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Publication Date

1978-02-01

Submitted to Journal of Mathematical
Physics

LBL-7418
Preprint

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February 1978

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A UNIFIED RADON INVERSION FORMULA[†]

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[†] This work was done with support from the U.S. Department of Energy

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ABSTRACT

A Radon inversion formula which holds in spaces of even or odd dimension n is obtained for functions which admit to a certain general decomposition. The inversion formula which is one member of a Gegenbauer transform pair is used to generate some interesting definite integrals involving special functions. Legendre and Tchebycheff transform pairs are discussed as special cases of the general result.

1. INTRODUCTION

The work reported here is part of an ongoing effort^{1,2} to obtain a better understanding of the integral equations which emerge in a natural way when studying the Radon transform on Euclidean space R^n . The theory of this transform is the foundation for an enormous number of diverse applications. These range from macroscopic to microscopic (astrophysics and molecular biology for example) and include medical applications as an intermediate case. In all of these applications the central aim is to obtain certain information about the internal structure of an object (or collection of objects) either by passing some probe (such as x-rays) through the object or by making use of the fact that the object itself is a self-emitting source, such as an organ in the body which contains a radioactive isotope or the interior of the earth when motions occur. Reviews of many of these applications may be found in the articles by Budinger and Gullberg,³ Gordon and Herman,⁴ and Brooks and DiChiro.⁵

The Radon transform is not new, having originated with a paper by Radon⁶ in 1917. Since that time various authors have contributed to an understanding of many technical aspects of the transform.⁷⁻¹² Our current purpose is to supplement this understanding with emphasis focused on the integral equation and special function aspects. The earlier work along these lines^{1,2} was primarily devoted to a study of even dimension. There were two main reasons for this. First, nearly all applications thus far have been associated with $n = 2$. Second, the inversion formula for even dimension is somewhat more challenging, involving a Hilbert transform which does not appear in the odd n case. By making use of some recent

work by Durand, Fishbane, and Simmons¹³ it was possible to do the Hilbert transform for the even case and obtain a rather convenient inversion formula for all even dimensions.²

In Sec. 3 we shall show that the corresponding result for odd n differs only by an overall multiplicative factor of -1 . And in Sec. 4, we simply write down a unification formula. In retrospect, this unification appears reasonable and with hindsight it is possible to say, with some enthusiasm, that the formula (17) which holds for both even and odd n is just what one would expect since there ought not to be such fundamental differences between even and odd n as n gets large. In this regard we observe that it is especially important to have an expression like (17) when extending the theory to very large n .

In Sec. 5 we take a close look at the special case $n=3$, which leads to a Legendre transform pair. In Sec. 6 the $n=4$ special case is discussed. In many respects this is similar to the $n=2$ case¹ with Tchebycheff polynomials of the first kind replaced by Tchebycheff polynomials of the second kind. Finally, in Sec. 7 we generalize results which appear in the special cases and find a rather interesting definite integral formula which appears to have been overlooked in the major tabulations.

2. THE RADON TRANSFORM

Let $F(x) = F(x_1, \dots, x_n)$ be a function of n real variables and let x be a vector in R^n . The Radon transform of F is given by⁷

$$f(\xi, p) = R\{F\} = \int F(x) \delta(p - \xi \cdot x) dx, \quad (1)$$

where ξ is a unit vector; p is a real number, $\xi \cdot x = \xi_1 x_1 + \dots + \xi_n x_n$, $dx = dx_1 \dots dx_n$, δ is the Dirac δ function, and the integral is over the entire space. For our purposes here it is assumed that F is a rapidly decreasing C^∞ function.¹⁴

Suppose $F(x)$ can be decomposed in the form

$$F(x) = G_\ell(r) S_{\ell m}(\hat{x}) \quad (2)$$

where $\hat{x} = x/|x|$, $r = |x|$, and $S_{\ell m}(\hat{x})$ is a real generalized spherical harmonic. (For a full discussion of the $S_{\ell m}$, see Hochstadt.¹⁵) Then it follows that f also admits to a similar decomposition,

$$f(\xi, p) = g_\ell(p) S_{\ell m}(\xi), \quad (3)$$

and g_ℓ is a rapidly decreasing C^∞ function. If g has negative argument, the defining equation is taken to be the symmetry condition,

$$g_\ell^{(k)}(-p) = (-1)^{\ell+k} g_\ell^{(k)}(p), \quad (4)$$

where

$$g_\ell^{(k)}(p) = \left(\frac{d}{dp}\right)^k g_\ell(p). \quad (5)$$

The functions G_ℓ and g_ℓ are related by² ($p \geq 0$)

$$g_\ell(p) = \frac{(4\pi)^\nu \Gamma(\ell+1)\Gamma(\nu)}{\Gamma(\ell+2\nu)} \int_p^\infty r^{2\nu} G_\ell(r) C_\ell^\nu\left(\frac{p}{r}\right) \left[1 - \frac{p^2}{r^2}\right]^{\nu-\frac{1}{2}} dr \quad (6)$$

where $\nu = \frac{1}{2}(n-2)$ and the dimensionality n may be even or odd. The functions C_ℓ^ν which appear in (6) are Gegenbauer polynomials of the first kind.¹³

