Solving additive word problems: Intuitive strategies make the difference

Katarina Gvozdic (katarina.gvozdic@gmail.com)
Paragraphe Lab, EA 349, University Paris 8, Department of Psychology,
2 Rue de la Liberté, 93526 Saint-Denis Cedex 02, France

Emmanuel Sander (emmanuel.sander@univ-paris8.fr)
Paragraphe Lab, EA 349, University Paris 8, Department of Psychology,
2 Rue de la Liberté, 93526 Saint-Denis Cedex 02, France

Abstract

Young children use informal strategies to solve arithmetic word problems. The *Situation Strategy First (SSF)* framework claims that these strategies prevail even after instruction. The present study was conducted with second grade students in order to investigate the persistence of intuitive, situation-based strategies, on word problems that do not involve dynamic temporal changes. This is challenging for the *SSF* framework, since the lack of this dimension might bypass intuitive strategies. The results revealed that intuitive strategies persist, are valid for these types of problems, and impact the problems' difficulty. Indeed problems that require the application of arithmetic principles remain hard, even though they have been practiced at school. These findings provide complementary evidence to how mental calculation strategies articulate with arithmetic word problem solving and call for the extension of the *SSF* framework.

Keywords:arithmetic word problems; problem solving; informal strategies; solution strategies; education.

Introduction

Even before instruction young children can solve arithmetic word problems by usinginformal strategies (Verschaffel& De Corte, 1997). These informal strategies reflect the situation described in the problem and preclude the flexible application of mathematical principles like commutativity, inversion or distributivity (Verschaffel & De Corte, 1997).During the early years of elementary school, children improve their numerical competencies and acquire certain mathematical principles, which could lead us to expect that newly acquired arithmetic competencies would take place over the informal strategies.

Indeed, numerous mental calculation strategies that schooled children develop to solve problems when presented in their arithmetic expression (*e.g.*'8 - 5=') have been documented (*e.g.* Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009). They are mainly determined by the arithmetic operation that provides the solution. For subtraction problems, the principal distinction bears between direct subtraction strategies in which the subtrahend is straightforwardly taken away from the minuend (*e.g.* in which '42 - 39 =' is solved by '42 - 39'), and indirect addition strategies in which the calculation consists in finding how much needs to be added to the minuend to reach the subtrahend (e.g. in which '42 - 39 =' is solved by '39 + . = 42'). In both of these strategies, the arithmetic *operation* that is used is subtraction, it is just the arithmetic *format* that is different (Campbell, 2008). In order to describe how students use the two strategies, Peters, DeSmedt, Torbeyns, Ghesquière, and Verschaffel (2013) provided empirical support for their *Switch* model. According to this model,students solve two-digit subtraction problems by switching between direct subtraction and indirect addition depending on the combination of the magnitude of the subtrahend and the numerical distance between the subtrahend and the minuend.

Brissiaud and Sander (2010) investigated how these mental calculation strategies articulate with the informal strategies students use on arithmetic word problem solving. They proposed a *Situation Strategy First* (*SSF*) framework which posits that the initial representation of a problem activates asituation-based strategy, both before and after instruction. Only when this strategy is not efficientthe representation of the problem may be modified and a set of arithmetic principles may be applied in order to provide an adequate solution in a more efficient way. In their experiments, each problem was presented to second and third grade students in two versions. The first version could be efficiently solved by mentally simulating the actions described in the problem - situation strategy problems (**Si-problems**). For example:

I. Luc is playing with his 42 marbles at recess. During the recess, he loses 3 marbles. How many marbles does Luc have now? [42 - 3 = .]

Problem I is an Si-problem because simulating the action of losing 3 marbles through mentally counting down from 42 is easy to perform(41 (1), 40 (2), 39 (3)). Thus, a situation-based solving strategy, modeling the described situation - the **Si-strategy -** is efficient.

For each Si-problem, a Mental Arithmetic counterpart was introduced (**MA-problem**). MA-problems are problems for which mental simulation is too costly to attain the result – thus for which the Si-strategy is not efficient. On the contrary, the use and application of arithmetic knowledge is efficient and makes the problem easy (**MA-strategy**). For example:

II. Luc is playing with his 42 marbles at recess. During the recess, he loses 39 marbles. How many marbles does Luc have now? [42 - 39 = .]

