A Manifestly Gauge-Invariant Approach to Quantum Theories of Gauge Fields

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Abstract

In gauge theories, physical histories are represented by space-time connections modulo gauge transformations. The space of histories is thus intrinsically non-linear. The standard framework of constructive quantum field theory has to be extended to face these *kinematical* non-linearities squarely. We first present a pedagogical account of this problem and then suggest an avenue for its resolution.

1 Introduction

As is well-known, for over 40 years, quantum field theory has remained in a somewhat peculiar situation. On the one hand, perturbative treatments of realistic field theories in four space-time dimensions have been available for a long time and their predictions are in excellent agreement with experiments. It is clear therefore that there is something "essentially right" about these theories. On the other hand, their mathematical status continues to be dubious in all cases (with interactions), including QED. In particular, it is generally believed that the perturbation series one encounters here can be at best asymptotic. However, it is not clear what exactly they are asymptotic to.

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This overall situation is in striking contrast with, for example, non-relativistic quantum mechanics. There, we know well at the outset what the Hilbert space of states is and what the observables are. In physically interesting models, we can generally construct the Hamiltonian operator and show that it is self-adjoint. We take recourse to perturbation theory mainly to calculate its eigenvalues and eigenvectors. Therefore, if the perturbation series turns out not to be convergent but only asymptotic, we know what it is asymptotic to. The theories exist in their own right and perturbative methods serve as approximation techniques to extract answers to physically interesting questions. In realistic quantum field theories, we do not yet know if there is an underlying, mathematically meaningful framework whose predictions are mirrored in the perturbative answers. When one comes to QCD, the problem is even more severe. Now, it is obvious from observations that the physically relevant phase of the theory is the one in which quarks and gluons are confined. And this phase lies beyond the grasp of standard perturbative treatments.

To improve this situation, the program of constructive quantum field theory was initiated in the early seventies. This approach has had remarkable success in certain 2 and 3 dimensional models. From a theoretical physics perspective, the underlying ideas may be summarized roughly as follows. Consider, for definiteness, a scalar field theory. The key step then is that of giving meaning to the Euclidean functional integrals by defining a rigorous version $d\mu$ of the heuristic measure "$[\exp(-S(\phi))]\prod_x d\phi(x)$" on the space of histories of the scalar field, where $S(\phi)$ denotes the action governing the dynamics of the model. The appropriate space of histories turns out to be the space $\mathcal{S}'$ of (tempered) distributions on the Euclidean space-time and regular measures $d\mu$ on this space are in one to one correspondence with the so-called generating functionals χ, which are functionals on the Schwarz space $\mathcal{S}$ of test functions satisfying certain rather simple conditions. (Recall that the tempered distributions are continuous linear maps from the Schwarz space to complex numbers.)

Thus, the problem of defining a quantum scalar field theory can be reduced to that of finding suitable measures $d\mu$ on $\mathcal{S}'$, or equivalently, "appropriate" functionals χ on $\mathcal{S}$. Furthermore, there is a succinct set of axioms –due to Osterwalder and Schrader– which spells out the conditions that χ must satisfy for it to be "appropriate," i.e., for it to lead to a consistent quantum field theory which has an associated Hilbert space of states, a Hamiltonian, a vacuum and an algebra of observables. This strategy has led to the rigorous construction of a number of interesting theories in 2 and 3 dimensions such as the $\lambda\phi^4$ and the Yukawa models and to an understanding of their relation to perturbative treatments. A more striking success of these methods is that they have led to a rigorous construction of the Gross-Neveau model in 3-dimensions which, being non-renormalizable, fails to exist perturbatively in the conventional sense.

These successes are remarkable. However, we believe that the framework has an important limitation: As in the heuristic, theoretical physics treatments [1, 2], it is based on the assumption that the theories under consideration can be considered to be *kinematically linear* [3, 4, 5]. That is, even though dynamical non-linearities are properly incorporated, it is assumed that the *space $\mathcal{S}'$ of histories is a vector space.* This assumption permeates the whole framework. In particular, in their standard form, all of the key Osterwalder-Schrader axioms use this property. Now, for unconstrained systems –such as the $\lambda\phi^4$-model– this assumption is not restrictive. However, for constrained systems it is generally violated. An outstanding example is provided by the Yang-Mills theory. Now, because of the presence of constraints, the system has gauge freedom and the space of physically distinct histories is provided by $\mathcal{A}/\mathcal{G}$, the space of connections on the Euclidean space-time modulo gauge transformations. In dimensions $d+1 > 2$, this is a highly non-linear space with complicated topology. Rather than facing this non-linearity squarely, one often resorts to gauge fixing, ignores global problems such as those associated with Gribov ambiguities, obtains a linear space and proceeds to apply the standard techniques of constructive quantum field theory. To an outsider the disparity between the "roughness" with which $\mathcal{A}/\mathcal{G}$ is steam-rollered into a linear space and the sophistication with which functional analysis is then used to construct measures seems rather striking. It is natural to ask if one can not modify the general framework itself and tailor it to the kinematical non-linearities of $\mathcal{A}/\mathcal{G}$.

The purpose of this contribution is to suggest an avenue towards this goal. (For earlier work with the same motivation, see [6].) We should emphasize, however, that ours is only an approach: we will not be able to present a definitive generalization of the Osterwalder-Schrader axioms. Furthermore, the key steps of our framework are rather loose. However, it *is* tailored to facing the kinematical non-linearities of gauge theories squarely from the very beginning.

None of the authors are experts in constructive quantum field theory. The main ideas behind this approach came rather from an attempt to construct quantum general relativity non-perturbatively. Consequently, certain notions from quantum gravity –such as diffeomorphism invariance– play a non-trivial role in the initial stages of our constructions. This is a strength of the framework in the context of diffeomorphism invariant gauge theories. Examples are: the Yang-Mills theory in 2 dimensions which is invariant under area-preserving diffeomorphisms and the Chern-Simons theories in 3 dimensions and the Husain-Kuchař model [7] in 4 dimensions which are invariant under all diffeomorphisms. In higher dimensional Yang-Mills theories, on the other hand, the action does depend on the background Euclidean or Minkowskian geometry; it is not diffeomorphism invariant. In the general program sketched here, this geometry *is* used in subsequent steps of our constructions. However, we expect that, in a more complete and polished version, it would play an important role from the beginning. It is clear that considerable work is still needed to make the framework tight and refined, tailored more closely to quantum Yang-Mills fields. As the title of the contribution indicates, our primary aim is only to suggest a new approach to the problem, thereby initiating a re-examination of the appropriateness of the "standard" methods beyond the kinematically linear theories.

This contribution is addressed to working theoretical –rather than mathematical– physicists. Therefore, the presentation will be somewhat pedagogical. In particular, *we will not assume prior knowledge of the methods of constructive quantum field theory.* We begin in section 2 with a summary of the idea and techniques used in this area and indicate in particular how the appropriate measure is constructed for the $\lambda\phi^4$ model. As indicated above, this discussion will make a strong use of the kinematical linearities of models considered. In section 3, we turn to gauge theories and show how certain recent advances in the development of calculus on the space of connections modulo gauge transformations can be used to deal with the intrinsic kinematical non-linearities. In section 4, we indicate how these techniques can be used for Yang-Mills theories. In particular, we will construct the 2-dimensional Euclidean Yang-Mills theory and indicate its relation to the Hamiltonian framework. In section 5, we summarize the main results, point out some of the strengths and the limitations of this approach and discuss directions for further work.

2 Kinematically linear theories

Several of the ideas we will use in the construction of measures on infinite dimensional non-linear spaces are similar (but not identical!) to those used in the linear case. Therefore, in this section we will recall from [3, 4, 8, 9] some well known results about measures on infinite dimensional linear spaces in the context of constructive quantum field theory.

Let us consider an Euclidean scalar field theory on flat Euclidean spacetime $M = \mathrm{I\!R}^{d+1}$. It is natural to choose as the space $\mathcal{SH}$ of *classical* histories of the theory the *linear* space of suitably regular (say C^2 and rapidly decreasing at infinity) scalar field configurations; $\mathcal{SH} = \{\phi(x)\}$. The dynamics of the theory is determined by the action functional S on $\mathcal{SH}$

$$S(\phi) = \int_{\mathrm{I\!R}^{d+1}} \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + V(\phi(x))\right) d^{d+1}x \ , \tag{2.1}$$

where $V(\phi(x))$ denotes the self-interaction potential (which is assumed to be bounded from below). In the "classical" theory [1] we are interested in *points*

[1] We used quotation marks in the word "classical" because, strictly speaking, the solutions of the Euclidean equations of motion do not play a direct role in classical physics; they have physical interpretation only in the semi-classical approximation.

in $\mathcal{SH}$ (i.e. particular histories) that correspond to the extrema of S, i.e. in solutions of the (Euclidean) equation of motion

$$\Delta\phi - \frac{\partial V}{\partial \phi} = 0 \ . \tag{2.2}$$

If V is cubic or higher order in ϕ the equation (2.2) is non-linear and we have an example of a dynamically non-linear theory with a linear space of histories. In our terminology, this is an example of a *kinematically linear* but *dynamically non-linear* theory. Theories of non-abelian gauge fields, on the other hand, are examples of theories in which non-linearities are present already at the kinematical level.