3. THE INVERSION FORMULA FOR ODD DIMENSION

The general inversion formula may be written as an integration over a unit sphere in ξ space,¹¹

$$F(x) = \int_{\Omega} f^*(\xi, \xi \cdot x) d\Omega_\xi, \quad (7)$$

where f^* is found from f by the equation

$$f^*(\xi, \xi \cdot x) = T f(\xi, p). \quad (8)$$

For odd n the operator T is defined by

$$f^*(\xi, t) = \frac{(-1)^{\frac{1}{2}(n-1)}}{2(2\pi)^{n-1}} \left(\frac{\partial}{\partial p} \right)^{n-1} f(\xi, p) \Big|_{p=t} \quad (9)$$

(For even n the operator T involves a Hilbert transform. This case is discussed in detail in Ref. 2.)

For the decompositions (2) and (4) it is straightforward to obtain the equation

$$G_\ell(r) = \frac{(-1)^{\frac{1}{2}(n-1)} M_\ell^\nu}{2(2\pi)^{n-1}} \int_{-1}^{+1} g_\ell^{(n-1)}(rt) C_\ell^\nu(t) (1-t^2)^{\nu-\frac{1}{2}} dt \quad (10)$$

by use of the Hecke-Funk theorem.¹⁵ In (10), the normalization factor M_ℓ^ν is given by

$$M_\ell^\nu = \frac{(4\pi)^\nu \Gamma(\ell+1) \Gamma(\nu)}{\Gamma(\ell+2\nu)} \quad (11)$$

By use of the symmetry properties of g_ℓ and C_ℓ^ν the limits of integration may be changed, $\int_{-1}^{+1} \rightarrow 2 \int_0^1$, and by the change of variable $t \rightarrow \frac{t}{r}$ we have

$$G_\ell(r) = \frac{(-1)^{\frac{1}{2}(n-1)} M_\ell^\nu}{(2\pi)^{n-1} r} \int_0^r g_\ell^{(n-1)}(t) C_\ell^\nu\left(\frac{t}{r}\right) \left(1 - \frac{t^2}{r^2}\right)^{\nu-\frac{1}{2}} dt. \quad (12)$$

But the integral $\int_0^r$ can be modified by making use of $\int_0^r = \int_0^\infty - \int_r^\infty$. This yields

$$G_{\ell}(r) = I_{\ell}^{\nu} - \frac{(-1)^{\frac{1}{2}(n-1)} M_{\ell}^{\nu}}{(2\pi)^{n-1} r} \int_r^{\infty} g_{\ell}^{(n-1)}(t) c_{\ell}^{\nu}\left(\frac{t}{r}\right) \left(1 - \frac{t^2}{r^2}\right)^{\nu-\frac{1}{2}} dt, \quad (13)$$

where

$$I_{\ell}^{\nu} = \frac{(-1)^{\frac{1}{2}(n-1)} M_{\ell}^{\nu}}{(2\pi)^{n-1} r} \int_0^{\infty} g_{\ell}^{(n-1)}(t) c_{\ell}^{\nu}\left(\frac{t}{r}\right) \left(1 - \frac{t^2}{r^2}\right)^{\nu-\frac{1}{2}} dt. \quad (14)$$

In the Appendix it is shown that (14) vanishes. Thus,

$$G_{\ell}(r) = \frac{(-1)^{\frac{1}{2}(n+1)} (-1)^{\nu-\frac{1}{2}} M_{\ell}^{\nu}}{(2\pi)^{n-1} r} \int_r^{\infty} g_{\ell}^{(n-1)}(t) c_{\ell}^{\nu}\left(\frac{t}{r}\right) \left(\frac{t^2}{r^2} - 1\right)^{\nu-\frac{1}{2}} dt, \quad (15)$$

or, making use of $\nu = \frac{1}{2}(n-2)$ and (11),

$$G_{\ell}(r) = \frac{\Gamma(\ell+1) \Gamma(\nu)}{2\pi^{\nu+1} \Gamma(\ell+2\nu) r} \int_r^{\infty} g_{\ell}^{(n-1)}(t) c_{\ell}^{\nu}\left(\frac{t}{r}\right) \left(\frac{t^2}{r^2} - 1\right)^{\nu-\frac{1}{2}} dt, \quad (16)$$

for odd n .

