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Abstract

Consider a mechanism design setting in which agents acquire costly information about an unknown, payoff-relevant state of nature. Information gathering is covert and the agents' information is correlated. We investigate conditions under which (i) efficiency and (ii) full surplus extraction are Bayesian incentive compatible and interim individually rational.

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1 Introduction

Consider a mechanism design setting in which agents acquire costly information about an unknown, payoff-relevant state of nature. Agents may acquire costly information covertly before accepting the mechanism. The agents' information is correlated. We investigate conditions under which (i) efficiency and (ii) full surplus extraction are Bayesian incentive compatible and interim individually rational.

A social choice rule suggests a profile of information acquisitions to agents and maps each agents' reported information to an outcome. A mechanism is a social choice rule together with a payment function that maps reported information to each agent's payment.

A social choice rule is ex post efficient if it selects an ex post efficient outcome after every realization of information given the suggested level of information acquisitions. It is ex ante efficient if it is ex post efficient and suggests a level of information acquisition that maximizes the sum of expected utilities net of information acquisition costs. A mechanism fully extracts surplus if its social choice rule is efficient[1] and each agent's expected surplus is zero. A social choice rule is implementable if it is a part of a mechanism that is Bayesian incentive compatible and interim individually rational. We investigate conditions under which (i) efficiency and (ii) full surplus extraction are implementable.

We provide two sufficient conditions for efficient implementation. First, when the set of agents' signals is large relative to the size of the set of states of nature, we show that efficient implementation is feasible for generic information structures. This result holds in settings with correlated information and either private or interdependent values. We also obtain a sufficient condition for efficient implementation when the set of agents' signals is not large. Our second sufficient condition for efficient implementation is the existence of a set of lotteries that have a particular property. We call a set of payment plans for an agent contingent on other agents' signals *semi-robust lottery* given a particular level of information acquisition by the agent if acquiring further information would not help the agent make a better choice from the set. When there exists such a lottery given the efficient level of information acquisition, we can use them as a menu of payments to induce an ex ante efficient level of information acquisition and an ex post efficient allocation.

When agents' information is independent, it is known that efficient implementation is ex post incentive compatible if and only if values are private (Bergemann and

[1] From here on, efficient means ex ante efficient when describing a mechanism or a social choice rule. We will explicitly write ex post efficient when we mean efficiency in that sense.

Valimaki [2], Stegeman [11]). With positively interdependent values, agents have an incentive to acquire more information than the socially optimal level (Bergemann, Shi and Valimaki [1], Bergemann and Valimaki [2]). Our conditions imply that efficient implementation under interdependent values and correlated information.

Next we turn to the question of full surplus extraction. A mechanism extracts full surplus from agents if it implements an efficient social choice rule and each agent's interim expected surplus is zero. We show that full surplus extraction is Bayesian incentive compatible and interim individually rational when we can find a *robust lottery*. Robust lottery is a semi-robust lottery which is fair, i.e., an agent's expected payoff from a robust lottery is always 0 after any realization of his private signal given that he acquires at least the suggested level of information.

Cremer and McLean [5] show that if agents' are costlessly endowed with correlated information, then full surplus extraction is Bayesian incentive compatible and interim individually rational. However, if information is costly, then neither full surplus extraction nor efficiency is assured under their condition (see Obara [9] and Bikhchandani [4].)

Parreiras [10] shows that full surplus extraction may fail when each agent obtains two kinds of information: his type and the informativeness of his type about the types of the others. The information acquisition in our model corresponds to this second type of information.

Obara [9] generalizes the well-known necessary and sufficient condition for full surplus extraction by Cremer and McLean [5] to the setting where agents can take actions to change the distribution of their payoff-relevant private signals. The condition in Obara [9] is different from our condition because private signals are not directly payoff relevant in our setting of pure information acquisition. Bikhchandani [4] shows that full surplus extraction fails if an agent can acquire costly information about other agents' types. Unlike in our paper, agents are fully and costlessly informed about their own type in [4].

The model is presented in section 2. We begin section 3 with examples showing the failure of efficient implementation (and therefore also of full surplus extraction). Sufficient conditions for efficient implementation are also provided. Sufficient conditions for full surplus extraction are in section 4. Some proofs are in an appendix.

2 Model

Consider a set of agents $N = \{1, 2, \ldots, n\}$, $n \geq 2$. The state of nature $\omega \in \Omega$ is not observable. Agent i takes an action a_i from $A_i = \{1, ..., K\}$ to acquire information about the hidden state variable $\tilde{\omega}$.[2] When agent i takes action $a_i = k$, agent i can observe k signals $\tilde{s}_i^{0,k} = (\tilde{s}_{i,1}, ..., \tilde{s}_{i,k})$ where each $\tilde{s}_{i,\ell}$ takes a value in finite set $\overline{S}_{i,\ell}$. More generally, let $\tilde{s}_i^{k,\ell} = (\tilde{s}_{i,k+1}, ..., \tilde{s}_{i,\ell})$ for $\ell > k$ be a sequence of agent i's $k+1$th signal to ℓth signal. For notational simplicity, we denote $\tilde{s}_i^{0,k}$ by $\tilde{s}_i^k$. Let $S_{i,k} = \overline{S}_{i,k} \bigcup \{\emptyset\}$, $S_i^k = S_{i,1} \times S_{i,2} \times \ldots \times S_{i,k}$ for $1 \leq k \leq K$ and $S_i^{k,\ell} = S_{i,k+1} \times S_{i,k+2} \times \ldots \times S_{i,\ell}$ for $1 \leq k < \ell \leq K$. If agent i takes action $a_i = k$, then his observed signal may be denoted as either $s_i^k = (s_{i,1}, \ldots s_{i,k}) \in S_i^k$ or $s_i = (s_{i,1}, \ldots s_{i,k}, \emptyset, \ldots, \emptyset) \in S_i^K$. The notation $\emptyset$ means that the agent does not observe that part of the K dimensional signal. Often we write S_i instead of S_i^K.

Let $q(\cdot)$ be the prior distribution over Ω and $p_i(\cdot|\omega)$ be the probability distribution over S_i conditional on $\tilde{\omega} = \omega$. Note that the distribution $p_i(\cdot|\omega)$ is not affected by agent i's action. Agent i's action only determines the range of signals that agent i can observe. Let p be a collection of such conditional distributions $[p_i(\cdot|\omega),\ \forall\omega, \forall i]$.