In quantum field theory, the interest lies not in particular histories satisfying (2.2) but in summing over all histories, i.e. in defining measures on $\mathcal{SH}$ that correspond to the heuristic expression [3, 4, 5]

$$d\mu(\phi) = \text{“}\frac{1}{Z} e^{-S(\phi)} \prod_{x\in \mathbb{R}^{d+1}} d\phi(x)\text{”} \ . \tag{2.3}$$

We will begin in section 2.1 with a brief review of how measures are constructed on infinite dimensional linear spaces. In section 2.2, these techniques will be applied to two illustrative examples in constructive quantum field theory; the massive free scalar field in $d+1$ dimensions and the $\lambda\phi^4$-model in 2 dimensions.

2.1 Integration on $\mathcal{SH}$

We will first present an "algebraic" approach to integration and then summarize the situation from a measure-theoretic viewpoint. In the algebraic approach, the main idea is to reduce the problem of integration over infinite dimensional spaces to a series of integrations over finite dimensional spaces by judiciously choosing the functions one wants to integrate.

In a linear space like $\mathcal{SH}$ the simplest functions that one can introduce are the linear ones. Let $\mathcal{S}$ denote the space of all test (or smearing) functions on $\mathbb{R}^{d+1}$, i.e. the space of all infinitely differentiable functions which fall off sufficiently rapidly at infinity. $\mathcal{S}$ is called Schwarz space. Its elements $e \in \mathcal{S}$ can be used to *probe* the structure of scalar fields $\phi \in \mathcal{SH}$ through linear functions F_e on $\mathcal{SH}$ defined by

$$F_e(\phi) = \int_{\mathbb{R}^{d+1}} \phi(x) e(x) \ d^{d+1}x \ ; \tag{2.4}$$

the test field e probes the structure of ϕ because it captures part of the information contained in ϕ, namely, the "component of ϕ along e". We can probe the behaviour of ϕ in the neighbourhood of a point in M by choosing test fields e_n supported in that neighbourhood.

To begin with, we want to define integrals of such simple functions. Let $e_1, ..., e_n$ denote arbitrarily chosen but fixed linearly independent probes. Consider the projection $p_{e_1,...,e_n}$ they define

$$\begin{array}{rcccl} p_{e_1,...,e_n} & : & \mathcal{SH} & \rightarrow & \mathbb{R}^n \\ & & \phi & \mapsto & (F_{e_1}(\phi), ..., F_{e_n}(\phi)) \ . \end{array} \tag{2.5}$$

Next, consider functions f on $\mathcal{SH}$ that depends on ϕ only through their "n-components" $F_{e_1}(\phi), ..., F_{e_n}(\phi)$, i.e. functions of the type

$$f(\phi) = \tilde{f}\,(F_{e_1}(\phi), ..., F_{e_n}(\phi)) \tag{2.6}$$

or equivalently

$$f = p^*_{e_1,...,e_n} \tilde{f} \ , \tag{2.7}$$

where $\tilde{f}$ is a (well-behaved) function on $\mathbb{R}^n$. The function f is said to be *cylindrical* with respect to the finite dimensional subspace $V_{e_1,...,e_n}$ (of $\mathcal{S}$) spanned by the probes $\{e_1, ..., e_n\}$. These are the functions we first want to integrate.

This task is easy because the cylindrical functions are "fake" infinite dimensional functions: although defined on $\mathcal{SH}$, their "true" dependence is only on a finite number of variables. Fix a (normalized, Borel) measure $d\mu_{e_1,...,e_n}$ on $\mathbb{R}^n$ and simply define the integral of f over $\mathcal{SH}$ to be the integral of $\tilde{f}$ over $\mathbb{R}^n$ with respect to $d\mu_{e_1,...,e_n}$:

$$\int_{\mathcal{SH}} f(\phi) d\mu(\phi) := \int_{\mathbb{R}^n} \tilde{f}(\eta_1, ..., \eta_n)\ d\mu_{e_1,...,e_n}(\eta_1, ..., \eta_n) \ . \tag{2.8}$$

Next, in order to be able to integrate functions cylindrical with respect to *any* finite dimensional subspace of the probe space, we select, for every collection $\{e_1, ..., e_n\}$ of linearly independent probes, and every $n \in \mathbb{N}$, a measure $d\mu_{e_1,...,e_n}$ on $\mathbb{R}^n$ and define the integral of that cylindrical function over $\mathcal{SH}$ by (2.8). This is the key technique in the algebraic approach.

The procedure seems trivial at first sight. There is, however, a catch. Since a function which is cylindrical with respect to a linear subspace V of $\mathcal{SH}$ is necessarily cylindrical with respect to a linear subspace V' if $V \subseteq V'$, the representation (2.6) is *not* unique. (Indeed, even for fixed V, the explicit form of (2.6) depends on the basis we use). For the left side of (2.8) to be well-defined, therefore, the finite dimensional measures $(d\mu_{e_1,...,e_n})$ must satisfy non-trivial consistency conditions; these serve precisely to ensure that the integral of $f(\phi)$ over $\mathcal{SH}$ is independent of the choice of a particular representation of f as a cylindrical function. When these consistency conditions are satisfied, $\mathcal{SH}$ is said to be equipped with a *cylindrical measure.*

Let us consider a simple but representative example of consistency conditions. Consider $e, \hat{e} \ \in \ \mathcal{S}$ and let $V_1,\ V_2$ be the one and two-dimensional spaces spanned by e and $e,\ \hat{e}$ respectively and f be the function on $\mathcal{SH}$, cylindrical with respect to V_1, given by

$$f(\phi) = \widetilde{f}_1\,(F_e(\phi)) := \exp\ [i\lambda \int_{\mathbb{R}^{d+1}}\ e(x)\phi(x)\ d^{d+1}x]\ . \tag{2.9}$$

This function is clearly cylindrical also with respect to V_2, i.e. it is a function of F_e and $F_{\hat{e}}$ that just happens not to depend on $F_{\hat{e}}$:

$$f(\phi) = \widetilde{f}_2\,(F_e(\phi), F_{\hat{e}}(\phi)) := \exp\ [i\lambda \int_{\mathbb{R}^{d+1}}\ e(x)\phi(x)\ d^{d+1}x]\ . \tag{2.10}$$

To obtain a cylindrical measure, therefore, f, $d\mu_e$,and $d\mu_{e,\hat{e}}$ must satisfy:

$$\int_{\mathcal{SH}}\ f(\phi)\ d\mu(\phi) = \int_{\mathbb{R}} e^{i\lambda\eta} d\mu_e(\eta) = \int_{\mathbb{R}^2} e^{i\lambda\eta_1} d\mu_{e,\hat{e}}(\eta_1, \eta_2)\ . \tag{2.11}$$

It is easy to see that this equality holds for our choice of f *and* for any other integrable function, cylindrical with respect to V_1, if and only if the measures satisfy the following consistency condition

$$d\mu_e(\eta) = \int_{\mathbb{R}} d\mu_{e,\hat{e}}(\eta, \hat{\eta})\ . \tag{2.12}$$

A natural solution to the consistency conditions is obtained by choosing all the $d\mu_{e_1,\ldots,e_n}$ to be normalized Gaussian measures. Then, the resulting $d\mu$ on $\mathcal{SH}$ is called a *Gaussian cylindrical measure*.

Associated with every cylindrical measure (not necessarily Gaussian) on $\mathcal{SH}$, there is a function χ on the Schwarz space $\mathcal{S}$ of probes, called the Fourier transform of the measure by analysts and the generating function (with imaginary current) by physicists:

$$\chi(e) := \int_{\mathcal{SH}}\ \exp\ [i \int_{\mathbb{R}^{d+1}} e(x)\phi(x) d^{d+1}x]\ d\mu(\phi) = \int_{\mathbb{R}} e^{i\eta} d\mu_e(\eta)\ . \tag{2.13}$$

To see that χ is the generating functional used in the physics literature, let us substitute the heuristic expression (2.3) of $d\mu$ in (2.13) to obtain:

$$\begin{aligned}\chi(e) \quad = \quad & \text{“}\frac{1}{Z} \int_{\mathcal{SH}}\ \exp[i \int_{\mathbb{R}^{d+1}} e(x)\phi(x) d^{d+1}x]\ \times \\ & \exp[-\left(\int_{\mathbb{R}^{d+1}} \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) + V(\phi(x))\right) d^{d+1}x] \prod_{x\in\mathbb{R}^{d+1}} d\phi(x)\text{”}\ .\end{aligned} \tag{2.14}$$

From the properties of these heuristic generating functionals discussed in the physics literature, one would expect χ to contain the complete information about $d\mu$. This is indeed the case.

In fact, such generating functionals can serve as a powerful tool to *define* non-Gaussian measures $d\mu$. This is ensured by a key result in the subject, the *Bochner theorem* [8, 9], which has the following consequence :

Let $\chi(e)$ be a function on the Schwarz space $\mathcal{S}$ satisfying the following three conditions :

$$\begin{array}{ll} (i) & \chi(0) = 1 \\ (ii) & \chi \text{ is continuous in every finite dimensional subspace of} \mathcal{S} \\ (iii) & \text{For every } e_1, ..., e_N \in \mathcal{S} \text{ and } c_1, ..., c_N \in \mathbb{C} \text{ we have} \\ & \sum_{i,j=1}^{N} \overline{c_i} c_j \chi(-e_i + e_j) \geq 0 \ . \end{array} \tag{2.15}$$

Then, there exists a unique cylindrical measure $d\mu$ on $\mathcal{SH}$ such that χ is its generating functional. Thus, functions χ satisfying (2.15) are generating functionals of cylindrical measures on $\mathcal{SH}$ and every generating functional is of this form. This concludes the "algebraic" part of our discussion.

