic MPC problem of standard form with altered constraint values.

2.5 A Network Traffic Control Example

In this section, the control of Available Bit Rate (ABR) traffic in an Asynchronous Transfer Mode (ATM) telecommunications network is considered as an example of applying our MPC method, *MPC4*. This is a multi-source, single-buffer congestion control problem.

2.5.1 ABR Congestion Control

The ABR service plays a central role in regulating telecommunications network traffic. The current standards for ATM traffic management are built on the foundation of a rate based (rate matching) flow control scheme, see details in (Imer, Compans, Başar & Srikant 2001). ABR traffic has a variable available total rate determined by the higher priority CBR, VBR traffic prevailing. The goal of ABR service congestion

control is to provide fairness among all links with a minimal cell loss ratio and maximal utilization of network sources. The control challenge is to regulate ABR source rates to utilize maximally the available capacity as it varies while respecting the requirement not to overflow buffer queues too frequently.

We apply MPC to the ABR control problem. The downstream ABR is stochastically varying and the limited buffer queue size provides an obvious state constraint. The dynamics consist of the queue's integral action, stochastic variation of the ABR and action delays in the response of controlled data sources to commands from the node. This problem has been studied in a similar formulation by a number of authors, (Mascolo 1999, Altman, Başar & Srikant 1999, Imer & Başar 1999, Imer, Compan, Başar & Srikant 2001). Our modelling framework and criterion are basically the same as in (Altman, Başar & Srikant 1999), explicitly taking delay into account, and treating the available bit rate service as an autoregressive(AR) process driven by white noise. The action delays and the AR process are incorporated into the states by augmenting the system dimension. Consequently, the system model is posed in the form of (2.2) and we introduce an explicit probabilistic constraint on the queue length.

2.5.2 MPC Problem Formulation

The ABR control problem is depicted in Figure 2.2. Main data streams are

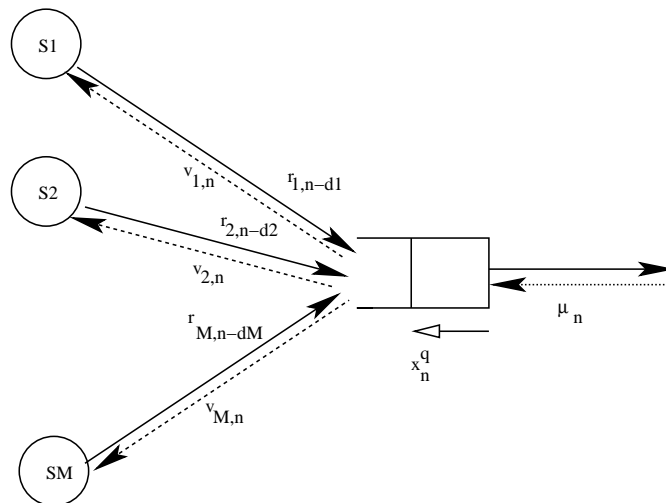


Figure 2.2: ABR control problem.

shown as solid lines, while network control data is shown as dotted lines. All signals are shown at the time of arrival or departure from the node. At time n at the node, it receives the current ABR, μ_n , from downstream and determines the source rate allocations, $v_{m,n}$, for each source $m = 1, \dots, M$. The arriving data rates from the sources are $r_{m,n} = v_{m,n-d_m}$, where the action delay d_m for each sources is due to the round trip transmission time from the node. The queue length is denoted q_n and evolves according to:

$$q_{n+1} = \text{sat}_{[0,1]}(q_n + \sum_{m=1}^M r_{m,n} - \mu_n),$$

where $\text{sat}_{[0,1]}(\cdot)$ is the saturation function and $[0, 1]$ corresponds to the range of the queue length from empty to full. It is presumed that the ABR is the sum with saturation of a known constant nominal service rate, μ , plus a p -th order autoregressive process, ξ_n .

$$\begin{aligned} \mu_n &= \text{sat}_{[0,1]}(\mu + \xi_n), \\ \xi_n &= \sum_{i=1}^p \alpha_i \xi_{n-i} + \phi_{n-1}, \end{aligned}$$

where $\{\phi_n\}$, an *i.i.d.*, $N(0, \sigma^2)$ distributed sequence, and $\alpha_i, i = 1, \dots, p$ are known coefficients.

The effective source rate $r_{m,n}$ is the response of data source m to a commanded rate or control action from the node d_m samples earlier. This action delay accounts for transmission time from the node to the source and return. Denoting the node control action for source m as $v_{m,n}$ we have,

$$v_{m,n-d_m} = r_{m,n}.$$

Define the centered variables

$$x_n^q = q_n - \bar{Q}, \quad u_{m,n} = v_{m,n} - \frac{1}{M}\mu,$$

where \bar{Q} is the nominal target mean queue utilization and μ is the mean ABR value.

To apply *MPC4* to this nonlinear system, we stick with the linear part of the system as the model running for the MPC controller. First introduce $M - 1$ new states

$$x_{n+1}^c = I_M x_n^c + \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix} \begin{pmatrix} u_{1,n} \\ u_{2,n} \\ \vdots \\ u_{M,n} \end{pmatrix}, \quad (2.17)$$

which integrate the differences between the control inputs and will be included in the criterion function to achieve fairness among the sources. Then this formulation of the ABR control problem may be written in state-space form as (2.2).

$$x_{n+1} = Ax_n + Bu_n + G\omega_n,$$

where the state $x_n = \begin{pmatrix} x_n^{dT} & x_n^{aT} & x_n^{cT} \end{pmatrix}^T$ with x_n^d the state of the transmission delays, x_n^a the state of the autoregressive ABR model. Vector u_n is the input vector appearing in (2.17). The detailed formulation is similar to that developed in (Yan & Bitmead 2002) except for the queue dynamics:

$$x_{n+1}^q = x_n^q + x_n^{d,1} - \text{sat}_{[-\mu, 1-\mu]}(\xi_n).$$

Note that, in this model for MPC, the saturation of the queue length is omitted, since the constraints should cope with the upper bound and the lower violation can be avoided by setting the nominal queue length \bar{Q} and the bound of constraints away from it. For simplicity in development of the state estimator, we assume that at each sample time the real value of ξ is available.

The objective function to be minimized is given by:

$$J = \frac{1}{N} E y_n \left\{ \sum_{i=1}^N \left[x_{n+i}^{qT} x_{n+i}^q + \delta \sum_{j=1}^M (u_{j,n+i} - \frac{1}{M} \text{sat}_{[-\mu, 1-\mu]}(\xi_{n+i+d_j}))^2 + \gamma x_{n+i}^{cT} x_{n+i}^c \right] \right\}. \quad (2.18)$$

where $\hat{\xi}$ is the predicted value of the AR process given by the Kalman filter time update part only. The first term in this criterion is the performance objective of keeping the queue close to an operating point. The second term tries to ensure that all available capacity is used. The third term manages fairness by making the average values of assigned rates the same.

Next a set of probabilistic constraints, related to the upper bound on the queue length, is introduced:

$$P_{y_n}(q_{n+i} \leq Q_{\max}) = P_{y_n}(x_{n+i}^q \leq Q_{\max} - \bar{Q}) \geq p_i, \quad (i = d_{\min} + 1, \dots, N). \quad (2.19)$$

This constraint says that to some small extent losing and re-transmitting a certain proportion of data is tolerable — p_i should be chosen close to 1. Integer d_{\min} denotes the

smallest delay among the sources. Due to the time delay, at time n states $x_n^q \cdots x_{n+d_{\min}}^q$ are not controllable, therefore, they should be constraint-free.

Now apply the method developed earlier by modeling \tilde{x}_{n+i}^q and $\{\xi_n\}$ as gaussian. The constraint (2.19) may be replaced by

$$\hat{x}_{n+i|n}^q \leq Q_{\max} - \bar{Q} - \beta_i^* \Sigma_{n+d_{\min}+1|n}^{q\frac{1}{2}},$$

where $\Phi(\beta_i^*) = p_i$, and $\Phi(\cdot)$ is the Standard Normal Distribution function. Note, that we have used the closed-loop covariance $\Sigma_{n+d_{\min}+1|n}^q$ for all values of i . As \tilde{x} involves the saturation of the AR process, its distribution is not normal but a truncated version of the normal distribution function. However, the constraints still work, because $\Sigma_{n+d_{\min}+1|n}^q$ is an overbound of the real covariance. Finally, the MPC problem becomes

ABR-MPC:

$$\begin{aligned} \min_u \quad J &= \frac{1}{N} E_{\mathcal{Y}_n} \left\{ \sum_{i=1}^N \left[x_{n+i}^{qT} x_{n+i}^q + \delta \sum_{j=1}^M (u_{j,n+i} - \frac{1}{M} \text{sat}_{[-\mu, 1-\mu]}(\xi_{n+i+d_j}))^2 \right. \right. \\ &\quad \left. \left. + \gamma x_{n+i}^{cT} x_{n+i}^c \right] \right\}, \\ \text{subject to:} \quad \hat{x}_{n+i|n}^q &\leq Q_{\max} - \bar{Q} - \beta_i^* \Sigma_{n+d_{\min}+1|n}^{q\frac{1}{2}}, \quad (i = d_{\min} + 1, \dots, N). \end{aligned}$$

And the control sent into the system should be $\text{sat}_{[-\frac{1}{M}, 1-\frac{1}{M}]}(u_n^j)$, since any real source rate can neither be negative nor exceed the maximum available service rate.

For the purpose of comparison, we also run the simulation for the stabilizing controller suggested in (Imer & Başar 1999) which is an infinite horizon LQG controller with the criterion function being the limit of (2.18) without saturation in the second term and the fairness term $\gamma x^{cT} x^c$. As proved in (Imer & Başar 1999), this approach can guarantee a stabilizing control law. The simulation result shows that fairness among the sources cannot be obtained by the second term in the criterion function. Another LQG controller is also simulated for the same system with the same criterion function (2.18) as in *ABR-MPC* and the controls are subject to the saturation. This receding horizon approach is a fair strategy. A comparison of these 3 controller will be presented next.

2.5.3 Simulation Results

Results of *ABR-MPC*

The simulation is of a single congested node accessed by three sources with delays $d_1 = 2, d_2 = 4, d_3 = 6$. The AR process is assumed to be 2nd order with parameters $\alpha_1 = 1.88, \alpha_2 = -0.8836$. Coefficients δ and γ are both set to .0001. The finite horizon should be taken bigger than the biggest action delay, so $N = 10$. The nominal queue length, \bar{Q} in the criterion function, is set to 2, this means the constraints are always active. The probabilistic constraint is $P(x_{n+i}^q \leq -1) \geq 0.9505$, with corresponding $\beta^* = 1.65$. The total simulation time is 10000 samples. Figure 2.3 is part of the plot of

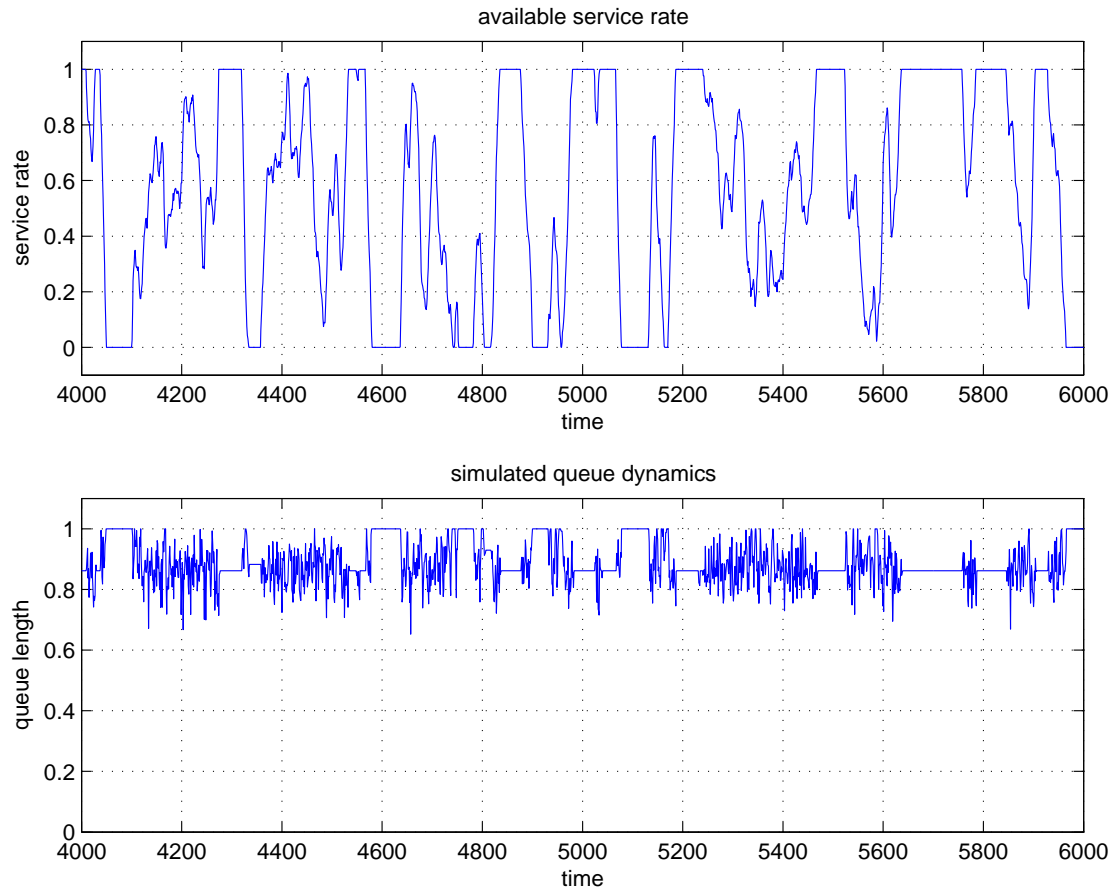


Figure 2.3: Available service rate and the queue length for the *ABR-MPC* control

the available service rate μ_n and the queue q_n . In this simulation result, there are about

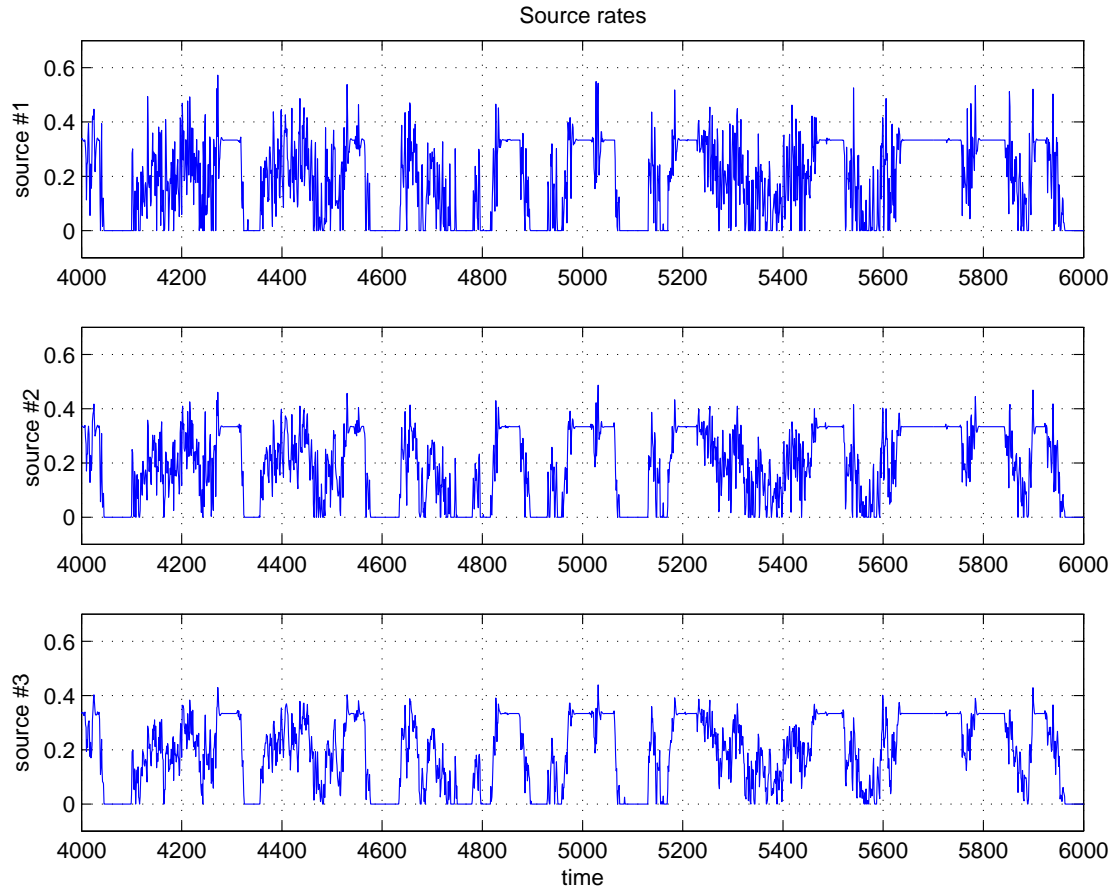


Figure 2.4: Source rates with *ABR-MPC*.

488 steps in 10000 overflowed, which satisfies the expected portion 5%. Figure 2.4 shows the plots of the source rates $r_{i,n}$. The means of $r_{1,n}, r_{2,n}, r_{3,n}$ are 0.1634, 0.1634, 0.1634 separately, indicating that fairness has been achieved. The variances of these source rates are 0.0223, 0.0198, 0.0189, which is decreasing with increasing round-trip delay from the node. This is because the lesser delayed source rate is more effective in regulating the queue length, and therefore is used more frequently to manage the overflow.

Using open-loop prediction covariance in *ABR-MPC*

Figure 2.5 depicts the simulated queue length of applying the open-loop covariances modified constraints in *ABR-MPC*. The overflowing rate is 1%, far less than the designed five percent and the 4.88% in Figure 2.3. In fact, the constraints here are overly

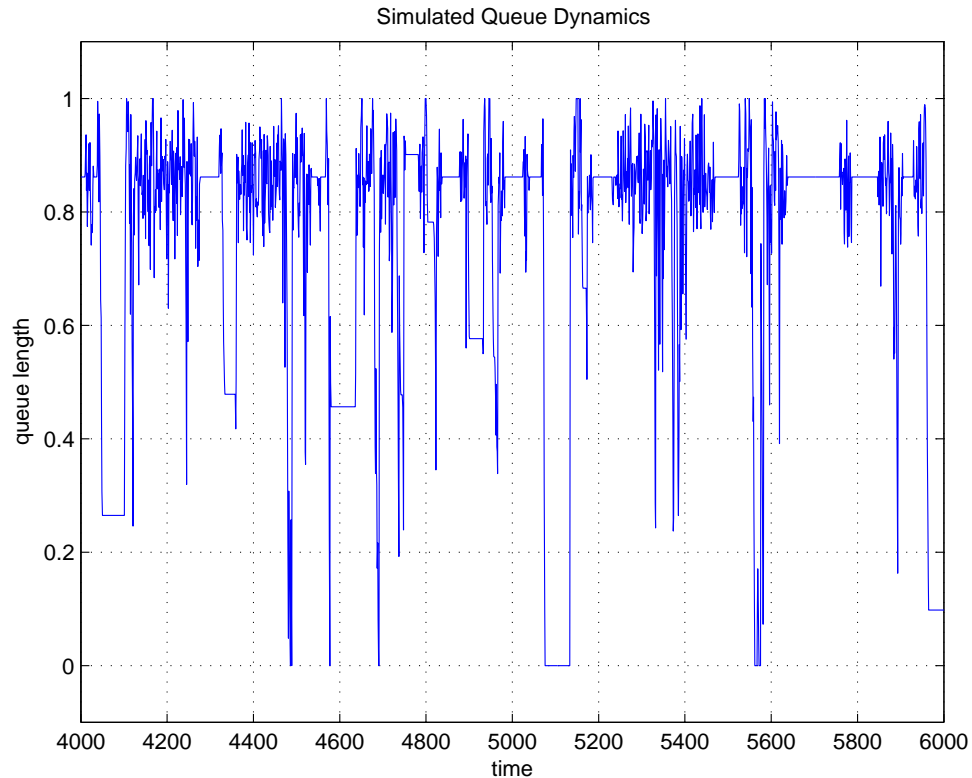


Figure 2.5: Queue length using open-loop covariance in *ABR-MPC*.

stringent so that the QP routine could not find a feasible solution for many steps and the source rates were set to 0. As a result, the mean source rates 0.1631, 0.1626, 0.1624 are lower than that of using closed-loop covariance. Also the result shows unfairness among the sources, since this strategy always puts tighter constraints on the source rates with larger delays. The mean queue length is 0.7204 which is lower than the 0.9004 in Figure 2.3. Obviously, this set of constraints gives conservative performance.

Comparison of existing approaches and *ABR-MPC*

The infinite horizon LQG controller proposed in (Imer & Başar 1999) and the receding horizon LQG controller are simulated with the same noise set applied to *ABR-MPC* for 10000 samples. The nominal queue length \bar{Q} is the only tuning parameter for these two strategies and is set to 0.65 and 0.85 respectively to achieve 5% overflowing

rate, the primary performance requirement. The means of the source rates given by the infinite horizon LQG are 0.1720, 0.1584 and 0.1562, therefore this is not a fair strategy. The receding horizon LQG is a fair strategy with the means of the sources rates 0.1634, the same as *ABR-MPC*, hence these two strategies almost have the same total data throughput.