The solution to problem II cannot be efficiently obtained by using the same procedure as for the first one; mentally simulating the action by counting down 39 marbles would be too costly. However the mental subtraction 42 – 39 is easy when the complement principle is mastered and leads to counting up from 39 to 42.

The findings revealed that even after instruction, the Si-problems remained systematically and significantly easier than the corresponding MA-problems. Furthermore, a higher use of informal strategies was observed on Si-problems, while arithmetic principles were almost exclusively used on MA-problems. For instance, when students succeeded to solve an Si-problem such as Problem I, they exclusively used a direct subtraction strategy, which is the Si-strategy in this case. However, when they succeeded to solve an MA-problem such as Problem II,even though Si-strategies were (scarcely)observed, they were solved to a greater extent by MA-strategies, such as looking for a missing addend in the previous example (‘39 + . = 42’). This was never observed for Si-problems: no child tried to solve a problem such as Problem I by the missing addend'3+ . = 42'.

Indeed, the arithmetic computations of both Si- and MA-strategies on subtraction problems are executed by the aforementioned mental computation strategies. The *Switch* model could accurately account for how the various arithmetic characteristics of the problems tested so far by the *SSF* framework yield a clear computational advantage for one strategy over another. However, the *Switch* model does not provide an explanation for why students fail to apply arithmetic principles, such as it is observed through the significantly lower success rates on MA-problems. Indeed, even though the *Switch* model accurately describes the numerical conditions that require a switch between direct subtraction and indirect addition, it does not account for the mental *re*-representation needed in order to make this switch when a presented strategy cannot be easily performed in the same format as the one it is presented in.

We propose that the attainment of a mental *re*-representation would reflect an underlying conceptual metaphor that guides the interpretation and application of arithmetic principles. Conceptual metaphors are based on everyday human experience. The underlying mathematical ideas are constructed through cognitive mechanisms called fictive motion, which refer to the conception of static entities in dynamic terms (Lakoff & Núñez, 2001). One of the main representations of arithmetic is object collection (Lakoff & Núñez, 2001). The most widespread conceptual metaphor of subtraction that can be drawn from itis "taking away" (Fischbein, 1989; Lakoff and Núñez, 2001). Alternatively, arithmetic can be considered as motion along a path (Lakoff & Núñez, 2001). The conceptual metaphor of subtraction that can be drawn from this conception is subtraction as a measuring stick (Lakoff & Núñez, 2001), or as "determining the difference" (Selter, Prediger, Nührenbörger & Hußmann, 2012). As Selter and collaborators (2012) pointed out, the "taking away" model might be more widespread, however seeing subtraction solely as "taking away" is too one-sided, and both models are required in order to be flexible in mental arithmetic.

We consider that the failure to apply arithmetic principles on MA-problems is due to a restrictive representation of arithmetic, an intuitive representation(such as the "taking away" model), which entails a limited interpretation of the arithmetic situation embedded in the problem statement. Such an extension of the *SSF* framework would also challenge the most commonly used classification of arithmetic word problems introduced by Riley, Greeno and Heller (1983). Their classification determines the difficulty of a problem based on the semantic category it belongs to, while the *SSF* framework puts emphasis on situation-based strategies and proposes thatthe efficiency of such strategies would be also a determining factor of difficulty.

Yet, all the subtraction problems that were tested by Brissiaud and Sander (2010) belonged to one same category of subtraction problems from Riley, Greeno and Heller's (1983) classification - change problems. These problems are dynamic in nature and describe an action with a temporal dimension,soliciting a mental simulation. However, the other problem categories do not involve this temporal dimension. They have been identified as more difficult than change problems, especially "compare" problems, in which a comparison between two quantities is involved and the question bears on the difference or on one of the compared quantities. It therefore remains an open issue if the mental simulation advocated by the *SSF* framework is still relevant for problems that do not unfold along a temporal dimension.