We now turn to the measure-theoretic part and ask if the above procedure for integrating cylindrical functions actually defines a genuine measure –i.e., a σ-additive set function (see below)– on $\mathcal{SH}$ or a related space. This issue is important because a proper measure theoretic understanding would enable us to integrate functions on $\mathcal{SH}$ that are genuinely infinite dimensional, i.e., depend on infinitely many probes. Indeed, the classical action is invariably such a function. It turns out that one *can* define genuine measures, but to do so, we have to extend the space $\mathcal{SH}$. For later convenience, we will proceed in two steps. First, we will present a "maximal" extension and arrive at a space $\overline{\mathcal{SH}}$ which serves as "the universal home" for all cylindrical measures on $\mathcal{SH}$. In practice, however, the space $\overline{\mathcal{SH}}$ is too large in that the measures of interest to constructive quantum field theory are generally supported on a significantly smaller subspace $\mathcal{S}'$ of $\overline{\mathcal{SH}}$ (which are still larger than $\mathcal{SH}$). Then, in the second step, we will discuss this "actual home," $\mathcal{S}'$, for physically interesting measures.

The universal home, $\overline{\mathcal{SH}}$, is simply the algebraic dual of $\mathcal{SH}$: the space of *all* linear functionals on the probe space $\mathcal{S}$. This space is "very large" because we have not required the maps to be continuous in any topology on $\mathcal{S}$. It is easy to check that $\mathcal{S}$ also serves as the space of probes for $\overline{\mathcal{SH}}$ and that, given a consistent family of measures $d\mu_{e_1,...,e_n}$ that defines a cylindrical measure $d\mu$ on $\mathcal{SH}$, it also defines a cylindrical measure, say $d\bar{\mu}$, on $\overline{\mathcal{SH}}$. ($\overline{\mathcal{SH}}$ is the "largest" space for which this result holds.) Now, for a cylindrical measure to define a genuine, infinite dimensional measure, it has to be extendible in the following sense. For a cylindrical measure, the measurable sets are all cylindrical, i.e., inverse images, under projections $p_{e_1,...,e_n}$ of (2.5), of measurable sets in $(\mathbb{R}^n, d\mu_{e_1,...,e_m})$. The measure of a cylindrical set is just the measure of its image under the projection. Now, for $d\bar{\mu}$ to be a genuine measure, it has to be σ-additive, i.e., the measure of a *countable* union of non-intersecting measurable sets has to equal the sum of their measures. Unfortunately, although the union of a finite number of cylindrical sets is again cylindrical, in general the same is *not* true for a countable number. Thus, the question is whether we can add countable unions to our list of measurable sets and *extend* $d\bar{\mu}$ consistently. It turns out that *every* cylindrical measure on $\overline{\mathcal{SH}}$ can be so extended. (This is in general *not* true for $\mathcal{SH}$; even the Gaussian cylindrical measures on $\mathcal{SH}$ may not be extendable to σ-additive measures thereon. In particular this happens for physically interesting measures.) This is why $\overline{\mathcal{SH}}$ can be regarded as the "universal home" for cylindrical measures.

Unfortunately, the space $\overline{\mathcal{SH}}$ is typically too big for quantum scalar field theories: The *actual* home of a given measure $d\bar{\mu}$ is generally a smaller space, say $\widehat{\mathcal{SH}}$, of better behaved generalized histories, in the sense that $\bar{\mu}(\overline{\mathcal{SH}} - \widehat{\mathcal{SH}}) = 0$. (More precisely, every measurable set U such that $U \subset \overline{\mathcal{SH}} - \widehat{\mathcal{SH}}$ has zero measure $\bar{\mu}(U) = 0$.) Since $\widehat{\mathcal{SH}}$ has a richer structure, it is most natural (and, in practice, essential) to ignore $\overline{\mathcal{SH}}$ and work directly with $\widehat{\mathcal{SH}}$.

The key result which helps one determine the actual home of a measure is the Bochner-Minlos theorem [9]. The version of this theorem which is most useful in scalar field theories can be stated as follows :
σ-additive measures on the space $\mathcal{S}'$ of continuous linear functionals on the probe space $\mathcal{S}$ (equipped with its natural nuclear topology $\tau_{(n)}$) are in one to one correspondence with generating functions χ on $\mathcal{S}$, satisfying the conditions (i), (iii) of (2.15) and

$$(ii')\chi \text{ is continuous with respect to } \tau_{(n)}. \tag{2.16}$$

(The space $\mathcal{S}'$ is of course the space of tempered distributions.) Since the topology on $\mathcal{S}$ used in (ii') is weaker than that in (ii), the Bochner-Minlos theorem expresses the general trend that the *weaker* the topology with respect to which the generating function χ is continuous, the *smaller* is the support of the resulting measure $d\bar{\mu}$.

This concludes our discussion of mathematical preliminaries. Let us summarize. One can define cylindrical measures on $\mathcal{SH}$ which enable one to integrate cylindrical functions. However, to integrate "genuinely infinite dimensional" functions, one needs a genuine measure. The algebraic dual $\overline{\mathcal{SH}}$ of the space $\mathcal{S}$ of probes is the univeral home for such measures in the sense that every cylindrical measure on $\mathcal{SH}$ can be extended to a genuine measure on $\overline{\mathcal{SH}}$. In practice, however, physically interesting measures have a much smaller support $\widehat{\mathcal{SH}}$ which, however, is larger than $\mathcal{SH}$. (Indeed, typically $\mathcal{SH}$ has *zero* measure.) The Bochner-Milnos theorem provides a natural avenue to constructing such measures.

2.2 Scalar field theories

For a measure $d\bar{\mu}$ to correspond to a physically interesting quantum scalar field theory, it has to satisfy (some version) of the Osterwalder-Schrader axioms [3, 4]. These axioms guarantee that from the measure it is possible to construct the physical Hilbert space with a well defined Hamiltonian and Green functions with the appropriate properties. They are based on the assumption that the actual home for measures that correspond to quantum scalar field theories is the space $\mathcal{S}'$ of tempered distributions. Thus, the appropriate histories for quantum field theories are distributional. In fact, typically, the measure is *concentrated* on genuine distributions in the sense that $\bar{\mu}(\mathcal{S}' - \mathcal{SH}) = 1$. This is the origin of ultra-violet divergences: while the measure is concentrated on distributions, the action (2.1) is ill-defined if the histories are distributional.

We will denote the measures on $\mathcal{S}'$ by $d\hat{\mu}$. The Osterwalder-Schrader axioms restrict the class of possible measures. The most important of these axioms are: Euclidean invariance and reflection positivity. The first requires that the measure $d\hat{\mu}$ be invariant under the action of the Euclidean group on $\mathbb{R}^{d+1}$. Reflection positivity is the axiom that allows to construct a physical Hilbert space with a non-negative self-adjoint Hamiltonian acting on it. Let θ denote the time reflection, i.e. reflection with respect to the hyperplane

$$(x^0 = 0, x^1, ..., x^d) \ . \tag{2.17}$$

Consider the subspace $\mathcal{R}^+$ of $L^2(\mathcal{S}', d\hat{\mu})$ of (cylindrical) functions of the form

$$f(\hat{\phi}) = \sum_{j=1}^{N} c_j e^{i\hat{\phi}(e_j^+)} \ , \tag{2.18}$$

where $c_j \in \mathbb{C}$ and e_j^+ are arbitrary probes with support in the $x^0 > 0$ half-space. Then, the reflection positivity of the measure is the condition that

$$< \theta f, f >_{L^2} = \int_{\mathcal{S}'} [(\theta f)(\hat{\phi})]^\star \, f(\hat{\phi}) \, d\hat{\mu}(\hat{\phi}) \ \geq 0 \ . \tag{2.19}$$

(If reflection invariance is satisfied for one choice of Euclidean coordinates, by Euclidean invariance of the measure, it is satisfied for any other choice.)

The Hilbert space and the Hamiltonian can then be constructed as follows. (2.19) provides a degenerate inner product $(.,.)$ on $\mathcal{R}^+$, given by

$$(f_1, f_2) = < \theta f_1, f_2 >_{L^2} \ . \tag{2.20}$$

Denoting by $\mathcal{N}$ the subspace of $(.,.)$-null vectors on $\mathcal{R}^+$ we obtain a Hilbert space $\mathcal{H}$ by taking the quotient $\mathcal{R}^+/\mathcal{N}$ and completing it with respect to $(.,.)$:

$$\mathcal{H} = \overline{\mathcal{R}^+/\mathcal{N}} \ . \tag{2.21}$$

The Euclidean invariance provides an unitary operator $\widehat{T}_t \ , \ t > 0$, of time translations on $L^2(\mathcal{S}', d\hat{\mu})$. It in turn gives rise to the self-adjoint contraction operator on $\mathcal{H}$:

$$e^{-Ht} \ , \ t > 0 \ , \tag{2.22}$$

where H is the Hamiltonian.

