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**Stability of resonances under singular perturbations**

by

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## Abstract

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We investigate the stability of complex numbers called resonances in certain open chaotic systems. In the context of waves scattered by a potential, scattering resonances are complex numbers that quantize exponential decay rates of the local energy. In the context of hyperbolic dynamical systems, Pollicott–Ruelle resonances quantize exponential decay rates of statistical correlations. We will show that resonances occurring in these two situations are stochastically stable. This theoretical result supports the possibility of observing resonances in experimental physics.

This dissertation consists of two independent chapters, based on the papers [Dr16a, Dr17].

Chapter 1 focuses on scattering resonances. We give a simple model for waves propagating through a localized disordered crystal with small typical scale of heterogeneity. Roughly speaking, our results show that resonances (hence propagating waves) are only weakly perturbed by the crystal. This chapter is organized as follows:

- We describe our model and state our theorems in §1.1. We relate these results to previous study in the idealized case of deterministic highly oscillatory perturbations, whose understanding is very important for the proofs.

- In §1.2, we give an overview of the theory of scattering resonances. We define them using the stationary Schrödinger resolvent and we relate them to local energy decay of waves. This makes Chapter 1 essentially self-contained.

- §1.3 studies the general perturbation theory of scattering resonances. We work with a parameter that typically quantifies the degree of oscillations of a random potential. When this parameter is small enough, we characterize locally scattering resonances as the zero set of a holomorphic function.

- In §1.4, we give a version of the Hanson–Wright inequality. This is a large deviation estimate for a quadratic form evaluated at random vectors with many entries. It is crucial in the rest of the proof. We also derive a modified version of the central limit theorem, based on Lindeberg's theorem.

- §1.5 is the core of the chapter. The tools of §1.4 show that the general theory developed in §1.3 applies to random highly oscillatory potentials. In particular, we can write their resonances as the zeroes of a random holomorphic function. We show that this random function can (roughly speaking) be written as a rescaled Gaussian – modulo negligeable terms. The variance of the Gaussian arises from large deviation effects, while the average of the Gaussian comes from constructive interference among oscillatory terms (an effect that was thoroughly studied in the context of deterministic highly oscillatory potentials).

- In §1.6, we prove the stochastic stability theorems. These are valid with high probability. We conclude by exhibiting an example (that appears with small probability) where the conclusions of the theorems do not hold.

Chapter 2 is a study of hyperbolic dynamical systems perturbed by a white noise. These form stochastic processes called kinetic Brownian motion. We use a recent microlocal approach to define Pollicott–Ruelle resonances via kinetic Brownian motion. This is a form of stochastic stability. The presentation is as follows:

- In §2.1, we state our stochastic stability result: the eigenvalues of the generator of the kinetic Brownian motion approach Pollicott–Ruelle resonances in the small white noise regime. We give an overview of recent results for random perturbations of the geodesic equation, and we describe related facts about the hypoelliptic Laplacian of Bismut.

- In §2.2, we recall a modern perspective on Pollicott–Ruelle resonances. It consists of seeing them as spectral quantities rather than dynamical one. This requires the semiclassical construction of anisotropic Sobolev spaces, which improve regularity in the contracting direction of the flow and lower it in the expanding direction. We give an axiomatic introduction to microlocal and semiclassical analysis.

- §2.3 presents the kinetic Brownian motion as a perturbation of the geodesic equation. We also describe its lift to the orthonormal frame bundle – a step required in §2.4.

- In §2.4, we prove a subelliptic estimate for the generator of the kinetic Brownian motion. An informal probabilistic statement is as follows: the white noise perturbation is not too large compared to the kinetic Brownian motion itself.

- §2.5 reformulates the subelliptic estimate in the context of anisotropic Sobolev spaces that are needed to define Pollicott–Ruelle resonances.

- In §2.6, we show that the subelliptic estimate allows to control high frequencies of the white noise perturbation; and that the low frequencies of the white noise perturbations can be treated as an absorbing potential. This enables us to prove the main theorem.

To my family and friends.

# Contents

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# Chapter 1

# Resonances for random highly oscillatory potentials

## 1.1 Introduction

Predicting the behavior of waves scattered by a rough media is a difficult practical problem: the propagation media is hard to know accurately, and it may contains defects and impurities that have a large impact on the diffusion. This motivates a general study of propagation of waves through random medias. It is a vast subject of research with many applications. We refer to the seminal paper of Anderson [An58] for the absence of diffusion of waves by certain models of condensed matter physics; Devillard–Dunlop–Souillard [DDS88] for oceanographic predictions; and to the monographs of Andrew–Phillips [AP98] and Fehler–Maeda–Sato [FS09] for applications in electromagnetism and seismography, respectively. Current mathematical research includes proofs of homogenization results and rigorous derivation of radiative transfer equations. We refer to the lecture note of Bal [Ba06] for a comprehensive introduction to theoretical aspects of waves in random media.

In this first chapter, we propose and study a simple model for waves scattered by a highly heterogeneous localized media in $\mathbb{R}^d$, $d$ odd. Specifically, the disordered media is assumed to create the random potential

$$V_N(x) \overset{\text{def}}{=} q_0(x) + \sum_{j\in[-N,N]^d} u_j q(Nx-j), \qquad N \gg 1, \quad x \in \mathbb{R}^d, \tag{1.1.1}$$

where $q_0,\ q \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ and $\{u_j\}_{j\in\mathbb{Z}^d}$ are bounded i.i.d. random variables on a probability space $(\Omega, \mathbb{P}, \mathscr{F})$, with mean $\mathbb{E}(u_j) = 0$ and variance $\mathbb{E}(u_j^2) = 1$. $V_N$ represents the potential created by a localized crystal $\{j/N,\ j \in [-N,N]^d\}$ plunged in the external field $q_0$, with sites $j/N$ each generating a potential $u_j q(Nx-j)$. The scale of heterogeneity of such crystals is $N^{-1}$. We will concentrate on discrete spectral quantities associated to the Schrödinger operator $-\Delta_{\mathbb{R}^d} + V_N$.

Deterministic versions of the operator $-\Delta_{\mathbb{R}^d} + V_N$ were studied in work of Borisov–Gadyl'Shin [BG06], Borisov [Bi05], Duchêne–Weinstein [DW11], Duchêne–Vukićević–Wein-

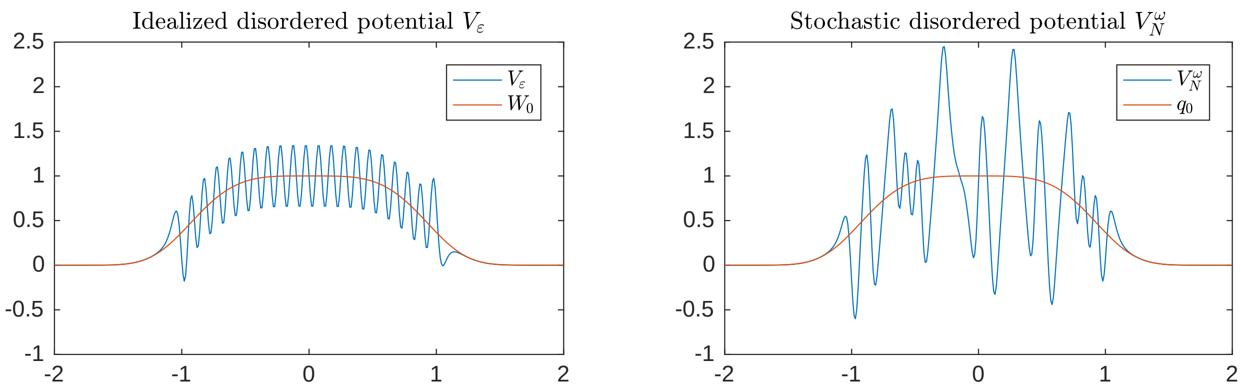


Figure 1.1: On the left, deterministic version of $V_N$ studied in [BG06, Bi05, DW11, DVW14, Dr15, Dr16b]. On the right, stochastic potential $V_N$. Here $N = 20$ and $W_0 = q_0$.

stein [DVW14] and ourselves [Dr15, Dr16b]. The present work extends part of these papers to the random case (1.1.1). For a pictorial comparison of the stochastic and deterministic version of $V_N$, see Figure 1.1.

Our analysis focuses on scattering resonances of $V_N$. Generally speaking, the set of resonances $\operatorname{Res}(\mathscr{V})$ of a potential $\mathscr{V} \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ is the set of poles of the meromorphic continuation to $\mathbb{C}$ of

$$R_{\mathscr{V}}(\lambda) = (-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda^2)^{-1} \ : \ C_0^\infty(\mathbb{R}^d, \mathbb{C}) \to C^\infty(\mathbb{R}^d, \mathbb{C}), \quad d \text{ odd}.$$

When $\mathscr{V}$ is real-valued, resonances of $\mathscr{V}$ in the upper half plane are in one-to-one correspondence with eigenvalues of $-\Delta_{\mathbb{R}^d} + \mathscr{V}$: $\lambda \in \operatorname{Res}(\mathscr{V})$ with $\operatorname{Im}\lambda > 0$ if and only if $\lambda^2$ is an eigenvalue of $-\Delta_{\mathbb{R}^d} + \mathscr{V}$. Other resonances play the role of generalized eigenvalues for this open system. Hence, resonances strikingly quantize the decay rates of waves scattered by $\mathscr{V}$: if $u$ is a sufficiently nice solution of $(\partial_t^2 - \Delta_{\mathbb{R}^d} + V_N)u = 0$, then $u$ admits the formal expansion

$$u(t,x) \sim \sum_{\lambda \in \operatorname{Res}(\mathscr{V})} u_\lambda(x) e^{i\lambda t}, \quad u_\lambda : \mathbb{R}^d \to \mathbb{C}. \tag{1.1.2}$$

– putting aside the issue of multiplicity and convergence of the expansion. Since for every $A > 0$, $\operatorname{Res}(\mathscr{V}) \cap \{\lambda \ : \ \operatorname{Im}\lambda \geq -A\}$ is a finite set, (1.1.2) admits a rigorous formulation in terms of exponential decay of local energy, see e.g. [DZ16d, Theorem 3.9] and Theorem 8 below for a simplified version. A comprehensive introduction to scattering resonances is found in [DZ16d, Chapters 2 and 3].

In this work, we localize precisely the eigenvalues and scattering resonances of the random Schrödinger operator $-\Delta_{\mathbb{R}^d} + V_N$, where $V_N$ is the chaotic potential given in (1.1.1). Our analysis lies within the effective media theory, which aims to replace rapidly varying terms by low frequency one. When $V_N$ is real-valued, our localization results transfer directly to qualitative information on the long-time behavior of waves, see e.g. the remark below Theorem 3.

### 1.1.1 Results

We recall that $q_0$, $q$ are two smooth compactly supported functions and that

$$V_N(x) \overset{\text{def}}{=} q_0(x) + \sum_{j \in [-N,N]^d} u_j q(Nx - j), \qquad N \gg 1, \quad x \in \mathbb{R}^d, \quad d \text{ odd}.$$

The potential $V_N$ has support contained in a fixed compact set and is uniformly bounded independently of $N$ and of the value of the $\{u_j\}$, see (1.4.1) and (1.4.2) below. Let $\hat{q}$ be the Fourier transform of $q$:

$$\hat{q}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^d} e^{-i\xi x} q(x) dx.$$

The influence of low frequencies of $q$ is well-described by the order of vanishing $m$ of $\hat{q}(\xi)$ at $\xi = 0$ (i.e., the largest integer such that $\hat{q}(\xi) = O(|\xi|^m)$ near 0). With this notation, we define

$$\gamma \overset{\text{def}}{=} \min(7/4, d/2 + m).$$

We recall that $\text{Res}(\mathscr{V}) \subset \mathbb{C}$ is the set of resonances of $\mathscr{V}$ and that we denote by $m_\lambda$ the multiplicity of a resonance $\lambda$. The set $\text{Res}(\mathscr{V})$ is discrete; and for any $A > 0$, there exists $B$ depending only on $A$, on the diameter of the support of $\mathscr{V}$ and on $|\mathscr{V}|_\infty$ such that

$$\text{Res}(\mathscr{V}) \cap \{\lambda : \text{Im}\,\lambda \geq -A, \ |\text{Re}(\lambda)| \geq B\} = \emptyset. \tag{1.1.3}$$

A proof of this fact is given in Lemma 1.2.4.

**Theorem 1.** *For any $R > 0$ such that $q_0$ has no resonance on $\partial\mathbb{D}(0, R)$, there exist $C, c > 0$ such that with probability $1 - Ce^{-cN^\gamma}$,*

$$\text{Res}(V_N) \cap \mathbb{D}(0, R) \subset \bigcup_{\lambda \in \text{Res}(q_0) \cap \mathbb{D}(0,R)} \mathbb{D}\left(\lambda, N^{-\frac{\gamma}{2m_\lambda}}\right). \tag{1.1.4}$$

*Conversely, if $\lambda \in \text{Res}(q_0) \cap \mathbb{D}(0, R)$ has multiplicity $m_\lambda$, then with probability $1 - Ce^{-cN^\gamma}$, $V_N$ has exactly $m_\lambda$ resonances in $\mathbb{D}\left(\lambda, N^{-\frac{\gamma}{2m_\lambda}}\right)$ – counted with multiplicity.*

An application of this theorem concerns local exponential decay for waves scattered by $V_N$. Assume that $q_0, q$ are real-valued and that $\text{Res}(q_0)$ is contained in $\{\text{Im}\,\lambda < -A\}$ for some $A > 0$ (this is satisfied for instance if $q_0 \geq 0$ and $q_0 \not\equiv 0$). Let $R$ be independent of $N$ such that for any $N$ and any event,

$$\text{Res}(V_N) \cap \{\text{Im}\,\lambda \geq -A\} \subset \mathbb{D}(0, R).$$

The bound (1.1.3) together with (1.4.1) and (1.4.2) guarantees that $R$ exists. Theorem 1 asserts that with probability $1 - O(e^{-cN^\gamma})$, resonances of $V_N$ are very close to resonances of $q_0$ in $\mathbb{D}(0, R)$, in particular that $\text{Res}(V_N) \cap \mathbb{D}(0, R) \subset \{\text{Im}\,\lambda < -A\}$. The characterization of resonances as quantized decay of waves (see Theorem 8) shows that with probability $1 - O(e^{-cN^\gamma})$, any solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ of

$$(\partial_t^2 - \Delta_{\mathbb{R}^d} + V_N)u = 0, \quad u(0, \cdot) \in C_0^\infty(\mathbb{R}^d, \mathbb{C}), \quad \partial_t u(0, \cdot) \in C_0^\infty(\mathbb{R}^d, \mathbb{C}), \quad d \geq 3 \tag{1.1.5}$$

decays faster than $e^{-At}$:

$$\forall M > 0, \quad \sup_{|x| \leq M} |u(x,t)| = o(e^{-At}).$$

A combination of the Borel–Cantelli lemma with Theorem 1 implies the following almost-sure, non-quantitative statement:

**Corollary 2.** *The set of accumulation points of* $\mathrm{Re}(V_N)$ *is* $\mathbb{P}$*-a.s. equal to* $\mathrm{Res}(q_0)$*.*

Since the vanishing potential has only a resonance at 0 in dimension 1 and none in higher dimension, Theorem 1 show that most resonances of $V_N$ with $q_0 = 0$ must exit every compact set as $N \to \infty$. Our next result gives a lower bound on the rate of escape:

**Theorem 3.** *Assume that* $q_0 = 0$ *and that* $V_N$ *is given by* (1.1.1)*. There exist* $C, c, A > 0$ *such that with probability* $1 - Ce^{-cN^\gamma}$*,* $V_N$ *have no resonance above the line* $\mathrm{Im}\,\lambda = -A\ln(N)$ *– apart from a single resonance in* $\mathbb{D}(0, N^{-\gamma/2})$ *when* $d = 1$*.*

By the same argument as in the remark below Theorem 1, if $d \geq 3$, $q_0 \equiv 0$ and $q$ is real-valued then solutions of (1.1.5) must locally decay like $N^{-At}$, with probability at least $1 - O(e^{-cN^\gamma})$.

