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Proceedings of the Annual Meeting of the Cognitive Science Society

Title

Coordinate-free Thinking

Permalink

<https://escholarship.org/uc/item/78j5p3bz>

Journal

Proceedings of the Annual Meeting of the Cognitive Science Society, 35(35)

ISSN

1069-7977

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Publication Date

2013

Peer reviewed

Coordinate-free Thinking

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Abstract

Thinking outside the box is a nice metaphor, but we are so used to putting things in boxes that it may seem essential to our way of thinking. This idea is enforced by the standard view of relations that says that relations always come in a certain order. There is, however, an alternative view on relations, the *antipositionalist view*, according to which the constituents of the complexes of a relation neither come in a certain order, nor do they occupy positions. Instead, a relation is conceived as a network of interrelated complexes. We show that this view facilitates a coordinate-free way of thinking and that it may thus have a heuristic value.

Keywords: Antipositionalism; relational complex; relation; substitution.

Introduction

Consider the following state ‘out there’:



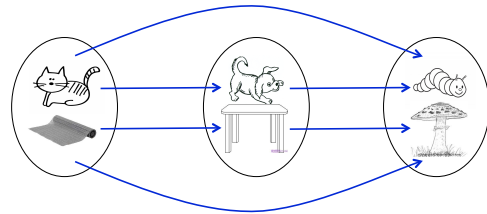
According to the *standard view* on relations, we have *two* distinct complexes here: *the cat’s being on top of the mat* and *the mat’s being under the cat*. So, in the first complex the cat comes first and in the second complex the mat comes first. However, in the state itself, there is no such order. This makes the standard view weak.

A better view on relations is the *positionalist view*. According to this view, each relation comes with a number of *positions* to which objects can be assigned. The state above can be expressed in a neutral way by assigning the cat to the position *Top* and the mat to the position *Bottom*. But for symmetric relations like the adjacency relation, we would get *two* indistinguishable complexes for a single state.¹ This makes also this view objectionable.

Kit Fine developed a radically different view on relations, the so-called *antipositionalist view* (Fine, 2000). In this view,

¹To see this, suppose that the adjacency relation has two positions *Next* and *Nixt*. Now let object *a* occupy the position *Next* and object *b* the position *Nixt*. Then switching the positions of the objects would yield another complex for the same state.

the objects of a complex neither come in a certain order, nor are objects assigned to positions. Instead, a relation consists of a network of the complexes interrelated by substitutions. We may illustrate this as follows.



If we substitute in the complex of the cat on the mat a dog for the cat and a table for the mat, we get the complex of the dog on the table. Furthermore, substituting in the last complex a caterpillar for the dog and a mushroom for the table gives the same result as substituting in the original complex the caterpillar for the cat and the mushroom for the mat.

Since the antipositionalist view rejects order and positions as fundamental, it is in fact a *coordinate-free view* on relations.

A detailed comparison of mathematical models for the different views on relations provide strong support for the claim that antipositionalism is the superior view on relations (Leo, 2008, 2010, 2013b). Nevertheless, the view does not seem to be in line with our ordinary way of thinking and of expressing ourselves. Our natural languages are linear and in many languages we exploit the linear order to our advantage to express factual situations concisely. We say, for example, “the cat is on the mat” and we know that by mentioning the cat *first* that he or she is on top. We may, however, be misled into thinking that the order is essential for the underlying relation.

This misconception is reinforced by standard logic, which also imposes an order that is not present in the represented states ‘out there’ in reality. In fact, standard logic functions like a distorting mirror. It does not faithfully represent reality. However, of an impeccable logic we expect that it can represent reality in a very pure and natural way. Only, such a logic does not yet exist.

In this paper a sketch will be given of a new logic of relations that matches well with a coordinate-free way of thinking. Furthermore some ideas will be given for utilization of the new logic.

A coordinate-free view of the world

Suppose you had to make clear that a block *a* is on top of another block *b*, but that you were not allowed to use expressions in which the order *a* and *b* are mentioned is relevant for

the meaning, and also that you are not allowed to use positions like *Top* and *Bottom*. In other words, you would have to give a *coordinate-free* account of the state.

Then what you could do is point at a situation where *a* is indeed on top of *b*. Now there is a big chance that someone else will not directly know what kind of relationship you mean between *a* and *b*. It might, for example, be thought that you wanted to express that *a* and *b* are close to each other. But it would help if you would point at a lot of other situations and in case there is a vertical placement between two objects you would say “This is another state of the same relation that can be obtained from the original state by substituting this object for *a* and that object for *b*,” and for dissimilar situations you would say “This state is not of the same relation.”