Finally, we can formulate the two axioms in terms of the generating functional. Let $E : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}$ denote an Euclidean transformation. Then the condition that the measure be Euclidean invariant is equivalent to demanding:

$$\hat{\chi}(e \circ E) = \hat{\chi}(e) \ , \ \text{for every } E \ . \tag{2.23}$$

Next, let us consider reflection positivity. From the definition of the generating functional $\hat{\chi}$ (2.13) (here with $\mathcal{SH} \equiv \mathcal{S}'$) it follows that the condition of reflection positivity is equivalent to

$$\sum_{i,j=1}^{N} \bar{c}_i c_j \hat{\chi}(e_j^+ - \theta e_i^+) \ \geq \ 0 \tag{2.24}$$

for all $N \in \mathbb{N}$, $c_1, ..., c_N \ \in \ \mathbb{C}$ and for all $e_1^+, ..., e_N^+ \ \in \ \mathcal{R}^+$.

To summarize, constructive quantum field theory provides an elegant and compact characterization of quantum field theory as a measure $d\hat{\mu}$ on $\mathcal{S}'$ or as a (generating) function $\hat{\chi}$ on the space $\mathcal{S}$ of test functions, satisfying certain conditions. So all the work can be focussed on finding (or at least proving the existence of) appropriate $d\hat{\mu}$ or $\hat{\chi}$.

To conclude this section, we will provide two examples of such measures.

The first example is that of a free, massive scalar field on $\mathbb{R}^{d+1}$. Note first, that, it follows from the discussion in section 2.1 that a measure $d\hat{\mu}$ is Gaussian if and only if its generating functional is Gaussian, i.e. if and only if

$$\hat{\chi}(e) = \exp[-\frac{1}{2}(e, \widehat{C}e)] \ , \tag{2.25}$$

where $\widehat{C}$ is a positive definite, linear operator defined everywhere on $\mathcal{S}$. $\widehat{C}$ is called the *covariance* of the resulting Gaussian measure $d\hat{\mu}$. The free massive quantum scalar field theory corresponds to the Gaussian measure $d\hat{\mu}_m$ with covariance

$$\left(\widehat{C}_m e\right)(x) = \int_{\mathbb{R}^{d+1}} (-\Delta + m^2)^{-1}(x,y) \ e(y) \ d^{d+1}y \ , \tag{2.26}$$

where the integral kernel is defined by:

$$(-\Delta + m^2)^{-1}(x,y) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \ d^{d+1}p \ \frac{e^{ip(x-y)}}{p^2 + m^2}.$$

Our next example is more interesting in that it includes interactions: the $\lambda\phi^4$ model in 2-dimensions. The classical action is now given by:

$$S(\phi) = \int_{\mathbb{R}^2} \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2}{2}\phi^2 + \lambda\phi^4 \right) d^2x \tag{2.27}$$

Therefore, the heuristic expression for the quantum measure is

$$\begin{aligned} d\mu_{\lambda,m} &= \frac{1}{Z_{\lambda,m}}\, e^{-\lambda\int_{\mathbb{R}^2}\hat\phi^4(x)d^2x}\, e^{-S_m(\hat\phi)} \prod_{x\in\mathbb{R}^2} d\hat\phi(x) \\ &= \frac{1}{\widetilde{Z}_{\lambda,m}} \left(e^{-\lambda\int_{\mathbb{R}^2}\hat\phi^4(x)d^2x}\right) d\hat\mu_m(\hat\phi)\ , \end{aligned} \tag{2.28}$$

where S_m is the action of the free field with mass m and $d\hat\mu_m$ denotes the (rigorous) Gaussian measure discussed above. The problem with this expression is that while $d\hat\mu$ is concentrated on distributional connections, the factor in the exponent is ill-defined for distributional $\hat\phi$.

One way of trying to make sense of (2.28) is by substituting the Gaussian measure $d\hat\mu_m$ by a regulated Gaussian measure with a smaller support on which the integrand is well-defined. This can be achieved, e.g., by replacing the covariance C_m in (2.26) with the cutoff covariance C_m^k, given in the momentum space by [4]

$$\widetilde{C}_m^k(p) = \frac{1}{(2\pi)^2}\frac{e^{-k(p^2+m^2)}}{p^2+m^2},\ k>0\ . \tag{2.29}$$

Then the resulting measure $d\mu_m^k$ "lives" on the space $\mathcal{L}$ of *infinitely differentiable* functions that grow at most logarithmically at infinity (more precisely as $\sqrt{\ln(|x|)}$). (This is an extremely useful property. Nonetheless, we cannot just use this measure as the physical one because, among other reasons, it does not satisfy reflection positivity.) However, the function

$$\lambda\int_{\mathbb{R}^2}\phi^4(x)d^2x \tag{2.30}$$

is still infrared divergent (almost everywhere with respect to $d\mu_m^k$). Therefore, we have to put an infrared cutoff by restricting space-time to a box of volume V. Of course if we now take the regulators away, $V\to\infty$ and $k\to 0$, then we return to the ill-defined expression (2.28). Therefore, the question is whether it is possible to change $\lambda\int_{\mathbb{R}^2}\phi^4(x)d^2x$ by an interacting term with the same leading order dependence on ϕ but such that the limit, when the regulators $V\to\infty,\ k\to 0$, exists, is non trivial and satisfies the Osterwalder-Schrader axioms.

In 2-dimensions the answer is in the affirmative if we substitute (2.30) by [4]

$$\lambda\int_{\mathbb{R}^2} :\phi^4(x): d^2x\ , \tag{2.31}$$

where $:\phi^4(x):$ denotes normal ordering with respect to the Gaussian measure $d\mu_m^k$

$$:\phi^4(x):\ =\ \phi^4(x) - 6C_m^k(0)\phi^2(x) + 3(C_m^k(0))^2\ . \tag{2.32}$$

The expression for the regulated generating functional then reads

$$\chi^{k,V}_{m,\lambda} := \frac{1}{\widetilde{Z}^{k,V}_{\lambda,m}} \int_{\mathcal{L}} \exp[i \int_V e\phi d^2x] \ \exp[-\lambda \int_V : \phi^4(x) : d^2x] \ d\mu^k_m(\phi) \ , \quad (2.33)$$

where $\mathcal{L}$ denotes the support of the measure (i.e., the space of C^∞ functions on $\mathbb{R}^2$ which grow at worst as $\sqrt{ln(|x|)}$). Finally, it can be now shown that the limit

$$\chi_{m,\lambda}(e) = \lim_{V\to\infty} \lim_{k\to 0} \chi^{k,V}_{m,\lambda}(e) \quad (2.34)$$

exists, is non-Gaussian and has the appropriate properties. Thus, while the extraordinarily difficult problem of rigorously constructing a quantum field theory is formulated succinctly in this approach as that of finding a suitable measure or generating functional, the actual task of finding physically interesting measures is correspondingly difficult.

Finally, note that the entire framework is "soaked in" kinematic linearity. The fact that the space $\mathcal{S}$ of probes and and home $\mathcal{S}'$ of measures are linear was exploited repeatedly in various steps.

3 Calculus on $\mathcal{A}/\mathcal{G}$

We now turn to gauge theories which are kinematically non-linear. Here, the classical space of histories is the infinite dimensional space $\mathcal{A}/\mathcal{G}$ of smooth connections modulo gauge transformations on the $d+1$ dimensional space-time M; $\mathcal{SH} = \mathcal{A}/\mathcal{G}$. We will assume the gauge group G to be a compact Lie group and, in specific calculations, take it to be $SU(2)$. In this section, we will summarize how one develops calculus on $\mathcal{A}/\mathcal{G}$ by suitably extending the arguments of section 2.1 [12, 15]. (Analogous methods for $d+1=2$ were first used in the second reference of [26]. For an alternative approach, see [6]). In the section 4, we will apply these ideas to a specific model along the lines of section 2.2

3.1 Cylindrical measures

The main idea ([11, 12]) is to substitute the linear duality between scalar field histories and test functions by the "non-linear duality" between connections and loops in M. This duality is provided by the parallel transport or holonomy map H, evaluated at a base point $p \in M$:

$$H : \ (\alpha, A) \to H_\alpha(A) = \mathcal{P}\exp(\oint_\alpha A.dl) \ \in \ G \ , \quad (3.1)$$

where α is a piecewise analytic loop in M and $A \in \mathcal{A}$. (For definiteness we will assume that A is explicitly expressed using (one of) the lowest dimensional representation(s) of the Lie algebra of G.) In view of this duality, loops will now be used to probe the structure of the space of connections. In the kinematically linear theories, the probes and the histories were objects of the same type; in the scalar field theories, for example, they were both functions on M. In gauge theories, the roles are played by quite different objects.

We will begin by specifying the precise structure of the space of probes. Fix a point $p \in M$ and consider only those loops α in M which are based at p. It is natural to define the following equivalence relation on the space of these loops:

$$\alpha' \sim \alpha \text{ iff } H_{\alpha'}(A) = H_\alpha(A), \ \forall A \in \mathcal{A} \ , \tag{3.2}$$

where $\mathcal{A}$ is the space of smooth connections on M. Denote the equivalence class by $[\alpha]_p$ and call it a (based) holonomic loop or a *hoop*. The set of all hoops, $\mathcal{HG}_p$, in M forms a group with respect to the product

$$[\alpha_1]_p \cdot [\alpha_2]_p = [\alpha_1 \circ_p \alpha_2]_p \ , \tag{3.3}$$

where $\alpha_1 \circ_p \alpha_2$ denotes the usual composition of loops at the base point p. For gauge theories, the hoop group $\mathcal{HG}_p$ will serve as the space of probes.

