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SAN DIEGO STATE UNIVERSITY

Calculus Students' Representation Use in Group-Work and Individual Settings

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Mathematics and Science Education

by

Dov Zazkis

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2013

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University of California, San Diego

San Diego State University

DEDICATION

This dissertation is dedicated to my supervisor, Dr. Chris Rasmussen. Whenever I thought I had written something good he always found a way to push me to think harder and make it better. I have grown immensely as a person and as a scholar thanks to his tutelage. I would also like to thank my mother, Dr. Rina Zazkis. Without her support I would not have kept my sanity. Finally, I would like to thank Dr. Janet Bowers for helping to facilitate my data collection.

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ABSTRACT OF THE DISSERTATION

Calculus Students' Representation Use in Group-Work and Individual Settings

by

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Doctor of Philosophy in Mathematics and Science Education

University of California, San Diego, 2013

San Diego State University, 2013

Chris Rasmussen, Chair

The study of student representation use and specifically the distinction between analytic and visual representations has fueled a long line of mathematics education literature that began more than 35 years ago. This literature can be partitioned into two bodies of work, one that is primarily cognitive and one that is primarily social. In spite of the large overlap in the results and foci of these two bodies of work they have tended to not inform one another. I bridge these two bodies of work by creating and implementing an analysis tool, referred to as the VAP-model, which can be used within both group-work and individual interview settings. This model allows focusing on how students in

both settings transition between representations during problem-solving and how these transitions fuel students' mathematical advancement. The VAP-model considers physical representations in addition to the analytic/visual distinction, which has driven much of the research on representation use. Physical representations are references to realistic or imaginable scenarios in mathematical problem-solving.

Fluency with multiple representations has been long considered an important aspect of deep understanding of mathematics in general and calculus in particular. This study examines the representation use of three calculus students from a technologically enriched calculus course in both in-class group-work settings and individual interview settings. This analysis reveals that students incorporate representations not required by given tasks and that, in spite of their common classroom experiences, the representations used in problem-solving vary widely across students. Comparing representation usage in the two settings showed similarities in how frequently each participant transitioned between modes of representation, which modes appeared within their reasoning and the types of questions in which particular shifts between representations occurred. This analysis also highlighted cross-setting similarities in how social roles facilitated shifts in representation use and in what ways these transitions occurred. Within both settings physical representations played an important role in students' thinking and were often introduced to reason about non-physical problems. Since in past studies this type of representation use was commonly subsumed under other categories this phenomenon was often overlooked.

CHAPTER 1:

Rationale and Motivation

The role of both imaginable physical scenarios and visualization in mathematics education has become more prominent in the last several decades. A mix of curricular changes, advances in technology and pedagogical advances has fueled these changes. As a result students entering calculus use these physical and visual representations to reason about mathematics more often than their counterparts from past decades. The research that helped establish that student's representation usage patterns now incorporate more visual and physical representations is fairly recent (e.g., George, 1999; Stylianou & Silver, 2004). Consequently, the specifics of how and for what purpose these representations are used and when shifts between representations occur is not well documented in the mathematics education literature. This study aims to add to what is known about the representation usage patterns of today's calculus students by examining students' problem-solving in a modern technologically enriched calculus classroom. This environment is reflective of some of the changes in calculus classrooms happening today, such as the use of technology and emphasis on non-notation based representations. The study explores how visual, analytic and physical modes of representation interact with one another during both individual problem-solving settings and in-class group-work settings. How what happens in one setting informs and frames what happens in the other setting is of particular interest.

Why Study Calculus?

Science, technology, engineering, and mathematics (STEM) graduates have been identified by policy makers (US Department of Commerce, 2011), major media (Baluja, 2011; Ojalvo, & Doyne, 2011) and education researchers (Eagan, Hurtado & Chang, 2010; Higher Education Research Institute (HERI) (2010), as crucial for the creation of economic growth and the future prosperity of America. All of these groups of stakeholders agree that there is a growing shortage of such graduates. In every higher education institution STEM students must complete a calculus sequence as part of their degree. Studies have pointed to calculus as a major stumbling block for potential STEM graduates (HERI, 2010; Seymour, & Hewitt, 1997). This stumbling block contributes to many students' decision to leave STEM disciplines. Improvements to calculus education have the potential to increase the number of STEM graduates and in turn have widespread positive economic and social impacts. There is a history of calculus improvement initiatives.

The calculus reform movement was a major push in curriculum development and education research that occurred in the United States during the late 1980's and early 1990's (Roberts, 1996). Many seminal works on students' thinking in calculus were published during this period (e.g., Heid, 1988; Thompson, 1994; Vinner, 1989) and these works continue to inform what is known about students' learning of calculus. However, the students entering calculus today differ from students that entered calculus 25 years ago. Modern students have been exposed to pre-calculus curricula that differ from those used 25 years ago. Curricula now incorporate graphing calculators and place much more emphasis on visually based reasoning (Knuth, 2000). Consequently, tertiary students

today are more inclined to use visualizations as part of their problem-solving processes (Stylianou, 2001; Stylianou & Silver, 2004) and to incorporate technology into their learning (Tall, Smith, & Peiz, 2008; Hoyles & Lagrange, 2010). Studying the thought processes of this new crop of students and how these processes interact with technology is thus important for informing what is known about student calculus learning and informing future improvements to calculus.

Understanding the role of physical and visual representations in calculus is particularly important because most of the students taking calculus are not mathematics majors. They are primarily science and engineering majors (Bressoud, Carlson, Mesa, & Rasmussen, 2013). These students will interact with real world data and have to interpret various graphical representations of this data throughout their careers. So studying calculus student's usage of visual and physical representations illuminates a part of their mathematics knowledge that is important for their future careers.

Overview of dissertation

Mathematics is inextricably linked to the representations of mathematical objects used to communicate and develop it. One of the significant ways in which curricula and mathematics as a discipline has changed is in its use of technology and the shifts in representation usage that come with it (Stewart, 1995; Knuth, 2000). The specific three representation types studied here are visual, analytic and physical representations.

Analytic representations utilize mathematical notation. Visual representations, on the other hand, utilize non-notation based mathematical objects, such as graphs and diagrams. Physical representations utilize reasoning about non-mathematical objects, which may or

may not be real world objects. A ubiquitous example of this within a calculus context involves reasoning about the motion of objects such as particles or vehicles. When these representations are used within mathematical problem-solving they will be framed as visual reasoning, analytic reasoning and physical reasoning.

Although there is growing evidence that students incorporate visual and physical representations in their tertiary mathematics problem-solving how these modes of representation interact with one another is poorly understood (George, 1999; Stylianou, 2001; Stylianou & Silver, 2004; D. Zazkis, in press). This dissertation examines a group of three students enrolled in a technologically enriched calculus class and studies how visual, analytic and physical modes of representation interact with each other to inform both individual problem-solving and group activities. It specifically concentrates on how the group and individual counter inform each other and helps to illuminate the role the three modes of representation play.

This chapter describes the motivation for the study and situates the study with respect to current literature. In particular, I highlight similarities and differences between this work and past research in terms of context and perspective. The chapter concludes with the research questions. The second chapter provides a detailed literature review of both technology in calculus and research on representations. The third chapter lays out the study setting, data and methods. It elucidates the theoretical perspective and the analytic tools used to examine the data. It also discusses the development of the analytic tools and how it is informed by the theoretical perspective and past research. The fourth chapter is an analysis of the three participants' individual use of visual, analytic and physical representations within individual problem-solving interviews. . The fifth chapter

analyzes the representation use of the same three students in a group problem-solving setting. The final results chapter, chapter six, examines patterns in representation and problem-solving behavior across both of these settings and how students' behaviors in one setting inform their behaviors in the other. The final chapter discusses the implications of this research and some possible future directions.

Visual Representations

The role of visualization in mathematics education has evolved over the last 25 years. A shift began occurring in the late 1980's due to curricular reform movements (NCTM, 1989; Roberts, 1996) and was further fueled by the availability of ever cheaper computing and graphing technology, which began to gain wide spread use in the mid 1990's. Curricula before this period, particularly in algebra, were fairly devoid of visual modes of representation and made very limited use of technology (Kaput, 1993; Kaput and Thompson, 1994; Leinhardt, Zaslavsky, & Stein, 1990). Although graphs and other visualizations were included in algebra textbooks, they were typically absent from the problems students were asked to complete. These visualizations were thus tangential to students' experiences with mathematics. The algebra curriculum leading up to calculus was primarily centered on developing a repertoire of computation techniques (Leinhardt, Zaslavsky, & Stein, 1990). Consequently, students entering tertiary institutions during this period showed significant resistance to implementing visual modes of representation (Dreyfus, 1991; Vinner, 1989). Vinner (1989) argued that this resistance was due to student experiences with algebra, which had painted mathematics as a symbol-pushing

discipline; visual approaches to tertiary mathematics tasks were not inline with students' previous experiences with mathematics.

Students entering tertiary institutions after this period do not show this same resistance and tend to use visual representations at rates similar to those of career mathematicians when working on novel tasks (George; 1999; Stylianou, 2001). The studies of George (1999), Stylianou (2001), and Stylianou and Silver (2004) point to students entering tertiary institutions using representations in very different ways than their counterparts from 20 years ago. Although there is growing evidence that modern students use visual arguments and visual representations more than past students, there is little in the literature that informs when these visualizations get used, for what purpose and how they interact with other modes of representation.

Physical Representations

The role of imaginable physical situations in mathematics curricula has undergone similar shifts in usage to those seen with visualization. Although word problems, which contextualize mathematics within imaginable scenarios, have been a part of mathematics curricula for as long as such curricula existed, these problems were often treated as end points for mathematical instruction rather than starting points for introducing mathematics (Kaput, 1993; Leinhardt, Zaslavsky, & Stein, 1990). That is, mathematical concepts were traditionally taught abstractly as finished products and procedures. A small handful of contextualized problems were often relegated to the end of a unit and far outnumbered by their strictly numerical counterparts. Freudenthal (1973) refers to this as an anti-didactical inversion, which is when the end results of the work of mathematicians,

typically standardized algorithms for solving particular types of problems, are taken as the starting points for mathematics education.

Mathematics curricula that begin with physical scenarios to motivate the reinvention of mathematical concepts are not as commonplace as the inclusion of visual approaches; however, shifts in this direction can be seen in a number of curricula. The most recent set of National Council of Teachers of Mathematics standards (NCTM, 2000) include recommendations for inclusion of more meaningful connections to real world based scenarios. Curricula such as Hughes-Hallett et al.'s (1994) calculus series and McCallum et al.'s (2006) algebra text motivate mathematical concepts through connections to imaginable real world-based scenarios.

The parallels between physical scenarios and visualization are not limited to curricular shifts. Technology can also play a role in connecting imaginable physical scenarios to mathematical representations of them. Although, the mathematics education literature is much less developed when it comes to the study of mathematical reasoning grounded in imaginable physical scenarios, it would not be surprising to see a similar shift in student problem-solving behavior to that seen with visualization.

The classroom that provided the data for this study often introduced calculus ideas through the use of physical scenarios. This study examines how physical representations interact with other representation modes. It provides valuable insights with regard to how students exposed to curricula that use physical scenarios approach problem-solving.

Cognitive vs. Social Perspectives

Much of the mathematics education literature on representational modes has focused on visual modes of representation and their relationship to analytic notation-based representations. The majority of this research has been conducted from cognitive perspectives focusing on individual learner's mental processes (e.g., Aspinwall & Shaw, 2002; Cox, 1999; Haciomeroglu, Aspinwall, & Presmeg, 2010, Krutetskii, 1976; Presmeg, 1985; Stylianou, 2001; Vinner, 1989; R. Zazkis, Dubinsky, & Dautermann, 1996). So students' use of visual and analytic representations is often framed as "visual thinking" and "analytic thinking". However, a small but growing cohort of researchers are using social perspectives to study this phenomena (e.g., diSessa, & Sherin, 2000; Cobb, 2002; Meira, 2002; Roth & McGinn, 1998). The cognitive and social approaches to research have remained largely disjoint. With the results from cognitive research on representation use not informing social research on representation use and vice versa.

A recent trend within the cognitive research on representation use has been to move away from categorizations of reasoners general tendencies to use particular representations (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010, Krutetskii, 1976; Presmeg, 1986) and instead model student thinking as a process facilitated by transitions between representations (e.g., Duval, 1999; D. Zazkis, in press; R. Zazkis, Dubinsky, & Dautermann, 1996). This is a move away from visual/analytic thinkers type categorizations of students and toward a treatment of all students as users of multiple representations. The visual/analytic thinker categories are effective for studying large collections of students who have large variations in their approach. However, it tends to hide subtler differences in student approaches, such as, the interplay between modes and

how one mode might inform another (D. Zazkis, in press). This interplay is increasingly important in light of studies that point to a large growth in students' use of multiple modes of representation within their problem-solving (George; 1999; Stylianou, 2001; Stylianou & Silver, 2004).

The approach within this work attempts to balance both social and cognitive perspectives without abandoning the results and methodologies that have grounded the cognitive work in this area. This is achieved by expanding and extending the use of a constructivist model of examining student mathematical thinking (R. Zazkis, Dubinsky, & Dautermann, 1996). This expansion extends the model's core-premise that individual mathematical advancement is facilitated by transitions between visual and analytic reasoning; the model developed in this work is built off the premise that the mathematical advancement of a group of students is also facilitated by these same transitions between modes of reasoning. This approach incorporates social considerations into the study of representations without abandoning the methods cognitive (constructivist) studies have utilized.

The Evolving Role of Technology

The curricular shifts that occurred through the 1990's did not occur separate from advances in technology. Although using computers to do mathematics is as old as computers, it was not until the birth of the personal computer in the 1970's that it became viable to use computers for educational purposes and it was not until the mid 1990's that technology became cheap enough to garner widespread use in American classrooms. Before the release of the Texas Instruments TI-83 calculator in 1996 it was difficult and

expensive to incorporate technology into classrooms. A particularly telling quote can be found in the work of Kaput and Thompson (1994) in their review of the first 25 years of technology in mathematics education research.

To use technology in mathematics education research also involves other practical complications, chief among which are the logistical problems of getting subjects and computers in the same location at the same time, and to provide teachers and students with appropriate computer and curricular materials (p. 681).

This quote was published only two years before the release of the TI-83 and stands in stark contrast to today's algebra curricula. What was once a "sparse but increasing use of computers and graphing calculators" (Romberg, Fennema, Carpenter, 1993, p. xi) is now a wide spread phenomena used for both visualization (graphing) and computation. The question of whether technology should be used in the classroom has been abandoned in favor of questions regarding how and for what purpose should technology be used (Tall, Smith, & Peiz, 2008). Today it is common for calculus textbook publishers to market online packages, which include interactive applets, online homework assignments and other tools, as part of their textbook.

The availability of technology for classroom, home and office use meant that many of the computational skills that were once solely done by hand could be relegated to a handheld device. The world for which mathematics education was preparing students was now one where technology would be available for calculation purposes. This meant that the mathematics knowledge and skill set needed for the work place was less calculation based. This allowed algebra curricula to broaden their narrow focus on computing and equation manipulation, to include work with multiple representations of

functions. This has elevated the importance of research on students' understandings of multiple representations and the connections between them (Knuth, 2000). Many curricula still primarily focus on pencil and paper calculations, but now also incorporate graphing units in which visual approaches cannot be supplanted with analytic approaches. The importance of studying student understanding of multiple representations stretches beyond work in algebra. The studies of George (1999), Stylianou (2001), and Stylianou and Silver (2004) point to tertiary students' increased usage of these multiple representations within their own problem-solving. So tertiary students' usage of multiple representations plays an important role in their mathematical thinking.

It is important to note that the role technology plays in classrooms goes beyond facilitating an increased focus on visual representation. Technology allows students to interact with mathematical objects in ways that would not be possible using only pencil and paper. Technology-facilitated changes in how students interact with mathematical objects can have profound effects on how students interpret and use mathematical objects and mathematical concepts (Sinclair, Healy, & Reis Sales, 2009; Tall, 2012).

Mathematical objects can now have dynamic representations and new types of mathematical objects can be created (Sinclair, Healy, & Reis Sales, 2009; Stewart, 1995). How technology is used in the classroom has much more influence on student learning than whether technology is used (Kaput & Thompson, 1994; Tall, Smith and Peiz, 2008; Hoyles & Lagrange, 2010). So when studying student learning in a technologically enriched environment it is important to document the nature of the technology and the role it is playing in classroom interactions. The next section provides a brief description

of some of the ways technology was utilized in the classroom that is the backdrop for this study.

The Setting for the Study

This study examines three students who were enrolled in a mainstream Calculus I class and their representation use patterns during in-class group-work and during a series of three one-on-one problem-solving interviews. The class made use of most of the major advances in learning technologies. This included: the assignment of online-video lectures as homework, web-based homework assignments, Geometer's Sketchpad® based applets, the use of web-based computer algebra systems (WolframAlpha), and instructor video-responses to particular difficulties that emerged during class. As such, the data from this study provides initial insights into what influences some of these technologies may have when carried through to a calculus setting. Studies of visualization and computation technologies and their effect on student learning have been part of the literature for a number of years, while studies of online videos are relatively scarce. However, for both the emerging areas of research and the areas that are well developed, a birds eye view of the literature reveals that how technology is implemented is more important than whether technology is implemented (Tall, Smith and Peiz, 2008; Hoyles & Lagrange, 2010).

The classroom that is the backdrop for this study utilized a number of pedagogical strategies that have been linked to student success (e.g., Boaler, 1998; Hoyles & Lagrange, 2010; Johnson & Johnson, 1992). This includes open ended group-work that often centered around a particular applet, regular formative assessments and student centered whole class discussions that often also centered around an applet. So, the

pedagogical strategies used in this classroom are consistent with strategies past studies have linked to improvements in student learning outcomes. This choice of backdrop helps provide a glimpse into the type of student representation patterns that may emerge from research informed implementations of technology in calculus classrooms.

Although applets used for visualization have been studied since the 1980's (Tall, 1986) the nature and interactivity of these applets has evolved due to both advances in technology and advances in research. The design and implementation of the applets used in this class has been informed by mathematics education literature and many of them have been created specifically for this class.

Focusing in on a specific applet

In order to help clarify the role applets may play in the evolution of student thinking and concept development, I describe one of the applets that was incorporated into the course. Discussing this applet as well as the instructor's intended use of it and what student understanding it is meant to support is intended to exemplify how technologies were used in the classroom in which this study takes place.

The applet discussed here is designed to help build students' intuitions and understandings regarding the relationship between the slopes of tangent lines, the derivative at a point and the derivative function. Difficulties reasoning about this relationship are well documented in the literature (e.g., Apsinwall & Shaw, 2002; Kung & Speer, in press; Nemirovsky & Rubin, 1992; Orton, 1983). Students' group interactions with the applet are examined in the fifth chapter of this dissertation. I first

describe how the applet works and then move to some understandings that students may gain from interacting with it.

The Tangent Intuition Applet (TIA) makes use of slope-widgets to help students make the connection between a function and its derivative. Each slope-widget consists of two connected points. The first point, the tangent point, bisects a small line segment, which is intended to act as a tangent. This point is colored red. The second point, which is aligned vertically with the first, controls the slope of the tangent through the first point. The second point is colored green. For each widget the slope through the red point corresponds to the height (y-value) of the second green point. Figure 1.1 shows how the slopes through the red (tangent) points take on different values as the green slope control points are placed in different locations.

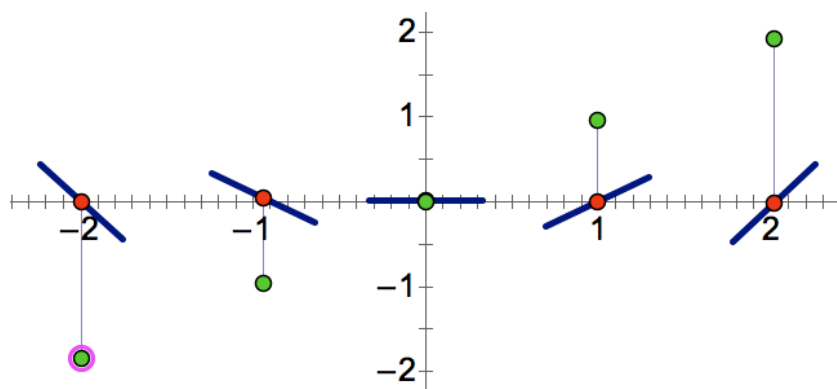


Figure 1.1: Slope-widgets

When the red tangent points are placed on the graph of a function, $f(x)$, and the green slope-control points are adjusted so that the slope through the tangent points corresponds to the slope of the function the green slope-control points correspond to points on $f'(x)$. When used in this way, the green slope-control point of particular widgets

can be thought of as the derivative at a point. This can be seen in Figure 2 where the red tangent points are placed onto the function $f(x)=x^2$ and the green points are adjusted until the slopes match the function. Here green slope-control points lie on the line $f'(x)=2x$, the derivative of $f(x)$.

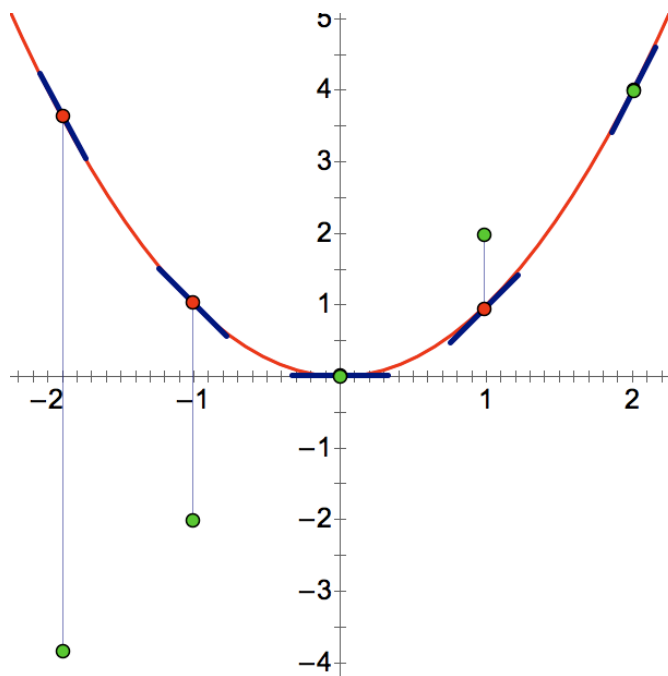


Figure 1.2: Slope-widgets on a function

How students interact with the tangent intuition applet and what intuitions and understandings they gain from these interactions have not been studied; it was created specifically for the classroom that is studied in this dissertation. Examining students' interactions with this applet and how these interactions inform student problem-solving behavior provides valuable insights into both students' thinking about calculus and how these behaviors relate to design features of the applet.

There are several important understandings that this applet may help build or endorse. When used individually, the slope-widgets may be used one at a time to gain intuition for slope at a point. For example, experience moving just the widget's red tangent point vertically while keeping the green derivative point static may help students gain intuition for why the derivative of $f(x_i)$ is equal to the derivative of $f(x_i) + c$. When multiple widgets are used concurrently they can map out the path $f'(x)$ takes relative to a particular function $f(x)$. This may help bridge a connection between the concept of derivative at a point and derivative of a function. This distinction is often lost when the concept is taught using traditional means, particularly since in English—unlike for example in Korean—both are given the same name, derivative (Park, 2012; Zandieh, 2002; Zandieh, & Knapp, 2006). Interacting with these widgets in relation to various functions may help build intuitions regarding how various functions are related to their derivatives. These intuitions and mental imagery regarding the operation of these widgets, which are introduced early in the course, may be invaluable later in the course when students work on graphing derivatives and second derivatives. The instructor/applet designer hopes that these intuitions can be in fact used to scaffold students' *guided reinvention* of algorithms for graphing the derivative of a function by hand (Freudenthal, 1991; Gravemeijer & Doorman, 1999; Rasmussen, Zandieh, King, & Teppo, 2005). Such work involving applets used to scaffold students' guided reinvention of calculus concepts is a blossoming area of undergraduate mathematics education research (Martin, Oehrtman, Swinyard, & Cory, 2012; Rasmussen, & Blumenfeld, 2007; Rasmussen, & Kwon, 2007).

Research Questions

Mathematical reasoning based on physical scenarios and/or visual models is much more prevalent in today's students than students from 25 years ago. However, although there are studies that point to students' increased usage of these modes of reasoning, little is known about how these modes of reasoning interact with each other and with analytic reasoning. How these modes of reasoning inform and build off of one another is poorly understood. Addressing the research questions that guide this study will help fill this gap in the literature. The research questions are:

- 1) How do individual students think about key calculus concepts during their problem-solving? In particular, how do they use representations to facilitate this thinking?
- 2) How do groups of students communicate about key calculus concepts? In particular, how do they use representations to facilitate this communication?
- 3) Within a calculus setting, in what ways do students' thinking in individual interview settings and their communication in group settings inform each other?

The subsequent chapter, chapter two, reviews the calculus education literature as well as the literature on student representation usage. The third chapter includes a discussion of the theoretical perspective and the analytic tools used to examine the data. Once the theoretical perspective, Sfard's (2006, 2008) commognitive perspective, is introduced the research questions will be revisited and explored relative to this perspective. The results chapters, chapters four, five, and six, address research questions

one, two, and three, respectively. The fourth chapter focuses on data from one-on-one problem-solving interviews. This provides a setting for studying individual problem-solving. The fifth chapter focuses on in-class student group-work. This provides a setting for studying each of the three modes of representation in a group setting. The sixth chapter coordinates the results of the other two analysis chapters and compares similarities and differences between the individual interview data and the group data.

CHAPTER 2:

On Calculus, Technology and Representations

This study follows a group of undergraduate students in a technology enhanced calculus course. The study itself focuses on the ways students think and interact around calculus concepts. Within this study I pay special attention to students' representation usage. The increased emphasis on multiple modes of representation is one of the fundamental differences between reform calculus curricula and their more traditional counterparts (Habre & Abboud, 2006; Hughes-Hallet et al. 1994). Conceptual fluency with calculus and in particular with multiple representations is an end goal of much of the calculus reform movement and is also a motivation for the incorporation of technology in this study. So it is natural to study representation usage in this context. Although technology itself is not a focus of this study per-se, its influence on student interactions, student approaches and student use of representations is a focus. There is a long history of research on technology in mathematics education. Exploring this literature helps situate the classroom that provides a backdrop for this study with respect to literature on technology use.

This literature review is organized into two main sections. The first section provides a background and situates the context of the study relative to other studies and reform movements related to teaching calculus. The second section delves deeper into the literature on representation use. Literature on students' understanding of calculus is also relevant for this study. Since it is deeply intertwined with both the context/history of

teaching calculus and representation use, I discuss it within each section rather than give it its own section.

Technology in Calculus Education Research

Technology has been used and studied in calculus classrooms since the early 1970s when the first desktop computers came to market (Holoein, 1971; O'Loughlin, 1976). The emergence of these studies coincides with a major paradigm shift in mathematics itself, which was also facilitated by the emergence and advancement of the personal computer (Stewart, 1995). Computers have changed the nature of what doing mathematics entails (Heid, 1997; Stewart, 1995; Hoyles, & Lagrange, 2010; Tall, Smith, & Peiz, 2008). The old adage that all one needs to do mathematics research is a pad of paper, a pencil and a wastebasket is now far removed from the realities of conducting modern mathematics research. Computer algebra systems (CAS), dynamic geometry software, graphing software and even computer programming abilities are now indispensable tools for many mathematicians. Consequently, these tools are making their way into mathematics classrooms and shifting the foci of curricula away from symbol pushing, which computers are ideally suited for, and toward a vision of mathematics that includes more work with visual representations and real world scenarios.

Education researchers are still working to understand what a modern calculus course that prepares students to function within this new mathematics paradigm can and should look like (Hoyles, & Lagrange, 2010; Tall, 2012; Tall, Smith, & Peiz, 2008; Thompson, Byerley, & Hatfield, in press). Creating and understanding these courses is a moving target; new technologies and ideas for what can be done in the classroom with

these technologies continue to emerge as research progresses. However, it is clear from the first three decades of research that technology is not a magic bullet. It is not a substitute for good teaching. A well thought out curriculum and a host of other factors contribute to a successful course (Tall, Smith, & Peiz, 2008; Hoyles, & Lagrange, 2010). A Mathematical Association of America committee on calculus reform (Roberts, 1996) observed that:

A calculus course cannot be modernized simply by finding a way to make use of graphing calculators or computers... Spelling checkers... will not make a good writer out of a poor one and computer algebra systems will not make a good mathematician out of a poor one, but efficient practitioners of the art will make intelligent use of all the tools that are available. (p.2)

Since the use of technology in calculus courses does not occur in a vacuum, research on technology use in calculus does not either. To discuss research on technology use in calculus and some of its results, it is useful to map out the types of studies that have emerged within the field as well as the types of other educational research and movements that have influenced it.

Procedural vs. Conceptual

Prior to the arrival of technology and the calculus reform movement, the main focus of a calculus course was to help students build a catalog of techniques they could use for differentiation and integration (Kung, & Speer, in press; Heid, 1997; Tall, Smith, & Peiz, 2008). When appropriate, static pictures of graphs were used to illustrate some of these phenomena, but the focus remained primarily on pencil and paper algorithmic techniques. Knowledge of these techniques and their implementation is often referred to

as *procedural knowledge* in the literature (Heid, 1997; Tall, Smith, & Peiz, 2008).

Calculus knowledge other than procedural knowledge is commonly referred to as *conceptual knowledge*. This however, is a category of exclusion. It includes all calculus knowledge that is not procedural knowledge. Such knowledge can include: graph interpretation and creation skills (Vinner, 1989; Zimmerman, 1991), knowledge of various representations and how to translate between them (Habre & Abboud, 2006; Hughes-Hallet, 1991, Zandieh, 2000), the ability to derive procedures from basic principles (Heid, 1988), the ability to tackle novel/nonroutine problems (Meel, 1996; Selden, Mason, & Selden, 1989), kinesthetic/physical interpretations of functions/graphs that describe motion (Botzer, & Yerushalmy, 2008; Nemirovsky, Tierney, & Wright, 1998), and the ability to translate word problems into calculus equations (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, & Carlson, 2012). Within the context of this study, I consider conceptual knowledge to include all of the above. However, this study focuses primarily on graphical and physical interpretations of calculus concepts.

As a general rule, research on technology in education since the late 1980s has focused on creating and studying technological tools and environments aimed at supporting students' conceptual knowledge in its many forms. Computers are useful tools for dealing with the procedural aspects of mathematics. In the modern computer age, it has become progressively more important for people to become comfortable with the parts of mathematics that computers cannot address—the conceptual parts (Fey, 1989; Thompson, Byerley, & Hatfield, in press). Computers therefore both provide some of the impetus for shifting to more conceptually oriented curricula, which includes some of the

moves toward more visual and physical representations discussed later in this chapter, and powerful tools for exploring the concepts associated with calculus.

The Calculus Reform Movement and the Current State of Calculus

In the late 1980s and early 1990s, there was a substantial amount of conversation regarding the state of calculus (Douglas, 1986; Ferrini-Mundy, & Graham, 1991; Hurley, Koehn, & Ganter, 1999). This was fueled by several considerations. First was a shortage of Science, Technology, Engineering, and Mathematics (STEM) graduates; traditional calculus was seen as a substantial barrier to people intending to pursue degrees in STEM fields (Douglas, 1986; Hurley, Koehn, & Ganter, 1999; Seymour, & Hewitt, 1997). Second was that the nature of calculus had changed (Buchberger, 1989; Douglas, 1986; Stewart, 1994). Computer Algebra Systems (CASs) were now available, meaning that many of the pencil and paper algorithms that formed the core of a calculus course could be tackled by computers instead of people. The Mathematical Association of America (MAA) published a report outlining a vision for a new “lean and lively” calculus (Douglas, 1986). The vision involved slimming down the curriculum; instead of a massive collection of calculation techniques, calculus would instead focus on a core set of calculus concepts. Technology was seen as an essential part of this vision (Ferrini-Mundy, & Graham, 1991). Several quantitative comparison studies during this period provided statistical evidence of the efficacy of a lean concept intensive calculus course (Heid, 1988; Hurley, Koehn, & Ganter, 1999; Palmiter, 1991). These studies showed that it was possible to use technology to run a “concepts first” calculus course that when

compared to a traditional course improved students' conceptual knowledge, yet did not have detrimental effects on students' procedural fluency.

In spite of these efforts, many calculus courses have remained unaffected by reform efforts. Instead of lean and lively curricula, popular calculus texts such as Stewart (2008), and Thomas, Weir, Hass and Gwardana (2005) stretch for over 1000 pages. A recent survey of 700 instructors at 212 universities and colleges revealed that only 19% of instructors utilized reform textbooks and only 15% encourage their students to use some form of computer algebra system during class time (Bressoud, Carlson, Mesa, & Rasmussen, 2013).

Some of the lack of impact from reform efforts has been attributed to a backlash against reforms (Hurley, Koehn, & Ganter, 1999; Tall, Smith, & Peiz, 2008). Anti-reform works claim that the calculus reform movement pushed for a “dumbed down” calculus, which replaces a formerly precise and rigorous discipline (e.g., Abain, 1998; Norwood, 1997; Wu, 1996, 1997). However, most of this anti-reform work has been based on the professional teaching experience of research mathematicians rather than empirical evidence from education research (Tall, Smith, & Peiz, 2008). Heid and Blume (2008) draw an analogy between the current state of reform efforts and Piddwell's (1939) mythical tale of the saber-tooth curriculum. In this tale, ancient cave dwellers continued to teach how to grab fish by hand long after fishing nets had been invented.

Technology in Calculus Education Research

Digital technologies have come a long way from their humble beginnings. The roles these technologies play in calculus education and the educational research

community's understandings of the influences these roles have on students' learning and understanding have made similar advancements. In order to organize a conversation regarding the evolution of this research area, it is useful to partition the various research and technological innovations that have driven this evolution forward into categories. I distinguish between influences external to education research, such as the continuing development of new technologies, and influences internal to education research. I organize my discussion of internal influences, which come from within the education research community, into three types of published work: small scale qualitative studies, which I refer to as small-n studies, large scale quantitative studies, which I refer to as large-n studies, and studies that develop new curricula. The relationship between these three subcategories of internal influences and the advancement of technology is illustrated in Figure 2.1. Small-n studies are in-depth, usually qualitative examinations of student thinking. They generally involve one-on-one or small group problem-solving interviews and illuminate the nuances of students' understandings of and difficulties with calculus. These often result in some kind of cognitive or social model of the phenomena examined. Due to the intense examination of individual thinking, these studies usually include only a small number of students.

Large-n studies are generally quantitative comparison studies. These usually take two treatment groups; for example, two classes that use different curricula and quantitatively compare measures of student performance, such as common exam items. The results of these studies are often statistics that inform claims about differences in understanding/performance between the two groups compared. These studies can provide

strong evidence that differences between treatment groups exist, but often provide little insight regarding the specific nuances and causes of these differences.

The final subcategory is studies that develop new curricular ideas. These studies attempt to create new approaches to teaching calculus, and often take the form of teaching experiments (Cobb, 2000). However, textbooks and other curricular materials influenced by previous research can also have large influences on how calculus is taught (e.g., Hughes-Hallet et al., 1994).

Now that I have outlined what I view as the major influences on technology in calculus education research, both internal and external, I discuss each of these in greater depth. I begin by briefly discussing the influence of technology and its evolution, which has been driven by forces outside of the education research community. This discussion provides a backdrop for discussing the research literature.

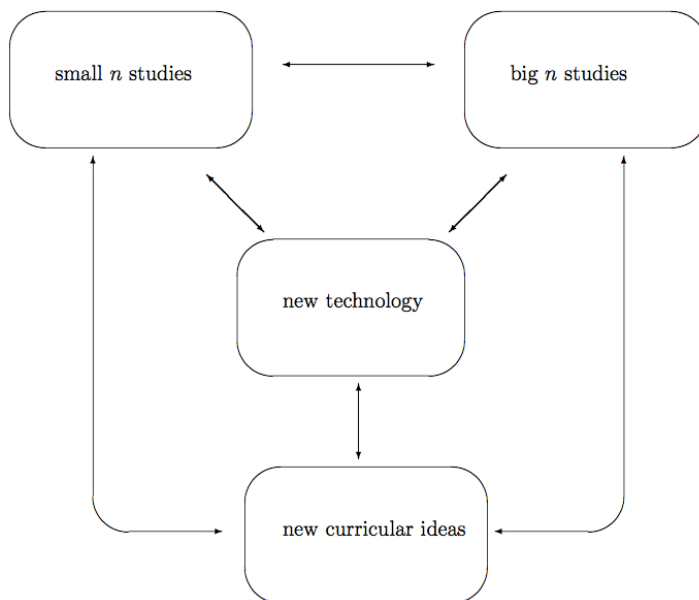


Figure 2.1: Relationship between types of research and technology

The Evolution of Technology for Education

In the early 1970s computing was in its infancy; the first desktop computers were just coming to market. The educational potential of these devices was not immediately obvious to many in the educational community. Early devices were heavy, expensive and operated solely through text-based interfaces. This made them quirky and unintuitive to use. The Mathematics Association, a United Kingdom based teachers' association, published the following statement: "It is unlikely that the majority of pupils in this age range will find [a computer] so efficient, useful and convenient a calculating aid as a slide rule or book of tables" (Mathematics Association, 1974, as cited in, Tall, Smith, & Peiz, 2008, p. 211). One way this quote can be interpreted from a modern perspective is to assume that those that authored it lacked foresight. However, I would argue that the ubiquity, functionality and usability of modern computing were all but impossible for most to imagine at the time. Looking up a value on a table may have genuinely been more convenient—and likely faster—than fumbling through a series of quirky text commands. Consequently, seeing educational potential for computing devices was a difficult proposition for many.

During the 1970s the United States was experiencing a backlash from the "New Math" movement of the 1960s. This backlash involved a strong push toward back-to-basics calculation and procedure-based mathematics. The field of mathematics education research was in its infancy. Work such as Erlwanger's (1973) seminal case-study, which showed that procedures-only mathematics education could result in a disconnect between mathematical objects and their meaning in a highly skilled student, were just starting to emerge as part of the educational research literature. Psychologists who used

mathematics as a context for their studies often treated mathematics knowledge as consisting solely of knowledge of mathematical procedures. Research on technology in education at the time reflected this procedure-centered mindset. Early studies of technology in education were primarily comparative studies. These examined the achievement differences on computation-centered exams between students taught according to a traditional paper and pencil calculation centered curriculum and a similar curriculum that used computers or hand held calculators as a sort of computational add on (see Kaput & Thompson, 1994; Heid, 1997 for reviews). These large-n studies generally showed either detrimental effects on computational skills or no statistically significant differences. Such studies only bolstered the view that technology, such as hand-held calculators and desktop computers, were not beneficial for mathematics instruction.

The latter part of the 1980s and early 1990s marked a turning point for research on technology in mathematics education. The field of mathematics education was in transition. Fueled by the work of Jean Piaget and Lev Vygotsky, the field shifted from research that concentrated on procedures based test outcomes to research that explored student thinking and the influences and nature of student-teacher and student-student social interactions.

Computers' computational capabilities had also improved. They were no longer solely tools for numerical computation. New classes of algorithms meant that computers were now capable of solving algebraic equations and computing integrals and derivatives. Programs designed to do such computation are referred to in the literature as CASs (Computer Algebra Systems). Changes in technology were not limited solely to what could be calculated. The introduction of high-resolution graphics meant that computers

could now be used for graphing and visualization and not just computation. Both the use of CASs (Heid, 1988) and the use of computerized graphing (Tall, 1986) were studied in calculus education contexts during this period.

The invention of the mouse in 1984 made interaction with computers much more intuitive and user friendly. The mouse also meant that one could interact with graphical mathematical objects in new ways. Early geometry graphic programs required the user to define and manipulate objects using written commands (cf. Leron & R. Zazkis, 1989; Yerushalmy, & Houde, 1986), which made testing conjectures and creating objects tedious (Tall, Smith, & Peiz, 2008). The advent of the mouse made it possible to create geometry programs that could facilitate clicking and dragging geometric and graphic objects (cf. Confrey, 1991; Confrey & Smith, 1994). Since a range of possible orientations and proportions could be easily and quickly explored with the use of a mouse, there was an increase in the exploration potential of these geometry programs (Tall, Smith, & Peiz, 2008).

The changes in both the general thrust of mathematics education and the capabilities of technology can be seen in research on technology in mathematics education. Mathematics education studies no longer assumed that technology should only be used as an adjunct to an existing curricula (Heid, 1988; Kaput, & Thompson, 1994; Tall, 1986). Heid (1988) in her influential study compared a concepts-first calculus class to a traditional class. The concepts-first calculus class spent 12 weeks of a 15 week semester studying calculus concepts with the aid of a CAS to do calculations and only spent the last 3 weeks of the semester on studying hand algorithms. Heid found that the concepts-first students performed comparably to the traditional class on algorithmic tasks,

but far outperformed the traditional class on conceptual questions. This study served as an existence proof that conceptual calculus learning can precede procedural learning with the aid of a computer. This qualitative study helped lay the groundwork for a new breed of technology facilitated mathematics education studies. These qualitative studies examined how the use of technology influenced and interacted with student thinking, with a focus on building cognitive models of student thinking (e.g., Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Beckmann, 1988; Martin, Oehrtman, Swinyard, & Cory, 2012; Tall, 1986).