Indeed, if the mental simulation of the problem was not solicited, then we could expect that the distinction between Si- and MA-problems among these categories would lose its relevance. It therefore remains an open issue if the mental simulation advocated by the *SSF* framework is still relevant for problems that do not unfold along a temporal dimension. If it would be demonstrated that the efficiency of the mental simulation influences a problem’s difficulty even when it does not develop along a temporal timeline, it would warrant a broader view of Si-strategies and provide a new criterion for the assessment of problem difficulty, not based only on the semantic category, but also on the efficiency of the Si- strategy.

Aim of the study

The purpose of the study was to demonstrate that the mental simulation of the arithmetic relations is not a mere consequence of a dynamic semantics of the problem, but an intrinsic property of arithmetic problem solving. Firstly, we conducted a longitudinal study in order to test the distinction between Si- and MA-problems in contexts less favorable for a mental simulation. Secondly, we conducted individual verbal reports in order to gather confirmatory evidence of situation-based strategies for Si-problems and a switch to non-situation-based strategies for MA-problems.

Table 1: Example of the problems for the number set (31, 27, 4)presented with different contexts

Problem categories		Si problems	MA problems
Comparison problems	D[b+ . =a]	There are 27 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?	There are 4 roses and 31 daisies in the bouquet. How many daisies are there more than roses in the bouquet?
	D[a- . =b]	There are 31 oranges and 27 pears in the basket. How many pears are there less than oranges in the basket?	There are 31 oranges and 4 pears in the basket. How many pears are there less than oranges in the basket?
	C[b+b'= .]	James has 27 marbles. Steve has 4 marbles more than James. How many marbles does Steve have?	James has 4 marbles. Steve has 27 marbles more than James. How many marbles does Steve have?
	C[a-b= .]	Anna has 31 euros. Susan has 4 euros less than Anna. How many euros does Susan have?	Anna has 31 euros. Susan has 27 euros less than Anna. How many euros does Susan have?
Equalizingproblems	E[b+ . =a]	There are 27 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?	There are 4 oranges and 31 pears in the basket. How many oranges should we add to have as many oranges as we do pears?
	E[a- . =b]	There are 31 roses and 27 daisies in the bouquet. How many roses should we take away in order to have as many roses as we do daisies?	There are 31 roses and 4 daisies in the bouquet. How many roses should we take away in order to have as many roses as we do daisies?
Combine problems	S[b+ . =a]	Mary has 27 euros in her piggybank and she has euros in her pocket. In total, Mary has 31 euros. How many euros does Mary have in her pocket?	Mary has 4 euros in her piggybank and she has euros in her pocket. In total, Mary has 31 euros. How many euros does Mary have in her pocket?
	S[b+b'= .]	There are 27 blue marbles and 4 red marbles in Marc's bag. How many marbles are there in Marc's bag?	There are 4 blue marbles and 27 red marbles in Marc's bag. How many marbles are there in Marc's bag?

Experiment 1

Method

Participants

269 second grade students from 13 classes in 7 schools from working-class neighborhoods participated in the study. The average age of the children in January, when the first test was passed, was 7.62 years (sd = 0.32, 138 girls).

Material

There were 8 addition and subtraction problem types belonging to 3major categories:

- compare problems: difference set(D[b+ . =a], D[a- . =b]) and compared set (C[b+b'= .], C[a-b= .]),
- equalizing problems(E[b+ . =a], E[a- . =b]),
- combine problems(S[b+ . =a], S[b+b'= .]).

The subtraction problems involved two numbers, *a* and *b* (*a*>*b*). The numerical values for *a* were either 42, 41, 33 or 31, while in order to differentiate between Si- and MA-problems the values for *b* were either kept small (3 or 4) or were close to *a* (39, 38, 29 or 27). To create Si-problems the small value of *b* was used for the C[a-b= .], while the *b* value close to *a* was used for D[b+ . =a], D[a- . =b], C[b+b'= .], E[b+ . =a], E[a- . =b]n S[b+ . =a], S[b-b'= .]. To create MA-problems the opposite *b* value was respectively used for each problem, since it would make the Si-strategy costly.