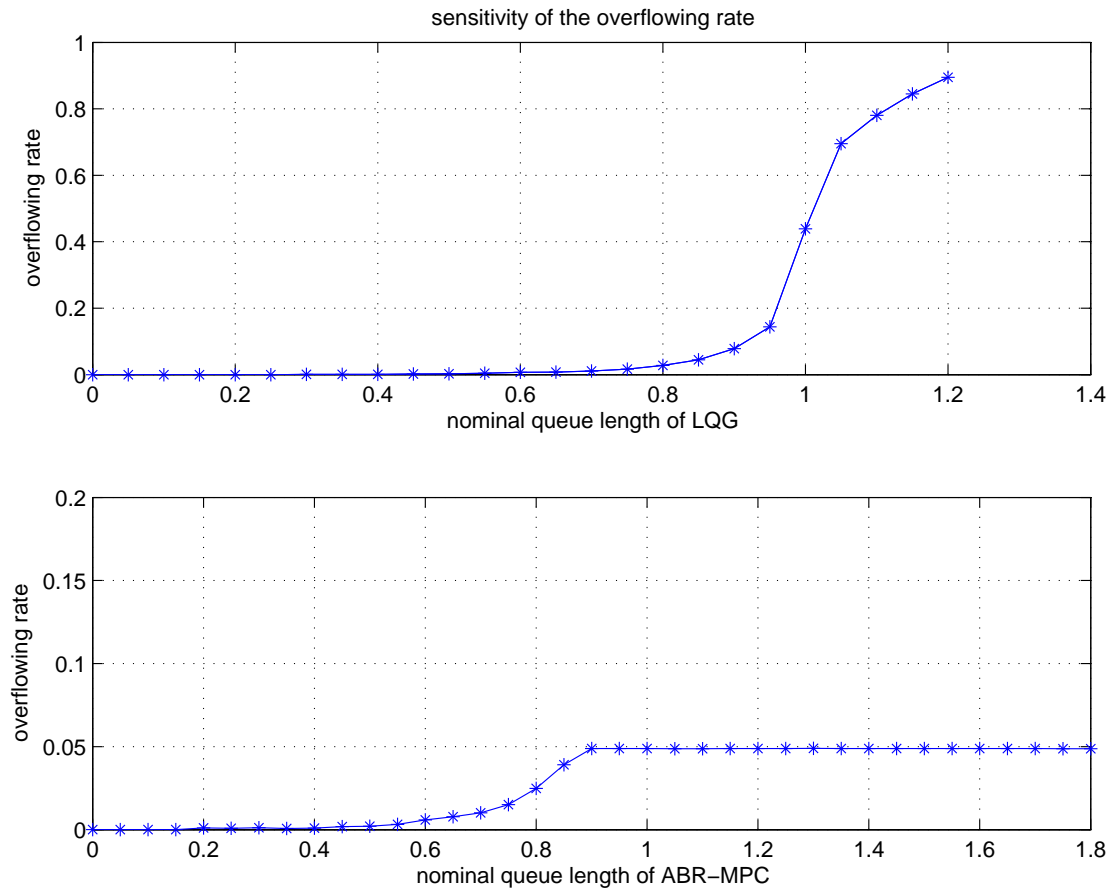


Figure 2.6: Sensitivity to nominal queue length (note the difference in scales).

Though the two controllers, receding horizon LQG and *ABR-MPC* have the same performance, LQG is sensitive to the nominal queue length \bar{Q} while *ABR-MPC* is robust to it. To show this, we simulated these two controllers for different values of \bar{Q} , 10000 samples for each. The result is shown in Figure 2.6, the plot of the overflowing rate for different nominal queue values. It can be seen that it is difficult for LQG to achieve the

exact 5% overflowing rate via tuning \bar{Q} . For *ABR-MPC*, the overflowing rate can be guaranteed by setting the nominal queue length great enough, i.e. make the constraints active which were designed corresponding to the given overflowing rate. Once some parameters of the system are changed (e.g. the covariance of the white noise), LQG might require re-tuning \bar{Q} which cannot be done explicitly; while *ABR-MPC* would require re-calculating and change of the corresponding terms in the constraints directly.

As mentioned earlier, in our simulation we use ξ_n in the optimization. However, this is not a realistic measurement. In practice, the available service rate μ_n should be at hand instead of ξ_n . Since μ_n is a nonlinear process, a trustable estimator needs to consider the conditional distribution of future μ s in order that the probabilistic constraints in *ABR-MPC* can be replaced with the deterministic ones. The detail is attainable but complicated and outside the scope of this dissertation.

2.6 Conclusion

This chapter has presented an approach to the treatment of state estimates in constrained MPC. The key ideas are:

- that the constraints should be converted to a probabilistic form,
- that the probabilistic constraint on the states should be replaced by a deterministic constraint on the controlled conditional mean state estimate and that this modified constraint should involve tighter stipulations reflecting the estimate covariance,
- that the closed-loop covariance should be used for the modification of all constraints along the horizon.

The resulting deterministic MPC problem is standard in its form but uses the state estimate as its starting state and introduces covariance information via the constraints. This is not a certainty equivalence controller.

The ideas are demonstrated on a recent telecommunications network control problem of interest and found to display an ease of use and effectiveness of application in this almost linear, constrained problem. A key feature leading to this facility was the

property of linear systems that the state estimates' covariances are independent of the control signal.

Further work remains to be tackled in extending the approach to a wide range of more obviously nonlinear problems. Equally, there would be considerable advantage to developing a method to over-bound simply probabilities of constraint violation with normal approximations.

The network example may also be expanded to include more nodes and the nodes operate in a coordinated way to avoid network congestion. In this problem, the MPC formulation posed in this chapter can be a good choice of the local controller. In addition, each node should also consider its neighboring nodes' low overflowing rate constraints and some necessary information should be exchanged locally between the neighbors. Instead of this problem, we study the coordination of a fleet of vehicles as an example of the multi-system coordinated control problem applying locally the constrained MPC in Chapter 3. But the ideas are very similar.

This chapter includes the reprints of the following papers:

Jun Yan, Robert R. Bitmead - *Incorporating State Estimation into Model Predictive Control and its Application to Network Traffic Control*, *Automatica*, Vol. 41, pp. 595-604, Apr. 2005.

Jun Yan, Robert R. Bitmead - *Model Predictive Control and State Estimation: A Network Example*, 15th IFAC World Congress in Barcelona, Jul. 2002.

The dissertation author was the primary author listed in these publications. And the co-author, Professor Bitmead, directed and supervised the research.

Chapter 3

A Constrained Model-Predictive Approach to Coordinated Control

3.1 Introduction

Constrained Model Predictive Control (MPC) is advanced as an effective approach to the design of coordinated control and communications for multiple autonomous systems. The central ideas are the use of local MPC at each distributed subsystem with the interaction managed via “no-collision” constraints between the subsystems being included into the MPC. Limited-bandwidth communication between the subsystems is available and its resource assignment is coupled to the achievable global system performance.

In considering distributed coordination, it is natural to identify tasks which are global, i.e. shared among all subsystems, in nature and those which are local or limited in their domain. Here we specify the global tasks to include those design elements which might sensibly be pre-computed and are known by all subsystems; reference trajectories, constraint specification, control law, observer structure and design, communications resource assignment, maps and other functional descriptions of the environment. By contrast, local tasks would be those associated with real-time and on-line calculation and the interaction between subsystems; the MPC control solution computation, observer calculation of state estimates, operation of communications links.

Our approach in this chapter is to explore the application of local constrained MPC control laws on the subsystems and to establish conditions on the constraints to permit guaranteed convergence to a local target trajectory or point. Thereby, if the global set of constraints is such as to permit this local satisfaction, then system stability and convergence follow. Properties of the constraint set flow from the specification of reference trajectories and of the communications bandwidth, as will be shown. This establishes a link between control objective and communication. To help fix ideas we focus on a vehicle coordination problem and demonstrate concepts using simulations of a one-dimensional example.

Design via constraints is an attractive approach to control and coordination. In particular, the recent interest in the application of MPC (Maciejowski 2002, Goodwin, Seron & Doná 2004) reflects the appeal of constraints as a design method for controlling systems. In spite of the approach to the solution of the constrained control via optimization with penalty or barrier functions, the attraction of MPC rests in the separation of constraints and the objective function. In vehicle control problems, recent work of (Gerdes & Rossetter 2001) indicates that common coordination tasks such as lane-keeping and distance-keeping are well captured by potential functions which may be interpreted as the action of constraints on the controlled driving dynamics.

The focus of this chapter is that coordinated control has a natural specification via imposed constraints between neighbors' actions. An MPC-like constrained control is a natural approach to developing local feedback control laws using communicated state information from the neighboring systems. We separate the global pre-computed tasks to include the determination of target (or reference) for each subsystem and use this to inform the constraint management and communication resource assignment. The sensing on board each subsystem will be only sensing its own state and some level of communication of this information is permissible to designated others. This helps crystallize the importance of the communication resource assignment task in coordinated control.

Distributed control has been identified as an effective approach to controlling the large-scale complex systems (Šiljak 1990), such as power systems, water distribution systems, etc. Decentralized constrained MPC has been considered for systems with

coupled local dynamics (Camponogara, Jia, Krogh & Talukdar 2002) and for vehicle formations with decoupled local dynamics (Richards & How 2004*a*, Keviczky, Borrelli & Balas 2004, Dunbar & Murray 2002, Dunbar & Murray 2004). In this chapter, we first discuss the formulation of a typical vehicle formation problem with decoupled local dynamics in Section 3.2. The formation tasks for the vehicles is to converge to their own target positions while avoiding collisions.

One advantage of the constrained MPC is the well established feasibility-guaranteeing-stability argument (Mayne, Rawlings, Rao & Scokaert 2000). Hence, one track of the decentralized MPC application is to partition a centralized stabilizing MPC into local subproblems assigned to the subsystems. If the sum of the local MPC solutions recovers or converges to the global, optimal MPC solution, then the nice global properties are inherited automatically. A set of conditions are given in (Camponogara, Jia, Krogh & Talukdar 2002) to achieve the global optimality via the convergence of the locally shared variables known to the different subsystems, which is guaranteed by the iterative local computation and exchange of information between the subsystems at each sampling instant. Practically, the computation and the information exchange is usually limited to once per sample, hence sending neighbors the predicted state trajectory at the previous sampling time becomes a natural choice of the shared state information. Local MPC constraints can be included to guarantee the global formation stability but not the global optimality, (Jia & Krogh 2001, Camponogara, Jia, Krogh & Talukdar 2002, Dunbar & Murray 2004). A discussion of the optimal partitioning of the global MPC problem can be found in (Motee & Sayyar-Rodsari 2003).

In this dissertation, we propose to deduce the convergence of the entire formation from assigning the local behaviors via the constrained MPC. No centralized MPC is considered here since the only interaction between the vehicles is the set of local no-collision constraints. The formation will have higher level of scalability and reconfigurability by changing only the local rules (the constraints and the criterion function of the local MPC) in flight without revisiting the global problem. In Section 3.3, we first study the deterministic formation problem and discuss the convergence conditions. For the case with stochastic disturbances, such as ambient physical disturbances and communication channel noises, the no-collision constraint of MPC is modified to accommodate

the uncertainty introduced by the disturbances using the idea presented in Chapter 2. String stability, which describes the propagation and amplification of the disturbances within the formation, is discussed. String stability can be guaranteed via decoupling the vehicles' closed-loop dynamics, i.e. using fixed reference trajectories in MPC cost functions and keeping constraints inactive for much of the time. Similar ideas are also discussed in (Richards & How 2004a), where the authors studied a decentralized collision avoidance MPC formulation. The local no-collision constraints are tightened according to the bounded disturbances and no-collision can be deduced from the local MPC and a feasible solution of the centralized MPC at the beginning of the entire operation.

In Section 3.4, the working problem is a simple 1-dimensional vehicle formation control problem. In each local MPC controller, the vehicles are decoupled in their local criteria while having the modified no-collision constraints coupling the neighbors. A leader-follower strategy with the vehicles coupled in the criterion functions is included for the purpose of comparison. Two performance indices are evaluated; string stability and inter-vehicle spacing. The relation between the communication channel noises and the control performance is clear from the simulations. This relation is also a guide to the design of the reference trajectories and the assignment of the communication bandwidth, which will be discussed in Chapter 4. This is the first appearance of the connection between control task, communication bandwidth and performance.

A two-dimensional version of no-collision constraint is introduced in Section 3.5 to point out the necessary modifications of the local MPC. The conclusions follow in Section 3.6. The detailed formulation of the simulation is given in the Appendix.

3.2 Multi-vehicle Coordination Problem

Vehicle formation control has become an active topic in the control community with many possible applications, for example, robotics (Balch & Arkin 1998, Desai, Ostrowski & Kumar 2001), unmanned aerial vehicles (Stipanović, Inalhan, Teo & Tomlin 2004) or spacecraft (Beard, Lawton & Hadaegh 2001), and automated highway systems (Varaiya 1993, Swaroop & Hedrick 1999), just to mention a few. In this paper, we consider a vehicle fleet consisting of p vehicles with concatenated state

$X_k = (x_k^{1T}, x_k^{2T}, \dots, x_k^{pT})^T$. The dynamics of the i th vehicle are given by

$$x_{k+1}^i = f^i(x_k^i, u_k^i) + w_k^i, \quad (3.1)$$

$$z_k^i = \pi^i(X_k) + v_k^i. \quad (3.2)$$

- x_k^i is the state vector, u_k^i is the local control, and w_k^i is the process disturbance impinging on the state of Vehicle i . Here the $f^i(\cdot)$ s need not all be the same. And this model (3.1) may also include a model describing the ambient disturbances.
- The measurement function $\pi^i(\cdot)$ captures the self-measurement of vehicle i and the communication link selections to Vehicle i and the nature of the communicated information, which could include the states of a subset of vehicles, the control values of a subset of vehicles, or the predicted future control and state values as in (Dunbar & Murray 2004).
- The noise term v_k^i models both the measurement noise and the inaccuracy of the data communicated via π^i . This latter part of v_k^i is due to finite bit-rates being assigned to the communication links carrying the data and is practically very well modelled by uniformly distributed white noise. The effects of channel noise could similarly be included.

We define the term *information architecture* to mean the complete set of assignments of bit-rates (including zero-bit rate links to represent un-communicated data) to communication links for data between vehicles and the selection of which data are communicated. Typically, the information architecture is constrained by channel bandwidth.

A typical vehicle formation task is described as: given a target formation $X^* = (x^{1*T}, x^{2*T}, \dots, x^{p*T})^T$, the local controllers should steer the entire formation so that $X_k \rightarrow X^*$ while avoiding collision. Some other performance requirements may also be added, such as the speed of convergence or energy usage.

For a large formation, a decentralized strategy is preferred so that the online computation speeds exceed the dynamics of the vehicles and we see this as dividing into two design phases:

1. The first part is the design of the information architecture, which determines the off-line shared information (such as a virtual leader (Leonard & Fiorelli 2001)) and

the necessary localized information flows for each vehicle to guarantee the fulfilment of the global task. It has been shown that Graph Theory tools lend themselves naturally in the information flow design problem (Muhammad & Egerstedt 2004). The stability of the formation can be directly related the algebraic properties of the information flow graphs as shown in (Fax & Murray 2002, Fax & Murray 2004, Moreau 2005). Perfect measurement is assumed in the works above. If MPC is used as the local controller, the design of the communicated information, $\pi^i(\cdot)$, may come directly from the coupled system dynamics (Camponogara, Jia, Krogh & Talukdar 2002), the formation task defined by the coupled criterion function (Dunbar & Murray 2004), and the coupled local constraints (Yan & Bitmead 2003, Richards & How 2004a). Especially, the structure of $\pi^i(\cdot)$ should permit the computation of the required prediction of the neighbors in a local MPC controller. The prediction of the neighbors is proven critical for the global stability (Camponogara, Jia, Krogh & Talukdar 2002, Jia & Krogh 2001, Dunbar & Murray 2004). When information is obtained by sensors or communication channels, there will be noises associated with the information. This raises another two design issues in the information architecture design — the design of the accuracy of the local information/prediction, which is directly related to the cost of the communication devices or sensors. To achieve the required quality of the local prediction, we should consider the design of the communication channel capacity and the design of good local estimators, which will be the main theme of Chapter 4.

In this chapter, we assume the architecture of the information flows is given. The net quantified effect of the specification of the information architecture is captured through the computed uncertainty of the global state, X_{k+j} , evaluated at time k and at Vehicle i using the sensed and communicated data. Denote this $\hat{X}_{k+j|k}^i$. The uncertainty may be modelled in a number of ways, notably;

- the support of the density of $\hat{X}_{k+j|k}^i$, or
- the covariance of $\hat{X}_{k+j|k}^i$.

The global state estimate and its information-architecture-dependent uncertainty will pass into the formulation of the collision avoidance MPC controller. In Sec-

tion 3.4, we will discuss the relation between the information accuracy and the control performance in a simple one-dimensional vehicle control problem. This provides a guide to the design issues in Chapter 4.

2. The second part is the design of the local controllers, which should lead the vehicles to their targets while avoiding collisions. MPC is an appealing approach because of its capability of constraint-handling, familiarity with its real-time implementation and the solid theoretical results on stability and feasibility (Mayne, Rawlings, Rao & Scokaert 2000). To achieve global stability via distributed MPC, one should pose condition on the discrepancy between the different versions of the locally shared prediction. In (Keviczky, Borrelli & Balas 2004), the authors suggested that the predicted neighbor's trajectory and the neighbor's own predicted trajectory should be stay close. Actual conditions are provided by (Jia & Krogh 2001) and (Dunbar & Murray 2004). In (Jia & Krogh 2001), the subsystems use the neighbors' predictions computed at the last sampling time and a contractive condition is imposed on the one-step ahead state to force the closed-loop system state to decrease to its equilibrium. In (Dunbar & Murray 2004), the vehicles exchange their optimizing future trajectories at each sampling time. The stability of the formation can be guaranteed by including explicitly a compatibility constraint between the exchanged trajectory and the trajectory under consideration in each local MPC controller. In (Camponogara, Jia, Krogh & Talukdar 2002), the authors showed that the global optimality can be achieved via iterative communication and local MPC computation at each sampling time. These works assumed perfect state information.

In this chapter, we consider the vehicles decoupled in their local cost functions by introducing to each vehicle an assigned target position. The vehicles are coupled via the no-collision constraints in their local MPC. These constraints incorporate the state estimates, $\hat{X}_{k+j|k}^i$, and the uncertainty characterization flowing from the information architecture. By applying the strategy in Chapter 2, the local constraints may be re-posed to accommodate the quantified uncertainty. Furthermore, these modified constraints relate the information quality and the control perfor-

mance as will be shown in Section 3.4.

3.3 Local Model Predictive Control Law

To accomplish the coordination task stated in Section 3.2, each vehicle should approach its own target state x^{i*} while complying with constraints. We propose to use constrained MPC as the local control law for this constrained problem. A typical local constrained MPC problem for the i th vehicle is:

$$\begin{aligned} \min_{u_{k,k}^i, \dots, u_{k+N-1,k}^i} J_k^i(x_k^i, u_{k,k}^i, \dots, u_{k+N-1,k}^i) &= \min F_k^i(x_{k+N}^i) + \sum_{j=0}^{N-1} l_k^i(x_{k+j}^i, u_{k+j}^i), \quad (3.3) \\ \text{subject to: } x_{k+j+1}^i &= f^i(x_{k+j}^i, u_{k+j}^i) + w_{k+j}^i, \\ u_{k+j}^i &\in U_{k+j}^i, \\ x_{k+j}^i &\in X^i(k+j), \\ x_{k+N}^i &\in X_f^i(k), \end{aligned}$$

where the criterion function consists of the sum of step costs $l_k^i(\cdot, \cdot)$ and a terminal state cost $F_k^i(\cdot)$. The feasible set of controls is the compact, convex set U_k^i and the feasible set of states is the closed, convex set $X^i(k)$, which should be determined online by the available information π_k^i . The terminal state constraint set $X_f^i(k)$ is the key to guaranteeing the stability of the MPC problem. The time index k in these constraint sets indicates that they can be time dependent. (The initial state x_k^i should be given by some estimator/observer based on z_k^i .)

3.3.1 Formulation and Stability Analysis of the Deterministic Case

In the deterministic case first, the process noises w_k^i and the measurement noises v_k^i in (3.3) are zero. A thorough stability discussion of existing constrained MPC approaches can be found in (Mayne, Rawlings, Rao & Scokaert 2000). The MPC problem in this paper is slightly different because the target state x^{i*} may be outside the constraint sets X^i and X_f^i . This may occur in the vehicle coordination problem.

Consider the time-invariant case first, meaning that $X^i(k)$ and $X_f^i(k)$ become X^i and X_f^i in the local MPC problem (3.3).

Definition 1. Functions $\bar{l}^i : R^n \times R^m \rightarrow R$ and $\bar{F}^i : R^n \rightarrow R$, where n and m are the dimensions of x_k^i and u_k^i ; $\bar{l}^i > 0$, $\bar{F}^i > 0$ except $\bar{l}^i(x^{i*}, u^{i*}) = \bar{F}^i(x^{i*}) = 0$; $\bar{l}^i(x^i, u^i) \uparrow \infty$ as $\|(x^i, u^i) - (x^{i*}, u^{i*})\| \uparrow \infty$ and $\bar{F}^i(x^i) \uparrow \infty$ as $\|x^i - x^{i*}\| \uparrow \infty$.

Using the stationary running state and control constraint sets X^i and U^i , define the terminal state constraint set X_f^i to be the smallest positively-invariant set of the controlled states contained in the set of the \bar{F}^i -closest feasible points to the target state. Denote by U_f^i the set of controls which would maintain x_k^i in X_f^i .