We now investigate the speed of convergence of resonances of $V_N$ to resonances of $q_0$. The next statement requires some preparation. We assume below that $\lambda_0$ is a resonance of $q_0$ with a rank-one residue and no other Lorenz coefficients:

$$\begin{gathered} \mathrm{Rank}\, \frac{1}{2\pi i} \oint_{\lambda_0} (-\Delta_{\mathbb{R}^d} + q_0 - \lambda^2)^{-1} d\lambda = 1, \\ \forall k \geq 1, \quad \frac{1}{2\pi i} \oint_{\lambda_0} (\lambda - \lambda_0)^k (-\Delta_{\mathbb{R}^d} + q_0 - \lambda^2)^{-1} d\lambda = 0 \end{gathered} \tag{1.1.6}$$

Such resonances are called simple, and are rather generic: any non-zero resonance of $q_0$ that have geometric multiplicity equal to 1 in the sense of [DZ16d, (3.2.4)] is simple. In dimension 1 and for real-valued potentials with $0 \in \mathrm{Res}(\mathscr{V})$, $\lambda_0 = 0$ is also simple. When $\lambda_0$ is simple, the residue of $(-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda^2)^{-1}$ at $\lambda_0$ has a lot of structure: we can find $f,\ g \in C^\infty(\mathbb{R}^d, \mathbb{C})$ such that

$$\begin{gathered} (-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda_0^2) f = 0, \quad (-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda_0^2)^* g = 0, \\ (-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda^2)^{-1} - i\frac{f \otimes g}{\lambda - \lambda_0} \ \text{ is holomorphic near } \lambda_0. \end{gathered}$$

We refer to Lemmas 1.2.2 and 1.2.3 for proofs of these facts.

If $\Sigma$ is a symmetric non-negative matrix, we denote by $\mathscr{N}(0,\Sigma)$ the multivalued Gaussian distribution centered at 0 with covariance matrix $\Sigma$. If $\varphi \in C^\infty(\mathbb{R}^d,\mathbb{C})$ has real-part $\varphi_1$ and imaginary part $\varphi_2$, we define $\Sigma[\varphi]$ as the $2\times 2$ symmetric, nonnegative matrix

$$\Sigma[\varphi] \overset{\text{def}}{=} \int_{[-1,1]^d} \begin{pmatrix} \varphi_1(x)^2 & \varphi_1(x)\varphi_2(x) \\ \varphi_1(x)\varphi_2(x) & \varphi_2(x)^2 \end{pmatrix} dx. \tag{1.1.7}$$

If $\Sigma[\varphi]$ is non-degenerate, we say that a complex-valued sequence of random variables $Z_j$ converges in distribution to $\mathscr{N}(0,\Sigma[\varphi])$ if the multivariate random variable $(\mathrm{Re}(Z_j),\mathrm{Im}(Z_j))$ converges in distribution to the multivariate normal distribution centered at 0 with covariance matrix $\Sigma[\varphi]$. If $\Sigma[\varphi]$ is degenerate then

$$\int_{\mathbb{R}^d}\varphi_1(x)^2dx\int_{\mathbb{R}^d}\varphi_2(x)^2dx - \left(\int_{\mathbb{R}^d}\varphi_1(x)\varphi_2(x)dx\right)^2 = 0.$$

Hence (if, say, $\varphi_2 \not\equiv 0$), there exists $\alpha \in \mathbb{R}$ such that $\varphi_1 = \alpha\varphi_2$. In this situation, we say that $Z_j$ converges in distribution to $\mathscr{N}(0,\Sigma[\varphi])$ if the multivariate random variable $(\mathrm{Re}((1+i\alpha)^{-1}Z_j),\mathrm{Im}((1+i\alpha)^{-1}Z_j))$ converges in distribution to $\mathscr{N}\left(0,\int_{[-1,1]^d}\varphi_1(x)^2dx\right)\otimes\delta_0$. The definition for $\varphi_1 \not\equiv 0$ is analogous.

We will distinguish the three following cases:

- **Case I:** $d=1$ or $3$ and $\int_{\mathbb{R}^d} q(x)dx \neq 0$;
- **Case II:** $d=1$ and $\int_{\mathbb{R}} q(x)dx = 0$, $\int_{\mathbb{R}} xq(x)dx \neq 0$ and $(f\cdot g)' \not\equiv 0$ on $[-1,1]$;
- **Case III:** all other cases.

**Theorem 4.** *Under the above notations, there exist $C,c>0$ such that the following is satisfied. For every $N$, there exists $\lambda_N$ a complex-valued random variable, such that*

$$\mathbb{P}(\lambda_N \in \mathrm{Res}(V_N)) \geq 1 - Ce^{-cN^{1/4}}$$

*and*

- *In Case I,*
$$\frac{N^{d/2}(\lambda_N-\lambda_0)}{i\int_{\mathbb{R}^d} q(x)dx} \xrightarrow{\mathrm{d}} \mathscr{N}(0,\Sigma[fg]).$$
- *In Case II,*
$$\frac{N^{3/2}(\lambda_N-\lambda_0)}{i\int_{\mathbb{R}} xq(x)dx} \xrightarrow{\mathrm{d}} \mathscr{N}(0,\Sigma[(fg)']).$$
- *In Case III,*
$$N^2(\lambda_N-\lambda_0) \xrightarrow{\mathbb{P}\text{-a.s.}} \frac{i}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{|\xi|^2}d\xi \cdot \int_{[-1,1]^d} f(x)g(x)dx.$$

When $q_0$ and $q$ are real-valued, the case of resonances lying on the imaginary axis is of special interest. It allows to study eigenvalues of $-\Delta_{\mathbb{R}^d} + q_0$: $\lambda_0 \in i(0,\infty) \cap \mathrm{Res}(q_0)$ if and only if $\lambda_0$ is an eigenvalue of $-\Delta_{\mathbb{R}^d} + q_0$, see for instance §1.2. It also allows to observe the emergence of eigenvalues from the edge of the continuous spectrum in the context of random perturbations. This phenomena was captured first for small perturbations in a pioneering work of Simon [Si76]. It was observed for highly oscillatory perturbations in [BG06, Bi05, DW11, DVW14, Dr15, Dr16b]. When $q_0$ is real-valued and $\lambda_0 \in i\mathbb{R}$, we can pick $f = \overline{g}$ in (1.1.6) and we obtain a refinement of Theorem 4:

**Corollary 5.** *Under the above notation, assume that $q$, $q_0$ are real-valued and that $\lambda_0 \in i\mathbb{R} \cap \mathrm{Res}(q_0)$. Then there exist $C, c > 0$ such that the following is satisfied. For every $N$, there exists $\lambda_N$ a random variable with values in $i\mathbb{R}$ such that*

$$\mathbb{P}(\lambda_N \in \mathrm{Res}(V_N)) \geq 1 - Ce^{-cN^{1/4}}$$

*and*

- *In Case I,*
$$\frac{N^{d/2}(\lambda_N - \lambda_0)}{i\int_{\mathbb{R}^d} q(x)dx} \xrightarrow{\mathrm{d}} \mathscr{N}(0,\sigma^2), \quad \sigma^2 \overset{\mathrm{def}}{=} \int_{[-1,1]^d} |f(x)|^4 dx.$$

- *In Case II,*
$$\frac{N^{3/2}(\lambda_N - \lambda_0)}{i\int_{\mathbb{R}} xq(x)dx} \xrightarrow{\mathrm{d}} \mathscr{N}(0,\sigma^2), \quad \sigma^2 \overset{\mathrm{def}}{=} \int_{[-1,1]} \left( \left(|f|^2\right)'(x) \right)^2 dx.$$

- *In Case III,*
$$N^2(\lambda_N - \lambda_0) \xrightarrow{\mathbb{P}\text{-a.s.}} \frac{i}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{q}(\xi)|^2}{|\xi|^2} d\xi \cdot \int_{[-1,1]^d} |f(x)|^2 dx.$$

*If in addition $\lambda_0^2$ is an eigenvalue of $-\Delta_{\mathbb{R}^d} + q_0$, then $\lambda_N^2$ is an eigenvalue with probability $1 - Ce^{-cN^{1/4}}$.*

Let us consider the example $d = 1$, $q_0 \equiv 0$ and $q$ real-valued. The potential $q_0$ has a single resonance $\lambda_0 = 0$, with constant resonant states $f = \overline{g} = 1/\sqrt{2}$, see (1.2.2) below. Theorem 4 shows that $V_N$ is likely to have a resonance $\lambda_N$ near 0, which in addition belongs to $i\mathbb{R}$. If $\int_{\mathbb{R}} q(x)dx \neq 0$, we fall in Case I: $\lambda_N$ is roughly at distance of order $N^{-1/2}$ from 0; precisely,

$$\frac{2N^{1/2}\lambda_N}{i\int_{\mathbb{R}} q(x)dx} \xrightarrow{\mathrm{d}} \mathscr{N}(0,1).$$

We observe that $\mathrm{Im}\,\lambda_N > 0$ with probability asymptotically equal to $1/2$: with probability roughly $1/2$, $\lambda_N^2$ is an eigenvalue that emerges from the edge of the continuous spectrum of

$-\Delta_{\mathbb{R}}$. If in contrast $\int_{\mathbb{R}} q(x)dx = 0$, we fall in Case III and $\lambda_N$ is at distance of order $N^{-2}$ from 0. Precisely, if $Q$ is the compactly supported antiderivative of $q$, then

$$N^2\lambda_N \xrightarrow{\mathbb{P}\text{-a.s.}} \frac{i}{2}\int_{\mathbb{R}} Q(x)^2 dx.$$

In particular, $V_N$ is likely to have a unique eigenvalue $\lambda_N^2 \sim -\frac{1}{4N^4}\int_{\mathbb{R}} Q(x)^2 dx$, which emerges from the edge of the continuous spectrum of $-\Delta_{\mathbb{R}}$. This is clearly connected with the aforementioned results in the context of highly oscillatory potentials.

### 1.1.2 Interpretation and comments

Theorem 1 involve the exponent $\gamma$, which depends on $d$ and $m$. This dependence comes from large deviations: when $\{u_j\}_{j\in\mathbb{Z}^d}$ takes unlikely values, the potential $V_N$ differs significantly from a purely oscillatory one. This slows down the speed of convergence of resonances of $V_N$ to resonances of $q_0$. Such large deviations happen less often in higher dimensional crystals, because the number $N^d$ of sites grows with the dimension. Their effect is reduced when $m$ is large (that is, $q$ contains few low frequencies), because in such cases $q$ is inherently oscillatory, independently of the values of the random sequence $\{u_j\}_{j\in\mathbb{Z}^d}$. This explains the dependence of $\gamma$ on $d$ and $m$. In §1.7, we show on an example that Theorem 3 does not hold if one does not remove an event of exponentially small probability.

In the idealized (deterministic) case, the works of Borisov and Gadyl'shin [BG06, Bo07], Duchêne–Weinstein–Vukićević [DVW14] and Drouot [Dr15, Dr16b] show that the difference between resonances of deterministic highly oscillatory potentials and their weak limit is of order $N^{-2}$. This is due to constructive interference between oscillatory terms. This effect is still present here. However it is not always the leading effect: it can be overcome by large deviations, see the three cases in Theorem 4. These are of generally of order $N^{-d/2-m}$ – $m$ is the order of vanishing of $\hat{q}(\xi)$ at $\xi = 0$. This explains why the transition between stochastic and deterministic corrections generically happens when $d/2 + m$ becomes greater than 2. This also explains why the speed of convergence of resonances of $V_N$ cannot be faster than $N^2$, even when $d$ is very large.

Because of Theorem 4, if $d + m/2 \geq 2$, leading terms in approximations of simple resonances result from deterministic corrections. In this situation, the analogy with [DVW14, Dr15] is at its strongest and we can derive an effective potential:

$$V_{\text{eff}}(x) \overset{\text{def}}{=} q_0(x) + \frac{1}{N^2}\cdot\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{|\xi|^2}d\xi \cdot \mathbb{1}_{[-1,1]^d}(x).$$

Indeed, $V_{\text{eff}}$ is a small perturbation of $q_0$; and for the purpose of approximating resonances with simple of $V_N$, $V_N^{\text{eff}}$ is better that $q_0$: near any resonance $\lambda_0$ of $q_0$, $V_N^{\text{eff}}$ has a simple resonance $\lambda_N^{\text{eff}}$, that satisfies

$$\frac{\lambda_N - \lambda_N^{\text{eff}}}{\lambda_N - \lambda_0} \xrightarrow{\mathbb{P}\text{-a.s.}} 0$$

– see for instance [Dr15, Lemma 3.3]. An effective potential in the above sense does not exist if $d/2 + m < 2$. Indeed, Theorem 4 implies that such a potential would be given by a distribution

$$\frac{1}{N^d} \int_{\mathbb{R}^d} q(x)dx \cdot \sum_{j \in [-N,N]^d} u_j \delta_{j/N},$$

which does not belong to $L^\infty$.