In the 1990s, computer technology was still relatively expensive and not particularly portable. Computers in schools and universities were limited to computer labs, which were facilities often not designed with teaching in mind. Kaput and Thompson (1994), in their review of research on technology in education research, noted that the logistical problems associated with getting subjects and computers in the same place at the same time was a barrier to both researchers, who wished to study technology in mathematics education, and instructors, who wished to use it. Additionally, mathematics software was also prohibitively expensive, which meant that even if students did have a home computer, which was often not the case, they would not be able to use mathematics software on them. Additionally, even though research was often able to show positive learning outcomes (e.g., Heid, 1988; Palmiter, 1991), students were not able to use these same technologies in subsequent courses.

The 2000s brought with them hardware innovations, software innovations, the spread of the Internet and a significant reduction in the cost of personal computing. College students in Western countries were now expected to have access to a computer

and often traveled with portable computing devices such as laptops and tablets. Coupled with the low cost of (some) mathematics computing software, digital technologies now provided practical mathematical thinking and exploration tools for a new generation of students. WolframAlpha is one example of a CAS available freely over the Internet. Its existence tremendously simplified the logistical issues that used to plague courses that utilized CAS software and allowed such instruction to be used in large (100 people +) classes (Chappell, & Killpatrick, 2003). This also means that students could use a CAS for homework, studying and for activities not associated with a calculus course.

Technology has become so ubiquitous and inexpensive that there are now choices in terms of which piece of hardware or software to use. Studies have emerged that compare such technologies (e.g., Conners, & Snook, 2001; Fisher, & Lucas, 2012).

The spread of the Internet also created new opportunities for the creation of software environments for studying students' mathematical reasoning. In the 1980s and 1990s, researchers often spent years creating and refining entire applications that provided environments for mathematical exploration (e.g., Yerushalmy, & Houde (1986) work with Geometric Supposer). It was often difficult to adjust how the technology worked in a timely manner in response to observations of students interacting with it. The 2000s also brought with them tools for short-cuts and for quickly creating miniature environments for mathematical exploration. Open source libraries of premade Java-code as well as programs like Geometer's Sketchpad (GSP)—which allow one to quickly create mathematical objects and define particular mathematical relationships—meant that it was now fairly simple to create catered small applications (applets) that targeted exploration of specific mathematical concepts. In the case of GSP-applets, students can

now modify the applet themselves if they want the applet to work in a slightly different way. This means that the control of educational technology tools is now no longer the domain of a small number of education researchers with programming abilities—the control of these tools can now be at least partially handed over to students. This new breed of technology likely supports and constrains student thinking in ways that the technology of years past could not.

The Internet has given birth to streaming video libraries, which can provide asynchronous lectures. These libraries provide access to a diverse collection of online lecture content that facilitates both flipped classrooms, such as the one that will be studied here, and classes that take place completely online. Currently only 3% of calculus classes in the USA have some kind of online distance learning component (Bressoud, Carlson, Mesa, & Rasmussen, 2013).

In the modern computer age, computers are a ubiquitous part of every day life. Gone are considerations of whether or not to use technology in class. In tertiary education scenarios, students bring their laptops and tablets into class and have them available while they are working at home regardless of the instructor's emphasis or de-emphasis of technological tools. These students find out about free CASs like WolframAlpha regardless of whether the instructor encourages their use. In other words, these technologies are making their way into calculus courses regardless of instructors' approval or disapproval. To revisit the saber-tooth curriculum analogy, the younger generation is using nets to catch fish regardless of whether they were taught only to catch fish by hand.

Modern research regarding technology in education takes the use of technology as a given. The questions now become how the curriculum needs to be adjusted, in what way or ways technologies should be used and what effects these uses have on students' understandings. The focus has turned to questions regarding the educational affordances of technology. "How can I approach calculus topics differently now that technological tools are available?", "How can I use technology to support my students' concept development?", and "How does the use of technology support or constrain students' development of particular calculus topics?" are all important questions, educational researchers and calculus instructors need to ask.

Research Synthesis

In my discussion of research on technology in calculus education, I start by discussing small-n studies that do not necessarily incorporate technology. These provide a basis for understanding student difficulties with calculus concepts and some of the reasons for these difficulties. I concentrate primarily on derivative—the central concept in most introductory courses in calculus—since most of my data collection centers around this concept and students' interactions with applets and classroom environments intended to facilitate the learning of it. Then I transition to discussing big-n studies and how they are influenced by these small-n studies. I conclude with discussing several curriculum development studies and situating this study's setting within the literature discussed in this review.

Small-n studies about the derivative concept

There are many layers to the concept of derivative. Zandieh (2000) decomposes the learning of this concept into three layers: ratio, limit and function. The first layer, ratio, involves an understanding of slope and how it is computed. This is an essential connection between a secant line and the difference quotient. Without it the difference quotient cannot be seen as a ratio of y-values to x-values. This initial layer is in itself difficult for students to grasp, particularly because it involves reasoning about the covariation of two quantities. Such covariational reasoning has been shown to be a major hurdle for calculus students (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Orton, 1983). Next comes an understanding of the limiting process as the secant lines approximate the tangent line. This involves getting students familiar with limits and is often students' first exposure to the concept of infinity in a mathematical context. Infinity is itself a difficult concept to grasp and students often find inappropriate metaphors for dealing with infinite processes (Oehrtman, 2002, 2009). The final layer of Zandieh's framework involves applying the limiting process from the second layer to every point along the original function; this forms the derivative, which is itself a function. Referring to Zandieh's framework, Kung and Speer (in press) commented that, "If you stop to think how complex the ideas are in each of these stages, it's no wonder that some students give up trying to understand what's going on and focus instead on mastering procedures!" (p. 12).

To add to the complexity of the derivative concept at least in English, unlike in say Korean, the same word, derivative, is used to describe both the point interpretation (the slope of a function at a point), and the derivative function itself (Zandieh, 2002; Zandieh, & Knapp, 2005; Park, 2012). This means that the single word derivative can

simultaneously be used to refer to multiple layers of the concept or any individual layer. Zandieh and Knapp (2006) argue that this metonymy both provides a powerful reasoning tool, wherein problems situations can be viewed in multiple related ways, and a problematic stumbling block, when the related layers of the concepts are disconnected from each other in students' minds.

In each layer of Zandieh's (2000) framework the concept can be represented symbolically, graphically, verbally and kinesthetically. Each of these representations looks very different and translating between them can be particularly difficult for some students (Aspinwall, Shaw, & Presmeg, 1997; Aspinwall, & Shaw, 2002; Vinner, 1989). However, becoming familiar with non-symbolic representations is a hallmark of conceptual understanding of a concept (Aspinwall, 1994; Hughes-Hallett, 1991; Vinner, 1989; Zandieh, 2000; Zimmerman, 1991). Since symbolic representations are in general what procedures act on, the other types of representations of calculus ideas — graphic, kinesthetic and verbal — are associated with conceptual knowledge. Multiple representations of concepts and problems are a hallmark of reform curricula. Calculus reformers in the early 1990s coined the phrase “rule of three” to refer to three different mathematical representations: analytic (symbolic), graphical, and numerical (Hughes-Hallett et al., 1994).

One of the guiding principles is the ‘Rule of Three,’ which says that whenever possible, topics should be taught graphically and numerically, as well as analytically. The aim is to produce a course where the three points of view are balanced, and students see each major idea from several angles. (p. 121)

Later, verbal representation was added to what is now known as the ‘rule of four’ (Kung, & Speer, 2012). These representations loosely line up with those used by Zandieh (2000). Numerical was likely not included in Zandieh’s framework since she was analyzing students’ personal conceptions of derivative and not their instruction. The kinesthetic considerations, which are included in Zandieh’s framework but not Hughes-Hallet, stem from the work of Nemirovsky and Monk (e.g., Monk, 1992; Monk, & Nemirovsky, 1995; Nemirovsky, 1994; Schnepf, & Nemirovsky, 2001; Nemirovsky, Tierney, & Wright, 1998). These studies generally involve examining students’ use of motion tracking devices to make connections between physical motion and Cartesian graphs used to record this motion.

The ability to reason with and interpret graphs and other visual representations is considered by many to be essential for understanding the concept of derivative (e.g., Tall, 1991; Vinner, 1989; Zimmerman, 1991). Zimmerman (1991) wrote that, “visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject” (p. 136). However, Vinner (1989) observed that even students that receive instruction that targets their ability to work with graphs often try to avoid this mode of reasoning. He, as well as several other authors, noted an aversion by students to the use of graphical representations (e.g., Hughes-Hallet, 1991; Zimmerman, 1991). This observation has been attributed by several authors to years of schooling that ingrain the image of mathematics as a symbol-pushing discipline (Vinner, 1989; Zimmerman, 1991).

In spite of the power of visual reasoning, several studies have shown that students struggle with interpreting and understanding graphs and their derivatives (Carlson, 1997,

1998; Carlson, Oehrtman, & Engelke, 2010; Nemirovsky, & Rubin, 1992; Orton, 1983; Kung, & Speer, in press). Nemirovsky, & Rubin (1992) noted a general tendency for student generated derivative graphs to look suspiciously like the graph of the original function. There is also a general tendency for students to struggle with connecting the value of a slope of a graph with the height of the derivative function (Kung, & Speer, in press; Nemirovsky, & Rubin, 1992; Orton, 1983). Although some graphs of functions do look much like the graphs of their derivatives (e.g., $\sin(x)$ and e^x) it is generally false that the graph of a derivative looks like the graph of its parent function. An understanding of how a graph and its derivative are linked involves considerations of rate and slope rather than visual assessments of whether two graphs look similar. The Tangent Intuition Applet (TIA) discussed in Chapter 1 is intended to provide students with a link between a function and its derivative. The widgets used within the applet directly link the properties of a function to the value of its derivative. This link allows for a new way of thinking about and approaching the visualization of how a function and its derivative are linked.

On the other hand, some authors have argued that particularly vivid visual reasoning can occasionally impede students' abilities to solve problems (Aspinwall, Shaw, & Presmeg, 1997, Presmeg, 1992). This phenomenon occurs when a particular cognitive image is inappropriate and prevents mathematical generalization. The authors of these studies argue that those seeking to implement visualization heavy reform curricula need to be acutely aware of the limitation of such visualizations, since for particular students some imagery can "obscure more than it explains" (Aspinwall, Shaw, & Presmeg, 1997, p. 314). Aspinwall and Shaw (2002) further explored differences in students' visual versus analytic reasoning. They found that students with analytic

preferences were unable to approach graphical problems where there was no simple analytic function into which a visually represented graph could be translated. The take away message from this research is that both visual and analytic forms of reasoning are essential tools for approaching calculus problems and that one should not necessarily be privileged over the other.

Big-n studies

Kaput and Thompson (1994) in their review of technology in mathematics education research implied that large comparative studies were slowly becoming obsolete since they tended to reveal little about students' thought processes. This implication, particularly in calculus education research, has not turned out to be the case. While qualitative studies, through small scale teaching experiments and models of student thinking, help develop new curricular ideas and approaches, large scale comparative studies take over providing a statistical significance of the explored ideas. These large-n studies are important tools for associating particular curricular visions with tangible results in terms of measurable student learning outcomes.

Tall, Smith and Peiz (2008) partitioned big-n comparative studies into two subcategories—superficial and non-superficial. Superficial studies tended to control variables. For example, the variable use/non-use of a particular technology would be set as the only difference between the two courses being compared. Non-superficial studies used the affordances of technology to implement reform curricula, which often could not be implemented in the same way without the technology. In other words, there are

tangible differences in content emphasis, content ordering and pedagogy between the two courses being compared.

In general, superficial comparative studies tend to show no statistically significant differences between the two classes being compared (Connors, & Snook, 2001; Cunningham, 1992; Judson, 1990; Melin-Conejeros, 1992). For example, Judson (1990) used the CAS Maple to explore application problems in a business calculus class. Both the experimental and control groups attended the same lecture and received the same homework assignments. The experimental group differed from the control group only in receiving supplemental Maple-based instruction on some specific applications. There was no significant difference between the two groups on conceptual, application or procedural questions on the common final exam.

Fisher and Lucas (2012) is an interesting superficial comparative study that defies the trend in terms of not finding differences between control and experimental groups. Fisher and Lucas compared two business calculus classes that implemented the same curriculum and were taught by the same instructor. The experimental group used iPads (a flat tablet style computer) and the control group used laptop computers. The researchers found that the iPad group outperformed the computer group on assessments of conceptual knowledge. The researchers conjecture that this difference was caused by the tablets being more conducive to sharing during group activities. The researchers observed that screen sharing and passing the device around was common in the experimental group but almost non-existent in the control group. Other studies which compared two technologies used for the same purpose, such as Connors and Snook's (2001) comparison of two

different calculators (TI-89 and HP 48G) have not had similar statistically significant results.

Non-superficial studies tend to show statistically significant improvements between experimental and control groups (Chappel, 2006; Chappel, & Killpatrick, 2003; Ellison, 1994; Heid, 1988; Hurley, Koehn, & Ganter, 1999; Palmiter, 1991). Most of these studies fall into two types within the research literature. In the first type are CAS based studies. These studies use a CAS to facilitate some reordering of the curriculum in which concepts and applications are covered at the beginning of the semester with the CAS handling the calculations; pencil and paper calculations are pushed to later in the semester (Chappel, 2006; Chappel, & Killpatrick, 2003; Heid, 1988; Palmiter, 1991). One such study, Heid (1988), was discussed in more detail earlier in this review. The second type of study uses some form of graphic technology to facilitate an approach that focuses on graphic representations (Ellison, 1994; Tall, 1990). Both these types of implementations of technology in the classroom have shown statistically significant improvements in conceptual knowledge when compared to traditional courses.

New Curricular Approaches to Mathematics That Rely on Technology

In some cases technology can facilitate a new way of approaching or understanding a concept or mathematical topic. Technology allows the teaching and learning of that topic to be approached in a novel way. One such approach is Tall's (1991, 1997, 2012) graphic approach to calculus. Part of this approach involves dealing with the concept of continuity of a function via the concept stretching. This approach is based on

the observation that the image of a continuous graph ‘pulls flat’ when the vertical scale is fixed but the horizontal scale is stretched whilst maintaining a fixed size viewing-screen. A discontinuous function does not have this property. This stretching approach to the topic relies on digital representations of a function that can simulate this stretching. The approach gives students a more versatile metaphor to work with than the “draw without lifting your pencil” approach (Tall, 1991, 2012). The drawing metaphor is limited in its ability to address functions like $y = \sin(1/x)x$, which oscillates an infinite number of times on a finite interval and hence cannot be drawn by hand.

Another approach to the teaching of calculus, which is not based on the graphic capabilities of computers, is Thompson, Byerley and Hatfield’s (in press) conceptual approach to calculus. This approach relies on discrete approximations of derivatives and integrals, which are developed concurrently. The approach is in essence a discrete approximation of difference quotients and Riemann sums. The ability to quickly calculate these otherwise tedious sums is facilitated by computers. In this way, many of the core ideas of calculus—such as the fundamental theorem of calculus and the meaning of differentiation and integration—are developed without notions of infinity. Limits are avoided by taking small but non-infinitesimal intervals. These develop the idea of *essentially equal to* and *close enough for practical purposes*. Thompson et al. argue that formal notions of infinity belong in an analysis class and not in calculus.

Technologically facilitated novel approaches to teaching calculus often have a difficult time gaining widespread acceptance. Tall (2012) reflecting on technology in calculus reform, said, “After reform projects have attempted a range of different approaches using technology, what has occurred is largely a retention of traditional

calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation” (p. 2-3).

Situating The Context of This Study Within the Literature

This dissertation studies a Calculus I class that incorporates technology in three ways: (a) CASs are used to aid in calculation; (b) applets are used to aid students in interacting with and visualizing concepts; and (c) online lectures are used to supplement in class instruction (flipped classroom). The first use, CASs, have been well documented in the literature (Chappel, & Killpatrick, 2003; Heid, 1988; Hurley, Koehn, & Ganter, 1999; Palmiter, 1991). This is what Buchberger (1989) refers to as a black box use of technology, since the underlying calculations being used by the computer are hidden from view. In general, the use of CAS to shift a course's curriculum toward calculus concepts and away from procedures, which Tall, Smith and Peiz (2008) refer to as a non-superficial use of technology, leads to improved learning outcomes when compared to traditional classes. Here the black box uses of the technology allow students to work with the outcomes of calculations before exploring with the inner workings of the calculations. When students later learn the hand calculations for which the technology was used the black box becomes a grey box (Buchberger, 1989; Olive, & Katie, 2010). Since it is already well documented in the literature, this use of technology is not a focus of this study, but it does constitute an important aspect of the classroom being studied.

The second use, the applets, come in varied forms. Some applets present novel approaches to interacting with and thinking about particular calculus concepts. Other applets simply provide interactive visual instantiations of concepts. For example, the Tangent Intuition Applet (TIA) discussed in Chapter 1 constitutes a novel approach to

thinking about the connection between a function and its derivative. Since the applet is transparent in terms of how it relates function and derivative, this is what Buchberger (1989) refers to as a white box use of technology. Exploring the effects of this type of applet on student understanding of calculus concepts, at both individual and group levels, contributes to research literature on such novel pedagogical approaches. Since these applets are designed to address specific student difficulties documented in the literature, studying the effects of these applets adds to this literature. The TIA is designed to address difficulties students have with connecting the graph of a function and the graph of its derivative (Apsinwall, & Shaw, 2002; Nemirovsky, & Rubin, 1992; Orton, 1983; Vinner, 1989). This connection is essential to understanding the three layers of the derivative concepts within the graphic mode of representation (Zandieh, 2000; Zandieh, & Knapp, 2006).

The last use of technology, the flipped classroom, is a relatively novel approach to teaching. Current literature on flipped classrooms primarily lies outside the domain of tertiary mathematics education. Researchers who have explored this model in other content disciplines, such as economics and engineering, have shown positive results in terms of student learning outcomes (Large, & Platt, 2000a, 2000b; Toto, & Nyguen, 2009; MacIsaac, 2010). Little is currently known about instructional strategies associated with this mode of teaching. However, flipped classrooms are not a focus of my research.

Literature on Representation Use

Much of the research on mathematical representation use has been conducted from what Roth & McGinn (1998) refer to as mental perspectives. These are also

commonly referred to as cognitive perspectives and tend to focus on modeling, documenting and/or predicting individual cognitive processes. Included under this label is work in cognitive science, cognitive psychology and education research conducted from constructivist perspectives. All of these treat representation use as either indicative of an internal mental process or at least treat these representations as facilitators of such a process. These studies have a long history that dates back to at least the mid 1970s (e.g., Krutetskii, 1976). Research that moves away from these perspectives and includes social considerations within representation research has a shorter history (Cobb, 2002; Meira, 2002; Roth & McGinn, 1998; Sawyer & Berson, 2005). This social research includes work conducted from a variety of perspectives such as sociocultural theory, activity theory and the emergent perspective.

Some researchers studying symbolic and diagrammatic tools, particularly those studying the role of representations' as social mediators, have abandoned the word "representation" in favor of the word "inscriptions." (e.g., Cobb, 2002; Meira, 1995; Roth & McGinn, 1998). Part of the impetus for this move is that the word representation, in reference to these diagrammatic and symbolic tools, can be easily conflated with mental representations, which occur inside the mind and are non-observable. Other researchers working from cognitive perspectives have made a similar move for compatible reasons in referring to these representations as "external representations" or "multiple external representations" (e.g Cox, 1999; Meyer, 2000; Nistal, VanDooren, Elen, Clarebout, & Verschaffel, 2009). I do not make either of these moves, because these mathematical representations are both internal and external entities. Instead, I keep

the word representation and emphasize that the word does not reference something that exists purely in the mind.

The words “inscription” and “external” are intended to separate symbolic and diagrammatic tools that exist on paper from those that may exist inside the mind. This is at times a fuzzy boundary. For example, one can discuss the slope of a function without the presence of any mathematical representations of that function. This references attributes of Cartesian graph representations of that function without creating or referring to a physically present Cartesian graph. The utterance itself is external and can be heard by all the interlocutors involved in the conversation. However, since the representation is not physically present but images of it are being referenced, it is not wholly external. The images being referenced are internal. The reference to a representation therefore is both internal and external. So within this work I do not adopt either “external representations” or “inscriptions”.

Duval (1999) emphasized that mathematics is inextricably tied to the symbolic, diagrammatic and discursive entities we use to reason about and communicate mathematics when he wrote that, “There is no direct access to mathematical objects but only to their representations”(p.23). There are often multiple ways to represent the same mathematical object. This includes multiple representational systems that can be used to represent the same object (e.g., a graph, equation and table can represent the same function) and different expressions within the same representational system (e.g., 46656 , 216^2 , 36^3 , $5 \cdot 7 \cdot 31 \cdot 43 + 1$ all represent the same quantity). These different representations can have profound effects on what features of the object are salient (R. Zazkis & Liljedahl, 2004) how we interact with that object (Bridger & Bridger, 2001; Sinclair,

Healy & Reis Sales, 2009) and the types of solutions to problems we can generate (Borba & Confrey, 1996; Meira, 2002). Additionally, although there are a number of culturally accepted and used representational systems within mathematics, students and mathematicians can create their own novel representations while working on mathematics tasks (Meira, 2002; Stylianou & Silver, 2004). Since all mathematics is tied to representations all mathematics education research is in some way tied to representations; we cannot discuss mathematical ideas without the words, symbols and diagrams they represent. However, it is not practical to discuss an entire research field within a literature review. Instead, I focus on some major themes in the literature that inform the role representations play within mathematics learning. These are:

- 1) Classifying types of representations
- 2) Transitions between representations
- 3) Representations and concept interaction
- 4) Student invented representations
- 5) Representations as mediation tools
- 6) Representations and imaginable physical scenarios

The first three of these themes have been primarily explored from cognitive perspectives and the last three from social perspectives. After each of the above themes is discussed I summarize some of the general trends in this literature and situate this study with respect to these trends.

Classifying Types of Representations

One way to partition standardized representations that get used within mathematics is to distinguish between analytic (notation-based) and visual (diagram-based) representations. Since in much of secondary school and early college the mathematical inscriptions used are primarily either Cartesian or equation based, this distinction references two main forms of representations that get used throughout much of this mathematical content. This distinction between types of representations and specifically how it manifests itself in students' problem-solving has fueled a large amount of research in mathematics education, cognitive psychology and cognitive science. In general, this distinction has been used in research that examines regularities with respect to which of these two types of representations particular students use within their problem-solving. Such research has been subsumed under a number of headings including, learning style, cognitive style, visualization research and multiple external representation research, just to name a few. However, these headings are often not used in a consistent way so the boundaries between these approaches to studying representations are rather fuzzy (Coffield, Moseley, Hall, & Ecclestone, 2004).

Additionally, how much representation use is assumed to be internal/external to the mind varies among researchers and is not always explicitly laid out within research. On one end of the spectrum are those falling under the banner of cognitive styles. These researchers treat the use of various representations as tied to variations in how different people's minds process information. That is, it is assumed that those who think primarily by manipulating mental/verbal objects prefer to work with analytic symbol-based representations and those who think by manipulating visual images prefer

diagrammatic/graphical representations. On the other end of the spectrum are those who view representation use as the product of social enculturation. This view holds the other extreme that an individual's representation use is the product of a lifetime of social experiences with representations and their current social surroundings and not indicative of any intrinsic processing differences. I take the middle ground and assume that both differences in processing and social considerations contribute to patterns in students' representation use.

Visual vs. Analytic thinkers

Within mathematics education much of the cognitive representation research is informed by the work of Krutetskii (1976). In his study of 35 school children he hypothesized that students had distinct reasoning styles and that these styles affected both how they approached problems and which problems they were able to successfully address. Krutetskii established a distinction between analytic thinkers, who prefer to reason in verbal and logical ways, visual thinkers, who prefer the use of visual and spatial reasoning, and harmonic thinkers, who regularly employ both types of reasoning in their mathematical work. He studied students' work on a variety of tasks which were classified as visual, average or mental. The visual tasks "are solved relatively easily via visual-pictorial means if the correlation of the given elements in the problem is expressed visually" (Krutetskii, 1976, p. 156). Cognitive tasks have relatively easy solution via analytic means, via for example algebraic manipulation or reasoning about definitions. Average tasks have relatively equal opportunity for solution by both visual and analytic means. He found that students who gravitated toward a particular type of solution

strategy on average problems also performed better on problems that favored that strategy. In other words, analytic thinkers tend to perform better on cognitive tasks and visual thinkers perform better on visual tasks.

Krutetskii's research, in particular his distinction between analytic, visual and harmonic thinkers, served as a basis for a long line of research studies that is still active (e.g., Apsinwall & Shaw, 2002; Haciomeroglu, Aspinwall, & Presmeg, 2010; Presmeg, 1985, 1986). This distinction has been studied in relation to a variety of foci such as mathematical giftedness (Presmeg, 1986), differences in sex (George, 1999), teachers' explanations (Owens, 1999; Grey, 1999) and cultural differences (Presmeg & Bergsten, 1995).

Presmeg (1986) in particular studied the connection between analytic, visual and harmonic thinkers and mathematical giftedness. She found that high performing students tended to be analytic thinkers. In spite of the results of Presmeg's early research, the importance of visualization was not lost on education researchers and some curriculum developers. A large group of researchers working in the early 1990's pushed for the inclusion of more visualization in mathematics curricula (e.g., Eisenberg & Dreyfus, 1991; Healy & Hoyles, 1996; Hughes-Hallet, 1991; Zimmerman, 1991). Presmeg (2006) later wrote that much of the results of her earlier study could be explained by the fact that she was studying high school students and that much of what it means to be high performing in a high school algebra context is linked to proficiency with analytic tasks. In other words, the tasks in her study had an analytic bias.

Although not all researchers studying visual/analytic-thinking use Krutetskii's constructs, the terminology used often aligns with his framework. For example, Vinner

(1989) made a distinction between graphical thinking and analytic thinking that parallels Krutetskii's (1976) distinction between visual and analytic. Schnotz (2002) made a similar distinction between depictive (visual) and descriptive (analytic) representations. Nonvisualizers (Presmeg, 1986, 1992) is another term used which is synonymous with analytic thinkers. I will use the terms visual and analytic throughout this dissertation in order to avoid confusion when referencing authors who use terminology that is largely isomorphic.

Researchers working in the 1980's and early 1990's observed that students often resisted visual modes of reasoning (Eisenberg & Dreyfus, 1991; Healy & Hoyles, 1996; Vinner, 1989). Healy and Hoyles (1996) wrote that, "students, unlike mathematicians, rarely exploit the considerable potential of visual approaches" (p. 67). Vinner (1989) and others theorized that this resistance was due to students' previous mathematical experiences, which in middle and high school tended to focus on the symbol pushing aspects of the discipline (Leinhardt, Zaslavsky, & Stein, 1990; Romberg, Fennema, Carpenter, 1993). Students therefore viewed mathematical success as proficiency with symbol pushing. This definition of success gave students the impression that reasoning with imagery was a less accessible and desirable solution path.

The late 1980's and early 1990's were a period of change. The calculus reform movement was in full swing (Roberts, 1996) and the first of a series of National Council of Teachers of Mathematics (NCTM, 1989) standards that pushed for more visualization in high school curricula was released. Many scholars were also on board with these shifts in the importance of including more visualization in curricula (e.g., Healy & Hoyles, 1996; Vinner, 1989). Zimmerman (1991) wrote that, "visual thinking is so fundamental

to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject” (p. 136). Similar, views were expressed by mathematics education researchers working in other areas, such as proof (e.g., Healy & Hoyles, 1996). Although there has been a large push to incorporate more visualization into curricula some research has shown that the effects of visualization on student reasoning are not always positive. Aspinwall, Shaw and Presmeg (1997) examined a phenomenon they termed “uncontrollable mental imagery” in which a students’ strong preference for visual approaches hindered his/her ability to approach particular tasks that were better suited for analytic approaches. Aspinwall and Shaw (2002) documented a similar phenomenon where preference for notation-based approaches hindered student’s abilities on tasks that had relatively simple visual solutions. An important thing to draw from this literature is that reliance on one mode of representation can hinder a student’s abilities regardless of what that mode is, and that fluency and the ability to translate between representations is an important aspect of calculus knowledge.

The calculus reform movement, and changes in high school and middle school curricula caused a shift in the status of visual approaches. Before these curricular changes took place students would encounter Cartesian graphs when first being introduced to algebra and plotting lines and then would not work with graphs again until calculus (Leinhardt, Zaslavsky, & Stein, 1990). When graphs were included in instruction they were often used as a means to an analytic end. The results of mathematical tasks took the form of numbers and equations. So for example, the generation of a particular graph was not a common solution to a task, but an equation was.

In 1996 the Texas instruments TI-83 calculator came to market and was almost immediately put into use in high school algebra curricula. In the United States these curricula, which had previously been largely devoid of visualization, now had units on reasoning with and about graphs of functions. The incorporation of the TI-83 calculator into these curricula meant that visual representations of function had become a non-trivial component of algebra education. This non-trivial use of visual representations meant that the primary reason theorized for students' resistance to visualization—the removal of visualization in pre-tertiary mathematics education—was now a less significant factor. Many curricula still had an analytic focus, however, some tasks within this curricula now required reasoning about graphs and generating graphs as solutions. So it was no longer possible to complete all tasks without using graphs.

Stylianou (2001) argued that the observations of students' resistance to visualization made in the early 1990's (e.g., Dreyfus, 1991; Vinner, 1989) had changed in the subsequent decade. Studies of tertiary mathematics students had begun to emerge that pointed to students using both standardized visualizations, such as Cartesian graphs, and their own invented diagrammatic methods at rates comparable to research mathematicians (George, 1999; Stylianou, 2001; Stylianou & Silver, 2004).

There is growing evidence that today's students use visualization as part of their reasoning processes (George, 1999; Stylianou, 2001; Stylianou & Silver, 2004) and that it serves as a powerful problem-solving tool for them (Hähkiöniemi, 2006; Presmeg & Balderas-Cañas, 2001). However, these results are relatively new. Consequently, how and when students use this mode of reasoning as well as how modes of reasoning interact with each other is underexplored. In the 1990's conducting such research was moot. Since

it was well documented that very little usage of visualization was occurring, there was little to be gained by exploring a type of reasoning that rarely occurred. However, now that there is evidence that this resistance to visualization is no longer the case expanding what is known about visual modes of reasoning, which are now an important part of students' reasoning tool box, is an important step toward better documenting and developing modern students' mathematical thinking. This is part of the impetus for some of the work discussed later in this chapter.

Task and representation alignment

As noted earlier, some of Krutetskii's (1976) work involved classifying tasks in terms of which representations were best suited to solving them and similar assessments of tasks were used by authors when studying the prevalence of visual/analytic thinkers in various populations (e.g., Presmeg, 1986, 1992; Knuth, 2000). This classification has also been a theme within the cognitive science and cognitive psychology literature on representation use (e.g., Cox, 1999; Meyer, 2000). These studies often involve the creation of models that attempt to account for why particular representations are chosen by particular students for particular tasks (e.g., Gilmore & Green 1984; Meyer, 2000; Sparrow, 1989; Vessey, 1991; Wickens and Andre, 1990). Common to all of these models are (a) the properties of the representation; (b) the match between the task and affordances of the representation and (c) the effects of within-subject factors such as prior knowledge/experience and cognitive style. The results of this research commonly point to students choosing non-optimal representations to approach particular tasks. However,

these results are based on the assumption that researchers themselves are able to determine which representations are optimal for which tasks.

Star and Newton's (2009) work highlights that even experts often make use of representations that researchers consider non-optimal. Additional comparisons of experts and novices have revealed that under some circumstances the representations generated and used by students can be more sophisticated than those generated by working scientists (Roth & McGinn, 1998). So what researchers consider optimal approaches are not always the approaches that experts use and in some cases novices may exhibit more sophisticated representation use than experts. These studies call into question some of the reasons for studying use/non-use of optimal representations in novices.

A recent development in this research is the use of a choice/no-choice methodology in which students are prescribed which representation to use on some tasks and allowed to make their own choices on others (Acevedo Nistal, Van Dooren, Clarebout, Elen, Verschaffel, 2010). This methodology is borrowed from research in solution strategies (Siegler & Lemaire, 1997) and allows documenting how representations used interact with both the task presentation and how students perform using particular representations. This research methodology has potential to gain insights as to task and individual factors that contribute to representation use, but may be ill-suited for studying tasks in which multiple representations are commonly used while solving a single task, such as those commonly found in calculus.

Transitions between representations

In algebra and pre-calculus the function concept is often explored using three representations of function—tables, Cartesian graphs and equations. Researchers have argued that the fluent use of these three representations, and particularly translation between them, is essential for a deep understanding of algebra and provides a solid foundation for those entering calculus (Confrey & Smith, 1994; Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990). Research on the connection between translation between Cartesian and equation notations has found that even when students are competent at translating between visual and analytic notation when this translation is the task itself, they often resist making such connections when the task does not specify which representations to use (e.g., Eisenberg & Dreyfus, 1991; Knuth, 2000; Vinner, 1989).

Starting in the mid 1990's several authors began exploring mathematical advancement as a process facilitated by transitions between mathematical representations (Duval, 1999; Rasmussen, Nemirovsky, Olszewski, Dost, & Johnson, 2004; Stylianou, 2002; D. Zazkis, in press; R. Zazkis, Dubinsky, & Dautermann, 1996). In particular, R. Zazkis, et al. (1996) developed a model that focuses on the interaction between visual and analytic modes of reasoning known as the Visualization/Analysis model (VA-Model). This model, which was originally developed in the context of students' thinking about group theory, partitions student thinking into two complementary processes—visualization and analysis. These processes line up with Krutetskii's (1976) distinctions between visual and analytic thinkers. The VA-model posits that the development of analytic and visual modes of reasoning complement one another. The visualization and

analysis processes build on one another as students' ability to reason about mathematics develops. This perspective implicitly treats all students as harmonic reasoners.

Although the use of this model is not as widespread in the education literature on representation as models consistent with Krutetskii's (1976) framework, a growing number of scholars are shifting toward using it (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010; Stylianou, 2002; D. Zazkis, in press). Of note is that Presmeg, who has a long history of utilizing models that align with Krutetskii's (1976) framework (e.g., Presmeg, 1985, 1986, 1991; Aspinwall, Shaw & Presmeg, 1997), is shifting to subscribing to the VA-model (e.g., Presmeg, 2001; Haciomeroglu, Aspinwall, & Presmeg, 2010). As studies emerge that point to an increase in harmonic thinkers (Goerge, 1999; Stylianou, 2001; Stylianou & Silver, 2004) the explanatory power provided by Krutetskii's (1976) framework is diminished. In other words, if the majority of students are harmonic thinkers there is little to be gained by the distinction between visual and analytic thinkers.

Representations and Concept Interaction

The specific inscriptions used to symbolize a mathematical object can affect the features of that object that are salient, how one can interact with that object and what connections are made with other objects. Any particular representation of a mathematical object brings to the forefront some properties while hiding others. Although it has long been acknowledged that visual and analytic representations have distinct advantages and disadvantages when compared to one another (e.g., Cox, 1999; Dreyfus, 1991; Krutetskii, 1976) some of the most salient demonstrations of the power of representations

to highlight some properties while obfuscating others come from work on other distinctions between representations. These, rather than the analytic/visual distinction which has grounded much of this review thus far, are discussed here.

Representations of number provide a backdrop that makes the power of representations to highlight/obfuscate properties particularly salient. For example, 216^2 and $5 \cdot 7 \cdot 31 \cdot 43 + 1$ are two representations of the same mathematical object. However, when represented as 216^2 the property of being a perfect square comes to the forefront and other properties such as its remainder when divided by 43 are obfuscated. Similarly when represented as $5 \cdot 7 \cdot 31 \cdot 43 + 1$ its properties with respect to division by 43 are more salient but other properties are obfuscated. R. Zazkis and Liljedahl (2004) introduced the construct of transparent and opaque representations of a property to describe this notion. They argued that part of students' difficulty with understanding the prime number concept stems from a lack of transparent representation of primes. The transparency construct can also be applied in a function context, where Cartesian (visual) representations are quite effective for noticing general trends and poor for constructing exact calculations. For example, when $f(x) = x \cdot \sin(1/x)$ is represented as a Cartesian graph the fact that it oscillates from positive to negative an infinite number of times as it approaches zero is transparent. This property is not immediately evident when the function is represented as " $f(x) = x \cdot \sin(1/x)$."

Another powerful illustration of the power of representations to highlight some properties and obfuscate others comes from the work of Sinclair, Healy and Reis Sales (2009). They studied students' interactions with function using two types of representations—Cartesian graphs and dynagraphs. Dynagraphs are dynamic computer

facilitated representations of graphs in which both the x and the y-axes are horizontally configured. Students interact with these by dragging the input along the x-axis, which causes an output to move along the y-axis. Sinclair and her colleagues found that students interacted with dynagraphs in very different ways than with Cartesian graphs.

Specifically, when an experimental and control group were presented the same set of functions, the experimental group, which interacted only with dynagraphs, grouped functions into very different categories in terms of similarity than the control group, which interacted with these functions using their Cartesian representations. This result helps highlight that the properties of these functions that were most salient differed as a result of the representation used. Bridger and Bridger's (2001) work with mapping diagrams garnered similar results. They argued that mapping diagrams, which represent function by associating points along two vertically oriented axes, are better suited for representing the function concept as an association between two sets.

Student Invented Representations

Researchers have studied students' self-evaluations of representation use and shown that such evaluation has a positive effect on representational competence (e.g., Hammer, Sherin, & Kolpakowski, 1991; diSessa, & Sherin, 2000; Larson, 2009; Uesaka & Manalo, 2006). Although some of this work focuses on institutionalized representations, such as Cartesian graphs and function notation, not all student representation use is constrained to these (Uesaka & Manalo, 2006). The work of diSessa and his colleagues studies classrooms where students are asked to create their own representations of motion (Hammer, Sherin, & Kolpakowski, 1991; diSessa, & Sherin,

2000). This work is predicated on the view that the representations used in mathematics are socially constructed and the construction of representations is itself a mathematical activity. In this work as students are asked to critique and suggest changes to their invented representations they gain what is referred to in this work as meta-representational competence. This is competence in answering what particular advantages particular representations have over others for answering particular questions and what makes one representation more desirable than another. Unlike much of the cognitive science and cognitive psychology literature, which takes expert evaluations of what an appropriate representation is and then examines whether students make that same evaluation, diSessa and his colleagues explore how and why students determine that one representation is more desirable than another. In general, this and similar work have shown that discussions of the affordances and utility of representations are linked with increased representational fluency and better choices on the part of students with respect to which representation gets used for what problem (Hammer, Sherin, & Kolpakowski, 1991; diSessa, & Sherin, 2000; Uesaka & Manalo, 2006). Additionally, this research has shown that student critical capabilities in evaluating representations are both rich and generative. Even young children are able to coordinate multiple simultaneous criteria when making evaluations as well as generate new criteria as a discussion progresses. Furthermore, students' criteria for evaluating representations are often linked to the design of these representations. That is, rather than treating a particular representation as given, their critique often includes suggested changes that can account for particular shortcomings of representations.

Representations as Mediation Tools

Although standardized mathematics representations have institutionalized meanings and conventions of use, students who are in the midst of an acculturation to these meanings and conventions do not necessarily hold these same meanings (Cobb, 2002; Meira, 1995, 1998). Additionally, since interaction with these representations is how these students undergo this acculturation, their meanings for and uses of representations are non-static (Cobb, 2002; Sawyer & Berson, 2005; Rasmussen & Marrongelle, 2006). Furthermore, as Roth and McGinn (1998) highlight, the use and meaning of particular representations may vary across different communities of experts. So the institutionalized meanings and uses for representations that mathematicians hold do not necessarily align with experts in other fields.

Earlier I discussed research that highlights that the use of particular representations affects which attributes of mathematical objects are salient (R. Zazkis & Liljedahl, 2004; Sinclair, Healy & Reis Sales, 2009). So the specific representations that are used by a student have a profound affect on the very ways in which mathematical activities evolve (Meira, 1995; Pea, 1987). The work of Meira (1995, 1998, 2002) with various physical devices, which were representations of linear equations, illustrates this affect well. He was able to show that the representation of linear function students were assigned to work with had profound affects on how they interacted with mathematical ideas and on the nature of their solutions. Also, features that experts might consider to be superficial attributes of these representations often had large effects on student interaction. So students may treat representation attributes that are superficial to experts as important. Additionally, the uses and meaning of representations change as student interact with

them (Cobb, 2002; Meira, 2002; Roth & McGinn, 1998). So student meanings for representations evolve. Consequently, students may attend to very different attributes of these representations at different points in time.