Thus, if the target were feasibly reachable and there exists a control u^{i*} satisfying $x^{i*} = f^i(x^{i*}, u^{i*})$, then $X_f^i = \{x^{i*}\}$ and $U_f^i = \{u^{i*}\}$. If x^{i*} were feasibly reachable but no holding control as above existed, then X_f^i would be a neighborhood around x^{i*} and U_f^i the set of controls which kept X_f^i positively-invariant. If x^{i*} were not feasibly reachable, then X_f^i would be a positively-invariant neighborhood of feasible points as close as possible to x^{i*} and U_f^i the set of controls which hold x_k^i in X_f^i .

Theorem 2. If the MPC problem (3.3) satisfies the following conditions:

1. U^i is compact and convex, X^i is closed and convex;
2. the step cost and terminal cost functions are $l^i(x_k^i, u_k^i) = \bar{l}^i 1_{\{x_k^i \notin X_f^i, u_k^i \notin U_f^i\}}$, $F^i(x_k^i, u_k^i) = \bar{F}^i 1_{\{x_k^i \notin X_f^i\}}$; where $1_{\{\cdot\}}$ is the indicator function;
3. $f^i(\cdot)$, $\bar{l}^i(\cdot, \cdot)$, and $\bar{F}^i(\cdot)$ are continuous;

then, provided an initial feasible solution exists, the MPC controller yields a feasible control law $\{u_k^i(x_k^i)\}$ and trajectory $\{x_k^i\}$ with $\limsup_{k \rightarrow \infty} \{x_k^i\} \subset X_f^i$.

Proof: To show that the closed-loop system converges to the terminal constraint set X_f^i , we first to show that the criterion function is a strictly decreasing function when the initial state is in the region $X^i \setminus X_f^i$.

Suppose the state at time k is any $x_k^i \in X^i \setminus X_f^i$, the MPC problem (3.3) yields an optimizing control sequence $\mathbf{u}_k^{\mathbf{i}, \text{opt}} = \{u_{k,k}^{i, \text{opt}}, u_{k+1,k}^{i, \text{opt}}, \dots, u_{k+N-1,k}^{i, \text{opt}}\}$, the corresponding state trajectory is $\mathbf{x}_k^{\mathbf{i}, \text{opt}} = \{x_{k+1,k}^{i, \text{opt}}, x_{k+2,k}^{i, \text{opt}}, \dots, x_{k+N,k}^{i, \text{opt}}\}$, and the associated cost is

$$J^{i, \text{opt}}(x_k^i) = J^i(x_k^i, \mathbf{u}_k^{\mathbf{i}, \text{opt}}) = \sum_{j=0}^{N-1} l^i(x_{k+j,k}^{i, \text{opt}}, u_{k+j,k}^{i, \text{opt}}) + F^i(x_{k+N,k}^{i, \text{opt}}).$$

Since the solution $(\mathbf{u}_k^{i,opt}, \mathbf{x}_k^{i,opt})$ is feasible, it follows that $x_{k+N,k}^{i,opt} \in X_f^i$ and by the definition of X_f^i , there is a u_{k+N}^{i*} such that $x_{k+N+1,k}^i = f^i(x_{k+N,k}^{i,opt}, u_{k+N}^{i*}) \in X_f^i$.

Define

$$\tilde{\mathbf{u}}_{\mathbf{k}+1}^i = \{u_{k+1,k}^{i,opt}, u_{k+2,k}^{i,opt}, \dots, u_{k+N-1,k}^{i,opt}, u_{k+N}^{i*}\}, \tilde{\mathbf{x}}_{\mathbf{k}+1}^i = \{x_{k+2,k}^{i,opt}, x_{k+3,k}^{i,opt}, \dots, x_{k+N,k}^{i,opt}, x_{k+N+1,k}^i\},$$

and their associated cost

$$\tilde{J}^i(x_{k+1,k}^{i,opt}, \tilde{\mathbf{u}}_{\mathbf{k}+1}^i) = \sum_{j=1}^{N-1} l^i(x_{k+j,k}^{i,opt}, u_{k+j,k}^{i,opt}) + l^i(x_{k+N,k}^{i,opt}, u_{k+N}^{i*}) + F^i(x_{k+N+1,k}^i).$$

Note that, $l^i(x_{k+N,k}^{i,opt}, u_{k+N}^{i*}) = F^i(x_{k+N+1,k}^i) = F^i(x_{k+N,k}^{i,opt}) = 0$, it follows that

$$J^{i,opt}(x_k^i) - \tilde{J}^i(x_{k+1,k}^{i,opt}) = l^i(x_k^i, u_k^{i,opt}) > 0.$$

The MPC strategy takes $u_k^i = u_k^{i,opt}$, thus, $x_{k+1}^i = x_{k+1,k}^{i,opt}$. It follows that $(\tilde{\mathbf{u}}_{\mathbf{k}+1}^i, \tilde{\mathbf{x}}_{\mathbf{k}+1}^i)$ is a feasible solution for the MPC problem for the next step and by optimality, the optimized cost $J^{i,opt}(x_{k+1}^i) \leq \tilde{J}^i(x_{k+1,k}^{i,opt}) < J^{i,opt}(x_k^i)$. Therefore,

$$J^{i,opt}(x_{k+1}^i) - J^{i,opt}(x_k^i) \leq \tilde{J}^i(x_{k+1,k}^{i,opt}) - J^{i,opt}(x_k^i) < -l^i(x_k^i, u_k^{i,opt}).$$

By induction, we have

$$0 \leq J^{i,opt}(x_{k+1}^i) < J^{i,opt}(x_0^i) - \sum_{j=0}^k l^i(x_j^i, u_j^{i,opt})$$

hold for arbitrarily large k . This requires that the series $\sum_{j=0}^k l^i(x_j^i, u_j^{i,opt})$ converges (to some value no larger than $J^{i,opt}(x_0^i)$), therefore, $l^i(x_j^i, u_j^{i,opt})$ converges to 0. By the definition of $l^i(\cdot, \cdot)$, (x_k^i, u_k^i) converges to the set $X_f^i \times U_f^i$. ■

The asymptotic stability implies that x_k^i converges to X_f^i as $k \rightarrow \infty$. This means that if the target x^{i*} is inside the constraint set X^i , then x_k^i will converge to a neighborhood of x^{i*} . If x^{i*} is outside the constraint set, then x_k^i will converge to the set X_f^i , the closest feasible points to the target. In the vehicle formation problem, this implies the following corollary.

Corollary 1. *For a p -vehicle formation with each vehicle running an MPC controller that satisfies the conditions of Theorem 2, each vehicle in the formation will either converge to its target x^{i*} , or to the set of the closest points to the target defined by X_f^i . In particular, if the terminal state constraint set is $X_f^i = x^{i*}$, then the local asymptotic stability implies the global fleet formation stability.*

For the time-varying case, the time-dependence of the criterion function and the constraint sets may invalidate the proof from the time-invariant case. Since the tail of the last optimizing sequence might not be feasible for the new problem and, even if feasible, the new criterion function might not be decreasing.

If the cost function in the MPC problem is fixed, then one may still have formation stability by managing the time-varying constraint sets $X^i(k)$ and $X_f^i(k)$ as in the following theorem.

Theorem 3. *If there exists a finite time T so that $\forall k \geq T$:*

1. $x^{i*} \in X^i(k+j)$, $X_f^i(k)$ is a positively invariant neighborhood of x^{i*} ; [respectively $X_f^i(k) = \{x^{i*}\}$, if this singleton is positively invariant];
2. the cost function stays the same after time T , i.e. $l_k^i(\cdot, \cdot) = l_T^i(\cdot, \cdot)$ and $F_k^i(\cdot) = F_T^i(\cdot)$;
3. $X^i(k) \subseteq X^i(k+1)$;

then, the MPC controller can stabilize x_k^i to a neighborhood of the target x^{i} [respectively to the target x^{i*}] asymptotically.*

The proof is similar to the proof of Theorem 2. The Condition 3 guarantees the feasibility of the previous optimizing trajectory for the next step of optimization. The formation stability can be achieved via two-stage manipulation. First, the vehicles should move to have the targets exposed to them by a certain time T . After that time, if the local MPC of each vehicle satisfies Theorem 3, then the entire formation will converge to the target formation.

Condition 3 in Theorem 3 requires non-shrinking state constraint set so that the feasibility and stability can be guaranteed. In case that, at some time k , one vehicle were assigned with an overly large $X^i(k)$, Condition 3 would require all other vehicles to leave this large set $X^i(k)$ intact. This may lead to limited behavior of the rest of the formation or even may cause infeasibility at other vehicles. Hence, this condition should be guaranteed by careful off-line design of the reference trajectories, which might itself be a difficult problem. The following condition may be used to replace the Condition 3:

(3') define $C^i(k)$ to be some convex set such that $\{x_{k+2,k}^{i,opt}, x_{k+3,k}^{i,opt}, \dots, x_{k+N,k}^{i,opt}\} \in C^i(k)$ from the optimization at time k , the state constraint set at time $k+1$ should satisfy $C^i(k) \subseteq X^i(k+1)$.

Comparing to the Condition 3, Condition 3' allows the real-time change (shrinking) of the state constraint set $X^i(k)$ and the convergence of vehicle i to the target x^{i*} is guaranteed as long as the set $C^i(k)$ is preserved from time k to time $k+1$, which can be less demanding than Condition 3. Condition 3' should be guaranteed online by posing the reservation of $C^i(k)$ as constraints to the neighboring vehicles, but this will raise the load of the real-time communication and the computation.

3.3.2 Formulation of the Stochastic Case

Disturbances to the measurement of the neighboring vehicles' states may exist and usually are caused by the communication finite-bit quantization effect and channel noises. Random variables with compact support can be a good model for this type of disturbances. In this case, instead of the exact π_k^i , the available measurement at the i th vehicle is

$$z_k^i = \pi_k^i + v_k^i.$$

Thus, state estimators for neighbors' states based on the collection of z_k^i are necessary. If the process noise w_k^i is also in the picture, each vehicle needs a self-estimator as well. The local MPC problem must be posed with these estimates.

If the estimates were used to replace the true values in the constrained MPC problem, then the optimizing control sequence could guarantee that the constraints were satisfied only by the estimated state process. If such a control law were applied to the vehicle formation, then the disturbance could drive the real state to violate the constraints even though the estimated state satisfied the constraints.

In order to use the estimates in the MPC problem, the constraints for the deterministic case should be tightened to reflect the necessary stand-off to buffer the uncertainty. We propose the same strategy as in (Yan & Bitmead 2003) which applied

the MPC formulation posed in (Yan & Bitmead 2005) to the vehicle formation control problem.

The MPC problem is formulated in a stochastic way:

$$\begin{aligned} \min_{u_k^i, \dots, u_{k+N-1}^i} J^i(x_k^i, u_k^i, \dots, u_{k+N-1}^i) &= \min E_{\mathcal{Z}_k^i} \left\{ \sum_{j=0}^{N-1} l_k^i(x_{k+j}^i, u_{k+j}^i) + F_k^i(x_{k+N}^i) \right\}, \\ \text{subject to: } x_{k+j+1}^i &= f^i(x_{k+j}^i, u_{k+j}^i) + w_{k+j}^i, \\ u_{k+j}^i &\in U^i, \\ P_{\mathcal{Z}_k^i}(x_{k+j}^i \in X^i(\pi_k^i)) &= 1, \tag{3.4} \\ P_{\mathcal{Z}_k^i}(x_{k+N}^i \in X_f^i) &= 1. \tag{3.5} \end{aligned}$$

The set $\mathcal{Z}_k^i = \{z_0^i, z_1^i, \dots, z_k^i, u_0^i, u_1^i, \dots, u_k^i\}$ is the collection of measurements and controls collected by Vehicle i up to time k . The *almost sure* state constraints (3.4) and (3.5) are posed in conditional probability given \mathcal{Z}_k^i , since the disturbances are modelled as random variables with compact support.

In Chapter 2, it was shown that, for a linear system with linear inequality constraints and normally distributed noise signals, the probabilistic state constraints can be recast as deterministic constraints on the predicted states. Hence, the probabilistically constrained MPC can be recast as a deterministic quadratic programming problem for which efficient algorithms exist. The same procedure is applicable with noises of compact support and almost sure constraints. The inequality constraint values are recast from probabilistic constraints on x_{k+j}^i to deterministic constraints on $\hat{x}_{k+j|k}^i$, the state predictions. These modified constraints are tighter than the originals and involve the state estimate's support or covariance. This provides a direct connection between information architecture and control performance. The example of Section 3.4 will illustrate this.

3.3.3 String Stability

Theorems 1 and 2 deal with the asymptotic stability of the constrained MPC problem in the absence of disturbance on the states or measurement. (Swaroop & Hedrick 1996) defined string stability of a platoon of vehicles to be the re-convergence of the entire platoon to their normal operating positions after an isolated disturbance to each

of the vehicles. Similarly, (Yanakiev & Kanellakopoulos 1996) require diminution of such disturbances along the one-dimensional vehicle string from the lead.

In our formulation, disturbances enter through state process noise and through communications quantization and channel noises, which have the prospect of introducing disturbances throughout the fleet. Thus the propagation of these disturbances needs consideration.

Theorem 4. *If the disturbances w_k^i, v_k^i do not cause the MPC no-collision constraints to become active then the fleet state X_k converges to the neighborhood of X^* given by the decoupled controlled response of the individual vehicles to their self-noises.*

Proof: If the no-collision MPC constraints are inactive, then $u_k^i = u_k^i(\hat{x}_{k|k}^i) = u_k^i(\hat{x}_{k|k}^i(w_k^i, v_k^i))$. Thus, x_k^i depends only on self-noises and follows a decoupled response. ■

This theorem gives some insight into stability of the fleet. Provided disturbances are not so large as to cause the strong dependence on (uncertain) positions of neighbors, there is no effect. Likewise if the stationary positions X^* are feasibly achievable with inactive state constraints in the limit, then there will be a time beyond which sufficiently small disturbances will not propagate. There are clear ties to the concept of weak coupling systems and the string instability of (Swaroop & Hedrick 1996). Here the key observation is that the condition of Theorem 4 for inactivity of no-collision constraints ties together fleet performance embodied in X^* and the information architecture captured through the tightened constraints, which reflect the support or covariance of the state estimate distribution. These inter-relations will become more evident in the next section.

3.4 Application to a 1-D Vehicle Formation Problem

3.4.1 1-D Vehicle Formation Problem

In this section, we consider a simple vehicle formation problem that consists of three locomotives on a single track operating with communications links and seeking to maintain a one-dimensional formation.

Dynamics: The locomotives' dynamics are identical and are described by a

simple integrator without process noise.

$$x_{k+1}^i = x_k^i + u_k^i, \quad i = 1, 2, 3,$$

where x_k^i is the absolute position of Vehicle i , and u_k^i represents its controlled velocity.

Information Architecture: Each vehicle has perfect knowledge of its own state and control. This information is communicated to the immediately following vehicle only so that,

$$z_k^i = \pi_k^i + \begin{pmatrix} \nu_k^i \\ \mu_k^i \end{pmatrix} = \begin{pmatrix} x_k^{i-1} \\ u_k^{i-1} \end{pmatrix} + \begin{pmatrix} \nu_k^i \\ \mu_k^i \end{pmatrix}, \quad i = 2, 3,$$

where ν_k^i and μ_k^i are white noises with $\nu_k^i \sim U[-\frac{V}{2}, \frac{V}{2}]$ and $\mu_k^i \sim U[-\frac{U}{2}, \frac{U}{2}]$, $U, V > 0$, capturing the communication noises and the communication bandwidth assignment.

Local MPC Problems:

$$\text{MPC-1:} \quad u_k^1 = \arg \min_{u_k^1} (x_{k+1}^1 - r_{k+1}^1)^2 \quad (3.6)$$

$$\text{MPC-2\&3:} \quad u_k^i = \arg \min_{u_k^i} E_{\mathcal{X}_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \quad (3.7)$$

$$[\text{no-collision}] \quad \text{subject to: } P_{\mathcal{X}_k^i} (x_{k+1}^i \leq x_{k+1}^{i-1}) = 1, \quad i = 2, 3. \quad (3.8)$$

The local control task is that each vehicle tracks its own reference signal r_k^i subject to the almost sure no-collision constraints, except for the first vehicle which has no constraints. The cost function is a conditional expectation and the reference signal in this problem replaces the target x^{i*} . For this coordination problem, the horizon $N = 1$ is sufficient and for simplicity, there is no constraint posed on the control u^i . Although the noise signals have uniform distribution rather than gaussian distribution, the ideas of Chapter 2 still apply to convert the probabilistic constraint (3.8) to deterministic.

Reference Trajectories: We consider two problem formulations which are distinguished by their reference specification. Define the fleet reference trajectory,

$$r_k^* = 250k, \quad 0 \leq k \leq 3,$$

where distance, r_k^* , is measured in kilometers and time, k , in hours. The two strategies are

- Leader-Follower Strategy: $r_k^1 = r_k^*$, $r_k^2 = x_k^1 - d_1$, $r_k^3 = x_k^2 - d_2$.
- Fixed-Reference Strategy: $r_k^1 = r_k^*$, $r_k^2 = r_k^1 - d_1 = r_k^* - d_1$, $r_k^3 = r_k^2 - d_2 = r_k^* - d_1 - d_2$.

Note that fixed-reference differs from the leader-follower strategy through the provision of absolute reference trajectories in place of relative positioning. This change will be shown to affect string stability through the achievement of decoupled control when constraints are inactive, as described in Theorem 3.

Performance indices: To evaluate these local controllers, the following performance indices are considered:

- String stability: we consider the effect of accumulating noises in the vehicle's closed-loop trajectory. The standard deviation of $detrend(x^i) = x^i - trend(x^i)$ is the index for string stability, where $trend(x^i)$ in this simulation is the best line fit of the resulting x^i . If this quantity is decreasing from Vehicle i to Vehicle $i + 1$ by a factor less than one, then the formation has string stability.
- Inter vehicle spacing: this implies efficiency and is preferred to be small in many circumstances. For example, in an automated highway system, the smaller distance between vehicles the higher is the efficiency of using the road.

3.4.2 Design Parameters and Simulation

This simple one-dimensional coordination problem exposes a number of design variables and, importantly for the purpose of this paper, their simple and conceptual interaction to achieve system performance. These design parameters are: control structure, examined in terms of leader-follower or fixed-reference design; information architecture, connected to communication link assignment, which is fixed here; target fleet structure, specified by separations d_1 and d_2 ; communication resource allocation, captured by noise bounds U and V .

As shown in the Appendix, the closed-loop states under leader-follower control

are:

$$\begin{aligned} x_{k+1}^1 &= r_{k+1}^*, \\ x_{k+1}^2 &= r_{k+1}^* - \max\left\{d_1, \frac{U+V}{2}\right\} + (\nu_k^2 + \mu_k^2), \end{aligned} \quad (3.9)$$

$$\begin{aligned} x_{k+1}^3 &= x_{k+1}^2 - \max\left\{d_2, \frac{U+V}{2}\right\} + (\nu_k^3 + \mu_k^3) \\ &= r_{k+1}^* - \max\left\{d_1, \frac{U+V}{2}\right\} - \max\left\{d_2, \frac{U+V}{2}\right\} + (\nu_k^2 + \mu_k^2 + \nu_k^3 + \mu_k^3). \end{aligned} \quad (3.10)$$

Under fixed-reference control, the closed-loop states are:

$$\begin{aligned} x_{k+1}^1 &= r_{k+1}^*, \\ x_{k+1}^2 &= r_{k+1}^* - \max\left\{d_1, \frac{U+V}{2} - (\nu_k^2 + \mu_k^2)\right\} \end{aligned} \quad (3.11)$$

$$\begin{aligned} x_{k+1}^3 &= \min\left\{r_{k+1}^3, x_{k+1}^2 - \frac{U+V}{2} + (\nu_k^3 + \mu_k^3)\right\} \\ &= r_{k+1}^* - \max\left\{d_1 + d_2, \frac{U+V}{2} + \max\left\{d_1, \frac{U+V}{2} - (\nu_k^2 + \mu_k^2)\right\} - (\nu_k^3 + \mu_k^3)\right\}. \end{aligned} \quad (3.12)$$

For demonstration, we first consider two choices of design parameters to illustrate the achievable performance of these two strategies with fixed $U = V = 80$.

Case I: Minimum Distance Task. That is to keep the three trains as close as possible. The parameters should be $d_1 = d_2 = 0$ and the fixed references $r^2 = r^3 = r^1$. The resulting trajectories of the two strategies are given in Figure 3.1. In this task, the two strategies give the same controller for each vehicle and hence their resulting plots are identical.