Theorems 1-4 show that resonances of $V_N$ in compact sets are very close to resonances of $q_0$ for $N$ large (with probability exponentially close to 1). The potential $V_\# = V_N - q_0$ is nowhere small, but is weakly small – with high probability. In the proofs, it is treated as a singular perturbation of $q_0$, in an abstract framework due to Golowich–Weinstein [GW05]. A different singular perturbation of $q_0$ was studied by Zworski [Zw15], who obtained resonances as viscosity limits. The results presented here show a form of of stochastic stability of scattering resonances; this reinforces the possibility of observing them in physical situations. Some other stochastic stability results regarding stochastic stability of resonances were obtained in the context of hyperbolic dynamical systems, see for instance Dyatlov–Zworski [DZ15] and Chapter 2.

### 1.1.3 Further relation to existing work

To the best of our knowledge, this is the first treatment of eigenvalues and resonances for random highly oscillatory Schrödinger operators. The closest work is possibly Klopp [Kl16], where a semiclassical Weyl law for large one-dimensional discrete ergodic systems is derived. The potentials considered there can be seen as a high amplitude version of the potentials considered here; specifically, after rescaling, Klopp's potential takes the form

$$N^2 \sum_{j \in [-N,N]} u_j q(Nx - j), \ x \in \mathbb{Z}, \ \ \mathbb{E}(u_j) = 0, \ \mathbb{E}(u_j^2) = 1, \ \ q \in C_0(\mathbb{Z}, \mathbb{R}).$$

For one-dimensional deterministic highly oscillatory potentials (HOPs), Borisov–Gadyl'-shin [BG06] and Borisov [Bi05] gave necessary and sufficient conditions for the existence of a bound state. Duchêne–Vukićevik–Weinstein [DVW14] derived an explicit formula for a small effective potential, created by the constructive interference of oscillatory terms. We developed new techniques in [Dr15, Dr16b] to extend the aforementioned work in higher dimensions. We obtained a full expansion for eigenvalues and resonances of HOPs, and a refined formula for the effective potential. The techniques developed in [Dr15, Dr16b] happen to be robust enough to handle here the case of random HOPs.

On a somewhat unrelated note, Duchêne–Raymond [DR16] obtained homogenization results for large HOPs in dimension 1. Dimassi [Di16] and Dimassi–Duong [DD17] used the effective Hamiltonian method of Gérard–Martinez–Sjöstrand [GMS91] to count resonances and eigenvalues of semiclassical rescaled HOPs in any dimension $d$. They obtained a nice Weyl law in the semiclassical limit, related to papers of Klopp [Kl12, Kl16] and Phong [Ph15a, Ph15b].

### 1.1.4 Structure of the proof

The proof is presented as follows. In §1.2, we review some well-known facts about scattering resonances and spectral theory. This makes the proof of the theorems stated in §1.1.1 essentially self-contained. Specifically, we first show that the Schrödinger resolvent $(-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda^2)^{-1} : C_0^\infty(\mathbb{R}^d) \rightarrow C^\infty(\mathbb{R}^d)$ meromorphically continues to the whole complex plane, with poles defining scattering resonances. This approach comes with useful estimates on the norm of $R_{\mathscr{V}}(\lambda)$ on standard Sobolev spaces. In the context of real-valued potentials, we investigate the relation between resonances and rates of decay of scattered waves. The expansion of waves in terms of resonances is proved; this result is the most spectacular manifestation of resonances in nature. Finally, we give an overview of trace-class operators and Fredholm determinants. These tools were originally introduced in the context of scattering theory to count resonances of potentials in large bounded sets. The reader familiar with the theory of scattering resonances is invited to skip §1.2.

The present work uses Fredholm determinants to study (for the first time) a form of stochastic stability of scattering resonances. The proofs of Theorems 1 and 3 start with a deterministic part, §1.3. We consider a scale of Banach spaces of functions $\mathscr{H}^{-s}$, $s > 0$, where highly oscillatory elements have small norms. These spaces were first introduced in work of Golowich–Weinstein [GW05]. They allow us to show that the random potential $V_N$ can be considered as a highly oscillatory perturbation of $q_0$, with high probability. We are then able to modify the arguments of [Dr15] (which studied the stability of resonances under deterministic highly oscillatory perturbations) to prove Theorems 1 and 3.

Theorem 4 is more difficult. It relies on serious modifications of our previous work in the (idealized) case of deterministic highly oscillatory perturbations [Dr15]. We first show that resonances of $V_N$ near a resonance $\lambda_0$ of $q_0$ satisfy a local characteristic equation of the form

$$\lambda = \lambda_0 + a_1(V_\#, \lambda) + a_2(V_\#, \lambda) + ..., \qquad V_\# \overset{\text{def}}{=} V_N - q_0,$$

In the above, the terms $a_k(V_\#, \lambda)$ are $k$-multilinear in $V_\#$. The terms $a_k(V_\#, \lambda)$ are shown to be negligible when $k \geq 3$. Theorem 4 requires a precise evaluation of $a_1(V_\#, \lambda)$ and $a_2(V_\#, \lambda)$, which is performed in §1.4. The central limit theorem will show that $a_1(V_\#, \lambda)$ is a stochastic term induced by large deviations, generally of order $N^{-d/2-m}$ – $m$ being the order of vanishing of $\hat{q}(\xi)$ at $\xi = 0$. Thus, it is a term produced by the low frequencies of $V_N$. In the limit $N \rightarrow \infty$, the bilinear term $a_2(V_\#, V_\#)$ happens to be of deterministic nature: it is created by constructive interference between high frequencies of $V_N$. Using techniques of [Dr16b], we prove that $a_2(V_\#, V_\#)$ is of order $N^{-2}$. To conclude the proof, we compare $a_1(V_\#)$ to $a_2(V_\#, V_\#)$:

$$d = 1,\ m \leq 1 \text{ or } d = 3,\ m = 0 \ \Rightarrow\ a_2(V_\#, V_\#) = o(a_1(V_\#)) \text{ with high probability;}$$
$$\text{every other case } \Rightarrow\ a_1(V_\#) = o(a_2(V_\#, V_\#)) \text{ with high probability.}$$

This explains the transition observed in Theorem 4. In Cases I and II, the stochastic effects (due to large deviations, and carried by $a_1(V_\#)$) dominate. In Case III, the deterministic effects (due to constructive interference and carried by $a_2(V_\#, V_\#)$) dominate.

## 1.2 Mathematical theory of scattering resonances

In this section, we recall basic results – with their proofs – about scattering resonances of bounded, compactly supported potentials. These facts, which will be needed below, include:

- Definition of resonances as poles of the meromorphic continuation of the resolvent $(-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda^2)^{-1}$;
- Existence of arbitrarily large spectral gaps;
- Structure of the residue of $(-\Delta_{\mathbb{R}^d} + \mathscr{V} - \lambda)^{-1}$ at simple resonances (see (1.1.6));
- Expansion of scattered waves in terms of resonances.

Unless precised otherwise, $\mathscr{V} \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$, $d$ odd, and we write $\Delta$ for $\Delta_{\mathbb{R}^d}$. The standard $L^2$-norm on $\mathbb{R}^d$ is denoted $|\cdot|$.

### 1.2.1 Meromorphic continuation of $(-\Delta + \mathscr{V} - \lambda^2)^{-1}$

Let $\mathscr{V} \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$, $d$ odd, $\lambda \in \mathbb{C}$ and consider the equation

$$(-\Delta + \mathscr{V} - \lambda^2)v = u, \quad v \in L^2. \tag{1.2.1}$$

This equation is associated with the continuous quadratic form $Q(v) = |\nabla v|^2 + \langle \mathscr{V} v, v\rangle - \lambda^2 |v|^2$ on the Sobolev space $H^1$. In particular,

$$\mathrm{Re}(Q(u)) = |\nabla v|^2 + \langle \mathrm{Re}(\mathscr{V})v, v\rangle - \mathrm{Re}(\lambda^2)|v|^2 \geq |\nabla v|^2 + (\mathrm{Im}(\lambda)^2 - \mathrm{Re}(\lambda)^2 - |\mathscr{V}|_\infty)|v|^2.$$

It follows that $Q$ is coercive for $\lambda$ in the cone $\mathrm{Im}\,\lambda > |\mathscr{V}|_\infty^{1/2} + |\mathrm{Re}(\lambda)|$. In this situation, the equation (1.2.1) admits a unique solution $v \in H^1$ – in the sense of distribution. Since this solution satisfies $-\Delta v = f - \mathscr{V} v + \lambda^2 v \in L^2$, elliptic regularity for $\Delta$ shows that $v$ lies in $H^2$. This defines a resolvent operator

$$R_{\mathscr{V}}(\lambda) : u \in L^2 \mapsto v \in H^2$$

for complex numbers $\lambda$ in the above cone. The family of operators $R_{\mathscr{V}}(\lambda)$ is holomorphic in $\lambda$ (in the sense that the pairing $\langle R_{\mathscr{V}}(\lambda)u, v\rangle$ depends holomorphically on $\lambda$). The most standard definition of scattering resonances of $\mathscr{V}$ passes trough the meromorphic continuation of the function $\lambda \mapsto R_{\mathscr{V}}(\lambda)$ to the whole complex plane $\mathbb{C}$. We start with the case $\mathscr{V} \equiv 0$:

**Theorem 6.** *The family of operators $R_0(\lambda) : L^2 \to H^2$ – well defined for $\mathrm{Im}\,\lambda > 0$ – extends uniquely to a meromorphic family of operators*

$$R_0(\lambda) : C_0^\infty(\mathbb{R}^d, \mathbb{C}) \to \mathscr{D}'(\mathbb{R}^d, \mathbb{C}), \qquad \lambda \in \mathbb{C}.$$

*In addition, $R_0(\lambda)$ has a unique (simple) pole at $\lambda = 0$ when $d = 1$ and $R_0(\lambda)$ has no poles for $d \geq 3$.*

*Proof.* For $d = 1$, an explicit computation shows that

$$(-\partial_x^2 - \lambda^2)\frac{i}{2\lambda}e^{i\lambda|x-y|} = \delta(x-y).$$

Hence the kernel of $R_0(\lambda)$ is the function

$$(x,y) \mapsto \frac{i}{2\lambda}e^{i\lambda|x-y|}. \tag{1.2.2}$$

It is meromorphic with a (unique) simple pole at $\lambda = 0$, which shows the statement when $d = 1$. Note that the residue of the kernel at 0 is given by $if(x)g(y)$ where $f = \overline{g} = 1/\sqrt{2}$; in particular 0 is a simple resonance of $\mathscr{V} \equiv 0$ in dimension 1, with residue $if \otimes g$. We now work with $d \geq 3$. The kernel of $R_0(\lambda)$ can be written as an oscillatory integral: $R_0(\lambda)$ is the Fourier multiplier with symbol $\xi \mapsto (|\xi|^2 - \lambda^2)^{-1}$ – which belongs to $L^\infty$ when $\operatorname{Im}\lambda > 0$. Therefore,

$$R_0(\lambda, x, y) = K(\lambda, x-y), \quad K(\lambda, x) \overset{\text{def}}{=} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{e^{i\xi x}}{|\xi|^2 - \lambda^2}d\xi.$$

We think of the integral defining $K(\lambda, x)$ as an oscillatory integral – see for instance Grigis–Sjöstrand [GS94, §1]. We introduce polar coordinates $\xi = r\theta$, so that

$$K(\lambda, x) \overset{\text{def}}{=} \frac{1}{(2\pi)^d}\int_0^\infty \int_{\mathbb{S}^{d-1}} \frac{e^{ir\theta x}}{r^2-\lambda^2}d\sigma(\theta)r^{d-1}dr = \frac{1}{2(2\pi)^d}\int_{\mathbb{R}}\int_{\mathbb{S}^{d-1}} e^{ir\theta x}d\sigma(\theta)\frac{r^{d-1}dr}{r^2-\lambda^2}dr \tag{1.2.3}$$

(the order of integration is arbitrary because of oscillatory integral formalism). We now give a formula for the integral over $\mathbb{S}^{d-1}$: using rotation invariance, we can assume that $x = |x|e_1$. We write $\theta = (\theta_1, \sqrt{1-\theta_1^2}\cdot\theta')$ so that

$$d\sigma(\theta) = \sqrt{1-\theta_1^2}^{\,d-2}\cdot\left(d\theta_1^2 + d\sqrt{1-\theta_1^2}^{\,2}\right)^{1/2}d\sigma'(\theta') = \sqrt{1-\theta_1^2}^{\,d-3}d\theta_1 d\sigma(\theta'),$$

$$\int_{\mathbb{S}^{d-1}} e^{ir\theta x}d\sigma(\theta) = \int_{-1}^1\int_{\mathbb{S}^{d-2}} e^{ir|x|\theta_1}\sqrt{1-\theta_1^2}^{\,d-3}d\sigma'(\theta')d\theta_1$$

$$= |\mathbb{S}^{d-2}|\cdot(1+\partial_k^2)^{\frac{d-3}{2}}\int_{-1}^1 e^{ik\theta_1}d\theta_1 = |\mathbb{S}^{d-2}|\cdot(1+\partial_k^2)^{\frac{d-3}{2}}\left(\frac{e^{ik}-e^{-ik}}{ik}\right), \qquad k = r|x|.$$

A direct computation shows that

$$(1+\partial_k^2)^{\frac{d-3}{2}}\left(\frac{e^{ik}-e^{-ik}}{ik}\right) = e^{ik}F(k) + e^{-ik}F(-k), \quad F(k) = c_d e^{-2ik}(-\partial_k)^{\frac{d-3}{2}}\left(\frac{e^{2ik}}{ik^{\frac{d-1}{2}}}\right).$$

The constant $c_d \in \mathbb{C}$ above is a dimensional constant, which is allowed to change in the lines below. We observe that the function $F$ is rational, with a simple pole at $k = 0$, of multiplicity exactly $d-2$. It follows that we can split (1.2.3) in two integrals:

$$K(\lambda, x) = c_d\int_{\mathbb{R}} e^{ir|x|}F(r|x|)\frac{r^{d-1}dr}{r^2-\lambda^2}dr + c_d\int_{\mathbb{R}} e^{-ir|x|}F(-r|x|)\frac{r^{d-1}dr}{r^2-\lambda^2}dr.$$

We see these as contour integrals. The first contour is deformed the first one to $\operatorname{Im} r = N \gg 1$ and the second one to $\operatorname{Im} r = -N \ll -1$. This creates two residues at $r = \pm\lambda$, and the resulting integral is shown to decay like $e^{-N|x|}$ because of the term $e^{-r|x|}$. We consequently obtain

$$\begin{aligned} K(\lambda, x) &= c_d \oint_{\lambda} e^{ir|x|} F(r|x|) \frac{r^{d-1} dr}{r^2 - \lambda^2} dr - c_d \oint_{-\lambda} e^{-ir|x|} F(-r|x|) \frac{r^{d-1} dr}{r^2 - \lambda^2} dr \\ &= c_d e^{i\lambda|x|} F(\lambda|x|) \lambda^{d-2}. \end{aligned} \tag{1.2.4}$$

This is an entire function of $\lambda$. It follows that $R_0(\lambda)$ continues meromorphically from $C_0^\infty(\mathbb{R}^d)$ to $\mathscr{D}'(\mathbb{R}^d)$. □

In the next lemma, we show that $R_0(\lambda)$ is in fact bounded from $L^2_{\mathrm{comp}}$ to $H^2_{\mathrm{loc}}$, and we give precise estimates on the operator norm.