Addition problems S[b+b'= .] and C[b+b'= .] involved two numbers, *b* and *b'*. Both numbers had the same characteristics as *b* for subtraction problems, while the unknown value was equivalent to *a*. To create Si-problems the *b* value close to *a* was presented first, while the small *b* value (*b'*) was presented second. To create MA-problems they were presented in the opposite order.

Thus the numbers involved in the data and the solution are (31, 27, 4), (33, 29, 4), (41,38, 3), and (42, 39, 3).

Note that the number size was not the determining factor in the Si *vs.* MA-problem distinction. In the Si- versions one problem had the *b* value close to *a*, while others had small *b* values[1]. Also the second experiment was conducted to support this, by directly investigating students' strategy use.

[1] Furthermore, in the princeps study that introduced the *SSF* framework, the small *b* values, and the ones close to *a* were equally present in the Si- and MA-versions of the problems.

Four different contexts were used for the wording of the problems: marbles, euros, flowers and fruits.

Table 1 provides examples of each problem category.

Design

Children solved a total of 8 problems created by combining the 8 problems categories in either their Si- or MA- version. Each student therefore solved 4 Si-problems and 4-MA-problems.To control for the impact of position, numerical sets and context, 8 different problem sets were created. Another 8 problem sets were 'mirror' sets in which the Si-version of one problem would be presented in its MA counterpart, while the MA-problem would be presented in its Si-counterpart. Thus, 16 groups of problem sets were created altogether and counterbalanced across classrooms.

Procedure

The experiment was composed of two sessions. The first conducted in January and the second one, strictly identical to the first one, 6 months later, in June. It was administered in the students' classrooms. Each child received an 8 pages booklet. There was a square in the middle of each page in which they wrote their answer. Each problem was read aloud twice to the whole classroom and children had one minute to write down the number that was the solution.

Scoring

The solutions provided by the children were scored with 1 point when the numerical answer was exact, or within the range of plus or minus one of the exact value, in order to take into account mistakes in counting procedures. Any other answer received 0 points. The average of the sum of the scores on Si- and MA-problems was used as the dependent variable and analyses on these scores were carried out.

Results

A first analysis was conducted in order to compare children's average success rates on Si- and MA-problems at the beginning of the year, followed by a second set of analyses in order to compare the success rates on the problems at the end of the school year. A third analysis bore on the progression over the year.

Repeated measure ANOVAs, with the 'Si- *versus* MA-problems' variable (further referred to as Problem type)as within-participant independent variables, were conducted for each session. The analyses of the scores obtained in January showed a highly significant main effect of Problem type on performance (F(1, 268)=98.39, p< 0.001,η^2=0.11).Table2displays the average success rates. Indeed the Si-problems had a 19.57% higher success rate than MA-problems.

In June, there was a significant difference in performance between the two times of testing on Si-problems (F(1,268)= 86.39, p<.001, η^2=0.06) and on MA-problems (F(1,268)=36.58, p<.001, η^2=0.05). Yet, in accordance with our hypotheses, the results still revealed a highly significant main effect of Problem type on performance in June (F(1,268)=119.57, p<.001, η^2=0.13).As displayed in Table2, the Si-problems had a 24.38% higher success rate than MA-problems in June (experiment 2).

Table 2: Average success rates

Averagesuccess rate	January	June
Si-problems	47.86%	64.53%
MA-problems	28.25%	40.15%

Indeed, after performing a repeated measure ANOVA with the Problem type and the times of testing as within-participant independent variables, the results confirmed that there was a significant main effect of Problem type (F(1,268)=171.64, p<.001, η^2=0.12) and a main effect of the Time of testing (F(1,268)=106.19, p<.001, η^2=0.05), but most importantly there was no interaction between the two variables (F(1,268)=3.51, p>.1, η^2=0.001). Thus, as hypothesized, despite the progress made on each problem type throughout the year, the gap in performance persisted between Si- and MA-problems.