As there is no cross-agent restriction such as budget balance, we can focus on one agent without loss of generality. Throughout our analysis, we fix a_{-i} and write the random variable representing other agents' signals as $\tilde{s}_{-i} = (\tilde{s}_j)_{j \neq i}$ rather than $\tilde{s}_{-i}^{a_{-i}} = (\tilde{s}_j^{a_j})_{j \neq i}$, and focus on agent i's incentive constraints. This should not create any confusion.

Agent i's belief about $\tilde{\omega}$ given that agent i (selected $a_i = k$ and) observed $\tilde{s}_i^k = s_i^k$ is:

$$d_i(\omega|s_i^k) \;=\; \frac{q(\omega)p_i(s_i^k|\omega)}{\sum_{\omega'} q(\omega')p_i(s_i^k|\omega')}.$$

Agent i's belief about the other agents' signals $\tilde{s}_{-i}$ given $\tilde{s}_i^k = s_i^k$ is:

$$h_i(s_{-i}|s_i^k) \;=\; \sum_{\omega \in \Omega} d_i(\omega|s_i^k)\Pi_{j \neq i} p_j(s_j|\omega).$$

Let X be a compact set of outcomes. Agent i's monetary transfer is denoted $t_i \in \Re$. Agent i's cost of information acquisition $c_i(a_i)$ is non-decreasing in a_i, i.e., $c_i(k+1) \geq c_i(k) \geq 0$ for $k = 1, ..., K-1$. Agent i's utility function over outcome x, money transfer t_i, and information acquisition decision a_i is quasi-linear

$$u_i(x, \omega) - t_i - c_i(a_i)$$

[2] We use $\tilde{x}$ for a random variable and x for its realization.

where $u_i : X \times \Omega \to \Re_+$ is agent i's type-dependent continuous utility function on X. Each agent has a large enough supply of the money commodity so that the budget constraint is not binding. Agent i's induced utility of outcome x conditional on (s, a) is

$$V_i(x, s, a) = E[u_i(x, \tilde{\omega})|\tilde{s} = s, a]. \tag{1}$$

An *information structure* is a set of types, a set of signals, a set of action profiles, a joint probability distribution over types and signals: (Ω, S, A, q, p), where $S = S_1 \times \cdots \times S_n$ and $A = A_1 \times \cdots \times A_n$. A *mechanism design problem* is an information structure together with an outcome set, utility functions, and cost functions: $(\Omega, S, A, q, p, X, u, c)$ where $u = (u_1, \ldots, u_n)$ and $p = (p_1, \ldots, p_n)$.

A *social choice function* $f : S \to X$ maps agents' (reported) signals to outcomes in X and a *payment function* $t_i : S \to \Re$ maps agents' signals to a transfer from agent i to the mechanism designer. A *social choice rule* is the pair (a, f). Since the revelation principle holds in this environment, we focus on direct mechanisms without loss of generality. A *mechanism* is triple (a, f, t) where $t = (t_1, \ldots, t_n)$.

The mechanism designer and agents play the following game. First, the mechanism designer proposes a mechanism (a, f, t). Next, each agent i covertly chooses an information acquisition level $\ell \in \{0, 1, 2, \cdots, K\}$,[3] observes the corresponding private signal $\tilde{s}_i^\ell$, then decides whether to accept the mechanism or not, and if he decides to accept, reports his signal to the mechanism designer (announcement is simultaneous).[4] The mechanism designer implements the outcome $f(s)$ and collects transfers $t(s)$ based on the reported signals s. We assume that the amount of information each agent acquires and whether or not each agent accepts the mechanism is not observable to the other agents. If an agent does not participate, his payoff, ignoring any information acquisition cost, is zero.

We consider a pure-strategy perfect Bayesian equilibrium, where agents are sequentially rational given their subjective belief computed via Bayes' rule at all private histories. Since we are interested in efficiency and full surplus extraction, the focus is on mechanisms in which every agent always accepts the mechanism in equilibrium without loss of generality.[5]

[3]Note that an agent may choose not to gather any information by selecting $\ell = 0$.

[4]We do not consider sequential information acquisition (Gershkov and Szentes [8]).

[5]To be precise, the mechanism needs to specify f and t off-the-equilibrium path where some agents did not accept the mechanism. Since before deciding to participate an agent does not observe participation decisions of others, and in the equilibria of interest all agents will participate, the agent need not explicitly consider the possibility that some other agent may not participate. Thus we omit a detailed description of the mechanism with some non-participation. For concreteness, one may assume that if one or more agents do not accept the mechanism, then the ex post efficient outcome

Bayesian Incentive Compatibility

Bayesian incentive compatibility requires that each agent gathers exactly the amount of information specified by the mechanism designer and truthfully reports his signal. Suppose that the mechanism designer wants to implement $a = (a_i, a_{-i})$, with $a_i = k$. Without loss of generality, f and t are measurable with respect to S_i^k. Assume that each agent $j \neq i$ gathers information a_j and suppose that agent i gathers information $\ell \in \{0, 1, 2, \ldots, K\}$.

The mechanism (a, f, t), with $a_i = k$ satisfies the incentive compatibility constraint if for all $\ell \in A_i$,

$$
\begin{aligned}
& E\left[V_i(f(\tilde{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^k, \tilde{s}_{-i}), (k, a_{-i})) - t_i(\tilde{s}_i^k, \tilde{s}_{-i}) | (k, a_{-i})\right] - c_i(k) \qquad (2)\\
\geq \; & E\left[\max\left\{\max_{\hat{s}_i^k \in S_i^k} E\left[V_i(f(\hat{s}_i^k, \tilde{s}_{-i}), (s_i^\ell, \tilde{s}_{-i}), (\ell, a_{-i})) - t_i(\hat{s}_i^k, \tilde{s}_{-i}) \middle| \tilde{s}_i^\ell = s_i^\ell, (\ell, a_{-i})\right], 0\right\} \middle| (\ell, a_{-i})\right] \\
& - c_i(\ell).
\end{aligned}
$$

Constraint (2) takes into account that, if the agent selects $a_i = \ell \neq k$, he may lie or may not accept the mechanism after observing a realization of $\tilde{s}_i^\ell$. We do not need to consider a deviation to report a null signal for the first k signals as such deviation can be punished by arbitrarily large t_i when implementing $a_i = k$. Moreover, because f and t are measurable with respect to S_i^k, the agent cannot benefit from reporting non-null signals in the set $S_i^{k,\ell}$, $\ell > k$.