**Lemma 1.2.1.** *For any $\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ with support in the open ball $B(0, R)$, there exists a constant $C > 0$ such that*

$$|\rho R_0(\lambda)\rho|_{L^2 \to H^j} \leq \frac{C\langle\lambda\rangle^j e^{2R(\operatorname{Im}\lambda)_-}}{|\lambda| + (d-1)}, \tag{1.2.5}$$

$$|\rho R_0(\lambda)\rho|_{H^{-j} \to L^2} \leq \frac{C\langle\lambda\rangle^j e^{2R(\operatorname{Im}\lambda)_-}}{|\lambda| + (d-1)}. \tag{1.2.6}$$

*Proof.* We begin with two observations. First, that it is enough to prove (1.2.5) for $j = 0$ and 2: an interpolation argument would imply it for all $j$ in $[0, 2]$. Second, (1.2.5) implies (1.2.6). Indeed, the adjoint of $\rho R_0(\lambda)\rho$ is $\rho R_0(-\overline{\lambda})\rho$ – this can be checked for $\operatorname{Im}\lambda > 0$ and meromorphic continuation to $\mathbb{C}$. Since the RHS of (1.2.5) is invariant under $\lambda \mapsto -\overline{\lambda}$, (1.2.6) is simply dual to (1.2.5).

We start the technical details with $d = 1$. The kernel of $\rho R_0(\lambda)\rho$ is explicitly given by

$$(x, y) \mapsto \frac{i}{2\lambda} \rho(x) e^{i\lambda|x-y|} \rho(y).$$

Therefore, if $\operatorname{supp}(\rho) \subset [-R, R]$ then

$$\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} \left| \frac{i}{2\lambda} \rho(x) e^{i\lambda|x-y|} \rho(y) \right| \leq \frac{2R|\rho|^2_\infty}{2|\lambda|} e^{2R(\operatorname{Im}\lambda)_-}.$$

Schur's lemma implies that (1.2.5) holds for $j = 0$. The bound for $j = 2$ follows from a very similar argument, observing that the kernel of $\Delta\rho R_0(\lambda)\rho$ is given by

$$(x, y) \mapsto \frac{i}{2\lambda} \left( \rho''(x) + 2i\lambda\rho'(x)\operatorname{sgn}(x-y) - \lambda^2\rho(x)\operatorname{sgn}(x-y) \right) e^{i\lambda|x-y|} \rho(y) + \delta(x-y)\rho(x)\rho(y).$$

The first term in the kernel is treated by Schur's lemma; the second term is the kernel of the operator $\rho^2\mathrm{Id}$, which is uniformly bounded independently of $\lambda$. Hence (1.2.5) holds for $j = 2$. An interpolation argument yields (1.2.5) for any $j \in [0, 2]$.

To treat the case $d \geq 3$, we first show an identity relating the free resolvent to the wave propagator. We start with the following identity: for $\operatorname{Im} \lambda > 0$,

$$\frac{1}{|\xi|^2 - \lambda^2} = \int_0^\infty \frac{\sin(t|\xi|)}{|\xi|} e^{it\lambda} dt. \tag{1.2.7}$$

Using that both sides are holomorphic functions of $\lambda$ and $|\xi|$, it suffices to prove (1.2.7) for $\lambda \in i(0, \infty)$ and $\xi \neq 0$ – the unique continuation principle will show that the identity holds for all $\lambda$ with positive imaginary part and $\xi \in \mathbb{R}^d$. With $\lambda = is$, $s > 0$ and $\xi \neq 0$, we have:

$$\int_0^\infty \frac{\sin(t|\xi|)}{|\xi|} e^{it\lambda} dt = \frac{1}{|\xi|} \operatorname{Im} \int_0^\infty e^{t(i|\xi| - s)} dt = \frac{1}{2i|\xi|} \left( \frac{-1}{i|\xi| - s} - \frac{1}{i|\xi| + s} \right) = \frac{1}{|\xi|^2 + s^2}.$$

This shows (1.2.7) (after holomorphic continuation). Using the spectral theorem, we now define the operator

$$U(t) = \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} : L^2 \to L^2.$$

This propagator has $L^2$-norm bounded by $t$, and the identity (1.2.7) shows that the operators $R_0(\lambda)$ and $U(t)$ are related through

$$R_0(\lambda) = \int_0^\infty U(t) e^{it\lambda} dt, \quad \operatorname{Im} \lambda > 0$$

(the integral converges for the topology of bounded operators on $L^2$ because of $\operatorname{Im} \lambda > 0$ and $|U(t)|_{L^2 \to L^2} \leq t$). Below we will write $| \cdot |_{\mathscr{B}}$ for the norm of an operator from $L^2$ to itself. The propagator $U(t)$ is related to the wave equation as follows: if $u_1 \in C_0^\infty(\mathbb{R}^d)$ then $u(t) = U(t)u_1$ solves

$$(\partial_t^2 - \Delta)u = 0 \quad u(0) \equiv 0, \quad \partial_t u(0) \equiv u_1.$$

The strong Huygens principle shows that $\rho U(t) \rho = 0$ for all $\rho$ with support in $B(0, R)$ and $t \geq 2R$. Hence,

$$\rho R_0(\lambda) \rho = \int_0^{2R} e^{i\lambda t} \rho U(t) \rho dt, \quad \operatorname{Im} \lambda > 0.$$

Since the integral on the RHS is realized over a bounded set, both sides are meromorphic functions of $\lambda$.

We are now ready to prove the bound (1.2.5). Integrating by parts,

$$\rho R_0(\lambda) \rho = \int_0^{2R} e^{i\lambda t} \rho U(t) \rho dt = -\frac{1}{i\lambda} \int_0^{2R} e^{i\lambda t} \rho \partial_t U(t) \rho dt.$$

Since $\partial_t U(t) = \cos(t\sqrt{-\Delta})$ is bounded on $L^2$ by 1, we deduce that

$$|\rho R_0(\lambda)\rho|_{\mathscr{B}} \leq \frac{2R}{|\lambda|} e^{2R(\operatorname{Im}\lambda)_-}.$$

This shows the bound (1.2.5) away from $\lambda = 0$. For $d \geq 3$ and $j = 0$, using that $R_0(\lambda)$ is an entire family of operators, the maximum principle implies that

$$|\rho R_0(0)\rho|_{\mathscr{B}} = \sup_{|u|=|v|=1} \langle R_0(0)u, v\rangle \leq \sup_{|u|=|v|=1} \sup_{|\lambda|=1} \langle \rho R_0(\lambda)\rho u, v\rangle \leq \sup_{|\lambda|=1} |\rho R_0(\lambda)\rho| \leq C.$$

This concludes the proof of the lemma when $j = 0$.

The statement for $j = 1$ comes from bounds on the $L^2 \to H^1$ norm of $U(t)$. We have

$$|U(t)|_{L^2\to H^1} \leq |U(t)|_{\mathscr{B}} + |\sqrt{-\Delta}U(t)|_{\mathscr{B}} = |t| + 1,$$

which leads to

$$|\rho R_0(\lambda)\rho|_{\mathscr{B}} \leq \int_0^{2R} e^{(\operatorname{Im}\lambda)_- t}(|t|+1)dt \leq Ce^{2R(\operatorname{Im}\lambda)_-}.$$

For $j = 2$ we note that

$$\begin{aligned} |\rho R_0(\lambda)\rho|_{L^2\to H^2} &\leq |\rho R_0(\lambda)\rho|_{\mathscr{B}} + |\Delta\rho R_0(\lambda)\rho|_{L^2\to L^2} \\ &\leq |\rho R_0(\lambda)\rho|_{\mathscr{B}} + |\rho\Delta R_0(\lambda)\rho|_{\mathscr{B}} + |[\Delta,\rho]U(t)\rho|_{\mathscr{B}}. \end{aligned}$$

The first and the third term are bounded because of the case $j = 0, 1$. To control the second term, we observe that $\Delta R_0(\lambda) = -\mathrm{Id} - \lambda^2 R_0(\lambda)$ (this can be checked for $\operatorname{Im}\lambda > 0$ then entire continuation to $\mathbb{C}$). This proves the bound when $j = 2$. □

We now show that the resolvent of Schrödinger operators with smooth compactly supported potentials always extend meromorphically to $\mathbb{C}$. We write $L^2_{\mathrm{comp}}$ for the space of $L^2$ functions with compact support, and $H^2_{\mathrm{loc}}$ for the space of locally $H^2$ functions.

**Theorem 7.** *The family of operators $R_{\mathscr{V}}(\lambda) : L^2 \to H^2$ – well defined for $\operatorname{Im}\lambda$ sufficiently large – extends uniquely to a meromorphic family of operators*

$$R_{\mathscr{V}}(\lambda) : L^2_{comp} \to H^2_{loc}, \qquad \lambda \in \mathbb{C}.$$

*Proof.* The proof of the theorem when $\mathscr{V} \not\equiv 0$ uses the Lippman-Schwinger principle. It takes the form of a simple formula:

$$\operatorname{Im}\lambda \gg 1 \;\Rightarrow\; R_V(\lambda) = R_0(\lambda)(\mathrm{Id} + \mathscr{V}R_0(\lambda)\rho)^{-1}(\mathrm{Id} - \mathscr{V}R_0(\lambda)(1-\rho)), \tag{1.2.8}$$

where $\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ is equal to 1 on $\mathrm{supp}(\rho)$. To check (1.2.8), we first observe that $|\mathscr{V}R_0(\lambda)\rho| < 1$ if $\operatorname{Im}\lambda$ is sufficiently large – this follows from Lemma 1.2.1. Thus, Id +

$\mathscr{V} R_0(\lambda)\rho$ is invertible by a Neumann series. Therefore,

$$(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}(\mathrm{Id} - \mathscr{V} R_0(\lambda)(1-\rho)) = \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k(\mathrm{Id} - \mathscr{V} R_0(\lambda)(1-\rho))$$
$$= \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k - \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k\mathscr{V} R_0(\lambda) + \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k\mathscr{V} R_0(\lambda)\rho$$
$$= \mathrm{Id} - \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k\mathscr{V} R_0(\lambda) = \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda))^k = (\mathrm{Id} + \mathscr{V} R_0(\lambda))^{-1}.$$

Hence, (1.2.8) is valid if $R_{\mathscr{V}}(\lambda) = R_0(\lambda)(\mathrm{Id} + \rho R_0(\lambda))^{-1}$, which is immediate.

We now meromorphically continue the operator $(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}$. Because of analytic Fredholm theorem, we shall verify that $\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho$ is a Fredholm operator of index 0 on $L^2$, or equivalently that the operator $\mathscr{V} R_0(\lambda)\rho$ is compact on $L^2$ – see e.g. [DZ16d, Appendix C] for general theory of analytic families Fredholm operators. This follows from the mapping property of $R_0(\lambda) : L^2_{\mathrm{loc}} \to H^2_{\mathrm{loc}}$, which proves that $\mathscr{V} R_0(\lambda)\rho$ maps $L^2$ to $H^2$.