The relation could in this way be grasped without using a specific order for the objects and also without using positions. In this setting, we understand the relation by explicitly using the notion of substitution. A very interesting question is how much of the world we could understand in terms of substitutions.

I consider substitution as a primitive kind of operation. We have, for example, a clear understanding of what it means to substitute Romeo for Adam in Adam’s loving Eve. What is less clear, however, is how important the notion of substitution is for our understanding of the world.

When you think about Adam’s loving Eve and of Romeo’s loving Juliet, you probably do not think explicitly in terms of substitutions. Rather, you think in terms of the love relation applied in a certain way to the persons involved. This is in line with the standard view and the positionalist view on relations, which considers a relation as something with ‘holes’ in which things can be put.

As I argued before, the standard view and the positionalist view on relations are wrong.² But this does not mean that the views are also harmful. In Leo (2010), an explicit justification was given for using positional representations. Moreover, positional representations are very practical.

Nevertheless, realizing that on a fundamental level there are no positions in relations is a liberating thought. It opens the door to explore new ways of learning and of looking at the world.

Because substitution may be an essential ingredient for any truly intelligent system, the insights are likely also relevant for the field of artificial intelligence. One of the challenges will be to investigate whether and how a general notion of substitution can be implemented in artificial systems.

Developing a logic of relations

The motivation for developing a logic of relations is that we like to have a formal framework that captures the essence of ‘real’ relations. In this section, a short description is given of a new logic in which we can reason about relations in a coordinate-free way. Although the technical details are kept

²Views based on thematic roles are also objectionable; they cannot handle certain cyclic relations. (cf. Fine (2000, p. 17, note 10)).

to a minimum, some basic knowledge of predicate logic will be assumed.

Let us start with the love relation. In predicate logic this relation is represented by a binary predicate symbol, say *L*, and the state of Adam’s loving Eve is represented by a formula like *L(a, e)*. The order of the arguments *a* and *e* play a role for the interpretation. But it makes no sense to say that in the state itself Adam comes first. The key question is: *How can we get rid of the order?*

The main idea

As in predicate logic, the new logic also has *terms* and *formulas*, where the terms represent entities in the world (or in our domain of discourse) and the formulas represent assertions about these entities. For example, a term may represent Adam and a formula may represent the assertion that Adam loves Eve.

To prevent problems with the order, we will in our new logic not accept terms $F(x_1, \dots, x_n)$ and formulas $P(x_1, \dots, x_n)$ for any $n \geq 2$. At the same time we do not want to lose anything of the expressive power of predicate logic. Fortunately this is possible.

The main idea for a new logic of relations is to use terms to represent not only individuals, but also all kinds of complex entities ‘out there’ and to build formulas from the terms only with equality and normal logical connectives and quantifiers. So, we will get rid of predicates altogether—with the exception of equality.

The terms may, for example, not only represent persons like Adam and Eve, but also complexes like Adam’s loving Eve and they may even represent substitutions like the one from Adam’s loving Eve to Romeo’s loving Juliet. Formally, for these entities the terms might look like:

$$a, e, L_{ae}, s$$

In addition, we have terms like $\text{src}(s)$, representing the *source* of the substitution *s*, i.e. the complex of Adam’s loving Eve, $\text{tgt}(s)$, representing its *target*, i.e. Romeo’s loving Juliet, and $s(r)$, representing the substitution of Adam by Romeo in the original complex.

As we said, the formulas represent assertions about the terms. To express in a formula that Adam loves Eve we could say

$$L_{ae} = L_{ae}$$

This might look as something that is trivially true, but this is not the case since we do not assume that all terms have an interpretation. It is similar to what we have in natural languages with non-referring terms like ‘Vulcan’, ‘ether’, ‘Santa Claus’, and ‘5 loves Eve’, and in arithmetic with terms like ‘ $\frac{1}{0}$ ’. The way equations are interpreted in our logic guarantees that if $t = t'$, then both *t* and *t'* have an interpretation. In other words, for an equality assertion to be true, the existence of the interpretation of the terms is required.

Because we need existence assertions so often, we abbreviate the formula above as

$$E! L_{ae}$$

Now if we would like to express that Eve loves Adam as well, then this might require a much larger formula involving src , tgt , $=$, logical connectives and quantifiers. For this reason, it is convenient to introduce abbreviations like:

$$E! L_{ae}[a \mapsto e, e \mapsto a]$$

Here $L_{ae}[a \mapsto e, e \mapsto a]$ stands for the result of substituting e for a and a for e in the complex L_{ae} .