Selecting $\ell = 0$ and non-participation gives a payoff of 0 on the right-hand side of (2), implying agent i's interim individual rationality constraint along the equilibrium path:

$$
E[V_i(f(s_i^k, \tilde{s}_{-i}), (s_i^k, \tilde{s}_{-i}), (k, a_{-i})) - t_i(s_i^k, \tilde{s}_{-i}) | \tilde{s}_i^k = s_i^k, (k, a_{-i})] \;\geq\; 0, \quad \forall s_i^k. \quad (3)
$$

Consider a mechanism design problem $(\Omega, S, A, q, p, X, u, c)$ and a social choice rule (a, f). If there exists t such that (2) is satisfied for each agent i then (a, f) can be *implemented* in this mechanism design problem.

An ex post efficient social choice function given $a \in A$ is $f_a^* : S \to X$ such that

$$
f_a^*(s) \;\in\; \arg\max_{x \in X} \sum_{i=1}^n \left[V_i(x, s, a) - c_i(a_i)\right], \qquad \forall s \in S.
$$

As mentioned earlier, f_a^* does not depend on $\tilde{s}_i^{k,\ell}$, $\ell > k$, when $a_i = k$. Let

$$
V(a, s) \;\equiv\; \sum_{i=1}^n \left[V_i(f_a^*(s), s, a) - c_i(a_i)\right]
$$

for participating agents is implemented. Non-participating agents obtain a zero expected payoff.

be the ex post maximized surplus given (a, s) and let $V(a) = E[V(a, \tilde{s})|a]$ be the ex ante maximum social welfare given $a \in A$. Then, a^* is an ex ante efficient information acquisition level if

$$a^* \in \arg\max_{a \in A} V(a).$$

The ex post efficient social choice function associated with a^* is $f^*_{a^*}$. In the efficient social choice rule, $(a^*, f^*_{a^*})$, (i) agents acquire information level a^* so that the ex post optimal social surplus minus the informational cost is maximized and (ii) for each realization of the agents' information an ex post efficient outcome is implemented *provided that* each agent i acquires information level a^*_i. If agent i deviates to $a_i \neq a^*_i$ to , an ex post efficient outcome for (a_i, a^*_{-i}) need not be implemented.

We consider two possible objectives for the mechanism designer: efficiency or surplus maximization. The two objectives need not be in conflict and are simultaneously satisfied if the mechanism designer is able to implement an efficient social choice rule and extract the entire surplus.

In section 3 we investigate conditions under which efficient implementation is feasible, i.e., sufficient conditions for $(a^*, f^*_{a^*})$ to be Bayesian incentive compatible and interim individually rational. In section 4 we investigate conditions under which full surplus extraction is feasible, i.e., efficient implementation with zero expected surplus for each agent is possible.

3 Efficient Bayesian Implementation

In this section, we provide several sufficient conditions for efficient implementation by pure strategy perfect Bayesian equilibrium.

Clearly efficient implementation is not possible when the signal spaces of other agents are too small. An obvious example is the one where other agents do not observe any signal (i.e. S_{-i} is a singleton). In such a case, the only agent with private information would announce any signal that implements the best allocation for him, which may not be necessarily socially optimal. In fact, efficient implementation may not be possible in general when $|S_{-i}|$ is smaller than $|\Omega|$, even if we put aside the issue of implementing the efficient information acquisition. The following example illustrates this point.

Example 1: Suppose that $n = 2$, $|\Omega| = |S_1| = \{0, 1, -1\}$ and $|S_2| = \{a, b\}$. Agent 1 observes the state of nature perfectly. The probability of $s_2 = a$ is $\Pr(s_2 = a|\omega = -1) = 0.25, \Pr(s_2 = a|\omega = 0) = 0.5, \Pr(s_2 = a|\omega = 1) = 0.75$ respectively. This is a simple auction problem: one indivisible object needs to be allocated between agent 1 and agent 2. Agent 1's reservation value for the object is \$10 when $\omega = 1$ or $\omega = -1$, and \$9 when $\omega = 0$. Agent 2's reservation value is \$20 when $\omega = 1$ or $\omega = -1$, and \$0 when $\omega = 0$. For ex post efficient implementation, the object must go to agent 2 when $\omega = 1$ or $\omega = -1$ and to agent 1 when $\omega = 0$. For agent 1 to reveal the true state, agent 1's expected reward $E\left[-t_1(s_1, s_2)|s_1\right]$ must be larger than \$10 given $\omega = 1$ or $\omega = -1$. However, this means that $E\left[-t_1(s_1, s_2)|s_1 = 0\right]$ exceeds \$10 as well, since $\Pr(s_2|\omega = 0) = 0.5 \Pr(s_2|\omega = -1) + 0.5 \Pr(s_2|\omega = 1)$ for any s_2. Hence agent 1 has an incentive not to reveal $s_1 = \omega = 0$. $\triangle$

Now we show that efficient implementation is feasible when $|\Omega| \leq |S_{-i}|$, for any mechanism design problem for a generic choice of (q, p). Let a^* be the efficient level of information acquisition for a given mechanism design problem $(\Omega, S, A, p, q, X, u, c)$. We focus on agent i without loss of generality.

Proposition 1 *Suppose that $|\Omega| \leq |S_{-i}|$. Then, for any mechanism design problem with a generic choice of (p, q), the ex ante efficient level of information acquisition a_i^* and the ex post efficient allocation $f_{a^*}^*$ can be implemented.*

Proof: Define agent i's transfer $t_i : S_i^k \times S_{-i} \to \Re$ so that the following conditions are satisfied.

$$E\left[t_i(s_i^k, \tilde{s}_{-i})|\omega, a_{-i}^*\right] = E\left[-\sum_{j \neq i} u_j(f_{a^*}^*(s_i^k, \tilde{s}_{-i}), \omega)\Big|\omega, a_{-i}^*\right], \ \forall \omega \in \Omega.$$

Such t_i exists generically by the assumption $|\Omega| \leq |S_{-i}|$.

To verify that $a_i^* = k$ and $f_{a^*}^*$ can be implemented with transfer t_i, first suppose that agent i chooses $a_i^* = k$. Agent i's expected payoff conditional on (a_i^*, s_i^k) and reporting $\widehat{s}_i^k$ is

$$\sum_{\omega} E\left[u_i(f_{a^*}^*(\widehat{s}_i^k, \tilde{s}_{-i}), \omega) - t_i(\widehat{s}_i^k, \tilde{s}_{-i})\Big|\omega, a_{-i}^*\right] d(\omega|s_i^k) = \sum_{\omega} E\left[\sum_{j=1}^n u_j(f_{a^*}^*(\widehat{s}_i^k, \tilde{s}_{-i}), \omega)\Big|\omega, a_{-i}^*\right] d(\omega|s_i^k).$$

Note that this is the expected social welfare (plus information acquisition cost) of the allocation $f_{a^*}^*(\widehat{s}_i^k, \cdot)$ given s_i^k. Since $f_{a^*}^*(s_i^k, \cdot)$ is ex post efficient by assumption when $\tilde{s}_i^k = s_i^k$, it must maximize such expected social welfare given s_i^k, i.e., the above expression is maximized when $\widehat{s}_i^k = s_i^k$. Hence, given that he selected $a_i^* = k$, it is optimal for agent i's to truthfully report his signal.