Sawyer and Berson (2005) observed that representations play an important role in facilitating the communication of groups of students. They observed that students often began conversations with talk that focused on specific inscriptions on notebooks, what they called mediated talk. After the meanings and referents of these representations were clarified students commonly transitioned to conversational talk, in which student eye gaze was directed at each other and references to the representations tended to be verbal. One possible interpretation of these results is that representations shift from primarily external to primarily internal as mathematical activity advances. As mentioned earlier, this dual internal/external use provides some of the impetus for my not adopting the term “inscriptions” within this work.

The work of Rasmussen and Marrongelle (2006) in differential equations categorized a different kind of shift in the role representations play. They coined the term *transformational record* to describe a particular shift in how a representation is used. Here students’ reasoning on novel tasks appropriate a representation that is initially introduced by an instructor to document student reasoning. The representation shifts from a record of reasoning to a tool for reasoning. Larson (2009) observed similar shifts in how representations are used when studying representations that students themselves create. He observed that symbols initially created by students to document the rotations of a triangle were later used in combination to create an algebraic system.

Representations and Imaginable Physical Scenarios

During the early history of mathematics, mathematics and physics were part of the same discipline, natural philosophy. Those who developed what would today be referred to as mathematics and those developing what today would be referred to as physics were often the same people. Isaac Newton developed much of what today is known as calculus in the context of studying the movement of celestial objects. In the 17th century, the study of natural philosophy was partitioned into subfields. Mathematics became a tool for those studying physics to use. Consequently, mathematics was used to inform physics, but the converse was less the case. Mathematics is now a separate discipline. At least for mathematicians, the methods and results of mathematics are no longer intimately linked with the physical contexts that originally precipitated much of their creation. These can now be regularly explored outside of physical contexts. The same is not necessarily true for professional users of mathematics.

Roth and Bowen's (2001, Roth, 2009) studies of working scientists reveal that although scientists are very adept at reasoning about Cartesian graphs in their own work, they have difficulty parsing the meaning of graphs related to their field but outside their specific research foci. When they do discuss graphs from their own data they situate this reasoning with respect to the context of the graph and often begin by discussing the situation the graph describes rather than its mathematical properties. For them, a graph is a representation of a situation and carries little meaning apart from this situation. Unfamiliar situations, or situation-free graphs, thus provide difficulty. So for these users of mathematics physical situations are so deeply interwoven with graphical representations that they are incapable of considering the graphical representations

separately from physical situations. Williams and Wake's (2007) study has similar results. In their studies of mathematics usage in various workplaces, they found that mathematical tools and representations often function as "black boxes" that disguise the underlying mathematics. This research points to the dangers of treating graphs (as well as other representations) as if they carry intrinsic meaning and value apart from their use; even experts who use these representations have little meaning for them in unfamiliar or context free situations. These results, however, are based on work outside the mathematics classroom; so it may be advantageous to not use them as prescriptions for situating all mathematical activity within familiar world-based contexts.

Such imaginable physical scenarios do play an important role in mathematics education. Examples of imaginable physical scenarios can be found throughout the curriculum, from situating addition and subtraction problems in terms of acquiring and losing apples to using the movements of a car to make sense of calculus. Imaginable physical scenarios can also be found in studies of mathematics students. These are often used as contexts/motivation for introducing/developing mathematical ideas (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Gravemeijer, & Doorman, 1999; Nemirovsky, Tierney, & Wright, 1998; Rasmussen & King, 2000; Wawro, Rasmussen, Zandieh, & Larson, 2013) or backdrops to word problems after mathematical ideas have already been introduced (e.g., Carlson, et al. 1998; Moore, & Carlson, 2012; D. Zazkis, 2012).

A number of studies have introduced students to concepts using a physical scenario but not examined physical thinking after its initial introduction (Meira, 1995, 1998, 2002; Rasmussen et al., 2004; Rasmussen & Nemirovsky, 2005). Rasmussen et al. (2004) in particular, were able to show a rich interplay between physical, visual and

analytic reasoning when students worked with a physical device known as a water wheel. However, by not following the students in the study beyond their initial exposure with the device the study was unable to document whether the observed interplay between physical, analytic and visual reasoning continued beyond this initial introduction.

Such long-term explorations of physical reasoning are, however, part of the literature. It has been observed that when students are introduced to an idea within an imaginable scenario, they often reason in reference to that scenario when encountering mathematics tasks that are disjoint from it (e.g., Larsen, Johnson, & Bartlo, 2013; Speiser & Walker, 1994; Wawro et al., 2013; D. Zazkis, in press). Wawro and her colleagues noted that when linear algebra students were introduced to the concept of span using an imagined scenario known as the “magic carpet ride sequence” (Wawro, Rasmussen, Zandieh, Sweeney, & Larson, 2012) students often reasoned with this scenario when working on context free problems about span (Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011; Wawro, Rasmussen, Zandieh, & Larson, 2013).

The observations of students using imaginable scenarios to inform their mathematical activity when working on context free mathematics tasks often occur in Realistic Mathematics Education (RME) inspired research (e.g., Larsen, Johnson, & Bartlo, 2013; Wawro, Rasmussen, Zandieh, & Larson, 2013). The RME instructional design heuristic holds that imaginable or actual scenarios should serve as starting points for mathematics instruction (Freudenthal, 1973; Gravemeijer, 1999; Rasmussen, & King, 2000). This design heuristic has been gaining increasing traction within the research in undergraduate mathematics education community. RME inspired research studies classrooms that introduce mathematical concepts through these imaginable scenarios.

Students' mathematization of these scenarios allows them to recreate mathematical ideas for themselves with the guidance of an instructor. Students may eventually move to what Gravemeijer (1999) refers to as formal mathematical activity, where these scenarios are no longer explicitly referenced. However, since mathematics is seen as a "human activity" rooted in interactions with the world mathematical objects are not seen as separable from the scenarios that precipitated their creation. Pirie and Kieren (1994) observed that students who commonly participate in formal mathematical activity commonly reintroduce physical scenarios into formal mathematics problems when encountering difficulty. So the physical scenarios used to introduce mathematical content are never completely replaced by formal mathematical notions.

In other work in which physical reasoning plays an important role, Hanna and her colleagues (Hanna, 1989; Hanna & Jahnke, 1999, 2002) introduced physical thinking into proof context. Unlike the RME work which used physical scenarios as a starting point, this work introduces physical scenarios into already formalized settings. Additionally, Zandieh's (2000; Zandieh & Knapp, 2005) derivative framework parses the components of what the mathematical community treats as part of an understanding of derivative. It includes the physical position/velocity/acceleration context as part of what it means to understand the concept. It hence treats reasoning with this imaginable scenario as part of the mathematics itself. In the same way that functions can be expressed analytically (as equations) and visually (as graphs), they can also be expressed in terms of physical scenarios such as position/ velocity/ acceleration contexts. A physical representation of function can inform thinking about analytic and visual representations

of that function and does not need to be included in the problem statement to be used by a student.

Although, physical reasoning has been an important theme in a host of education research discussed here (e.g., Hanna & Jahnke, 2002; Meira, 1995; Rasmussen et al., 2004; Roth & Bowen, 2002) it has largely been avoided in the cognitive research on representation (Presmeg, 2006; Hanna & Sidoli, 2007). The cognitive research often subsumes physical reasoning under the analytic and visual reasoning categories. This study, which coordinates the analysis of individuals and groups, and builds off of these cognitive models, does not subsume physical reasoning under these categories.

Summary of Representation Research

Much of the research on student representation use can be partitioned into two categories—one cognitive and one social. Research within each of these categories tends to ignore the methods and results of the other. Cognitive approaches have a long history and far outnumber studies which take into account the social roles of representations (Meirna, 2002; Roth & McGinn, 1998; Cobb, 2002). The cognitive approaches, in general, treat students' observable representation use as indicative of some kind of internal cognitive process. This research treats visual and analytic representations as indicative of visual or analytic internal cognitive processes. One common approach to such research is to classify students in terms of which of these cognitive processes are more dominant. These classifications of students as visual/analytic/harmonic thinkers have a long history which goes back at least as far as Krutetskii's (1976) work and continues to be used today (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010). This type

of research has helped document that there has been a large shift in the prevalence of the use of visual representations within students' thinking. This branch of representation research also includes work that models student thinking about representations as developing via a dialectic relationship between visual and analytic thinking (Duval, 1999; Stylianou, 2002; D. Zazkis, in press; R. Zazkis, Dubinsky, & Dautermann, 1996). This modeling approach has seen an increase in use since its first emergence in the mid 1990's. Researchers using this approach have been able to document how analytic and visual approaches inform each other.

Social approaches to studying representation use are relatively new. These approaches treat representations as having situated use-oriented meanings that evolve through social activity. In its short history, this research has been able to establish a series of important results regarding representation use. These results include that the meaning of representations shifts over time, that children are able to generate, compare, refine and choose amongst competing representations and that novices may under certain circumstances display more sophisticated representation uses than experts (Cobb, 2002; diSessa, & Sherin, 2000; Meira, 1995, 1998, 2002; Roth & Bowen, 2001; Roth 2009; Roth & McGinn, 1998). Although some of these studies have worked to coordinate individual and social units of analysis when studying representation (e.g., Cobb, 2002; Roth & Bowen, 2001), these analyses have largely ignored the research on representations conducted from cognitive perspectives when doing so.

Situating This Study Within the Literature

Researchers examining representation use through social lenses (e.g., activity theory and sociocultural theory) have proposed shifts in how researchers approach the study of representations in mathematics education (e.g., Meira, 2002; Roth & McGinn, 1998). Much of this proposed shift involves abandoning the methods and models of cognitive studies. This study instead takes social considerations into account in a different way, which does not completely abandon the models and methods developed within cognitive perspectives. This is achieved through a modification of frameworks and models for studying individual advancement for use in a group setting. Such modification allows the frameworks to be applied to both group-work and individual interviews and allows exploration and coordination of both individual and social roles during mathematical activity. This coordination allows social and cognitive analyses, which have remained largely separate from each other, to be bridged in a way that does not abandon either set of results and methods.

Another important difference between this and some past research involves the treatment of imaginable physical scenarios. These are treated as staples of mathematical reasoning rather than scaffolds for formal mathematical activity. This treatment of physical scenarios has roots in the instructional design theory of RME, Zandieh's (2000) derivative framework and Hanna and her colleague's work on physical proof (Hanna, 1989; Hanna & Jahnke, 1999, 2002

CHAPTER 3:

Methodology

Sfard's (2006, 2008) commognitive perspective is the theoretical perspective that grounds this study. After briefly revisiting the motivations for this study I summarize this perspective and the reasons for choosing it in relation to my research questions.

Following this summary, I shift to the data sources, the participants and how they were selected. Detailing the data sources provides the background necessary for discussing the analytic tools used to examine the data. One particular analytic tool, the VAP-model, was created during the course of this study. I discuss the process that facilitated its creation and how it related to the visualization/analysis model (VA-model) that it evolved from (R. Zazkis, Dubinsky, & Dautermann, 1996). Krummheuer's (2007, 2011) participation framework, which facilitated the analysis of participants' social roles within the data, is also discussed. Details regarding how the analytic tools used in this study are operationalized in subsequent chapters and how this relates to the theoretical perspective are included within discussions of each of the analytic tools.

Revisiting the Motivation for the Study

A number of studies of tertiary students conducted two decades ago found that students often resisted using non-analytic methods of approaching mathematics problems (Dreyfus, 1991; Presmeg & Bergsten, 1995; Healy & Hoyles, 1999; Vinner, 1989). The authors of these studies commonly attributed this resistance to high school curricula that

almost exclusively treated mathematics as a symbol-pushing discipline. Since many students at that time, particularly those who were high performing, resisted visualizing (Presmeg, 1985, 2006), researchers who studied representations classified students in terms of which solution methods they commonly exhibited, visual methods (visual thinkers), analytic methods (analytic thinkers) or in rare cases both (harmonic thinkers). The students who used visualization in their problem-solving (both harmonic thinkers and visual thinkers) were rare enough that it made sense to categorize them as different from analytic thinkers.

More recent studies have found that the tendency to gravitate to strictly analytic reasoning is no longer prevalent (George; 1999; Stylianou, 2001; D. Zazkis, in press). It has been argued that a shift in American high school curricula away from a symbol pushing focus is at the heart of this phenomenon. That is not to say that curricula do not still place a sizable emphasis on symbol pushing, only that the inclusion of units on graphing and visual interpretations of functions is now commonplace in algebra curricula.

The observation that most students can now be classified as harmonic thinkers is based on the analytic/visual/harmonic distinction. However, in situations where most students are classified in the same way a classification system loses much of its explanatory power. Studying representation use in settings where all students are harmonic thinkers requires new analytic tools that account for how different modes of thinking interact and inform each other, not just that multiple modes are used. This allows exploration of how harmonic thinking manifests itself differently within various students' problem-solving process.

Additionally, I argue that a shift to theoretical perspectives that take into account both social interactions and individual use is an important direction in which research on representation use can be expanded. Most of the studies of representation phenomena to date, including my own, have been focused on individuals in isolated interview or exam settings (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010, Krutetskii, 1976; Presmeg, 1986; D. Zazkis, in press; R. Zazkis, Dubinsky, & Dautermann, 1996). The research that has studied representation use within social perspectives has been largely disjoint from the work conducted through cognitive perspectives (e.g., Cobb, 2002; Meira, 1995; Roth & McGinn, 1998). Of note is that even the work of Cobb (2002) on representations focuses only on classroom use of representations and not individual cognition. This is particularly illuminating with regard to the tendency in representation research to separate social and cognitive analyses since Cobb is well known for work which coordinates both social and cognitive approaches (e.g., Cobb & Yackel, 1996).

Theoretical Perspective

It is useful to begin a discussion around theoretical perspective by revisiting the research questions as they appeared in the first chapter. Since the theoretical perspective needs to frame all three research questions reexamining them will help motivate my reasons for choosing Sfard's (2006, 2008) commognitive perspective as the lens through which I analyze the data.

- 1) How do individual students think about key calculus concepts during their problem-solving? In particular, how do they use representations to facilitate this thinking?
- 2) How do groups of students communicate about key calculus concepts? In particular, how do they use representations to facilitate this communication?
- 3) Within a calculus setting, in what ways do students' thinking in individual interview settings and their communication in group settings inform each other.

On the surface the first two research questions in this study, as they appear above, can be interpreted as being grounded in different theoretical perspectives. The first question is focused on thinking. Thinking is in general treated as a cognitive phenomenon that occurs inside the head. Sfard (1998) describes such perspectives as *acquisitionist*. That is, they are grounded in a view of learning in terms of the acquisition metaphor, which treats knowledge as something that is accrued through interactions with the world. Within education research such perspectives are often attributed as having roots in the work of Jean Piaget and his constructivist perspective. Sfard (2008) argues that acquisitionist perspectives have much older roots in language used to talk about learning. Phrases such as “Building up his knowledge base” and “Cramming for tomorrow’s exam,” are grounded in the acquisition metaphor. They refer to knowledge as something that can be stuffed into or organized inside the mind. The use of these types of phrases has a long history that predates the work of Piaget.

The second research question, as it appears above, is focused on communication. Communication is a social phenomenon, in the sense that it happens between people. Sfard (1998) describes perspectives that focus on the social organization of learning as *participationist*. She argues that such perspectives are rooted in the learning as participation metaphor. In general, education research that is grounded in this metaphor has roots in Marxist theories and the work of Vygotsky and treats learning as akin to an enculturation process in which socially organized ideas and norms of behavior get passed down from one generation to the next.

So the first research question is acquisitionist, the second is participationist and the third question seeks to coordinate the two perspectives. In order for this dissertation to be a cohesive body of work it would be useful to frame it with a theoretical perspective that is able to frame all three questions. However, the dichotomy between participationist and acquisitionist perspectives is more than a shift in unit of analysis. These perspectives view learning as occurring through fundamentally different processes.

Whereas acquisitionists view the individual development as proceeding from personal acquisitions to the participation in collective activities, strong participationists reverse the picture and claim that people go from the participation in collectively implemented activities to similar forms of doing, but which they are now able to perform single-handedly. (Sfard, 2006, p. 157)

Both perspectives by focusing on one type of phenomena leave out or have little in terms of explanatory power for addressing other phenomena. For example, acquisitionist perspectives tend to treat the strong influences of social setting on human behavior as ancillary and hence have trouble explaining setting dependent differences in

problem-solving behavior. Participationists, on the other hand, tend to black-box individual cognition which occurs and deals only with observable human behavior. Thus, a participationist perspective, which views learning as inherently tied to social setting, has little in the way of informing human behavior in novel social settings.

Several authors have argued for approaches that would in essence narrow the gap between participationist and acquisitionist perspectives (Lave, 1993; Cobb & Yackel, 1996; Forman, 2003; Sfard, 1998; Yackel & Rasmussen, 2002). Sfard (1998) in particular, discussed the dangers of choosing one perspective in lieu of the other. There have been several approaches to bridging this gap. One approach is to expand the types of phenomena considered but not the lens itself. With this approach the basic underlying metaphor and the terminologies associated with it remain relatively unchanged, but more phenomena that were previously ancillary are treated as relevant. Lave (1993) speaks about cognition plus when referring to perspectives that append social considerations to acquisitionist discourse. This approach, by keeping the underlying metaphors and terminologies, does not necessarily bridge the gap between the two perspectives. However, it does increase the overlap of phenomena considered within them. Not re-thinking the terminologies used within each perspective means that researchers with acquisitionist roots still metaphorically speak a different language to researchers with participationist roots.

Another possible approach to bridging participationist and acquisitionist discourse is to use both metaphors simultaneously and then make connections between the two perspectives through a mapping that associates compatible constructs. Cobb and Yackel's (1996) emergent perspective takes this approach. It associates participationist constructs,

such as sociomathematical norms, with their acquisitionist counterparts, such as mathematical beliefs and values. This approach allows both acquisitionist and participationist questions to be addressed. “Each [perspective] provides a backdrop against which to consider the other.” (Yackel & Rasmussen, 2002). However, when coordinating both acquisitionist and participationist perspectives both sets of constructs/terminology are kept intact. These constructs still keep their metaphorical entailments. So a construct like sociomathematical norms is still participationist and a construct like mathematical beliefs is still acquisitionist. The coordination preserves both metaphors but assures that they are both considered and coordinated. This coordination means that neither participation in collective activities nor individual development are taken as the root causes of learning. Instead, the collective and the individual are seen as evolving in tandem through a reciprocal relationship.

Sfards (2006, 2008) commognitive perspective takes a different approach in bridging the gap between acquisitionist and participationist perspectives. In order to facilitate this bridging she regards thinking and communicating as different forms of the same action. Thinking is regarded as an internal communication where the thinker is playing the role of all the interlocutors. This process is not regarded as wholly internal. Notation, gestures and even outward verbal utterances may be part of this process. However, these are not intended for an audience. Communicating, which includes speech, gestures, body positioning, eye contact and the like, is regarded as an outwardly expressed thinking process that occurs between individuals. In order to emphasize this view on the likeness of thinking and communicating, Sfard created a single term, *commognition*, a hybrid of communication and cognition, to refer to both compatible

processes simultaneously. Commognition, thinking, reasoning and communicating are thus treated as synonyms under her commognitive lens. The terms refer to essentially the same processes, but differ only in the number of observable interlocutors involved. The creation of a new term, commognition, allows researchers the ability to discuss their data while simultaneously using both metaphors. When a researcher uses a phrase like, “the interviewee’s commognition” she does not specify whether the communication being discussed is internal or external and thus the phrase is not constrained by the entailments of one of the metaphors. The phrase allows there to be no distinction made between the interviewee’s self-communication (thinking) and the communicative actions intended for the interviewer.

Revisiting the research questions through a commognitive lens we see that the first and second research questions, which initially appeared to be situated in terms of different lenses, now reference compatible phenomena and differ only in the setting (e.g., number of observable interlocutors, available technology, positioning of the researcher relative to the observed, etc.) in which this phenomena occurs. Similarly the third research question now compares compatible phenomena across two different settings.

Commognition in the two settings cannot necessarily be examined with identical analytic tools. The group setting means that commognition can vary in terms of which interlocutors are involved at any given moment and the roles they play within the commognition also varies as a function of time. Specific analytic tools intended to classify these roles are thus important to addressing the second research question. Krummheuer’s (2007, 2011) participation framework is used to facilitate such a classification and is discussed later in this chapter. The types of variations in interlocutor

roles are not as complex in a one-on-one interview setting. Thus the analytic apparatus needed to make sense of variations in interlocutor roles in a one-on-one setting do not need to be as elaborate as those needed in a group setting. Krummheuer's framework will still be used in the analysis of one-on-one interviews, but the role it plays in the analysis of these data is less central than its role in analyzing the group data.

It is worth mentioning that, unlike in interviews conducted within acquisitionist perspective that often minimize the role of the interviewer, the commognitive perspective treats the interviewers commognitive role as an important influence on an interviewee's commognition. This example helps illustrate that the perspective does not simply redefine terms as equivalent and then conduct analyses in the same ways they were conducted under acquisitionist or participationist lenses. Instead, the ways in which both participationist and acquisitionist questions are examined is shifted to maintain consistency across settings and foci. Sfard's approach morphs both perspectives into a single entity that can accommodate multiple units of analysis. How this perspective informs the analysis of data will be expanded upon further after the participants, setting and data sources are discussed.

Participants and Setting

Setting Design

A technologically enriched Calculus I course taught at a large university in the southwestern United States served as the setting for this study. The course instructor, Dr.

Byne¹, is a renowned education researcher who specializes in how students' development of mathematical understandings can be supported through advanced technologies. She is also a talented instructor and recently received a departmental teaching award. In order to provide background to this study it is important to acknowledge the countless hours that she invested in the design and development of this course. I appreciate the opportunity she gave me to take a small part in this development process.

During the development of her course Dr. Byne regularly and systematically observed three different calculus classes taught by three different instructors. Two of these courses were taught using a standard lecture style. The third was taught using a flipped classroom model in which students were assigned online video lectures as homework (primarily through www.khanacademy.org/) and spent the entirety of the class period working on associated instructor-designed problems in small groups with the support of teaching assistants (TAs). I also regularly observed this flipped classroom.

The two standard courses were used to gain insight regarding the varied ways particular calculus topics could be approached through lecture. Observations of these courses were used as inspiration for applet design. Dr. Byne has designed and refined over a dozen applets for use in the calculus course. Some of these applets are intended to add to her lectures by providing animated visualizations of calculus concepts. Other applets, such as the Tangent Intuition Applet (TIA) discussed in Chapter 1, are intended to serve as tools students can use to make connections between concepts and reason through problems. Many of these applets have gone through multiple iterations with the

¹ This is a pseudonym

² During later versions of the course, after the data collection for this study was

aid of comments from myself and other colleagues from the mathematics department. Informal trials with calculus students have provided an additional avenue for refinement.

Dr. Byne and I have also administered a series of four in-class surveys to students in the flipped classroom. These surveys included questions about study habits, how the course compares to other courses students have taken in college, and open-ended questions regarding how students feel about the course structure. The surveys allowed us to track changes in students' study behavior and opinions (Bowers & D. Zazkis, 2012). These data have helped the instructor make informed decisions regarding how to structure and run her own classroom.

The flipped classroom that Dr. Byne and I regularly observed also provided useful information about students' reasoning with calculus ideas and how students interact within a flipped classroom. In the observed flipped classroom, Dr. Byne regularly followed and interacted with a group of students that worked together throughout the semester. Tracking their struggles with problems assigned during class provided her valuable insights into the nature of these students' struggles with calculus. I also regularly interacted with and observed a different small group in the same class. The discussions we had about the differences in-group dynamics, study habits and approaches our respective groups took provided valuable insights for both of us. Chief among these is that students in the flipped classroom, because all lectures are online short videos on specific topics, rarely make connections between various topics in calculus. Such connections are not part of the content of the videos and the students we have observed have not shown evidence of making these connections for themselves. Consequently, these students view calculus as a series of loosely related techniques for manipulating

equations into other equations rather than an interconnected web of ideas and techniques. For example, the class did not discuss why the slope of a position graph provided information about velocity or how the shape of a graph related to the shape of its derivative. One particularly telling example surfaced through a student comment. One of the students Dr. Byne observed commented that she had been finding derivatives for the last several weeks of class, was sick of the topic and wanted to know when the class would be moving on to something else. In other words, she was unaware that the first course in calculus was primarily about derivative and assumed that the class would soon move to a different unrelated topic that she would struggle less with. She was eagerly waiting for this move to happen. This student's comment is an indication that the overall structure and purpose of calculus was not evident to her.

Such observations form the motivation for how Dr. Byne implements the use of online videos in her class. They were less central a learning resource than in the flipped classroom observed and were instead used in parallel to in-class instruction. Rather than placing all content learning online, her class discussed topics in class. In some cases these topics were preluded by assigned online videos and in other cases the videos provided supplemental instruction after a topic was already introduced.

Setting

The Calculus I course that provided a setting for this study took place in Summer 2012 at a large southwestern university over a 10 week period. The class met four times per week, with each class session lasting one hour and 30 minutes. The course roughly followed a standard curriculum found in Anton, Bivens and Davis (2012), but was

supplemented with applets used for in-class activities. Additionally, the class made use of several physical scenarios and discrete data from these scenarios when introducing major topics such as derivation and integration. These additions to the curriculum laid out in Anton et al. (2012) make the curriculum implemented more inline with reform calculus curricula, such as Hughes-Hallet et al. (1994).² The addition of physical scenarios to motivate calculus ideas helps avoid what Freudenthal (1973) refers to as an anti-didactical inversion, which is when the end results of the work of mathematicians, for example standardized algorithms for solving particular types of problems, are taken as the starting points for mathematics education.

Additionally, the class made use of online videos as part of the homework. The class did not follow a flipped model per-se (e.g., Bowers & Zazkis, 2012; Large & Platt, 2000a, 2000b; Toto & Nyguen, 2009), since new content was still introduced in-class and the online-videos did not substitute lecture entirely. The structure of the class was consistent with what Garrison and Vaughan (2008) refer to as blended learning, which is a “thoughtful fusion of face-to-face and online experiences” (p. 5).

A typical class day was split into segments. The class typically began with a 5 minute quiz to assess students’ understanding of either an assigned video-lecture or concepts covered in the previous day’s class. This quiz was followed by a short discussion of the solution to the previous day’s quiz. The next segment of class typically involved a small group problem-solving task. Each group had 2-5 people, was formed of self selected peers and the task was typically 15-40 minutes. These tasks were used both

² During later versions of the course, after the data collection for this study was completed, the instructor chose to switch to the Hughes-Hallet et al. text, because she felt it was more reflective of her approach and teaching philosophy.

for introducing new topics and further exploring topics covered in the previous nights' video-lecture assignment. The small group problem-solving tasks were often based around a particular applet. The small groups were structured so that most groups had at least one person with a laptop or tablet. A set of Apple iPads was also made available to groups that did not have a computer. These iPads allowed every small group to incorporate applets into their discussion. The day-to-day observation of this segment of class provided the data on group interactions with technology and students' transitions between representational modes in social settings.

The next segment of class was dedicated to reviewing solutions to assigned problems. This segment involved discussing both the solutions to the previous night's homework assignment and the solutions to the group-work problems. The final segment of class involved introducing a new topic, elaborating on an earlier topic, or alluding to online-video lecture content and pointing to what students should pay particular attention in the video-lectures.

Participant Selection

Participants were chosen from the mainstream Calculus I described in the previous section. Although the class had an enrollment cap of 60 students, the class began with 79 students. The additional students were added by the instructor due to a high demand for the course and a large number of students requesting to be let into the course.

Students were informed of this study on the first day of class when I explained the nature and goals of the study. After this explanation students were given an informed consent form in which they indicated whether they would (a) allow their assignments,

quizzes and exams to be used as part of this study and (b) participate in in-class observation and one-on-one problem-solving interviews. I refer to students who select both (a) and (b) as “potential participants” and the students who were selected for the study as “full participants.” Approximately half the class (43) agreed to be full participants and an additional 15 students agree to just (a). There were no students who only agreed to (b). The three full participants were selected from the 43 potential participants. This selection was based on both group dynamics and scores on a pretest. How the three participants were chosen will be discussed further after a discussion of the pre-test.

The Calculus Concept Inventory (CCI) exam was administered on the first day of class to all students (Epstein, 2007). The CCI is a validated 22 question multiple-choice instrument, is based on an extensive body of mathematics education research and evaluates a broad spectrum of major understandings, representational abilities, and reasoning abilities needed in Calculus I (Epstein, 2007). The CCI served to both inform Dr. Byne regarding the class’ knowledge of mathematics, served as the pretest of students’ knowledge for this study and helped me choose a focus group of three students whose knowledge of calculus is comparable to that of the class as a whole.

The class had a mean score of 5.87 of a possible 22 (26%) with a standard deviation of (2.4). All the questions on the CCI exam are multiple-choice and have 5 possible answers. Thus, an average of 5.87 is only slightly better than what would be expected if all students randomly chose answers. The first page of the exam also provided data with regard to students’ past history with calculus. 27 of the class’s 79 students had taken a college calculus course before (34%). This high number of students retaking

calculus is not surprising given that the calculus class was offered in the summer and was therefore not highly populated by freshmen.

Students in the class were allowed to choose their own small groups. Students who agreed to be potential participants in the study were asked to form groups with each other. This selection was intended to maximize the number of small groups that could participate in the study; since a small group with just one person who did not volunteer for the study could not have its conversations video-recorded for the purposes of this study. These groups of potential participants were also asked to sit in a particular section of the classroom to make it easier to observe multiple groups when selecting a group to participate in the study.

I spent the first two weeks of class, which were dedicated to a review of pre-calculus algebra skills and Cartesian-graph interpretation, observing all potential small groups. I regularly observed 10 groups that ranged in size from 2 to 4 people during this period. Several potential participants had chosen to form groups with non-participants and were thus eliminated from the pool. Several other groups of potential participants situated themselves in locations that were surrounded by groups of non-participants. These groups were difficult to observe for this reason and were thus not part of the 10 groups I observed. Dr. Byne and her teaching assistant, Jagger³, also interacted with the groups during group-work. Their input was also accounted for during the selection process. Groups that were working cooperatively and communicating mathematical ideas, as opposed to working separately and then comparing answers, were given priority.

³ This is a pseudonym

Based on my observations and with input from Dr. Byne and Jagger, a group of three students was selected after the first week of class. A major contributing factor to my decision to choose this particular group stemmed from their rich in-class discussions. These involved them communicating ideas as they were forming; many other groups of potential participants tended to work in isolation and then compare answers and only discuss their solution process in cases where their answers were different. The chosen group of three was observed and recorded for three days prior to the start of the limit unit. These three days give me time to change my mind and choose a different group if issues arose, which was not the case. The students were given the gender preserving pseudonyms, Ann, Brad and Carson. Brad and Ann had unsuccessfully attempted to complete a calculus course with a different instructor the previous semester. They had also worked on homework together in that class, and thus had already established some rapport. Carson was taking calculus for the first time. Ann, Brad and Carson had CCI scores of 4, 6 and 7, respectively. Their average score (5.67) was thus on par with the class average of 5.87. They were thus, at least with respect to the CCI, on par with the rest of the class as a whole.

The students chosen also represented a diversity of intended majors and ethnic backgrounds. Ann was an intended mathematics major, Brad intended to complete an economics degree and Carson was completing a degree in public health. Ann is Hispanic, Brad is Caucasian and Carson is Asian-American.

Data Sources

The majority of data come from two sources: observation of in-class focus group activities and individual problem-solving interviews with members of the focus group. The individual problem-solving interviews target my first research question, the observation of focus group target the second, and comparison of the analyses from both these sources addresses the third.

Designing the Interviews

I conducted a series of three one-on-one clinical interviews with each of the three participants in the study (Clement, 2000; Ginsburg, 1981). Each interview protocol went through a series of drafts. These drafts were initially refined through discussion with Dr. Chris Rasmussen, my dissertation supervisor. After Dr. Rasmussen and I were satisfied with the content of the interview protocol the interviews were internally-piloted on a set of two students from the class. These interviews were conducted within the time frame of the actual study. The two students had volunteered to be full-participants in the study but were not chosen during the selection process. The two pilot students were chosen with respect to the perception Dr. Byne and I had of their abilities. This perception was based on both exam/quiz grades and their in-class interactions. They were chosen so that one of the pilot students was higher performing than the study participants and the other was lower performing. This range of abilities allowed me to refine the difficulty of the interview tasks. Tasks that are too straightforward would yield quick solutions and little insight into students' commognitive process. Tasks that were too difficult would lead to students quickly getting stuck and giving up on the task, which would also yield little

insights into students' commognitive processes. These interviews served as a mechanism for further refining the interview protocol. Dr. Rasmussen and I met after the internal-pilot interviews to further refine and adjust the protocols using the internal pilot data before I conducted the actual interviews analyzed in this study.

The tasks came from a variety of sources. Some tasks were borrowed from the calculus education literature, some were modified from tasks found in the literature and others were novel tasks that I created specifically for the study. The specific tasks in each of the interviews are located in Appendix A. The goal was to generate a set of tasks that would be novel for students yet accessible enough for them to not get stuck on all of them. The tasks also varied greatly in presentation, specifically which representations were used to express the task and which representations the required solution was supposed to be presented in. These variations will be discussed in more detail in the categorizing transitions section later in this chapter.

In-Class Observation

The in-class focus group observations consisted of video-recordings of group discussions as well as field notes I took during these sessions. During data collection it quickly became apparent that it was not feasible for me to remain an observer during this process. My presence during focus group activities meant that it was natural for students to ask me questions and also natural from me to ask them to clarify their approaches during focus-group discussions. The observation of the focus group occurred without any intentional design on my part. Since I use the commognitive perspective, my presence would have been considered an influence on students' commognition regardless of the

centrality of my role in the discourse. So my contributions to the work of the group do not conflict with my chosen theoretical perspective.

Data Transcription and Reduction

The utterances in video recordings of the interviews of all three participants in this study were fully transcribed. Short descriptions of notations and gestures that accompanied verbal utterances were selectively added to these transcripts to aide with clarifying the nature and underlying meaning of students' commognition. These were added in sections of the data where the utterances themselves without accompanying gestures and notation would have been especially ambiguous. For example, when a student used a phrase such as "it looks like this" the hand gesture and/or what the student was pointing at was included along with the transcript. Although, there was likely data lost in not fully documenting every gesture, this loss was considered acceptable given that a full analysis of all gestures within these data was beyond the scope of this study.

Given the volume of in-class video it was impractical to transcribe all of it. Instead, I elected to transcribe only four selected days of class. These were days where concepts and major ideas were first introduced. The day derivative was first introduced through the ball drop problem, the two days the TIA was used in class to introduce the graphing of derivative and the day the TIA was used to introduce the concept of second derivative were transcribed. I deemed these days of class as particularly relevant in informing the individual problem-solving behaviors I observed during the interviews. These data were transcribed in a similar manner to the interviews. However, the nature of video recording multiple actors in a classroom environment meant that gesture and

notation data were often not captured in the video recordings. So the descriptions of gestures within the group transcripts are sparser than the equivalent descriptions in the interview data. How these data were analyzed will be discussed further in this chapter after the theoretical model is introduced.

Analytic Tools and Constructs

The analytic tool used to explore the data from both the interviews and group evolved out of R. Zazkis et al.'s (1996) VA-model. In this section I will first discuss the VA-model and then move to discussing the creation of the VAP-model and how the commognitive perspective informs it.

The VA-Model

The VA-model (R. Zazkis, Dubinsky, & Dautermann, 1996) is rooted in constructivism, specifically Dubinsky's Action Process Object Schema (APOS) theory (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Dubinsky & McDonald, 2001). The VA-model is therefore acquisitionist in nature and focused on student thinking. The model partitions mathematical thinking into two modes, visual thinking and analytic thinking, and views the development of visual and analytic modes of thinking as complementary, rather than disjoint processes (see Figure 3.1). The modes build on one another as students' mathematical thinking advances. As students progress, their ability to translate between these modes becomes more common and the connections between the modes become stronger. In other words, the model contends that a back-and-forth relationship between the modes of thinking develops over time, and as it does, the transition between modes becomes progressively more natural for students to make.

Figure 3.1 illustrates this process via a path through successive levels of visualization and analysis in which the distance between visualization and analysis decreases as the levels advance. The VA-diagram is a discrete representation of a continuous process. That is, there are no discrete specified levels of visualization and analysis that a student progresses through. However, the model contends that learning occurs as students' transition between representations, and as they do this transition becomes more natural.

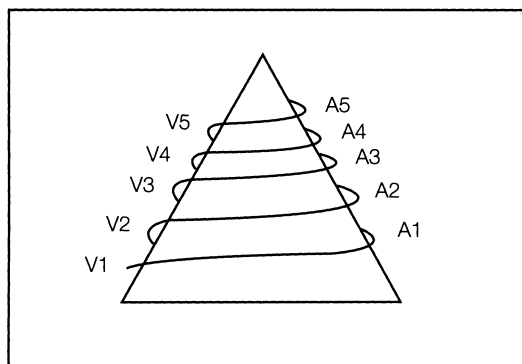


Figure 3.1: The VA-Model Diagram (from R. Zazkis et al. (1996))

Developing the VAP-Model

I adopted the commognitive perspective in order to facilitate a compatible analysis of both the group and interview data. This perspective, not constructivism, grounds the VAP-model. I will hence replace my use of the word thinking, with the more encompassing term commognition when discussing the VAP-model.

All mathematical commognition is tied to the representations used to communicate mathematics and these representations can naturally be partitioned into analytic (notation based) representations and visual (non-notation based representations). So a natural way to code the data was to partition students' commognition two these two

categories. During initial coding stages I attempted to code interview data by just distinguishing between analytic and visual. There were many works that utilized similar distinctions that informed this initial approach (e.g., Aspenwall & Shaw, 2002; Krutetskii, 1976; Zandieh, 2000; R. Zazkis et al., 1996). All of these research studies operationalized the distinction between analytic and visual in compatible ways. So it was possible to code the data using the analytic/visual dichotomy without having to subscribe to a particular model. So this approach seemed reasonable when initially coding the data. Two realizations very quickly surfaced. First, all the students in my study made frequent use of both analytic and visual reasoning and often move quickly back and forth between the two. They would thus all be classified as harmonic thinkers if I were to use a classification such as Krutetskii's (1976) or Presmeg's (1986, 1992). So subscribing to a model similar to these was not desirable; a categorization in which all data is classified in the same way has little explanatory power.

The second realization was that the visualization/analysis dichotomy was not going to suffice for classifying students' commognition. There were multiple instances where the commognition was based on physical objects. These instances occurred even on problems that were not specifically presented in terms of a physical scenario. Students would occasionally add a physics setting in. Such commognition did not qualify as visual or analytic because both visual and analytic involves reasoning with purely mathematical objects. A code "physical" was created to describe these instances. Zandieh's (2000; Zandieh & Knapp, 2006) model of the derivative concept has laid some of the ground work for approaching position/velocity/acceleration as part of the derivative concept and hence part of student commognition in calculus settings. The "physical" code was used to

code both students' reasoning about physical objects external to them, such as balls, particles and cars, and commognition about their own physical motions. An initial attempt was made to distinguish between external objects, such as particles, and participants referring to their own motion. However, keeping track of this distinction became tedious and I deemed it not important enough to continue making it.

The realization that all the participants in the study could be classified as harmonic thinkers, drove me away from distinctions made by Krutetskii (1976) and Presmeg (1986, 1992). Since all students were in the same category this distinction was not useful for detailing the differences between each of the participants commognitive processes. These differences, at least the ones that became apparent from applying the visual/analytic/physical coding scheme were differences in how, when and how often each student transitioned between each of the modes. This realization motivated me to extend the VA-Model, which added a physical mode.

The addition of a physical mode is not the only way in which I extended the VA-model. My extension also expands the unit of analysis. I had decided to coordinate cognitive and social analyses before I collected data. Consequently, I collected both rich individual problem-solving data and rich group-work data from the participants. The VA-model was, however, developed through a constructivist perspective and hence focused on individual thinking. My use of the VAP-model in contrast is based around a commognitive perspective to facilitate the coordination of group and individual centered data. The use of a common theoretical framework and a common coding of commognitive actions across both individual and group data facilitated the co-ordination of analysis across these settings.

Elucidating the VAP-Model

The VAP-model borrows many of its assumptions from the VA-model that it extends. The VA-model diagram shows levels that spiral up a triangle as thinking advances. Visualization and analysis become closer to each other as one moves to more advanced levels. The VAP-model diagram is a tetrahedron, to accommodate the addition of a physical mode. The path between modes also spirals up with levels getting closer to each other. As with the VA-model getting closer represents the transitions between modes becoming more natural as more transitions occur. However, in the VA-diagram there is an orderly path that moves from visualization to analysis and back. In the VAP-diagram the path moves upward between three modes, but does so through a process that can include all three or just two of the modes of reasoning. This aspect of the diagram signifies that the transitions between visual, analytic and physical modes do not follow a specified sequence.