To conclude, we must show that $(\mathrm{Id}+\mathscr{V} R_0(\lambda)\rho)^{-1}$ is well-defined as an operator $L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}$ – so that we can multiply it on the left by $R_0(\lambda)$. For any $u \in L^2_{\mathrm{comp}}$, $u(\lambda) = (\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}u$ is a meromorphic vector-valued function of $\lambda$, and

$$\mathrm{Im}\,\lambda \gg 1 \quad \Rightarrow \quad u(\lambda) = \sum_{k=0}^{\infty}(-\mathscr{V} R_0(\lambda)\rho)^k u.$$

This formula shows that $\mathrm{supp}(u(\lambda)) \subset \mathrm{supp}(\rho) \cup \mathrm{supp}(u)$ for $\mathrm{Im}\,\lambda \gg 1$. The unique continuation principle shows that this holds for all $\lambda$ (that are not poles of $u(\lambda)$). This establish that

$$(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1} : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}. \tag{1.2.9}$$

Since $R_0(\lambda) : L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$ and $\mathrm{Id} - \mathscr{V} R_0(\lambda)(1-\rho) : L^2_{\mathrm{comp}} \to L^2_{\mathrm{comp}}$, we deduce that for any $\lambda \in \mathbb{C}$,

$$R_0(\lambda)(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}(\mathrm{Id} - \mathscr{V} R_0(\lambda)(1-\rho)) : L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}.$$

This mapping property, together with (1.2.8), provides the meromorphic continuation of $R_{\mathscr{V}}(\lambda)$ from $L^2_{\mathrm{comp}}$ to $H^2_{\mathrm{loc}}$. □

The poles of $R_{\mathscr{V}}(\lambda)$ are the scattering resonances of $\mathscr{V}$. Because of (1.2.8), resonances of $\mathscr{V}$ are either poles of $R_0(\lambda)$ or of $(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}$ – and we recall that $R_0(\lambda)$ is holomorphic for $d \geq 3$ and has a single pole at $\lambda = 0$ when $d = 1$. This gives a more practical characterization of resonances. In particular, we can prove (1.1.3): for any $A > 0$ and $\mathscr{V}$ with $\mathrm{supp}(\mathscr{V}) \subset B(0,R)$, there exists $B$ depending only on $A$, $R$ and $|\mathscr{V}|_\infty$ such that

$$\#\mathrm{Res}(\mathscr{V}) \cap \{\lambda : \mathrm{Im}\,\lambda \geq -A,\ |\mathrm{Re}(\lambda)| \geq B\} = 0. \tag{1.2.10}$$

Indeed, apart from the special case of 0 in dimension 1, resonances of $\mathscr{V}$ all occur as poles of $(\mathrm{Id}+\mathscr{V}R_0(\lambda)\rho)^{-1}$. According to Lemma 1.2.1, the operator $\mathscr{V}R_0(\lambda)\rho$ satisfies the bound

$$|\mathscr{V}R_0(\lambda)\rho|_{\mathscr{B}} \leq C_R|\mathscr{V}|_\infty \frac{e^{2RA}}{B}, \quad \operatorname{Im}\lambda \geq -A, \quad |\operatorname{Re}(\lambda)| \geq B.$$

where $\mathrm{supp}(\mathscr{V}) \subset B(0,R)$ and $C$ depends only on $R$. If $B$ is large enough depending only on $|\mathscr{V}|_\infty$, $A$ and $R$ then the RHS is bounded by $1/2$, hence $\mathrm{Id}+\mathscr{V}R_0(\lambda)\rho$ is invertible by a Neumann series and $\lambda$ is not a resonance. This shows that the above claim holds.

When $\mathscr{V}$ is real valued, the operator $-\Delta_{\mathbb{R}^d}+\mathscr{V}$ is selfadfjoint on $L^2$ and semibounded from below:

$$\langle(-\Delta_{\mathbb{R}^d}+\mathscr{V})u,u\rangle \geq -|\mathscr{V}|_\infty|u|^2.$$

By standard Kato perturbation theory, the spectrum of $-\Delta_{\mathbb{R}^d}+\mathscr{V}$ is equal to $[0,\infty)$ modulo a discrete set of (negative) eigenvalues. Hence, the operator $R_{\mathscr{V}}(\lambda)$ can be meromorphically continued to $\operatorname{Im}\lambda > 0$ by the spectral theorem. The unique continuation principle implies that the poles $\lambda^2$ of $R_{\mathscr{V}}(\lambda)$ with positive imaginary part are all the eigenvalues. This shows the one-to-one correspondence between resonances of $\mathscr{V}$ in the upper half-plane and negative eigenvalues of $-\Delta_{\mathbb{R}^d}+\mathscr{V}$.

One of the main results of this work is Theorem 4, which works in the context of simple resonances (i.e. resonances with simple residues and no other Lorenz coefficients), see (1.1.6). The next lemma gives a precise description of the residue at simple resonances.

**Lemma 1.2.2.** *Assume that $\lambda_0$ is a simple resonance of $\mathscr{V}$. Then there exist two smooth functions $f$ and $g$ such that*

$$(-\Delta+\mathscr{V}-\lambda_0^2)f=0, \qquad (-\Delta+\mathscr{V}-\lambda_0^2)^*g=0,$$

*and a holomorphic family of operators $\lambda \mapsto L_{\mathscr{V}}^{\lambda_0}(\lambda)$ such that*

$$R_{\mathscr{V}}(\lambda)^{-1} = \frac{if\otimes g}{\lambda-\lambda_0} + L_{\mathscr{V}}^{\lambda_0}(\lambda). \tag{1.2.11}$$

*If in addition $\lambda_0 \in i\mathbb{R}$ and $\mathscr{V}$ is real-valued, then we can take $g=\overline{f}$ in (1.2.11).*

*Proof.* We recall that $R_{\mathscr{V}}(\lambda)$ maps $C_0^\infty(\mathbb{R}^d)$ to $C^\infty(\mathbb{R}^d)$. Hence, so does

$$\Pi_0 \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\lambda_0} R_{\mathscr{V}}(\lambda)d\lambda. \tag{1.2.12}$$

Since $\Pi_0$ has rank one, there exists a distribution $g$ and a smooth function $f$ such that $\Pi_0 = if\langle\overline{g},\cdot\rangle$. The relation $(-\Delta+\mathscr{V}-\lambda_0^2)\Pi_0=0$ – coming from the identity $(-\Delta+\mathscr{V}-\lambda^2)R_{\mathscr{V}}(\lambda)=\mathrm{Id}$ – forces $(-\Delta+\mathscr{V}-\lambda_0^2)f=0$. The same argument applied to $R_{\mathscr{V}}(\lambda)^*$ (which is a Schrödinger resolvent because it is equal to $R_{\overline{\mathscr{V}}}(-\overline{\lambda})$) shows that $g$ is smooth and $(-\Delta+\mathscr{V}-\lambda_0^2)^*g=0$. Therefore,

$$\Pi_0 = if\otimes g, \quad f,\ g \in C^\infty(\mathbb{R}^d), \quad (-\Delta+\mathscr{V}-\lambda_0^2)f = (-\Delta+\mathscr{V}-\lambda_0^2)^*g=0. \tag{1.2.13}$$

If in addition $\mathscr{V}$ is real-valued and $\lambda_0$ belongs to $i\mathbb{R}$, then $\Pi_0 = \Pi_0^*$. Hence, we can take $g = \overline{f}$ in (1.2.13). This ends the proof. □

The next lemma gives fundamental examples of simple resonances. Although it is not needed in the proofs of the new theorems of this thesis, they show that simple resonances are generic – see [KZ95].

**Lemma 1.2.3.** *Let $\lambda_0 \in \mathrm{Res}(\mathscr{V})$. If either $(i)$ or $(ii)$ is satisfied:*

*$(i)$ $0 \neq \lambda_0$ and $\lambda_0$ has geometric multiplicity equal to one:*

$$\mathrm{Rank}\, \frac{1}{2\pi i} \oint_{\lambda_0} R_{\mathscr{V}}(\lambda) 2\lambda d\lambda = 1;$$

*$(ii)$ $d = 1$, $\mathscr{V}$ is real-valued and $\lambda_0 = 0$;*

*then $\lambda_0$ is a simple resonance of $\mathscr{V}$.*

*Proof.* Assume that $(i)$ is satisfied. Since $R_{\mathscr{V}}(\lambda)$ forms a meromorphic family of Fredholm operators, there exist finite rank operators $A_1, ..., A_K$ such that modulo holomorphic operators near $\lambda_0$:

$$R_{\mathscr{V}}(\lambda) = \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \lambda_0^2)^k}.$$

The definition of $\Pi_0$ in (1.2.12) forces $A_1 = 2\lambda_0 \Pi_0$; by assumption, $\mathrm{Rank}(A_1) = 1$; and since $\lambda_0 \neq 0$, $\Pi_0$ has rank one. In addition, $(-\Delta + \mathscr{V} - \lambda^2) R_{\mathscr{V}}(\lambda) = \mathrm{Id}$. Hence, again modulo holomorphic operators near $\lambda_0$,

$$\begin{aligned} 0 &= (-\Delta + \mathscr{V} - \lambda^2) R_{\mathscr{V}}(\lambda) \\ &= (-\Delta + \mathscr{V} - \lambda_0^2) \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \lambda_0^2)^k} - (\lambda^2 - \lambda_0^2) \sum_{k=1}^{K} \frac{A_k}{(\lambda^2 - \lambda_0^2)^k} \\ &= \sum_{k=1}^{K} \frac{(-\Delta + \mathscr{V} - \lambda_0^2) A_k}{(\lambda^2 - \lambda_0^2)^k} - \sum_{k=0}^{K-1} \frac{A_{k+1}}{(\lambda^2 - \lambda_0^2)^k} = \sum_{k=1}^{K} \frac{(-\Delta + \mathscr{V} - \lambda_0^2) A_k - A_{k+1}}{(\lambda^2 - \lambda_0^2)^k}. \end{aligned}$$

This proves $(-\Delta + \mathscr{V} - \lambda_0^2) A_k = A_{k+1}$. Similarly $A_k(-\Delta + \mathscr{V} - \lambda_0^2) = A_{k+1}$ and therefore $(-\Delta + \mathscr{V} - \lambda_0^2) : \mathrm{Range}(A_1) \to \mathrm{Range}(A_1)$ is nilpotent. Since $\mathrm{Rank}(A_1) = 1$, $(-\Delta + \mathscr{V} - \lambda_0^2) A_1 = 0$. In particular, $A_2 = ... = A_K = 0$, which shows that $\lambda_0$ is a simple resonance.

Assume that $(ii)$ is satisfied. The operator $-\Delta + \mathscr{V}$ is selfadjoint, with absolutely continuous spectrum equal to $[0, \infty)$. Therefore, if $s_0$ is sufficiently small then

$$0 < s < s_0 \;\Rightarrow\; |R_{\mathscr{V}}(is)|_{\mathscr{B}} = \frac{1}{s^2}. \tag{1.2.14}$$

This bound shows that there exist finite rank operators $A_1,\ A_2$ such that modulo holomorphic operator near 0,

$$R_{\mathscr{V}}(\lambda) = \frac{iA_1}{\lambda} - \frac{A_2}{\lambda^2}.$$

We must show that $A_2 = 0$; and that $A_1$ has rank one. Because of the identity $(-\Delta + \mathscr{V} - \lambda^2)R_{\mathscr{V}}(\lambda) = \mathrm{Id}$, we see that $(-\Delta + \mathscr{V})A_j = 0,\ j = 1, 2$: the range of $A_j$ is contained in the kernel of $-\Delta + \mathscr{V}$. Since $d = 1$ and $\mathscr{V}$ is compactly supported, this implies that any element in the range of $A_j$ is affine outside a compact set. In addition, for any $\psi$ with compact support, (1.2.14) shows that

$$|\rho A_2 \psi| = \lim_{s\to 0^+} |s^2 \rho R_{\mathscr{V}}(is)\psi| \leq |\psi|.$$

In particular, $A_2$ is bounded on $L^2$. Since $d = 1$ and $\mathscr{V}$ is compactly supported, the condition $(-\Delta + \mathscr{V})A_2 = 0$ implies that any element in the range of $A_2$ is affine outside a compact set; and in $L^2$. Thus it vanishes outside a compact set. Since the equation $(-\Delta + \mathscr{V})u = 0$ is an ODE, any solution vanishing on an open set vanishes everywhere. Hence $\mathrm{Range}(A_2) = 0$, implying $A_2 = 0$.

We now show that $A_1$ has rank one. Observe that the kernel of the operator $(\partial_x \pm i\lambda)R_0(\lambda)\rho$ is given by

$$-\frac{1}{2}\left(\mathrm{sgn}(x-y) \pm 1\right) e^{i\lambda|x-y|}\chi(y).$$

In particular, it vanishes if $\mp x$ is larger than a fixed constant depending only on the support of $\chi$. Let $u \in C_0^\infty(\mathbb{R}^d)$ and define $u_s = sR_{\mathscr{V}}(is)u$. Note that $u_s \to A_1 u$ in $H^2_{\mathrm{loc}}$. Because of (1.2.8), $u_s \in \mathrm{Range}(R_0(is))$; hence $(\partial_x \mp s)u_s(x) = 0$ for $\mp x$ sufficiently large. Hence, $\partial_x A_1 u = 0$ (in $H^1_{\mathrm{loc}}$) for $\mp x$ sufficiently large. This shows that every element in the range of $A_1$ is constant at infinity. Since it is also contained in the kernel of $(-\Delta + \mathscr{V})$ (which forms the set of solutions to an ODE), the range of $A_1$ has dimension at most 1, which ends the proof. □

Examples of non-simple resonances include non-zero resonances with geometric multiplicity at least 2 (see [DZ16d, Theorem 2.4 and 3.7]); and the zero resonance when $d \geq 2$ and $\mathscr{V}$ is real valued, see for instance [DZ16d, Theorem 3.20].

Simple resonances are structurally stable – i.e. if $z \mapsto \mathscr{V}(z) \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ is analytic, then simple resonances of $\mathscr{V}(z)$ depend analytically on $z$. In general, (non-simple) resonances are not structurally stable; but one can show that if $z \mapsto \mathscr{V}(z)$ is continuous, then they move along continuous path in $z$. Corollary 2 reinforces this fact: it applies to a singular perturbation of $q_0$ and to all resonances of $q_0$. However, the refinement provided by Theorem 4 requires the simplicity assumption.

### 1.2.2 Resonance expansion for scattered waves

The most spectacular manifestation of resonances in nature arises in connection with waves scattered by real-valued potentials. This interpretation comes back to the work of Lax–

Phillips [LP67]. In the context of waves in bounded domains, waves can be expanded in terms of eigenvalues. This shows that scattering resonances are the complex analog of eigenvalues in unbounded domains.