Next, we show that the agent does not have an incentive to acquire a different level of information. Suppose that i chooses $a_i = \ell \neq k$ and uses a reporting strategy

$\sigma_i : S_i^\ell \to S_i^k$. Then agent i's ex ante expected payoff is given by

$$\sum_{s_i^\ell} \sum_{\omega} E\left[u_i(f^*_{a^*}(\sigma_i(s_i^\ell), \tilde{s}_{-i}), \omega) - t_i(\sigma_i(s_i^\ell), \tilde{s}_{-i}) \middle| \omega, a^*_{-i} \right] d(\omega|s_i^\ell) \Pr(\tilde{s}_i^\ell = s_i^\ell | \ell) - c_i(\ell)$$

$$= \sum_{s_i^\ell} \sum_{\omega} E\left[\sum_{j=1}^{n} u_j(f^*_{a^*}(\sigma_i(s_i^\ell), \tilde{s}_{-i}), \omega) \middle| \omega, a^*_{-i} \right] d(\omega|s_i^\ell) \Pr(\tilde{s}_i^\ell = |\ell) - c_i(\ell)$$

Since $f^*_{a^*}$ does not take into account agent i's true action $(= \ell)$, it may not maximize the social welfare conditional on (ℓ, a^*_{-i}). Thus, the above payoff is bounded above by $V(\ell, a^*_{-i}) + \sum_{j \neq i} c_j(a^*_j)$. However, this is less than $V(k, a^*_{-i}) + \sum_{j \neq i} c_j(a^*_j)$ as $a^*_i = k$ is ex ante efficient by assumption, which agent i can achieve by choosing k and reporting his signal truthfully. This completes the proof. ∎

Remark 1

(i) Bergemann and Valimaki [2] proved that in a model with independent information acquisition, efficient ex post incentive compatible implementation is possible only under private values.[6] Proposition 1 proves that under the weaker requirement of perfect Bayesian incentive compatibility, efficient implementation is possible under interdependent values and correlated information.

(ii) This mechanism is a variation of the expected externality mechanism (d'Aspremont and Gerard-Varet [7]). In the mechanism in [7], the expected externality caused by agent i's report can be evaluated without knowing agent i's true type due to independent information. Information may be correlated in our setting, but we can still evaluate agent i's expected externality without knowing agent i's true type. Since the state of nature $\tilde{\omega}$ is a sufficient statistic for the expected externality, we can evaluate the expected externality and charge it to agent i state by state through other agents' reports if $\tilde{s}_{-i}$ is informative enough about the state of nature.

Next, we turn to conditions under which efficient implementation is possible even if $|\Omega| > |S_{-i}|$. The payment $t_i : S_i^k \times S_{-i} \to \Re$ is a *semi-robust lottery given* $a_i = k$ if

$$\sum_{s_{-i}} t_i(s_i^k, s_{-i}) h(s_{-i}|s_i^k, s_i^{k,K}) < \sum_{s_{-i}} t_i(\widehat{s}_i^k, s_{-i}) h(s_{-i}|s_i^k, s_i^{k,K}) \tag{4}$$

for any s_i^k, $\widehat{s}_i^k \neq s_i^k \in S_i^k$ and any $s_i^{k,K} \in S_i^{k+1,K}$. It is optimal for agent i to choose $t_i(s_i^k, \cdot)$ from a set of lotteries $\left\{ t_i(\widehat{s}_i^k, \cdot) : \widehat{s} \in S_i^k \right\}$ when $a_i = k$ and $\tilde{s}_i^k = s_i^k$. Furthermore, this optimal choice of lottery would not change even if agent i acquired more information than $a_i = k$.

[6] See also Bergemann, Shi and Valimaki [1] and Bergemann and Valimaki [3].

The payments $t = (t_1, \ldots, t_n)$ are *semi-robust lotteries given* $a = (a_1, \ldots, a_n)$ if each t_i is a semi-robust lottery given a_i.

Proposition 2 *Suppose that there exist semi-robust lotteries given the ex ante efficient level of information acquisition* a^* *for a mechanism design problem. Then the ex ante efficient information acquisition level* a^* *and the ex post efficient allocation* $f^*_{a^*}$ *can be implemented.*

Proof: Let t_i be a semi-robust lottery given a_i^*. We show that (a_i^*, f_{a^*}) can be implemented by using monetary transfer $t_i^* = bt_i$ for some very large $b > 0$. (In fact, any allocation $f : S \to X$ can be implemented with such t_i^*).

Participation constraints. We can assume without loss of generality that t_i^* is non-positive. Hence agent i would accept the mechanism $(f^*_{a^*}, t_i^*)$.

Deviation to acquire more information. We show that acquiring more information does not help agent i. Suppose that $a_i^* = k < K$ and agent i chooses action $a_i = K$ and observes $\tilde{s}_i^K = (s_i^k, s_i^{k,K})$. As t_i is a semi-robust lottery for $a_i^* = k$, the expected amount of transfer from agent i increases strictly by announcing $\widehat{s}_i^k$ instead of s_i^k, whatever agent i's additional information $s_i^{k,K}$ may be. Agent i may gain some payoff by announcing $\widehat{s}_i^k$ instead of s_i^k through changing the final allocation $f^*_{a^*}(s)$. But, because X is compact and i's utility function is continuous on X, we can choose b large enough so that this effect is outweighed by the expected loss in transfers. Because $f^*_{a^*}$ is measurable with respect to S_i^k, it is optimal for agent i's to truthfully report s_i^k after information acquisition action $a_i = K$ is taken; announcement of additional signals $s_i^{k,K}$ will not change the implemented outcome. Agent i does not gain anything by acquiring more information. Since acquiring information is costly, a deviation to acquire more information (and possibly lying) is not profitable. The same proof applies to the case when agent i takes action $a_i = \ell \in \{k+1, \ldots, K\}$. Since agent i does not have incentive to lie about the first k signals even when he is most informed ($\ell = K$), he does not have incentive to lie when he is less informed ($\ell < K$).