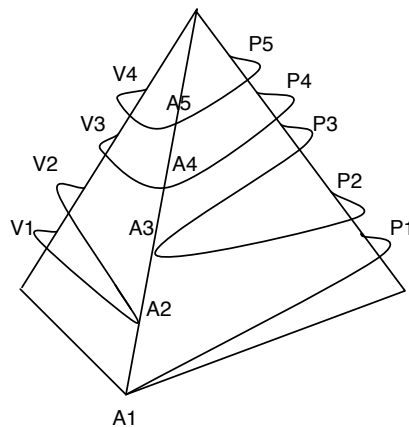


Figure 3.2: The VAP-Model Diagram

Students' transitions between the three modes of reasoning are of particular interest because they inform how students use multiple modes of representation in conjunction with each other. Much like the VA-model it is build on, the VAP-model contends that these modes inform each other, but what this informing looks like, when it happens or how such transitions can be fostered by instruction are not predicted by the model, they are instead research questions to be explored.

Since the VAP-model is based in a commognitive perspective some terminology and assumptions do not get carried over. Specifically the terms visual thinking/visualization and analytic thinking/analysis do not encompass both group and individual uses of representations. When the VAP-model is used these terms are substituted with the terms analytic reasoning, visual reasoning and physical reasoning. The word reasoning, unlike thinking, does not imply an individual cognitive process and can be used to apply to both group and individual contexts. An assumption carried over from the commognitive perspective is that the commognitive interactions that occur in a group setting are compatible phenomena to the commognition that occurs during one-on-one interviews. Although this assumption is not unique to the commognitive perspective, it is not part of the VA-model, which was developed within a constructivist perspective. The two settings differ in terms of the particular role observable interlocutors play relative to each other, but the process of analyzing the commognition can be, the same. In this study the analysis with respect to the VAP-model is the same across group and interview settings. However, an additional analytic tool is used to parse students' conversational roles in group settings.

Operationalizing the VAP-Model

The power of the analytic, visual and physical categories is that they are tied to representations and hence can be applied to multiple entities that use those representations, not just students' commognitive actions. In addition to students' commognitive actions the categories can be applied to a task/problem, an entire solution strategy, a student's general propensity when approaching a series of tasks, etc. Although such cross-unit implementation of the analytic/visual distinction has been used to classify tasks and student work in a single study (e.g., Krutetskii, 1976), to my knowledge it has not been used to coordinate individual centered and group centered units of analysis. I intentionally avoid using the distinction between analytic, visual and physical to classify students in terms of a preferred commognition strategy. Some prior work that used the VA-model classified students in this way (e.g., Haciomeroglu, Aspinwall, & Presmeg, 2010). However, I see this classification as inconsistent with the VA-model. The model implicitly contends that all students use multiple modes of commognition. So I see the use of categorizations like analytic thinker, visual thinker and physical thinker, as inconsistent with the spirit of the VAP-model.

It is not uncommon for work on visualization to not provide explicit definitions for analytic and visual, and instead leave these implicit. (e.g., George, 1999, Haciomeroglu, Aspinwall, & Presmeg, 2010; Zimmerman, 1991). However, this lack of explicit definitions is not always the case. The VA-model provided explicit definitions for visualization and analysis. In order to develop the definitions used in the VAP-model it is useful to begin by discussing the definitions used by the VA-model. The definition used by R. Zazkis et al. (1996) for visualization is:

Visualization is an act in which an individual establishes a strong connection between an internal construct and some thing to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively an act of visualization may consist of the construction on some external medium such as paper, chalk board or computer screen, of objects or events that the individual identifies with object(s) or process(es) in her or his mind. (p.441)

It should be noted that although the above definition uses the phrase “gained through the senses,” in practice the definition only gets applied to a single sense, vision. R. Zazkis et al.’s definition for analysis is:

An act of analysis or analytic thinking (we will use the two terms interchangeably) is mental manipulation of objects with or without the aid of symbols. (p. 442)

The word “object” in both of these definitions is consistent with Dubinski’s Action, Process, Object, Schema theory (APOS) (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Dubinsky & McDonald, 2001). Notice that both of these definitions are phrased from an acquisitionist perspective and center on visualization and analysis as types of mental actions. These mental actions happen inside the mind and act on mental objects. This interpretation is inconsistent with the commognitive perspective which views thinking and communicating as different sides of the same coin. The commognitive perspective views thinking as a process that may occur inside the mind, however this internal process is viewed as a type of self-communication rather than the construction, manipulation and consolidation of mental objects.

However, what is common across both perspectives is how one would identify analytic/visual/physical. One cannot observe processes that occur strictly inside the mind.

The identification of analytic/visual/physical within data occurs based on observable phenomena such as gestures, utterances and inscriptions. So it is possible to borrow the main thrust of these definitions and modify the aspects of them that refer to mental processes. In general R. Zazkis et al. and other authors define analytic in a way that captures commognitive actions that center on notation-based mathematical objects. These may or may not be manipulated by hand. Similarly visual is used in a way that captures phrases and actions that center on non-notation based mathematical objects. Both the above definitions and the implicit definitions used by other authors are consistent with this interpretation. Using this general sense of visual and analytic a commognitive action that is indicative of, for example, the analytic mode is portable with respect to the unit of analysis. That is, the same commognitive action will be interpreted as analytic regardless of which setting it occurs in. A commognitive action like uttering the phrase “solve this system of equations” can be viewed as indicative of analytic reasoning regardless of the setting it is uttered in. The same can be said about the portability of notions of visual/physical.

Additionally, tasks are a type of written communication between the author(s) of the task statement and the student interpreting the task. These tasks are therefore a form of commognition. Since the visual/analytic/physical labels are tied to mathematical representations and tasks use these representations to communicate mathematical content, the labels can be applied to tasks in similar ways to how they are applied to student commognition. Thus we can discuss a task, which for example, is presented with analytic notation and requires a visual solution (such a task is discussed in the Categorizing Transitions section of this chapter).

Since I use the categories visual, analytic and physical to describe tasks, and student commognition in both interview and group-work settings my definitions need to be neutral with respect to what unit of analysis they are applied to. This use of a common cross-setting categorization helps to facilitate cross-setting comparisons:

- 1) Analytic reasoning makes reference to use or manipulation of notation-based mathematical objects, such as equations, variables and formulae. The notation itself does not necessarily have to be actually present for something to be considered analytic; phrases or gestures that indicate the notation suffice. Some common words and phrases that would indicate this type of commognition are: equation, variable, solve for, derive. It can also be indicated by gestures that refer to notation, such as pointing at equations and gestures that aid the manipulation of notation.
- 2) Visual reasoning makes reference to the use or manipulation of non-notation based mathematical objects such as graphs and diagrams. These objects do not necessarily have to be drawn or explicitly manipulated for something to be coded as visual. Instead phrases or gestures that indicate a non-notation based object will suffice. Some common words and phrases that indicate this type of commognition are: slope, tangent, steeper, draw, line and x-axis. It can also be indicated by gestures that refer to graphs, such as pointing at graphs, tracing graphs and gestures that mimic the motions of graphs.
- 3) Physical reasoning makes reference to real or imagine non-mathematical objects or settings that are used to reason about mathematics. Common words and phrases used to indicate this type of commognition are particle, accelerates, stops, moves and turns around. The physical object informing the mathematics does not have to exist.

Notice that this definition of physical includes imaginable scenarios that have no relation to actual physical objects. So for example a reference to the magic carpet ride setting discussed in Wawro et al. 2012) would be classified by this definition as physical even though a magic carpet is by no means a physical object. The word physical here reflects the fact that the model was developed in a calculus setting and the representations students in this study referred to were based in physics settings.

When focusing on an individual these definitions are inline with the definitions R. Zazkis et al. use. So phrases that would be identified by R. Zazkis et al.'s framework as indicative of analytic thinking or visual thinking would be identified as indicative of analytic and visual reasoning, respectively, using these new definitions. However, the lack of specificity with regard to the unit being analyzed allows the new definitions to be applied to units of analysis other than individuals.

It is important to mention the multi-referential nature of certain mathematical terms and gestures. As noted by Zandieh and Knapp (2006) in mathematics often a single term is used to refer to multiple representations of a mathematical object. For example the phrase “sine of x ” is multi-referential. It can refer to both the specific equation, $y=\sin(x)$, the graph of that equation, or even some kind of periodic physical motion. In these instances surrounding gestures and phrases can occasionally clarify the mathematical representation being referenced. For example the phrase “graph of sine of x ” has a clear referent to a visual object. Additionally, the phrase “sine of x ” uttered in conjunction with a up and down finger movements that mimics the shape of a sine graph would also clarify the type of representation the phrase “sine of x ” was referring to. However, in some cases it is not clear what representation is being referenced. In these

cases it is important to acknowledge this ambiguity when analyzing the representations within a particular data set. During the analysis of the data a “neutral” code was created to label data in which it was not clear what representation was being used. The multi-referential phrases that the neutral code was applied to occasionally played a role during students’ transitions between modes. Examples of this phenomenon are explored further in subsequent chapters.

Categorizing Transitions

Transitions in Tasks

The VAP-model informs my approach to student mathematical reasoning, and this reasoning often happens in the setting of mathematics tasks, which may favor the use of one representation over another. In line with the commognitive perspective, I view a task as a form of communication between the authors of the task and the students engaged in it. Task can therefore be thought of as a form of commognition, much like a mathematical question asked by a teacher.

Krutetskii (1976) categorized tasks in terms of which mode of solution was most accessible. Developing such a categorization of tasks involves documenting possible solutions to a task as well as what representations are used within these solutions and then evaluating which of these approaches is the most straight forward. This categorization can be rather subjective and using a similar coding scheme to categorize tasks for this study would be much more complicated, since my model uses 3 modes rather than two. Instead, I choose to categorize questions in terms of which

representations where needed to understand the question and in which representations the solution was requested. This type of categorization treats tasks in ways that align with how students' commognition is coded, since it examines only the representations present. This approach has roots in the work of Aspinwall (Aspinwall & Shaw, 2002; Haciomeroglu, Aspinwall, & Presmeg, 2010), who presented students with a graph and asked them to create a related graph (of either a derivative or anti-derivative function). Students who consistently translated the graph into analytic notation before solving the task were classified as analytic thinkers and students who never did so were classified as visual thinkers. In other words, those that solved the problem in the representation it was presented in were classified as preferring that representation and those that switched representations were categorized as preferring the representation they switched to.

My classification of tasks and student solutions to these tasks follow the same basic premise but is adjusted to accommodate the possibility of multiple representations being part of the problem statement and the fact that a task can be presented in one mode and require a solution in another.

In order to help clarify the above point, it is useful to explore how the representation modes in the model relate to the framing of a mathematical task, as well as possible solutions to that task. This clarification is particularly important because translation between representation modes is often an element the task itself. That is, there are some calculus tasks that cannot be solved without using more than one of the representations in the model. Let us consider the following task: "If $f(0)=1$, $f'(0)=1$, $f(3)=7$, $f'(3)=-1$ and $f''(3)=-1$, sketch a possible graph of $f(x)$." The task is stated in terms of the analytic mode, since the information about the function is given symbolically, and

the answer is supposed to be provided in a graphical mode. Note that in order to complete the task a student is required to interpret the information about the function presented analytically and translate this information into visual (graphical) information. Solving the task requires moving from one edge of the VAP-diagram to another.

It can also be the case that translation between modes is not necessarily required in order to complete the task. However, spontaneous transitions between representation modes may occur anyway during students' problem-solving. For example, if a student is given the following integral to solve $\int_{-3}^3 x\sqrt{9-x^2} dx$, she may solve it using standard methods, such as u-substitution. This approach makes use of the analytic mode. The problem can, however, be solved by reasoning about the shape of the graph of $x\sqrt{9-x^2}$. The graph has a 180° rotational symmetry about the origin (odd function). Therefore every region above the x-axis has a corresponding region below the axis on the other side of the y-axis. Since the bounds of integration are symmetric with respect to the origin, the integral evaluates to zero. Even though the problem is stated in symbolic/analytic terms and requires a symbolic/numerical answer, the second solution makes extensive use of the graphical mode. If a student solves the task in this way, her thought process moves from one edge of the VAP-diagram to another, but this transition is not specifically required by the problem itself.

I refer to transitions between modes that are not required by the task, as *unprompted transitions*. Further, I refer to transitions that are part of the problem itself, that is, when a problem is stated in one mode and requires an answer stated in another, as *prompted transitions*. Note that prompted transitions are an attribute of a task and

unprompted transitions are an attribute of a solution. So it is possible to have an unprompted transition occur within the setting of a prompted transition problem. This phenomenon is similar to how some researchers have categorized students who translate graphical tasks into function notation before solving them as analytic thinkers (e.g., Aspinwall & Shaw, 2002; Haciomeroglu, Aspinwall, & Presmeg, 2010). However, one crucial difference is that I do not use the presence of unprompted transitions to a particular representation to categorize a student's overall thinking as preferring that type of representation.

Transitions between modes in inter-student commognition

Transitions between representation modes can also occur within students' commognition. When students communicate in groups this commognition typically occurs in turns, where one student's commognition is followed by another's. Within this group setting transitions between modes can occur both within a particular student's turn and between turns. *Within turn transitions* occur when a student's commognition switches between two of the modes in the VAP-model during a single turn. *Between turn transitions* occur when one student responds to another student's commognition which ended in a particular mode with commognition which is in a different mode.

Within turn and between turn transitions are not unlike the prompted and unprompted transitions discussed in the previous section. These categorize the representations used to facilitate a student's commognition relative to the representations that were used prior to it. In the case of within turn and between turn transitions the prior representations are those that facilitated a previous student's commognition. In the case

of prompted and unprompted transitions the previous representation were present in the task. All these constructs highlight different ways in which transitions between modes occur within data. These transitions are mediated by the various social roles students take on during collective activity. The model used to document these roles is discussed in the next section.

The different roles students play within a conversation

Although the VAP-model's distinction between visual, analytic and physical, is useful for parsing some aspects of student commognition, because it was developed in the setting of individual interviews it is not suited for parsing differing social roles within collective activity. In order to address this aspect of student commognition in a group setting I adopted Krummheuer's (2007, 2011) participation framework. Krummheuer situates his framework within Sfard's commognitive perspective and Lave and Wenger's (1991) notions of legitimate peripheral participation. It is thus consistent with my chosen theoretical perspective.

Krummheuer's framework categorizes both those directly involved as speakers within collective activities and those who are present but not speaking. The non-speakers are classified in one of three categories that are determined by how accessible they are to a speaker's utterance. Krummheuer categorizes the person or people being directly addressed by the speaker as *conversation-partners*, those who are tolerated by the speaker but not directly addressed as *over hearers*, and those who are excluded by the speaker as *eavesdroppers*. These roles naturally change as those involved in a collective activity take turns speaking.

Krummheuer uses the construct of formulation and content as building blocks for his framework. These are used to operationalize his classification of speakers. However, Krummheuer does not explicitly provide definitions for either of these constructs. So in order to operationalize the formulation and content constructs I adopted my own definitions that are in line with Krumheuer's use of these notions. In his work Krummheuer utilizes Toulmin's (1969) categorization of elements of an argument. This partitions elements of an argument into data, claim, warrant, and backing. The claim is the result being justified, the data is information used to assert the truth of this justification, the warrant is used to link the data to the claim, and the backing is used to support the legitimacy of the warrant. Krumheuer implicitly treats content as consisting of each of these elements. I will make this treatment the explicit definition. So introducing new content to a collective mathematical activity involves introducing data, claim, warrant, and/or backings that have not already been stated.

Krummheuer's formulation construct is tied to how mathematical content is expressed. The construct can be operationalized in various ways. When operationalizing what kinds of expressions are considered different/equivalent it is important to take into account the nature of the students and subject matter being analyzed. For example, Krummheuer in his study of first grade children treats $10+3$ and 13 as different formulations. Within his analysis of interactions in a first grade classroom treating $10+3$ and 13 as different from one another makes sense, since the students in his study view these two entities as different. However, in a calculus setting it is safe to assume that students treat $10+3$ and 13 as equivalent expressions.

One possible heuristic for assessing the grain-size of the formulation construct is to assess how students themselves express mathematical content. Changes in formulation can be identified when students themselves see two expressions of mathematical content as equivalent or when a difference in how something is expressed causes miscommunication among them. Changes in formulation are related to Cobb and Yackel's (1996) notions of sociomathematical norms. These norms include what expressions are treated as mathematically different/equivalent and how particular concepts are communicated. However, as Cobb and Yackel (1996) point out, sociomathematical norms evolve over time. So within such an operationalization of formulation two expressions that are treated as different at one point in time may be treated as equivalent at a later point. One constraint of such an operationalization is that it leads to an evolving notion of formulation that may change throughout the course of the semester. An additional constraint of this heuristic is that within a conversation it may be difficult to assess what students themselves see as different/equivalent.

Another possible approach to operationalizing formulation within this data set is to make it equivalent to the VAP-model's operationalization of representation. This operationalization has the advantage of maximizing the overlap between the implementation of Krummheuer's framework and the VAP-framework. However, certain within representation shifts in how content is expressed are overlooked by this approach. For example moving between what Monk (1992) refers to as "point-wise" and "across time" interpretations could be considered a change in formulation under certain operationalizations. An example of this phenomenon within a visual representation mode is a shift from considering the slope of a function at specific points to considering the

shape of the entire function. Operationalizing formulation as equivalent to representation mode would overlook such a change in approach.

I choose to bridge the two approaches to operationalizing the formulation construct mentioned above. So I define changes in representation mode as changes in formulation and also use student's own approaches as evidence for what they see as different formulations. The definition allows for overlap of the VAP-model and Krummheuer's framework, but does not limit Krummheuer's framework to the operationalization of the VAP-framework.

Returning to the categorizations of speakers' roles, Krummheuer classifies speakers in terms of whether they take responsibility for the content and/or the formulation of their utterances. These two binary criteria lead to four categories. *Authors* are responsible for both the formulation and content of their utterance. *Relayers* are responsible for neither the form nor the content of their utterance. They simply re-convey another speaker's utterance, or their interpretation of it. A *ghostee* introduces new content into a conversation but formulates the content in a way that is compatible with previous utterances. A *spokesman*, on the other hand, reformulates existing content in a new way.

Since Krummheuer's categorizations of roles within a conversation were developed in the context of cooperative collective activities, in which all participants are working toward a common goal, his framework occasionally miss-aligns with the roles of statements within non-cooperative activities. Interviews are non-cooperative in the sense that the interviewer and the interviewee are working toward separate goals. The interviewer seeks to gain insights into the interviewee's commognition and the interviewee is attempting to solve the interview tasks. So, in order to facilitate analysis of

interview data that is compatible with Krummheuer's framework the same building-block constructs, formulation and content, were used to analyze these interviews. The use of common building-block constructs allows direct comparison with the data across both settings without the need to fully adopt Krummheuer's framework when analyzing interview data. How the implementation of this analysis is discussed further in the subsequent chapters.

Summary

This chapter elucidated the commognitive perspective as well as the reasons for choosing it as the theoretical framing for the study. It also described the participant selection process. This discussion was led to a discussion of the analytic tools. The VAP-model evolved out of R. Zazkis et al.'s (1996) VA-model. It is used here to facilitate analysis of the representations that facilitate student commognition. Krummheuer's framework is used to analyze social roles within interview and group-work data. Within group-work settings the model is used directly and within interview settings the building-block constructs that operationalize the model's categorization are used. The use of compatible frameworks to analyze both interview and group data facilitates the comparison of these two settings in the final analysis chapter.

CHAPTER 4:

Transitions Between Modes of Representation in One-On-One Interviews

This chapter addresses the first research question, “How do individual students think about key calculus concepts during their problem-solving? In particular, how do they use representations to facilitate this thinking?” The subsequent chapter will conduct a compatible exploration of student reasoning in a group problem-solving context. The final analysis chapter will compare the reasoning that occurs in these two contexts. In line with the commognitive perspective, the interviewer is treated as an integral part of the social environment in which student reasoning is occurring. Consequently, the influence the interviewer’s communication has on students’ commognition is an important part of the analysis. However, the examination of the interviewer’s role is primarily discussed in the last analysis chapter, chapter 6, where it is compared to the social roles individual participants play in group settings.

Derivative is a concept that can be understood and explored using a variety of representations, all of which are treated by the mathematical community as an important part of the concept (Zandieh, 2000). Understanding derivative with respect to all of these representations as well as how they relate to each other is essential for a rich understanding of the concept (Dreyfus, 1991; Zandieh, 2000; Zimmerman, 1991). Exploring what modes of representation arise during student problem-solving, and how students move from reasoning with one representation to another, illuminates an important aspect of their thinking about derivative. It gives insights into which

representations of derivative are prominent in their thinking and which connections between representations help facilitate their thinking.

It is not possible to communicate mathematical concepts without representations of those concepts (Duval, 1999). Which representations are present within discussions about derivative and how they interact with each other illuminates an important aspect of this communication. Since Sfard's commognitive perspective treats communication and thinking as different instantiations of the same phenomena, through a commognitive lens the representations that facilitate communication about derivative are also the representations that facilitate thinking.

Data and Analysis

The data discussed in this chapter come from one-on-one clinical interviews which center on individual students' problem-solving. Each of the three one-on-one clinical interviews involved students' work on approximately 7 tasks. Some of these tasks were borrowed or modified from existing tasks in the literature and others were created specifically for this study. The tasks themselves and notes regarding their origins and development can be found in appendix A. The interviews ranged from 40-75 minutes. Students took as much time as they felt they needed to work on the tasks. They were given the option of moving on to a subsequent task if they felt they were stuck and not making progress and were also allowed to return to a previous task. Only one of the three students, Ann, actually returned to a task during an interview.

The tasks themselves, individual phrases and accompanying gestures were coded using the VAP-model's framework as visual, analytic, physical or neutral/verbal. This

coding allowed me to categorize which representation modes were used to facilitate student commognition and how this usage related to students' understanding of underlying calculus concepts. I view representation usage as inseparable from commognition. That is, it is not possible to reason about mathematics apart from some representation of the mathematical objects being reasoned about. The next section details how the representation modes used to communicate tasks to students relate to the representation modes used to facilitate students' commognition when exploring these tasks. The rest of the chapter details the specifics of some of the tasks and then delves deeper into the analysis of some of these tasks using the VAP-Model. Although all tasks were coded, the examples and transcripts included here were chosen because they revealed something interesting about student reasoning or were representative of a larger pattern that occurred in the data. Short summaries of student work on each task, which included a description of students' solution process, the representations used to facilitate their reasoning, and the interviewer/interviewee's social roles were created during the analysis. These summaries allowed me to notice larger patterns in the data and assess how common particular phenomena were within the data.

Representations in Tasks and Their Solutions

This section provides an overview of how representations present in task statements relate to the representations present in student solutions. As a reminder to the reader, since I consider the tasks themselves to be a form of commognition between the authors of the task and the student, the VAP-framework, which is used to code individual students' commognition, is applied here to the tasks as well. Applying the coding in this

way facilitates a comparison of representation use in the tasks and representations used to facilitate student commognition about them. Table 4.1 documents which transitions (if any) were explicitly required by each of the interview tasks and the transitions between representation modes students used in their solutions. Visual, analytic and physical are indicated with V, A and P, respectively.

As discussed in the methods chapter, a task can be stated in one mode and require a solution in another. So the type column in Table 4.1 indicates what representation modes were included in the setup of the problem to the left of the colon and what mode(s) the requested solution was expected to be in on the right. So for example, “AP:A” indicates that a task was stated in terms of both the analytic and physical modes and that the required solution was analytic in nature. The "AP" portion of the code to the left of the colon indicates that the task statement included some physical scenario, such as the motion of a particle, and an analytic description of that motion, such as a position/time equation. The "A" portion to the right of the colon indicates that the requested solution was in the form of a number or equation, such as a velocity/time equation or the speed of a particle at a particular time.

The student solution cells only indicate which modes arose during student solutions and not which order the modes arose. These cells also ignore any influence the interviewer may have had on representation use. Both the order in which modes occurred and the influence the interviewer may have had on student solutions are too complex to be properly addressed in the table, which is a fairly blunt analytic tool. It was difficult to distinguish between students' re-reading of a problem and their actual work on that problem. So determining which mode came first for example, was difficult since often

there was no clear starting point. Additionally, since students commonly moved back in forth between modes multiple times during a solution, it was difficult to determine order. ‘V’ ‘A’ and ‘P’ simply occur in Table 4.1 in the order they appear in the name of the model. Although in calculus contexts it is possible to use various different physical representations to facilitate reasoning (e.g., position/velocity/acceleration, temperature/change in temperature, population/birth-death rate), the students in this study only incorporated unprompted transitions to position/velocity/acceleration-based representations. So any unprompted transitions to physical reasoning seen in the table are transitions to position/velocity/acceleration-based representations.

The influence of the interviewer was also difficult to determine. In some cases the interviewer’s influence on student representation use was more obvious than others. There were enough instances where students shifted representation use after an interviewer comment or question, but the actual shift in use was not clearly linked to the interviewer’s utterance. A decision was made to ignore the influence of the interviewer within the table and further explore these instances in the subsequent section.

One important thing to notice from Table 4.1 is that most of the instances where students made an unprompted transition involved the addition of a visual or physical mode. The instances that incorporated unprompted transitions to the visual mode typically involved drawing a graph to aid with their reasoning. Unprompted transitions to the physical mode typically involved reasoning about a graph as if it were describing the motion of some particle or vehicle, which was not part of the specified problem. In all cases this addition involved a position-velocity scenario, which Zandieh's framework (2000; Zandieh& Knapp, 2005) treats as part of the derivative concept.

A specific transition to the physical mode is explored in more detail below. The addition of an unprompted analytic component was rare but did occur in the data. The unprompted analytic component took the form of suggesting that a graph (or part of a graph) appeared similar to a known analytic function and then finding a related function analytically before translating back to the visual mode. This behavior, which typifies analytic thinkers in Presmeg and her colleague's work, was fairly uncommon in this data set.

Table 4.1 documents what modes were used for particular questions but does not detail the specifics of how representation modes inform one another during the problem-solving process.

(I remind the reader that the tasks themselves are included in appendix A.) The VAP-model contends that these transitions between modes are central to students' development. The next section details particular sets of transitions used to solve tasks and further explores the relationship between representation use and students' understanding of derivative.

Table 4.1: Prompted and Unprompted Transitions.

	Task #	Task Type	Ann	Brad	Carson
Interview 1	1	AP:A	VAP	VAP	VAP
	2	AP:A	VAP	VAP	VAP
	3 ⁴	V:V	V	V	VAP
	4 ⁵	V:V	V	V	VP
	5	VP:V	VP	VP	VP
Interview 2	1	AP:P	VAP	VAP	VAP
	2	V:V	VP	V	VP
	3	V:V	VP	V	VP
	4	V:V	VAP	VA	VAP
	5	V:V	VAP	VA	VAP
	6	AP:A	VAP	VAP	VAP
	7	AP:A	VAP	VAP	VAP
Interview 3	1 ⁶	AP:P	VAP	VAP	VAP
	2	AP:A	VAP	VAP	VAP
	3 ⁷	VP:V	VAP	VAP	VAP
	4	V:V	VP	V	VP
	5	VP:A	VAP	VAP	VAP
	6	VP:V	VP	VP	VP
	7	VP:V	VAP	VP	VP

One observation that can be made by looking at Table 4.1 is that the visual mode appears in every one of the student cells. These appearances often took the form of student generated graphs to aide with reasoning. The visual mode appearing in every solution means that even when students were given tasks that did not prompt for the visual mode students used the visual mode to reason about the tasks. Another finding that supports the predominance of visual reasoning in the interview data is that of the 637 observed transitions between representation modes that occurred within the interview

⁴ These are the two derivative sketching tasks shown in Figure 4.1.

⁵ This is the anti-derivative task shown in Figure 4.2.

⁶ This is the tree task shown in Figure 4.3.

⁷ This is the temperature task shown in Figure 4.4.

data 535 of these transitions involved the visual mode (84%). If the modes of reasoning were approximately evenly distributed we would expect transitions that involve the visual mode to make up approximately two thirds of the observed transitions. They occur much more often. Consequently, if I were to use the classifications of overall reasoning used by past authors, all three of the students in this study would be classified as visual thinkers (e.g., Aspinwall & Shaw, 2002; Haciomeroglu, Aspinwall, & Presmeg, 2010, Krutetskii, 1976; Presmeg, 1986). Such results would have been unlikely to happen 20 years ago given students' well documented resistance to visual reasoning during that period (e.g., Eisenberg & Dreyfus, 1991; Healy & Hoyles, 1996; Vinner, 1989; Zimmermann, 1991). However, this observation is inline with work that points to the opposite trend in modern students (George, 1999; Stylianou, 2001; Stylianou & Silver, 2004).

Another important observation that can be made is that Carson used the physical mode in each of his solutions. So he incorporated the mode in every one of his solutions regardless of whether it was prompted for. If we were to extend Krutetskii's (1976) framework to include physical thinking it would be natural to conclude that Carson was a harmonic thinker, because of his use of both physical and visual representations within his problem-solving. However, note that Krutetskii's (1976) framework treats only students who extensively use both analytic and visual modes in their problem-solving as harmonic. An expansion of the framework to include physical thinkers would necessitate the creation of three new types of harmonic thinkers, physical/visual-harmonic, physical/analytic-harmonic and students that regularly incorporate all three modes. I however, do not classify the students in this study using such a framework, and only do so here to elucidate how other frameworks could have been modified to accommodate the

data in this study. Additionally, I am unconvinced that it is possible to reason about mathematics solely in a physical mode. Insights gained when relating a mathematical situation to a physical one generally need to be related back to a visual or analytic mathematics context in order to complete most mathematics tasks. The physical mode is intentionally defined in this study to refer to non-mathematical objects. If a task can be completed without relating the solution to some kind of mathematical object that task is unlikely to be mathematical in nature.

The data in the table provide a very coarse view of students' solution strategies. However, even this coarse view of students' solutions highlights the variation in student solution strategies. In 8 of the 19 tasks the representations used to facilitate student solutions were not identical across the three participants. So participants varied in terms of which representations they used on particular tasks. It should also be noted that unprompted transitions within students' solutions can be seen within the table. Any task which does not incorporate all three representation modes into its task statement provides an opportunity for students to incorporate an additional mode of representation into their solution. Two thirds of student solutions that could have shown such unprompted transitions, that is, tasks that did not incorporate all three modes, showed unprompted transitions. For example in their work on tasks 2 and 3 in interview 2 both Ann and Carson made unprompted transitions to physical reasoning but Brad did not. So in the majority of tasks students are incorporating modes of representation that are not explicitly required by the question in order to facilitate their solutions. In other words, the multiple representations in the VAP-model are playing an important role in facilitating students'

commognition and the modes that get used within a solution are not only the modes that appear in the task statement.

Tasks

Each interview involved several mathematics tasks. This section presents tasks for which student solutions are discussed in detail in the results section of this chapter. Each task discussed was included in the chapter because analysis of student work on the task using the VAP-model was able to reveal something surprising about student reasoning, the task was representative of a larger trend or because student work revealed something interesting about the role of the classroom environment.

The derivative sketching task seen in Figure 4.1a is a graphing task that asks students to sketch the graph of a derivative given a particular original function. The task is similar to a task discussed by Aspinwall and Shaw (2002) that asked students to sketch the derivative of a continuous symmetric ‘saw-tooth’ graph that alternated between a slope of negative one and one. The graph in Figure 4.3a, unlike Aspinwall and Shaw’s graph, alternates between several different slopes. These graphs have no simple translation into analytic notation. They therefore discourage an analytic approach. Aspinwall and Shaw observed that students they classified as analytic thinkers had difficulty with the saw-tooth task. As presented, the task below does not prompt any transitions between modes of representation since it can be solved using only visual reasoning.

Figure 4.1b shows a task similar to that of 4.1a. Both tasks had the same prompt. The task in 4.1b, unlike the task in 4.1a, is not a saw tooth graph and instead has

smoothed transitions. It was intentionally designed to have a similar shape to that of 4.1b. The related tasks were chosen to highlight differences in how participants thought about angular vs. differentiable graphs. Both tasks were given back-to-back during interview one, which occurred after the Tangent Intuition Applet (TIA) was used in class.

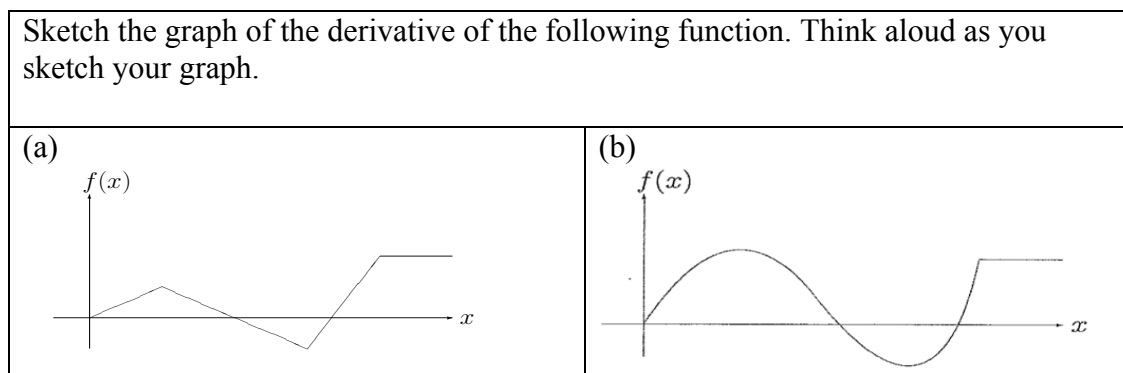


Figure 4.1: Derivative Sketching Tasks

In addition to the graphing derivative tasks students were also asked to do an anti-derivative task. This question was asked before students had encountered the concept of anti-derivative in class. Students were given a graph of a parabola, told that that graph was the derivative of some function and then asked to think about the shape of the original function. In order to reason about this task, students had to use their knowledge of graphing derivatives, much of which was facilitated by their work with the TIA, and to invent their own techniques for thinking about anti-derivative. Although similar tasks have been studied in the context of students' understanding of anti-derivative, I am unaware of research that studied student work on such tasks before they had been formally exposed to the concept of anti-derivative (Haciomeroglu, Aspinwall, & Presmeg, 2010, Orton, 1983b).

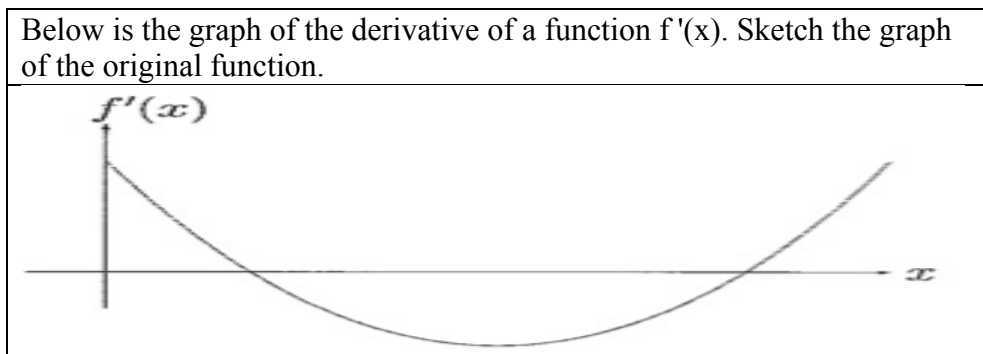


Figure 4.2: The anti-derivative task

The Tree Task, presented in Figure 4.3, is a modification of a task used in Zandieh and Knapp (2006). Both of these tasks present a function that describes a physical situation and ask students to reason about analytically what described attributes of the function mean in terms of the described physical situation. The task used in Zandieh and Knapp was based in a temperature context. The tree task instead uses the growth of a tree as a context. The tree context can be viewed as analogous to the position/acceleration/velocity of a particle, with the position of the top of the tree above the ground corresponding to the position of the particle. This task presents information about a function analytically and asks for this information to be interpreted in terms of a physical situation. It therefore prompts a transition from analytic to physical modes.

$f(t)$ gives height of a tree in meters as a function of time in years since the tree was planted. Interpret what each of the following means in terms of the tree:
 $f(0)=1$, $f(4)=15$, $f'(4)=3$, $f''(4)=-3$.

Figure 4.3: The Tree Task

The Temperature Task was borrowed from Carlson, Jacobs, Coe, Larsen and Hsu (2002). It is a task stated in terms of both visual and physical modes and requires a visual

answer. As such, it prompts a transition from physical to visual modes. The task is of particular interest because although the calculus is situated within a physical scenario, the scenario used is not the typical position/velocity/acceleration context. This physical temperature context is atypical in most calculus curricula. In particular, the group of three students in this study had not been exposed to temperature-based examples in the course (both in class and on homework) at the time the task was given. Carlson et al.'s use of the temperature context to study students' understanding of calculus concepts is not unique. The study of Zandieh and Knapp (2006) also contains a temperature based calculus problem.

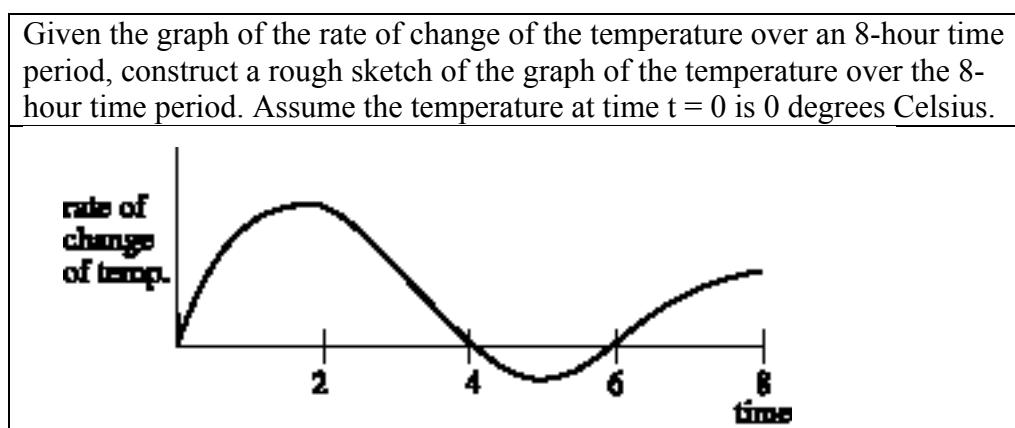


Figure 4.4: The Temperature Task

Representations in Visual Tasks

Derivative sketching tasks

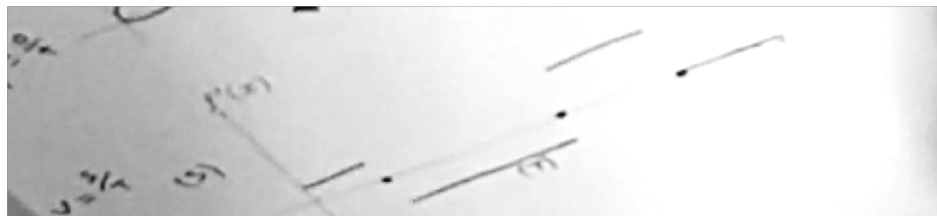
As mentioned above, the majority of unprompted transitions in students' work were transitions to the physical or visual modes. This section examines a series of visual tasks. That is task that can be understood and solved solely in the visual mode. Several

student solutions that involved unprompted transitions are discussed and these are compared to student solutions that do not make unprompted transitions.

Below is the transcript of Carson working on the saw-tooth derivative sketching task (Figure 4.3a). This episode is an example of an unprompted transition to the physical mode and is fairly representative of what these transitions look like within Carson's work. This task is stated in graphic terms and requires a graphic (visual) solution but the transcript details a solution that does not stay solely within the confines of visual reasoning. This transcript is fairly representative of Carson's unprompted transitions to the physical mode, which can be found in all his work on visual tasks.

Excerpt 4.1a:

Carson [00:22:49]: Alright, so I know that the derivative is the slope and I took physics so I know that this is distance [writes a d under x axis] over, no that's wrong this is time [crosses out d and writes t under the x axis]. This is time over distance, which is you speed. Speed is distance over time. So this is time and this is your speed [labels axis on derivative function] and so as your distance...I'm sorry... So this is constant so you know that velocity is constant. So your velocity is something like this [draws short horizontal line segment above x -axis] and then later when it hits this tip it's at zero [marks a dot on the x -axis after previously drawn segment]. And then later when it's decelerating. Ya this is a negative speed so the graph. And it's a straight line so you know it would be something like this [draws a horizontal line under the x -axis]. And then again at this point it's zero [draws another dot on the x -axis after the second segment]. And then um accelerating. But this time it's more this is steeper. So it would be higher because your velocity would be faster. Your speed would be faster. So it would be like this [draws a third horizontal line segment above the x -axis higher than the first]. And then again right here it's zero [draws another dot on the x -axis]. And then now this one your distance isn't changing. Since your distance isn't changing. This equation [$s=d/t$] looks like zero over time. So the rest of the graph would look like this [draws a fourth line along the x -axis.]



Notice that Carson's solution makes use of all three edges of the tetrahedron in the VAP-diagram. He moves back and forth between the physical and visual edges several times. Each time he discusses a physical scenario before pausing to sketch an accompanying portion of the derivative graph. Although both his utterances and inscriptions are forms of commognition, the inscriptions are centered on the visual mode of representation and his utterances center on the physical mode.