Let us now turn to connections. For simplicity, we will first consider the space $\mathcal{A}/\mathcal{G}_p$ where $\mathcal{G}_p$ is the subgroup of gauge transformations which are equal to the identity $1_G \in G$ at p. For each $\alpha \in \mathcal{HG}_p$ the holonomy H_α defines a G-valued function on $\mathcal{A}/\mathcal{G}_p$, i.e. a G-valued function on $\mathcal{A}$ which is invariant under $\mathcal{G}_p$-gauge transformations. These functions are sufficient to separate the points of $\mathcal{A}/\mathcal{G}_p$. That is, for every $[A_1]_p \neq [A_2]_p$ there exists a $\alpha \in \mathcal{HG}_p$ such that $H_\alpha(A_1) \neq H_\alpha(A_2)$, where $[A]_p$ denotes the $\mathcal{G}_p$ equivalence class of the connection A. In this sense, hoops play the role of non-linear probes for histories $[A]_p \in \mathcal{A}/\mathcal{G}_p$. Finally, note that each smooth connection $A \in \mathcal{A}$ defines a homomorphism H of groups,

$$H_{.}(A): \ \mathcal{HG}_p \to G; \ \ \alpha \to H_\alpha(A), \tag{3.4}$$

which is smooth in an appropriate sense ([16]).

As in section 2.2, we can now define cylindrical functions and cylindrical measures on $\mathcal{A}/\mathcal{G}_p$ using the "probe functions" F_α defined by $F_\alpha([A]_p) := H_\alpha(A)$. Our first task is to introduce the analogs of the projections (2.5). For this, we need the notion of strongly independent hoops. Following [12], we will say that hoops $[\beta_i]_p$ are strongly independent if they have representative loops β_i such that each contains an open segment that is traced exactly once and which intersects other representatives at most at a finite number of points. The notion of strong independence turns out to be the appropriate non-linear analog of the linear independence of probes used in section 2.1. In particular, we have the following result. Given n strongly independent hoops, $[\beta_i]_p, i = 1, ..., n$, there exists a set of projections, $p_{\beta_1,...,\beta_n} : \mathcal{A}/\mathcal{G}_p \to G^n$, given by:

$$p_{\beta_1,..,\beta_n}([A]_p) = (H_{\beta_1}(A), .., H_{\beta_n}(A)) \tag{3.5}$$

which are surjective [12].

Now, a function f on $\mathcal{A}/\mathcal{G}_p$ is called *cylindrical* with respect to the subgroup of $\mathcal{HG}_p$ generated by $\beta_1, .., \beta_n$ if it is the pull-back by $p_{\beta_1,..,\beta_n}$ of a (well-behaved) complex valued function $\tilde{f}$ on G^n

$$f([A]_p) = \tilde{f}(H_{\beta_1}(A), .., H_{\beta_n}(A)). \tag{3.6}$$

Consider now a family $\{d\mu_{\beta_1,..,\beta_n}\}$ of (positive, normalized) measures on G^n, one for each set $\{\beta_1, .., \beta_n\}$, $n \in \mathbb{N}$ of strongly independent hoops. As in section 2.1, this family of measures on finite dimensional spaces allows us to define in a unique way a *cylindrical measure* $d\mu$ on the infinite dimensional space $\mathcal{A}/\mathcal{G}_p$, provided that appropriate consistency conditions are satisfied. These conditions are again a consequence of the fact that the representation (3.6) of f as a cylindrical function is not unique. If the consistency conditions are satisfied, then the family $(d\mu_{\beta_1,..,\beta_n})$ defines a unique cylindrical measure $d\mu$ on $\mathcal{A}/\mathcal{G}_p$ through

$$\int_{\mathcal{A}/\mathcal{G}_p} d\mu([A]_p) f([A]_p) := \int_{G^n} d\mu_{\beta_1,..,\beta_n}(g_1, .., g_n)\tilde{f}(g_1, .., g_n) \tag{3.7}$$

where f is the pull-back of $\tilde{f}$ as in (3.6).

In the linear case, Gaussian measures on $\mathbb{R}^n$ provided a natural way to meet the consistency conditions. In the present case, one can use the Haar measure on G^n. More precisely, the consistency conditions are satisfied if one chooses [12]

$$d\mu^0_{\beta_1,..,\beta_n}(g_1, .., g_n) = d\mu_H(g_1)..d\mu_H(g_n)\ , \tag{3.8}$$

where $d\mu_H$ is the normalized Haar measure on G. This family $(d\mu^0_{\beta_1,..,\beta_n})$ leads to a measure cylindrical $d\mu^0$ on $\mathcal{A}/\mathcal{G}_p$ which is in fact invariant under the action of diffeomorphisms [2] of M.

3.2 "Universal home" and "actual home" for measures

As in the linear case, not every cylindrical measure on $\mathcal{A}/\mathcal{G}_p$ is a genuine, infinite dimensional measure. It turns out, however, that they can be extended to genuine measures on a certain completion, $\overline{\mathcal{A}/\mathcal{G}_p}$. This comes about as follows. Since $\mathcal{A}/\mathcal{G}_p$ is in one to one correspondence with the set of smooth homomorphisms from $\mathcal{HG}_p$ to G [16], a natural analog of the algebraic dual $\overline{\mathcal{SH}}$ of the space $\mathcal{S}$ of linear probes is now

$$\overline{\mathcal{A}/\mathcal{G}_p} := Hom(\mathcal{HG}_p, G), \tag{3.9}$$

[2] Additional solutions of the consistency conditions leading to diffeomorphism invariant measures on $\mathcal{A}/\mathcal{G}_p$ were found by Baez in [13]. The resulting measures are sensitive to certain kinds of self-intersections of loops. Such measures are relevant for quantum gravity, formulated as a dynamical theory of connections [18],[19].

i.e., the set of *all* homomorphisms (without any continuity condition) from the hoop group to the gauge group [12]. Elements of $\overline{\mathcal{A}/\mathcal{G}_p}$ will be denoted by $\bar{A}$ and called *generalized connections*. Just as $\mathcal{S}$ continues to serve as the space of probes for $\overline{\mathcal{SH}}$ in the linear case, the group $\mathcal{HG}_p$ continues to provide non-linear probes for the space $\overline{\mathcal{A}/\mathcal{G}_p}$ of generalized connections. Furthermore, as in the linear case, every consistent family $(d\mu_{\beta_1,..,\beta_n})$ of measures on G^n that defines a cylindrical measure $d\mu$ on $\mathcal{A}/\mathcal{G}_p$ through (3.7) also defines a measure $d\bar{\mu}$ on $\overline{\mathcal{A}/\mathcal{G}_p}$ by

$$\int_{\overline{\mathcal{A}/\mathcal{G}_p}} \bar{f}(\bar{A})d\bar{\mu}(\bar{A}) = \int_{G^n} \tilde{f}(g_1,..,g_n)d\mu_{\beta_1,..,\beta_n}(g_1,..,g_n) \tag{3.10}$$

where $\bar{f}$ is an arbitrary cylindrical function on $\overline{\mathcal{A}/\mathcal{G}_p}$ with respect to the subgroup of $\mathcal{HG}_p$ generated by $\beta_1,..,\beta_n$; $\bar{f}(\bar{A}) = \tilde{f}(\bar{A}(\beta_1),..,\bar{A}(\beta_n))$. Finally, every measure $d\bar{\mu}$ on $\overline{\mathcal{A}/\mathcal{G}_p}$ defined as in (3.10) can be extended to a σ-additive measure on $\overline{\mathcal{A}/\mathcal{G}_p}$ and is thus a genuine, infinite dimensional measure [9, 12, 14].

From a physical point of view, however, we should still factor out by the gauge freedom *at* the base point p. A generic gauge transformation $g(.) \in \mathcal{G}$ acts on $\overline{\mathcal{A}/\mathcal{G}_p}$ simply by conjugation

$$(g \circ \bar{A})(\beta) = g(p) \cdot \bar{A}(\beta) \cdot g^{-1}(p) \tag{3.11}$$

The physically relevant space $\overline{\mathcal{A}/\mathcal{G}}$ is therefore the quotient of $\overline{\mathcal{A}/\mathcal{G}_p}$ by this action. The classical space of histories $\mathcal{A}/\mathcal{G}$ is naturally embedded in $\overline{\mathcal{A}/\mathcal{G}}$ through

$$[A] \to \{gH_{.}(A)g^{-1},\ g \in G\} \tag{3.12}$$

where $[A]$ denotes the gauge-equivalence class of the connection A. By integrating gauge-invariant functions on $\overline{\mathcal{A}/\mathcal{G}_p}$ with the help of $d\bar{\mu}$ we obtain a measure on $\overline{\mathcal{A}/\mathcal{G}}$ that we will denote also by $d\bar{\mu}$. This measure is of course just the push-forward of the measure on $\overline{\mathcal{A}/\mathcal{G}_p}$ under the canonical projection $\pi : \overline{\mathcal{A}/\mathcal{G}_p} \to \overline{\mathcal{A}/\mathcal{G}}$. Again, these measures $d\bar{\mu}$ on $\overline{\mathcal{A}/\mathcal{G}}$, associated with consistent families $(d\mu_{\beta_1,..,\beta_n})$, are always extendible to $\sigma-$additive measures. Thus, for gauge theories, $\overline{\mathcal{A}/\mathcal{G}}$ serves as the universal home for measures (for which the traces of holonomies are measurable functions) in the same sense that $\overline{\mathcal{SH}}$ is the universal home for measures in the linear case.