Deviation to acquire less information. Suppose that agent i chooses $a_i < k$, i.e., acquires less information than he would in equilibrium. First, observe that the assumption (regarding t_i) implies that $h_i(\cdot|s_i^k) \neq h_i(\cdot|\widehat{s}_i^k)$ for any s_i^k and $s_i^k \neq \widehat{s}_i^k$. Of course there are many such pairs of $\left(s_i^k, \widehat{s}_i^k\right)$ such that $s_i^{k-1} = \widehat{s}_i^{k-1}$, where the only kth signals are different. This means that agent i cannot announce the right k digit signals with probability 1 if he acquires less information than k signals. Hence the expected transfer from agent i given $a_i < k$ would be strictly more than when $a_i = k$. Again we can choose b large enough so that this expected loss outweighs any gain from saving the cost of information acquisition. Therefore this type of deviation is not profitable either. ∎

We provide a sufficient condition on the information structure for existence of a semi-robust lottery.

Let $\hat{S}_i^k \subset S_i^k$ be any subset of the set of agent i's first k signals. We say that $s_i^k \in \hat{S}_i^k$ is *separated from* $\hat{S}_i^k$ if $h\left(\cdot|s_i^k\right)$ can be represented by a convex combination of $h\left(\cdot|\widehat{s}_i^k, s_i^{k,K}\right), \forall \widehat{s}_i^k \in \hat{S}_i^k, \forall s_i^{k,K} \in S_i^{k,K}$ only by placing zero weight on $h\left(\cdot|\widehat{s}_i^k, s_i^{k,K}\right)$ for any $\widehat{s}_i^k \neq \hat{S}_i^k$ and $s_i^{k,K} \in S_i^{k,K}$.

This condition is equivalent to the existence of a hyperplane that separates $h\left(\cdot|s_i^k\right)$ from $h\left(\cdot|\widehat{s}_i^k, s_i^{k,K}\right)$ with $\widehat{s}_i^k \neq s_i^k$ strictly and from $h\left(\cdot|s_i^k, s_i^{k,K}\right)$ weakly. More precisely, this condition means that there exists $\mu : S_{-i} \to \mathbb{R}$ such that

$$\sum_{s_{-i}} \mu\left(s_{-i}\right) h\left(s_{-i}|s_i^k\right) \geq \sum_{s_{-i}} \mu\left(s_{-i}\right) h\left(s_{-i}|\widehat{s}_i^k, s_i^{k,K}\right), \qquad \forall \left(\widehat{s}_i^k, s_i^{k,K}\right) \in \hat{S}_i^k \times S_i^{k,K}$$

where the inequality is strict for any $\widehat{s}_i^k \neq s_i^k \in \hat{S}_i^k$.

We show that there exists a semi-robust lottery given $a_i = i$ for an information structure if we can order S_i^k as $s_i^k(1), s_i^k(2), ...$ so that $s_i^k(r)$ can be separated from $\{s_i^k(r), ..., s_i^k(|S_i^k|)\}$ for any r.

Proposition 3 *Suppose that S_i^k can be ordered as $s_i^k\left(1\right), s_i^k\left(2\right), ...$ so that $s_i^k\left(r\right)$ is separated from $S_i^k / \left\{s_i^k\left(1\right), ..., s_i^k\left(r-1\right)\right\}$ for each $r = 1, 2, ..., \left|S_i^k\right|$. Then there exists t_i, a semi-robust lottery given k.*

The proof of this proposition is in an appendix.[7]

4 Full surplus extraction

Full surplus extraction occurs in a mechanism design problem if it is Bayesian incentive compatible and interim individually rational for agents to acquire the ex ante efficient information level and truthfully report their signals while the mechanism designer implements the ex post efficient rule and collects transfers such that each agent's expected utility is zero. We need to modify the semi-robust lotteries of section 3 to obtain full extraction.

[7]The proof also provides a somewhat technical necessary and sufficient condition for existence of semi-robust lottery.

A *robust lottery given k* for agent i is a payment $\pi_i : S_i^k \times S_{-i} \to \Re$ such that

$$\sum_{s_{-i}} h_i(s_{-i}|s_i^k, s_i^{k,K})\pi_i(s_i^k, s_{-i}^k) = 0, \quad \forall s_i^k,\ s_i^{k,K}, \tag{5}$$

$$\sum_{s_{-i}} h_i(s_{-i}|s_i^k, s_i^{k,K})\pi_i(\hat{s}_i^k, s_{-i}^k) > 0, \quad \forall \hat{s}_i^k \neq s_i^k,\ s_i^{k,K}. \tag{6}$$

If the left-hand side of (4) equals zero, then the semi-robust lotteries t_i are robust lotteries. Note that a robust lottery is also a Cremer-McLean full extraction lottery (but not vice versa). The set of robust lotteries is a cone.

Proposition 4 *Consider an information structure (Ω, S, A, q, p). Suppose that for each agent i and any $a_i = k$, robust lotteries π_i given k exist. Then for each mechanism design problem on this information any social choice rule can be implemented such that each agent's expected surplus is zero.*

Proof: Let (a, f) be a social choice rule for a mechanism design problem on this information structure. Let π_i be a robust lottery given $a_i = k$ for agent i. Define

$$t_i(s_i^k, s_{-i}) = \lambda\pi_i(s_i^k, s_{-i}^k) + g_i(s_i^k, s_{-i}), \qquad \forall s_i^k,\ s_{-i},$$

where $\lambda > 0$ and g_i is a function that satisfies $g_i(s_i^k, s_{-i}) \leq V_i(f_i(s_i^k, s_{-i}), (s_i^k, s_{-i}), (k, a_{-i}))$. It is straightforward to check that the mechanism (a, f, t) satisfies (3).

By choosing λ sufficiently large we can ensure that for all s_i^k and $\hat{s}_i^k \neq s_i^k$,

$$\begin{aligned}
& E[V_i(f(\tilde{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^k, \tilde{s}_{-i}), (k, a_{-i})) - t_i(\tilde{s}_i^k, \tilde{s}_{-i})|\tilde{s}_i^k = s_i^k, (k, a_{-i})] \\
= \ & E[V_i(f(\tilde{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^k, \tilde{s}_{-i}), (k, a_{-i})) - g_i(\tilde{s}_i^k, \tilde{s}_{-i})|\tilde{s}_i^k = s_i^k, (k, a_{-i})] \\
\geq \ & E[V_i(f(\hat{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^k, \tilde{s}_{-i}), (k, a_{-i})) - g_i(\hat{s}_i^k, \tilde{s}_{-i})|\tilde{s}_i^k = s_i^k, (k, a_{-i})] \\
& - \lambda \sum_{s_{-i}} h_i(s_{-i}|s_i^k)\pi_i(\hat{s}_i^k, s_{-i}^k) \\
= \ & E[V_i(f(\hat{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^k, \tilde{s}_{-i}), (k, a_{-i})) - t_i(\hat{s}_i^k, \tilde{s}_{-i})|\tilde{s}_i^k = s_i^k, (k, a_{-i})],
\end{aligned}$$

where the inequality follows from (6). Thus, if agent i chooses $a_i = k$, he has no incentive to lie.