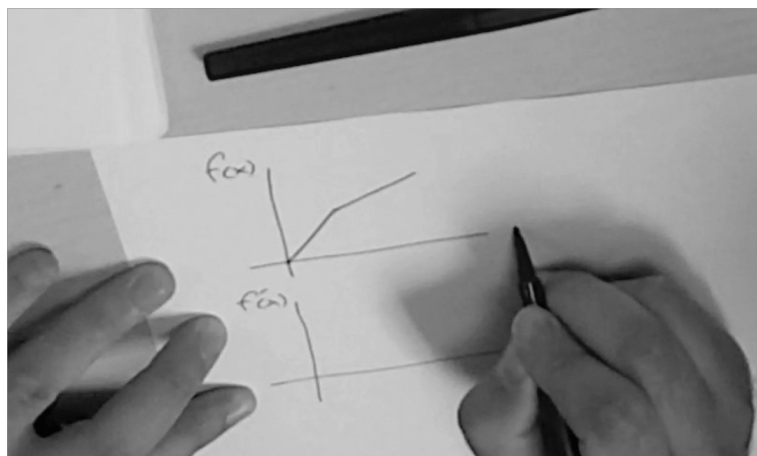
Near the end of his solution he briefly moves to the analytic edge of the VAP-diagram when he reasons that the horizontal portion of the graph corresponds to "zero over time." This type of brief use of analytic reasoning was present in Carson's solutions of three of the seven visual tasks. This use of analytic reasoning appears only in his reasoning about constant sections of position graphs. This pattern in Carson's reasoning occurs regardless of whether he introduced the position context himself or if it was part of the original problem.

The connections that Carson makes between physical and visual contexts led to some interesting artifacts. The given graph cannot represent the position graph of a physical object. Physical objects cannot instantaneously change directions and so it does not make sense to discuss the physical interpretation of what happens at those points. Carson, however, does not abandon the physical-visual link. This link leads him to

conclude erroneously that there are zeros at the points where the graph instantaneously switches direction. This phenomenon is similar to one noted in Aspinwall, Shaw and Presmeg (1997), which they termed uncontrollable mental imagery, which is where visual images associated with students' visual interpretations interfere with their analytic interpretations. Further questioning by the interviewer revealed more about the nature of Carson's thinking regarding non-differentiable points of a function.

Excerpt 4.1b:

Interviewer [00:24:47]: I just want to sort of explore that idea a little bit more because it's really interesting to me. Can I ask you to do something similar to this. So pretty much exactly the same setup but I'm going to have something like this. So that's a straight line that's another straight line. I'm just curious what happens if that's my set up instead.



Carson [00:23:32]: Yeah I understand what you're asking for. So this is time and this is distance, this is your speed distance. Speed is distance over time [writes $s=d/t$] velocity is speed over time writes $[v=s/t]$. So the starting point would be the same and since it's accelerating at a constant speed I would give it something like this [draws a short segment parallel to the x-axis] and then now it's traveling at a slightly slower pace. So I would say it's like this [Draws a second segment closer to the x-axis]



Interviewer [00:26:12]: And then what happens at that corner point in this case

Carson [00:26:18]: In this case since it's still accelerating I wouldn't give it a zero because you don't necessarily have to stop to slow down. Were as in this case [points at previous paper] you have to stop for like a split second to change your direction. So since it's not changing direction I wouldn't bring it down to a zero velocity.

Further questioning revealed that Carson's translation into the physical mode only appears to affect his ability to deal with sudden transitions from increasing to decreasing or vice-versa. Consequently, his errors are limited to several discrete points. So, his use of the connection between visual and physical modes appears to help him more than it hinders. Carson's solution to the smooth derivative sketching task (Figure 4.3b) used similar transitions between modes to that of the solution detailed above. So it is not discussed in detail here.

Ann and Brad, in contrast to Carson, solved both the derivative-sketching tasks while reasoning only in a visual mode and thus their thinking on these tasks stayed within the confines of the VAP-diagram's visual edge. However, Ann's work on the task revealed several interesting artifacts left over from her work with the TIA:

Excerpt 4.2a:

Ann: [00:25:38] When we did the sketchpad it was easy, because you'd pull it until it was straight you know (two hand motion mimicking the "slope widgets"). But I don't have a pulley thing like on that sketchpad. So I don't know where the little dot would be. But since this is a negative slope it would have been pulled down and this one was a positive so it would have been pulled up so that's why I know it has to go something like that [...] So this is a slope of like I would say one [traces the first segment of the given function and then draws a horizontal line above x-axis for the derivative function]. And this is a negative slope, I feel like it's equal so this would be negative one. This is right here [traces the second segment of the given function and then draws horizontal line below x-axis]. This is negative one but it's a long one. And I feel like this is steeper so it would be somewhere up here [traces the third segment of the given function and then draws horizontal line above x-axis higher than first horizontal line segment].. And then it's zero [traces the last segment of the given function and then draws horizontal line along x-axis].

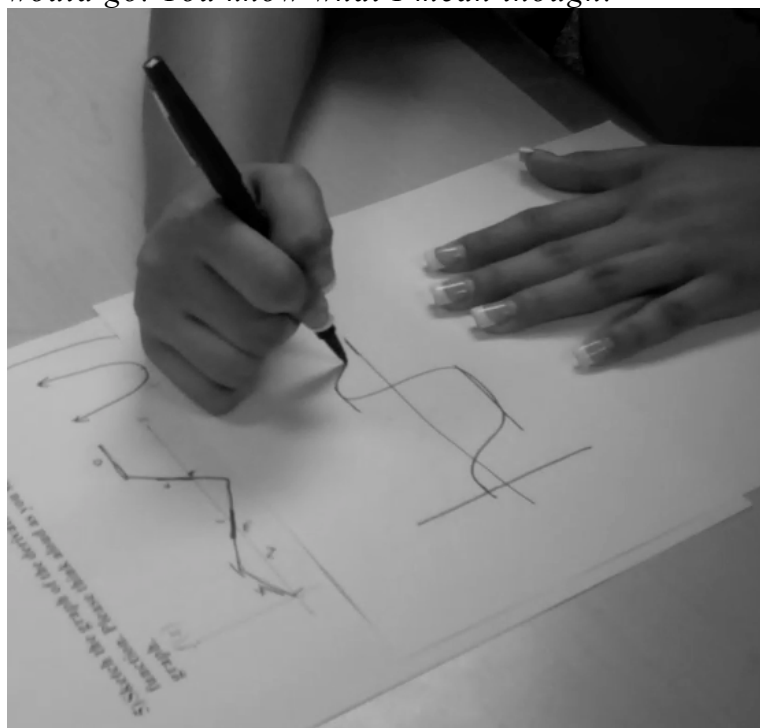


Notice that Ann's solution does not include the points at zero, which Carson included after reasoning that one must stop to change directions. She also explicitly traces the given function. I argue that for her this action mimics the mini-tangents in the TIA.

She then uses these mini tangents to reason about the height of the corresponding section of the derivative graph. She used this strategy on several other graphing tasks over the course of the study and it, for the most part, was rather effective. However, note that the TIA does not actually draw the function for its user. Instead it creates a set of points along the derivative function, which the user then has to figure out what to do with. For continuous differentiable functions simply drawing a smooth continuous line through the points given by the applet will generate an appropriate graph. However, this strategy does not necessarily work on non-differentiable functions. The strategy can be seen in the continuation of Ann's transcript.

Excerpt 4.2b:

Ann: [00:25:38] ...So I'm guess it would go like.. Not exactly like that maybe a little bit smoother like connecting them in the areas that they would go. You know what I mean though.



Notice that when Ann draws a smooth curve through the segments using her knowledge of the applet she is using a technique that works well on differentiable functions. However, the given function is not differentiable at several points. So connecting the segments is incorrect. This error is an artifact of working with the applet. However, similar to what happened with Carson's use of physical reasoning, relating this problem back to the applet helps Ann more than it hinders.

The task described above was given to Ann during interview one. When a similar task was given to Ann during the second interview this applet-based error was not present. Linear sections in her solution corresponded to constant sections of her solution and non-linear sections were dealt with using the TIA heuristic. So although her initial error may have been an artifact of work with the applet, further exposure to the material was able to eliminate the effects of this artifact. The modes of representation used on the related task in interview two were much the same as the ones used in the solution detailed above and stayed within the confines of visual reasoning.

To highlight how powerful Ann's applet-inspired method was for her it is useful to discuss her solution to the smoother derivative sketching task (Figure 4.3b). Since this task included only one non-differentiable point the error from her solution to the saw-tooth derivative sketching task (Figure 4.3a) did not carry over to the solution to the smooth derivative sketching task.

Excerpt 4.3:

Ann: [00:23:00] This is zero and this is zero [draws tangent lines at maxima and minima] and this is clearly zero [traces last segment]. And this is negative this is positive and this is positive [adds to more tangent lines at decreasing and increasing sections]. I know that this has a slope of like one and then it goes zero [draws horizontal lines at one and zero of the derivative graph]. And then it goes negative

[draws line under x-axis of the derivative graph]. A see I don't want to end up with the same graph. Okay. Here the slope is like I would say that this is like a one [draws horizontal line at one of the derivative graph] this is zero [draws line along x-axis] and this is like a negative one [draws horizontal line under x-axis] and this is zero [draws horizontal line along x-axis] and this is steeper I feel like it's like a negative two and this is zero.



This is a, like I'm trying to explain to you what I'm doing now. I did this because this is how my partner [Brad] does it. Or X or Y or A or B or whatever his name is in this. This is how I always saw him doing it. He would do this and be like I think this slopes is like one and I think this slopes is like zero. And then he would just like connect them.



And I feel like he would always get it right and so I'm trying to understand. But then like applied to this thing where it goes like this and then we have like this.

Notice that in this case the applet-inspired technique that caused errors with the saw tooth graph is no longer an issue since she is not dealing with only one non-

differentiable point. The technique based on her work with the tangent intuition applet works much in the same way it did with the previously discussed task. She first creates a series of tangents along the function, then she reasons about the slope of each of these to determine the corresponding height of the derivative graph. Once she has determined the heights she connects them to create a smooth sketch of the derivative. She explicitly makes reference to her work with the group in class as a catalyst for her use of this technique and that it was developed as an imitation of her groupmates work on similar tasks. However, it is interesting to note that neither of her groupmates used similar techniques while working on both of the derivative sketching tasks during the interviews. Brad did briefly use a similar technique during group-work, which will be explored further in the next chapter. He abandoned this technique during the interviews. So although Ann credits her work with the group as a catalyst for her use of the technique, when working individually she is the only one who uses it. Apart from her shift in how she dealt with non-differentiable points in the second and third interviews, Ann's reasoning on visual tasks in which she did not use physical or analytic reasoning is fairly well represented by her work on the above two tasks.

The above transcript of Ann was coded as her exclusively using the visual mode. There are several phrases in the transcript that would be ambiguous without her gestures and actions for context. For example, if the phrase "*This is zero and this is zero*" were looked at without her gesture it would be difficult to determine which representation she was using. The same phrase could be used in reference to analytic representations when setting an equation to zero. The phrase can also be used in reference to a physical context where "this is zero" can refer to a point where a particle is momentarily stationary (zero

speed) or at the same location it started (zero position). However, here the phrase is used in conjunction with graphing gestures, which clarifies that “this is zero” refers to zero slope.

The anti-derivative task

The anti-derivative task was given to students before they were formally exposed to the concept of anti-derivative in class. Although it is possible that the participants encountered the concept of anti-derivative in previous coursework, since graphing anti-derivatives is not a commonly found topic in introductory calculus curricula, it is likely that the anti-derivative task was their first experience with graphing anti-derivatives. So student work on the task involved them inventing their own techniques for dealing with anti-derivative graphs. Let us begin by examining Brad’s solution.

Excerpt 4.4:

Brad [00:33:58]: So from here on the original the slope is increasing [starts at origin and draws an curve with positive slope] until it crosses zero and then it begins decreasing [draws a maxima and curves the function so it has a negative slope] and then it hits zero and after it hits zero it goes back up [draws a minima and curves the function so it has a positive slope].

until it hits zero again, the rate of the slope because here the slope is increasing till it crosses zero.

Brad’s work on this task was coded as visual. What is astounding about this transcript is that it occurs over less than a minute. Brad was able to almost instantaneously re-adjust his thinking about derivative functions and work the problem backward. That is, relating the height of a function with the slope of the anti-derivative instead of relating the slope of a function to the height of its derivative. Unlike in Ann’s

work, in which she explicitly made reference to the TIA, it is not clear that Brad's quick solution to this task was informed by his work with the applet. However, his in class work with the applet was the primary means through which he was introduced to the concept of graphing derivatives.

Ann's solution to the anti-derivative task, which was also coded as purely visual, is interesting to compare to Brad's because it highlights that different students' visual reasoning about the same problem can look very different:

Excerpt 4.5a:

Ann: [00:30:30] Well this one is this [draws two graphs which look like $y=x^2$ and $y=x^3$ on the same coordinate plain off to the right away from the space to put solutions in] So I'm guessing it's this. But I'm trying to understand if I was just give this [pointing at $y=x^2$ like graph] how could it go to here [pointing at $y=x^3$ like graph]. Because I think it would just be this one [Draws x^3 like shape solution part of paper]. But I want to just try to think like [draws a parabola that is above the x-axis but touches the axis at its apex]. So from here to here is positive this is positive [run pen along the $y=x^3$ graph she drew] and this is positive [points at x^2 graph she drew]. I guess I just get a little confused like how am I supposed to think about this. From me this is negative [points at the part of the given graph under the axis] and this is positive [pointing at the given graph above the x-axis]. This is a decreasing slope and this is an increasing slope [running pen along $y=x^3$ function she drew]. But I just know that these two go together. But to me this is not a decreasing slope [points at the apex of the parabola she drew]. but then. All I can think is whenever were' doing it. I would say that this is a really high slope and at some point it gets to zero. And that would have been if I drew these and then reversed them. And this graph wasn't there. I'd go ya this is a really steep slope and then at some point it gets close to zero from this point and then it gets really steep again but I would have drawn it more like that because it hits zero or close to zero. I would have drawn it like that. So that's why I'm a little confused.



At this point Ann had drawn two familiar corresponding graphical shapes one which looks much like an x -cubed graph and the other much like an x -squared graph. She reasoned that she knew these two graphs corresponded to each other and used the slope-height relationship to reiterate why these two shapes corresponded. Notice that even though she is reasoning about two functions, one of which is the derivative of the other, she never explicitly identifies these functions in terms of their analytic references. Several authors in past work have identified analytic thinkers as doing something similar in terms of relating a visual problem to known analytic relationships between functions. Analytic thinkers often relate a given graph to a known analytic function reason about its derivative and then use that to inform the drawing of the derivative (e.g., Aspinwall and Shaw, 2002). However, the phenomena here is different in that it stays within a visual representation mode. When Ann says, “well this one is this” she is referring to two graphical objects that she has drawn, one of which is similar to the graph of the given

function. So she is using known graphical relationships between two functions to reason about this problem as opposed to using a known analytic relationship.

Even if she had explicitly named the two graphical objects 'x-squared' and 'x-cubed' instead of saying “well this one is this”, her commognition would have still been classified as visual. Using the specific name of a function can reference both its analytic and its visual representation. So, for example, a statement like "the derivative of sine is cosine" can both express a specific relationship between two analytic objects, namely $y=\sin(x)$ and $y=\cos(x)$, and can express the relationship between two periodic graphs. The specific representation referenced is clarified by accompanied gestures and notation.

Returning to Ann's solution, Ann's reasoning about the known relationship reveals a difference between the known functions and the one she is presented with. The familiar parabola stays above the x-axis and thus has non-negative height values, which corresponds to a cubic function that has non-zero slope. The presented function has a section below the x-axis. Its anti-derivative therefore must have a corresponding segment with negative slope. The cubic function she drew does not have such a segment. She resolves this issue in the continuation of the transcript:

Excerpt 4.5b:

Or maybe it doesn't hit zero. That's why it's like that. So maybe it's more like this [readjusts graph to have a negative section].



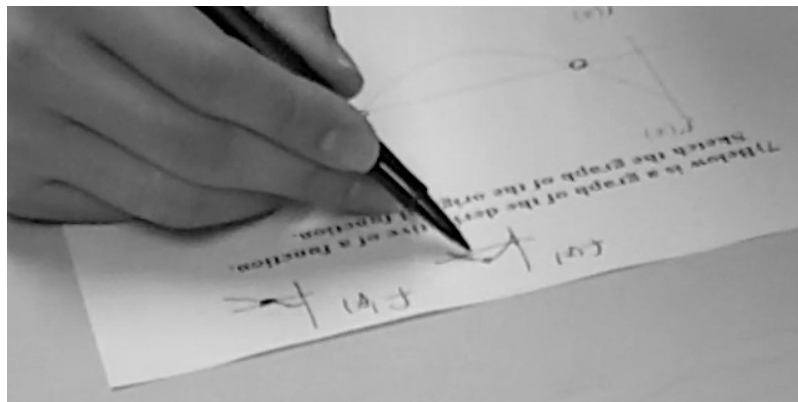
But then this would be a negative slope which is why it would be down here and this would be a positive slope.

In the remainder of the episode Ann comes to the realization that the negative portion of the given graph must correspond to a negative slope portion of the anti-derivative graph and adjusts her graph accordingly. The transcript above continues for several more minutes. Ann spends this time convincing herself that her solution is in fact correct by implementing the same TIA inspired technique described in her solution to the two derivative sketching tasks. Although her solution is much less direct than the one implemented by Brad, it demonstrates her ability to manipulate visual objects in order to solve the task. Unlike algebraic manipulations that are based on a rigid set of rules Ann is familiar with, these visual manipulations involve figuring out the rules in the moment.

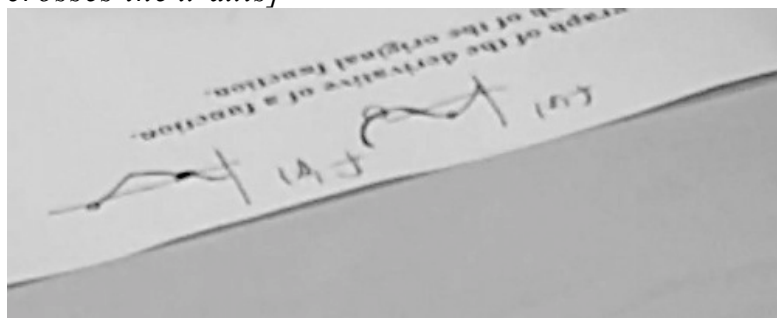
Unlike Ann and Brad's Solutions, which used only the visual mode, Carson's solution involved both visual and physical modes and thus moved between the physical and visual edges of the VAP-diagram.

Excerpt 4.6a:

Carson[00:30:47]: ...Well I'll just do an example for myself. So.[mumbles][draws negative parabola and a corresponding derivative graph on the top of the page]



ya so these my zeros when the graph reaches a steady slope [points at vertex of parabola he drew]. Okay and then this [circles negative portion of the graph] would look like [adds an increasing section to the parabola to make it look like a cubed function and a corresponding line to extend the derivative graph] It's a zero right there [circles minima of the cubed graph he drew and the point where the derivative crosses the x-axis]



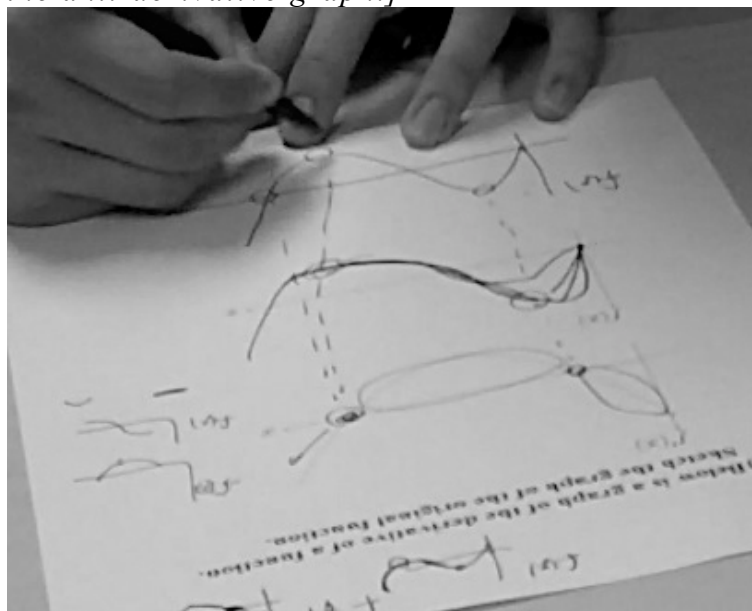
At this point Carson had established the general shape of the graph and had only used the visual mode of representation to arrive at his conclusion. Carson reasoned with sections of functions that he pieced together. He first drew a maxima before sketching a corresponding negative slope on the derivative graph. Then he adjoined a minima to the previous maxima before adjoining a positive slope to the derivative graph. So although the final shape resembled a graph of the cubic function, the process of sketching it involved adjoining two parabola-shaped graphs. Both this solution and Ann's solution used commognition about known graphic objects in order to solve the anti-derivative task.

However, Carson's strategy of adjoining several familiar pieces differed from Ann's, which involved finding a familiar function that mimicked the entire shape of the graph and then adjusting it. Carson continued working on the task for several minutes. He continually adjusted the shape of the anti-derivative graph to correspond to the shape of the derivative graph. After this process was complete he was asked to elaborate on the correspondence between the graphs:

Excerpt 4.6b:

Interviewer[00:34: 47]: Okay, and can you tell me what point line up with what points?

Carson[00:34:50]: Okay, So I know that since this is zero [connects zero of original graph to maxima of anti-derivative graph] I wrote down this. So this is another zero I wrote this [draws dotted line connecting the second zero of the original graph to the zero of the anti-derivative graph] No that's wrong, its right here [draws dotted line connecting the second zero of the original graph to the minima of the anti-derivative graph.]



So I know that there are my zeros which is why there are right here and so since it's increasing pretty fast then it slows down there this slows down to zero [given derivative function] and then this is a

negative slope the whole time [pointing his function] which is why this is in the negative portion of the graph. But it accelerates and then it slows down right back down to zero, which is what this is and then it increases again which is why I drew the increase. This is wrong [crosses out point where the function he drew crosses the x-axis] Yeah [puts cap back on pen]

Notice that when Carson is justifying his solution he incorporates physical representation. This shift to the physical mode was not specifically prompted by the task or the interviewer's follow up question. However, it is clear that a shift in Carson's discussion of the problem occurred after this interaction with the interviewer. What the cause of this shift in representation mode usage was is unclear.

Carson had already answered the interviewer's question with regard to the correspondence between the graphs when he began to further elaborate on the relationship using the physical mode. It is interesting that Carson does not incorporate the physical mode earlier in his solution. One possible explanation may be that he is comfortable moving from a position to an acceleration context but not vice versa. So he may have needed the anti-derivative function to already be drawn in order to reason about the relationship between the two graphs in terms of a position-acceleration context.

Also notice that Carson does not prelude his shift to the physical mode in any way. He simply shifts to discussing the two graphs as if they are position and velocity graphs and does not feel obliged to indicate which graph is being thought of in which way. He treats the position-velocity context as an implicit part of what it means to work with function in a calculus context (Zandieh, 2000; Zandieh & Knapp, 2005). Ann also exhibited similar behavior when making unprompted transitions to physical reasoning.

One important thing to note is that even though all of the participants worked in the same group in the same class and where introduced to graphing functions in the same classroom milieu, they each had very different approaches to the anti-derivative task. Also, of note is that each of the students successfully solved this novel task that centered around a concept, anti-derivative, that they had not yet encountered. This speaks to how flexible they are in their approaches and thinking. As a reminder these students were chosen to be fairly representative of the class as a whole.

Representations in Physical Tasks

The temperature task

Some of the interview tasks were not presented solely in a visual mode. The tasks discussed in this section involved some kind of physical context. None of these were purely about the physical situation itself. The tasks presented here thus all involve multiple modes.

During the coding process several instances occurred where it was difficult to apply a single code to the data. Unlike the instances discussed earlier, where the representation being used was ambiguous because of the multi-representational nature of certain mathematical terms, the phenomena I discuss here involves a different kind of ambiguity. This ambiguity occurs when verbal utterances reference one mode of representation and the accompanying gestures, which usually clarify what mode is being referenced, indicate a different mode. In these cases the transcript was double coded with both the mode indicated by the utterance and the mode indicated by the gesture. The

following transcript is an example of where such a double coding was used. It is an excerpt of Carson working on the Temperature Task. The excerpt is presented without any description of gestures to emphasize the prevalence of the physical mode:

Excerpt 4.7:

Carson[00:16:46]: Um like this. Um because it's slowing down to get to reach a flat part. And so this one is and this one is slowing down to get to a flat part... I don't know this is weird. And then it speeds back up...

Apart from the two uses of the phrase “a flat part,” which is a reference to a particular part of the drawn graph that has zero slope, Carson’s utterances indicate he is primarily working in a physical mode. The references to slowing down and speeding up indicate a physical position-velocity context that is grounding the discussion. However, as he was uttering the phrases in the above transcript he was simultaneously drawing an anti-derivative graph. So the utterances which indicate a physical mode were accompanied by the creation of a graphical object. Consequently, this section was double coded. Carson’s transcript was double coded only three times in the first interview but was double coded nine and seven times during the second and third interviews, respectively. This usage pattern may be an indication that he became more comfortable using simultaneously multiple modes of representation during the course of the semester. The VA-model predicts that transitions between modes of commognition become more natural for students to make as they become more comfortable with mathematical ideas. The VAP-model that extends the VA-model carries through this assumption. Although the evidence here is far too tentative to validate the model, it is worth mentioning that it is

consistent with it. In addition I believe, much like Zazkis et al. (1996), that the value of a model within education research lies in its usefulness in making sense of phenomena, rather than its accuracy.

All of the instances of double coding occurred only in Carson's interviews and all of these double codes involved both visual and physical modes. This double coding may occur due to the overlap between these modes. It is possible to have a graphical object that describes a physical motion. The elements of these objects simultaneously provide both physical information about the object and visual information such as slope. So it is possible to trace a graph and attend to the motion it describes as the graph changes.

However, similar links can exist between other pairs of modes. For example one can refer to a point where the graphs of two functions intersect (visual) as the solution to a system of equations (analytic). Also one can refer to specific terms in an equation (analytic) in terms of their physical instantiation. So, for example, if $d=30t+10$ is a position time function that describes the motion of a particular particle, '30' is the velocity of that particle. However, the students in this study did not make such connections between modes. Perhaps this type of data could be seen in studies of students in more advanced calculus courses.

One important thing to notice about the above excerpt is that although the temperature problem is stated in terms of a temperature-based physical scenario, Carson uses a more familiar physical scenario, a moving particle, to ground his commognition. So it is not that Carson prefers to have a physical scenario to help him reason about problems. It is that he uses a specific physical scenario, the position-velocity scenario, to help him reason about visual problems.

A similar phenomenon was observed in Roth's (2004) studies of scientists' interpretations of graphs. He observed that the scientists could not provide standard interpretations of graphs which depicted situations outside of their research field even in cases where these graphs were structurally equivalent to those that they had used and interpreted within their own research. In both Roth's study and this one the interpretation of graphs is tied to a particular physical situational referent without which the graph itself has little interpretable meaning for the person examining it. I interpret Carson's shift to discussing the graph in terms of a position/velocity scenario as an attempt to avoid a situational referent in which he has little intuition.

The tree task

Each interview included at least one task that was presented in the analytic and physical modes and required either analytic or physical solutions. There were five such tasks total. In all of these tasks the participants drew a graph to aid with their reasoning. The next episode is of Ann working on the Tree Task. Her work on this task is an interesting instance of an unprompted transition to a visual mode. The task forces a transition between analytic and physical modes. However, Ann does not use only physical and analytic modes.

Excerpt 4.8:

Ann [00:03:30]: Alright. So I have to draw a graph. You're just asking me what this means right?

Interviewer [00:04:10]: No, I'm just asking you if you can tell me, F or four equals 15, what does that mean is happening with the tree.

Ann [00:04:23]: Okay, and F of t give height of the tree in meters as a function of time. So this is time [labels x axis with "time"] I want to say and this is meters [labels y axis with "m"]. And so four and zero is

I four [marks 15 on the y-axis and 4 on the x axis] well that's not. You know what I mean it's one [places a point at (4,15) and (0,1) on the graph]. and then this is four-three, four and negative three. This is F of x. This is prime. This is F double prime. Okay. This is saying that at zero years, or when they very first start it was one meter. And then after four years it was 15 meters. So it's going up.

The Tree Task is a problem that prompts for a transition between analytic and physical modes. Ann however, makes an unprompted transition to a visual mode. In fact, Ann specifically asked if the question was asking for a graph, she was told that it did not and then she proceeded to draw a graph anyway. It is only after she plots the points (0,1) and (4,15) that she concludes that these points indicate the height of the tree at particular points in time. Here Ann translates from analytic notation that gives information about particular points to a graphic representation of those points. After these points are graphed she translates the visual information into information about the tree. So the problem prompted a transition between analytic and physical modes, but Ann mediated this transition through an unprompted transition to a visual mode. Brad and Carson also had unprompted transitions to the visual mode when working on this task. In fact, in every problem that was stated in terms of the physical and analytic modes students incorporated an unprompted transition to the visual mode. So at no point in all three interviews did any of the students make sense of a physical context without the use of visual representation. Their problem-solving processes always involved the creation of some kind of diagram or graph to make sense of physical situations.

Discussion

The representations used to facilitate student commognition about derivative are an important aspect of their thinking about the concept. The VAP-model used to frame the analysis in this chapter was able to reveal important characteristics of this thinking. Most importantly, the modes of representation present in a task statement do not necessarily align with the modes of representation students incorporate into their solution strategies. Students often make unprompted transitions to physical and visual modes. When these transitions do happen a rich back and forth between modes is common where students move back and forth between modes in quick succession and inform one mode of representation with the other.

The physical mode in particular, was a valuable tool for Carson who regularly switched to thinking about graphs and their derivative graphs in terms of a position-velocity scenario (e.g., Excerpts 4.1a/b and 4.6b). These transitions often created a rich back-and-forth relationship between his visual and his physical representations. In several instances it became difficult to label a particular instance with just one mode of representation (Excerpt 4.7). This difficulty occurred when Carson's verbal utterances were indicative of a physical mode and his gestures simultaneously related these to the interpretation or construction of visual mathematical objects. These instances were double coded and correspond to being at the tip of the tetrahedron in the VAP-diagram, between the visual and physical edges.

Relating calculus ideas to general physical contexts was not what drove Carson's commognition. The specific physical context used to introduce the concept of derivative in class, the relationship between position and velocity, was what grounded his reliance

on physical representation in problem-solving. When an alternative temperature context was provided as a physical grounding for a particular task, Carson chose to reinterpret the problem in terms of a position-velocity scenario in order to ground his thinking (Excerpt 4.7). This context was so interwoven with his thinking about derivative that he often switched to discussing position and velocity without mentioning that he was reinterpreting context free functions. He interpreted a derivative function as a velocity function and its parent function could be interpreted as a position function whenever it was convenient and did not feel that he needed to justify or even indicate that he was making this move (Excerpt 4.6b). He treated these contexts implicitly as if they were part of the derivative concept.

The central role physical modes of representation played within Carson's problem-solving could not have been elucidated using frameworks such as those of R. Zazkis et al. (1996) or Krutetskii (1976). In these frameworks physical modes would have been subsumed under visual or analytic modes, depending on which accompanying representations occurred either in succession to or coincided with physical ones. The lens provided by the VAP-model helped draw out the importance of the physical mode and frame its relationship to both analytic and visual modes in Carson's problem-solving.

The visual mode was particularly important to both Ann and Brad . They incorporated this mode into every task solution regardless of whether a task prompted for visual representation. It allowed them to make sense of both physical scenarios incorporated in tasks and analytic notation. In particular, During Ann's work on the tree problem she gravitated to drawing a graph of the tree's growth even after she clarified that drawing such a graph was not part of what the problem was asking for (Excerpt 4.8).

Only after she drew a graph consistent with the analytic information provided in the task was she able to interpret the tree's growth pattern. Her thinking about the tree's growth was facilitated by this transition to the visual mode. She was only able to move from the analytic edge of the VAP-diagram to the physical edge through the visual edge rather than moving directly from analytic to physical mode. This mediation of the transition between physical and analytic through visual representation would have been difficult to observe and document without the VAP-model and does not fit within the scope of frameworks that focus solely on visual and analytic modes. Other frameworks, which do not have mechanisms for dealing with physical representation, would have likely documented this as a transition from analytic to visual and could have ignored the physical grounding of the solution.

It is worth noting that all three students in the study moved back-and-forth between the visual and physical edges of the VAP-diagram more often than any other pairs of edges. Much of this reasoning would have been labeled as visual in other frameworks. However, I argue that visual reasoning and physical reasoning are different from one another. The former deals with exclusively mathematical objects while physical reasoning deals with non-mathematical objects. The role these non-mathematical objects play in students' thinking plays an important role in their mathematical problem-solving. Although I believe that unlike analytic and visual modes of reasoning, it is not possible to work on mathematics tasks solely using physical reasoning, the role physical reasoning plays in students' problem-solving is often central to their problem-solving.

Studies of students' representation use often do not differentiate physical representations from other modes of representation. Those studies that do inform how

physical reasoning is related to the other modes tend to present students with physically grounded scenarios and examine how students are able to interpret these mathematically (e.g., Carlson et al., 2002; Zandieh & Knapp, 2006). The phenomenon shown here within Carson's work is the opposite; Carson moves from mathematics contexts to physical reasoning in order to make sense of the mathematics, rather than using visual or analytic reasoning to make sense of physical scenarios. Descriptions of this phenomenon add to what is known about the relationship between physical reasoning and the other modes.

CHAPTER 5:

Using the VAP-model to Study Group Problem-solving

This chapter addresses the second research question, “How do groups of students communicate about key calculus concepts? In particular, how do they use representations to facilitate this communication?” So in this chapter I shift to examining a different data set, in-class group-work, but maintain my focus on students’ commognition around the concept of derivative and how it is informed by representation use. In addition to the VAP-framework used to code representation use in the previous chapter, I also incorporate a classification of the various roles students and I play relative to each other during this group-work. This classification is borrowed from Krummheuer (2007, 2011). The subsequent chapter will relate the analyses in this chapter to those of the previous chapter, which focused on individual interviews.

Group-work occurred in every class period that was not an exam or review, so during data collection I accumulated over 20 hours of group-work data. It was not feasible to transcribe and analyze all of these data so a subset of the data was chosen for analysis. The particular days chosen for inclusion were days that introduced various aspects of the derivative concept. Specifically, I examine the data from the day in which the concept of derivative was introduced through the ball drop problem and the three days in which notions of graphing derivative were introduced using the Tangent Intuition Applet (TIA). The ball-drop problem was discussed after the class finished covering limits. The TIA was used at several points throughout the semester. It was used on two

consecutive days the week following the ball-drop problem and then was revisited a month later when discussing second derivatives. Both the ball drop and TIA activities are discussed in further detail in the group tasks section of this chapter. Central ideas related to derivative were introduced on these days and these introductions set up subsequent work with the concept. These days were deemed particularly important because many of the trends in student approaches to derivative began during these introductions. Thus, examining these days allows me to make richer connections between the patterns found in individual interviews and patterns found in the group activities. These connections between the two settings are discussed in the next chapter.

This chapter begins with a brief recapitulation of the nature of focus group problem-solving activities in this study. I then revisit the VAP-model and discuss the similarities and differences between the coding of the focus group data and the coding of individual interviews. This comparison of settings involves adapting the VAP-diagram for use in a social context, and revisiting VAP-model constructs specific to social contexts. I also revisit Krummheuer's (2007, 2011) classification of roles that can be played within a discussion. Then I shift to discussing particular episodes within the data in more detail. In order to facilitate this discussion I describe the nature and purpose of some of the group activities examined in the chapter. That discussion sets up the subsequent analysis of more fine-grained phenomena within the group data and how these phenomena are informed by representation usage and the various roles played within discussions.

The subsequent chapter will compare representation use and social roles in the group and interview data. This will include both a top-level comparison of results from

this chapter and chapter four, and a more detailed comparison of selected episodes from the two data sets. This comparison will involve the introduction of some additional episodes not already discussed in this chapter or chapter four.

Collection of Group-Work Data

Every non-review class period incorporated some amount of group-work. This group work took up between 20 and 50 minutes of class time and involved an activity intended to get students to work with or develop a calculus concept or procedure. Some of these activities involved work on multiple problems provided on a work sheet and others involved a single problem that was put up on the board.

The classroom had lecture style seating. I situated the focus group at the front of the classroom. This location allowed me to transition easily from recording group activities to recording the instructors' lecture. These were both recorded with the same small HD camera. I partitioned each segment of class into separate video files and took field notes with regard to what the content of each of these files was. After each class these files were downloaded onto my computer and labeled so that particular sessions and activities could later be easily found. Four selected days of activities were then transcribed and analyzed using the VAP-framework. One day revolved around the ball-drop problem and the other three utilized the TIA for introducing various aspects of graphing derivative. Because the VAP-model was developed within my examination of interview data the coding scheme needed to be expanded slightly in order to perform some of the analyses here. Labels for individual speakers needed to be added to allow examination of how shifts in representational mode occur in a multi-person conversation. Additionally, since the class activities unlike the interviews, were conducted with the

use of technology, specifically the online applets, a technology code was created in order to document technology use within the data.

As discussed in the methodology section, it was difficult to not interact with the group during group-work. I often asked them what they were thinking when they got quiet as well as clarifying questions. The group often asked me mathematical questions. I often avoided answering these questions directly, since I was interested in being a participant observer more so than a participant. During multiple instances, in order to clarify my role to the focus group, I responded to their questions with instructions such as, “That’s a good question, why don’t you ask Dr. Byne that” or directed members of the group to ask each other. Such actions were intended to get the group to operate more like groups that I was not observing.

Although the transcription of the data followed a similar process to that of the interviews, the nature of the data meant that fewer gestures and inscriptions were caught on tape. It was often difficult to quickly redirect the camera at the hands and notes of the speaker. This information was included within the transcript whenever it was available within the video data. However, these descriptions were not as detailed as those found in the interviews because of the constraints of the video-recordings. Therefore, there are more neutral codes used in the focus group data than in the interview data; there was less gesture-based information to disambiguate the representational referents of particular verbal utterances.

Using the VAP-Model Within a Group Environment

Research on representation use in mathematics that focuses on individual students in interview or test environments has been largely disconnected from research which studies representation use in multi-person environments (e.g., Bishop, 1989; Hana & Sidoli, 2007; Presmeg, 2006; Roth & McGinn, 1998). This disconnect is highlighted by a quote from Presmeg's (2006) review of visualization research:

Effective pedagogy that can enhance the use and power of visualization in mathematics education is perhaps the most pressing research concern at this period: very few studies have addressed this topic since Presmeg (1991) reported the results of her study of classroom aspects that facilitate visualization. (p.227)

Contrary to Presmeg's claim a number of such studies have been conducted since 1991 (e.g., Cobb, 2002; diSessa, & Sherin, 2000; Meira, 1995, 1998; Roth, 2001). A major finding of this research is that discussion, refinement, and comparison of representations facilitates more thoughtful use of representations (both visual and analytic) within problem-solving. The exclusion of this research from Presmeg's review is a reflection of the disconnect between social focused and cognitive focused research on representation use.

One of the contributions of this dissertation is in linking representation studies in both social and individual settings. The link is facilitated by the creation of a model which can be used to interpret the role transitions between representations play in advancing the mathematical activity of both a group of students and individual students.

This chapter examines how students influence each other's representation use rather than focusing on how instructor actions and attitudes affect representation usage.

When using the VAP-model to study social settings the same basic premise that framed the model in an individual centered context applies; translations between modes of representation advance mathematical commognition. However, in a group setting there are multiple actors who communicate in turns. The translations between modes then happen both within a single actor's turn and between actor's turns. So, for example, one actor can communicate in an analytic mode and the responding actor can reason in the visual mode. This communication pattern constitutes a *between turn transition*. *Within turn transitions* occur when an actor transitions between modes within a single sequence of gestures and utterances that is uninterrupted by other actors. These within turn transitions are similar to those observed in the previous chapter, which had long sections of transcript involving a single actor. The VAP-diagram used to model a groups work on a problem or set of problems is similar to the diagram used in an individual setting. Both diagrams illustrate a path between modes that spirals up as mathematical reasoning advances. However, in order to account for multiple actors multiple tetrahedrons are used to represent each actor's contributions to a conversation.