For the sake of simplicity, we state (and prove) the resonance expansion in the following situation: $d = 1$; initial data $(u_0, u_1)$ with $u_0 \equiv 0$; $\mathscr{V}$ has no eigenvalues (their contribution is clear anyway); and every non-zero resonance of $\mathscr{V}$ has geometric multiplicity equal to 1 (hence all resonances of $\mathscr{V}$ are simple, see Lemma 1.2.3). We denote $\Pi_{\lambda_0}$ the (possible) residue of $R_{\mathscr{V}}(\lambda)$ at $\lambda_0$ – which is an operator of rank 1:

$$\Pi_{\lambda_0} = \frac{1}{2\pi i} \oint_{\lambda_0} R_{\mathscr{V}}(\lambda) d\lambda.$$

**Theorem 8.** *Assume that* $\mathscr{V} \in C_0^\infty(\mathbb{R}, \mathbb{R})$ *and that* $u(t, x)$ *is the solution of*

$$\begin{cases} (\partial_t^2 - \Delta + \mathscr{V})u = 0, \\ u(0, x) = 0, \\ \partial_t u(0, x) = u_1 \in H^2_{comp}. \end{cases}$$

*Then for any* $A > 0$,

$$u(t, x) = i\Pi_0 u_1(x) + 2i \sum_{\substack{\lambda_0 \in \mathrm{Res}(\mathscr{V}), \\ \mathrm{Im}(\lambda_0) > -A}} e^{-i\lambda_0 t} \Pi_{\lambda_0} u_1(x) + E_A(t, x),$$

*where the above sum is realized over a finite set and for any* $L$, *there exists a constant* $C$ *such that*

$$\sup_{x \in [-L, L]} |E_A(x)| \le C e^{-At}.$$

We will need the following lemma, which is a refinement of (1.2.10):

**Lemma 1.2.4.** *Assume that* $|\mathscr{V}|_\infty \le M$ *and* $\mathrm{supp}(\mathscr{V}) \subset (-L, L)$. *For any* $A > 0$, *the number of resonances of* $\mathscr{V}$ *in*

$$\left\{ \lambda : \mathrm{Im}\, \lambda \ge -A - \frac{1}{4L} \ln(1 + |\,\mathrm{Re}\, \lambda|) \right\}$$

*is bounded by a constant depending only on* $A$, $M$ *and* $L$.

*Proof.* Let $\delta = \frac{1}{4L}$. It suffices to prove that there are no resonances of $\mathscr{V}$ in

$$\{\lambda : \mathrm{Im}\, \lambda \ge -A - \delta \ln(1 + |\,\mathrm{Re}\, \lambda|),\ |\,\mathrm{Re}\, \lambda| \ge B\} \tag{1.2.15}$$

for $B$ sufficiently large depending only on $M, L$ and $A$. For $\lambda$ in the set (1.2.15) and $\rho$ with support in $(-L, L)$ equal to 1 on $\mathrm{supp}(\mathscr{V})$,

$$|\mathscr{V} R_0(\lambda)\rho| \le \frac{CMe^{2L(A+\delta \ln(1+|\,\mathrm{Re}\, \lambda|))}}{|\lambda|} \le CMe^{2LA} \cdot (1 + |\,\mathrm{Re}\, \lambda|)^{-1/2}.$$

In particular, if $\delta = 1/4L$ and $B = (CMe^{2LA})^2$ (which depends only on $A, M$ and $L$) then the RHS is strictly less than by 1. The same argument as in the proof of (1.2.10) allows us to conclude. $\square$

The next result concerns the absence of embedded eigenvalues in the continuous spectrum; for a statement in higher dimension, see [DZ16d, Theorem 3.30].

**Lemma 1.2.5.** *If $d = 1$ and $\mathscr{V}$ is real valued, then* $\mathrm{Res}(\mathscr{V}) \cap \mathbb{R} \subset \{0\}$.

*Proof.* Assume that $\mathscr{V}$ has a resonance $\lambda_0 \in \mathbb{R} \setminus 0$. As in the beginning of the proof of Lemma 1.2.3, we can write modulo holomorphic operators

$$R_{\mathscr{V}}(\lambda) = \sum_{k=1}^{K} \frac{(-\Delta + \mathscr{V} - \lambda_0^2)^{k-1} A_1}{(\lambda^2 - \lambda_0^2)^k}$$

for some finite rank operator $A_1$ and some integer $K$ such that

$$(-\Delta + \mathscr{V} - \lambda_0^2)^{K-1} A_1 \neq 0, \qquad (-\Delta + \mathscr{V} - \lambda_0^2)^{K} A_1 = 0.$$

Let $0 \neq u \in \mathrm{Range}((-\Delta + \mathscr{V} - \lambda_0^2)^{K-1} A_1)$. Then, at least in the sense of distributions, $u$ satisfies: $(-\Delta + \mathscr{V} - \lambda_0^2)u = 0$; and $u$ belongs to the Sobolev space $H^{-2K}_{\mathrm{loc}}$. Using elliptic regularity, $u$ is smooth. In addition,

$$\lim_{\lambda \to \lambda_0} (\lambda^2 - \lambda_0^2)^K R_{\mathscr{V}}(\lambda) = (-\Delta + \mathscr{V} - \lambda_0^2)^{k-1} A_1.$$

By the Lippman-Schwinger formula (1.2.8) and the fact that $\lambda_0$ is not a pole of $R_0(\lambda)$, $\lambda_0$ must be a pole of order $K$ of $(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}$. In particular, if $B_1$ is such that $(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1} \sim B_1/(\lambda^2 - \lambda_0^2)^K$ near $\lambda_0$, we deduce that

$$\lim_{\lambda \to \lambda_0} (\lambda^2 - \lambda_0^2)^K R_{\mathscr{V}}(\lambda) = R_0(\lambda_0) B_1 (\mathrm{Id} - (1 - \rho) R_0(\lambda_0) \mathscr{V}).$$

Recalling that $(\mathrm{Id} + \mathscr{V} R_0(\lambda)\rho)^{-1}$ maps $L^2_{\mathrm{comp}}$ to itself, we conclude that the range of $A_1$ is contained in the range of $R_0(\lambda_0)$ on $L^2_{\mathrm{comp}}$. In particular, the explicit formula for the kernel (1.2.2) shows that there exists $a_\pm \in \mathbb{C}$ with

$$u(x) = a_\pm e^{\pm i\lambda_0 x}, \qquad \pm x \gg 1.$$

Since $\lambda_0 \in \mathbb{R}$ and $\mathscr{V}$ is real-valued, $\overline{u}$ is also outgoing. We can compute the Wronskian of $u$ and $\overline{u}$ for $\pm x \gg 1$:

$$W[u, \overline{u}] = \mathrm{Det} \begin{pmatrix} a_\pm e^{\pm i\lambda_0 x} & \overline{a_\pm} e^{\mp i\lambda_0 x} \\ \pm i\lambda_0 e^{\pm i\lambda_0 x} & \mp i\lambda_0 e^{\mp i\lambda_0 x} \end{pmatrix} = \mp 2i\lambda_0 |a_\pm|^2, \qquad \pm x \gg 1.$$

Since $u$ and $\overline{u}$ solve the same ODE, the Wronskien is constant. This implies that $a_\pm = 0$, hence $u \equiv 0$ (because $u$ solves an ODE and vanishes for large $x$). This is a contradiction. $\square$

*Proof of Theorem 8.* Under the above assumptions and notations, $u(t) = u(t, \cdot)$ is given by

$$u(t) = U(t)u_1, \quad U(t) \stackrel{\text{def}}{=} \frac{\sin(t\sqrt{P_{\mathscr{V}}})}{\sqrt{P_{\mathscr{V}}}}, \quad P_{\mathscr{V}} \stackrel{\text{def}}{=} -\Delta + \mathscr{V}$$

(observe that since $\mathscr{V}$ has no eigenvalues, the spectrum of $P_{\mathscr{V}}$ is $[0, \infty)$, in particular $\sqrt{P_{\mathscr{V}}}$ is well defined). Stone's formula for the spectral measure of $P_{\mathscr{V}}$ is

$$dE_\lambda = \frac{1}{2\pi i}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda)) \cdot 2\lambda d\lambda.$$

This is well defined because 0 is the only possible pole of $R_{\mathscr{V}}(\lambda)$ for $\lambda \in \mathbb{R}$, see Lemma 1.2.5. Using that $\sin(t\lambda)(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))$ is even while $\cos(t\lambda)(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))$ is even, we obtain

$$\begin{aligned} u(t) &= \frac{1}{2\pi i}\int_0^\infty \frac{\sin(t\lambda)}{\lambda} \cdot (R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 \cdot 2\lambda d\lambda \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{\pi i}\int_\varepsilon^\infty \sin(t\lambda) \cdot (R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 \cdot d\lambda \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \sin(t\lambda)(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda. \end{aligned}$$

Define $\mathscr{C}_\varepsilon = \{\varepsilon e^{i\theta}, \theta \in [-\pi, 0]\}$ (oriented for $\theta$ from $-\pi$ to 0); and let $\Sigma_\varepsilon = \mathbb{R} \setminus [-\varepsilon, \varepsilon] \cup \mathscr{C}_\varepsilon$ (oriented from $-\infty$ to $\infty$). Then

$$u(t) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi}\int_{\Sigma_\varepsilon} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda - \lim_{\varepsilon \to 0^+} \frac{1}{2\pi}\int_{\mathscr{C}_\varepsilon} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda.$$

We compute the second term: since $R_{\mathscr{V}}(\lambda) = \Pi_0/\lambda$ modulo holomorphic terms,

$$\frac{1}{2\pi}\int_{\mathscr{C}_\varepsilon} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda = \frac{1}{2\pi}\int_{\mathscr{C}_\varepsilon} \frac{2\Pi_0}{\lambda}u_1 d\lambda + O(\varepsilon) = i\Pi_0 u_1 + O(\varepsilon).$$

This gives the contribution of the zero resonance:

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi}\int_{\mathscr{C}_\varepsilon} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda = i\Pi_0 u_1.$$

We now study the term

$$\frac{1}{2\pi}\int_{\Sigma_\varepsilon} e^{-it\lambda}(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))u_1 d\lambda,$$

which accounts for all other resonances – observe that this term does not depend on $\varepsilon$ for $\varepsilon$ sufficiently close to 0. For $B, \delta > 0$ we define $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$, where:

$$\gamma_1 \stackrel{\text{def}}{=} (-\infty, B] \cup [B, \infty)$$
$$\gamma_2 \stackrel{\text{def}}{=} [-B, -B - i(A + \delta \ln(1+B)) \cup [B, B - i(A + \delta \ln(1+B))$$
$$\gamma_3 \stackrel{\text{def}}{=} \{s - iA - i\delta \ln(1+|s|) : s \in [-B, B]\}.$$

The number $\delta > 0$ is fixed and sufficiently small so that there exists $B_0 > 0$ with

$$\forall B \geq B_0, \ \#\mathrm{Res}(\mathscr{V}) \cap \{\lambda : |\operatorname{Re}(\lambda)| \geq B, \ \operatorname{Im}\lambda \geq -A - \delta \ln(1 + |\operatorname{Re}(\lambda)|)\} = 0 \tag{1.2.16}$$

(see Lemma 1.2.4). We work with $B \geq B_0$ – below $B$ is bound to go to $+\infty$. When deforming the contour $\Sigma_\varepsilon$ to the contour $\gamma$, the contribution of non-zero resonances is given by a finite sum over residues:

$$\sum_{\lambda_0} \frac{1}{2\pi} \oint_{\lambda_0} e^{it\lambda_0} \frac{2\Pi_{\lambda_0}}{\lambda} d\lambda = \sum_{\lambda_0} \frac{1}{2\pi} \oint_{\lambda_0} e^{-it\lambda_0} \frac{2\Pi_{\lambda_0}}{\lambda} d\lambda = 2i \sum_{\lambda_0} e^{-it\lambda_0} \Pi_{\lambda_0} u_1.$$

The above sums are realized over $\lambda_0 \in \mathrm{Res}(\mathscr{V})$ with $\operatorname{Im}\lambda_0 \geq -A - \delta\ln(1 + |\operatorname{Re}\lambda_0|)$. The condition (1.2.16) shows that it does not depend on $B \geq B_0$. In particular, the contribution of these resonances is given by

$$2i \sum_{\substack{\lambda_0 \in \mathrm{Res}(\mathscr{V}), \\ \operatorname{Im}\lambda_0 > -A}} e^{-it\lambda_0} \Pi_{\lambda_0} u_1 + O(e^{-At}),$$

where the term $O(e^{-At})$ does not depend on $B \geq B_0$. (note that since the term $O(e^{-At}$ is induced by a finite-rank operator, it lives in a finite dimensional space and one does not need to precise the norm in which it is measured). It remains to show that the integral over $\gamma$ is $O(e^{-At})$ in $L^\infty_{\text{loc}}$. To this end, we fix $\rho \in C_0^\infty((-L, L))$ and we set

$$E_k(t) = \left| \frac{1}{2\pi} \int_{\gamma_k} e^{-it\lambda} \rho(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda)) u_1 d\lambda \right|_\infty.$$

We observe that

$$R_{\mathscr{V}}(\lambda)(-\Delta + \mathscr{V}) = \lambda^2 R_{\mathscr{V}}(\lambda).$$

Using that $H^1$ embeds in $L^\infty$ when $d = 1$, and that $\rho R_{\mathscr{V}}(\lambda)\rho$ is bounded from $H^1$ to $L^2$ by $Ce^{2L(\operatorname{Im}\lambda)_-}$,

$$\begin{aligned} E_k(t) &\leq C \left| \int_{\gamma_k} e^{-it\lambda} (R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda)) u_1 d\lambda \right|_{H^1} \\ &\leq C \int_{\gamma_k} \frac{e^{t\operatorname{Im}\lambda}}{|\lambda|^2} \left| \rho(R_{\mathscr{V}}(\lambda) - R_{\mathscr{V}}(-\lambda))(-\Delta + \mathscr{V}) u_1 \right|_{H^1} d\lambda \\ &\leq C \int_{\gamma_k} \frac{e^{(t-2L)\operatorname{Im}\lambda}}{|\lambda|^2} \left| (-\Delta + \mathscr{V}) u_1 \right| d\lambda \leq C |u_1|_{H^2} \int_{\gamma_k} \frac{e^{(t-2L)\operatorname{Im}\lambda}}{|\lambda|^2} d\lambda. \end{aligned}$$

Hence, for $t \geq 2L$, we have

$$E_1(t) + E_2(t) \leq \frac{C|u_1|_{H^2}}{B}, \quad E_3(t) \leq C|u_1|_{H^2} e^{-(t-2L)A},$$

where the constant $C$ does not depend on $B$. We can then take $B = e^{At}$ to conclude $E_k(t) = O(e^{-At})|u_1|_{H^2}$ for $t \geq 2L$. This ends the proof. □

This expansion applies to partial differential equations. It shows that locally, waves scattered by $\mathscr{V}$ decay exponentially. Such a result cannot be applied directly to non-linear PDEs in asymptotically Euclidean spaces, because one still has to deal with the propagation of the waves. We would like to mention some recent development in asymptotically hyperbolic spacetimes, specifically the exterior of the de-Sitter black holes. Dyatlov [Dy11a, Dy11b, Dy12] and Vasy [Va13] defined resonances for such spacetimes and studied their distributions. They obtained a resonance expansion for scattered waves. This expansion holds globally in the black-hole exterior, because the energy eventually crosses the black hole and cosmological horizons and cannot come back. Hintz and Vasy [HV16] used these results to prove the global stability of slowly rotating Kerr-de Sitter black holes.