Next, consider $\ell < k$. For any $s_i^\ell,\ \hat{s}_i^k \neq s_i^k = (s_i^\ell, s_i^{\ell,k})$, (6) implies that

$$\sum_{s_{-i}} h_i(s_{-i}|s_i^\ell, s_i^{\ell,k})\pi_i(\hat{s}_i^k, s_{-i}^k) > 0.$$

Therefore,

$$\sum_{s_{-i}} h_i(s_{-i}|s_i^\ell)\pi_i(\hat{s}_i^k, s_{-i}^k) = \sum_{s_{-i}} \sum_{s_i^{\ell,k}} \Pr[s_i^{\ell,k}|s_i^\ell] h_i(s_{-i}|s_i^\ell, s_i^{\ell,k})\pi_i(\hat{s}_i^k, s_{-i}^k)$$

$$
\begin{aligned}
&= \sum_{s_i^{\ell,k}} \Pr[s_i^{\ell,k}|s_i^\ell] \sum_{s_{-i}} h_i(s_{-i}|s_i^\ell, s_i^{\ell,k}) \pi_i(\hat{s}_i^k, s_{-i}^k) \\
&> 0. \qquad (7)
\end{aligned}
$$

Thus, for any $\ell < k$ and $s_i^\ell \in S_i^\ell$

$$
\max_{s_i^k \in S_i} \Big[E[V_i(f(s_i^k, \tilde{s}_{-i}), ((s_i^\ell, \tilde{s}_i^{\ell,k}), \tilde{s}_{-i}), (k, a_{-i})) - g_i(s_i^k, \tilde{s}_{-i}) | \tilde{s}_i^\ell = s_i^\ell, (\ell, a_{-i})] - \lambda \sum_{s_{-i}} h_i(s_{-i}|s_i^\ell) \pi_i(s_i^k, s_{-i}^k) \Big] < c_i(\ell) - c_i(k)
$$

where the inequality follows from (7) and by taking λ large enough. If agent i chooses $a_i = \ell < k$ and reports some $s_i^k \in S_i^k$, then even after taking into account that agent i is reimbursed $c_i(k)$ towards information acquisition costs but incurs only $c_i(\ell)$, agent i's expected payoff is negative after observing any $s_i^\ell \in S_i^\ell$. If, instead, after observing $\tilde{s}_i^\ell = s_i^\ell$ he gathers additional information signal $\tilde{s}_i^{\ell,k}$, interim individual rationality guarantees that his expected payoff is non-negative. Even if he makes information acquisition decisions sequentially, agent i cannot benefit by gathering less information than $a_i = k$. Hence, (2) is satisfied for $\ell < k$.

Finally, if $\ell > k$ then for all s_i^k

$$
\begin{aligned}
& E[\max_{\hat{s}_i^k \in S_i^k} [V_i(f(\hat{s}_i^k, \tilde{s}_{-i}), (\tilde{s}_i^\ell, \tilde{s}_{-i}), (\ell, a_{-i})) - g_i(\hat{s}_i^k, \tilde{s}_{-i}) - \lambda \pi_i(\hat{s}_i^k, \tilde{s}_{-i})] | \tilde{s}_i^k = s_i^k, (k, a_i)] \\
= \; & E[V_i(f(s_i^k, \tilde{s}_{-i}), (\tilde{s}_i^\ell, \tilde{s}_{-i}), (\ell, a_{-i})) - g_i(s_i^k, \tilde{s}_{-i}) - \lambda \pi_i(s_i^k, \tilde{s}_{-i}) | \tilde{s}_i^k = s_i^k, (k, a_i)] \\
= \; & E[V_i(f(s_i^k, \tilde{s}_{-i}), (s_i^k, \tilde{s}_{-i}), (k, a_{-i})) - g_i(s_i^k, \tilde{s}_{-i}) | \tilde{s}_i^k = s_i^k, (k, a_i)]
\end{aligned}
$$

where the first equality follows from the fact that $\lambda E[\pi_i(\hat{s}_i^k, \tilde{s}_{-i}) | \tilde{s}_i^k = s_i^k] < 0$ if $\hat{s}_i^k \neq s_i^k$ can be made arbitrarily small by choosing λ large enough and the second equality follows from (5). Thus, after selecting $a_i = k$ and observing any realization of signal $\tilde{s}_i^k$ agent i does not gain by gathering additional information $\ell > k$ at cost of $c_i(\ell) - c_i(k)$. ∎

Recall that a^* is the ex ante efficient information acquisition level and $f_{a^*}^*$ is the ex post efficient rule associated with a^* for a mechanism design problem.

Corollary 1 (Full surplus extraction.) *If for each agent i robust lotteries given a_i^* exist, then the mechanism $(a^*, f_{a^*}^*, t^*)$ where*

$$
E[t_i^*(\tilde{s}_i^{a_i^*}, \tilde{s}_{-i}) | \tilde{s}_i^{a_i^*} = s_i, a_i^*] = E[V_i(f_{a^*}^*(\tilde{s}_i^{a_i^*}, \tilde{s}_{-i}), (\tilde{s}_i^{a_i^*}, \tilde{s}_{-i}), a^*) | \tilde{s}_i^{a_i^*} = s_i^{a_i^*}, a^*]
$$

is implementable.