In order to test the hypotheses problem per problem, univariate ANOVAs, with the Problem type variable, were conducted for each of the eight problem categories and showed that almost all of the Si-problems were significantly easier than the corresponding MA-problems both in January and in June: D[a - . =b], C[b + b'= .], C[a - b= .], E[b + . =a], E[a - . =b] and S[b + . =a] (3.843 <F(1, 267)<72.501, p< .01, 0.01<η^2<0.20. The D[b+ . =a] seemed to be particularly hard in January when no difference was observed (F(1,267)=0.23, p>.1,η^2=0.001) (27.6% success rate on Si- and 25% on MA-problems), but the Si- *versus* MA-distinction was valid at the end of the year (F(1,267)=15.63, p<.001, η^2=0.06)(47% success rate on Si- and 24% on MA-problems). The single exception for which no difference was observed on either time of testing was the combine superset problem S[b+b'= .] (January F(1,267)=2.69, p>.1, η^2=0.01, June F(1,267)=1.223, p>.1, η^2=0.005), for which a difference was observed in the expected direction but not confirmed by the test (75% and 83% success rate on Si-problems and 66% and 77% on MA-problems, in January and June respectively).

Discussion

The results revealed that the distinction between Si- and MA-problems remain relevant for subtraction and addition word problems that do not evolve along a temporal time-line. Our study shows that indeed, problems efficiently solved by direct modeling strategies remain easier for students even after they acquired more advanced skills in mathematics at the end of the year. The progression between the two sessions did not obliterate the distinction between Si-and MA-problems. The similar progression on Si- and MA- problems might be explained by the advances children made in computational execution of the calculations, or regarding their general comprehension skills.

A second experiment was conducted in order to provide confirmatory evidence that the difference in difficulty between Si- and MA-problems actually results from the preferential use of Si-strategies when they are efficient and for the lack in the application of arithmetic principles when this strategy is inefficient.

Table 3 : solving strategies for each problem category (with an example of the number set (31, 27, 4))

Problem category	Si-problems				MA-problems			
	% correct responses with described strategy	Si-strategy	Non-SI-strategies		% correct responses with described strategy	Si-strategy	Non-SI-strategies	
		Direct modelling strategy	MA-strategy	Other		*Direct modelling strategy*	MA-strategy	Other
		27+ . =31	31-27= .	31- . =27		4+ . =31	31-4= .	31- . =4
D[b + . =a]	66.67%	*92.86%*	*0%*	*7.14%*	19.05%	*50%*	*50%*	*0%*
		31- . =27	31-27= .	27+ . =31		31- . =4	31-4= .	4+ . =31
D[a - . =b]	55.56%	*50%*	*0%*	*50%*	33.33%	*16.67%*	*83.33%*	*0%*
		31-4= .	4+.=31	31- . =4		31-27= .	27+ . =31	31- . =27
C[a - b= .]	47.62%	*100%*	*0%*	*0%*	19.05%	*50%*	*50%*	*0%*
		27+ . =31	31-27= .	31-.=27		4+ . =31	31-4= .	. +4=31
E[b + . =a]	47.62%	*100%*	*0%*	*0%*	38.10%	*50.00%*	*25.00%*	*25.00%*
		31- . =27	31-27= .	27+ . =31		31- . =4	31-4= .	4+ . =31
E[a - . =b]	66.67%	*71.43%*	*14.29%*	*14.29%*	38.10%	*12.50%*	*75%*	*12.50%*
		27+ . =31	31-27= .	31- . =27		4+ . =31	31-4= .	31- . =4
S[b + . =a]	47.62%	*90%*	*0%*	*10%*	33.33%	*28.57%*	*71.43%*	*0%*
		27+4= .	4+27= .			4+27= .	27+4= .	
S[b + b'= .]	85.71%	*100%*	*0%*		90.48%	*5.26%*	*94.74%*	
		27+4= .	4+27= .		23.81%	4+27= .	27+4= .	
C[b + b'= .]	61.90%	*100%*	*0%*			*0%*	*100%*	

Experiment 2

We collected additional information concerning the strategies children actually use when solving Si- and MA-problems. We asked them to solve problems and then to describe their solving strategy aloud. We predicted that the solution strategies which directly model the problem would be predominant for Si-problems but that alternative strategies would emerge for MA-problems.

Method

Participants

42 Grade 2 students from 4 classes in2 different schools from working-class neighborhoods participated in the study. The test occurred in June and the average age of the children on the test was 7.93 years (sd = 0.26, 23 girls). None of the participants participated in the previous experiment.