It is therefore natural to ask if there is a "non-linear" analog of the Bochner theorem. The answer is in the affirmative. In fact the arguments are now simpler because both $\overline{\mathcal{A}/\mathcal{G}_p}$ and $\overline{\mathcal{A}/\mathcal{G}}$ are compact Hausdorff spaces (with respect to the natural Gel'fand topologies). To see this explicitly, let us now restrict ourselves to the gauge group $G = SU(2)$. In this case, the Mandelstam identities imply that the vector space generated by traces of holonomies T_α is closed under multiplication. This in turn implies that the entire information about the measure is contained in the image of the "loop transform:"

$$\chi(\beta) = \int_{\overline{\mathcal{A}/\mathcal{G}}} \bar{T}_\beta([\bar{A}])\ d\bar{\mu}([\bar{A}])\ , \tag{3.13}$$

where $\bar{T}_\alpha[\bar{A}] = \frac{1}{2}\mathrm{tr}\bar{A}[\alpha]$ is the natural extension to $\overline{\mathcal{A}/\mathcal{G}}$ of the "trace of the holonomy (or Wilson loop) function" on $\mathcal{A}/\mathcal{G}$. Note that (3.13) is the natural non-linear analog of the Fourier transform (2.13). The generating function χ is again a function on the space of probes which now happens to be the hoop group $\mathcal{HG}$ rather than the Schwarz space $\mathcal{S}$. From the normalization and the positivity of $d\bar{\mu}$ it is easy to see that $\chi(\alpha)$ satisfies

$$\begin{aligned} &(i) \qquad \chi(p) = 1 \\ &(ii) \qquad \sum_{i,j=1}^{N} \bar{c}_i c_j [\chi(\beta_i \circ \beta_j) + \chi(\beta_i \circ \beta_j^{-1})] \geq 0,\ \forall \beta_i \in \mathcal{H}\ , \end{aligned} \tag{3.14}$$

where p denotes the trivial (i.e., identity) hoop, $c_i \in \mathbb{C}$, are arbitrary complex numbers and $N \in \mathbb{N}$ is an arbitrary integer. Finally, the Riesz-Markov theorem [17] implies that every generating functional χ satisfying $\sum_i c_i \chi(\beta_i) = 0$ whenever $\sum_i c_i T_{\beta_i} = 0$ is the loop transform of a measure $d\bar{\mu}$ so that there is a one to one correspondence between positive, normalized (regular, Borel) measures on $\overline{\mathcal{A}/\mathcal{G}}$ and generating functionals on $\mathcal{HG}_p$ [11]. This result is analogous to the Bochner theorem in the linear case.

As in section 2.2, a given measure $\bar{\mu}_0$ on $\overline{\mathcal{A}/\mathcal{G}}$ may be supported on a smaller space of better behaved generalized connections (support or actual home for $d\bar{\mu}_0$). Indeed, just as $\overline{\mathcal{SH}}$ is "too large" for scalar field theories, we expect that $\overline{\mathcal{A}/\mathcal{G}}$ is "too large" for Yang-Mills theories. To see this, note that in (3.14), no continuity condition was imposed on the generating functional $\chi_{\bar{\mu}_0}$ (or, equivalently, the generating functionals are assumed to be continuous only with the discrete topology on $\mathcal{HG}_p$). Now, in Yang-Mills theories (with $d+1 > 2$), a background space-time metric *is* available and it can be used to introduce suitable topologies on $\mathcal{HG}_p$ which would be much weaker than the discrete topology. It would be appropriate to require that the Yang-Mills generating functional χ be continuous with respect to one of these topologies. Now, from our experience in the linear case, it seems reasonable to assume that the following pattern will emerge: the weaker the topology with respect to which χ is continuous, the smaller will be the support of $d\bar{\mu}$. Thus, we expect that, in Yang-Mills theories, physically appropriate continuity conditions will have to be imposed and these will restrict the support of the measure considerably. What is missing is the non-linear analog of the Bochner-Minlos theorem which can naturally suggest what the domains of the physically interesting measures should be.

The situation is very different in diffeomorphism invariant theories of connections such as general relativity. For these theories, diffeomorphism invariant measures such as $d\bar{\mu}^0$ are expected to play an important role. There are indications that the generating functionals of such measures will be continuous only in topologies which are much stronger than the ones tied to background metrics on M. Finally, since $d\bar{\mu}^0$ is induced by the Haar measure on the gauge group, it is very closely related to the measure used in the lattice gauge theories. Therefore, it is possible that even in Yang-Mills theories, it may serve as a "fiducial" measure –analogous to μ_m^k of the scalar field theory– in the actual construction of the physical measure. We will see that this is indeed the case in $d+1=2$.

4 Example : Quantum Yang Mills Theory in two dimensions

In this section we will show how the mathematical techniques introduced in section 3 can be used to construct Yang-Mills theory in 2 dimensions in the cases when the underlying space-time M is topologically $\mathbb{R}^2$ and when it is topologically $S^1 \times \mathbb{R}$. In particular, we will be able to show equivalence rigorous [3] between the Euclidean and the Hamiltonian theories; to our knowledge this was not previously demonstrated. The analysis of this case will also suggest an extension of the Osterwalder-Schrader axioms for gauge theories. Due to space limitations, however, we will present here only the main ideas; for details see [22].

4.1 Derivation of the continuum measure

We wish to construct the Euclidean quantum field theory along the lines of section 2.2. As indicated in section 3.2, the analog of the generating functional $\chi(e)$ on $\mathcal{S}$ is the functional $\chi(\alpha)$ on the hoop group $\mathcal{HG}_p$. In the present case, the heuristic expression of $\chi(\alpha)$ is:

$$\text{``}\chi(\alpha) = \int_{\mathcal{A}/\mathcal{G}} T_\alpha([A])\; e^{-S_{\rm YM}([A])} \prod_{x\in M} d[A](x)\text{''}\ , \qquad (4.1)$$

where $[A]$ is the gauge equivalence class to which the connection A belongs, $T_\alpha([A])$ is the trace of the holonomy of A around the closed loop α and S_{YM} is the Yang-Mills action. To obtain the rigorous analog of (4.1), we proceed, as in section 2.2, in the following steps: i) regulate the action using suitable cut-offs; ii) replace $\mathcal{A}/\mathcal{G}$ by a suitable completion thereof; iii) introduce a measure on this completion with respect to which the regulated action is measurable; iv) carry out the integration; and, v) take the appropriate limits to remove the cut-offs.

[3] Similar results were obtained via the heuristic Fadeev-Popov approach in [23].

To carry out the first step, we will use a lattice regularization. Introduce a finite, square lattice of total length L_x and L_τ in x and τ direction respectively, and lattice spacing a, using the Euclidean metric on M. Thus we have imposed both infrared and ultraviolet regulators. There are $(N_x+1)(N_\tau+1)$ vertices on that finite lattice in the plane (and $(N_\tau+1)N_x$ in the cylinder) where $N_x a := L_x,\ N_\tau a := L_\tau$. Denote the holonomy associated with a plaquette $\Box$ by $p_\Box$. There are no boundary conditions for the plane while on the cylinder we identify [24] the vertical link variables l associated with the open paths starting at the vertices $(0,\tau)$ and $(a\ N_x,\tau)$. The regularized action is then given by $S_{\text{reg}} = \beta S_W$ where $\beta = 1/(g_0 a^2)$, with g_0, the bare coupling constant, and S_W is the Wilson action given by:

$$S_W := \sum_{\Box}[1 - \frac{1}{N}\text{Re tr}(p_\Box))] \ , \tag{4.2}$$

where Re tr stands for "real part of the trace." Finally, we note certain consequences of this construction. Choose a vertex and call it the base point p. One can show that in two dimensions the based plaquettes form a complete set of loops which are independent in the sense that one can separately assign to each holonomy associated with a plaquette an arbitrary group element. This means that they can, in particular, be used as independent integration variables on the lattice. They can also be used to define projection maps of (3.5).