The corollary follows by taking $g_i(s_i^{a_i^*}, s_{-i}) = V_i(f_i(s_i^{a_i^*}, s_{-i}), (s_i^{a_i^*}, s_{-i}), a^*)$ in the proof of Proposition 4. In fact, a stronger result follows almost immediately from Proposition 4: Even if agents take information gathering decisions sequentially (but without knowledge of other agents' signal realizations) rather than simultaneously, existence of robust lotteries implies that full surplus extraction is possible. If agent i is asked to gather $a_i^* = k$ then he can either pay $c_i(k)$ and observe $\tilde{s}_i^k = (\tilde{s}_{i,1}, \ldots, \tilde{s}_{i,k})$ simultaneously or sequentially, i.e. first pay $c_i(1)$ and observe $\tilde{s}_{i,1}$, then pay $c_i(2) - c_i(1)$ and observe $\tilde{s}_{i,2}$, and so on until he observes $\tilde{s}_{i,k}$. If robust lotteries exist then full surplus extraction with sequential information gathering by agents can also be implemented. Note, however, the social choice rule $(a^*, f^*_{a^*})$ that is implemented is ex ante efficient among simultaneous (and not sequential) information acquisition mechanisms.

Here is a characterization result for robust lotteries. The necessary and sufficient condition below is stronger than the sufficient condition of Proposition 3. This is not surprising as robust lotteries are semi-robust lotteries but not vice versa.

Proposition A: (BIKHCHANDANI [4]). *Robust lotteries exist given k for agent i iff the set of linear combination of beliefs $h_i(\cdot|s_i^k, s_i^{k,K})$, $\forall s_i^{k,K}$ does not intersect the convex hull of beliefs $h_i(\cdot|\hat{s}_i^k, s_i^{k,K})$, $\forall \hat{s}_i^k \neq s_i^k$ and $\forall s_i^{k,K}$.*

Proposition 5 *If $|S_i^K| \leq \min\{|\Omega|, |S_{-i}|\}$ then robust lotteries for agent i exist for generic probability distributions q and p.*

Proof: If $|S_i^K| \leq |\Omega|$ and $|S_i^K| \leq |S_{-i}|$ then for generic probability distributions q and p and any $k = 1, 2, \ldots, K$ the vectors $[h_i(\cdot|s_i^k, s_i^{k,K})]$ for all $(s_i^k, s_i^{k,K})$ are linearly independent. Therefore, the condition in Proposition A is satisfied. ∎

If, instead of assuming that agents' signals are distributed independently conditional of ω, we assume that $(\tilde{w}, \tilde{s}_i, \tilde{s}_{-i})$ are jointly distributed then the restriction on the size of Ω can be dropped.

Proposition 6 *If $|S_i^K| \leq |S_{-i}|$ then robust lotteries for agent i exist for generic joint probability distributions over $(\tilde{w}, \tilde{s}_i, \tilde{s}_{-i})$.*

Proof: Once again, for any $k = 1, 2, \ldots, K$ the vectors $[h_i(\cdot|s_i^k, s_i^{k,K})]$ for all $(s_i^k, s_i^{k,K})$ are linearly independent. Therefore, the condition in Proposition A is satisfied. ∎

5 Appendix: Omitted proofs

First we prove a lemma that is used to prove Proposition 3. Consider the following condition on the information structure.

Condition B: If $\lambda : S_i^k \times S_i^k \times S_i^{k,K} \to R_+$ satisfies

$$\sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k, s_i^{k,K}\right) h\left(\cdot | s_i^k, s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(s_i^k, \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right) \tag{8}$$

and

$$\sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k, s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(s_i^k, \widehat{s}_i^k, s_i^{k,K}\right) = 1 \tag{9}$$

for any s_i^k, then $\lambda\left(s_i^k, \widehat{s}_i^k, s_i^{k,K}\right)$ must be 0 for any $s_i^k, \widehat{s}_i^k \neq s_i^k$ and $s_i^{k,K}$.

The following lemma shows that this condition is equivalent to the existence of semi-robust lottery for every $s_i^k \in S_i^k$ given $a_i = k$.

Lemma 1 *Condition B is satisfied if and only if there exists a semi-robust lottery given k, i.e., there exists $t_i : S_i^k \times S_{-i} \to \mathbb{R}$ that satisfies*

$$\sum_{s_{-i}} t_i(s_i^k, s_{-i}) h(s_{-i} | s_i^k, s_i^{k,K}) < \sum_{s_{-i}} t_i(\widehat{s}_i^k, s_{-i}) h(s_{-i} | s_i^k, s_i^{k,K})$$

for any s_i^k, $\widehat{s}_i^k \neq s_i^k, \in S_i^k$ and for any $s_i^{k,K} \in S_i^{k+1,K}$.

Proof: There exists such $t_i : S_i^k \times S_{-i} \to \mathbb{R}$ if and only if the LP below has a feasible solution:

$$\min_{t^1} 0$$

s.t.

$$\sum_{s_{-i}} t^1(\hat{s}_i^k, s_{-i}) h(s_{-i} | s_i^k, s_i^{k,K}) - \sum_{s_{-i}} t^1(s_i^k, s_{-i}) h(s_{-i} | s_i^k, s_i^{k,K}) \geq 1, \quad \forall s_i^{k,K},\ \forall s_i^k,\ \forall \hat{s}_i^k \neq s_i^k.$$

Its dual is

$$\max_{\lambda \geq 0} \sum_{s_i^k} \sum_{\hat{s}_i^k \neq s_i^k} \sum_{s_i^{k,K}} \lambda(\hat{s}_i^k, s_i^k, s_i^{k,K})$$

s.t.

$$\sum_{s_i^{k,K}} \sum_{\hat{s}_i^k \neq s_i^k} \lambda\left(\hat{s}_i^k, s_i^k, s_i^{k,K}\right) h\left(s_{-i} | s_i^k, s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k \neq s_i^k} \lambda\left(s_i^k, \hat{s}_i^k, s_i^{k,K}\right) h\left(s_{-i} | \hat{s}_i^k, s_i^{k,K}\right), \quad \forall s_i^k,\ \forall s_{-i}. \tag{10}$$

The LP has a feasible solution iff in every feasible solution to the dual $\lambda(\hat{s}_i^k, s_i^k, s_i^{k,K}) = 0$ for all s_i^k, for all $\hat{s}_i^k \neq s_i^k$, for all $s_i^{k,K}$. Summing (10) over s_{-i} we have

$$\nu(s_i^k) \equiv \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k \neq s_i^k} \lambda\left(\hat{s}_i^k, s_i^k, s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k \neq s_i^k} \lambda\left(s_i^k, \hat{s}_i^k, s_i^{k,K}\right) \geq 0, \quad \forall s_i^k.$$

Let $M > \nu(s_i^k)$ for all s_i^k and select any $\lambda\left(s_i^k, s_i^k, s_i^{k,K}\right) \geq 0$ for each s_i^k so that

$$\begin{aligned} \sum_{s_i^{k,K}} \lambda\left(s_i^k, s_i^k, s_i^{k,K}\right) &= M - \nu(s_i^k) > 0 \\ \Longrightarrow \quad \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k} \lambda\left(\hat{s}_i^k, s_i^k, s_i^{k,K}\right) &= \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k} \lambda\left(s_i^k, \hat{s}_i^k, s_i^{k,K}\right) = M > 0 \end{aligned}$$