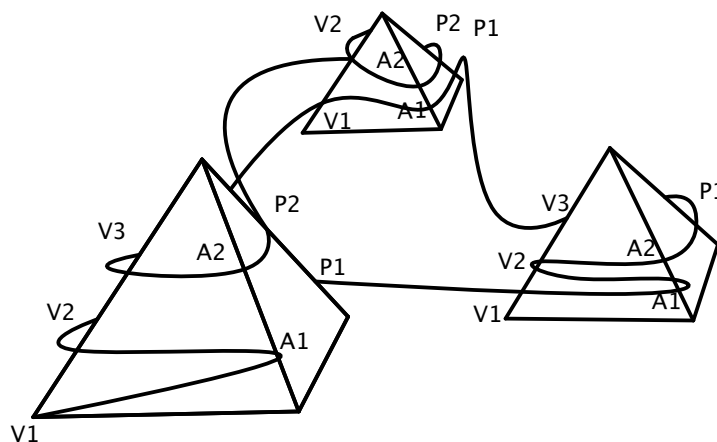


Figure 5.1: VAP Group diagram

The model provides a diagrammatic tool that can be used to keep track of representation use patterns in group settings. It also provides an approach to analyzing how representations get used within a conversation to advance mathematical reasoning. The model allows for transitions to occur both within as single actor's turn and between actors' turns, but does not assert which order or between which modes these transitions occur. This is similar to how the single actor VAP-diagram in the previous chapter helped illuminate how transitions between modes of representation facilitated the advancement of mathematical thinking, but not which modes were present or how they build off of each other.

The transitions between modes in a group setting, as well as how they inform the advancement of mathematical activity, are examined in this chapter. As a reminder, there are no actual discrete levels associated with the three representation modes. The levels in the diagram are a discrete representation of what is viewed as continuous mathematical advancement.

The transitions are operationalized by examining the sequence of codes that occur within the data. So if one actor's actions (gestures and utterances) are coded with one representation mode and the next actor's subsequent actions are coded with a different mode the resulting transition between modes is classified as a *between turn transition*. If either of the two actors' commognition was coded as neutral, meaning it was not possible to determine the representation used within their utterance, then the instance was not considered a between turn transition. So only transitions from one of the three modes in the model, visual, analytic and physical, to a different mode in the model was considered. Similarly, *within turn transitions* are operationalized in terms of changes in modes within a single actor's turn. In the cases where the representation used was unclear neutral codes were assigned. Cases where neutral codes were assigned were not considered as within turn transitions.

In addition to coding for representation mode and labeling the speaker, it is also important to keep track of the different roles participants played within a conversation. To facilitate the analysis of these roles I adopted an extension of Krummheuer's (2007, 2011) participation framework. This framework is reiterated in the next section.

Revisiting Krummheuer's Participation Framework.

Krummheuer's framework documents the different roles actors can play within a collective activity. The framework uses formulation and content as building-block constructs. As a reminder to the reader I operationalize content in terms of Toulmin's (1969) categorization of elements of an argument. Introducing new content to a collective activity involves adding one or more of these elements to an argument (i.e. data, claim,

backing, warrant). Formulation is operationalized in terms of both the VAP-model's change in representation and in terms of how content is treated within a conversation. So changes in representation mode are considered changes in formulation. Changes in formulation also occur when students themselves see two expressions of mathematical content as non-equivalent or when a difference in how something is expressed causes miscommunication among them.

Krummheuer classifies people's utterances in terms of whether or not the speaker is responsible for the content of an utterance and whether or not the speaker is responsible for the formulation of that utterance. These two binary criteria lead to four roles that a speaker can play within a conversation. An *author* is responsible for both the content and formulation of an utterance. A *relayer* is responsible for neither content nor formulation and simply revoices or validates existing mathematical content without changing its formulation. A *ghostee* is responsible for the content but not the formulation. So a ghostee expresses new content using a formulation present in a previous utterance. A ghostee role is not necessarily always intentional on the part of the actor. It may occur as a misinterpretation of existing content. Finally a *spokesman* is responsible for the formulation but not the content itself. So a spokesman adopts the idea of a preceding utterance and then tries to formulate this idea in a different way.

The VAP classifications and Krummheuer's classifications categorize different phenomena. The VAP classifications categorize representation and Krummheuer's classifications categorize social roles. In spite of these differing foci there is some overlap between the two. Part of this overlap is due to how I have operationalized the formulation construct. However, other operationalizations of the formulation construct would also

cause overlap. To clarify the overlap it is useful to discuss it in terms of one of Krummheuer's classifications of speaker roles. If a speaker formulates an idea in the visual mode that was previously expressed in an analytic mode she is both shifting between representation modes and acting as a spokesman when expressing this new formulation. That is not to say that all reformulations involve shifts in representation mode. One can shift formulations within a single mode. For example, moving from "point-wise" and "across time" approaches (Monk, 1992). So changes in formulations do not always involve changes in representation modes, but changes in representation mode are changes in formulation. However, Krummheuer's notion of content is independent of the VAP-framework since one can introduce new content with or without using the same representation mode that framed the previous content.

Within each of the four participant roles outlined by Krummheuer I add an additional dimension, which accounts for whether an utterance is a question or statement. I define a statement to be an utterance in which the speaker takes ownership of the expressed idea and its formulation and a question if the speaker does not take ownership of at least one of these aspects. Each of Krummheuer's four categories can be applied to both questions and statements. Questions, much like statements allow those participating in collective activity to influence the direction of a conversation. Questions allow participants to add new formulations or ideas to a collective activity without explicitly taking ownership of the expressed ideas/formulations. So for example a spokesman's question may re-express an idea in a new formulation but not directly take ownership of this formulation. Instead, the question may present this new formulation and prompt others to evaluate the legitimacy of it. In contrast, a spokesman's statement treats the new

formulation as legitimate and does not specifically prompt others to evaluate it. The distinction between a question and a statement is not always clear-cut. It is not always clear whether a speaker is taking ownership of the content and formulation of their utterance. In cases where the distinction between a question and statement are fairly unambiguous it is useful to distinguish between the two.

Finally, Krummheuer's framework categorizes non-speaker roles in relation to collective activity. His categories of eavesdropper, over-hearer and co-hearer, classify those present but not directly involved in the collective activity. That is not to say that those taking on non-speaker roles do not influence a group's commognition, only that their commognitive roles leave little in the way of observable data that can be used to parse this influence. These three categories differ in terms of how non-speakers are positioned relative to what is being said. Co-hearers are included in a conversation but not directly addressed, over-hearers are neither included nor excluded and eavesdroppers are excluded from a conversation but are still able to hear it.

Group Activities

This section describes the two group activities that are examined in this chapter. The ball drop problem and the TIA investigation occurred on consecutive weeks and were used to introduce the concept of derivative to the class. The class was also assigned online videos after the introduction of the ball drop problem. These videos introduced the computational aspects of derivative and focused on basic computation rules (e.g., quotient rule, chain rule, power rule) while accompanying group activities were intended to give students experience with these analytic tools.

Ball drop problem

The Ball drop problem centered on a video produced by the Michigan institute of technology physics department. The video involved a ball being dropped 1.75 meters above the ground against a background that had horizontal line markers located $\frac{1}{4}$ meters apart. It can be found at (<http://youtu.be/xQ4znShlK5A>). A strobe light that flashed 10 times per second and a long exposure camera was used to produce a picture of this event. The picture below illustrates the position of the ball at discrete points in time along its trajectory. Students were given a work sheet, which included a copy of this picture and the following group-work questions:



Ball drop questions:

- 1) What is the average rate at which the ball falls during the trip?
- 2) Is the ball falling faster during the first or second half of its (albeit short) journey?
- 3) What is the average rate of speed during each half of the journey?
- 4) What is the average rate of speed as the ball hits the ground?

Figure 5.2: Ball drop problem

Tangent intuition applet

The TIA was introduced the week after the ball drop problem. Students worked on tasks related to this applet for two consecutive class sessions and revisited it a month later when second derivative was introduced. The TIA activity began with Dr. Byne demonstrating the TIA on the function $y=x^2$. This activity was aimed at fostering connections between the derivative concept and its graphical interpretations as well as fostering intuitions regarding the relationship between a function and its derivative. During the introduction of the activity Dr. Byne emphasized this goal when she said:

*Alright what are we really doing when we take the derivative?
This is what you need to understand as opposed to just merely
do. It's easy. It's two x, There is no question about that. But
why? What does that mean?*

As a reminder to the reader the TIA involves a set of slope-widgets which are comprised of two vertically aligned points (See Figure 5.3). One point has a line segment through it, which acts as a tangent, and the slope of this line is controlled by the height of the other point. The tangent points are placed along the function and then the slope-control points are adjusted until the tangents line up with the slope of the function. These then are miniature instantiations of derivative at a point. Using several of the slope-widgets in conjunction with each other allows students to map out discrete points along the trajectory of the derivative function.

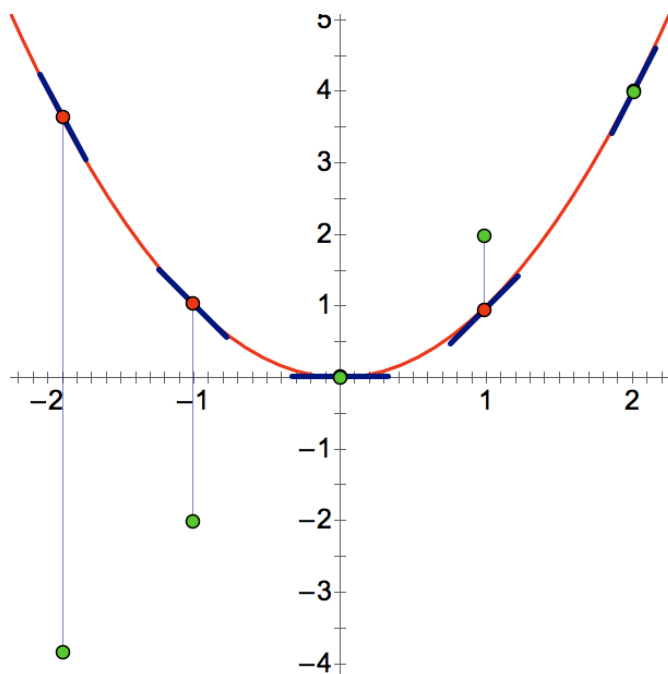


Figure 5.3: The tangent intuition applet

The specific problems given to students during their initial work with the TIA are presented in Figure 5.4. These were provided both on a work sheet and were also made available through a Geometers' sketchpad file. This file allowed students to work on the problems using the slope-widgets. Students were asked to both graph the derivative of each of the given functions and to find the equations of each of the derivatives. They were told that they could work in whatever order they wanted. Consequently, some students began by graphing all of the functions and others began by taking derivatives.

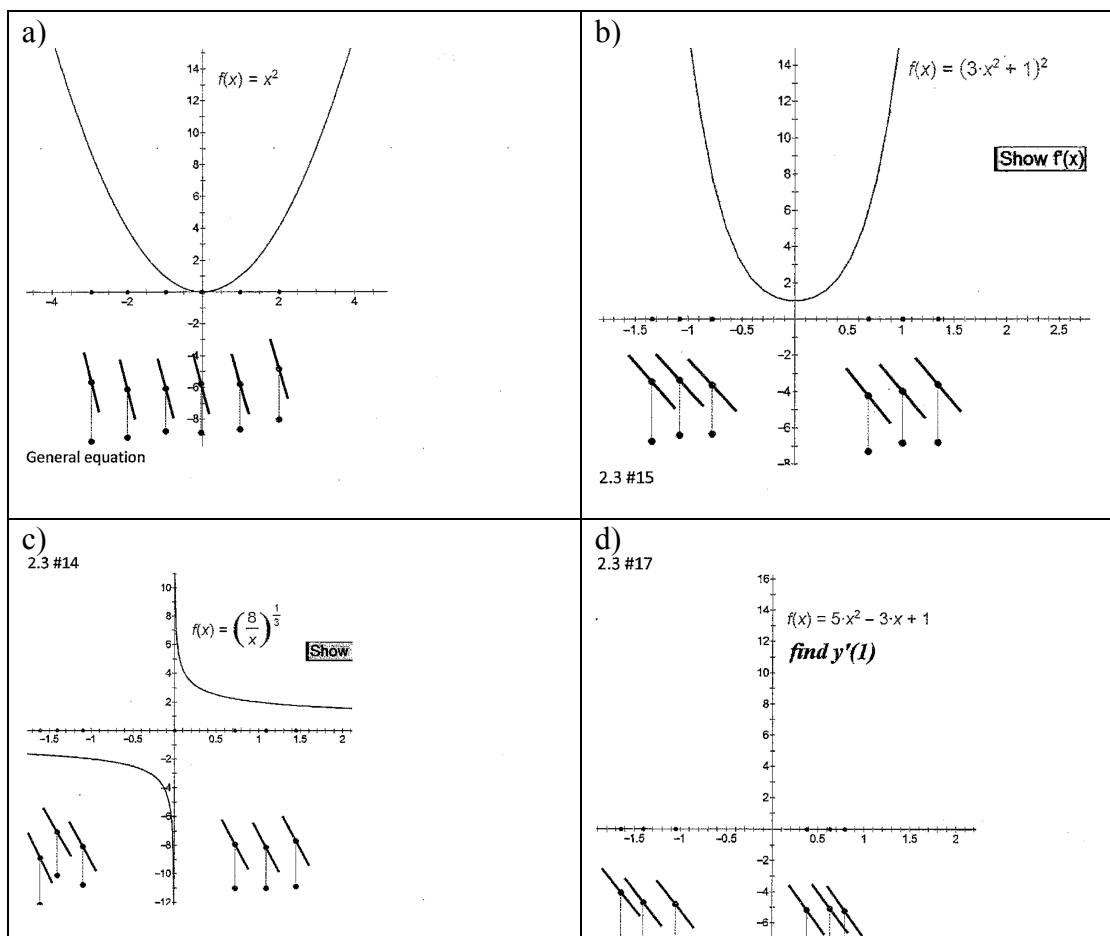
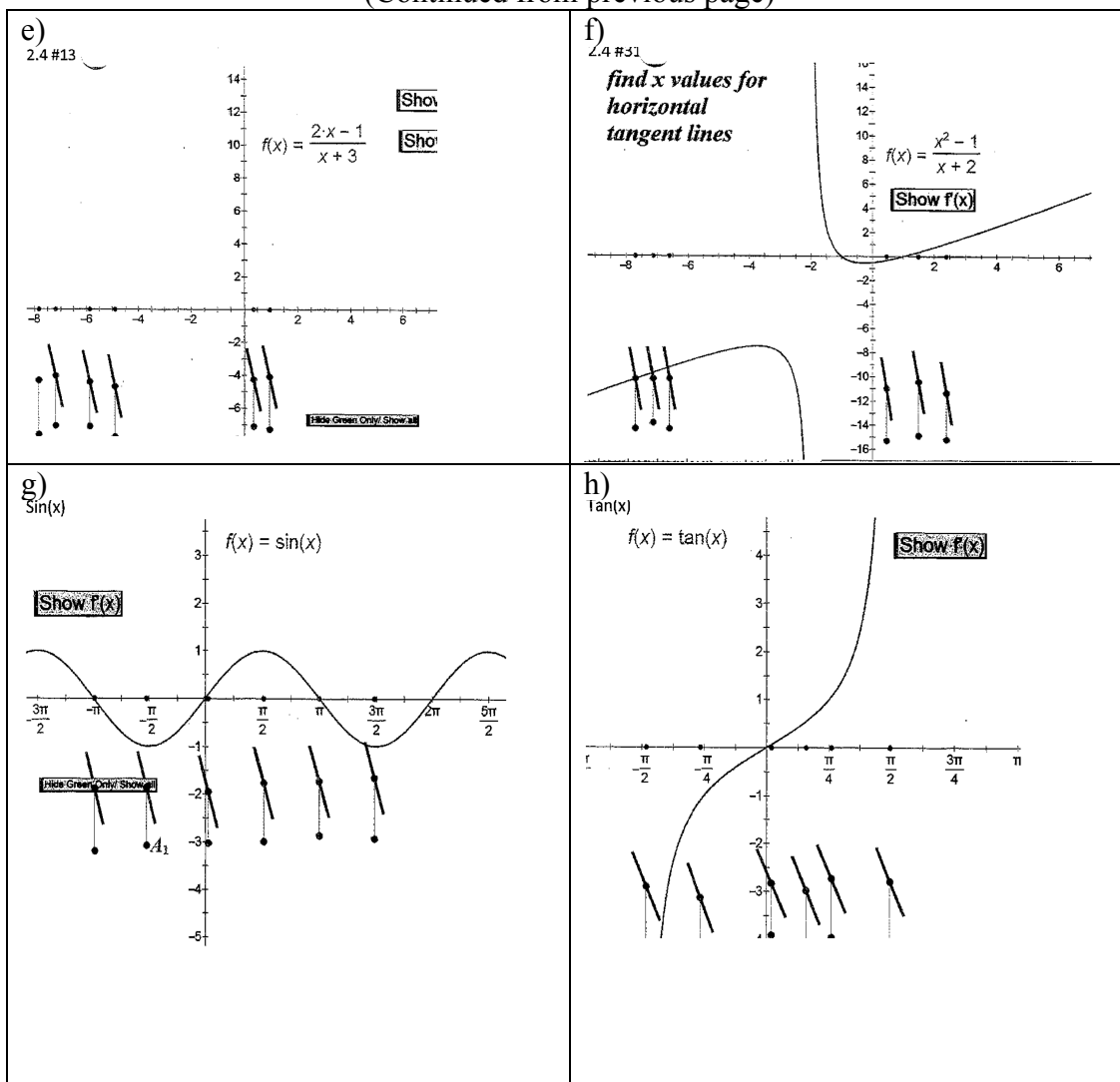


Figure 5.4: Tangent intuition applet tasks.

(Continued from previous page)



The TIA was also used to introduce the concept of second derivative later in the semester. Group-work episodes associated with second derivative are discussed in the subsequent chapter and compared to interview data involving second derivative.

Results

How students' use various representations of derivative within their problem-solving and how they translate between representations are an important aspect of how they communicate about derivative. Although both within turn and between turn transitions occurred in the group data, within turn transitions were more common. The data contained 28 within turn transitions between representational mode and only 3 between. So in general, a single actor's turn, which ends with the use of a particular representation mode, is typically followed by another actor's turn, which begins in that mode. The social norm that responses generally begin with the same representation as the utterances being responded to appears to be important for maintaining coherent communication between students. The occurrences of these types of transitions were split between Carson (11) and Ann (18). All but two of the transitions in the data involved the visual mode. So transitions both to and from analytic and physical reasoning usually involved moving from or to visual reasoning. Within Brad's communications only one within turn transition occurred. As determined by his grade in the course, Brad was the weakest of the three students so his lack of transitions between modes should not be surprising. Fluency with multiple representations of derivative is important for a rich understanding of calculus (Duval, 1999; Vinner, 1989; Zandieh, 2000) and the number of transitions between modes, which occur within students' communications about derivative, may be used as a very rough proxy for this fluency.

In the following I discuss examples of both between turn and within turn transitions. Then I discuss a third type of transition in which a switch between representational modes is buffered by several neutral turns, which are ambiguous with

respect to which representation is being used. Due to how formulation was operationalized in this study some shifts in formulation are not necessarily shifts in representation mode. I discuss an example of this phenomenon to illustrate what such changes in formulations look like within group-work data. The final results section attends to the role technology played within group-work and how this role differed among study participants.

Between Turn Transitions in Representation

As mentioned previously, between turn transitions between representations were rare in the group-work data. It is however useful to see an example of a between turn transition between modes. The following excerpt occurred when the group was working on the ball drop problem:

Excerpt 5.1:

Carson [00:00:00]: Then you have your rise over run formula [inaudible] times that and that's going to be negative 16 minus negative one which turns out to be negative 15. And then your run is four minus one which turns out to be three. [Has formula written down and points to each number in the calculation as he discusses it.] And then you divided it and have it, it turns out to be negative five. Does that make sense?

Ann [00:00:29]: So the slope is the average rate of change? [Gestures with her thumb toward a Cartesian plane in front of her] See I did not know that. I'm not connecting it all.

Notice here that Carson authors a calculation used for solving part one of the ball drop problem. His explanation involves the implementation of a particular formula. He discusses the specific symbolic mathematical objects used in the calculation and points to each one as it is mentioned in the discussion. This segment was thus coded as Carson using the analytic mode. If he were instead pointing at a graph while discussing each of

the quantities in his calculation, then much of that section would have been coded using a visual code. However, such pointing did not occur.

Ann responds to Carson by inquiring about the graphical interpretation of the calculation and gestures toward a Cartesian plane while doing so. This response was thus coded as her using the visual mode. Since Carson's analytic turn was followed by Ann's visual one the turn was deemed a between turn transition. Ann's question attempts to clarify the meaning of the idea that Carson authored. In taking this role she reformulates Carson's idea in terms of its graphical interpretation, and therefore takes a spokesman role. However, she does not take ownership of this formulation. Instead, she inquires about the legitimacy of her visual interpretation of Carson's statement.

When the ball drop data was expressed as a position vs. time graph Carson treated the connection between slope and the ball's speed as an implicit part of the underlying scenario. Ann on the other-hand was not cognizant of this connection until Carson explicitly used an analytic formula for measuring slope. Ann then needed to clarify if the relationship between visual, analytic and physical interpretations of the problem that she inferred from Carson's calculation was in fact what was grounding Carson's calculation. Although, Ann does not explicitly use the analytic mode when she says, "So the slope is the average rate of change?" – since she is trying to clarify Carson's analytic reasoning this link is implied.

Within Turn Transitions

As mentioned above, the within turn transitions occurred much more often than between turn transitions. Since these are the predominate means through which shifts between representation modes occurred it is useful to illustrate what these transitions look

like with an example. The following is taken from the second day of work on the TIA and illustrates an example of a within turn transition. Here Carson asks a question when working with sine (Figure 5.4g):

Excerpt 5.2:

[00:01:11]Carson: Okay, lets see, so this is the sine and you have to measure the slope at each point. The slope of up here is zero, slope of down there is zero {max and min} this is a decreasing slope so it's going to be decreasing up until here. except you don't find the amplitude of the slope [up and down hand motion along the y axis] how do you figure that out?

[00:01:42]Dov: Like, it's hard for you to figure out quite how steep it is?

[00:01:45]Carson: Yeah,

[00:01:47]Dov: Well if you were to draw yourself a mini-tangent and then you could probably figure out how much up and how much over your moving. But this is more an exercise in figuring out rough shapes. I mean if you are doing it on paper.

[00:02:06]Carson: Okay, so at the middle of line is where the highest velocity is [points to where the sin function crosses x-axis] so that's why you know that that's the highest point. Because it goes slow [points at minima] it goes fast [crosses axis] and then it slows down [maxima]. Awesome.

Carson begins in an author role expressing his approach to graphing the derivatives of functions in a visual mode. He introduces new content regarding the steepness of the function and adds in a physical formulation. Carson states and demonstrates his method, which he had used extensively the previous day. This method helps determine when the derivative is positive, negative and zero. He then switches from the author of a statement to the author of a question when he inquires about how to determine the specific amplitude of the derivative function and not just whether it is

above, below or on the x-axis. The response to Carson's query reformulates Carson's approach in terms of the TIA and how it can be used to determine the amount of slope, not just whether a slope is positive or negative. This response makes a reference to using the TIA; the applet was often referred to as the "mini-tangent applet" in class. However, much like Carson's original utterance, the response remains in a visual mode. Carson's response however does not make use of the TIA. Instead, he reframes the problem of graphing by relating the given function to a position function and reasons about the related velocity function. In much the same way he did in the individual interviews discussed in the previous chapter, he begins to move back and forth between visual reasoning and physical reasoning in order to graph the function. This turn includes a within turn transition between visual and physical reasoning. Instead of moving from, "this is a positive slope" to "how positive is this slope," which would have remained in the visual mode, he reformulates the slope as the velocity of some imagined moving particle and moves between visual and physical reasoning. Although both Brad and Ann played the roles of over-hearers during this episode, their commognition, as determined by their utterances and gestures both before and after the above episode remained in a visual mode. So Carson's within turn transition to the physical mode did not catalyze Brad and Ann's usage of this type of communication.

It is unclear exactly why Carson began to utilize physical reasoning in the above episode. I believe this usage may be linked to the fact that the concept of derivative was originally introduced in this context and his approach was inspired by his work with the ball drop task. In Carson's work on the ball drop task he was able to draw connections between the given physical situation and the slope formula. So his use of physical

reasoning here may be a natural extension of his approach to the ball drop problem, which occurred a week earlier. As can be seen in the episode in the previous section, Ann did not make these same connections between visual, analytic and physical representations during the ball drop task. This lack of connection making can be inferred from her utterance “See I did not know that. I'm not connecting it all.” Her lack of connection making may be contributing to her not exhibiting the same representation behavior as Carson.

Transitions That Occur After Neutral Codes

Commognition that was coded as neutral caused particular instances within the transcript to not be coded as between or within turn transitions, these instances nevertheless played important roles in facilitating transitions within an overall conversation. The following episode occurred on the first day of work on the TIA and is an example of this phenomenon:

Excerpt 5.3:

[00:15:31]Carson: slope is zero here. Zero here and zero here [moving along sin and pointing at the maxima and minima] zero here. This is decreasing, increasing, decreasing, [running pen along function saying increasing when slope is positive and decreasing when negative] No wait. Decreasing increasing.... Yea so that's why I thought [sketches a drawing that looks like $1/6 \cdot \cos(x)$].

[00:16:34]Brad: Isn't this just negative cosine [pointing at C's graph]

[00:16:44]Carson: No, or is that [inaudible]. Yeah it actually is.

[00:16:51]Brad: The derivative of cosine, which is sine.

[00:16:56]Dov: But they gave you sine.

[00:17:01]Brad: The derivative of sine is negative cosine [writes $\sin(x) \rightarrow -\cos(x)$].

[00:17:10]Carson: Yeah I think it's negative cosine... Well it's positive [pointing to Brad's notation] .

[00:17:26]Dov: Negative cosine goes through negative one.

[00:17:32]Carson: Cosine is negative sine and ... is that right?

[00:17:35]Brad: Is cosine negative sine and sine positive cosine?[writes $-\cos(x) \rightarrow \sin(x)$ and changes the $-$ in his previous notation to a $+$]

[00:17:43]Carson: Yeah, Alright tangent is one over cosine squared [writes $\tan(x) \rightarrow 1/\cos^2(x)$]

Notice that in the beginning of the transcript Carson takes the role of author and is reasoning in the visual mode. He reasons about the slope of the given graph of a $\sin(x)$ function and sketches a derivative graph. He is attentive to where the graph has zero slope, where it is increasing and where it is decreasing, but does not explicitly focus on the magnitude of the positive and negative slopes. This focus leads him to sketch a derivative graph that has the correct basic shape but not the correct amplitude. Brad responds in a spokesman role, when he recasts Carson's sketch as negative cosine. The phrase "isn't this" indicates that Brad sees his utterance as recasting Carson's work. He is responsible for the reformulation of the sketch as negative cosine (form) but not the sketch itself (content). This reformulation recasts the sketch as a known mathematical object, rather than just a sketch. It is unclear whether Brad made this connection because of the shape of Carson's sketch, because of the label $y=\sin(x)$ on the original graph, or a combination of the two.

Regardless of what facilitated Brad connecting Carson's sketch to cosine, notice that Brad phrases his contribution as a question, rather than a statement, and in doing so

seeks Carson's evaluation of his idea. Brad's utterance was coded as neutral. Negative cosine can refer to both a visual (graphical) object and an analytic one. It is not clear whether Brad is using a visual or analytic mode of representation to facilitate his commognition in this case. Although, Brad's pointing gesture can be clearly seen in the video recording of the episode his gesture does not disambiguate the representation used. Carson's response, "Yeah it actually is," is also neutral for the same reasons. This response validates Brad's commognition but does not change its form or content. Carson is hence functioning as a relayer.

Up until this point I have been functioning as an over-hearer. I was privy to the happenings of the mathematical activity but was not addressed, or expected to contribute. In stating, "But they gave you sine," I took a ghostee role, keeping Carson and Brad's formulation but highlighting the original content of the problem. I kept the response neutral with respect to representation mode. Since the work sheet had both the graph of sine and the notation " $y=\sin(x)$ " it was not clear which of the two was referenced by my statement.

Carson and Brad then switch to the analytic mode and attempted to recall the derivative rules they had learned the previous week. What is noteworthy here is that the commognition switched from visual, when Carson was reasoning about the graph of $\sin(x)$, to neutral, when the resulting objects were labeled as negative cosine. The commognition subsequently switched to the analytic mode when Carson and Brad began to frame the problem in terms of memorized analytic rules. This switch to analytic reasoning occurred after several neutral turns. Although a new formulation was used, it may be the case that Carson and Brad did not have a specific representation in mind

during these neutral turns, or were simultaneously referencing both visual and analytic representations. A transition in what representation was used occurred. However, this switch was not labeled as a within turn or between turn transition, only a change in formulation. It may be the case that the neutral turns facilitated the conversation's transition between modes or that Brad and Carson were reasoning in a particular mode that was just not observable within the data.

In either case the transition to analytic reasoning was not solely tied to a single task. In the last line of the transcript Carson begins to reason about the next problem on the work sheet, $\tan(x)$, in the analytic mode (Figure 5.4h). In the several tasks they had worked on previously Carson and Brad's commognition was coded as either neutral or visual. The subsequent several minutes were coded as either analytic or neutral. So this episode represents a shift in Carson and Brad's overall approach to the tasks and is not isolated solely to work with $\sin(x)$.

It is also interesting to note that on the subsequent day of class at the beginning of group-work Carson said "Well we did it yesterday. You basically just memorize the formula." The utterance was an indication that most of the previous day had been about working with rules for taking derivatives and not necessarily about graphing functions. Carson's perspective is interesting given that visual reasoning codes far outnumbered analytic reasoning codes during the analysis of the previous day. It may be the case that even though Carson spent most of the previous day working with graphs in a visual mode he still saw his activity as being about building fluency and intuition about analytic notation as its end goal. The work of Vinner (1989) and others had tied such a tendency to overall resistance to visual reasoning. Carson's statement may be related to this

tendency. Here the resistance to visual reasoning is no longer prevalent, but the bulk of mathematical activity is still viewed as targeting analytic mathematical reasoning. Counter intuitively, Presmeg (1991) observed that instructors who treated visual reasoning as something that could be dispensed after it had served its initial purpose, were more effective at promoting visual reasoning than instructors who continued to incorporate such reasoning throughout their instruction. It is thus interesting to observe a compatible attitude toward visual reasoning within Carson's work.

Commognition Within A Single Mode

The previous section focused on transitions between representation modes. However, there were large sections of transcript in which students used only a single mode to facilitate their commognition. As discussed above, it is important to note that staying within one representation mode does not mean that the formulation used remains constant. The previously discussed episode was an example, however in it the conversation eventually transitioned between representations.

Below is an episode that illustrates a change in formulation that is not a change in representation mode. In it Brad and Ann discuss graphing derivative function within a visual mode. The episode occurred during the first day of work with the TIA. Ann is inquiring how Brad and Carson manage to sketch the derivative graphs much quicker than she does. At the time this episode started she had been struggling with the tasks for several minutes while Carson and Brad worked without her. Carson and Brad had worked for about a minute with the TIA at the beginning of the session and had switched to reasoning about graphs of derivatives without the applet. It took a particularly long time

to line up each slope-widget with the given graph, so they had abandoned the applet in favor of pencil and paper techniques.

Before the beginning of the episode Ann had been struggling with making connections between the slope of a function and its derivative. She had not used the applet at all and had instead attempted to solve the given problems without first working on the applet. In her work she repeatedly kept trying to relate the slope of the original function to the slope of the derivative, which is a common error that has been well documented in the literature (e.g., Apsinwall & Shaw, 2002; Kung & Speer, in press; Nemirovsky & Rubin, 1992; Orton, 1983). However, Ann had resisted using the applet stating that she would not be allowed to use it during exams and wanted to learn how to graph derivatives on her own. The episode below begins with a discussion of $y=(8/x)^{1/3}$ (Figure 5.4c) and then shifts to a discussion of $\tan(x)$ (Figure 5.4h).

Excerpt 5.4:

[00:11:48]Ann: Um Yeah, but then now I'm kind of confused as to how you guys are just like getting it right away. You know without solving.

[00:11:55]Brad: For those ones they are both going to zero [points at both the negative and positive ends of the $y=(8/x)^{1/3}$ function].

[00:11:59]Ann: But why how do you know that?

[00:12:05]Brad: It's because of the slopes. Because everyone's sloping...

[00:12:16]Ann: But then like for this one.

[00:12:20]Brad: For which one?

[00:12:23]Ann: For this like how it's that way [$\tan(x)$]. Like how would you know?

[00:12:27]Brad: See that one is like... Okay I might have got that wrong [presses show function on iPad]... so. Okay so it's sloping down

like the rate of the slope is decreasing, like that's why you have a parabola shape, so it picks back up when the slope increases. So the slope is greater down here but it's coming down to its lowest sloping point and then it goes back up as it hits the steeper part.

[00:13:27]Ann: This doesn't look like a negative slope to me [points at negative slope on the $\sec^2(x)$ graph]. You know it's going like that [runs pen along $\tan(x)$]. So it's.

[00:13:36]Brad: Well the slope is decreasing, that's what this shows [runs pen along first half of $\sec^2(x)$] the rate of the slope is decreasing. Then the rate of the slope increases [runs pen along second half of $\sec^2(x)$]. Does that make sense?

[00:13:47]Ann: Yeah. No Yeah, I think I'm still confused.

[00:13:50]Brad: You just got to keep telling yourself the tangent is the rate of the slope.

In the above episode Ann is the author of a question with regard to how Brad and Carson are connecting graphs of functions to their derivatives. Brad responds by taking an author role and demonstrating the process, albeit briefly, on a function that he and Carson have already graphed. He points to the negative end and the positive end of the x-axis and highlights that the slopes are close to zero at these points, which is why the derivative of the function is close to zero. Ann asks an additional question to redirect the conversation to a function she was struggling with, $y=\tan(x)$, and its derivative function, $y=\sec^2(x)$. Brad demonstrates the same technique on this problem. He reasons that the slope is positive, then it becomes less positive and then more positive again. He takes the relationship between the slope and the parabola-like shape of the derivative function, which has large positive values at both its negative and positive ends and small positive values, as understood. Ann however, does not make the connection between the slope of the function and the magnitude of its derivative. Instead, she points at the negative branch

of the derivative function, and says, “this does not look like a negative slope to me”. Her utterance indicates her assumption that a negative sloping derivative function should correspond to a negative sloping parent function. Brad’s subsequent reiteration of his commognition does not address Ann’s underlying issue. She repeatedly attempts to take on a relayer role in order to make sense of Brad’s commognition, but is unable to formulate Brad’s commognition in a way that is consistent with her conceptions of the problem.

What is interesting here is that even though they are reasoning using the same representational mode they are attending to different things. Brad attends to the slope of the original function and relates the slope to the height of the derivative function, however he does not use the word, height. He keeps referring to the slope of the original function but does not explicitly mention which attribute of the derivative function this slope corresponds to. Ann attends only to the slope and is thus not able to parse how a large positive slope in the original function corresponds to a segment with a large negative slope in the derivative function. She is framing her commognition in a different way than Brad. Her framing attends to only one attribute, slope, while Brad’s framing coordinates slope and magnitude. Their use of compatible terminology is masking that they are attending to different things. Consequently, Ann is not able to parse Brad’s communication and Brad is unable to parse Ann’s, but they both appear unaware of why they do not understand each other. In line with the operationalization of formulation this difference in what they attended to was coded as a difference in formulation.

Also it is worth noting that even though in the episode Ann and Brad were discussing $\tan(x)$ and $\sec^2(x)$, Brad identifies the shape of $\sec^2(x)$ as parabolic. It is

unclear whether Brad did not notice which function he was actually looking at or whether he made a connection to a similarly shaped function. However, it is worth mentioning that when working with strictly analytic representations making connections between functions with similar attributes in terms of magnitude is much more difficult. His use of the visual mode, regardless of whether the connection to parabolas was intentional or not, facilitated his discussing $\tan(x)$ in a similar way to x^3+x .

Additionally, notice that during the interaction, Brad briefly assumes he may have erred in his solution and uses the “show function” button on the applet to verify that he has indeed produced the correct graph. It is unclear whether Brad’s momentary doubt regarding his solution stems from his communication with Ann or is centered on other factors. After Brad verifies the correctness of his solution he chooses to use the paper image of the function and its derivative, which he had already produced, rather than the compatible image produced by the applet. The usage of the applet during this activity is discussed further in the subsequent section.

Use Of The Tangent Intuition Applet

The use of a technology code when examining the data allowed me to keep track of when, how and how often the three participants used or referred to technology. Although, the TIA activity was constructed around the applet and intended to help students build intuitions for how graphs of functions and graphs of their derivatives relate to each other, the applet itself was not a core part of the group's interactions. Only 46 of the 418 student turns coded in the TIA transcripts (11%) involved some reference to or usage of technology. Here I discuss some of the overall patterns in technology usage

during this session and examine particular instances where technology was used, one of which appeared in the previous section (Excerpt 5.4).

For Carson and Brad, the applet served as a verification platform. They worked with it briefly at the beginning of group-work and then checked back periodically after having graphed several tasks by hand to evaluate the correctness of their work. This approach made little use of the slope-widgets themselves. Instead the “show function” button was used to display the graph of the derivative. This type of usage of the applet can be seen in the last transcript discussed in the previous section, where Brad verified that his answer was correct before discussing how he arrived at it using pencil and paper.

In that transcript Brad attempts to explain the approach to graphing derivatives that he and Carson had adopted during group-work. Brad’s explanation centers on the slope of the function as he moves along it. His commognition utilizes words such as increasing and decreasing as he runs his pen along the original function, which is indicative of what Monk (1992) refers to as an across time interpretation of function. That is, he is treating the function as something that continually changes as he moves along it. The applet, in contrast, evaluates the slope of the function at discrete points. Thus, it corresponds with what Monk refers to as a point-wise interpretation. During Brad’s explanation he chooses not to implement the TIA or its widgets. There are a number of possible explanations for this non-usage. It may be the case that he at no point thought of the tasks in terms of the applet. It may be the case that he began with a point-wise conception and abandoned the applet as he moved to an across time conception. Or it may be the case that the gestures he used on paper during his explanation would have affected the touchscreen’s display and interfered with his explanation. Regardless of

which of these explanations we adopt, it is notable that the slope-widgets were not a theme within Brads communication. This non-usage can be interpreted as an indication that the slope-widgets were not, at that point in time, a part of his commognition about graphing derivative.

Ann, in contrast to Brad and Carson, did not use the applet at the beginning of the session. She instead attempted to work directly with graphs using pencil and paper techniques. She began interacting with the applet 20 minutes after the group problem-solving session began. This interaction occurred after several unsuccessful attempts to get Brad and Carson to explain their approach to her and a little bit of encouragement from my end. When she did begin to interact with the TIA, she did not initially use the slope-widgets, but instead kept reasoning in ways similar to her commognition at the beginning of the session and only used the applet to verify whether her intuitions were correct, which they often were not. This usage is similar to the usage of the applet demonstrated by Brad, when he attempted to explain his reasoning.

After several minutes of unsuccessful attempts Ann began using the slope-widgets in the applet. The following is a transcript taken from her use of the applet to reason about sine (Figure 5.4g). Carson and Brad had already worked this problem 15 minutes earlier and were eavesdroppers through out this episode and were therefore not directly involved in her work on it.

Excerpt 5.5:

[00:31:45]Dov: It's kind of a tough one because it has a lot of waves.

[00:31:47]Ann: Yeah but I mean, I think I'm starting to like. Okay, I would probably find the graph most likely. Because remember yesterday when we were talking about slopes and it was making all of those lines and you would just connect them and that would be the

graph of the slope. And right now I'm pretty much trying to find like off of the slope. Aww what did I do? So then isn't that kind of what I'm trying to do? So then from this I would [places slope-widgets on sin graph at max min and zeros] think the slope went the same thing [tracing out cosine shape with her pen through the green points from right to left with her pen]?

[00:32:32]Dov: Well the green dots and the red aren't lining up.

[00:32:36]Ann: So the slope is like this [makes an angular up-down-up-down right to left motion with her pen through the green dots and then back the other way]. Like this [makes the same angular motion through the point from left to right and then from right to left]

[00:32:44]Dov: Do you think it's like really angular like you are doing it?

[00:32:46]Ann: Like lines?

[00:32:48]Dov: No, Yeah like when you are doing it you did this. [imitates Ann's hand motions.]

[00:32:56]Ann: Oh no, I did each of these it's like lines [repeats hand motions]

In the above segment Ann uses the slope-widgets on maxima, minima and zeros of the sine function. She reasons in a visual mode using her hand to trace an angular saw tooth shape through the green slope control points that sit on the derivative function. She has not yet hit the show-function button and therefore has not verified the exact shape of the derivative. Since she is taking responsibility for the formulation and content of the solution she is taking an author role. I take on the role of relayer in mimicking her hand motions with the same formulation and representation mode. However, I am asking a question in this relayer role. In doing so I am attempting to verify that the angular nature of her tracing is an intentional part of her commognition. Ann responds by verifying my conjecture.