What I presented here is only an example. It is just to give an impression of the *logic of relations*. In Leo (2013a) a more detailed and formal description of the syntax and semantics of the logic is given.

Axiomatization

For the logic of relations, a rather straightforward axiomatization can be given. Here, we do not give the more general axioms, but only axioms that say something specific about complexes and substitutions.

The logic has two constants:

$$\text{src}, \text{tgt}$$

The constants will be interpreted as partial functions that give the source and the target of a substitution.

For convenience, we give a few definitions:

$$\begin{array}{ll} E! t : & t = t \\ t \simeq t' : & E! t \vee E! t' \rightarrow t = t' \\ t \text{ is a complex} : & \exists s (t = \text{src}(s)) \\ t \text{ in } t' : & \exists s (\text{src}(s) = t' \wedge E! s(t)) \end{array}$$

The first formula, $E! t$, says that (the interpretation of) the term t is defined. The formula $t \simeq t'$ expresses *weak equality* between the terms t and t' , i.e. if either t or t' is defined, then so is the other and their contents are the same. The formula ‘ t is a complex’ is true if and only if t is the source of a substitution. And the last formula, t in t' , expresses that t is an object that belongs to t' , which is the case if t' is the source of a substitution for which $s(t)$ is defined.

The axioms are as follows :

Source and target axioms

Any substitution has a source and a target:

$$\forall s (E! \text{src}(s) \leftrightarrow E! \text{tgt}(s))$$

The target of a substitution is a complex as well:

$$\forall s (E! \text{tgt}(s) \rightarrow \text{tgt}(s) \text{ is a complex})$$

Constituents axiom

For any complex, all substitutions are defined for the same objects:

$$\forall x \forall s (x \text{ in } \text{src}(s) \rightarrow E! s(x))$$

Extensionality of substitutions axiom

A substitution is uniquely determined by what objects are substituted for the constituents of a complex:

$$\forall s, s' (\text{src}(s) = \text{src}(s') \wedge \forall u (s(u) \simeq s'(u)) \rightarrow s = s')$$

Identity of substitutions axiom

Substituting for each object of a complex the same object results in the same complex:

$$\begin{array}{l} \forall x (x \text{ is a complex} \rightarrow \exists s (\text{src}(s) = x \wedge \text{tgt}(s) = x \wedge \\ \forall u (u \text{ in } x \rightarrow s(u) = u)) \end{array}$$

Composition of substitutions axiom

Substitutions can be composed like partial functions:

$$\forall s, s' (\text{tgt}(s) = \text{src}(s') \rightarrow \exists s'' (\text{src}(s'') = \text{src}(s) \wedge \text{tgt}(s'') = \text{tgt}(s') \wedge \forall u (s''(u) \simeq s'(s(u))))))$$

Furthermore, to make deductions we use *modus ponens* as our single rule of inference:

$$\text{from } \varphi \text{ and } \varphi \rightarrow \psi, \text{ infer } \psi$$

With these axioms and this rule of inference we have a powerful system to reason about all kinds of relational structures in a natural way.

Advantages of the logic of relations

With this design of the new logic we get the same expressive power as that of first-order predicate logic. In addition, it has some significant advantages compared to predicate logic:

1. We got rid of the artificial order of the arguments of a relation. The logic allows us to express relations in a neutral way. This applies not only to everyday relations like the love relation, but to mathematical relations as well. We speak of the *less-than* relation and the *greater-than* relation, but it would be more natural and correct to say that there is just a *single* strict ordering relation. In the logic of relations this single relation can be formulated in a convincing way.
2. The logic of relations seems conceptually simpler than predicate logic. The formulas only make use of terms, logical connectives, quantifiers and equality. Predicate symbols do not occur at all—except equality. I consider this a significant advantage. The logic allows us to talk in a purely ‘logical’ way about the things that are ‘out there’.

3. Substitution—the basic operation of the new logic—is a very intuitive notion. In my view it is more elementary than assuming that arguments of a relation come in a specific order or that relations have fixed positions to which arguments can be assigned.
4. In the logic of relations, certain complex properties of objects can be expressed very concisely. For example, that objects a and b have exactly the same relations can be expressed as

$$\forall x (x \text{ is a complex} \rightarrow E! x[a \mapsto b, b \mapsto a])$$

In predicate logic we need in some cases an infinite number of formulas for this.

5. In principle, the logic of relations allow complexes to have any number of objects, including infinitely many. This is not the case for normal predicate logic.