Material & Design

The same material and design was used as in the first experiment.

Concerning the evaluated strategies, if we take the D[a- . =b] problem as an example, the Si- strategy used to solve it is to start from the largest presented quantity (31) and to double-count downwards until the second quantity is reached: in the Si-problem this would not be costly. The students would describe their solving process as starting from 31 and counting down30(**1**), 29(**2**), 28(**3**), 27(**4**), bearing the answer 4, and noted by the experimenter as 31- . = 27.Yet using the same Si-strategy of double-counting downward in the MA-problem to get from 31 to 4 is a costly procedure. When students would use this strategy they would describe the same solving process: starting at 31 and counting down30 (**1**), 29(**2**), 28(**3**), ... 5(**26**), 4(**27**), bearing the answer 27 and noted by the experimenter as 31- . = 4.

Nevertheless, when applying arithmetic knowledge we can easily know that taking away 4 from 31 provides the correct numerical answer to this MA-problem. One of the possible descriptions of the students' solving process would be to start from 31 and take away 4, with the result no longer being the number of times they counted down, but the number they reached. This Non-Si-,mental arithmetic strategy (MA-strategy) would be noted as '31- 4 = .'.

Procedure

The procedure was identical to the first experiment, except that the test was conducted individually in the school library and that after writing down the numerical answer, the student was asked to explain aloud how he or she found the solution. The possible strategies were established beforehand and there was no ambiguity in their coding. The strategies that the students reported were classified according to table 3 into Si-strategies when the strategy directly modeled the wording of the problem, or into Non-

Si-strategies when the strategy that the student described did not directly model the problem.

Scoring

For both Si- and MA-problems, we computed a score of Si-strategies (Si-score) and Non-Si-strategies (Non-Si-score). If a pupil provided a correct answer and explained a strategy, the nature of this strategy was assessed and contributed 1 point to either the Si-strategy score or the Non-Si-strategy score of the problem type. No points were attributed if a student did not provide a correct response and/or did not describe any strategy after providing the correct answer (only 7.5% of the correct responses were not accompanied by a strategy description). Given that children solved 4 problems of each type, the scores ranged from 0 to 4.

Results

The experiment replicated the previous findings, confirming that Si-problems were easier for children than MA-problems. The success rates were 67.2% and 41.5% respectively, and the variance analysis revealed that this difference was significant ($F(1,41)=17.86$, $p<.001$, $\eta^2=0.13$).

Table 3 shows strategy use for each problem category and the disparities between the two kinds of strategies, among students that provided the right numerical solution and described a strategy.

We further performed two variance analyses using the Si-strategy score and the Non-Si-strategy score as the dependent variables, and Problem type as the within factor variable. The average scores are presented in table 4.As expected, both differences were significant. Si-strategies were used significantly more on Si-problems ($F(1,38)=79.1$, $p<.001$, $\eta^2=.5$), as well as Non-Si-strategies on MA-problems ($F(1,38)= 20.06$, $p<.001$,$\eta^2=.2$).

Table 4 : Si- and Non-Si-score for solving strategies

Problem type	Si-score	Non-Si-score
Si-problem	2.05	0.31
MA-problem	0.26	1.13

Discussion

The variance analyses confirmed that solution strategies which directly model the situation were predominant for Si-problems and that solution strategies which required arithmetic knowledge were predominant for MA-problems. These findings suggest that the selected strategy drives the difference in performance on top of the problem category or factors such as mental calculation competences.

General Discussion and Conclusion

The experiments conducted in the present study account for the spontaneous and intuitive modeling of the situation described in arithmetic word problems, which leads to a primary use of situation strategies and the application of arithmetic principles only when the first one is too costly. The significant difference that was observed between Si- and MA-problems fits with the previous findings on change problems (Brissiaud & Sander, 2010), and confirms that this problem distinction is not specific to problems that evolve along a temporal timeline. These findings complement the traditional classification of arithmetic word problems according to which problem difficulty depends mostly on the problem category. They also provide evidence that situation strategies are not only tied to the semantic wording of a problem, but could be a fundamental property solicited by arithmetic problems.

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