The next step is to find the appropriate extension of $\mathcal{A}/\mathcal{G}$ and a measure thereon. For this, we note that since $\overline{\mathcal{A}/\mathcal{G}}$ of section 3.2 is the space of all homomorphisms from $\mathcal{HG}_p$ to G, for any given lattice, the Wilson action (4.2) can be regarded as a bounded cylindrical function $\bar{S}_W$ on $\overline{\mathcal{A}/\mathcal{G}}$. Therefore, this function is integrable with respect to the measure $d\bar{\mu}^0$. Since the push-forward of $d\bar{\mu}^0$ under the projections $p_{\beta_1,\dots,\beta_n}$ of (3.5) is just the Haar measure on G^n, using for β_i the plaquettes, we have:

$$\begin{aligned} \chi^{a,L_x,L_\tau}\ (\alpha) \ &:= \ \frac{1}{Z}\int_{\overline{\mathcal{A}/\mathcal{G}}} \bar{T}_\alpha(\bar{A})\ e^{-\beta\bar{S}_W}[\bar{A}]\ d\bar{\mu}^0 \\ &= \ \frac{1}{NZ(a,L_x,L_\tau)}\prod_l \int_G\ e^{-\beta S_W}\ \text{tr}(\prod_{l\in\alpha} H_l)\ d\mu_H(H_l) \end{aligned} \tag{4.3}$$

for any loop α contained in the lattice. Here $d\mu_H$ is the Haar measure on G and the partition function Z is defined through $\chi(p) = 1$ where, as before, p denotes the trivial hoop. Note that (4.3) is precisely the Wilson integral for computing the vacuum expectation value of the trace of the holonomy. Thus, the space $\overline{\mathcal{A}/\mathcal{G}}$ and the measure $d\bar{\mu}^0$ are tailored to the calculations one normally performs in lattice gauge theory.

We have thus carried out the first four steps outlined above. The last step, taking the continuum limit, is of course the most difficult one. It has been carried out for the general case $G = SU(N)$ [22]. For simplicity, however, in what follows, we will restrict ourselves to the Abelian case $G = U(1)$.

Consider the pattern of areas that the loop α creates on M and select simple loops β_i that enclose these areas. It can be shown that α can be written as the composition of the β_i and the completely horizontal loop c at "future time infinity". Let $k(\beta_i)$ and k be the effective winding numbers of the simple loops β_i, $i = 1, .., , n$ and of the homotopically non-trivial loop c respectively, in the simple loop decomposition of α (that is, the signed number of times that these loops appear in α). Define $|\beta_i|$ to be the number of plaquettes enclosed by β_i. Then, if we set

$$K_n(\beta) := \int_G [\exp -\beta(1 - \mathrm{Re}(g))] \ \ g^n \ d\mu_H(g), \tag{4.4}$$

the generating functional (on the plane or the cylinder) becomes

$$\chi(\alpha) = \prod_{i=1}^{n} [\frac{K_{k(\beta_i)}(\beta_i)}{K_0(\beta_i)}]^{|\beta_i|} \tag{4.5}$$

for $k = 0$ (and, in particular, for any loop in the the plane), and it vanishes identically otherwise. We can now take the continuum limit. The result is simply:

$$\chi(\alpha) = \lim_{a\to 0} \chi^{a,L_x,L_\tau}(\alpha) = \exp[-\frac{g_0^2}{2}\sum_{i=1}^{n} k(\beta_i)^2 \mathrm{Ar}(\beta_i)]. \tag{4.6}$$

if $k = 0$, and $\chi(\alpha) = 0$ if $k \neq 0$, where $\mathrm{Ar}(\beta)$ is the area enclosed by the loop β.

A number of remarks are in order. i) We see explicitly the area law in (4.6) (which would signal confinement in $d + 1 > 2$ dimensions). The same is true for non-Abelian groups. ii) As in the case of the $\lambda\phi^4$-model in 2-dimensions, no renormalization of the bare coupling g_0 was necessary in order to obtain a well-defined limit. This is a peculiarity of 2 dimensions. Indeed, in higher dimensions because the bare coupling does not even have the correct physical dimensions to allow for an area law. iii) It is interesting to note that the above expression is completely insensitive to the fact that we have not taken the infinite volume limit, $L_x, L_\tau \to \infty$ on the plane, and $L_\tau \to \infty$ on the cylinder. (The only requirement so far is that the lattice is large enough for the loop under consideration to fit in it.) Thus, the task of taking these "thermodynamic" limits is quite straightforward. iv) In higher dimensions, one can formulate a program for constuction of the measure along similar lines (although other avenues also exist). This may be regarded as a method of obtaining the continuum limit in the lattice formulation. The advantage is that if the limit with appropriate properties exists, one would obtain not only the Euclidean "expectation values" of Wilson loop functionals but also a genuine measure on $\overline{\mathcal{A}/\mathcal{G}}$, a Hilbert sapce of states and a Hamiltonian.

Finally, let us compare our result with those in the literature. First, there is complete agreement with [26]. However our method of calculating the vacuum expectation values, $\chi(\alpha)$, of the Wilson loop observables is somewhat simpler and more direct. Furthermore, it does not require gauge fixing or the introduction of a vector space structure on $\mathcal{A}/\mathcal{G}$ and we were able to treat the cases $M = \mathbb{R}^2$ and $M = S^1 \times \mathbb{R}$ simultaneously. More importantly, we were able to obtain a closed expression in the $U(1)$ case. Finally, the invariance of $\chi(\alpha)$ under area preserving diffeomorphisms is manifest in this approach; the huge symmetry group of the classical theory transcends to the quantum level. This important feature was not so transparent in the previous rigorous treatments (except for the second paper in [26]).

4.2 A proposal for Constructive Quantum Gauge Field theory

In this section, we will indicate how one might be able to arrive at a rigorous, non-perturbative formulation of quantum gauge theories in the continuum. The idea of course is to define a *quantum theory of gauge fields* to be a measure $d\bar{\mu}$ on $\overline{\mathcal{A}/\mathcal{G}}$ –whose support may be considerably smaller than $\overline{\mathcal{A}/\mathcal{G}}$– satisfying the analogs of the Osterwalder-Schrader axioms. These must be adapted to the kinematical non-linearity of gauge theories. We will discuss how the key axioms can be so formulated. As one would expect, they are satisfied in the 2-dimensional example discussed above. Using them, we will arrive at the Hamiltonian framework which turns out to be equivalent to that obtained from canonical quantization.

The two key axioms in the kinematically linear case were the Euclidean invariance and the reflection positivity. These can be extended as follows. Let $E: \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ denote an Euclidean transformation. We require:

$$\chi(E \circ \alpha) = \chi(\alpha) \tag{4.7}$$

for all E in the Euclidean group. The axiom of reflection positivity also admits a simple extension. Let us construct the linear space $\mathcal{R}^+$ of complex-valued functionals $\Psi_{\{z_i\},\{\beta_i\}}$ on $\overline{\mathcal{A}/\mathcal{G}}$ of the form:

$$\Psi_{\{z_i\},\{\beta_i\}}(\bar{A}) = \sum_{i=1}^{n} z_i \ \bar{T}_{\beta_i}[\bar{A}] = \sum_{i=1}^{n} z_i \ \bar{A}[\beta_i] \ , \tag{4.8}$$

where $\beta_i \in \mathcal{HG}$ are independent in the sense of the previous subsection and have support in the positive half space, and z_i are arbitrary complex numbers. Then a measure $d\bar{\mu}$ on $\overline{\mathcal{A}/\mathcal{G}}$ will be said to satisfy the reflection positivity axiom if:

$$(\Psi, \Psi) :=< \theta\Psi, \Psi >:= \int_{\overline{\mathcal{A}/\mathcal{G}}} (\theta\Psi(\bar{A}))^\star \Psi(\bar{A}) \ d\bar{\mu}(\bar{A}) \geq 0 \ , \tag{4.9}$$

where θ, as before, is the "time-reflection" operation [4]. The remaining Osterwalder-Schrader axioms can be extended to gauge theories in a similar manner [21, 22], although a definitve formulation is yet to emerge.

[4] This formulation is for the case when the gauge group is $U(1)$, or $SU(2)$. For

Let us now consider the 2-dimensional example for $G = U(1)$. Since the generating functional (4.6) is invariant under all area preserving diffeomorphisms, it is, in particular, invariant under the 2-dimensional Euclidean group on the plane [5] The verification of reflection positivity requires more work. We will simultaneously carry out this verification *and* the construction of the physical Hilbert space. Let us apply the Osterwalder-Schrader algorithm, outlined in section 2.2, to construct the physical Hilbert space and the Hamiltonian. First, a careful analysis provides us the null space $\mathcal{N}$: its elements are functionals on $\overline{\mathcal{A}/\mathcal{G}}$ of the type:

$$\tilde{\Psi}[\bar{A}] = (\sum_{i=1}^{n} z_i \ \bar{T}_{\beta_i}[\bar{A}]) - (\sum_{i=1}^{n} z_i \chi(\beta_i)) \ \bar{T}_p[\bar{A}] \ , \tag{4.10}$$

provided all β_i are homotopically trivial. The physical Hilbert space $\mathcal{H}$ is given by the quotient construction, $\mathcal{H} = \mathcal{R}^+/\mathcal{N}$. On the plane, it is simply the linear span of the trivial vector $T_p = 1$; $\mathcal{H}$ is one-dimensional.