### 1.2.3 Trace class operators and determinants

We give a brief overview to the general theory of determinants of trace-class operators. For a complete introduction, we refer to [DZ16d, Appendix B]. This theory will be needed in the proof of Theorem 3. Let $A$ be a compact operator on a Hilbert space $\mathscr{H}$. The spectrum of $A$ is described by a sequence of non-negative eigenvalues $\lambda_j(A)$ converging to 0. The operator $(AA^*)^{1/2}$ is a compact selfadjoint operator and its spectrum is a sequence of non-negative eigenvalues $s_j(A)$ decreasing to 0.

Thanks to the min-max principle for selfadjoint operators, there exists a very useful characterization of singular values:

$$s_j(A) = \min\{|A - K|_{\mathscr{H} \to \mathscr{H}}, \ K : \mathscr{H} \to \mathscr{H}, \ \text{Rank}(K) \leq j\}.$$

It implies the following fundamental inequalities:

$$\begin{gathered} A, \ B \text{ compact } \ \Rightarrow \ s_{j+k}(A+B) \leq s_j(A) + s_k(B), \quad s_{j+k}(AB) \leq s_j(A)s_k(B); \\ A \text{ compact }, \ B \text{ bounded } \ \Rightarrow \ s_j(AB) \leq s_j(A)|B|. \end{gathered} \tag{1.2.17}$$

These two inequalities are often used to give precise estimates on the singular values of pseudo-differential operators. Concretely, assume that $(X, g)$ is a compact Riemannian manifold and $P : L^2(X) \to H^s(X)$ for some $s > 0$. Let $\Delta_g$ be the Laplacian operator on $X$. The operator $(\text{Id} - \Delta_g)^{s/2}P$ is a bounded operator on $L^2(X)$. Hence,

$$\begin{aligned} s_j(P) &= s_j((\text{Id} - \Delta_g)^{-s}(\text{Id} - \Delta_g)^s P) \leq C s_j((\text{Id} - \Delta_g)^{-s}) \\ &= C\lambda_j((\text{Id} - \Delta_g)^{-s}) = C(1 + \lambda_j(-\Delta_g))^{-s/2} \end{aligned}$$

The last equality comes from the fact that $\Delta_g$ is self-adjoint. The Weyl law gives precise bounds on the eigenvalues of $-\Delta_g$: $\lambda_j(-\Delta_g) \leq Cj^{2/d}$. It follows that

$$\exists C > 0,\ \forall j > 0,\ s_j(P) \leq Cj^{-s/d}, \tag{1.2.18}$$

where $C$ depends only on the norm of $(\mathrm{Id} - \Delta_g)^{s/2} P$ on $L^2$. This estimate also applies when $P : L^2 \to H^s$ is a pseudodifferential operator on $\mathbb{R}^d$, with the following growth property:

$$\exists \chi \in C_0^\infty(\mathbb{R}^d), \quad P\chi = P,\ \chi P = P. \tag{1.2.19}$$

Indeed, if $L$ is large enough so that $\mathrm{supp}(\rho) \subset (-L, L)^d$, then $P$ induces an equivalent operator on the torus $(\mathbb{R}/(L\mathbb{Z}))^d$. Since $(\mathbb{R}/(L\mathbb{Z}))^d$ is a compact manifold, (1.2.18) follows. We give a particularly important example in the context of scattering resonances:

**Lemma 1.2.6.** *Let $K$ be a compact subset of $\mathbb{C}$, with $0 \notin \mathbb{C}$ if $d = 1$. There exists $C > 0$ such that for all $\lambda \in K$, $\sum_{j=0}^\infty s_j((\mathscr{V} R_0(\lambda)\rho)^d) \leq\leq C$.*

*Proof.* Let $K_{\mathscr{V}}(\lambda) = \mathscr{V} R_0(\lambda)\rho$. This operator satisfies (1.2.19) and maps $L^2$ to $H^2$, hence $s_j(K_{\mathscr{V}}(\lambda)) \leq Cj^{-2/d}$. Since $s_j(K_{\mathscr{V}}(\lambda))$ is a non-increasing sequence,

$$\sum_{j=0}^\infty s_j(K_{\mathscr{V}}(\lambda)^d) \leq d \sum_{j=0}^\infty s_{dj}(K_{\mathscr{V}}(\lambda)^d).$$

Apply now (1.2.17): $s_{dj}(K_{\mathscr{V}}(\lambda)) \leq s_j(K_{\mathscr{V}}(\lambda))^d$. It follows that

$$\sum_{j=0}^\infty s_j(K_{\mathscr{V}}(\lambda)^d) \leq d \sum_{j=0}^\infty s_j(K_{\mathscr{V}}(\lambda))^d \leq C \sum_{j=0}^\infty j^{-2} \leq C,$$

where $C$ depends only on the norm of $K_{\mathscr{V}}(\lambda) : L^2 \to H^2$. Since $\lambda$ is restricted to a compact set, the lemma is proved. □

There are other important inequalities due to Weyl: if $A$ is compact on $\mathscr{H}$ and the complex number $\lambda_j(A)$ are ordered such that $|\lambda_j(A)|$ decreases to 0, then for any $n \geq 0$,

$$\sum_{j=0}^n |\lambda_j(A)| \leq \sum_{j=0}^n |s_j(A)|, \qquad \prod_{j=0}^n (1 + |\lambda_j(A)|) \leq \prod_{j=0}^n (1 + s_j(A)).$$

In particular, if the sum (resp. product) on the right converges, than the sum (resp. product) on the left converges. Equivalently, if $s_j(A)$ is summable, then one can define

$$\mathrm{Tr}(A) \overset{\text{def}}{=} \sum_{j=1}^\infty \lambda_j(A), \qquad \mathrm{Det}(\mathrm{Id} + A) \overset{\text{def}}{=} \prod_{j=0}^\infty (1 + \lambda_j(A)).$$

These quantities are called the Fredholm trace of $A$ and Fredholm determinant of $\mathrm{Id} + A$. Operators such that $s_j(A)$ is summable are called trace-class, and we write

$$|A|_{\mathscr{L}} \stackrel{\text{def}}{=} \sum_{j=0}^{\infty} s_j(A).$$

Fredholm determinants generalize the standard determinants of matrices. In particular, if $A$ is a trace-class operator, then

$$\mathrm{Id} + A \text{ is invertible} \iff \mathrm{Det}(\mathrm{Id} + A) = 0.$$

This property becomes very interesting when $A(\lambda)$ is a family of trace-class operators, depending analytically on $\lambda$, because it shows that the set of points such that $\mathrm{Id} + A(\lambda)$ is not invertible is equal to the nodal set of the analytic function $\lambda \mapsto \det(\mathrm{Id} + A(\lambda))$.

Complex analysis theorems on zeroes of analytic functions can then be applied in the context of non-selfadjoint spectral theory – in particular for scattering resonances. Nevanlinna theory (asymptotic of the number of zeroes of entire functions of order 1 in large balls) led Zworski [Zw87] to give precise asymptotic in large balls on the number of resonances of potentials – see also [Fr97] and [Si00]. In higher dimension, Zworski [Zw89] established sharp upper bounds for the number of resonances in large balls. A lower bound is known generically [DV15] but the validity of this bound for all potentials is still open. Simpler complex analysis results about the localization of zeroes (Rouché's theorem, Hurwitz's theorem) are central to the stability analysis of resonances, see Drouot [Dr15, Dr16b] and Zworski [Zw15].

In many concrete examples, we need to consider compact operators that fail to be trace-class. This is for instance the case when $P = (\mathrm{Id} - \Delta_g)^{-1}$ on a compact Riemannian manifold $(X, g)$ of dimension $d \geq 2$: the Weyl law shows that $\sum_{j=0}^{\infty} s_j(P) = \infty$. However, (1.2.18) shows that there exists a constant $C > 0$ such that

$$\sum_{j=0}^{\infty} s_j(P)^d \leq C \sum_{j=1}^{\infty} j^{-2} < \infty.$$

In particular, $P$ is not trace-class, but $P^d$ is trace-class. This motivates the definition of a Banach space $\mathscr{L}^p$ made of operators $A$ such that $A^p \in \mathscr{L}$. The following trick allows to define modified Fredholm determinants for operators $A \in \mathscr{L}^p$. Let

$$\psi_p(z) = (1+z) \exp\left( -z + \frac{z^2}{2} - ... + \frac{(-z)^{p-1}}{p-1} \right) - 1.$$

We observe that $\mathrm{Id} + \psi_p(A)$ is invertible if and only $\mathrm{Id} + A$ is invertible. In addition, $\psi$ is an entire function such that $\psi(z) = O(z^p)$ near 0. The inequality (1.2.17) implies that $\psi_p(A) \in \mathscr{L}^1$. Hence we can define $\mathrm{Det}(\mathrm{Id} + \psi_p(A))$, which has the property

$$\mathrm{Id} + A \text{ is invertible} \iff \mathrm{Det}(\mathrm{Id} + \psi_p(A)) = 0.$$

The theory of Fredholm determinants and modified Fredholm determinants was developed in work of Grothendieck [Gr56] and Simon [Si77]. In dimension greater than 1, it is generally difficult to apply modified Fredholm determinants for counting scattering resonances in large compact sets, because there is no Nevanlinna theory of entire functions with order hihger than one. They are however very useful to study stability results, as performed here.

## 1.3 Deterministic theory of perturbation of resonances

### 1.3.1 Functional framework

Let $H^s$ be the standard scale of Sobolev spaces on $\mathbb{R}^d$. The functional framework relevant here is given by a scale of Banach space $\mathscr{H}^{-s}$, introduced in Golowich–Weinstein [GW05]. The associated norm $|\cdot|_{\mathscr{H}^{-s}}$ is defined on smooth functions $\mathscr{V} \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ as the $H^s \to H^{-s}$ operator norm of the multiplication operator by $\mathscr{V}$:

$$|\mathscr{V}|_{\mathscr{H}^{-s}} \overset{\text{def}}{=} |\mathscr{V}|_{H^s \to H^{-s}} = \sup_{f \in H^s} \frac{|\mathscr{V} f|_{H^{-s}}}{|f|_{H^s}} = \left|\langle D\rangle^{-s} \mathscr{V} \langle D\rangle^{-s}\right|_{\mathscr{B}} \tag{1.3.1}$$

– where $|\cdot|_{\mathscr{B}}$ is the operator norm for linear operators on $L^2$. Intuitively, a highly oscillatory function is small in $\mathscr{H}^s$. The norm on the spaces $\mathscr{H}^s$ cannot be easily computed. In the next lemma, we compare the $\mathscr{H}^{-s}$-norm by norms with more standard Hilbert spaces. The bilinear structure of the hilbertian norm will be useful later.

**Lemma 1.3.1.** *Fix $s > 0$.*

*(i) If $s > d/2$, then there exists $C > 0$ such that for any $\mathscr{V} \in C_0^\infty$, $|\mathscr{V}|_{\mathscr{H}^{-s}} \leq C|\mathscr{V}|_{H^{-s}}$.*

*(ii) If $0 < s \leq d/2$, then for any $s' > d/2$, there exists $C > 0$ such that*

$$\mathscr{V} \in C_0^\infty \;\Rightarrow\; |\mathscr{V}|_{\mathscr{H}^{-s}} \leq C|\mathscr{V}|_\infty^{1-s/s'} |\mathscr{V}|_{H^{-s'}}^{s/s'},$$

*Proof.* If $s > d/2$, then the Sobolev space $H^s$ is an algebra. Therefore, there exists $C > 0$ such that for any $\mathscr{V}$, $f$ in $H^s$, $|\mathscr{V} f|_{H^s} \leq C|\mathscr{V}|_{H^s}|f|_{H^s}$. The corresponding dual inequality reads $|\mathscr{V} f|_{H^{-s}} \leq C|\mathscr{V}|_{H^{-s}}|f|_{H^s}$. Part $(i)$ follows now from the definition (1.3.1).

Assume now that $0 < s \leq d/2$. By interpolation theory, for any $s' > d/2$, $|\mathscr{V}|_{\mathscr{H}^{-s}} \leq |\mathscr{V}|_{\mathscr{H}^0}^{1-\theta} |\mathscr{V}|_{\mathscr{H}^{-s'}}^{\theta}$ where $s = \theta s'$. The $\mathscr{H}^0$-norm of $\mathscr{V}$ is controlled by $|\mathscr{V}|_\infty$, and the $\mathscr{H}^{-s'}$ -norm of $\mathscr{V}$ is controlled by $|\mathscr{V}|_{H^{-s'}}$ because of $(i)$. This implies $(ii)$. □

### 1.3.2 Perturbation theory

In this section, we develop a perturbation theory for resonances, which is based on analytic Fredholm theory. The results are stated without reference to Fredholm determinants, but the proof use them in a fundamental way.

For $\mathscr{V} \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ and $\rho \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ equal to 1 on $\mathrm{supp}(\mathscr{V})$, we define $K_{\mathscr{V}}(\lambda) = \mathscr{V} R_0(\lambda)\rho$. In Lemma 1.3.2 and 1.3.3 below, $\mathscr{V}, \mathscr{V}_0, \mathscr{V}_1$ denote three smooth compactly supported functions with support in $[-M, M]^d$ and bounded uniformly by $M$; and the constants $C$ depend uniformly in $M$.