In the above segment Ann uses the TIA to establish the location of several points along the derivative function. She uses a point-wise interpretation of function, because she is not treating these discrete points along the derivative's trajectory as quantities that vary with time. In doing so she is taking a connect-the-dots strategy where she simply takes a linear path through the points on the derivative function and assumes that it corresponds to the overall shape of it. The derivative function is however not angular, like her hand motion. Her assumption that it is angular is based around her limited interactions with the applet and the instruction she had received before group-work began. She explicitly makes reference to the instruction and the connect the dots strategy when she says “Because remember yesterday when we were talking about slopes and it was making all of those lines and you would just connect them and that would be the graph of the slope.” Although she uses the word “yesterday”, the previous day's class was not about graphing functions and she is likely referencing the instruction she received at the beginning of the session earlier in the day. Her connect-the-dots interpretation may be linked to the example of x^2 that Dr. Byne chose to use to demonstrate the applet. Since the derivative of x^2 is $2x$, a straight line, the connect-the-dot strategy may have been falsely inferred from it. Later in the transcript Ann attempts to verify her work by moving the slope-widgets closer to one another. This action helps her verify that the graph of the derivative of sine is less angular than she had originally assumed.

Discussion

In this chapter I examined students' group-work on the ball drop problem and graphing tasks associated with the Tangent Intuition Applet. I used the VAP-framework

to classify students' representation use and Krummherer's (2007, 2011) participation framework to classify the roles played during these collective activities. Additionally, keeping track of when technology was used during these sessions, facilitated tracking patterns and differences in technology use and how these patterns related to students' commognition.

Shifts in representation use within a collective activity can occur both within a single turn and between turns. In this data set within turn transitions were far more common than between turn transitions. I discussed examples of both types of transitions and observed that almost all such transitions involved either shifting to or from visual reasoning. So visual reasoning played a central role when linking to other modes of reasoning.

The between turn transitions found in this data set were initiated by spokesman's questions. That is, each between turn transition was phrased as a question which sought an author's feedback regarding whether his/her expressed idea could be reformulated or interpreted using a different mode. The within turn transitions, which were much more common, varied with respect to the participatory roles taken on by the students. In some cases the representation used within a whole discussion shifted as a result of these within turn transitions. In other cases, such as Carson's within turn transition to the physical mode (Excerpt 5.2), the shift in representation had no observable influence on the representation usage of the other members of the group. So in general student commognition remained in a single mode until within turn transitions initiated shifts of a representation mode the group was using. These within turn transitions did not always cause shifts in which representation other participants used. However, transitions in what

representations were used to facilitate communications rarely occurred without them. In the two cases where a shift in representation use was not initiated by within turn transitions, the between turn transitions occurred took the form of spokesman's utterances. That is, the formulation changed but not the content. So in both between turn transitions and within turn transitions some part of the previous utterance, either the representation used or the content, was present at the beginning of the responder's utterance. This social norm appears to be important for facilitating group communication. Without it shifts in both what is being talked about (content) and how it is being talked about (representation/formulation) would occur simultaneously. Such abrupt shifts would disconnect what is communicated in a particular turn and the response to that communication.

Several authors have mentioned that social factors influence which representations get used within student solutions, but not explicitly provided detail regarding these factors (Dreyfus, 1991; Presmeg, 2006). Research that has focused on social factors influence representations has tended to examine how particular representations, such as Cartesian graphs, acquire meaning through collective activity and use, rather than focusing on connections between modes (e.g., Roth, 2001, 2009). The observations in this chapter help clarify how transitions between representations occur within group-work settings. Namely, the primary means through which a student can shift the representation used to facilitate a discussion is to begin to use this representation within their utterances in order to prompt their group mates to do the same.

Another phenomenon that was documented in this chapter's analysis is the important role of the communication that is ambiguous with respect to representational

referent. This type of communication can facilitate transitions in representation mode usage. Thus, not all shifts in which representation is used within a collective activity are between or within turn transitions. In an episode involving Brad and Carson discussed above (Excerpt 5.3), their transition from approaching tasks visually to approaching them analytically occurred after several turns that were coded as neutral. In these turns the representational ambiguity of mathematical terms such as “sine” and “cosine” made it difficult to determine if Brad and Carson were identifying a sketched graph as a known graphical (visual) entity or reasoning in an analytic mode. It very well may be that neither is the case and the representational ambiguity allowed them to reference both modes simultaneously or that one of them was reasoning in a different mode from the other, without them being aware of it. In any case, Carson and Brad’s transition to the analytic mode, which continues through onto the next problem, occurred after several turns that were coded as neutral. So transitions between representational modes do not always occur in response to within turn transitions. They can occur through representationally ambiguous utterances rather than within turn transitions. This phenomenon is further explored in the next chapter, which compares Brad and Carson’s work on sine with the TIA to their individual interview work on similar tasks. This subsequent chapter shows that when working on his own Brad uses analytic reasoning to approach periodic functions. This pattern in Brad’s individual work supports the assumption the sine group-work episode discussed in this chapter was an example of Brad intentionally moving the conversation in an analytic direction without explicitly using analytic phrases to steer the communication.

It was also evident that even though Ann, Brad and Carson received the same instruction and had the same exposure to technology during class, the ways in which they utilized the TIA were disparate. Brad and Carson used the applet as a verification platform and made little use of the slope-widgets. Their commognition treated derivatives as functions that continually changed across time, and thus the applet, which plots discrete points of the derivative function, may have been incompatible with the ways in which they were approaching derivative.

Ann, in contrast, resisted using the applet for a prolonged period of time and only adopted its use after several failed attempts with other methods, which included attempting to get Brad and Carson to elucidate their approach. Ann's thinking was centered in a point-wise interpretation of derivative and was hence more compatible with the applet. When she finally did use the applet she carefully used the slope-widgets to plot points along the derivative function. However, she had falsely inferred a connect-the-dot strategy from her instruction and thus sketched functions that were more angular than the actual derivative functions.

The usage of Krummherer's (2007, 2011) framework, and in particular his formulation construct, helped highlight shifts other than shifts in representation mode. This construct was operationalized here in terms of both changes in representational mode and changes in how content was expressed. The episode where Brad attempted to explain his strategy is an example of a shift in formulation that was not a shift in representation mode (Excerpt 5.4). Krummherer's framework was able to highlight that Ann was using point-wise language and focusing only on slope, while Brad was using across time language and was coordinating both slope and amplitude. As was shown in

the analysis of this episode, this shift in formulation unlike shifts in representation was harder for both Ann and Brad to notice. Ann and Brad functioned as if they were talking in compatible ways, and in fact used overlapping terminology, but neither was able to parse the other's commognition. Unlike shifts in representation usage, which refer to different mathematical objects and are hence more straightforward for participants in collective activity to notice, certain kinds of shifts in formulation may go completely unnoticed.

CHAPTER 6:

Cross-Setting Comparisons of Representation Usage

The previous two chapters examined students' commognition in one-on-one interview and group problem-solving settings. This chapter compares similarities and differences across these two settings in order to gain a richer picture of students' commognition and address the third research question, "Within a calculus context, in what ways do students' thinking in individual interview settings and their communication in group settings inform each other?" This question is addressed in two ways. First, the results of the previous two chapters are compared to illustrate the similarities and links between representational behaviors in both settings. Then, in order to delve deeper into the connections between these two settings, episodes that deal with compatible content in both settings are examined and compared sequentially. In order to facilitate this sequential comparison of group and interview data I introduce a pair of interview tasks involving periodic functions that were not addressed in the previous chapters. Students' work on these tasks is then compared to their group-work on related tasks, focusing on similarities in social roles across the two settings. Chapter seven summarizes the findings of the analysis chapters, discusses some of the implications of these findings and discusses future research.

Comparing the Use of Representations in Group and Interview Settings

Students used multiple modes of representation in both the group-work and interview settings and in both settings transitioned between the modes they used during problem-solving. Unlike many studies conducted about 20 years ago, which found that students using multiple representations (harmonic reasoners) were rare, this study is inline with more recent work, which found that modern students often use multiple representations within their problem-solving and often transition between modes during work on a single problem (George; 1999; Häikiöniemi, 2006; Stylianou, 2001; Stylianou & Silver, 2004).

The study adds to what is known about how and how often transitions between modes of reasoning serve to advance mathematical activity. There were patterns in both settings with respect to which modes played a central role. In the interview data all students used visual reasoning on all problems regardless of whether those problems prompted for the use of visual reasoning in some way. Of the observed transitions between modes in the interview data 84% involved the visual mode. Within the group-work data a compatible pattern was observed; the vast majority of both between turn and within turn transitions involved visual reasoning (93%). In other words, within both the interview and group-work data transitions from physical to analytic reasoning, or vice versa, were less common.. So visual reasoning played a central role within both data sets.

Visual reasoning often played a mediating role in students' commognition. An illustrative example of this role occurred in Ann's work on the Tree Task (Figure 4.3). The task described a physical situation and provided information about that situation in the form of analytic expressions that students were asked to interpret in terms of the

physical situation. In the episode (Excerpt 4.8) described in detail in Chapter four Ann asks if she is supposed to draw a graph to solve the problem, she is told she does not, and then she proceeds to draw one anyway. Although the tree task itself prompted for a transition between analytic and physical representations, Ann used visual reasoning to mediate this transition as opposed to transitioning directly from analytic to physical reasoning. Similar uses of visual reasoning to mediate between other modes of reasoning were documented in the work of both Brad and Carson. In many of these episodes visual reasoning was used to qualitatively assess the problem before formulating a solution. These observations are consistent with those of Hähkiöniemi (2006) who noted that, “increase, steepness, horizontalness and tangent representations seemed to be those tools that the students used for thinking about the derivative qualitatively without calculating anything. Students used these tools literally for thinking and not just for carrying out algorithms.” (p. 76). The mediating role played by visual reasoning helps explain the dominance of the visual mode within representational transitions.

Relating these results back to the VAP-model, the transitions between the visual edge of the diagram to the other two edges appear, at least within the data in this study, to be much richer and more numerous than the links between the analytic and physical edges. Additionally, in situations where a student uses both analytic and physical reasoning, it is common for transitions between these two modes to be mediated through the visual edge of the diagram.

There were also patterns in occurrence and frequency of different modes. Within the group data Brad rarely transitions between modes. This tendency was also present within the interview data. Brad used fewer representation modes within his problem-

solving than Ann and Carson. For example, when solving visual tasks within the interview it was common for him to approach visual tasks using only the visual mode, where Ann and Carson commonly used physical reasoning to solve such tasks. Brad's work on periodic functions – described later in this chapter – is a notable exception to his avoidance of multiple modes of reasoning; there he used analytic reasoning. The VAP-model contends that transitions between representations facilitate mathematical advancement. Since Brad was the weakest of the three students, it should not be surprising that he exhibited fewer transitions between modes than the other two participants.

Within the group-work Carson was the only student who made use of physical reasoning while working on the TIA. This activity did not prompt for usage of physical reasoning since each of the functions in the activity was presented in equation (analytic) and graphical (visual) form, and the solutions required were graphs and equations. Ann made use of physical reasoning during her interviews, but this usage was not as extensive as Carson's. Carson's use of the physical mode was prevalent within the interviews. Unlike Brad and Ann, Carson made use of physical reasoning when working on every task in the interviews. This reasoning was tied specifically to the position/velocity/acceleration scenario. In research literature a variety of studies consider tasks that require translating between physical scenarios and equations or graphs (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Roth, 2001; Zandieh & Knapp, 2006). In these studies the physical scenario is predetermined by the task itself. These studies have documented the use of visual and analytic reasoning to make sense of these given physical scenarios. What is documented here in Carson's work and less prevalent in the

literature is the opposite phenomenon. That is, Carson uses reasoning about physical scenarios he himself introduces to inform his thinking about mathematical contexts, rather than using visual/analytic reasoning to inform thinking about a problem-prescribed physical scenario. Although such phenomena, where students introduce physical scenarios to aide their reasoning about mathematical tasks, have been reported within other mathematical subject matter, such as linear algebra (Wawro et al., 2011, 2013) and group theory (Larsen, 2013; Larsen, Johnson, & Bartlo, 2013), I am unaware of similar phenomena being documented in calculus contexts. Carson's tendency to reason about derivative using the position/velocity/acceleration context is so predominant that he was observed translating other physical contexts into position/velocity/acceleration contexts during his solution process. Since Zandieh's (2000) framework of what the mathematical community considers part of the derivative concept includes the physical position/velocity/acceleration scenario, it makes sense for students to use this scenario to inform thinking about other aspects of the derivative concept, such as its graphical (visual) and analytic interpretations. The data in this study help illuminate what physical reasoning looks like in a calculus setting.

Ann was the only one of the three students to use the slope-widgets in the Tangent Intuition Applet (TIA) when working in class. Both Brad and Carson used the show function attributes of the applet to verify their answers, but did not use the slope-widgets to reason about any of the TIA tasks. Consequently, Ann actually made reference to the applet during her interview when working with graphing tasks. She also used a methodology that mimicked the TIA's slope-widgets during her interviews, which were conducted in a technology-free environment. Her method involved drawing a series of

small tangents along a function and then reasoning about their slope. Both Brad and Carson's approaches involved across time interpretations to function, while Ann's methods often utilized point-wise interpretations.

Although every student in this study used multiple modes of representation and transitioned between modes of representation within their thinking in both group-work and interview settings, their patterns of representation usage were very different. These differences include dissimilar patterns in how often particular modes of representation were used within their reasoning, which modes were used for which problems and even the nature of reasoning with particular representations. These differences occurred even though they worked together throughout the semester and were all exposed to the same classroom milieu. Some of these differences can be attributed to differences in how the students interacted within in-class group activities and other differences are likely due to experiences that occurred long before the calculus course that provided a backdrop for this study. One important thing that emerged from the data in this study is that although all three students would be classified as harmonic reasoners using frameworks such as Krutetskii's (1976), how their reasoning interacted with representations varied a lot. These variations cannot be captured well with categorizations of which modes are expressed. Noticing these variations requires frameworks that document transitions between modes such as the VAP-framework. These frameworks are, however, built upon the foundations laid out by frameworks that categorized students in terms of which modes of representation were expressed or preferred.

Comparing Social Roles in Interview and Group-work

The social interactions between the interviewer and interviewee were not a significant theme in chapter four, which examined the interviews. However, the analysis of the group-work data, and particularly the use of Krummheuer's (2007, 2011) framework, sensitized me to the important role that my own actions played within the interviews. When I revisited the interview data with social roles in mind many similarities between my role as an interviewer and the group member's social roles during class emerged. Here I discuss this role and compare social roles across these two settings using episodes that address compatible mathematical subject matter. In order to facilitate this analysis, I briefly introduce a pair of tasks that is not included in chapter 4. Here these tasks are analyzed primarily with respect to participants' social roles and how they affect representation use. I also revisit Krummheuer's (2007, 2011) framework and discuss how it is used here. Since Krummheuer's constructs were created within a collective mathematical activity setting, as opposed to interview settings, how the constructs are used needs to be adapted to suit an interview setting.

Tasks

As discussed in Chapter four, which focused on the interviews, each of the three one-on-one interviews consisted of approximately 7 tasks. Although all of these were coded and analyzed, in order to avoid repetition not all of this analysis was included in that chapter. Here I briefly discuss an additional pair of related tasks, which I refer to as the Periodic Function Tasks. Some important patterns in student commognition that occurred during group-work on the TIA tasks also manifested themselves on the Periodic

Function Tasks. However, these patterns were not salient to me until after the analysis of the group-work data was completed. Hence, focusing on these particular tasks helps elucidate what is gained by coordinating analyses of representation in both individual interview and group-work settings. The particular pair of tasks is presented below in Figure 6.1:

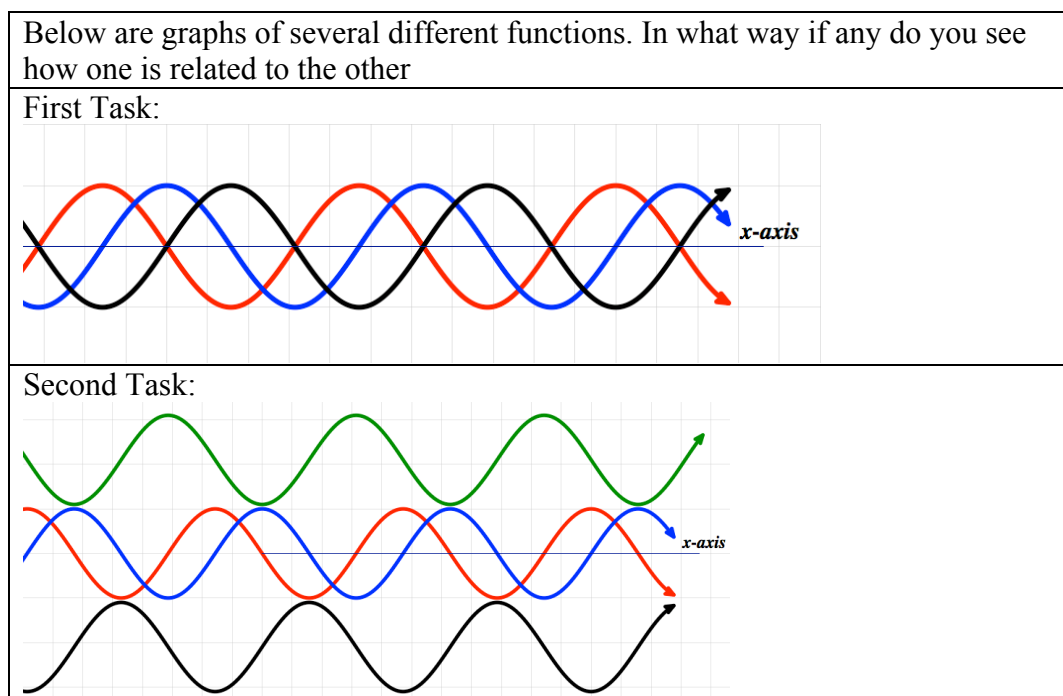


Figure 6.1: Periodic Function Tasks

The two tasks were given back-to-back during the second interview. Both tasks had the same instruction, which was to describe the ways in which each one of the functions related to the others. Notice that in both of these tasks although well known periodic functions are pictured, they are not labeled as $\sin(x)$, $\cos(x)$, etc. This feature of the task is intentional and allows students to make their own connections. Also notice that there is no y-axis in both graphs in Figure 6.1. If students wanted to relate the given periodic functions to known analytic functions, they would need to choose a location to

place the y-axis themselves. Although, choosing a location to place a y-axis in order to relate the graphed functions to known analytic ones is arbitrary and does not effect the solution, it was not necessarily clear that students would be able to make that assessment themselves. The task allowed for exploration of this aspect of students' commognition.

Categorizing Social Roles In Interview Settings

Krummheuer's (2007, 2011) categories of social roles were developed in cooperative collective activity settings in which actors are working toward a common goal. In an interview setting it is not uncommon for the interviewer to not be directly responsible for advancing the mathematical activity and hence not working toward the same goal as the interviewee. Although interviews may take on various forms, it is common for an interviewee to be primarily responsible for advancing the mathematical activity, and the interviewer to attempt to elicit information about how this advancement is occurring and to affect the direction of the activity. The differing actor roles in cooperative and non-cooperative collective activities means that Krummheuer's framework is not well suited for parsing roles within some instances of non-cooperative activities, such as one-on-one interviews. Specifically, how some constructs are operationalized causes misalignment between the names of the constructs and actual social roles being played within non-cooperative activities.

However, Krummheuer's notions of content and formulation, which he uses as building blocks for his framework, can be carried over to interview settings. My operationalization of these constructs is also carried over to the analysis here. My use of these common building block constructs to analyze interview data allows for comparison

of interview and group-problem-solving data. The formulation and content constructs are used to discuss similarities and differences between social roles and behavior in interview and in group-work settings. In the case of the group-work Krummheuer's framework is explicitly used and in the case of the interview data the building block constructs are used. The use of content and formulation constructs facilitates a comparison to Krummheuer's framework and a discussion of why the framework's categorizations do not capture some of the nuances of interview utterances. The cross-setting comparison is done on student work with compatible mathematical content so that similar commognition is compared across both settings.

Social Roles And Periodic Functions

In this section the interview data and group data come from work with derivatives of periodic functions. Specifically, the two periodic function tasks are discussed in relation to the work with sine during the TIA group-work. The two periodic function tasks introduced in the previous section have relatively accessible solutions using both analytic and visual reasoning. The analytic route involves associating the given graphical objects with analytic functions and relying on memorized rules to determine the relationship between the functions. The visual route involves reasoning about the slope of each of the functions and how it relates to the amplitude of their derivative functions. Analytic and visual solutions were present in both the group and the interview data.

In the following I discuss Brad and Carson's work on one of the periodic function tasks and compare it to their work with graphing the sine function during the TIA group-work. Ann's work will also be discussed later in this section. Unlike the task involving

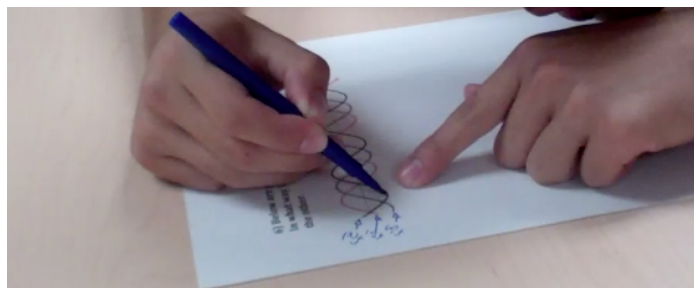
the TIA, both of the periodic function tasks are presented in only the visual mode. The TIA task on the other hand had both analytic notation and the graph of a function when problems were originally presented. Let us begin by examining Carson's solution to the first periodic function task:

Excerpt 6.1a:

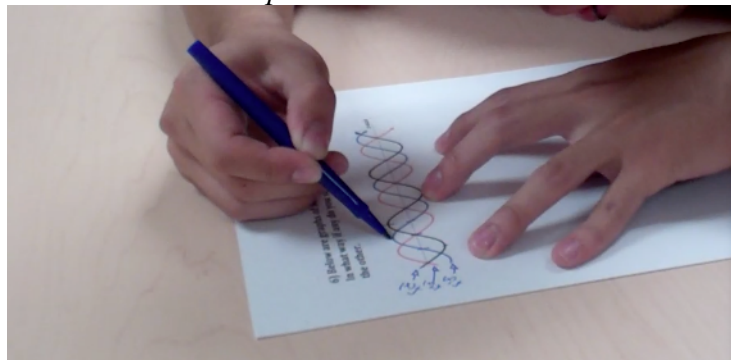
[00:35:40] Carson: Well I would first label the black one as my f of x and then I can see that the blue one is f prime and then the red one is f double prime.

[00:36:14] Interviewer: So could you tell me a little bit about how you just did that.

[00:36:18] Carson: So if you're looking at the black one you can see that this



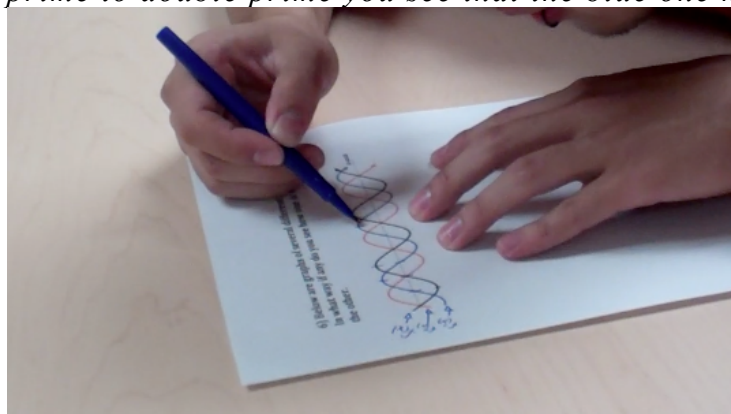
is your, right there and there are your critical points and then later this is your point of inflection and then for f prime you can see that there and there and there turn out to be your roots and then the max I mean the critical point



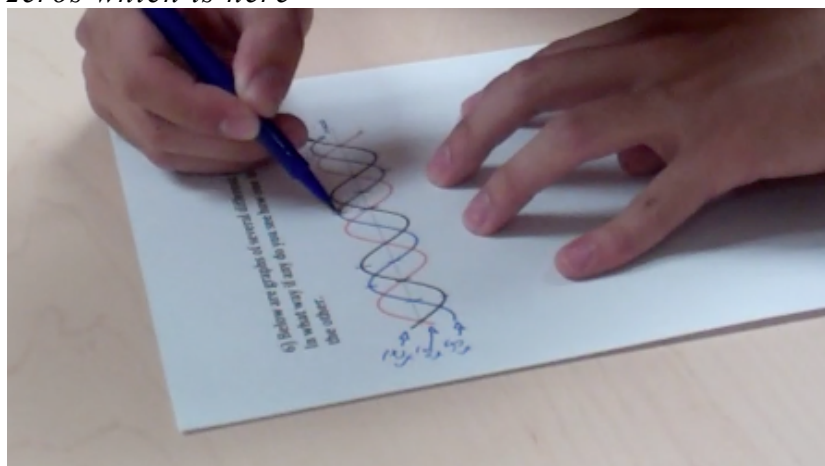
turns out to be there which is the critical point, which is the point of inflection from



your f of x . And then so if you want to look at the red, the red from prime to double prime you see that the blue one here your point of



here's your critical point and then here is your other critical point and this is your point of inflection. So then from blue to red you have your zeros which is here



which is you know from your stationary point. And then later here is your second stationary point or critical point. And this is your point of inflection, the blue one so this is your next, then that turns out to be your critical point. Of the second derivative. Yeah.

Carson begins in an author role introducing a solution (content) and expressing it in terms of analytic notation to identify derivative relationships. This solution provided little detail other than Carson's final outcome. The interviewer's prompt for further explanation, "So could you tell me a little bit about how you just did that," does not add a new formulation or new content to the conversation. This prompt is therefore classified under Krummheuer's framework as the interviewer taking a relay role. However, as discussed earlier, Krummheuer's framework is not designed with interviews in mind. So even though the interviewer's utterance would be labeled as him taking on a relay role, it is a prompt for Carson to relay, rather than an act of relaying itself.

The interviewer's prompt asks Carson to elaborate on his reasoning. This elaboration remains in a visual mode as opposed to the analytic notation he used to label the functions in his initial stating of his solution. This change in formulation but not in content means that Carson is taking a spokesman role when he elaborates on his previously stated solution. Carson reasons that the maxima and minima of the black function correspond to zeros of the blue function and that the points where the function changes concavity correspond to the maxima and minima of the red function.

Also notice, that he has not identified the function as a known graphical object at this point, and has only worked with the functions as generic sketches of periodic functions. In the continuation of the transcript this behavior shifts after an interviewer question:

Excerpt 6.1b:

[00:37:54]Interviewer: Alright, are there any other ways you could have approached the task and come to the same conclusion?

[00:38:06]Carson: Is there any other way that I could have approached... I mean well this looks like a sine cosine graph so I would write myself the equation f of x equals I'm going to say sine. And there is also a cosine in here so I guess you can write f of x equals cosine and then and then there's another one which is which looks like x is negative sine [writes $f(x)=-\sin(x)$].

[00:38:50]Interviewer: Wait, are all these F 's the same F ?

[00:38:55]Carson: No, but this line and this line and this line, This [points a $f(x)=-\sin(x)$] would be your black one, and this [writes $f'(x)=\cos(x)$] would be your blue one and this [$f''(x)=\sin(x)$] would be your red one. And so there relationship with each other is, the derivative of sine is cosine and the derivative of cosine is negative sine. And so since this red one right here looks the most like sine I would give this label f of x to sine. And then later...

The interviewer's question about other possible approaches does not add a new formulation or new content to the discussion. Thus, if we were to apply the definition stated in Krummheuer's framework, the interviewer would be taking on a relayer role. However, the interviewer in the above transcript is not relaying previously constructed content and formulations. Instead, the interviewer inquires about what an alternative formulation of Carson's solution might look like. Alternative solution paths naturally entail different formulations of the solutions. So although the interviewer does not introduce a new formulation, he does inquire whether Carson is able to come up with an alternative formulation. By doing so the interviewer initiates Carson's introduction of a new formulation. In response to the interviewer's question above Carson identifies the graphical objects in the problem, recasts them as their analytic referents and begins to use known derivation rules to reason about the problem. The interviewer's question sparked Carson's shift to analytic commognition.

During the course of this study, particularly when I was initially developing the VAP-model, my analysis treated the interviewer questions as background noise. At that stage I planned to analyze group-work and coordinate this analysis with the analysis of the interviews but had not yet done such analysis or coordination. Analysis of the group-work data caused a shift in my approach and theoretical perspective. I adopted the commognitive perspective due to its compatible treatment of both group and individual centered data. The commognitive perspective treats the interviewer question as part of the commognitive interaction. Although the commognitive perspective is not the only theoretical perspective to treat interviewer questions in this way, this perspective as well as the analysis of group-work data provided the impetus for revisiting the interviewer's social roles within the interviews. Revisiting the data helped highlight some of the ways in which the operationalization of Krumheuer's categories misalign with interviewer utterances. It also helped elucidate the similarities between social roles in the interview and group-work settings.

As stated above, episodes where similar mathematical content was explored within group-work and interview data were compared to facilitate the analysis of social roles in this chapter. The above is such an episode. Now let us revisit a comparable episode that occurred during group-work when Carson and Brad were working on graphing sine during the TIA group-work session (Excerpt 5.3). This episode was previously analyzed in Chapter five:

Excerpt 6.2:

[00:15:31]Carson: slope is zero here. zero here and zero here [moving along sin and pointing at the maxima and minima] zero here. this is decreasing, increasing, decreasing, [running pen along function saying increasing when slope is positive and decreasing when

negative] No wait. Decreasing increasing.... Yea so that's why I thought [sketches a drawing that looks like $1/6 \cdot \cos(x)$].

[00:16:34]Brad: Isn't this just negative cosine [pointing at C's graph]

[00:16:44]Carson: No, or is that [inaudible]. Yeah it actually is.

[00:16:51]Brad: The derivative of cosine, which is sine.

[00:16:56]Dov: But they gave you sine.

[00:17:01]Brad: The derivative of sine is negative cosine [writes $\sin(x) \rightarrow -\cos(x)$].

[00:17:10]Carson: Yeah I think it's negative cosine... Well it's positive [pointing to Brad's notation] .

[00:17:26]Dov: Negative cosine goes through negative one.

[00:17:32]Carson: Cosine is negative sine and ... is that right?

[00:17:35]Brad: Is cosine negative sine and sine positive cosine?[writes $-\cos(x) \rightarrow \sin(x)$ and changes the – in his previous notation to a +]

[00:17:43]Carson: Yeah, Alright tangent is one over cosine squared [writes $\tan(x) \rightarrow 1/\cos^2(x)$]

In the analysis within the previous chapter, Brad's first utterance in this segment was identified as him taking on a spokesman role because he was responsible for the reformulation of the sketch as negative cosine (form), but not the sketch itself (content). Notice that here too the commognitive interaction shifts to discussion of periodic functions in terms of known analytic referents. As noted in the previous chapter, this shift in representation mode is mediated through several neutral turns, in which it is not clear which mode of commognition is being utilized.

Brads question in the above transcript “Isn't this just negative cosine”, and the interviewer question in the previous transcript, “are there any other ways you could have approached the task?” point to similar roles. In both cases a commognitive interaction that begins with Carson reasoning in the visual mode about slope shifts to an analytic commognitive interaction. So the prompt for a new formulation in the interview and the addition of a new formulation in the group-work both catalyzed similar shifts in representation usage.

As was noted in the analysis in the previous chapter the shift to using analytic commognition found in the group interaction carried through to the subsequent task. This pattern was also present in Carson’s work on the periodic function tasks during the interview. He continued to use analytic commognition in his work on the second periodic function task. The following is the continuation of the transcript from Carson’s one-on-one interview that begins when he starts the second periodic function task. The remainder of his work on the first periodic function task (not presented here) consisted of him reasoning in an analytic mode and then associating this reasoning with the graphic function objects in the problem:

Excerpt 6.1c:

[00:46:43]Carson: Okay well I know that. So if I just drew a line from right here. Start my graph [draws y-axis].



And then this is like [mumbles]. This would be my negative sine and this would be my cosine. So blue is my negative cosine red is my negative sin. This is actually positive cosine I don't know why I had a negative. And then later my green is your well your green is your negative cosine plus a variable. Which I'm going to give.. So this is like here [adds dotted line down the line $y=-1$ and $y=1$ and around the max and min of the green and black functions respectively] this is that is this guy.

Carson continues to work on the second periodic function task using the same representation mode as he used at the end of the previous task. In both the individual interview and the group-work the shift to analytic commognition (change in formulation) and the subsequent work in that mode was brought about by interactions with an interlocutor. In the case of the group-work, the interlocutor continued to interact with Carson using analytic commognition. In the case of the interview the interviewer stepped back and only asked clarifying questions that were neutral with respect to representation mode and did not cause shifts in which mode he was using.

Also notice that within his work on the second periodic function tasks Carson draws a y-axis in order to establish which of the functions should receive which label. This move occurs right after Carson reads the problem statement. He seems to treat the move as a logical first step in his analytic approach to the task. However, a similar y-axis

was not drawn during his work with the first periodic function task. There he treated the first point where the red and black functions intersect as the location of the y-axis, but never made this move explicit. It is unclear why he created a y-axis in the second task but not the first.

It is also important to compare how Brad worked on the periodic function tasks during the one-on-one interviews to the group-work episode.

Excerpt 6.3:

[00:25:36]Brad: Well these look like. They look like sine and cosine graphs but just with different.. different like attributes or different numbers in them. [long pause]

[00:26:23]Interviewer: So how are sine and cosine graphs related?

[00:26:30]Brad: They are each other's derivative.

[00:26:39]Interviewer: So could you sort of help me figure out what's the derivative of what in that picture?

[00:26:44]Brad: So the derivative of sine is. So sine the derivative is negative cosine and cosine the derivative is sine. So sine starts at zero and goes to one. So this would look like the regular sine graph [orange]. Which would also be

[00:27:16]Interviewer: Which color are you pointing at right now?

[00:27:19]Brad: This one this would be sine. And then this one looks like negative cosine {blue}. Instead of one it starts at negative cosine. And then this one looks like it would be negative sine. Because it starts at zero but then it goes negative.

[00:27:58]Interviewer: So what color is the derivative of what other color.

[00:28:06]Brad: So this is. This is the derivative of cosine because the derivative of cosine is sine. So that means that this one is f prime of sine because the derivative of sine is negative cosine.

[00:28:40]Interviewer: And the black one?

[00:28:41]Brad: The derivative of negative sine would be cosine. It would positive. The derivative of negative sine would be positive cosine. That's not...{pause} Oh it's negative cosine. So the sine of negative cosine. The derivative of negative sine would be negative cosine. That's not right because that's...So the derivative of negative sine would be negative f prime cosine.

Brad begins the task by relating the shape of the given graphs to sine and cosine functions. He took a similar approach to sine within the group setting, which is what sparked his and Carson's shift to using analytic commognition to reason about sine. In both settings he uses miss-remembered analytic rules about the relationships between sine and cosine, stating that the derivative of cosine is sine. The opposite is true. Within the group-work setting Carson highlighted Brad's mistake, which allowed them to cooperatively arrive at the correct answer. Within the interview setting, the interviewer did not correct Brad, and instead asked questions about Brad's commognition. Without a similar intervention to that which occurred in the group setting Brad continued to reason using misremembered rules.

It is important to mention that the data from the TIA activity and the second interview from which the transcripts of Carson and Brad were presented occurred 6 weeks apart. So Brad's difficulty with remembering and applying derivative rules for periodic functions are a phenomenon that extended long beyond his initial exposure to these rules. This difficulty is interesting given that in both the interview and the group-work he is the one pushing the commognitive interaction in an analytic direction. So even though he pushes the conversation in the group-work settings in an analytic direction and uses analytic commognition to reason about the problem in an interview setting he has trouble with utilizing analytic reasoning, which includes difficulty with remembering and

applying derivation rules. Brad's push toward analytic commognition in both settings is interesting, given that Brad is arguably better able to reason about derivative graphs in a visual mode. So the representation he tends to gravitate toward when dealing with periodic functions is not necessarily the representation he is most fluent in working with.

Within some of the work on representation Brad's initial approach of discussing these functions in terms of their analytic referents, if it persisted across a number of tasks, would be taken as an indication that he was an analytic thinker (e.g., Aspinwall & Shaw, 2002; Aspinwall, Shaw, & Presmeg, 1997). However, this behaviour did not persist across multiple interview tasks. It was noted in chapter four that Brad incorporated the visual mode into every one of his solutions. The same cannot be said about his use of the analytic mode, which was only incorporated into a hand full of visual tasks. In fact, his tendency to approach visual tasks using analytic functions was restricted to tasks that involved periodic functions (e.g., Excerpt 6.3). His work on the anti-derivative task discussed in chapter four is an example of his approaching visual tasks using visual commognition, and is consistent, in terms of representation usage, with his approaches to non-periodic functions throughout the study (Excerpt 4.4). The observation that Brad's analytic approaches to visual tasks are isolated to periodic functions within these data speaks to regular usage patterns within his commognition that relate to the presentation of tasks. However, it is unclear what fosters this particular approach to periodic functions or why the approach appears to be limited to only this class of functions.

Returning to Carson, it is interesting that in both the group data and the individual data Carson continues to use analytic commognition after he made the shift to it. In every task given during the one-on-one interviews Carson at some point incorporated the visual

mode. Introducing visual reasoning was usually his initial approach, as was the case with the above two transcripts (Excerpts 6.1 & 6.3). Given this tendency to incorporate the visual mode early and often, his shift to and continued usage of analytic commognition is curious. One possible explanation for this shift is linked to momentum. That is, if a particular commognitive strategy is working, he may tend to continue to use it until he encounters difficulty. However, this momentum was not apparent in his usage of the physical mode, which he often switched to spontaneously regardless of whether he appeared to be struggling (e.g., Excerpts 4.1, 4.6b & 5.2). So it may be that a momentum-based explanation of his behavior is unfounded or that momentum affects different modes of commognition differently in different people.

Another possible explanation for Carson's behavior is the influence of those around him. Both Brads' utterance "Isn't this just negative cosine" and the interviewer's utterance, "are there any other ways you could have approached the task?" may have been interpreted by Carson as indications that those involved in the collective activity wanted him to explore the problem using an analytic approach. Similar explanations of social behavior where a student solves a particular problem in a particular way – based on his/her personal interpretation of expectations – can be found in the literature (e.g., Sfard, 2009). So Carson's shift to and continued usage of analytic commognition may have more to do with the influence of others than Carson's own problem-solving tenancies. Although Carson did demonstrate shifts to analytic commognition within his work that were not linked to the utterances of others, these were always brief and involved reasoning about sections of position graphs which corresponded to sections corresponding to zero velocity (e.g. Excerpt 4.1a). Sustained transitions to analytic

methods were always preluded by other utterances that he may have interpreted as pushing in an analytic direction. Thus, the above transcripts may be interpreted as initial tentative evidence that preferred modes of commognition may be abandoned in favor of non-preferred modes in response to social pressures.

Without the comparison of the group-work and interview data Carson's prolonged usage of analytic reasoning may have been interpreted as indicative of representation behavior that he exhibited in isolation. The comparison of the two episodes as well as his behavior on other tasks in this study help reveal that his shift to and continued usage of analytic reasoning is mediated by social interaction. Such prolonged usages of analytic reasoning did not occur without the direct influence of others. The comparison of the two settings reveals an important socially mediated attribute of Carson's representation use that would not have been apparent without the direct comparison of both settings. It helps establish that student representation use is socially situated and that the roles others play in influencing which representations facilitate student commognition should not be ignored in analysis of representation behavior.

The influence of others on student representation use is however non deterministic. The following episode in which Ann is working on the periodic function task reveals that her response in terms of representation is very different from Carson's:

Excerpt 6.4:

[00:25:40] Ann: We'll do back right here. And blue and orange. I'm just deciding right now, I know this, this blue one right here is f' prime of x [writen $f'(x)$] and I know this because these are zeros [points at the first min on black and first max on orange, which are vertically aligned] where the slope are zero so it would be crossing at zero between. So the blue has to be f' prime. And that's where I get confused on which one is the acceleration graph and which one is the original. So this would be speeding up I don't know. I Think it's the orange is

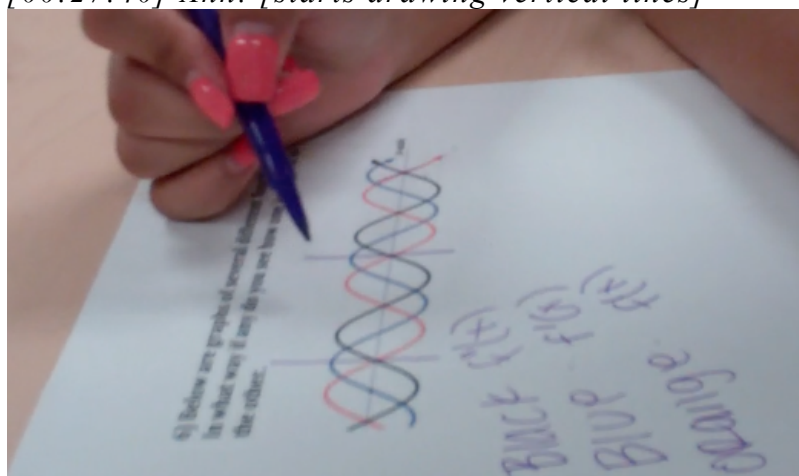
the original. Only because the endings. Like this would mean that it's speeding up continuously and this is just showing that speeding up. So that's what I would think

[00:26:53]Interviewer: Could you talk to me a little bit more about that.