Remarks:

- Both strategies avoided collision. Vehicle 1 sits right on r^1 while Vehicle 2 and 3 try to stay close to Vehicle 1. However, due to the effect of the constraints, they are forced back to avoid collision.
- String instability occurs in both strategies. As shown in the figure, the trajectories of Vehicle 2 and Vehicle 3 look like random walks with a trend. We compute the standard deviation of $detrend(x^i)$ as an index of the string stability. The results are 0 for Vehicle 1, 31.1779 for Vehicle 2, and 51.2069 for Vehicle 3. For larger formations, this quantity will keep increasing and the resulting behavior of the

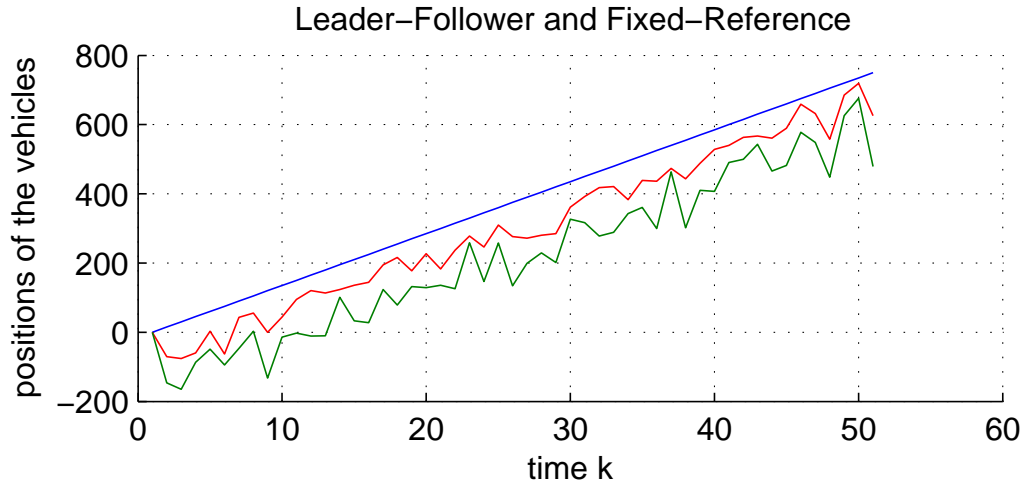


Figure 3.1: Identical result of both strategies for the minimum distance task.

later vehicles is unacceptable. Another explanation of the string instability is that a higher degree of the string instability implies a larger amount of consumed control energy. Therefore, adding control penalty terms in the MPC criterion function may relieve the string instability with the possible trading off the ability of tracking.

Case II: Fixed Separation Task. That is to have the three vehicles travelling a certain distance apart. The parameters are $d_1 = d_2 = 160$ and the fixed references $r^2 = r^1 - d_1$ and $r^3 = r^2 - d_2$. The resulting trajectories of the two strategies are given in Figure 3.2. As seen in the plots, string instability still exists in the leader-follower strategy with the same $\detrend(x^i) = 0, 31.1779, 51.2069$ for Vehicle 1, 2, and 3. The fixed-reference strategy achieved string stability with $\detrend(x^i) = 0, (i = 1, 2, 3)$.

In this framework, string instability occurs when the controls introduce estimates, and hence communication noises, into the closed-loop behavior. In the leader-follower strategy, the controls of Vehicle 2 and 3 always do so; while the fixed-reference strategy can achieve string stability when the constraints are inactive. This can be inferred from the closed-loop system equations given earlier. For leader-follower strategy, the closed-loop behaviors of Vehicles 2 and 3 are governed by (3.9) and (3.10) and the communication noises are added onto the Vehicles' trajectories no matter the choices of d_1 and d_2 . For fixed-reference strategy, the closed-loop behaviors of Vehicles 2 and 3 are governed by (3.11) and (3.12). By increasing d_1 and d_2 , the effect of string instability

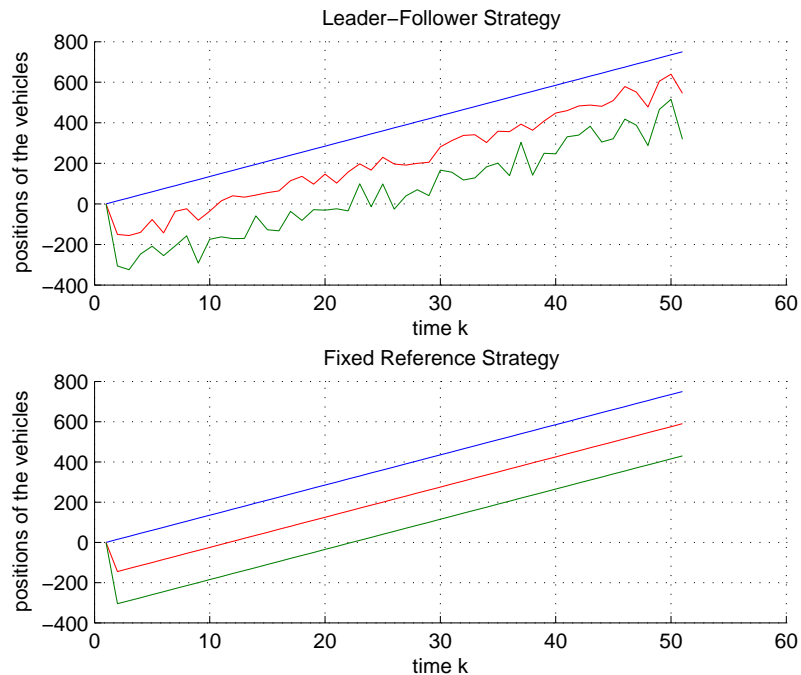


Figure 3.2: Comparison of the two strategies in the fixed separation task with $d^i > U + V$.

diminishes and when $d_1, d_2 \geq U + V$, the resulting closed-loop trajectories of Vehicles 2 and 3 are their assigned reference trajectories r_k^2 and r_k^3 .

The simulation results of the fixed-reference strategy in Case I and Case II showed that string stability and small inter-vehicle spacing cannot be achieved simultaneously. The reason is the large noise bounds U and V that correspond to low-quality communications and low communication cost. In Case I, the zero inter-vehicle spacing and the large communication noises keep the modified no-collision constraint always active and string instability is always present. In Case II, we separate reference signals from each other in order to maintain the constraints inactive. Though string instability can be removed, the separations d_1 and d_2 are large. Thus, to achieve string stability and small separations simultaneously we must increase the communication accuracy and we consider the following case.

Case III: Fixed Separation Task with High-quality Communications.

To increase the accuracy of the local information, we set $U = V = 10$ and $d_1 = d_2 = 20$. The result is shown in the Figure 3.3. The vehicles are able to stay close while string

stability is achieved. This high performance is due to the high bandwidth communication.

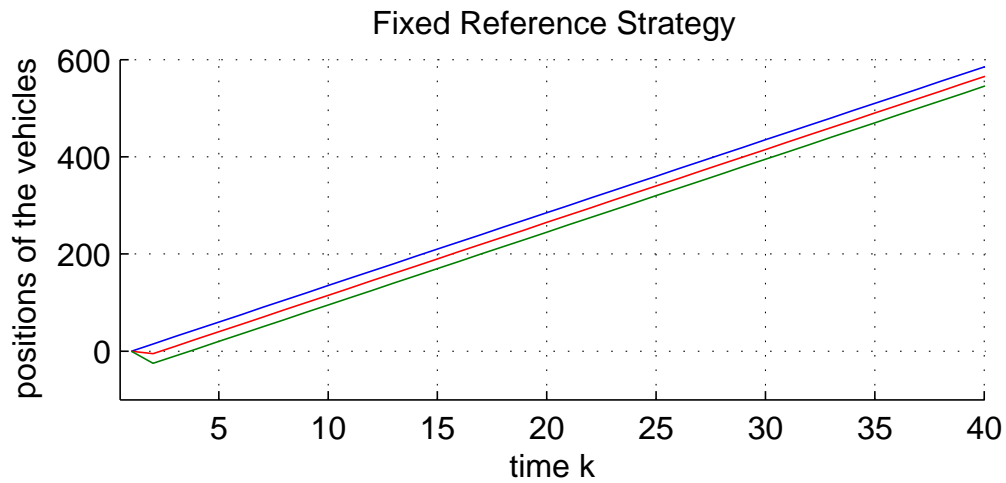


Figure 3.3: Comparison of the two strategies in the fixed separation task.

Summary

1. In both minimum-distance and fixed-separation tasks, the constrained MPC approach in both strategies guarantees the no-collision requirement.
2. With the presence of disturbances, as stated in the Theorem 3, the fixed-reference strategy can completely remove string instability by choosing the proper references that deactivate the constraints. The closed-loop dynamics are decoupled. The determination of these proper references needs off-line work or online supervisor action.
3. The leader-follower strategy always display string instability with disturbance.
4. The inactivity of the constraints in the fixed-reference MPC relates the communication cost to the control performance.

3.5 A 2-D No-collision Constraints

In this section we formulate a two dimensional version of the vehicle no-collision constraint for the local MPC of vehicle i . Consider the no-collision constraint with a large probability between vehicle i and vehicle j :

$$P(\text{no collision between vehicle } i \text{ and } j) \geq p_1 \gg 0. \quad (3.13)$$

Our goal is to reach an explicit separation posed on the predicted positions of the two vehicles:

$$\| \hat{y}_{i,k+l|k}^{v_i} - \hat{y}_{i,k+l|k}^{v_j} \| > \rho_{ij,l}, \quad (3.14)$$

where $\hat{y}_i^{v_j}$ is the predicted position of vehicle j known to vehicle i with prediction error covariance Σ_i^j (both known and computed by the cross-estimator run by vehicle i). Vehicle i 's own predicted position $\hat{y}_i^{v_i}$ is to be determined by its MPC control sequence. However, its covariance can be known if this covariance is independent of the control, as discussed in Chapter 2 this is always true for the linear systems. Therefore, we need to derive the proper $\rho_{ij,l}$ so that (3.14) implies (3.13).

Consider these two vehicles as in Figure 3.4, we choose $\rho_{ij,l}$ as the following:

1. Define ellipses $E_i^{v_j} : (y - \hat{y}_i^{v_j})^T (\Sigma_i^j)^{-1} (y - \hat{y}_i^{v_j}) \leq \chi_p^2$ and $E_i^{v_i} : (y - \hat{y}_i^{v_i})^T (\Sigma_i^i)^{-1} (y - \hat{y}_i^{v_i}) \leq \chi_p^2$ such that $P(y^{v_j} \in E_i^{v_j}) = P(y^{v_i} \in E_i^{v_i}) = p_1$ where $p_1 = \frac{(1+p_0)}{2}$. These two ellipses can be regarded as the region of uncertainty related to the predicted values.
2. The circumscribed circle of $E_i^{v_j}$ is $C_i^{v_j}$ centered at $\hat{y}_i^{v_j}$ with radius $r_{ji} = \sqrt{\lambda_{max}(\chi_p^2 \Sigma_i^j)}$ where $\lambda_{max}(\cdot)$ means the maximum eigenvalue. Similarly, the circumscribed circle of $E_i^{v_i}$ is $C_i^{v_i}$ centered at $\hat{y}_i^{v_i}$ with radius $r_{ii} = \sqrt{\lambda_{max}(\chi_p^2 \Sigma_i^i)}$. When the prediction errors are isotropic, the ellipses are identical to the circles.
3. Define

$$\rho_{ij} = r_{ji} + r_{ii} = \sqrt{\lambda_{max}(\chi_p^2 \Sigma_i^j)} + \sqrt{\lambda_{max}(\chi_p^2 \Sigma_i^i)}, \quad (3.15)$$

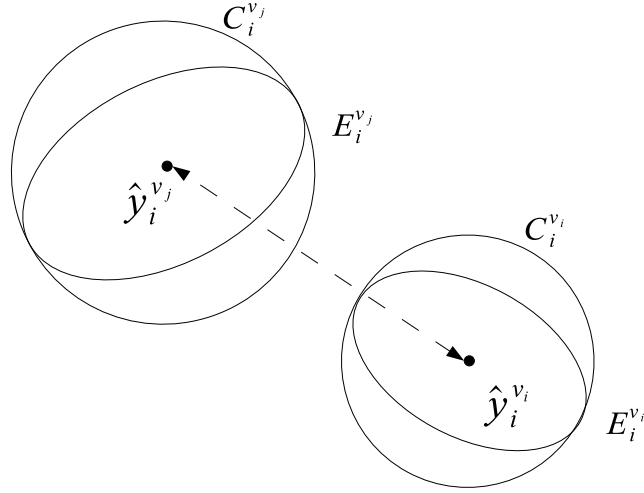


Figure 3.4: Illustration of the deterministic no-collision constraint between two vehicles.

hence if the constraint (5.12) is assured, the two circles in Figure (3.4) do not overlap. Therefore,

$$\begin{aligned}
 P(\text{collision}) &< P(y^{v_j} \notin C_i^{v_j} \text{ or } y^{v_i} \notin C_i^{v_i}) \leq P(y^{v_j} \notin E_i^{v_j} \text{ or } y^{v_i} \notin E_i^{v_i}) \\
 &\leq P(y^{v_j} \notin E_i^{v_j}) + P(y^{v_i} \notin E_i^{v_i}) = 2(1 - p_1), \\
 P(\text{no collision}) &= 1 - P(\text{collision}) > 2p_1 - 1 = p_0.
 \end{aligned}$$

That is, whenever the constraint (3.14) holds, (3.13) must hold. The chosen parameters for (3.14) make it a sufficient but not necessary condition for (3.13).

Note that the disturbances in a practical vehicle formation have uniform distribution or, more generally, bounded distribution, this control solution works because, by choosing the right covariance of the model gaussian noises, the large probability constraint (3.13) can be replaced by the almost sure (with probability one) constraint as

discussed in the earlier sections. Then in step one, p_1 becomes 1 and the choice of $\rho_{ij,l}$ (3.15) obviously guarantees no-collision.

The deterministic no-collision constraint (3.14) makes the local MPC problem a nonlinear and non-convex problem. Though the QP routine is not applicable any longer, when dealing with this kind of problem, we should first consider some other optimization tools such as *fmincon* or the mixed integer programming as suggested in (Richards & How 2004b) instead of discarding the MPC strategy completely.

3.6 Conclusion

In this paper, we examine the application of constrained model-predictive approach to a coordinated control problem. The direct inclusion of constraints makes the design easy and the global properties can be deduced from the local properties that are guaranteed by the local constrained MPC controllers.

We discussed the multi-vehicle coordination problem as a working example. The local constrained MPC controllers have fixed reference trajectories in their criterion functions and direct inclusion of no-collision constraints coupling the vehicles' dynamics. For deterministic problems, the asymptotic stability of each vehicle can be guaranteed via the local constrained MPC controller formulations and the global formation stability follows as summarized in Theorems 1 and 2. When stochastic disturbances are in the picture, the constraints of MPC should be modified to reflect the uncertainty so that collisions can be prevented. Theorem 3 provides a way to completely reject disturbance via decoupling the vehicles' closed-loop dynamics. As shown in the 1-D problem, the inclusion of fixed references and modified constraints is proven to be able to decouple the vehicles in their closed-loop dynamics via keeping the modified constraints inactive and thereby to achieve string stability. The relation revealed between the control performance and the communication disturbance is a guide to design.

This chapter includes the reprints of the following papers:

Jun Yan, Robert R. Bitmead - *A Constrained Model-Predictive Approach to Coordinated Control*, Automatica, Submitted, Apr. 2006.

Jun Yan, Robert R. Bitmead - *Coordinated Control and Information Architecture*, 42nd IEEE Conference on Decision and Control, in Maui, Hawaii, USA, Dec. 2003.

The dissertation author was the primary author listed in these publications. And the co-author, Professor Bitmead, directed and supervised the research.

3.7 Appendix

3.7.1 Fixed-Reference MPC Formulation and Solution

The local MPC problem for Vehicle 1:

$$\min_{u_k^1} J(x_k^1, u_k^1) = \min_{u_k^1} (x_{k+1}^1 - r_{k+1}^1)^2, \quad (3.16)$$

with the solution $u_k^1 = -x_k^1 + r_{k+1}^1$.

The local MPC problem for Vehicle 2 or 3:

$$\left. \begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= \min_{u_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \\ \text{subject to: } P_{\mathcal{Z}_k^i}(x_{k+1}^i \leq x_{k+1}^{i-1}) &= 1, \quad i = 2, 3, \end{aligned} \right\}. \quad (3.17)$$

To change (3.17) into a deterministic problem involving estimates, first we choose the unbiased estimates of the other vehicle at Vehicle i to be

$$\begin{aligned} \hat{x}_{k|k}^{i-1} &= [1 \ 0] z_k^i, \\ \hat{u}_{k|k}^{i-1} &= [0 \ 1] z_k^i, \\ \hat{x}_{k+1|k}^{i-1} &= \hat{x}_{k|k}^{i-1} + \hat{u}_{k|k}^{i-1} = [1 \ 1] z_k^i. \end{aligned}$$

The corresponding estimate errors are

$$\begin{aligned} \tilde{x}_{k|k}^{i-1} &= x_k^{i-1} - \hat{x}_{k|k}^{i-1} = -\nu_k^i, \\ \tilde{u}_{k|k}^{i-1} &= u_k^{i-1} - \hat{u}_{k|k}^{i-1} = -\mu_k^i, \\ \tilde{x}_{k+1|k}^{i-1} &= x_{k+1}^{i-1} - \hat{x}_{k+1|k}^{i-1} = -\nu_k^i - \mu_k^i. \end{aligned}$$

Since $\nu_k^i \sim U[-\frac{V}{2}, \frac{V}{2}]$, $\mu_k^i \sim U[-\frac{U}{2}, \frac{U}{2}]$, it follows that $\tilde{x}_{k+1|k}^{i-1} \in [-\frac{U+V}{2}, \frac{U+V}{2}]$.

Hence:

$$P_{\mathcal{Z}_k^i}(x_{k+1}^i \leq x_{k+1}^{i-1}) = 1, \Leftrightarrow x_{k+1}^i \leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}.$$

The $\frac{U+V}{2}$ term is the necessary stand-off due to the uncertainty introduced by using $\hat{x}_{k+1|k}^{i-1}$. Therefore, the local MPC problems (3.17) for Vehicles 2 and 3 become

$$\begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= \min_{u_k^i} (x_{k+1}^i - r_{k+1}^i)^2, \\ \text{subject to: } x_{k+1}^i &\leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}, \quad i = 2, 3, \end{aligned} \quad (3.18)$$

Note that, in this transformation, the lower bounds of ν , μ , and $\hat{x}_{k+1|k}^{i-1}$ are of importance.

The solution of (3.18) can be easily computed:

$$u_k^2 = \min \left\{ -x_k^2 + r_{k+1}^2, -x_k^2 + \hat{x}_{k+1|k}^1 - \frac{U+V}{2} \right\}, \quad (3.19)$$

$$u_k^3 = \min \left\{ -x_k^3 + r_{k+1}^3, -x_k^3 + \hat{x}_{k+1|k}^2 - \frac{U+V}{2} \right\}. \quad (3.20)$$

3.7.2 Leader Follower MPC Formulation and Solution

The criterion function in the leader-follower MPC problem for Vehicle 2 and 3 should be changed to:

$$\begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= E_{\mathcal{X}_k^i} [x_{k+1}^i - (x_{k+1}^{i-1} - d_{i-1})]^2, \\ \text{subject to: } P_{\mathcal{X}_k^i} &(x_{k+1}^i \leq x_{k+1}^{i-1}) = 1, \quad i = 2, 3. \end{aligned}$$

The criterion function is posed as a conditional expectation due to the inaccuracy of x^{i-1} at Vehicle i . By the same process, this MPC problem is equivalent to:

$$\begin{aligned} \min_{u_k^i} J(x_k^i, u_k^i) &= [x_{k+1}^i - (\hat{x}_{k+1|k}^{i-1} - d_{i-1})]^2, \\ \text{subject to: } x_{k+1}^i &\leq \hat{x}_{k+1|k}^{i-1} - \frac{U+V}{2}, \quad i = 2, 3, \end{aligned}$$

and the solution is:

$$u_k^2 = \min \left\{ -x_k^2 + \hat{x}_{k+1|k}^1 - d_1, -x_k^2 + \hat{x}_{k+1|k}^1 - \frac{U+V}{2} \right\}, \quad (3.21)$$

$$u_k^3 = \min \left\{ -x_k^3 + \hat{x}_{k+1|k}^2 - d_2, -x_k^3 + \hat{x}_{k+1|k}^2 - \frac{U+V}{2} \right\}. \quad (3.22)$$

Chapter 4

Cross-estimator Design with Known Coupled Linear Feedback

4.1 Introduction

As discussed in Chapter 3 in a multi-system coordination context, each local controller requires knowledge of its neighbors' state information that is usually derived from state-estimators. We call this type of state-estimator for neighbors the *cross-estimator*. This chapter is dedicated to the minimum covariance cross-estimator design problem with known coupled linear feedback control laws. The working problem of this chapter is a simple, but generalizable, two-vehicle formation. The local control laws are assumed to be linear feedbacks of both state vectors (actually the local state estimates). This kind of mixed feedback control is usual in the decentralized coordinated control context (Šiljak 1990).

In a classical information architecture, each vehicle needs the state measurements and the control values of the other to calculate a shared common state estimate, e.g. Kalman filter (Anderson & Moore 1979). This chapter studies a non-classical type. The control values cannot be transmitted, instead the control laws (i.e. feedback gains) are known to both vehicles. Each vehicle runs a Kalman filter to estimate its own state and needs to design an observer for the states of the other vehicle (cross-estimator). The key issues are the calculation of observer gains and covariance matrices

of the observer errors. With the non-standard information structure, the cross-estimator design problem can also be regarded as a state estimation problem with unknown inputs, which has been studied sporadically throughout last thirty years, e.g. (Meditch & Hostetter 1971, Johnson 1975, Akhenak, Chadli, Maquin & Ragot 2004). The main tack is to assume the observer structure and to use the available knowledge of the control, such as a model capturing the control commands.

In this chapter, we assume that the cross-estimators take a similar form to a Kalman filter except for the exact control values and the observer gain matrices. As only control laws are given, the observers use the feedback of the best estimate at hand. Because of the mixed structure of the controllers, the observer errors depend on each other as well as the Kalman filter errors. As a result, the covariance matrix calculation of each observer error is much more complicated than that of a Kalman filter. Our method is to write down an augmented error system including the two Kalman filter errors and the two cross-estimator errors. The covariance matrix of the augmented system has a clean form and linear matrix inequality techniques can help to find a related optimal observer gain that minimizes a function of the estimation error covariance while taking care of the stability requirement in steady state. Details are developed in Section 4.2 following by a scalar example in Section 4.3.