4. THE UNIFICATION

In Ref. 2 the result for even n differed from (16) by a negative sign. Hence, it is possible to write a unified formula, valid for even or odd n . In terms of $\nu = \frac{1}{2}(n-2)$,

$$G_{\ell}(r) = \frac{(-1)^{2\nu+1} \Gamma(\ell+1) \Gamma(\nu)}{2\pi^{\nu+1} \Gamma(\ell+2\nu) r} \int_r^{\infty} g_{\ell}^{(2\nu+1)}(t) C_{\ell}^{\nu}\left(\frac{t}{r}\right) \left[\frac{t^2}{r^2} - 1\right]^{\nu-\frac{1}{2}} dt \quad (17)$$

Equations (17) and (6) constitute a Gegenbauer transform pair.

5. SPECIAL CASE $n = 3$.

If $n = 3$ (or $\nu = \frac{1}{2}$) the Gegenbauer transform pair becomes the Legendre transform pair. (Here x is a real variable and not a vector.)

$$g_{\ell}(s) = 2\pi \int_s^{\infty} x G_{\ell}(x) P_{\ell}\left(\frac{s}{x}\right) dx, \quad (18)$$

$$G_{\ell}(x) = \frac{1}{2\pi x} \int_x^{\infty} g_{\ell}''(t) P_{\ell}\left(\frac{t}{x}\right) dt. \quad (19)$$

Note that if $2\pi x G_{\ell}(x)$ is replaced by $G(x)$ and if $g_{\ell}(s)$ is replaced by $g(s)$ we obtain the pair

$$g(s) = \int_s^{\infty} G(x) P_{\ell}\left(\frac{s}{x}\right) dx, \quad (20)$$

$$G(x) = \int_x^{\infty} g''(t) P_{\ell}\left(\frac{t}{x}\right) dt. \quad (21)$$

It is of interest to try a direct verification that (21) does indeed satisfy (20). When (21) is substituted into (20) we obtain

$$\int_s^{\infty} dx P_{\ell}\left(\frac{s}{x}\right) \int_x^{\infty} dt g''(t) P_{\ell}\left(\frac{t}{x}\right), \quad (22)$$

and by changing the order of integration (22) becomes $(0 < s \leq t)$

$$\int_s^{\infty} dt g''(t) \int_s^t dx P_{\ell}\left(\frac{s}{x}\right) P_{\ell}\left(\frac{t}{x}\right). \quad (23)$$

We designate the integral on the right by $K_{\ell}^{\nu}(t,s)$ with $\nu = \frac{1}{2}$,

$$K_{\ell}^{\frac{1}{2}}(t,s) = \int_s^t P_{\ell}\left(\frac{s}{x}\right) P_{\ell}\left(\frac{t}{x}\right) dx = t-s. \quad (24)$$

After substituting (24) into (23) and doing one integration by parts we have

$$-\int_s^{\infty} g'(t) dt = g(s) \quad (25)$$

as required.

6. SPECIAL CASE $n = 4$.

If $n = 4$, or equivalently $\nu = 1$, the Gegenbauer transform pair becomes the Tchebycheff pair

$$g_\ell(s) = \frac{4\pi}{\ell+1} \int_s^{\infty} x^2 G_\ell(x) U_\ell\left(\frac{s}{x}\right) \left(1 - \frac{s^2}{x^2}\right)^{\frac{1}{2}} dx, \quad (26)$$

$$G_\ell(x) = \frac{-1}{2\pi^2(\ell+1)x} \int_x^{\infty} g_\ell'''(t) U_\ell\left(\frac{t}{x}\right) \left(\frac{t^2}{x^2} - 1\right)^{\frac{1}{2}} dt. \quad (27)$$

If, as in the previous section, we attempt a direct verification that (27) satisfies (26) we find, after changing the order of integration, that we must evaluate the integral

$$K_\ell^1(t,s) = \int_s^t x U_\ell\left(\frac{s}{x}\right) U_\ell\left(\frac{t}{x}\right) \left(1 - \frac{s^2}{x^2}\right)^{\frac{1}{2}} \left(\frac{t^2}{x^2} - 1\right)^{\frac{1}{2}} dx. \quad (28)$$

The result is

$$K_{\ell}^1(t,s) = \frac{\pi}{4} (\ell+1)^2 (t-s)^2. \quad (29)$$

After doing two integrations by parts the verification is immediate.