[00:26:58]Ann: because acceleration is speeding up and from, which one okay blue.. so that's the f prime and it would be speeding up for ever and ever and ever. I think because I mean, I'm pretty sure if it's speeding up it's going to keep going positive [makes a motion like the blue graph will continue to increase indefinitely] for ever and ever I guess it's kind of chewed off. And this one {pointing at orange} Even though it's going down it's speeding up the original because it would just be this just shows that it's going for ever. Oh no I'm confusing myself.

[00:27:32]Interviewer: Take your time.

[00:27:40] Ann: [starts drawing vertical lines]



[Silently points at various parts of the graph]

Okay from this, I'm going from, well I do those line to try and just focus on that [hands in a boxing motion that cuts off portions outside the lines]. Because it was confusing me. So this this says that the original has the negative slope till here [pointing at orange] Oh wait, oh wait, I lie. This has a negative slope [blue], which means it has to be this the orange one. Oh wait this has a positive slope, I'm confusing myself. So it has to be the black one and at this point it switches to a negative till here [point where orange crosses x axis] where it hits again. And then. I wish I really could have separated them because this is so confusing. From the beginning again. The blue is the f so this I telling the slope. Okay from here to here where it crosses it a

positive slope. So that's that which makes me think that it's the black one is f of x [crosses out prime next to where she wrote black] and then the blue one doesn't cross until here so it has a negative slope until here which would make sense with the black because it's negative. And then it's positive again so ya the black has to be f of x and the orange f prime or f double prime or whatever. The black is the original, the blue follows the slope of the black and all that's left is orange and I'm not very good at telling acceleration graphs. So that's were I would leave that.

[00:29:36] Interviewer: Okay, is there a different way you could have approached this task?

[00:29:41] Ann: Um, I don't know. Because I mean the f .. what I've noticed in squiggly graph lines is that f of x and f prime of x are similar there just kind of, they actually look like the same graph but separated, like one stays put and one has moved, over a little bit so that's how I could have just looked at it if that... but I didn't think that was a very smart way. So then that's why I got into like looking at the slopes. And I know that acceleration looks completely different. So that's how I could have looked at that.

Notice that in Ann's transcript the interviewer has a similar prompt to the one that caused Carson, to shift to analytic commognition, "is there a different way you could have approached this task?" This prompt, much like the prompt in Carson's transcript, asks about an alternative formulation without actually introducing one and thus is similar to Brad's spokesman role in the TIA transcript (Excerpt, 6.2). However, in Ann's case this same prompt did not catalyze her connecting the given functions to known periodic functions or commognizing in an analytic mode. She mentions that she could have taken an alternative approach but does not elaborate on or attempt to use this approach. She continued to commognize using the same modes observed before the prompt. So although the interviewer's prompts in both interviews were comparable, these prompts

caused an abrupt shift in representation behavior only in Carson's work. So the influence of others on representation use varies among students.

Revisiting the transcript in more detail, much like Carson's initial approach, Ann uses analytic labels to aide in identifying derivative relationships, but does not in fact reason about this relationship in terms of the notation itself. Unlike Carson, her approach makes use of physical and visual representations. This approach is interesting, since within the data Carson's use of physical reasoning is much more common than Ann's use of this type of reasoning. In the above she explicitly moves back and forth between reasoning about slopes and reasoning about speed/ position. When she states, "I get confused on which one is the acceleration graph and which one is the original", she implies that any derivative relationship can be thought of in terms of a position/velocity/acceleration situation, and that this situation is an implicit part of understanding derivative. She further states, "I'm not very good at telling acceleration graphs." This statement implies that she sees the process of moving from a position to a velocity graph as different from the process of moving from a velocity to an acceleration graph. Her difficulty with second derivative was also evident in her work with the concept in class. In contrast Carson was able to quickly adapt methods used when reasoning about first derivative when working with second derivative.

Another interesting phenomena within this episode involves Ann's avoidance of the names of functions. Instead of referring to the given graphs of functions as sine/cosine graphs, which is what both Brad and Carson did, she refers to them as "squiggly graph lines." Similar behavior was also displayed in her work with the anti-

derivative task (Figure 4.2). There she referred to parabolas and cubic functions using pointing gestures saying “this one is this” (Excerpt 4.5a).

Also notice that Ann adds vertical lines to the graphs. On the second periodic function tasks Carson used a similar vertical line as a y-axis to help in applying analytic labels. However, for Ann these lines serve a very different purpose. She uses them to aide in considering only a single cycle of the given graphs and uses her hands to block out parts that remain outside of this cycle. Here she is not specifically using the lines as a y-axis. However, this spontaneous usage of vertical lines is based on an implicit assumption that one can start comparing periodic functions at any point along their cycle in order to establish derivative relationships. Neither Carson nor Ann consider more than a single cycle when working on these tasks. Brad did not exhibit this kind of behavior.

Ann was not part of the group interaction around sine during the work on the TIA. She was working independently of both Brad and Carson during that time and hence was not exposed to thinking about graphing derivatives using analytic techniques. Hence, it should not be surprising that she did not switch to reasoning analytically about this problem when prompted to do so.

Discussion

Student representation use is an important aspect of how students think about derivative. Their usage of representations within a group setting and their usage of representations in individual centered settings inform each other. In this chapter I discussed the connections between group-work and one-on-one interview data in two ways. The first involved revisiting representation use patterns found in the previous two

chapters, which allowed for a comparison and consolidation of what is known about each of the student's representational behavior. The comparison painted a richer and more complete picture of the students' representation use. The second comparison involved delving back into the data to analyze the social roles and social influences on representation use. This analysis established that some of the representation behavior exhibited by students is closely tied to social influences, that there are similarities in how these influences manifest across settings and that the influences affect each student differently.

The comparison of group-work and interview data revealed that all three students' representation usage patterns were highly similar across both settings. This correspondence included similarities in (a) how often particular representations were used, (b) how often each student transitioned between representations, and (c) what types of transitions were present. A direct comparison of episodes with compatible mathematical content was also able to show similarities in how social roles catalyzed shifts in representation use. These shifts were facilitated by prompts for or introductions of alternative formulations. More specific details of these cross-setting patterns are discussed below.

Carson made extensive use of both physical and visual reasoning. His physical reasoning, which was based in a position/acceleration/velocity scenario, was often used to reason about tasks that were presented in purely mathematical contexts, as well as tasks that were presented with respect to different physical scenarios. Although translating between physical scenarios and mathematical representations of those scenarios has been examined extensively in the literature, the introduction of physical

reasoning into purely mathematical problems is not as well documented. The documentation of this phenomena here informs what this type of reasoning looks like and may provide a foundation for research into how such behavior can be fostered in students.

Carson's use of analytic reasoning was also interesting. He made brief uses of this reasoning when working with segments with zero velocity. However, he only made prolonged use of analytic reasoning on non-analytic tasks when he was prompted to do so by others. This observation helped establish that which modes of representation he uses in his problem-solving are highly influenced by interactions with others. The compatible roles of Brad in the group-work setting and the interviewer in the interview setting brought about Carson's continued use of analytic reasoning and this reasoning continued onto the subsequent task. Such uses of analytic reasoning were often triggered by explicit prompts from others. This socially mediated shift in representation is particularly interesting in comparison to some of Carson's remarks during group-work, which indicated that he saw much of the in class work with graphs as serving an analytic end. In spite of this expressed view, when provided with tasks that could be approached with both analytic and visual methods, he chose to use visual reasoning until prompted to do otherwise.

Ann's work did not exhibit the same socially mediated shifts in representation use. In response to the same prompts that shifted Carson's use of representational modes she acknowledged the existence of alternative approaches, however, she did not use these approaches. She largely avoided analytic reasoning in both the group-work and interview data. This avoidance extended to avoiding names such as "sine" and "x-squared" when she discussed the graphs of functions. Through out both her group-work and her

interviews she made extensive use of visual reasoning. However, unlike Carson and Brad, this reasoning often revolved around point-wise rather than across time interpretations of functions. Ann also made use of physical reasoning. However, this physical reasoning was far less prevalent than Carson's uses of it. Her use of physical reasoning however still provides additional illustrations of the phenomenon of introducing physical scenarios to help with reasoning about purely mathematical tasks.

Brad's approach during the interview paralleled his approach within the group-problem-solving. In both situations he was the one initiating work in an analytic direction. However, in an interview setting, without Carson to interject corrections to Brad's misstated analytic rules, Brad's ability to advance the mathematical activity was stifled. In the group-work Brad contributed to the overall direction of the problem-solving activity by introducing alternate framings. Within the interview setting the same framing was present but the same type of progress was not made toward a solution. Brad used fewer shifts between representation modes than the other two participants. This pattern was evident in both the interview and group-work data. What reasoning he did use seemed to be closely tied to the task itself. He used the analytic mode only in dealing with analytic graphing tasks and avoided using analytic reasoning in other graphing tasks.

The comparison of the group-work and interview data pointed to the important role social surroundings play in influencing the direction of student solutions. If the group data were analyzed without comparison to the interview data, Brad's difficulty with applying derivation rules may have been masked by Carson's corrections. If the interview data were analyzed without the group data, the influence of the interviewer on the mode of commognition Carson used may have been masked without the comparison to the role

played by Brad within group-work. In both cases the comparison of the two settings revealed more about student thinking about derivative than would have been the case if only one set of data sources was considered.

Chapter 7:

Summary and Discussion

In this final chapter, I discuss the implications and results of this work as well as future research directions. The discussion begins with a brief recapitulation of the research questions and short descriptions of the results of my investigation. Next I discuss some of the study's limitations. I then shift to discussing how this work contributes to the field. This discussion begins with a discussion of the VAP-model and its role in bridging analyses of group and interview data and then moves to the discussion of a the role physical reasoning plays within mathematics education. I conclude with several remarks about possible future directions.

Research questions and results

As a reminder to the reader, the research questions that guided this work are:

- 1) How do individual students think about key calculus concepts during their problem-solving? In particular, how do they use representations to facilitate this thinking?
- 2) How do groups of students communicate about key calculus concepts? In particular, how do they use representations to facilitate this communication?
- 3) Within a calculus setting, in what ways do students' thinking in individual interview settings and their communication in group settings inform each other?

The first research question was addressed by using the VAP-framework to examine a series of three one-on-one interviews with each of the study's participants. I found that the students often incorporated modes of reasoning within their solutions that were not required by the question and the modes of reasoning used in problem-solving varied across study participants. This result helped highlight that although all the participants would be classified as harmonic reasoners using Krutetskii's (1976) framework, the modes of reasoning manifested themselves in different ways in each of the students. As such, my analysis focused on analyzing students' transitions between different modes of reasoning rather than identifying students according to their representation preferences. In particular, students often introduced visual and physical reasoning to make sense of tasks. Visual reasoning was commonly introduced to make sense of the structure of problems and physical reasoning was commonly introduced to aid with relating graphs of functions to graphs of their derivatives. Physical reasoning in particular, was grounded in the position/velocity/acceleration context and was even introduced to make sense of other physical contexts. This phenomenon, where a non-mathematical situation is introduced to reason about a purely mathematical problem, is discussed in greater detail later in this chapter.

The second research question was addressed by examining selected days of group-work using both the VAP-framework and Krummheuer's social role framework. This analysis was able to highlight ways in which students transitioned between modes of reasoning during group problem-solving. Transitions between actors' turns were rare. Most transitions between representations occurred within a particular turn. So when

communicating the mode of representation that was used at the end of one utterance is almost always the mode of representation used at the beginning of the responder's utterance. This social norm appears to be important for students functioning in environments where the use of multiple representations is encouraged (or expected). Additionally, most of these transitions involved the visual mode. Much like with the interviews, transitions between the three modes of reasoning centered on the visual mode and varied in regard to how often they showed up in different students. These transitions were not constrained to a particular social role. Shifts in representation usage occurred within each of the speaking roles (except the relayer role, which revoices existing content and formulations and by definition cannot have such a shift). A phenomenon in which the representational ambiguity of particular mathematical terminology caused changes in formulation that were not necessarily changes in representation use was also highlighted. These representationally ambiguous statements helped illustrate that changes in formulation could facilitate changes in representation use, even though these changes in formulation were not necessarily changes in representation use themselves.

The third research question was addressed by comparing data across group-work and individual interview settings. This comparison showed similarities in representation usage across the two settings. These similarities included how frequently each participant transitioned between modes of reasoning, which modes appeared within their reasoning and the types of questions in which particular shifts between representations occurred. Additionally, this analysis highlighted cross-setting similarities in how social roles facilitated shifts in representation use. These were tied to Krummheuer's notion of formulation. Both requests for alternate formulations and introductions of such

formulations facilitated compatible shifts in representation use. These shifts in representation use affected more than just the initial task in which the shift occurred. Although there were similarities across settings in terms of how social roles facilitated transitions, not every student was affected in the same way. The same utterance that catalyzed a shift in representation within one student did not always affect other students' representation use in the same way. However, compatible utterances tended to cause compatible changes in a student's representation use regardless of the setting in which they occurred.

Limitations

This study followed a small group of students in one technologically enriched classroom. Although these data were sufficient to help establish patterns in representation use within this group, more data are needed to assess how similar patterns occur in larger populations and with students exposed to more traditional curricula. In particular, tentative links between how often students transitioned between representation modes and their abilities to work through calculus tasks were noted in these data. This link is both predicted by the VAP-model and not contradicted by these data. A significantly larger and more diverse sample would be needed to verify this assertion of the model.

An additional phenomenon observed within these data is the role physical scenarios play in aiding reasoning with purely mathematical problems. More data are needed to infer how common this phenomenon is among calculus students exposed to reform curricula that introduce calculus concepts in physical settings (e.g., Hughes-Hallet, 1994). Whether students exposed to more traditional approaches to calculus exhibit this

behavior is also an open question. Although analyses of these data were able to establish that not all students use this type of reasoning since it was not exhibited by Brad, it is not known how common it is for calculus students to reason in this way. Also, more detailed analysis of in-class data across multiple classrooms would be needed to trace the genesis of this phenomenon and to document the kind of pedagogical moves that foster it. Roth's (2001, 2009) studies of working scientists reveal similar thinking, which links familiar scientific phenomena to graph interpretation. It may be that the use of the physical mode observed in this study occurs for compatible reasons to the phenomena discussed by Roth. If this phenomenon is consistent with the one observed in Roth's studies, it may (or may not) be independent of instruction.

Contributions And Future Directions

My research contributes to a number of arenas. In terms of theory development, my contribution is in extending the VA-model (R. Zazkis, Dubinsky, & Dautermann, 1996) into the VAP-model. The VAP-model adds physical representations to the VA-model, which previously used only visual and analytic. In terms of methodology, my contribution is in bridging analysis of individuals and a group using the VAP-model. I also mention a potential contribution to curriculum and task design, and discuss the utility of a particular physical scenario.

The VAP-Model

The VAP-model developed in this work provides a tool for diagraming and modeling how transitions between representation modes facilitate mathematical

advancement. Part of the model's utility stems from its ability to classify various forms of communication. This communication includes but is not limited to: group communication, the communication that occurs within interviews, and the statements of tasks, which can be viewed as a form of communication between the author of the task and the student solving it. It also allows comparison across these differing forms of communication. Although the data in this study show that the representations that are needed to comprehend and solve a task do not limit which modes are used, they do provide an avenue for evaluating which transitions between modes are essential for solving a particular task. Fluency with multiple representations has long been acknowledged as necessary for a deep understanding of calculus (Vinner, 1989; Zandieh, 2000; Zazkis, in press; Zimmerman, 1991). The VAP-model provides an avenue for assessing not only the representations needed to understand a task and solve it, but also for documenting the direction of such transitions between representations. The model helped illustrate that representations do not get used in isolation from each other, with one representation getting used on one task - the converse was observed. Students often moved back and forth rather quickly between modes of reasoning to facilitate their problem-solving and the specific back and forth patterns used differed amongst students.

Bridging Individual And Group Analyses

Much of the research on student representation usage has fallen into two largely disjoint bodies of work, one that is primarily cognitive (e.g., Hanna & Sidoli, 2007; Presmeg, 2006) and one that is primarily social (e.g., Roth & McGinn, 1998). I attempt to bridge these two bodies of work in two ways. The first involves the creation and use of an

analysis tool that can be applied to both individual and group settings. The second is an attempt to tie this study to both of these bodies of work and use both to inform the phenomena observed here. Coordination of analysis of both individual and group data is not new. The work of Cobb and Yackel (1996) laid the groundwork for a number of such analyses (e.g., Stephan, Cobb, & Gravemeijer, 2003; Rasmussen, Wawro, & Zandieh, 2012; Yackel & Rasmussen, 2002). However, within the study of student representation use such coordination is not a common practice. The work here is an attempt to introduce this practice to the study of representations and in doing so help bridge the two largely disjoint bodies of work on student representation use. The introduction of the VAP-model proved useful in analyzing students' use of representations in both individual and group settings and in drawing a comparison across the settings.

VAP-Model's Further Potential

The model's utility is not limited by the uses demonstrated here. When designing curricular materials that incorporate multiple representations, there is a potential to only assign tasks that encourage particular transitions and not others. Using the model to assess which transitions are prompted by particular tasks can avoid this issue. An example of such assessment can be seen in Table 4.1 in chapter 4, which classified each task within the one-on-one interviews in terms of the representational mode(s) needed to understand a problem and the mode(s) needed to express its solution. Similar classifications of homework tasks can be used to insure that a diverse range of prompted transitions between modes is represented within students' assigned homework. Tasks such as The Tree Task (Figure 4.3 in Chapter 4), in which the required solution is a physical interpretation rather than a visual or analytic object, tend to be uncommon in

calculus curricula. The use of the model can help in injecting more such problems into curricula. So the VAP-model can be used as a tool to aid the development of curricular materials and is not just a tool for coordinating individual and group analyses such as those within this study.

On Reasoning With a Physical Scenario

The operationalization of physical reasoning in the VAP-model was able to highlight an important type of reasoning within the data that involves the introduction of non-mathematical contexts to reason about purely mathematical problems. Physical reasoning was specifically tied to the position/velocity/acceleration scenario, which Zandieh's (2000) framework identifies as part of what the mathematics community treats as part of the derivative concept. This type of reasoning was documented in the work of both Carson and Ann, although it occurred more often in Carson's work.

In his seminal work "Mindstorms", Papert (1980) coined the term "object to think with" when describing children's interactions with LOGO turtles and their introduction of them outside LOGO contexts to aide in problem-solving. The phenomenon described here is similar. However, rather than being an "object to think with", the position/velocity/acceleration context provided students with a *scenario to think with*. This construct is related to Rasmussen et al.'s. (2004; Rasmussen & Nemirovsky, 2005) construct *knowing-with*. However, their construct refers to a connection between mathematical notation and a physical scenario that is "attached to evanescent circumstances" (Rasmussen et al., 2004, p. 316). In contrast, a scenario to think with is used as an, often ubiquitous, mode of reasoning across multiple contexts. It is introduced

by the student in multiple problem situations and its contributions to student reasoning stretch far beyond the evanescent circumstances in which it was introduced.

Scenarios to think with similar to those described in this study have been described in other research. For example, Nunes, Schliemann and Carraher (1993) studied Brazilian children's arithmetic abilities in both a contextualized market situation, where the financial transactions in which children were involved were directly tied to their livelihood and school settings. They found that children often performed almost flawlessly in the market situation, where the children were dealing with the exchange of actual goods for money. However, these children often struggled when the isomorphic calculation was given in a school setting where the arithmetic was not tied to any actual objects. In the market situation, the children were performing calculations that were tied to tangible objects they could see and touch. These calculations involved them working with these objects (physical reasoning) and calculating quantities associated with them (analytic reasoning). In the school situation, the numbers involved in these same calculations were presented abstractly with no relation to physical objects. Since the students did not relate these back to the market situation, what they were involved in was only analytic reasoning. Without the back and forth relationship between physical and analytic reasoning, their performance on problems suffered.

One can imagine these children, either catalyzed by instruction or their own internal commognitive process, adapting a strategy where they dealt with these abstract situations by relating them to the familiar market situation. This strategy would transform the abstract arithmetic, with which they are uncomfortable, into the comfortable market situation. In my view, Carson and Ann's use of the physical mode within both group-

work and one-on-one interviews was such a move. They related abstract notations and abstract graphs into representations of a physical situation with which they were familiar. Imagining the motion of a particle and reasoning about how its position related to its velocity grounded their reasoning in ways similar to how the market grounded the reasoning of the Brazilian school children.

In both this study and the imagined situation where Brazilian school children use the market situation to think about class-work, the physical scenario came first and provided a context for thinking about abstract mathematics. Similar imaginable physical grounding for abstract mathematical concepts have been documented by Wawro et al. (Wawro, Zandieh, Sweeney, Larson, & Rasmussen, 2011; Wawro, Rasmussen, Zandieh, & Larson, 2013) in a linear algebra context, by Larsen et al. in a group theory context (Larsen, 2013; Larsen, Johnson, & Bartlo, 2013), by Rasmussen and King (2002) in a differential equations context, and by Roth (2001, 2009) in his study of working scientists' graph interpretations. Within Wawro et al.'s, Larson et al.'s, and Rasmussen and King's work, the instruction also began with the grounding scenario.

Much of this work is inspired by the RME instructional design heuristic. Following the prominent advocates for this perspective many of these researchers assume that these scenarios to think with will be replaced by *formal mathematical activity* (Gravemeijer, 1999). This formal mathematical activity no longer explicitly references these scenarios, although they are still regarded as important to students' understanding. I in contrast view these scenarios as a permanent fixture within student thinking. Roth and his colleagues' studies point to professional users of mathematics having meanings for mathematical representation that are intimately tied to scenarios to think with (Roth, 2001,

2009; Roth & Bowen, 2002; Roth & McGinn, 1998). Similarly, the data in this study do not show a drop off in the amount of physical reasoning shown in students. So it may be that scenarios to think with remain permanent fixtures within student thinking. Although it may be the case that scenarios to think with get replaced with new scenarios to think with as students advance; calculus students do not generally reason about addition and subtraction in terms of acquiring and losing apples.

The descriptions and analysis of physical reasoning found in this study provide a glimpse into what scenarios to think with look like when used in a calculus context. Further research is needed to understand the mechanisms that facilitate this phenomenon and whether it in fact remains a permanent fixture in students' thinking.

Concluding remarks

Heinzel, Star and Verschaffel (2009) in their commentary in a recent ZDM special issue on flexible and adaptive use of strategies and representations in mathematics education noted that there are many unanswered questions in literature on representation use. Unanswered questions include: which individual and non-individual centered factors influence representation use, the types of instruction that yield flexible and adaptive representation use and the creation of models that predict representation use. This dissertation has played a small part in narrowing some of these immense gaps in the literature by documenting the representation use patterns of both individuals and a group and comparing representation use in these two settings.

More research is needed to determine the specific connections between the classroom environment and students' representation use. More specifically, the social and

sociomathematical norms the instructor, Dr. Byne, establish in the classroom contributed to the patterns in students' representation use described in this study. She created a classroom environment where multi-representational approaches were valued. The value of and usage of multiple representations was evident in the work of the students in this study. The classroom norms that were established, the instructor actions that helped established these norms and how these norms influenced student representation use are all beyond the scope of this dissertation. Exploring these factors is an important future direction for research on representation use.

There is much left to do in this area and I hope to continue to contribute to research on student representation use. In the coming year, I will be involved in a large NSF project that seeks to document the representation use patterns of undergraduate mathematics majors, and how it relates to their performance on novel mathematics tasks. Although such analysis has been done with high school students (e.g., Presmeg, 1985), this project is the first to conduct a large-scale examination of representation use with mathematics majors. The experience I have gained in conducting the research in this dissertation will be a valuable stepping-stone for this future work.

Appendix A:

Interview Tasks

Interview One

0a) How would you describe to another student who has missed the last three weeks of class what a derivative is? (Feel free to use the space below to help communicate your idea to the student)

0b) Pat says that a derivative is the slope of a function at a point and Jaime says that a derivative is a function itself. Is your thinking more inline with Pat or more inline with Jaime?

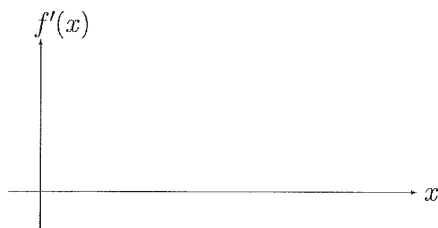
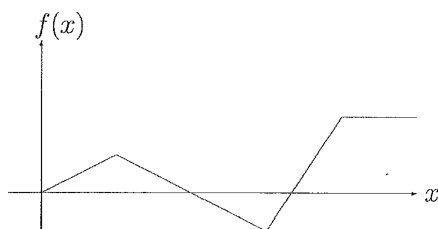
Strongly agree Pat 0 0 0 0 0 Strongly agree with Jaime

- 1) A biologist measures the Piranha population of a two-mile stretch of the Amazon River. She makes measurements twice a week for 6 weeks. Estimate the rate of change of the fish population on week 1.

Week Number	Population (in thousands of fish)
0	2.3
0.5	2.5
1.0	2.8
1.5	3.4
2	3.5
2.5	3.3
3	3.4
3.5	3.4
4	3.2
4.5	3.0
5	2.8
5.5	2.2

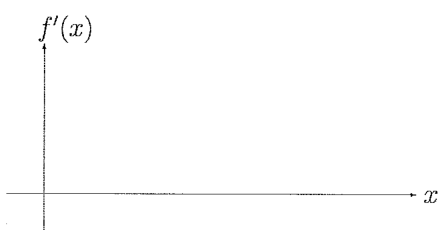
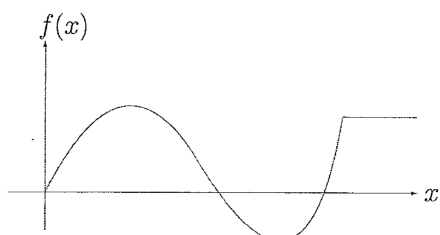
2) Suppose $f(t)$ is a function of time that gives you the outside temperature measured in $^{\circ}\text{F}$. At a particular time of the day, say 3:00pm, how would you interpret the derivative of f being 4 ($f'(3)=4$). (Task can borrowed from Zandieh and Knapp (2006))

3a) Sketch the graph of the derivative of the following function. Please think aloud as you sketch your graph.

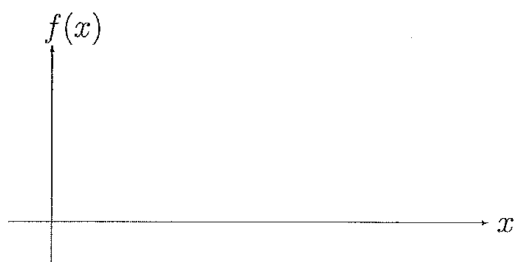
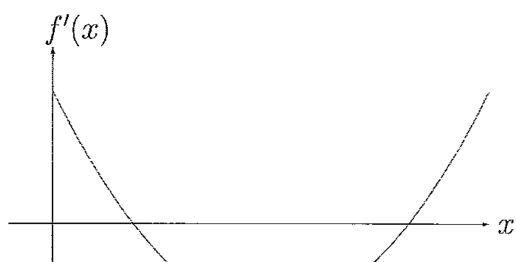


(Task is a modification of the saw-tooth task in Apsinwall and Shaw (2002))

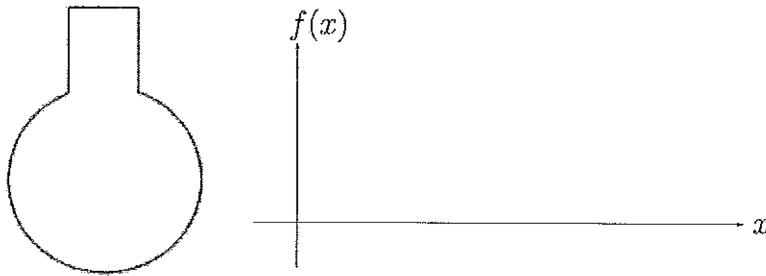
3b) Sketch the graph of the derivative of the following function. Please think aloud as you sketch your graph.



4) Below is the graph of the derivative of a function. Sketch the graph of the original function.



5) Imagine this bottle filling up with water. Sketch a graph of the height as a function of the amount of water.



(Task can borrowed from Carlson, Jacobs, Coe, Larsen, & Hsu, (2002).)

Interview Two

0) Just to get started tell me a little bit about your experiences with course so far?

- a) The use of technology?
- b) Professors' teaching style?
- c) Working with other students in class
- d) Working on problems in class.

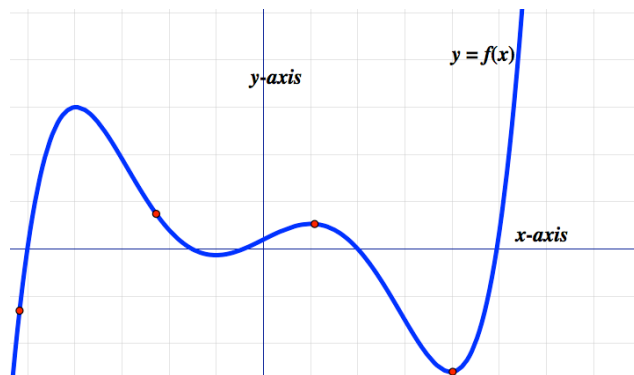
1) Suppose $f(x)$ is a function of time (in seconds) that gives the height (in meters) of a rock being thrown straight up. What does it mean in terms of the situation if $f'(3)=10$?

What does it mean if $f''(4)=-9.8$.

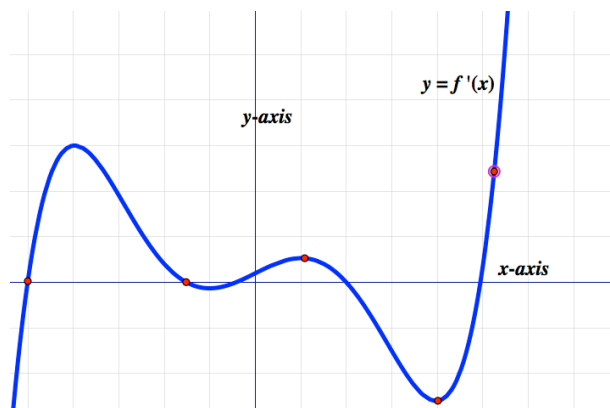
- a) What does the negative sign in front of the "9.8" indicate?
- b) Would you expect f'' to ever be positive? Why or why not?

c) What does $f'(3)=10$ and $f''(4)=-9.8$ mean in terms of the graph.

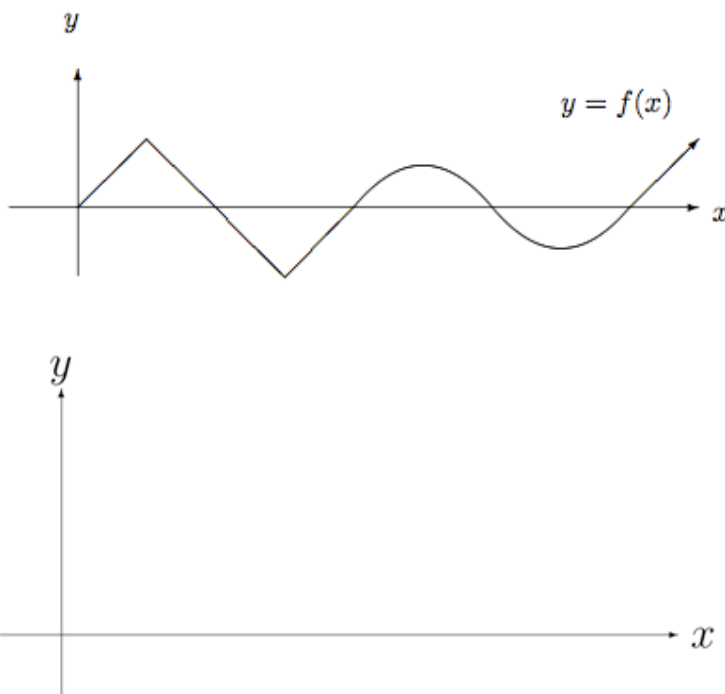
2) The following graph represents a function, $f(x)$. Several points are marked. For each point tell me what you can say about the value of the derivative and second derivative at that point. What do these tell you about the original function?



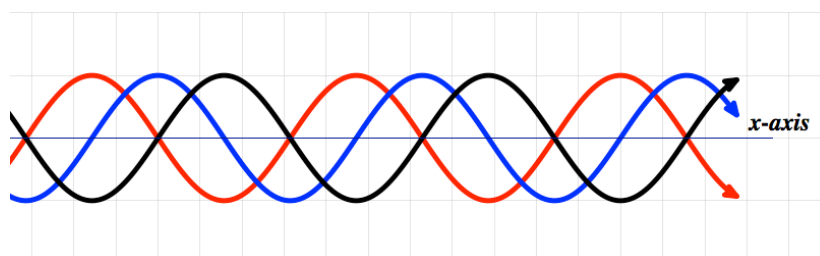
3) The following graph represents the derivative of a function, $f'(x)$. Several points are marked. For each point tell me what you can say about the original function at that point on the graph.



4) Sketch the graph of the derivative of the following function on the space provided below.

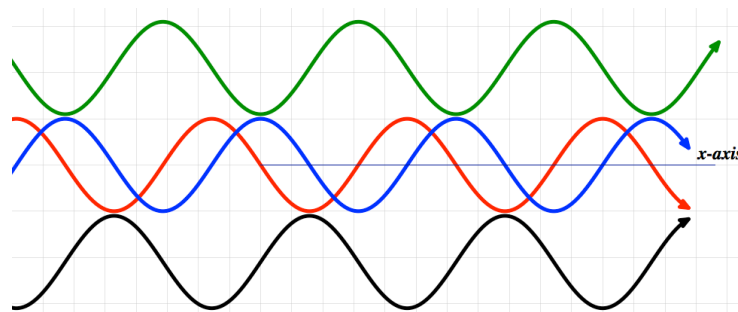


5a) Below are graphs of several different functions. In what way if any do you see how one is related to the other.



- Describe any relationships that you might see between any of these graphs in terms of the derivative.
- How do you know this?
- Are there other ways you could have approached this task.

5b) Below are the graphs of several functions. Describe any ways you see one function is related to one of the other three.



- a) Describe any relationships that you might see between any of these graphs in terms of the derivative.
- b) How do you know this?

6) The distance in miles a car had traveled is described by the function, $f(x)=x^3-6x+3$, where x stands for time in seconds. Determine the times, if any, the car was traveling 69 miles per hour during its journey.

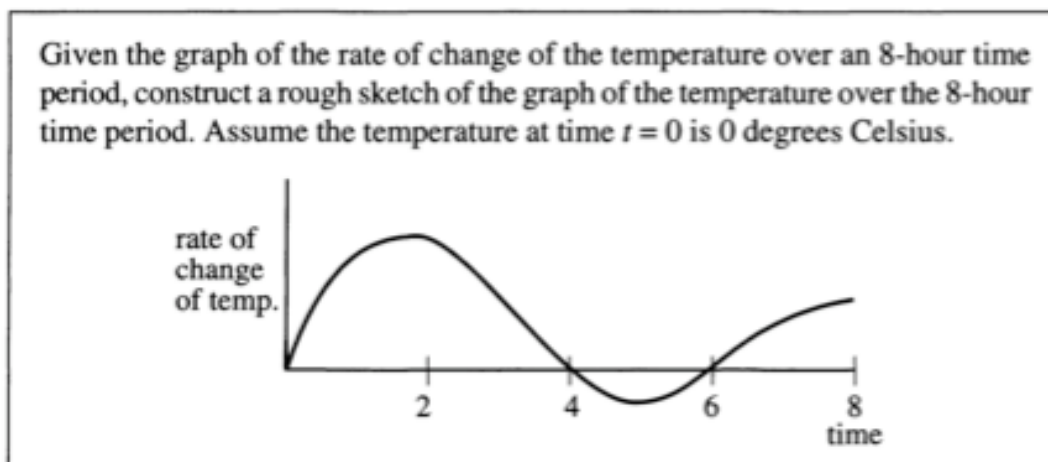
7) A plane flying horizontally at an altitude of 3 miles and a speed of 600 miles/hour passes directly over a radar station. Let the distance from the plane to the radar station be represented by z . What is the rate of change of the distance from the plane to the radar station with respect to time?

Interview Three

1) $f(t)$ gives height of a tree in meters as a function of time (in years) since the tree was planted. Interpret the what each of the following means in terms of the tree: $f(4)=15$, $f(0)=1$, $f'(4)=3$, $f''(4)=-3$. (Task is a modification of the temperature task in Zandieh and Knapp (2006))

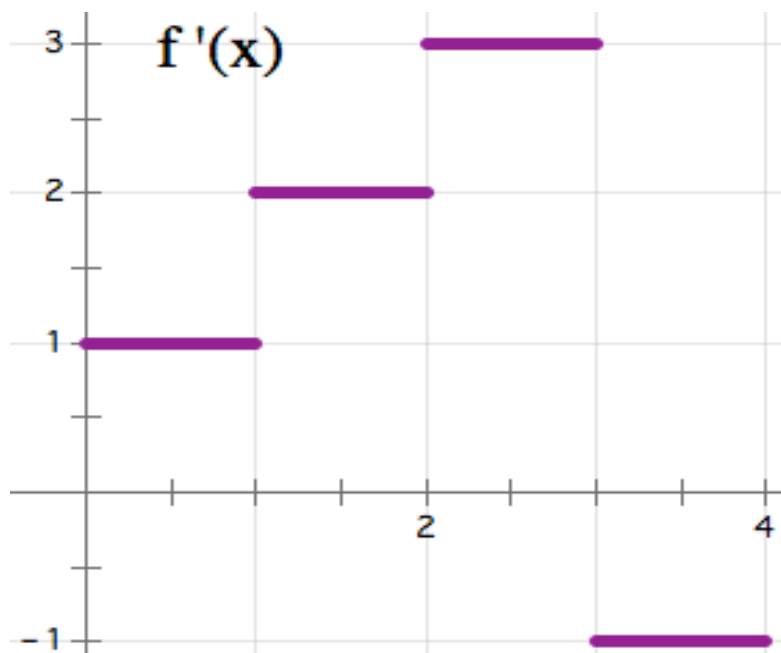
2) A rocket takes off 50 meters away from a viewing station and shoots straight up in the air. Find the rate of change of the angle when the rocket is 50 meters above the ground if the rocket travels at 100 m/s.

3)

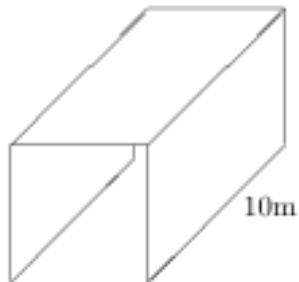


(Task can borrowed from Carlson, Jacobs, Coe, Larsen, & Hsu, (2002).)

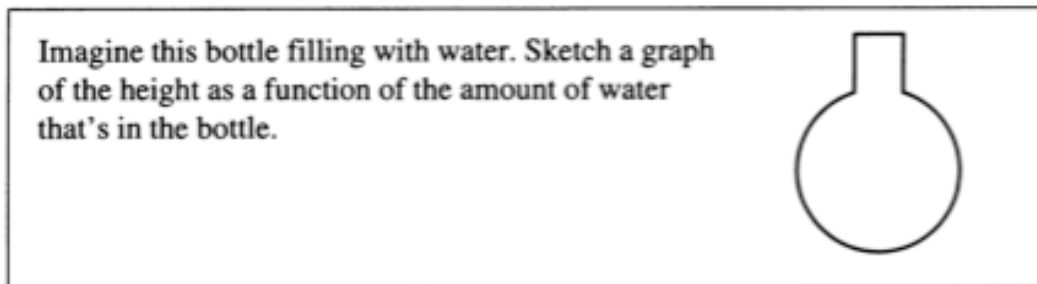
4) $f'(x)$ is drawn below. Sketch the graph of $f(x)$, assuming that $f(x)$ is continuous.



- 5) Jason has a 10m by 10m piece of cardboard he wants to bend into the shape below. How tall must the shape be to maximize its internal volume?

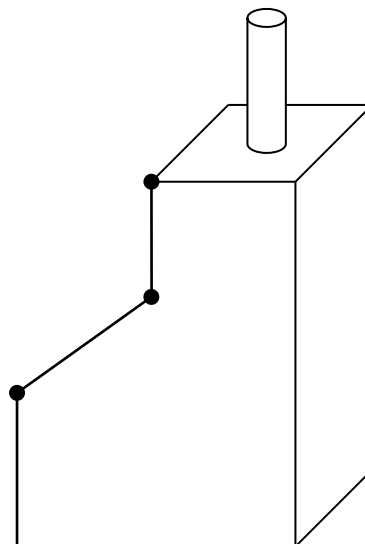


6)



(Task can borrowed from Carlson, Jacobs, Coe, Larsen, & Hsu, (2002).)

7) Water is poured into the following container at a constant rate. Sketch a possible graph to depict the height of water as a function of time. You do not need to label or give scales for the axis, but do highlight points A, B, and C on your graph.



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