4.2 Problem Formulation

This section is devoted to a simple two-vehicle formation problem. Our focus is to formulate a reasonable cross state estimator according to the non-classical information structure, which does not allow transmitting the control actions but instead assumes the state feedback control laws are known to all parties a priori.

The dynamics and the measurement of the vehicles can be described as follows:

Vehicle 1

$$x_{k+1}^1 = A_1 x_k^1 + B_1 u_k^1 + w_k^1, \quad (4.1)$$

$$y_{1,k}^1 = C_{11} x_k^1 + v_{1,k}^1, \quad (4.2)$$

$$y_{1,k}^2 = C_{12} x_k^2 + v_{1,k}^2. \quad (4.3)$$

Vehicle 2

$$x_{k+1}^2 = A_2 x_k^2 + B_2 u_k^2 + w_k^2, \quad (4.4)$$

$$y_{2,k}^1 = C_{21} x_k^1 + v_{2,k}^1, \quad (4.5)$$

$$y_{2,k}^2 = C_{22} x_k^2 + v_{2,k}^2. \quad (4.6)$$

Where x stands for the state, y for the measurement, w for the process noise with $w_k^i \sim N(0, Q_i)$ and v for measurement noise with $v_{j,k}^i \sim N(0, R_{ji})$. There are superscripts and subscripts throughout this chapter with superscripts meaning ‘of’ and subscripts meaning ‘at’; for example, $y_{2,k}^1$ is the measurement of vehicle 1 taken by (at) vehicle 2.

There are four state estimators based on different sets of measurements. The state estimators using y_1^1 and y_2^2 are standard Kalman filters,

Kalman Filter 1@1:

$$\begin{aligned} \hat{x}_{1,k+1|k}^1 &= A_1 \hat{x}_{1,k|k}^1 + B_1 u_k^1, \\ \Sigma_{1,k+1|k}^1 &= A_1 \Sigma_{1,k|k}^1 A_1^T + Q_1, \\ M_{11} &= A_1 \Sigma_{1,k|k}^1 C_{11}^T (C_{11} \Sigma_{1,k+1|k}^1 C_{11}^T + R_{11})^{-1}, \\ \hat{x}_{1,k+1|k+1}^1 &= \hat{x}_{1,k+1|k}^1 + M_{11} (y_{1,k+1}^1 - C_{11} \hat{x}_{1,k+1|k}^1), \\ \Sigma_{1,k+1|k+1}^1 &= \Sigma_{1,k+1|k}^1 - \Sigma_{1,k|k}^1 C_{11}^T (C_{11} \Sigma_{1,k+1|k}^1 C_{11}^T + R_{11})^{-1} C_{11} \Sigma_{1,k+1|k}^1. \end{aligned} \quad (4.7)$$

Kalman Filter 2@2:

$$\begin{aligned} \hat{x}_{2,k+1|k}^2 &= A_2 \hat{x}_{1,k|k}^1 + B_2 u_k^2, \\ \Sigma_{2,k+1|k}^2 &= A_2 \Sigma_{2,k|k}^2 A_2^T + Q_2, \\ M_{22} &= A_2 \Sigma_{2,k+1|k}^2 C_{22}^T (C_{22} \Sigma_{2,k+1|k}^2 C_{22}^T + R_{22})^{-1}, \\ \hat{x}_{2,k+1|k+1}^2 &= \hat{x}_{2,k+1|k}^2 + M_{22} (y_{2,k+1}^2 - C_{22} \hat{x}_{2,k+1|k}^2), \\ \Sigma_{2,k+1|k+1}^2 &= \Sigma_{2,k+1|k}^2 - \Sigma_{2,k+1|k}^2 C_{22}^T (C_{22} \Sigma_{2,k+1|k}^2 C_{22}^T + R_{22})^{-1} C_{22} \Sigma_{2,k+1|k}^2. \end{aligned} \quad (4.8)$$

The cross-estimators based on y_2^1 and y_1^2 are more interesting and the major difficulty is the lack of the knowledge between the two vehicles about the controls. Nevertheless, these two estimators take an observer form similar to a Kalman filter,

Estimator 1@2:

$$\begin{aligned}\hat{x}_{2,k+1|k}^1 &= A_1 \hat{x}_{2,k|k}^1 + B_1 \bar{u}_k^1, \\ \hat{x}_{2,k+1|k+1}^1 &= \hat{x}_{2,k+1|k}^1 + M_{21}(y_{2,k+1}^1 - C_{21} \hat{x}_{2,k+1|k}^1).\end{aligned}\tag{4.9}$$

Estimator 2@1:

$$\begin{aligned}\hat{x}_{1,k+1|k}^2 &= A_2 \hat{x}_{1,k|k}^2 + B_2 \bar{u}_k^2, \\ \hat{x}_{1,k+1|k+1}^2 &= \hat{x}_{1,k+1|k}^2 + M_{12}(y_{1,k+1}^2 - C_{12} \hat{x}_{1,k+1|k}^2).\end{aligned}\tag{4.10}$$

Remarks:

- The controls applied in these two estimators \bar{u}^1 and \bar{u}^2 are different from the real values of u^1 and u^2 . Suitable choices of \bar{u}^1 and \bar{u}^2 should be decided according to the off-line knowledge about u^1 and u^2 and the associated control laws together with the state cross-estimates.
- The observer gain matrices M_{21} and M_{12} are unknown at the moment. It will be shown soon that the gains determine the state error covariance matrices, i.e. the performance measure, of these two cross estimators. Hence the task of this section is seeking design values of M_{21} and M_{12} .
- The state estimate covariance has an effect on control performance. Studying the covariances off line will enable us to manage them (and hence the control performance) via adjusting/designing the proper information architecture (e.g. adjusting R_{ij} which is directly tied to the assigned communication channel capacity).

The key issue for *Estimator 1@2* and *Estimator 2@1* to work is the knowledge of the control. In this section, each vehicle will apply a state (estimate) feedback control law involving both its own and the other's state estimate. At this stage, we assume this control to be linear and, since coordination is involved, to include all local state estimates.

$$\begin{aligned}u_k^1 &= K_{11} \hat{x}_{1,k|k}^1 + K_{12} \hat{x}_{2,k|k}^2 + l_1, \\ u_k^2 &= K_{21} \hat{x}_{2,k|k}^1 + K_{22} \hat{x}_{1,k|k}^2 + l_2.\end{aligned}$$

The control gains K_{ij} and the additive constant vectors l_i are known to both of the vehicles. Then it is reasonable to apply the estimated control values

$$\begin{aligned}\bar{u}_k^1 &= K_{11}\hat{x}_{2,k|k}^1 + K_{12}\hat{x}_{2,k|k}^2 + l_1, \\ \bar{u}_k^2 &= K_{21}\hat{x}_{1,k|k}^1 + K_{22}\hat{x}_{1,k|k}^2 + l_2.\end{aligned}$$

Usually \bar{u}^i and u^i are not the same. Hence the ‘cross estimation’ will have a covariance larger than that of the Kalman filter, which should be minimized or at least bounded as per requirement.

To derive the covariance matrix of the estimates, an expression for the estimate errors should be derived first. As usual define $\tilde{x}_j^i = x^i - \hat{x}_j^i$, then

Filtering Error Equations:

$$\tilde{X}_{k+1} = \bar{M}_r \bar{A} \tilde{X}_k + \bar{M}_r \bar{B} w_k - \bar{M} v_{k+1}, \quad (4.11)$$

where

$$\begin{aligned}\tilde{X}_k &= \begin{bmatrix} \tilde{x}_{1,\cdot}^1 \\ \tilde{x}_{2,\cdot}^1 \\ \tilde{x}_{1,\cdot}^2 \\ \tilde{x}_{2,\cdot}^2 \end{bmatrix}, w_k = \begin{bmatrix} w_k^1 \\ w_k^2 \end{bmatrix}, v_{k+1} = \begin{bmatrix} v_{1,k+1}^1 \\ v_{2,k+1}^1 \\ v_{1,k+1}^2 \\ v_{2,k+1}^2 \end{bmatrix}, \bar{M}_r = (I - \bar{M}\bar{C}), \\ \bar{A} &= \begin{bmatrix} A_1 & 0 & 0 & 0 \\ -B_1 K_{11} & (A_1 + B_1 K_{11}) & -B_1 K_{12} & B_1 K_{12} \\ B_2 K_{21} & -B_2 K_{21} & (A_2 + B_2 K_{22}) & -B_2 K_{22} \\ 0 & 0 & 0 & A_2 \end{bmatrix}, \bar{B} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \\ \bar{M} &= \begin{bmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{21} & 0 & 0 \\ 0 & 0 & M_{12} & 0 \\ 0 & 0 & 0 & M_{22} \end{bmatrix}, \bar{C} = \begin{bmatrix} C_{11} & 0 & 0 & 0 \\ 0 & C_{21} & 0 & 0 \\ 0 & 0 & C_{12} & 0 \\ 0 & 0 & 0 & C_{22} \end{bmatrix}.\end{aligned}$$

Note that from (4.11), if the initial estimation errors have zero means, $E(\tilde{x}_{j,0|0}^i) = 0$, it follows that $E(\tilde{x}_{2,k|k}^1) = E(\tilde{x}_{1,k|k}^2) = 0$, meaning that *Estimator 1@2* and *Estimator 2@1* are unbiased estimators. Also note that the known constant terms l_1 and l_2 vanished in the filtering error equations (4.11).

Consider the steady state covariance $P = cov(\tilde{X})$, it follows that

$$P = \bar{M}_r \bar{A} P \bar{A}^T \bar{M}_r^T + \bar{M}_r \bar{B} Q \bar{B}^T \bar{M}_r^T + \bar{M} R \bar{M}^T, \quad (4.12)$$

$$\text{where } Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, R = \begin{bmatrix} R_{11} & 0 & 0 & 0 \\ 0 & R_{21} & 0 & 0 \\ 0 & 0 & R_{12} & 0 \\ 0 & 0 & 0 & R_{22} \end{bmatrix}.$$

First of all, it is required that *Estimator 1@2* and *Estimator 2@1* should be stable. If a feasible matrix P can be found for the following matrix inequalities to hold:

$$\begin{aligned} -P + \bar{M}_r \bar{A} P \bar{A}^T \bar{M}_r^T + \bar{M}_r \bar{B} Q \bar{B}^T \bar{M}_r^T + \bar{M} R \bar{M}^T &< 0, \\ P &> 0, \end{aligned} \quad (4.13)$$

then it provides an upper bound of the algebraic solution P of (4.12) and hence the estimators are stable. The “<” sign is used in (4.13) instead of “ \leq ” to avoid the semi-definite problem. Actually, the solution of (4.13) from a numerical solver is arbitrarily close to the solution of (4.12). Taking the Schur complement of (4.13) yields

$$\begin{bmatrix} -P & \bar{A} + \bar{M}C & B + \bar{M}D \\ \bar{A}^T + C^T \bar{M}^T & -P^{-1} & 0 \\ B^T + D^T \bar{M}^T & 0 & -I \end{bmatrix} < 0. \quad (4.14)$$

where $C = -\bar{C}\bar{A}$, $B = [\bar{B}Q^{\frac{1}{2}} \quad 0]$, $D = [-\bar{C}\bar{B}Q^{\frac{1}{2}} \quad R^{\frac{1}{2}}]$.

Multiplying (4.14) on the left by $T = \text{blockdiag}(P^{-1}, I, I)$ and on the right by $T^T = T$ yields the following linear matrix inequality (LMI).

Theorem 5. *Matrix L and symmetric matrix Y satisfying the LMI*

$$\begin{bmatrix} -Y & (Y - L\bar{C})\bar{A} & (Y - L\bar{C})\bar{B}Q^{\frac{1}{2}} & LR^{\frac{1}{2}} \\ \bar{A}^T(Y - \bar{C}^T L^T) & -Y & 0 & 0 \\ Q^{\frac{1}{2}}\bar{B}^T(Y - \bar{C}^T L^T) & 0 & -I & 0 \\ R^{\frac{1}{2}}L^T & 0 & 0 & -I \end{bmatrix} < 0, \quad (4.15)$$

exist if and only if $P = Y^{-1}$, $\bar{M} = PL$, and $\bar{M}_r = I - \bar{M}\bar{C}$ satisfy matrix inequality (4.13).

Note that this is a standard construction in optimal filtering derived using LMIs and minimization (Skelton, Iwasaki & Grieriadis 1998, Colaneri, Geromel & Locatelli

1997). The appeal of (4.15) versus (4.13) is that this latter inequality is linear in the matrix variables Y and L , which makes its solution and optimization tractable. From the perspective of formulating and solving non-classical information architecture control problems, however, an incipient problem arises through the inability to explore a block diagonal structure on the computed solution \bar{M} from (4.14) without also imposing such a structure on both Y and P . Evidently from the structure of \bar{A} a block diagonal P is not typically of interest. Indeed it is the cross-covariance between terms such as $\tilde{x}_{1,k}^1$ and $\tilde{x}_{2,k}^1$ that captures the information architecture. Without any structural condition on \bar{M} , the solution of minimizing $\text{tr}(Y^{-1})$ subject to (4.15) versus (4.13) would yield the classical, fully-shared-measurement Kalman filtering solution. To explore the development of an LMI approach to finding feasible solutions to the non-standard information architecture problem, we employ a result of (de Oliveira, Bernussou & Geromel 1999).

Lemma 1. *(de Oliveira, Bernussou & Geromel 1999) The following statements are equivalent.*

(i) *There exists a symmetric matrix $P > 0$ such that*

$$A^T P A - P < 0.$$

(ii) *There exist a symmetric matrix P and a matrix G such that*

$$\begin{bmatrix} -P & A^T G^T \\ GA & -G - G^T + P \end{bmatrix} < 0.$$

(iii) *There exist a symmetric matrix $Y = P^{-1}$ and a matrix G such that*

$$\begin{bmatrix} -G - G^T + Y & GA^T \\ AG^T & -Y \end{bmatrix} < 0. \tag{4.16}$$

Note that statement (iii) is added to the original Lemma and its equivalence to the statement (i) can be proven in the same way as in (de Oliveira, Bernussou & Geromel 1999). And by the equivalency between (i) and (iii), we establish the following theorem.

Theorem 6. *Matrices G , Y , and L satisfying*

$$\begin{bmatrix} -G - G^T + Y & G\bar{A} + LC & GB + LD \\ \bar{A}^T G^T + C^T L^T & -Y & 0 \\ B^T G^T + D^T L^T & 0 & -I \end{bmatrix} < 0, \quad (4.17)$$

yield $P = Y^{-1}$ and $\bar{M} = G^{-1}L$ which satisfy (4.13). Conversely, P and \bar{M} satisfying (4.13) provide $Y = G = P^{-1}$, $L = P^{-1}\bar{M}$ which satisfy (4.17).

Corollary 2. *If G and L are constrained to be block diagonal matrices in (4.17), then $P = Y^{-1}$ and $\bar{M} = G^{-1}L$ are also feasible in (4.13) with \bar{M} block diagonal.*

Corollary 3. *If matrices G , Y , and L , with G and L block diagonal conformably with \tilde{X}_k , can be found satisfying (4.17) then the state estimators (4.7)–(4.10), with the gains given by the block diagonal elements of $\bar{M} = G^{-1}L$, are stable and their covariances are bounded above by the corresponding diagonal blocks of P .*

By inspecting the finer structure of matrices \bar{A} and B , the matrix on the left hand side of (4.17) is linear in the unknowns Y , L , and G . A feasible solution from (4.17) will give us feasible values of P , M_{12} and M_{21} .

Furthermore, one may aim to seek a feasible solution of (4.17), which minimizes $\text{tr}(P) = \text{tr}(Y^{-1})$. To do this we introduce a new variable W such that

$$W > Y^{-1}, \quad (4.18)$$

and then minimize the $\text{tr}W$. The Schur compliment of (4.18) is

$$\begin{bmatrix} -W & I \\ I & -Y \end{bmatrix} < 0. \quad (4.19)$$

This yields the following convex LMI optimization problem to provide a solution for the observer gains for coordinated control with non-standard information structure.

Min Cov:

$$\min_{G,L,W,Y} \text{tr}W$$

subject to:

$$\begin{bmatrix} -G - G^T + Y & G\bar{A} + LC & GB + LD \\ \bar{A}^T G^T + C^T L^T & -Y & 0 \\ B^T G^T + D^T L^T & 0 & -I \end{bmatrix} < 0, \quad (4.20)$$

$$\begin{bmatrix} -W & I \\ I & -Y \end{bmatrix} < 0,$$

where G and L are block diagonal Y and W are symmetric.

Note that the covariances Σ_1^1 and Σ_2^2 from (4.7) and (4.8) are constant. Hence, once we get solutions G , Y and L from (4.20), the Kalman gains \bar{M}_{11} and \bar{M}_{22} will be replaced by M_{11} and M_{22} from (4.7) and (4.8).

4.3 Example

In this section, the results from Section 4.2 will be demonstrated with a *scalar* example. The notations remain the same as in Section 4.2 with only the matrix operation eased to the scalar calculation.

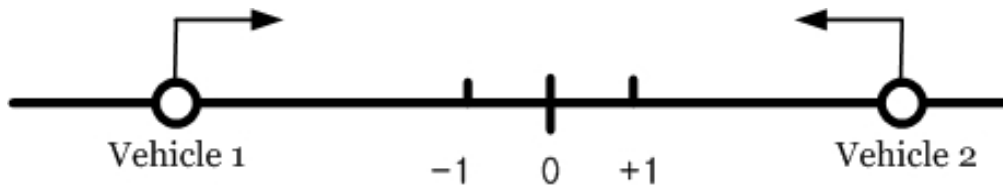


Figure 4.1: Two mobile beads cooperation task.

Consider two coordinated autonomous mobile beads (vehicles) on the real line described by

$$x_{k+1}^1 = x_k^1 + u_k^1 + w_k^1, \quad (4.21)$$

$$x_{k+1}^2 = x_k^2 + u_k^2 + w_k^2. \quad (4.22)$$

At each sampling time k , each solves the same optimization problem based on their local information:

$$\begin{aligned} \min_{u^i} J_k = & (x_{k+1}^1 + x_k^2)^2 + (x_{k+1}^1 + 1)^2 \\ & + (x_{k+1}^2 + x_k^1)^2 + (x_{k+1}^2 - 1)^2. \end{aligned} \quad (4.23)$$

The coordination tasks captured by J_k are:

- (i) to drive *vehicle 1* to -1 and *vehicle 2* to $+1$;
- (ii) to maintain the 2-vehicle formation symmetric about the origin, if the initial positions are symmetric.

Note that, emphasizing the estimation part of the problem, the coordination control task (ii) is merely illustrative.

This task is a simple one-step-ahead LQG problem. The solution is:

$$\begin{aligned} u_k^1 &= -\hat{x}_{1,k|k}^1 - \frac{1}{2}\hat{x}_{1,k|k}^2 + 1, \\ u_k^2 &= -\frac{1}{2}\hat{x}_{2,k|k}^1 - \hat{x}_{2,k|k}^2 - 1. \end{aligned}$$

Following the procedure in Section 4.2, the \bar{M} solved from (4.20) is

$$\bar{M} = \text{diag}(0.6751, 0.7379, 0.7379, 0.6751).$$

Its corresponding steady state prediction covariance is

$$P_{LMI} = \begin{bmatrix} 0.6276 & 0.1377 & -0.0209 & 0 \\ 0.1377 & 0.6725 & -0.0335 & -0.0209 \\ -0.0209 & -0.0335 & 0.6725 & 0.1377 \\ 0 & -0.0209 & 0.1377 & 0.6276 \end{bmatrix},$$

with $\text{tr}(P_{LMI}) = 2.6001$.

Since we can have M_{11} and M_{22} pre-computed from the Kalman filter equations (They both equal 0.6180 in this example.), we replace $\bar{M}(1, 1)$ and $\bar{M}(4, 4)$ with 0.6180. Thus, the gain matrix applied is:

$$\bar{M}_{app} = \text{diag}(0.6180, 0.7379, 0.7379, 0.6180).$$

The achieved steady state covariance matrix via applying \bar{M}_{app} is

$$P_{achv} = \begin{pmatrix} 0.6180 & 0.1608 & -0.0229 & 0 \\ 0.1608 & 0.6707 & -0.0312 & -0.0229 \\ -0.0229 & -0.0312 & 0.6707 & 0.1608 \\ 0 & -0.0229 & 0.1608 & 0.6180 \end{pmatrix},$$

with $tr(P_{achv}) = 2.5775$ which is less than $tr(P_{LMI}) = 2.6001$. This means that \bar{M}_{app} is closer to the optimal gain than \bar{M} . The reason that *Min Cov* problem does not give \bar{M}_{app} as its solution may be due to the feasibility of \bar{M}_{app} or due to the numerical errors.