7. AN INTEGRAL FORMULA

If a direct verification that (17) satisfies (6) is attempted, as was done with the $n = 3$ and $n = 4$ special cases, we are led to the formula ($0 < s \leq t$)

$$\begin{aligned} K_{\ell}^{\nu}(t,s) &= \int_s^t x^{2\nu-1} C_{\ell}^{\nu}\left(\frac{s}{x}\right) C_{\ell}^{\nu}\left(\frac{t}{x}\right) \left[1 - \frac{s^2}{x^2}\right]^{\nu-\frac{1}{2}} \left[\frac{t^2}{x^2} - 1\right]^{\nu-\frac{1}{2}} dx \\ &= \frac{\pi}{2^{2\nu-1}} \left[\frac{\Gamma(\ell+2\nu)}{\Gamma(\ell+1)\Gamma(\nu)} \right]^2 \frac{(t-s)^{2\nu}}{\Gamma(2\nu+1)}, \end{aligned} \quad (30)$$

where $\nu = \frac{1}{2}, 1, \frac{3}{2}, \dots$ and $\ell = 0, 1, 2, \dots$. The $\nu = 0$ case is obtained by multiplying (30) by $1/\nu^2$ and taking the limit as $\nu \rightarrow 0$.

A search through some of the more extensive sources for integral formulas^{16,17} seems to indicate that this result has been overlooked in the tabulations.

APPENDIX

We wish to show that the integral I_ℓ^v defined in (14) vanishes.

Consider the integral

$$J = \int_0^\infty g_\ell^{(n-1)}(t) c_\ell^v\left(\frac{t}{r}\right) \left(1 - \frac{t^2}{r^2}\right)^{v-\frac{1}{2}} dt. \quad (A1)$$

Now, for odd n (3, 5, 7, ...), $c_\ell^v(x)(1-x^2)^{v-\frac{1}{2}}$ is a polynomial of degree $\ell + n - 3$ in x . Let us designate this polynomial by $Q_{\ell+n-3}(x)$; then (A1) becomes

$$J = \int_0^\infty g_\ell^{(n-1)}(t) Q_{\ell+n-3}\left(\frac{t}{r}\right) dt. \quad (A2)$$

After doing $n-1$ integrations by parts this becomes

$$J = r^{1-n} \int_0^\infty g_\ell(t) Q_{\ell+n-3}^{(n-1)}\left(\frac{t}{r}\right) dt. \quad (A3)$$

Observe that the integrated part always vanishes since $g_\ell^{(k)}$ vanishes at ∞ , $Q_{\ell+n-3}(-x) = (-1)^\ell Q_{\ell+n-3}(x)$, and the symmetry condition (4) holds.

Since the polynomial $Q_{\ell+n-3}^{(n-1)}$ is of degree $\ell-2$ we redesignate it by $Q_{\ell-2}$ and observe that $Q_{\ell-2}(-x) = (-1)^\ell Q_{\ell-2}(x)$. If g_ℓ in (A3) is replaced by (6) the integral J is proportional to

$$\int_0^\infty dt Q_{\ell-2}\left(\frac{t}{r}\right) \int_t^\infty x^{2\nu} G_\ell(x) C_\ell^\nu\left(\frac{t}{x}\right) \left(1 - \frac{t^2}{x^2}\right)^{\nu-\frac{1}{2}} dx \quad (A4)$$

If the order of integration is reversed then

$$J \propto \int_0^\infty dx x^{2\nu} G_\ell(x) \int_0^x dt Q_{\ell-2}\left(\frac{t}{r}\right) C_\ell^\nu\left(\frac{t}{x}\right) \left[1 - \frac{t^2}{x^2}\right]^{\nu-\frac{1}{2}}. \quad (A5)$$

By the symmetry of the integrand the $\int_0^x$ integral can be written as $\frac{1}{2} \int_{-x}^x$ and by a change of variable $t \rightarrow tx$,

$$J \propto \int_0^\infty dx x^{2\nu+1} G_\ell(x) \int_{-1}^{+1} dt Q_{\ell-2}\left(\frac{xt}{r}\right) C_\ell^\nu(t) (1-t^2)^{\nu-\frac{1}{2}}. \quad (A6)$$

Since $Q_{\ell-2}$ is a polynomial of degree $\ell-2$ in t it follows by the orthogonality property of the Gegenbauer polynomials that $J = 0$. Hence, I_ℓ^ν defined in (14) vanishes.

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This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

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