What might seem a disadvantage of the logic of relations is that the formulas can be relatively long. However, using abbreviations like $t[u_1 \mapsto v_1, \dots, u_n \mapsto v_n]$ may solve this problem. We might even go a step further and let certain formulas look exactly like formulas of ordinary predicate logic, for example by writing $L(x, y)$ for $E! L_{ae}[a \mapsto x, e \mapsto y]$.

In conclusion, the logic of relations has the potential to represent reality more adequately than predicate logic.

Learning relations

The new logic of relations may influence the way we look at the world. But how do we normally ‘learn’ relations? Is it by substitution, by abstraction, by positional representations or via processes with a completely different logic? And what is the role of language in learning relations?

A psychological investigation of the way we learn relations would be extremely useful. It might, however, not be easy to develop experiments to determine how we learn all kinds of relations. It will require experts in cognitive psychology to design appropriate tests and experiments.

And there are also more questions to be asked, for example, how small children and animals learn relations. Answers to these questions might deepen our insight in fundamental aspects of the way we understand and represent the world. In addition, they might suggest new learning programs.

Some theoretical research has been done in this field. For example, in Tomlinson and Love (2006) a model of relational learning has been developed. But as far as I know, the role that the notion of substitution may play in learning relations has never been explicitly considered. A more explicit investigation could fit in nicely with the research field of the way we reason. Interesting research in this more general field has been done in Johnson-Laird (1983); Evans, Newstead, and Byrne (1993); Gentner and Smith (2012). In particular, work on analogical reasoning seems to be most relevant.

A related question is what is the best way for artificial intelligence systems to learn relations. There is encouraging re-

search with respect to developing algorithms for learning relations (Richards & Mooney, 1992; Heyer, Läuter, Quasthoff, Wittig, & Wolff, 2001; Katrenko & Adriaans, 2007). In some cases the goal is to build systems that ‘discover’ relations, and in other cases to find instances for which a given relation holds.

As we saw, the basic operation of the new logic is substitution. If we will be able to implement a general notion of substitution in an AI system, then it might perhaps be possible for such a system to learn a variety of relations by feeding it large sets of samples. As a result we might get sophisticated systems that are able to discover all kinds of subtle regularities in the world.

Impact of a coordinate-free view

The introduction of a coordinate system in the 17th century by René Descartes marked a major step forward for mathematics and physics. The idea is quite simple: to each point in the plane a pair of numbers is assigned. This made it possible to describe geometric shapes by algebraic equations. However, in the twentieth century coordinate-free treatments of certain geometric topics turned out to be simpler and more elegant. In particular this is the case for vector analysis and differential geometry.

The choice of a particular coordinate system often turns out to be irrelevant. In physics, this may have an underlying reason; around 1910, Albert Einstein introduced the principle of *general covariance*, according to which the basic laws of physics remain invariant under changes in frames of reference. From this, one should not immediately conclude that a completely coordinate-free formulation of the laws is always possible. It would also be misleading to call the coordinates used in physics artefacts.

For relations the situation is different. According to the antipositionalist view—the superior view on relations—a relation has on a fundamental level no positions and no order for the relata. The view is genuinely coordinate-free. We do not have the problem of having to deal with irrelevant details like choosing an order, since such details are simply lacking.

This observation may not immediately have an effect on how we perceive the world around us. However, I expect an impact in the longer term:

1. Becoming more acquainted with the new logic of relations may make us more familiar with the idea that relations around us are indeed networks of interrelated complexes. This may help us to ‘discover’ new relations and instances of relations in an easier way.
2. An interesting application of the new logic presented in this paper will be the development of a philosophically driven alternative for set theory—the standard foundation of mathematics. There is a substantial need for this, since we do not live in a world of sets, but in a world of things with relations between them. Although almost everything

can be coded in set theory, the coding is in some cases quite artificial.³

3. The new logic may be useful for developing a new programming language in which complexes and substitutions play a central role. Programs written in such *complex-oriented* programming languages may have a simpler internal structure. This would be of great interest.
4. Finally, I can imagine that some day an effect of this research may be found in websites, shops, airports, and cities: innovations in the design of such places may be inspired by what is for us the most *logical* way to relate things.

It will be obvious that to accomplish the goals mentioned above, more innovative and interdisciplinary research needs to be done—in particular by computer scientists, linguists, logicians, philosophers, and psychologists. But the results so far are promising.

Acknowledgments

This research was supported by a VENI grant from the Netherlands Organization for Scientific Research (NWO). I thank Niels van Miltenburg, Vincent van Oostrom, and Albert Visser for their comments on the paper.

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³An ordered pair $\langle a, b \rangle$, for example, may be coded in set theory as the set $\{\{a\}, \{a, b\}\}$.