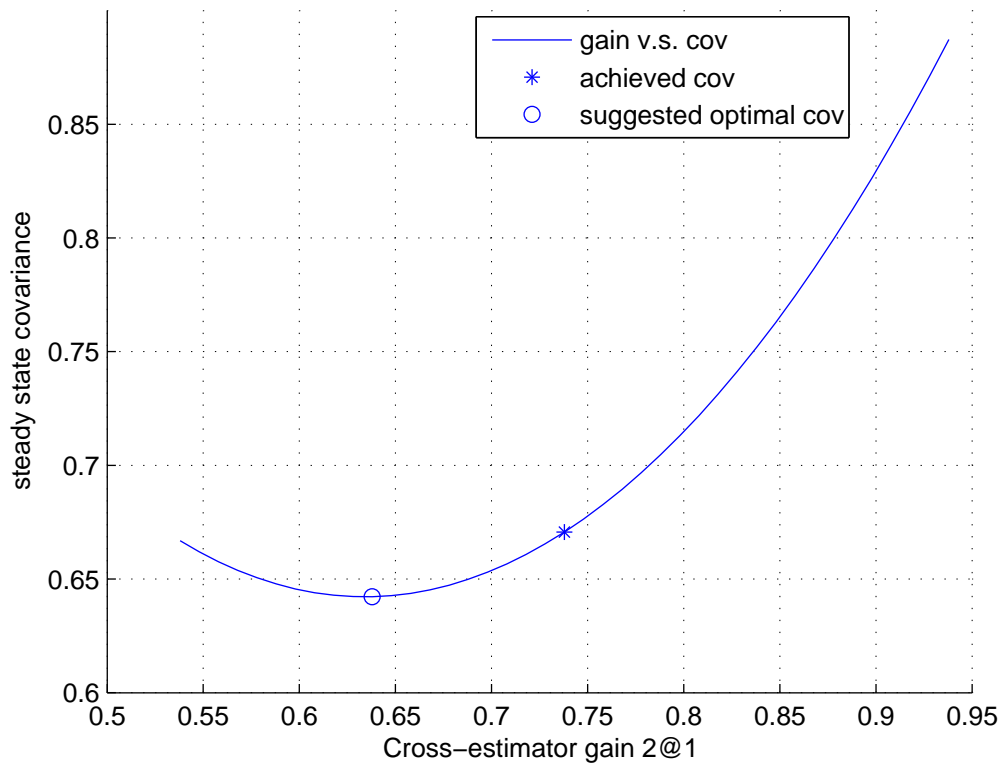


Figure 4.2: Error Covariances of *cross-estimator 2@1* depending on M_{12} and M_{21} .

To demonstrate the validity of our technique, covariances, in comparison with P_{achv} , were computed with various $\bar{M}_{app}(2, 2)$ and $\bar{M}_{app}(3, 3)$ values. The Figure 2 shows the Error Covariances of *Estimator 2@1* by changing the observer gains $M_{12} = M_{21}$. The

achieved steady state covariance of *Estimator 2@1* is 0.6569; the optimal covariance, suggested in Figure 2, is 0.6423.

4.4 Conclusion

This chapter studies the local cross-estimator design in the autonomous vehicle coordinated control problem. Though the non-classical information architecture complicates the structure of the local estimator, LMI techniques can help the design. The result makes the study of more design issues in the coordinated control context possible. Especially, the LMI formulation developed in this chapter can be easily modified to solve the cross-estimator design and the communication bandwidth assignment problem when constrained MPC is the local controller, which is the main theme of next chapter.

This chapter includes the reprints of the following paper:

Jun Yan, Keunmo Kang, Robert R. Bitmead - *State Estimation In Coordinated Control With A Non-Standard Information Architecture*, 16th IFAC World Congress in Prague, Jul. 2005.

The dissertation author was the primary author listed in these publications. And the co-author, Professor Bitmead, directed and supervised the research.

Chapter 5

Local Estimation and Communication Resource Assignment for Coordinated Control

5.1 Introduction

5.1.1 Main Ideas

From the discussion in Chapter 3, it is evident that the local controllers in multi-system coordination need neighbors' state information which should be provided by cross-estimators. In Chapter 4, we studied a cross-estimator design problem with a non-standard information structure, i.e. the linear control law is given but not the exact control values. In this chapter, we continue the cross-estimator design problem for a vehicle formation with each vehicle governed by linear dynamics and applying the constrained MPC discussed in Chapter 3. We intend to solve this design problem via modifying the LMI formulation developed in Chapter 4.

The modifications in this chapter are:

M1: Vehicles should exchange their control values. Usually, a constrained MPC is a

nonlinear feedback control law which immediately invalidates the LMI formulation. Furthermore, each local MPC controller needs cross-estimators (predictors) to provide its neighbors' state prediction throughout the horizon. We propose that the vehicles exchange their optimizing control sequence together with their current states. This information package suffices to construct the state prediction, although other choices may also be viable, such as the optimizing state trajectory (Dunbar & Murray 2004).

M2: The linkage between the prediction covariance and the control performance via the no-collision constraint as pointed out in Chapter 3 will be considered. In Chapter 3, we have shown how the prediction covariance affects the achievable control performance. This chapter will be focused on deriving feasible prediction covariances satisfying the over-bound posed by the control performance requirement.

M3: The prediction error covariance is partially determined by the measurement noises on the states and the controls. In this chapter, the measurement noises on the communicated cross-state and control information are solely induced by the limited communication capacity. Hence, by adjusting different numbers of bits assigned to each variable in the information package defined in *M1*, we can regulate the covariance to be feasible.

The modification *M2* is the pivot of this chapter. It actually defines two types of cross-estimator design and bandwidth assignment problem:

- Q1*. Minimum communication resource: for a given control performance requirement, what is the minimum required communication resource of the entire formation and the corresponding estimators/predictors and the bandwidth assignment strategy? This is the main theme of this chapter.
- Q2*. Minimum covariance: for a given amount of total communication resource, what is the best estimator/predictor and the corresponding bandwidth assignment strategy? By solving this problem, one can deduce the best achievable control performance. This problem has been considered in (Kang, Yan & Bitmead 2005) and the formulation is a simple modification from that of *Q1*.

In Section 5.2 we formulate our working problem: each vehicle uses the constrained MPC as its local controller to regulate the vehicle towards its target position while obeying the no-collision constraints posed by neighbors up to a certain future time. In Section 5.3, we first discuss the information package sent by each vehicle and a corresponding linear model to retrieve the proper state prediction. Then the detailed derivation of the final LMI problem (for $Q1$) is given. It is to minimize the total number of bits subject to three constraints:

1. that it guarantees a stable predictor;
2. that it satisfies the control performance requirement (an over-bound on the covariance);
3. that it respects the total bandwidth limit.

A numerical example is shown in Section 5.4.

5.1.2 Previous Literature Linkage

Communication link requirements may be derived directly from the vehicle formation tasks and constraints. Decentralized MPC strategies always require sharing the future state prediction among the neighbors. The usual way is to use the state prediction from the last optimization and special MPC constraints are used to guarantee the overall stability, (Camponogara, Jia, Krogh & Talukdar 2002, Dunbar & Murray 2004, Richards & How 2004*a*), or feasibility. In (Fax & Murray 2004) and (Moreau 2005), the formation problems are defined via the relative distances between the vehicles and graph theory tools are used to establish the communication links so that the stability of the entire formation is ensured. In these results, the communication links are assumed to provide perfect data transmission which needs infinite communication resources.

The effect of finite communication resources on control systems has been considered, with existing results focusing on minimum permissible communication resource assignment problem for a single system. In (Wong & Brockett 1997), state estimation problems based on quantized observations are considered and various necessary and sufficient conditions on communication data rate are provided for the existence of stable

and asymptotically convergent coder-estimator schemes. In (Nair & Evans 2003), the authors derived the lowest rate above which a coding and control law exists and guarantees the exponential stabilizability of a noiseless, linear time-invariant (LTI) system with random initial state. In (Tatikonda & Mitter 2004), the authors provided a lower bound (necessary condition), determined by the system dynamics solely, on the encoder-decoder channel capacity for the asymptotic observability and stabilizability of LTI systems with bounded initial states and possibly white, additive process noises. The necessary condition is also sufficient when the encoder has access to the control signals, while for an encoder knowing only control laws, a larger channel capacity is required.

The estimation problem for linear systems with measurement noise on both state and control variables is also known as errors-in-variables. In (Guidorzi, Diversi & Soverini 2002), the authors discussed the optimal filtering problem in an errors-in-variables framework. This chapter focuses on the feasible solution of the predictors that satisfies constraints posed on the prediction covariance posed by the formation. Moreover, the bandwidth assignment problem makes the measurement noise covariance also a design parameter rather than fixed.

5.2 The Local Model Predictive Controller

Consider a p -vehicle formation, the dynamics of the i th vehicle are governed by

$$x_{k+1}^{v_i} = A^i x_k^{v_i} + B^i u_k^{v_i} + w_k^{v_i}, \quad (5.1)$$

$$y_k^{v_i} = C_p x_k^{v_i}, \quad (5.2)$$

$$z_{i,k}^{v_i} = C_i^i x_k^{v_i} + v_{i,k}^i. \quad (5.3)$$

where x^{v_i} is the n -dimensional the state vector, u^{v_i} is the m -dimensional control vector, w^{v_i} is the process disturbance with normal distribution $N(0, Q_i)$, y^{v_i} consists of the state variables representing vehicle i 's position, $z_i^{v_i}$ is vehicle i 's own noisy state measurement with noise $v_{i,k}^i \sim N(0, R_i^i)$. Each vehicle has a Kalman filter providing its own state estimates.

This coordinated vehicle fleet can have various control tasks, such as formation transformation or maintenance, maximum territory covering etc. In this paper, we consider a vehicle formation maintenance task, which is also known as rigid body formation

when there are no disturbances. As in (Eren, Belhumeur, Anderson & Morse 2002, Eren, Belhumeur & Morse 2002), communication channels between vehicles are established so that the formation can keep the rigid body like motion via each vehicle's guaranteed relative position to its neighbors. If one considers the presence of the process disturbance, the measurement noise and communication error, the direct involvement of others' noisy state information into one's control law can make guaranteeing closed-loop performance a complicated problem. To ease the problem, the position and the trajectory of the centroid of the formation are provided to each vehicle and the formation is defined by each vehicle's relative position to the centroid which is again known to all vehicles. This framework is also known as the virtual leader (Beard, Lawton & Hadaegh 2001, Leonard & Fiorelli 2001). To illustrate our idea of bandwidth assignment, we study the problem in moving coordinates centered at the centroid with each vehicle trying to stay at their fixed nominal positions $y^{v_i^*}$. Here, we assume that each vehicle can keep up with the centroid's trend of movement very well and so neglect this part of the problem.

Due to the stochastic nature of the disturbances, the vehicles cannot stay right on their nominal positions. For a linear system with linear feedback control law applied, such as LQG (linear quadratic gaussian), the closed-loop performance can be measured by the controlled state covariance, which can be easily computed (Anderson & Moore 1989). In our vehicle problem, the elements associated with the vehicle position in this controlled covariance matrix define a controlled region in which the vehicle stays almost all the time, when the noises are unbounded, or never exceeds such a region, when the noises are bounded. If the vehicles have enough control power so that the controlled covariance equals the one-step-ahead Kalman prediction covariance and the process noise covariance is small enough, then by improving each vehicle's self measurement one can shrink the controlled regions such that they do not overlap each other and hence no-collision between vehicles will occur. However, when the control power is desired to be small, a control penalty term is usually added in the LQG criterion function, and improving the accuracy of self measurements cannot avoid overlapping controlled regions. Thus, the no-collision requirement must be considered directly and this leads us to a constrained control problem. MPC with constraints is our choice such that the no-collision constraint is considered over a certain future period. Furthermore, the future

no-collision constraints require the knowledge of neighboring vehicles' future trajectories. Communication channels should be established to send these future trajectories directly or to send data packages from which the trajectories can be reconstructed.

Usually a communication channel has only limited capacity which adds quantization errors to the transmitted data. We treat the communicated information to vehicle i as measurements:

$$z_{i,k}^j = x_k^j + v_{i,k}^j, \quad j \in I_i, \quad (5.4)$$

where I_i is the index set of vehicles (or neighbors of vehicle i) that are transmitting their information x_k^j to vehicle i and $v_{i,k}^j$ represents the quantization error due to the finite communication resource. Here we do not consider additional channel noises. Note that x_k^j is not the vehicle state vector, but a combined data package necessary for predicting vehicle j 's trajectory. The selection of x_k^j will be given in Section 5.3

Our choice of MPC controller at vehicle i for the formation control task with constraints and inaccurate information is the one proposed in Section 3.3.2:

$$\{u_{k,k}^{v_i}, \dots, u_{k+H-1,k}^{v_i}\} = \operatorname{argmin} E\{J(y_k^{v_i}, y^{v_i*}, H)\} \quad (5.5)$$

$$\text{subject to: } x_{k+l}^{v_i} = A^i x_{k+l-1}^{v_i} + B^i u_{k+l-1}^{v_i} + w_{k+l}^{v_i}; \quad (5.6)$$

$$y_{k+l}^{v_i} = C_p x_{k+l}^{v_i}, \quad (5.7)$$

$$P\{\text{No Collision with vehicle } j\} \geq p_0 \gg 0, \quad j \in I_i. \quad (5.8)$$

- The criterion function is used to minimize the expected distance between the vehicle i 's future position and its nominal position y^{v_i*} ;
- Vehicle dynamics up to the MPC horizon H are included as equality constraints;
- Large probability p_0 of no-collision between vehicle i and its neighbors up to the horizon is posed as a set of inequality constraints.

Due to the process and measurement noises, vehicle i obtains its own best state information throughout the horizon from a Kalman predictor and needs good state predictors for its neighbors' state prediction, which we call it cross-estimator whose study is one of the main tasks of this paper. Suppose we have the two types of predictors providing state prediction and the associated prediction error covariances, then the stochastic

MPC can be converted to a deterministic one on the state predictions (conditional mean processes) as illustrated in Chapter 2:

$$\{u_{k,k}^{v_i}, \dots, u_{k+H-1,k}^{v_i}\} = \operatorname{argmin} J(\hat{y}_{k|k}^{v_i}, y^{v_i^*}, H) \quad (5.9)$$

$$\text{subject to: } \hat{x}_{i,k+l|k}^{v_i} = A^i \hat{x}_{i,k+l-1|k}^{v_i} + B^i u_{k+l-1|k}^{v_i}; \quad (5.10)$$

$$\hat{y}_{k+l|k}^{v_i} = C_p \hat{x}_{k+l|k}^{v_i}, \quad (5.11)$$

$$\|\hat{y}_{i,k+l|k}^{v_i} - \hat{y}_{i,k+l|k}^{v_j}\| > \rho_{ij,l}, \quad j \in I_i. \quad (5.12)$$

The deterministic inequality (5.12) is the modified no-collision constraint saying that vehicle i should separate its own predicted position $\hat{y}_{i,k+l|k}^{v_i}$ at least $\rho_{ij,l}$ away from where it predicts that vehicle j will be $\hat{y}_{i,k+l|k}^{v_j}$. The separation ρ_{ij} is determined by the prediction error covariances $\Sigma_i^{v_i}$ of $\hat{x}_i^{v_i}$ and $\Sigma_i^{v_j}$ of $\hat{x}_i^{v_j}$ such that the small probability in (5.8) can be guaranteed. The resulting control law has the control performance tied to estimate quality due to the constraints in the way that bigger covariances lead to bigger ρ_{ij} and hence to conservative performance or even an infeasible formation task. This idea has been illustrated in a one-dimensional problem in Chapter 3.

Figure 5.1 shows the no-collision constraint between two vehicles in a two dimensional plane. The transformation from the constraint (5.8) to (5.12) has been shown in Section 3.5. Due to the change of notations, we briefly present the process here again and focus on the derivation of ρ_{ij} which is the key to relate the control performance and the predictor design.

From vehicle i 's perspective at some time in the horizon, it has predicted vehicle j 's position to be $\hat{y}_i^{v_j}$ with prediction error covariance $C_p \Sigma_i^{v_j} C_p^T$. Vehicle i can also compute its own position error covariance $C_p \Sigma_i^{v_i} C_p^T$ associated with $\hat{y}_i^{v_i}$. The exact value of $\hat{y}_i^{v_i}$ is to be determined according to the proper separation ρ_{ij} from $\hat{y}_i^{v_j}$ so that the constraint (5.8) can be guaranteed. We choose the ρ_{ij} as the following:

1. Define ellipses

$$E_i^{v_j} : (y - \hat{y}_i^{v_j})^T (C_p \Sigma_i^{v_j} C_p^T)^{-1} (y - \hat{y}_i^{v_j}) \leq \chi_p^2$$

and

$$E_i^{v_i} : (y - \hat{y}_i^{v_i})^T (C_p \Sigma_i^{v_i} C_p^T)^{-1} (y - \hat{y}_i^{v_i}) \leq \chi_p^2$$

such that $P(y^{v_j} \in E_i^{v_j}) = P(y^{v_i} \in E_i^{v_i}) = p_1$ where $p_1 = \frac{(1+p_0)}{2}$.

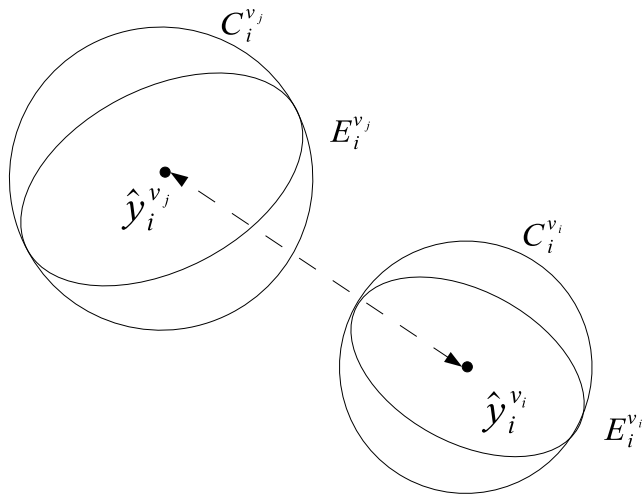


Figure 5.1: Illustration of the deterministic no-collision constraint between two vehicles.

2. The circumscribed circle of $E_i^{v_j}$ is $C_i^{v_j}$ centered at $\hat{y}_i^{v_j}$ with radius

$$r_{ji} = \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_i^{v_j} C_p^T)},$$

where $\lambda_{\max}(\cdot)$ means the maximum eigenvalue. Similarly, the circumscribed circle of $E_i^{v_i}$ is $C_i^{v_i}$ centered at $\hat{y}_i^{v_i}$ with radius

$$r_{ii} = \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_i^{v_i} C_p^T)}.$$

When the prediction errors are isotropic, the ellipses are identical to the circles.

3. Define $\rho_{ij} = r_{ji} + r_{ii}$, hence if the constraint (5.12) is assured, the two circles in

Figure (5.1) do not overlap. Therefore,

$$\begin{aligned} P(\text{collision}) &< P(y^{v_j} \notin C_i^{v_j} \text{ or } y^{v_i} \notin C_i^{v_i}) \leq P(y^{v_j} \notin E_i^{v_j} \text{ or } y^{v_i} \notin E_i^{v_i}) \\ &\leq P(y^{v_j} \notin E_i^{v_j}) + P(y^{v_i} \notin E_i^{v_i}) = 2(1 - p_1), \\ P(\text{no collision}) &= 1 - P(\text{collision}) > 2p_1 - 1 = p_0. \end{aligned}$$

That is, whenever the constraint (5.12) holds, (5.8) must hold. The chosen parameters for (5.12) make it a sufficient but not necessary condition for (5.8).

Note that the disturbances in a practical vehicle formation have uniform distribution or, more general, bounded distribution. This control solution works because, by choosing the right covariance of the model gaussian noises, the large probability constraints imply the real system will satisfy the constraints almost surely (with probability one).

The main task of this paper is to design good predictors of x^{v_j} , $j \in I_i$ that are run by vehicle i and provide relatively small covariances. Note that, in a physical vehicle formation, the bit rate of each communication channel is only finite. If an N -bits-per-sample communication channel is available between vehicle i and j , the allocation of these bits among the communicated variables will also change the covariance $\Sigma_{i,k}^j$. This bit-rate allocation problem is called the bandwidth assignment problem which can be integrated into the predictor design problem. We discuss these issues in the next section.

5.3 Local Cross-Estimators and Bandwidth Assignment

As shown in Chapter 3, the constraints in the MPC control law relate the formation control performance to the available communication resources, one design problem of interest is the $Q1$ stated in Section 5.1.1: for a given formation control performance specification what is the minimum overall communication resources and its allocation among the communicated data. In this section, we discuss this problem in a two-vehicle scenario (vehicle i and j) that is extensible to the more general multi-vehicle case. Each vehicle runs a Kalman filter as self state estimator and a cross-predictor of the other, which is indispensable to the MPC controller and is to be designed.

Predictors for classical linear systems can be designed via the powerful LMI technique. We intend to keep this design tool for the cross-predictor with a little complication added. The prediction along the control horizon needs information of the future control or state trajectory to be transmitted. We choose the control sequence solved from MPC and a simple linear model is developed in Section 5.3.1. Constraints posed by the formation performance and by the communication limit should be considered and are discussed in Section 5.3.2. The final LMI formulation is generated in Section 5.3.3.