Let us now consider the more interesting case of the cylinder. If one of β_i or β_j contains a homotopically trivial loop and the other does not, $(\theta\beta_i)^{-1} \circ \beta_j$ necessarily contains a homotopically non-trivial loop and the characteristic function (4.6) vanishes at that loop. Finally, if both loops are homotopically non-trivial, then $(\theta\beta_i)^{-1} \circ \beta_j$ necessarily contains a homotopically non-trivial loop unless the effective winding numbers of the non-trivial loops in β_i and β_j are equal. Therefore, the linear spans of T_α's with zero and non-zero k are orthogonal under $(.,.)$. The former was shown to coincide with the null space generated by T_p. To display the structure of the Hilbert space $\mathcal{H} = \mathcal{R}^+/\mathcal{N}$, let us introduce a horizontal loop γ at $\tau = 0$ with winding number one. Then, any loop with winding number $k = n$ can be written as the composition of γ^n and homotopically trivial loops. Therefore $\beta = \gamma^{-n} \circ \alpha$ has zero winding number. Finally, using $\theta\gamma = \gamma$, it is easy to show that $T_{\gamma^n \circ \beta} - \chi(\beta) T_{\gamma^n} \in \mathcal{N}$, implies that the Hilbert space $\mathcal{H}$ is the completion of the linear span of the vectors T_{γ^n} $n \in \mathbb{Z}$. The vectors $\psi_n := T_{\gamma^n}$ form an orthonormal basis in $\mathcal{H}$. Since the quotient of $\mathcal{R}^+$ by the null sub-space $\mathcal{N}$ has a positive definite inner-product, $(.,.)$, it is clear that the generating function χ satisfies reflection positivity. Finally, note that, since the loop γ probes the generalized connections $\bar{A}$ only at "time" zero, the final result is analogous to that for a scalar field where the Hilbert space constution can also be reduced to to the fields at "time" zero.

Our next task is to construct the Hamiltonian H. By definition, H is the generator of the Euclidean time translation semi-group. Let $\gamma(\tau) := T(\tau)\gamma$

$SU(N)$ $N > 2$, the product of traces of holonomies is not expressible as a linear combination of traces of holonomies. Therefore, the argument of the functionals χ contain $1, 2, ..., N-1$ loops. However, the required extension is immediate.

[5] On the cylinder the Euclidean group, of course, has to be replaced by the isometry group of the metric, that is, the group generated by the Killing vectors ∂_τ and ∂_x.

be the horizontal loop at time τ and set $\alpha = \gamma \circ \rho \circ \gamma(\tau)^{-1} \circ \rho^{-1}$, where ρ is the vertical path between the vertices of the horizontal loops. Then, we have $T_{\gamma(\tau)^n} = \chi(\alpha^n) T_{\gamma^n}$ as elements of $\mathcal{H}$, so that

$$(\psi_m,\ T(\tau)\psi_n) \ = \ \chi(\alpha^n)\delta_{n,m} = [\exp(-\tfrac{1}{2}n^2 g_0^2 L_x \tau)]\ \delta_{n,m} \quad (4.11)$$

$$=: \ (\psi_n, \exp(-\tau H)\psi_m)\ . \quad (4.12)$$

Finally, the completeness of $\{\psi_n\}_{n\in\mathbb{Z}}$ enables us to write down the action of the Hamiltonian H simply as:

$$H\psi_n = \frac{g_0^2}{2} L_x E^2 \psi_n,\ \ E\psi_n = -in\psi_n\ . \quad (4.13)$$

Finally, it is clear that the vacuum vector is unique and given by $\Omega(\bar{A}) = 1$. Thus, because the key (generalized) Osterwalder axioms are satisfied by our continuum measure, we can construct the complete Hamiltonian framework.

To conclude, we note that the 2-dimensional model can also be quantized directly using the Hamiltonian methods [25, 22]. The resulting quantum theory is completely equivalent to the one obtained above, starting from the Euclidean theory.

5 Discussion

In this contribution we first pointed out that, in its standard form [3], the basic framework of constructive quantum field theory depends rather heavily on the assumption that the space of histories is linear. Since this assumption is not satisfied in gauge theories (for $d+1 > 2$), a fully satisfactory treatment of quantum gauge fields would require an extension of the framework. We then suggested an avenue towards this goal.

The basic idea was to regard $\mathcal{A}/\mathcal{G}$ as the classical space of histories and to attempt to construct a quantum theory by suitably completing it and introducing an appropriate measure on this completion. To achieve this, one can exploit the "non-linear duality" between loops and connections. More precisely, if one uses loops as probes –the counterpart of test functions in the case of a scalar field– one can follow the general methods used in the kinematically linear theories and introduce the notion of cylindrical functions and cylindrical measures on $\mathcal{A}/\mathcal{G}$. The question is if these can be extended to genuine measures. The answer turned out to be in the affirmative: there exists a completion $\overline{\mathcal{A}/\mathcal{G}}$ of $\mathcal{A}/\mathcal{G}$ such that every cylindrical measure on $\mathcal{A}/\mathcal{G}$ can be extended to a regular, σ-additive measure on $\overline{\mathcal{A}/\mathcal{G}}$. Thus, we have a "non-linear arena" for quantum gauge theories; the discussion is no longer tied to the linear space of tempered distributions. We were able to indicate how the Osterwalder-Schrader axioms can be generalized to measures on the non-linear space $\overline{\mathcal{A}/\mathcal{G}}$. The key open problem is that of singling out *physically appropriate* measures.

The space $\overline{\mathcal{A}/\mathcal{G}}$ is analogous to the algebraic dual $\overline{\mathcal{SH}}$ of the Schwarz space one encounters in the kinematically linear case. Therefore, it is almost certainly too big for quantum Yang-Mills theories (although there are indications that it is of the "correct size" for diffeomorphism invariant theories such as general relativity.) That is, although measures which would be physically relevant for Yang-Mills theories could be well-defined on $\overline{\mathcal{A}/\mathcal{G}}$, their support is likely to be significantly smaller. In the kinematically linear case, the Bochner-Milnos theorem provides tools to find physically relevant measures and tells us that their support is the space S' of tempered distributions. The analogous result is, unfortunately, still lacking in our extension to gauge theories. Without such a result, it is not possible to specify the exact mathematical nature of the Schwinger functions of the theory –the "Euclidean expectation values" of the Wilson loop operators. This in turn means that we have no results on the analytic continuation of these functions, i.e., on the existence of Wightman functions. What we *can* formulate, is the notion of reflection positivity and this ensures that the physical Hilbert space, the Hamiltonian and the vacuum exists.

It is clear from the above discussion that our framework is incomplete. We need to introduce appropriately weak topologies on the space of hoops –the probe space– and find generating functionals $\chi(\alpha)$ which are continuous with respect to them. Only then can one have sufficient control on the nature of Schwinger functions. For the moment, $\overline{\mathcal{A}/\mathcal{G}}$ serves only as the "universal home" for the measures we want to explore. As we saw, this strategy was successful in the 2-dimensional Yang-Mills theory.

In higher dimensions, there are reasons to be concerned that $\overline{\mathcal{A}/\mathcal{G}}$ may be too large to play even this "mild" role. That is, one might worry that the elements of $\overline{\mathcal{A}/\mathcal{G}}$ are allowed to be so "pathological" that it would be difficult to define on them the standard operations that one needs in mathematical physics. For instance, $\overline{\mathcal{A}/\mathcal{G}}$ arises only as a (compact, Hausdorff) topological space and does not carry a manifold structure. For physical applications on the other hand, one generally needs to equip the domain spaces of quantum states with operations from differential calculus. Would one not be stuck if one has so little structure? It turns out that the answer is in the negative: Using projective techniques associated with families of graphs, one *can* develop differential geometry on $\overline{\mathcal{A}/\mathcal{G}}$. In particular, notions such as vector fields, differential forms, Laplacians and heat kernels are well defined [15]. Thus, at least for the purposes we want to use $\overline{\mathcal{A}/\mathcal{G}}$, there is no *obvious* difficulty with the fact that it is so large.

A more subtle problem is the following. A working hypothesis of the entire framework is that the Wilson loop operators should be well-defined in quantum theory. From a "raw," physical point of view, this would seem to be a natural assumption: after all Wilson loop functionals are the natural gauge invariant observables. However, technically, the assumption *is* strong. For example, if the connection is assumed to be distributional, the Wilson loop functionals cease to be well-defined. Therefore, in a quantum theory based on such a hypothesis, the connection would not be representable as an operator valued distribution. In particular, in the case of a Maxwell field, such a quantum theory can not recover the textbook Fock representation [27]. More generally, the representations that can arise will be *qualitatively* different from the Fock representation in which the elementary excitations will not be plane waves or photon-like states. Rather, they would be "loopy," concentrated along "flux lines." They would be related more closely to lattice gauge theories than to the standard perturbation theory. In 2 dimensions, this reprentation does contain physically relevant states. Whether this continues to be the case in higher dimensions is not yet clear. It is conceivable that the quantum theories that arise from our framework are only of mathematical interest. However, even if this turns out to be the case, they would still be of considerable significance since as of now there does not exist a single quantum gauge theory in higher dimensions. Finally, if it should turn out that loops are too singular for physical purposes, one might be able to use the extended loop group of Gambini and co-workers [20] as the space of probes. This group has the same "flavor" as the hoop group in that it is also well tailored to incorporate the kinematical non-linearities of gauge theories. However, the hoops are replaced by extended, smoothened objects so that Fock-like excitations are permissible.

To conclude, our main objective here is to revive intertest in manifestly gauge invariant approaches to quantum gauge theories, in which the kinematical non-linearities are met head on right from the beginning. There have been attempts along these lines in the past (see, particularly [21]) which, however, seem to have been abandoned. (Indeed, to our knowledge, none of the major programs for construction of quantum Yang-Mills theories is still being actively pursued.) The specific methods we proposed here are rather tentative and our framework is incomplete in several respects. Its main merit is that it serves to illustrate the type of avenues that are available but have remained unexplored.

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