Then (10) is equivalent to

$$\sum_{s_i^{k,K}} \sum_{\hat{s}_i^k} \lambda\left(\hat{s}_i^k, s_i^k, s_i^{k,K}\right) h\left(s_{-i} | s_i^k, s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\hat{s}_i^k} \lambda\left(s_i^k, \hat{s}_i^k, s_i^{k,K}\right) h\left(s_{-i} | \hat{s}_i^k, s_i^{k,K}\right), \quad \forall s_i^k, \ \forall s_{-i}$$

which is (8). Without loss of generality, let $M = 1$ and the lemma follows. ■

Proof of Proposition 3: We just need to show that the above assumption for S_i^k implies Condition B. Suppose that S_i^k can be ordered as assumed and that $\lambda \geq 0$ satisfies (8) and (9). Then the following equation holds for $s_i^k(1)$:

$$\sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k(1), s_i^{k,K}\right) h\left(\cdot | s_i^k(1), s_i^{k,K}\right) = \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(s_i^k(1), \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right). \tag{11}$$

We can define $g\left(s_i^k(1), s_i^{k,K}\right)$ and find $\eta > 0$ that satisfies

$$\frac{g\left(s_i^k(1), s_i^{k,K}\right) + \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k(1), s_i^{k,K}\right)}{\eta} = \Pr\left(\widetilde{s}_i^{k,K} = s_i^{k,K} | s_i^k(1)\right)$$

for every $s_i^{k,K}$. Add $g\left(s_i^k(1), s_i^{k,K}\right)$ to both sides of (11) as follows:

$$\begin{aligned} & \sum_{s_i^{k,K}} \left\{ \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k(1), s_i^{k,K}\right) + g\left(s_i^k(1), s_i^{k,K}\right) \right\} h\left(\cdot | s_i^k(1), s_i^{k,K}\right) \\ = & \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k \neq s_i^k(1)} \lambda\left(s_i^k(1), \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right) \\ & + \sum_{s_i^{k,K}} \left\{ \lambda\left(s_i^k(1), s_i^k(1), s_i^{k,K}\right) + g\left(s_i^k(1), s_i^{k,K}\right) \right\} h\left(\cdot | s_i^k(1), s_i^{k,K}\right). \end{aligned}$$

Divide both sides by η. Then we obtain

$$
\begin{aligned}
& \sum_{s_i^{k,K}} \Pr\left(\widehat{s}_i^{k,K} = s_i^{k,K} | s_i^k(1)\right) h\left(\cdot | s_i^k(1), s_i^{k,K}\right) \\
= \ & h\left(\cdot | s_i^k(1)\right) \\
= \ & \frac{1}{\eta} \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k \neq s_i^k(1)} \lambda\left(s_i^k(1), \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right) \\
& + \frac{1}{\eta} \sum_{s_i^{k,K}} \left\{\lambda\left(s_i^k(1), s_i^k(1), s_i^{k,K}\right) + g\left(s_i^k(1), s_i^{k,K}\right)\right\} h\left(\cdot | s_i^k(1), s_i^{k,K}\right).
\end{aligned}
$$

Since $s_i^k(1)$ is separated from S_i^k by assumption, this implies $\lambda\left(s_i^k(1), \widehat{s}_i^k, s_i^{k,K}\right) = 0$ for any $\widehat{s}_i^k \neq s_i^k(1)$ and $s_i^{k,K}$.

Note that $\sum_{s_i^{k,K}} \lambda\left(s_i^k(1), s_i^k(1), s_i^{k,K}\right) = 1$ by (9). This in turn implies $\lambda\left(\widehat{s}_i^k, s_i^k(1), s_i^{k,K}\right) = 0$ for any $\widehat{s}_i^k \neq s_i^k(1)$ and $s_i^{k,K}$ by using (9) again.

The rest of the proof is by induction. Suppose that for every $r = 1, 2, ..., m$, $\lambda\left(s_i^k(r), \widehat{s}_i^k, s_i^{k,K}\right) = \lambda\left(\widehat{s}_i^k, s_i^k(r), s_i^{k,K}\right) = 0$ for any $\widehat{s}_i^k \neq s_i^k(r)$ and $s_i^{k,K}$ by (9). For $s_i^k(m+1)$, we get

$$
\begin{aligned}
& \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k(m+1), s_i^{k,K}\right) h\left(\cdot | s_i^k(m+1), s_i^{k,K}\right) \\
= \ & \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k} \lambda\left(s_i^k(m+1), \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right) \\
= \ & \sum_{s_i^{k,K}} \sum_{\widehat{s}_i^k \in S_i^K / \left\{s_i^k(1), ..., s_i^k(m)\right\}} \lambda\left(s_i^k(m+1), \widehat{s}_i^k, s_i^{k,K}\right) h\left(\cdot | \widehat{s}_i^k, s_i^{k,K}\right).
\end{aligned}
$$

As in the first step, define $g\left(s_i^k(m+1), s_i^{k,K}\right)$ and $\eta > 0$ that satisfies

$$
\frac{g\left(s_i^k(m+1), s_i^{k,K}\right) + \sum_{\widehat{s}_i^k} \lambda\left(\widehat{s}_i^k, s_i^k(m+1), s_i^{k,K}\right)}{\eta} = \Pr\left(\widehat{s}_i^{k,K} = s_i^{k,K} | s_i^k(m+1)\right).
$$

Since $s_i^k(m+1)$ is separated from $S_i^K / \left\{s_i^k(1), ..., s_i^k(m)\right\}$, by exactly the same argument, we have $\lambda\left(s_i^k(m+1), \widehat{s}_i^k, s_i^{k,K}\right) = 0$ for any $\widehat{s}_i^k \in S_i^K / \left\{s_i^k(1), ..., s_i^k(m), s_i^k(m+1)\right\}$ and $s_i^{k,K}$. Thus, $\sum_{s_i^{k,K}} \lambda\left(s_i^k(m+1), s_i^k(m+1), s_i^{k,K}\right) = 1$ and $\lambda\left(\widehat{s}_i^k, s_i^k(m+1), s_i^{k,K}\right) = 0$ for any $\widehat{s}_i^k \neq s_i^k(m+1)$ and $s_i^{k,K}$ by (9). This proves the proposition. ■

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