5.3.1 Information Package and A Prediction Model

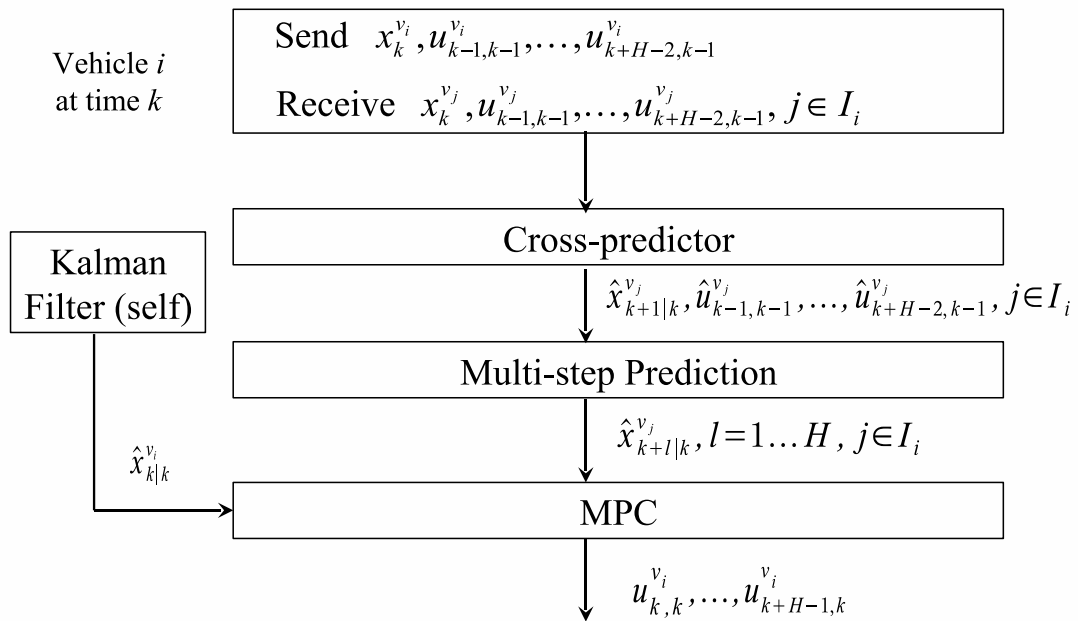


Figure 5.2: Vehicle Operations at time k .

Our interest is to have vehicle i calculate a reliable state prediction of vehicle j from current time k and along the horizon. Hence, either vehicle j 's own future state trajectory or its current state plus the future control sequence should be sent to vehicle i .

We choose sending the future control sequence as the control vector is usually of lower dimension than the state vector.

At each time k , each vehicle does the following as shown in Figure 5.2:

1. transmits its own current state and control sequence computed at time $k - 1$;
2. receives neighbors' current state measurements and control sequences formed at time $k - 1$;
3. predicts neighbors' future trajectories and computes the corresponding covariances;
- (3') estimates current self state, when necessary
4. uses the predicted neighbors' trajectories as constraints in its own constrained MPC and solve.

The control sequence communicated in Step One is needed since the prediction requires future control information. The transmission of control sequences derived at time $k - 1$ instead of those of time k is necessary when synchronization is required. For instance when two vehicles are mutually constraining their future states, none can derive the time k MPC control sequence from Step Four unless the others compute their control sequences first. The solution to this dilemma is to update the prediction of one's neighbors based on the time k state measurement and the communicated past control sequences up to time $k - 1$, $\{u_{t,t}^{v_j}, \dots, u_{t+H-1,t}^{v_j}; t = 0, 1, \dots, k - 1\}$. The state prediction of interest is $\{\hat{x}_{k+1|k}^{v_j}, \dots, \hat{x}_{k+H|k}^{v_j}\}$ that is needed to form vehicle j 's predicted position $\{\hat{y}_{k+1|k}^{v_j}, \dots, \hat{y}_{k+H|k}^{v_j}\}$. This requires information of controls $\{u_k^{v_j}, \dots, u_{k+n-1}^{v_j}\}$ which has a one-step lead on the available control information. As suggested in (Meditch & Hostetter 1971) and (Johnson 1975), the unknown controls should be characterized by a model. We assume the following model for the control sequences provided by the MPC at each time:

$$\begin{bmatrix} u_{k+1,k+1}^{v_j} \\ \vdots \\ u_{k+H,k+1}^{v_j} \end{bmatrix} = \begin{bmatrix} 0 & I_{m(H-1)} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_{k,k}^{v_j} \\ \vdots \\ u_{k+H-1,k}^{v_j} \end{bmatrix} + w_k^u, \quad (5.13)$$

where $w_k^u = [\xi_{1,k}^j, \dots, \xi_{H,k}^j]^T$ is modelled as a gaussian random vector with distribution $N(0, Q_u)$. The last $m \times m$ block of Q_u is chosen to be large since the corresponding control

$u_{k+H,k+1}^{v_j}$ is the new one entering the control horizon and we model it as a zero mean white noise with large covariance. This can be a good model if the change between two control sequences is guaranteed to be small via penalizing the change between them in the criterion function or including this type of behavioral constraints in the MPC formulation, as in (Dunbar & Murray 2004) where a slowly varying constraint on future state trajectories is included in the MPC.

The prediction sequence $\{\hat{x}_{k+2|k}^{v_j}, \dots, \hat{x}_{k+H|k}^{v_j}\}$ is a linear combination of $\hat{x}_{k+1|k}^{v_j}$ and $\{\hat{u}_{k+1}^{v_j}, \dots, \hat{u}_{k+H-1}^{v_j}\}$. Hence, we should derive these estimates first.

Note that (5.1) can also be written as

$$x_{k+1}^{v_i} = A^i x_k^{v_i} + B^i u_{k,k-1}^{v_i} + B \xi_{1,k-1}^j + w_k^i. \quad (5.14)$$

By stacking (5.14) and (5.13), we reach the following system:

$$x_{k+1}^j = F^j x_k^j + w_k^j; \quad (5.15)$$

$$z_k^j = x_k^j + v_k^j, \quad (5.16)$$

where $x_k^j = [x_k^{v_j T}, u_{k-1,k-1}^{v_j T}, u_{k,k-1}^{v_j T}, \dots, u_{k+H-2,k-1}^{v_j T}]^T$, $F^j = \begin{bmatrix} A^j & 0 & B^j & 0 \\ 0 & 0 & I_{m(H-1)} & \\ 0 & 0 & 0 & 0 \end{bmatrix}$,

$w_k^j = \begin{bmatrix} B \xi_{1,k-1}^j + w_k^{v_j} \\ w_{k-1}^u \end{bmatrix}$, and z_k^j is the measurement with gaussian measurement noise $v_k^j \sim N(0, R^j)$ and R^j has the following structure

$$R^j = \begin{bmatrix} \Sigma_j^{v_j} + R_x^j & 0 \\ 0 & R_u^j \end{bmatrix}, \quad (5.17)$$

where R_x^j and R_u^j are diagonal matrices representing the covariances of the quantization errors associated with the finite bit-rate communication of the elements in x_k^j . The first block in R^j has a $\Sigma_j^{v_j}$ term which represents that vehicle j 's the best available state information is provided by its self Kalman filter and has error covariance $\Sigma_j^{v_j}$. Hence the accuracy of the transmitted state is bounded below by this $\Sigma_j^{v_j}$

A Kalman predictor can be easily written down for the augmented system (5.15)

if R^j is known and the best prediction of interest is given by

$$\begin{bmatrix} \hat{x}_{k+2|k}^{v_j} \\ \vdots \\ \hat{x}_{k+H|k}^{v_j} \end{bmatrix} = \begin{bmatrix} A^j & 0 & B & 0 & \dots & 0 \\ (A^j)^2 & 0 & A^j B & B & \dots & 0 \\ \vdots & & & & & \\ (A^j)^{H-1} & 0 & (A^j)^{(H-2)}B & (A^j)^{(H-3)}B & \dots & B \end{bmatrix} \hat{x}_{k+1|k}^j. \quad (5.18)$$

This model facilitates solving the bandwidth assignment problem and the stable predictor gain design problem together with the LMI technique, as will be shown.

5.3.2 Three Constraints on Predictor Design

Our task is to design a stable predictor of the augmented system (5.15) and the bit-rate assignment strategy for x^j that satisfies the covariance constraint posed by the formation and the control constraint (5.12). This implies three types of LMI should be considered.

Constraint Representing Stable Prediction

Since the bit-rate assignment is not known a priori, the measurement noise covariance matrix R^j becomes a design variable and the standard Kalman filter/predictor design will not work for the augmented system.

We propose using the following predictor structure for (5.15):

$$\hat{x}_{i,k+1|k}^j = F^j \hat{x}_{i,k|k-1}^j + K_i^j (z_k^j - \hat{x}_{i,k|k-1}^j), \quad (5.19)$$

where K_i^j is the predictor gain. The corresponding prediction error covariance can be computed by

$$\Sigma_{i,k+1|k}^j = (F^j - K_i^j) \Sigma_{i,k|k-1}^j (F^j - K_i^j)^T + K_i^j R^j K_i^{jT} + Q^j. \quad (5.20)$$

A stable predictor means that the steady state covariance $P = \lim_{k \rightarrow \infty} \Sigma_{i,k+1|k}^j$ exists, where P is also the solution of the Lyapunov equation:

$$P = (F^j - K_i^j) P (F^j - K_i^j)^T + K_i^j R^j K_i^{jT} + Q^j. \quad (5.21)$$

Therefore, any feasible solution (K^j, R^j) of the following matrix inequality

$$-P + (F^j - K_i^j) P (F^j - K_i^j)^T + K_i^j R^j K_i^{jT} + Q^j < 0, P > 0 \quad (5.22)$$

provides a stable predictor (5.19) for the augmented system (5.15) and (5.16). This matrix inequality is equivalent to the following LMI via Schur complement:

$$\begin{bmatrix} -Y & L & Y & YF^j - L \\ L^T & -R^{j-1} & 0 & 0 \\ Y & 0 & -Q^{j-1} & 0 \\ F^{jT}Y - L^T & 0 & 0 & -Y \end{bmatrix} < 0, \quad (5.23)$$

where $Y = P^{-1}$ and $L = YK_i^j$.

Constraint Representing Formation Control Requirement

There are natural constraints on the state prediction error covariance Σ_i^j posed by the formation. Intuitively, when vehicle i and vehicle j are supposed to be close, vehicle i needs a clearer image of vehicle j . The constraint we propose here is

$$\sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_{i,k+l|k}^{v_i} C_p^T)} + \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_{i,k+l|k}^{v_j} C_p^T)} < s_{ij} \triangleq \|y^{v_i^*} - y^{v_j^*}\|. \quad (5.24)$$

The left hand side sum is the ρ_{ij} in (5.12) defined in Section 5.2. The right hand side is the distance between the nominal positions of vehicle i and j . Since Σ_i^i is provided by vehicle i 's self Kalman predictor, this is a constraint imposed on Σ_i^j . The purpose of (5.24) is to insure that the nominal position $y^{v_i^*}$ will not be infeasible for vehicle i too frequently. The explicit constraint on $\Sigma_{i,k+l|k}^j$ is

$$\chi_p^2 C_p \Sigma_{i,k+l|k}^{v_j} C_p^T \leq \lambda_{\max}(\chi_p^2 C_p \Sigma_{i,k+l|k}^{v_j} C_p^T) I < [s_{ij} - \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_{i,k+l|k}^{v_i} C_p^T)}]^2 I \triangleq W. \quad (5.25)$$

Furthermore, the multi-step-ahead state prediction covariance $\Sigma_{i,k+l|k}^{v_j}$ can be calculated in terms of the one-step-ahead covariance P and the process noise covariance Q . And in the steady state, we have $\Sigma_{i,k+l|k}^{v_j} = [I_n \ 0] [(F^j)^l P (F^{jT})^l] [I_n \ 0]^T + \Delta$ with $\Delta = [I_n \ 0] [\sum_{p=1}^{l-1} (F^j)^p Q^j (F^{jT})^p] [I_n \ 0]^T$. Finally, the constraint (5.24) represents another constraint imposed on P representing the formation:

$$GPG^T + \chi_p^2 C_p \Delta C_p^T < W, \quad (5.26)$$

or equivalently,

$$\begin{bmatrix} -W + \chi_p^2 C_p \Delta C_p^T & G \\ G^T & -Y \end{bmatrix} < 0, \quad (5.27)$$

where $G = \chi_p C_p [I_n \ 0] (F^j)^l$.

The choice of s_{ij} as an upper bound in (5.24) is an obvious one. Other bounds on Σ_i^j may be imposed from the formation control requirement. For example, the no-collision control constraint (5.12) being active too often may introduce more disturbance from vehicle i 's neighbors to its closed-loop behavior. This generates bad maneuvers such as sudden accelerating and braking. To avoid this scenario, one needs to compute a tighter upper bound than s_{ij} for Σ_i^j by scrutinizing the MPC solution without any active constraint (5.12) and the formation geometry.

Constraint Representing Communication Capacity Limit

The limited communication capacity means that the channel between vehicle i and j has only N bits per sample time to be assigned to transmit each element of x^j . The study of the bit-rate assignment problem devolves into the consideration of how to determine the specific values for $R_{x,l}^j$, the l th diagonal element of R_x^j . The key idea is the following. If we round binary numbers to the N th binary place, then the magnitude of the rounding error is 2^{-N} . Our technique is to capture the bit-rate limitations by modifying the measurement noise covariance to satisfy

$$\prod_{l=1}^n R_{x,l}^j \times \prod_{l=1}^{mH} R_{u,l}^j = C_0 \prod_{l=1}^n 2^{-2n_{x,l}} \times \prod_{l=1}^{mH} 2^{-2n_{u,l}} = C_0 2^{-2N}, \quad (5.28)$$

where C_0 is a positive constant and is determined by the square of the unit of the round off error and N is the total number of bits. For simplicity we set C_0 to be 1 henceforth. We model this as contributing to the measurement noises of these components; smaller bit-rate implying larger measurement noise.

Define $n_{x,l} = -\frac{1}{2} \log_2 R_{x,l}^j$ and $n_{u,l} = -\frac{1}{2} \log_2 R_{u,l}^j$ to be the number of bits associated with the states and the controls respectively. Then we have the following equation that is equivalent to (5.28):

$$\sum_{l=1}^n n_{y,l} + \sum_{l=1}^{mH} n_{u,l} = N. \quad (5.29)$$

The corresponding measurement noise matrix is

$$R^j = \begin{bmatrix} \Sigma_j^{v_j} + R_x^j & 0 \\ 0 & R_u^j \end{bmatrix},$$

$$R_x^j = \Sigma_j^{v_j} + \text{diag}\{2^{-2n_{x,l}}\}, R_u^j = \text{diag}\{2^{-2n_{u,l}}\}. \quad (5.30)$$

Note that, this is a nonlinear relationship between R^j and $n_{x,l}$, $n_{u,l}$.

5.3.3 Predictor Design and Bandwidth Assignment via LMI

Now, our design task becomes to derive the predictor gain K_i^j and the bit-rates $n_{x,l}$ and $n_{u,l}$ that together minimize the total number of bits N and to satisfy the three constraints (5.22), (5.27), and (5.28). Ideally, an optimal strategy should be derived by minimizing N subject to these constraints. However, the nonlinear relation (5.30) between the bits assignment and the measurement noise covariance makes the optimization difficult.

We propose to replace the original R^j with an upper bound R_a that is diagonal and its inverse is linear in elements of n_x and n_u in (5.23). Then an optimization problem using the LMI formulation can be derived and the solution will provide a feasible strategy, although not necessarily the optimal one. The main reason that we are interested in an over-bounding R_a is stated in the following lemma guaranteeing the feasibility of our solution.

Lemma 2. *If we have a feasible solution (P, K_i^j, R_a) of the following matrix inequality*

$$-P + (F^j - K_i^j)P(F^j - K_i^j)^T + K_i^j R_a K_i^{jT} + Q^j \leq 0, P > 0,$$

with $R_a \geq R^j$, then (P, K_i^j, R^j) is also a feasible solution of (5.22).

The proof is trivial, since $R_a \geq R^j$ implies $K_i^j R_a K_i^{jT} \geq K_i^j R^j K_i^{jT}$ and other terms in the two inequalities are the same.

Here, for clarity, we assume each vehicle knows its own current state perfectly, i.e. $\Sigma_j^{v_j} = 0$. The first block of R^j becomes R_x^j only. We choose R_a as

$$R_a^{-1} \triangleq \text{diag}\{1 + 2n_{x,1} \ln 2, \dots, 1 + 2n_{x,n} \ln 2, 1 + 2n_{u,1} \ln 2, \dots, 1 + 2n_{u,mH} \ln 2\}. \quad (5.31)$$

By applying the inequalities $2^p \geq 1 + p \ln 2$, $n \geq 0$ to (5.30), we have

$$\begin{aligned} R_x^{j-1} &= \text{diag}\{2^{2n_{x,l}}\} \geq \text{diag}\{1 + 2n_{x,l} \ln 2\}, \\ R_u^{j-1} &= \text{diag}\{2^{2n_{u,l}}\} \geq \text{diag}\{1 + 2n_{u,l} \ln 2\}, \\ R^{j-1} &\geq R_a^{-1}, \text{ i.e. } R_a \geq R^j. \end{aligned}$$

Note that, if $\Sigma_j^{v_j}$ is not zero, the choice of R_a becomes complicated but the idea still works.

Our final LMI formulation for the minimization of the total number of bits is summarized in the following theorem.

Theorem 7. *If a solution (L, Y, n_x, n_u) of the following optimization problem exists, Min N :*

$$\min_{L, Y, n_x, n_u} N \quad (5.32)$$

$$\text{subject to: } \begin{bmatrix} -W + \chi_p^2 C^T \Delta C_p^T & G \\ G^T & -Y \end{bmatrix} \leq 0, \quad (5.33)$$

$$\sum_{l=1}^n n_{x,l} + \sum_{l=1}^{mH} n_{u,l} \leq N, \quad (5.34)$$

$$\begin{bmatrix} -Y & L & Y & YF^j - L \\ L^T & -R_a^{-1} & 0 & 0 \\ Y & 0 & -Q^{j-1} & 0 \\ F^{jT}Y - L^T & 0 & 0 & -Y \end{bmatrix} < 0, \quad (5.35)$$

then the corresponding cross-predictor (5.19) with gain $K_i^j = Y^{-1}L$ and the measurement noise covariance $R_i^j = \text{diag}\{2^{-2n_{x,1}}, \dots, 2^{-2n_{x,n}}, 2^{-2n_{u,1}}, \dots, 2^{-2n_{u,mH}}\}$ is stable. The steady state covariance P_a is bounded by the LMI solution $P = Y^{-1}$.

Proof. Since $R_a > R^j$, by Lemma 1, the solution (P, K_i^j, R^j) from the optimization problem (5.32)~(5.35) makes (5.22) hold, i.e. the bandwidth assignment strategy and the predictor gain provide a stable predictor ($F - K_i^j$ has eigenvalues less than one) for the augmented system (5.15).

Furthermore, the achievable steady state predictor covariance P_e satisfies

$$P_a = (F^j - K_i^j)P_a(F^j - K_i^j)^T + K_i^j R^j K_i^{jT} + Q^j, \quad (5.36)$$

while the LMI solution satisfies

$$P > (F^j - K_i^j)P(F^j - K_i^j)^T + K_i^j R_a K_i^{jT} + Q^j, \quad (5.37)$$

which implies that there exists a positive definite matrix ΔP such that

$$P = (F^j - K_i^j)P(F^j - K_i^j)^T + K_i^j R_a K_i^{jT} + Q^j + \Delta P. \quad (5.38)$$

Subtract (5.36) from (5.38), we have

$$P - P_a = (F^j - K_i^j)(P - P_a)(F^j - K_i^j)^T + K_i^j (R_a - R^j) K_i^{jT} + Q^j + \Delta P. \quad (5.39)$$

Since $(F^j - K_i^j)$ is stable and $(R_a - R^j)$ is positive definite, the Lyapunov equation (5.39) implies $(P - P_a)$ is positive definite, i.e. P is an upper bound of P_a . \square

Remarks:

- The formulation so far focuses on a simple two-vehicle case. This can be extended to the multi-vehicle scenario by adding more constraints. The specific formulation depends on the information structure and here we discuss one example where vehicle i receives state and control information from η neighbors via η communication channels. The goal of stating the LMI is to decide the overall minimum communication resource, the η predictor gain matrices, and the bit assignment scheme for every variable communicated via every channel. The criterion function (5.32) remains the same, though N now means the sum of bits of η channels. The constraint on the sum of bits (5.34) should be replaced by

$$\sum_{j=1}^{\eta} \left(\sum_{l=1}^n n_{x,l}^j + \sum_{l=1}^{mH} n_{u,l}^j \right) \leq N.$$

The formation performance constraint (5.33) and the stable predictor constraint (5.35) should be posed on the predictor for each neighbor separately. In Section 5.4, we will provide the numerical result of a one-vehicle two-neighbors example with this information structure.

- The LMI formulation can be modified to solve the following problem: with fixed N , derive the predictor gain and bandwidth assignment strategy that minimize the steady state prediction covariance. The criterion becomes to minimize the trace of the covariance matrix. The total bits constraint (5.34) and the stable predictor constraint (5.35) remain the same. The formation performance constraint (5.33) degrades to a passive constraint: when the overall LMI has a feasible solution. The solution will not change if this constraint is removed since it poses an upper bound on the matrix being minimized. This problem has been studied in (Kang, Yan & Bitmead 2005).
- If in a time-dependent vehicle formation, the nominal distance between the vehicles may vary over a large range, then the formation performance bound W posed on the covariance matrix in (5.33) becomes time dependent, which invalidates the one-time off-line design. However, we may solve the LMI repeatedly for different W and store a table of the gains and the bits assignment schemes on the vehicle to avoid solving LMI on-line.
- The numbers of bits $n_{x,l}$ and $n_{u,l}$ of the LMI (5.32)~(5.35) are usually not integers. Rounding up n_x and n_u gives smaller R^j , and by lemma 2, we still have feasible solution. Moreover, a Kalman predictor for the augmented system (5.15) with the new R^j may provide smaller steady state covariance than that of the gain K_i^j solved from the LMI.
- In a multi-vehicle formation, we also hope that the LMI solution may tell us which variables need not be sent. Even in a large formation, two vehicles that are far away from each other need not establish a communication channel at all. This means that by preliminary knowledge (bounds) of these variables only, there exists a stable predictor that guarantees no violation of the formation performance constraint (5.33) on the covariance. Hence, we may also consider rounding down numbers with small decimal part in n_x and n_u instead of rounding up every number. This problem is outside the primary interest of this paper.

5.4 Example

For a numerical demonstration a three vehicle example will be given. The notations remain the same as in the previous sections except we use x and y for state variables in typical cartesian coordinates. Consider the following three vehicle cooperating task. The vehicles track a known 2 – D trajectory and each vehicle is controlled by a local MPC controller attempting to keep a specific formation with the neighboring vehicles as shown in Figure 5.3 with a no-collision constraint. The offsets between the vehicles are globally known. From now on, we import the dynamical model of HoTDeC(HOvercraft

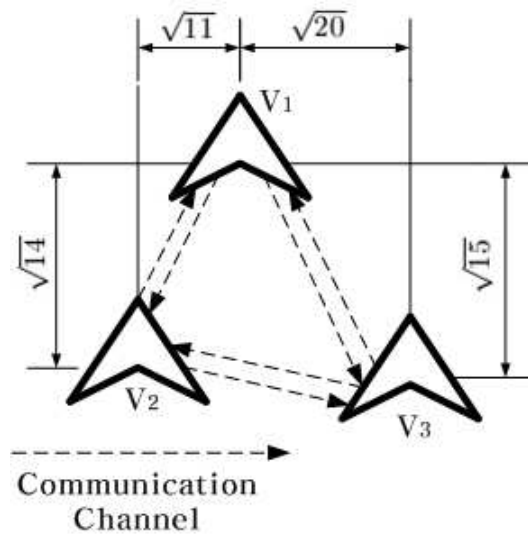


Figure 5.3: Three Vehicle Cooperation Task.

Testbed for DEcentralized Control) by the University of Illinois, Urbana-Champaign. More details are described in (Rubel 2004). The dynamical model of each vehicle is

assumed to be identical and the discretized model with added noises is

$$\begin{aligned}
X_{k+1} &= FX_k + w_k, \\
z_k &= X_k + v_k, \\
X_k &= [x_{p_k}, y_{p_k}, \theta_{p_k}, x_{v_k}, y_{v_k}, \theta_{v_k} U_{k-1,k-1} U_{k,k-1} U_{k+1,k-1}]^T, \\
U_{k-1,k-1} &= [u_{1k-1,k-1}, u_{2k-1,k-1}, u_{3k-1,k-1}], \\
F &= \begin{bmatrix} A_d & 0_{6 \times 3} & B_d & 0_{6 \times 3} \\ 0_{6 \times 6} & 0_{6 \times 3} & I & \\ 0_{3 \times 6} & 0_{3 \times 3} & 0_{3 \times 6} & \end{bmatrix}_{15 \times 15}, \quad w_k = [\xi_{1,k-1}^T B_d^T + w_k^{vT}, w_{k-1}^u]^T,
\end{aligned} \tag{5.40}$$

where w_k is given as described in (5.15). The first six elements in X_k represent position and velocity variables of x, y coordinates and of the angular motion of vehicles. The disturbance and measurement noises w_k^v, w_{k-1}^u and v_k are white and normally distributed.

$$\begin{aligned}
d_k &\sim N(0, Q), \quad Q = \text{diag} [1 \ 1 \ 1 \ 10 \ 10 \ 10], \\
w_{k-1}^u &\sim N(0, Q_u), \quad Q_u = \begin{bmatrix} Q_1^u & Q_2^{uT} \\ Q_2^u & Q_3^u \end{bmatrix}, \\
Q_1^u &= \begin{bmatrix} 0.0001 & 0 & 0 & 0.0006 & 0 & 0 \\ 0 & 0.0001 & 0 & 0 & 0.0006 & 0 \\ 0 & 0 & 0.2737 & 0 & 0 & 1.8240 \\ 0.0006 & 0 & 0 & 0.0038 & 0 & 0 \\ 0 & 0.0006 & 0 & 0 & 0.0038 & 0 \\ 0 & 0 & 1.8240 & 0 & 0 & 12.1574 \end{bmatrix}, \\
Q_2^u &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0065 & 0 & 0 & 0.0436 & 0 & 0 \\ 0 & 0.0065 & 0 & 0 & 0.0436 & 0 \\ 0 & 0 & 0.3699 & 0 & 0 & 2.4655 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
Q_3^u &= \text{diag} [0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5 \ 0.5], \\
v_k &\sim N(0, R^i), \quad i = 1, 2, 3.
\end{aligned} \tag{5.41}$$

The cross-estimators running at each vehicle will use (5.40) as a model. Our goal is to find the feasible predictor gains and the bandwidth assignment strategy that minimize the total number of bits for receiving neighbors' states and controls. Here we solve only for V_1 . Hence every superscript j on variables means 'of vehicle j ' and subscript 1 means 'computed at V_1 '. The other vehicles can be treated in the same way. The measurement noise covariance matrix $R_1^j (j = 2, 3)$ in (5.17), associated with measurement of Vehicle 2 and 3's states and control sequences at Vehicle 1 has the following structure

$$\begin{aligned} R_1 &= \text{blockdiag} \left[R_{1,states}^j, R_{1,controls}^j \right], \\ R_{1,states}^j &= \text{diag} \left[r_{x_p}^j, r_{y_p}^j, r_{\theta_p}^j, r_{x_s}^j, r_{y_s}^j, r_{\theta_s}^j \right], \\ R_{1,controls}^j &= \text{diag} \left[r_{u_{1k-1}}^j, r_{u_{2k-1}}^j, r_{u_{3k-1}}^j, r_{u_{1k}}^j, r_{u_{2k}}^j, r_{u_{3k}}^j, r_{u_{1k+1}}^j, r_{u_{2k+1}}^j, r_{u_{3k+1}}^j \right]. \end{aligned} \quad (5.42)$$

Here R_1 is the design variables to be determined from V_1 's cross-estimator design. In our example the prediction error covariance matrix $P_1^j = Y_1^{j-1}$ is a 15 by 15 symmetric matrix with diagonal elements corresponding to the state estimates

$$\hat{x}_p^j, \hat{y}_p^j, \hat{\theta}_p^j, \hat{x}_s^j, \hat{y}_s^j, \hat{\theta}_s^j, \hat{u}_{1k-1}^j, \hat{u}_{2k-1}^j, \hat{u}_{3k-1}^j, \hat{u}_{1k}^j, \hat{u}_{2k}^j, \hat{u}_{3k}^j, \hat{u}_{1k+1}^j, \hat{u}_{2k+1}^j, \hat{u}_{3k+1}^j,$$

and $\text{blockdiag} \left[R_{1,states}^j, R_{1,inputs}^j \right]$ is a 15 by 15 diagonal matrix with elements determined by the bit rates

$$n_{x_p}^j, n_{y_p}^j, n_{\theta_p}^j, n_{x_s}^j, n_{y_s}^j, n_{\theta_s}^j, n_{u_{1k-1}}^j, n_{u_{2k-1}}^j, n_{u_{3k-1}}^j, n_{u_{1k}}^j, n_{u_{2k}}^j, n_{u_{3k}}^j, n_{u_{1k+1}}^j, n_{u_{2k+1}}^j, n_{u_{3k+1}}^j. \quad (5.43)$$

Then denote by R_{a1} the approximated R_1 of (5.42) as in (5.31). Despite the stable predictor requirement, the other two LMIs stated in the $MIN N$ problem, Theorem 7, are:

- Vehicle Formation Constraints

From (5.25) and Figure 3, we have

$$\chi_p^2 C_p \Sigma_{1,k+3|k}^{v_2} C_p^T < [s_{12} - \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_{1,k+3|k}^{v_1} C_p^T)}]^2 I = \begin{bmatrix} 70.40 & 0 \\ 0 & 70.40 \end{bmatrix}, \quad (5.44)$$

$$\chi_p^2 C_p \Sigma_{1,k+3|k}^{v_3} C_p^T < [s_{13} - \sqrt{\lambda_{\max}(\chi_p^2 C_p \Sigma_{1,k+3|k}^{v_1} C_p^T)}]^2 I = \begin{bmatrix} 72.08 & 0 \\ 0 & 72.08 \end{bmatrix}, \quad (5.45)$$

where

$$C_p \Sigma_{1,k+3|k}^{v_1} C_p^T = \begin{bmatrix} 7.61 & 0 \\ 0 & 7.61 \end{bmatrix}, \quad C_p = [I_2, 0_{2 \times 13}],$$

$$s_{12} = 16.76, \quad s_{13} = 16.86, \quad \chi_p^2 = 9.21.$$

Here χ_p^2 corresponds to 99% no-collision probability.

- **Bandwidth Constraints**

As in (5.34), sum of all variables ($j = 2, 3$) in (5.43) must be less than N . The variable N is the objective that we want to minimize.

We set up this LMI problem as in Theorem 7 and solve it using `lmitool` in MATLAB with a block diagonal structure imposed on Y_1^j and L_1^j to yield predictor gain K_1^j , bit-rate assignments (5.43) and an upper bound $P_1^j = Y_1^{j-1}$ on the prediction error covariance. In addition, based on the calculated and rounded up bandwidth assignment (partially shown Table 1), we construct the new R^j and use it to compute the Kalman predictor covariance for (5.15) using the Discrete-time Algebraic Riccati Equation (DARE). This will give the best linear estimate for the same R^j .

Table 5.1: Bandwidth Assignment at Vehicle 1, states.

Bit-rate to Vehicle 1	Assigned bits for the states					
	x_p^j	y_p^j	θ_p^j	x_v^j	y_v^j	θ_v^j
Vehicle 2	19.87 (20)	19.87 (20)	0.02 (1)	18.76 (19)	18.76 (19)	0.02 (1)
Vehicle 3	6.41 (7)	6.41 (7)	0.02 (1)	6.50 (7)	6.50 (7)	0.02 (1)

Remarks:

- Our computation shows that a small difference in nominal separation, 0.1, between vehicles (s_{12}, s_{13}), leads to a dramatic increase in the requirement of the bit-rates (see Table 5.1), since vehicle 2 and 3 agree on all other parameters. A small change

Table 5.2: Bandwidth Assignment at Vehicle 1, controls.

Bit-rate to Vehicle 1	Assigned bits for the controls								
	$U_{k-1,k-1}^j$			$U_{k,k-1}^j$			$U_{k+1,k-1}^j$		
Vehicle 2	0.02 (1)	0.02 (1)	0.02 (1)	0.03 (1)	0.03 (1)	0.02 (1)	0.03 (1)	0.03 (1)	0.02 (1)
Vehicle 3	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)	0.02 (1)

of the nominal separation means a small change in the over-bound on the three-step-ahead prediction covariance in (5.44) and (5.45). The large change of the number of bits (e.g. 14 bits difference for x_p^j) is due to the small system matrices that make the change of the measurement noises covariance R^j diminish quickly in the open-loop prediction covariance evolution.

- Figure 5.4 shows three steady state predictors' (cross-estimator 2@1) error covariances computed by: (i) solving the LMI problem (marked by xs); (ii) solving the discrete time Lyapunov equation 5.21 with K_i^j solved from the LMI and R^j determined by the rounded-up numbers of bits solved from the LMI (marked by os); (iii) solving the Kalman predictor for cross-estimator 2@1 with R^j determined by the rounded-up numbers of bits solved from the LMI (marked by $\square s$). Clearly, the LMI solution provides an over-bound to the other two solutions. In this example, the vehicle state prediction error covariances of (ii) and (iii) are really close with the Kalman predictor (iii) being slightly better than the cross-estimator in (ii):

$$P_{archieved} = \begin{bmatrix} 1.0033 & * & * & * & * & * \\ * & 1.0033 & * & * & * & * \\ * & * & 1.6085 & * & * & * \\ * & * & * & 10.0452 & * & * \\ * & * & * & * & 10.0452 & * \\ * & * & * & * & * & 153.8227 \end{bmatrix},$$

$$P_{kp} = \begin{bmatrix} 1.0001 & * & * & * & * & * \\ * & 1.0001 & * & * & * & * \\ * & * & 1.6022 & * & * & * \\ * & * & * & 10.0447 & * & * \\ * & * & * & * & 10.0447 & * \\ * & * & * & * & * & 153.7319 \end{bmatrix},$$

but the gain matrix are quite different. It is conceivable that for a different system and the same measurement noise covariance the Kalman predictor shows more prominent improvement in the covariances. But this optimality is not important here, since the design goals, the bounds on the covariances of the predicted positions (5.44) and (5.45), have been achieved by the solution provided by the LMI. And better result can also come from a tighter R_a approximating R^j .

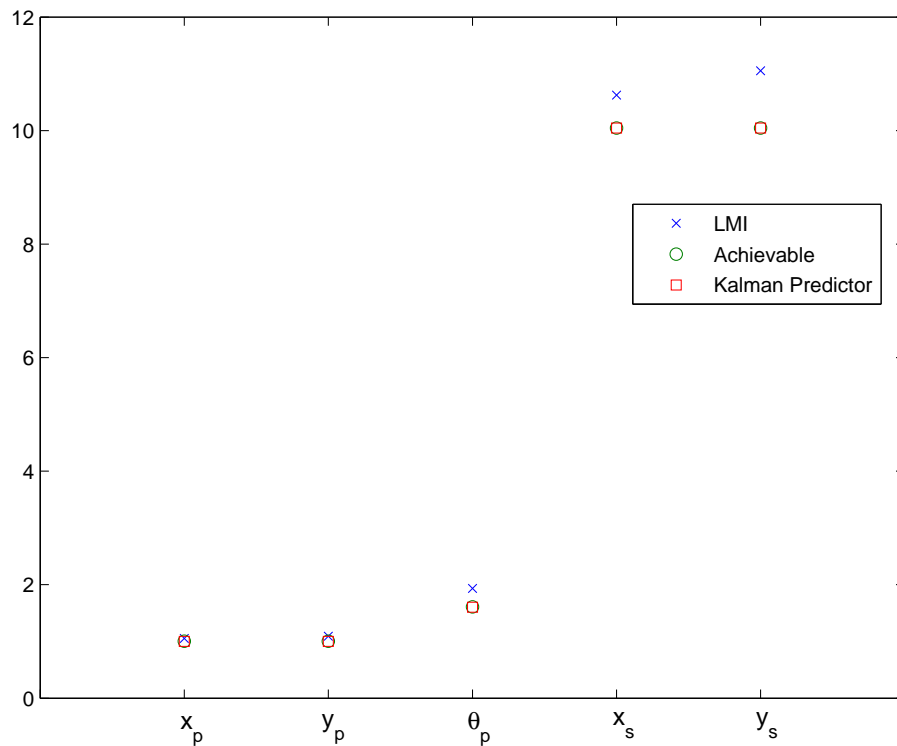


Figure 5.4: Prediction Error Covariances v.s. different estimator gains and R^j .

5.5 Conclusion

This chapter discussed the cross-predictor design and the bandwidth assignment problem in coordinated vehicle formation context using LMI.

The local MPC controller has no-collision constraints posed by neighbors' future states. Hence the vehicles exchange not only the current state but also the control sequence solved from MPC in order to predict others' states. Due to the synchronization problem, there may always be a one-step delay between the control sequence and the current state. A simple model is used to derive a proper control sequence starting at the current time for the prediction.

The solution of the LMI formulation summarized in Theorem 7 provides feasible bandwidth assignment strategy and stable predictor gains that satisfy the constraint on the prediction covariance posed by the formation. Due to the approximating relation between the measurement noise covariance and the bit-rate variables, the minimizing solution of the LMI problem provides only a feasible solution for the formation.

This chapter includes the reprints of the following papers:

Keunmo Kang, Jun Yan, Robert R. Bitmead - *Communication Design for Coordinated Control with a Non-Standard Information Structure*, 44th IEEE Conference on Decision and Control in Seville, Spain, Dec. 2005.

Jun Yan, Keunmo Kang, Robert R. Bitmead - *Local Estimation and Communication Resource Assignment for Coordinated Control*, Automatica, Submitted, Apr. 2006.

The dissertation author was the primary author listed in these publications. And the co-author, Professor Bitmead, directed and supervised the research.

Chapter 6

Conclusion

6.1 Conclusions

This dissertation has explored the following issues:

- the incorporation of the state estimate into constrained MPC formulations;
- the application of constrained MPC as the decentralized control law in coordinated control problems;
- the cross-estimator design and bandwidth assignment problems in coordinated control problems.

The main contributions are:

- The formulation of a constrained MPC with state estimation error covariance modifying the constraints;
- The identification of the connection between the control performance and the available communication resources via the constraints in MPC;
- The formulation of cross-estimator design and bandwidth assignment design via LMI tools.

Chapter 2 presented an approach to the treatment of state estimates in constrained MPC and the key steps are:

1. the constraints should be converted to a probabilistic form;
2. the probabilistic constraint on the states should be replaced by a deterministic constraint on the controlled conditional mean state estimate and that this modified constraint should involve tighter stipulations reflecting the estimate covariance. This is based on the property of linear systems that the state estimates' covariances are independent of the control signal;
3. the closed-loop covariance should be used for the modification of all constraints along the horizon.

The resulting deterministic MPC problem is standard in its form but uses the state estimate as its starting state and introduces covariance information via the constraints. This is not a certainty equivalence controller.

In Chapter 3, we examined the application of constrained MPC to a coordinated control problem. The direct inclusion of constraints makes the design easy and the global properties can be deduced from the local properties that are guaranteed by the local constrained MPC controllers. We discussed the multi-vehicle coordination problem as a working example.

The local constrained MPC controllers have fixed reference trajectories in their criterion functions and direct inclusion of no-collision constraints coupling the vehicles' dynamics. For deterministic problems, the asymptotic stability of each vehicle can be guaranteed via the local constrained MPC controller formulations and the global formation stability follows. When stochastic disturbances are in the picture, the constrained MPC formulation developed in Chapter 2 is suitable for collision avoidance. A sufficient condition is given providing a way to reject completely disturbances via decoupling the vehicles' closed-loop dynamics. As shown in the 1-D problem, the inclusion of fixed references and modified constraints is proven to be able to decouple the vehicles in their closed-loop dynamics via keeping the modified constraints inactive and thereby to achieve string stability. The relation revealed between the control performance and the communication disturbance is a guide to the design of the cross-estimator and the bandwidth assignment strategies.

In Chapter 4 and 5, we discussed the cross-estimator design and the bandwidth assignment problems. Chapter 4 focused on the cross-estimator design under a non-classical information structure, i.e. the estimator knows the control laws but not the exact control values. The significance of this chapter is the introduction of an LMI formulation that facilitates the cross-estimator design problem and can be extended to include the bandwidth assignment problem in Chapter 5. The vehicle formation in Chapter 5 applies the constrained MPC formulation proposed in Chapter 3. The main design problem is the minimum communication resources problem, which is to derive the cross-estimator gains and the corresponding bit-rates assignment strategy that minimize the total communication resources subject to the requirements of stable estimators, the control performance, and the limited communication resources.

These ideas are synthesized in the application of constrained control to systems with stochastic disturbances. The physical state constraints should be transformed into constraints on the state estimates with a standoff term to accommodate the state estimate quality, which is the task of Chapter 2. This type of constraint works as a bridge connecting the control performance and the information (state estimate) quality, which is presented in Chapter 3. The two directions of the bridge identify two design problems: (i) given the information quality what is the best control performance? This is briefly illustrated in Chapter 3; (ii) given a control performance requirement what is the minimum communication resources and what are the corresponding cross-estimators and the bandwidth assignment strategy? This is solved in Chapter 5.

6.2 Future Works

Further work remains to be tackled in extending the constrained MPC formulation developed in Chapter 2 to a wide range of more obviously nonlinear problems. Equally, there would be considerable advantage to developing a method to over-bound simply probabilities of constraint violation with normal approximations.

The ‘closed-loop covariance’ concept in constrained MPC is worth further exploring and should be considered as an analog to the receding horizon control idea which represents the closed-loop nature. In the more general cases, this term should be modified

to ‘closed-loop distribution’ and a proper choice is the key to guarantee the constraint satisfaction and yet avoid conservative performance.

A synthesized analysis of the coordinated control proposed in Chapter 3 and the information architecture design in Chapter 5 is also worth studying, such as a 2-D vehicle formation problem. It may help us to clarify:

- The application of the closed-loop covariance: in Chapter 3, we proposed to use the closed-loop covariance to modified the probabilistic constraints and in Chapter 5, we formulated the control performance imposing constraints on the open-loop cross-estimator error covariance. That is, we design the information architecture based on the constrained open-loop cross-estimator covariances and we apply closed-loop covariance in our control problem. Other choices may also be available, for example we may treat the self estimate covariance in the closed-loop way but the cross-estimate covariance in the open-loop fashion. A synthesized analysis may help to justify these strategies.
- The control performance requirement on the estimate covariance: in Chapter 5 we formulated a simple bound on the cross-estimate covariance matrix, which represents that no-collision constraint is not active all the time. In Chapter 3, we discussed the string stability and the activeness of the no-collision constraints. One may be interested in the relation between the rate of the constraints being active, the string stability, and the over-bound on the cross-estimator design. This question can only be answered in a more concrete framework.
- The establishment of communication links via the $MIN N$ LMI solution: in Chapter 5, we formulated the $MIN N$ LMI problem to provide the bandwidth assignment schemes, i.e. the bit-rates assigned to each variable. In Section 5.4, we discussed that by rounding up the numbers of bits, the cross-estimators are guaranteed to be better. However, the rounding-up strategy excludes the possibility of zero-bit channels (no channels), i.e. no matter how far away two vehicles are, they always need a channel to exchange information. It is worth developing a proper scheme to rounding-down certain numbers of bits with guaranteed performance, such as the bound imposed by the formation control requirement.

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