**Lemma 1.3.2.** *Let $R > 0$ such that $\mathscr{V}_0$ has no resonances in $\partial\mathbb{D}(0, R)$. There exist $\varepsilon_0, C > 0$ such that if $C|\mathscr{V}_1|_{\mathscr{H}^{-2}} \leq \varepsilon < \varepsilon_0$ then*

$$\mathrm{Res}(\mathscr{V}_1) \cap \mathbb{D}(0, R) \subset \bigcup_{\lambda \in \mathrm{Res}(\mathscr{V}_0) \cap \mathbb{D}(0,R)} \mathbb{D}\left(\lambda, \varepsilon^{1/m_\lambda}\right),$$
$$\lambda \in \mathrm{Res}(\mathscr{V}_0) \cap \mathbb{D}(0, R) \ \Rightarrow \#\mathbb{D}(\lambda_0, \varepsilon^{1/m_\lambda}) \cap \mathrm{Res}(\mathscr{V}_0 + \mathscr{V}_1) = m_\lambda.$$

*Proof.* The proof is based on a Fredholm determinant approach. To simplify the notations, we write $K_{\mathscr{V}}$ instead of $K_{\mathscr{V}}(\lambda)$ in this proof. We first deal with the case $d \geq 3$ and explain the modifications needed for $d = 1$ at the end.

For $d \geq 3$, let $\psi$ be the entire function given by

$$\psi(z) \overset{\text{def}}{=} (1 + z) \exp\left(-z + \frac{z^2}{2} - ... + \frac{(-z)^{2d}}{2d}\right) - 1.$$

We define the Fredholm determinant $D_{\mathscr{V}}(\lambda) = \det(\mathrm{Id} + \psi(K_{\mathscr{V}}))$. The function $D_{\mathscr{V}}(\lambda)$ is entire and its zeroes are exactly the resonances of $\mathscr{V}$ in $\mathbb{C}$, with their multiplicity – see [GLMZ05, Theorem 5.4]. We show below that

$$\sup_{\lambda \in \mathbb{D}(0,R)} |D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda)| \leq C\,|\mathscr{V}_1|_{\mathscr{H}^{-2}}\,. \tag{1.3.2}$$

We first observe that

$$\begin{gathered} D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda) = \int_{t=0}^1 \partial_t D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda) dt \\ = \int_{t=0}^1 D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda) \mathrm{Tr}\left((\mathrm{Id} + \psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1} \partial_t \psi(K_{\mathscr{V}_0+t\mathscr{V}_1})\right) dt. \end{gathered} \tag{1.3.3}$$

Since $\psi$ is an entire function with $\psi(z) = O(z^{2d+1})$ near $z = 0$, we can write $\psi(z) = \sum_{m=2d+1}^\infty a_m z^m$ with uniform convergence on bounded subsets of $\mathbb{C}$. Hence,

$$\partial_t \psi(K_{\mathscr{V}_0+t\mathscr{V}_1}) = \sum_{m=2d+1}^\infty a_m \frac{d}{dt} K^m_{\mathscr{V}_0+t\mathscr{V}_1} = \sum_{m=2d+1}^\infty a_m \sum_{j+\ell=m-1} K^j_{\mathscr{V}_0+t\mathscr{V}_1} K_{\mathscr{V}_1} K^\ell_{\mathscr{V}_0+t\mathscr{V}_1}. \tag{1.3.4}$$

The convergence is uniform in the space of trace-class operators: when $j + \ell \geq 2d$, either $K^j_{\mathscr{V}_0+t\mathscr{V}_1}$ or $K^\ell_{\mathscr{V}_0+t\mathscr{V}_1}$ is trace-class, hence Lemma 1.2.6 shows that

$$|\partial_t \psi(K_{\mathscr{V}_0+t\mathscr{V}_1})|_{\mathscr{L}} \leq \sum_{m=2d+1}^\infty |a_m|(m-1)C^m$$

– where $|\cdot|_{\mathscr{L}}$ denotes the trace-class norm. This series converges absolutely because $\psi$ is an entire function of order $2d$, therefore $|a_m|$ converge rapidly to 0 – see for instance [Dr15, (4.7)] for a precise statement. The cyclicity of the trace implies that

$$\mathrm{Tr}\left((\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}K^j_{\mathscr{V}_0+t\mathscr{V}_1}K_{\mathscr{V}_1}K^\ell_{\mathscr{V}_0+t\mathscr{V}_1}\right) = \mathrm{Tr}\left((\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}K^{j+\ell}_{\mathscr{V}_0+t\mathscr{V}_1}K_{\mathscr{V}_1}\right).$$

We combine this identity with (1.3.3) and (1.3.4) to get

$$\begin{aligned}\mathrm{Tr}\left((\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}\partial_t\psi(K_{\mathscr{V}_0+t\mathscr{V}_1})\right) &= \sum_{m=2d+1}^{\infty} m a_m \mathrm{Tr}\left((\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}K^{m-1}_{\mathscr{V}_0+t\mathscr{V}_1}K_{\mathscr{V}_1}\right)\\ &= \mathrm{Tr}\left((\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}\psi'(K_{\mathscr{V}_0+t\mathscr{V}_1})K_{\mathscr{V}_1}\right).\end{aligned}$$

Since $(1+\psi(z))^{-1}\psi'(z) = (1+z)^{-1}z^{2d}$, we obtain

$$\begin{aligned}D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda) &= \int_{t=0}^1 D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda)\mathrm{Tr}\left((\mathrm{Id}+K_{\mathscr{V}_0+t\mathscr{V}_1})^{-1}K^{2d}_{\mathscr{V}_0+t\mathscr{V}_1}K_{\mathscr{V}_1}\right)dt\\ &= \int_{t=0}^1 D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda)\mathrm{Tr}\left((\mathrm{Id}+K_{\mathscr{V}_0+t\mathscr{V}_1})^{-1}K^{2d-1}_{\mathscr{V}_0+t\mathscr{V}_1}K_{\mathscr{V}_1}K_{\mathscr{V}_0+t\mathscr{V}_1}\right)dt.\end{aligned}$$

Hence, $|D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda)|$

$$\leq \sup_{t\in[0,1]} \left|D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda)(\mathrm{Id}+K_{\mathscr{V}_0+t\mathscr{V}_1})^{-1}\right|_{\mathscr{B}} \cdot |K^{2d-1}_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{L}} \cdot |K_{\mathscr{V}_1}K_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{B}}. \tag{1.3.5}$$

We show that the RHS of (1.3.5) is uniformly bounded for $\lambda \in K$. If $p$ is the polynomial such that $\psi(z) = (1+z)e^{p(z)}$, then

$$D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda)(\mathrm{Id}+K_{\mathscr{V}_0+t\mathscr{V}_1})^{-1} = D_{\mathscr{V}_0+t\mathscr{V}_1}(\lambda)(\mathrm{Id}+\psi(K_{\mathscr{V}_0+t\mathscr{V}_1}))^{-1}\cdot e^{p(K_{\mathscr{V}_0+t\mathscr{V}_1})}. \tag{1.3.6}$$

The first factor in the RHS of (1.3.6) is controlled by [Dr15, Appendix 5.1] while the second factor is uniformly bounded by Lemma 1.2.1. The term $|K^{2d-1}_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{L}}$ in the RHS of (1.3.5) is also uniformly bounded because of Lemma 1.2.6; finally, $|K_{\mathscr{V}_1}K_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{B}}$ is controlled as follows:

$$|K_{\mathscr{V}_1}K_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{B}} \leq |K_\rho\langle D\rangle^2|_{\mathscr{B}}\cdot|\langle D\rangle^{-2}\mathscr{V}_1\langle D\rangle^{-2}|_{\mathscr{B}}\cdot|\langle D\rangle^2K_{\mathscr{V}_0+t\mathscr{V}_1}|_{\mathscr{B}} \leq C|\langle D\rangle^{-2}\mathscr{V}_1\langle D\rangle^{-2}|_{\mathscr{B}},$$

where the boundedness of $K_\rho\langle D\rangle^2$ and $\langle D\rangle^2K_\rho$ follow from Lemma 1.2.1. This shows (1.3.2).

The Fredholm determinant $D_{\mathscr{V}_0}(\lambda)$ has no zeroes on $\partial\mathbb{D}(0,R)$, hence there exists $t>0$ such that $|D_{\mathscr{V}_0}(\lambda)| > t$ for $\lambda \in \partial\mathbb{D}(0,R)$. Hence, if $C|\mathscr{V}_1|_{\mathscr{H}^{-2}} \leq \varepsilon$,

$$\lambda \in \mathbb{D}(0,R) \ \Rightarrow |D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda)| \leq \varepsilon. \tag{1.3.7}$$

If $\varepsilon$ is sufficiently small, the RHS is bounded by $t$. Rouché's theorem implies that

$$C|\mathscr{V}_1|_{\mathscr{H}^{-2}} \leq \varepsilon \ \Rightarrow \#\mathrm{Res}(\mathscr{V}_1+\mathscr{V}_0)\cap\mathbb{D}(0,R) = \#\mathrm{Res}(\mathscr{V}_0)\cap\mathbb{D}(0,R). \tag{1.3.8}$$

Let $\lambda_0 \in \mathbb{D}(0,R)$ be a resonance of $\mathscr{V}_0$, of multiplicity $m_{\lambda_0}$. We show that $\mathscr{V}_0 + \mathscr{V}_1$ has exactly $m$ resonances in the disk $\mathbb{D}(\lambda_0, \varepsilon^{1/m_{\lambda_0}})$ for $C$ sufficiently large, and $\varepsilon$ sufficiently small. There exists $r_0 > 0$ such that $\lambda_0$ is the only zero of $D_{\mathscr{V}_0}(\lambda)$ on $\mathbb{D}(\lambda_0, r_0)$. Hence, $|D_{\mathscr{V}_0}(\lambda)| > c_0|\lambda - \lambda_0|^{m_{\lambda_0}}$ for $c_0$ sufficiently small and $\lambda \in \mathbb{D}(\lambda_0, r_0)$. Because of this and (1.3.7), after possibly increasing the value of $C$,

$$\lambda \in \partial\mathbb{D}(0, \varepsilon^{1/m_{\lambda_0}}) \ \Rightarrow \ |D_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - D_{\mathscr{V}_0}(\lambda)| < |D_{\mathscr{V}_0}(\lambda)|.$$

Again, Rouché's theorem implies that $\mathscr{V}_0 + \mathscr{V}_1$ and $\mathscr{V}_0$ have the same number of zeroes in $\mathbb{D}(\lambda_0, \varepsilon^{1/m_{\lambda_0}})$ – i.e. $m_{\lambda_0}$. This fact, combined with (1.3.8), implies that all resonances of $\mathscr{V}_0 + \mathscr{V}_1$ in $\mathbb{D}(0,R)$ are confined in

$$\bigcup_{\lambda_0 \in \mathrm{Res}(\mathscr{V}_0)} \mathbb{D}(\lambda_0, \varepsilon^{1/m_{\lambda_0}}).$$

This concludes the proof of the lemma for $d \geq 3$.

When $d = 1$, the estimate (1.3.7) holds uniformly locally on $\mathbb{D}(0,R) \setminus 0$; however, unless $\int_{\mathbb{R}} \mathscr{V}(x)dx = 0$, the function $D_{\mathscr{V}}(\lambda)$ has an essential singularity at $\lambda = 0$. We introduce

$$d_{\mathscr{V}}(\lambda) \overset{\mathrm{def}}{=} \lambda \det(\mathrm{Id} + K_{\mathscr{V}}(\lambda)),$$

which is an entire function of $\lambda$, and whose zeroes are exactly the resonances of $\mathscr{V}$ counted with multiplicity – see [DZ16d][Theorem 2.6]. Since $\mathrm{Tr}(K_{\mathscr{V}}) = \frac{i}{2\lambda}\int_{\mathbb{R}} \mathscr{V}$ (see for instance the explicit formula (1.2.2) for the kernel of $R_0(\lambda)$ in dimension 1) and $\psi(z) = (1+z)e^{-z}$,

$$D_{\mathscr{V}}(\lambda) = \det((\mathrm{Id} + K_{\mathscr{V}})e^{-K_{\mathscr{V}}}) = \det(\mathrm{Id} + K_{\mathscr{V}})e^{-\mathrm{Tr}(K_{\mathscr{V}})} = \lambda^{-1} d_{\mathscr{V}}(\lambda) \exp\left(-\frac{i}{2\lambda}\int_{\mathbb{R}} \mathscr{V}\right).$$

It follows that

$$d_{\mathscr{V}}(\lambda) = \lambda D_{\mathscr{V}}(\lambda) \exp\left(\frac{i}{2\lambda}\int_{\mathbb{R}} \mathscr{V}\right).$$

Hence, to deal with $d = 1$, it suffices to replace $D_{\mathscr{V}}$ by $d_{\mathscr{V}}$ and essentially show

$$\lambda \in \mathbb{D}(0,R) \ \Rightarrow |d_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) - d_{\mathscr{V}_0}(\lambda)| \leq C\varepsilon. \tag{1.3.9}$$

By the maximum principle, (1.3.9) holds on $\mathbb{D}(0,R)$ if it holds on $\partial\mathbb{D}(0,R)$. The estimate (1.3.7) works when $d = 1$ and $\lambda$ is away from 0 – for instance $\lambda \in \partial\mathbb{D}(0,R)$. Hence, (1.3.9) holds if we can show

$$\left| \int_{\mathbb{R}} \mathscr{V}_0 + \mathscr{V}_1 - \int_{\mathbb{R}} \mathscr{V}_0 \right| = O(\varepsilon).$$

This follows from the condition $|\mathscr{V}_1|_{\mathscr{H}^{-2}} = O(\varepsilon)$: if $\rho \in C_0^\infty$ is 1 on $\mathrm{supp}(\mathscr{V})$,

$$\left| \int_{\mathbb{R}} \mathscr{V}_0 + \mathscr{V}_1 - \int \mathscr{V}_0 \right| = \left| \int_{\mathbb{R}} \mathscr{V}_1 \rho^2 \right| \leq |\mathscr{V}_1 \rho|_{H^{-2}} |\rho|_{H^2} \leq |\mathscr{V}_1|_{\mathscr{H}^{-2}} |\rho|_{H^2}^2 = O(\varepsilon).$$

This concludes the proof for $d = 1$. □

Fix $\lambda_0 \in \mathbb{C}$ and assume that $\lambda_0 \in \operatorname{Res}(\mathscr{V}_0)$ is simple. Hence, there exist $f, g \in C^\infty(\mathbb{R}^d)$ and a holomorphic family of operators $L_{\mathscr{V}_0}^{\lambda_0}(\lambda)$ near $\lambda_0$ such that

$$R_{\mathscr{V}_0}(\lambda) = L_{\mathscr{V}_0}^{\lambda_0}(\lambda) + i\frac{f \otimes g}{\lambda - \lambda_0}. \tag{1.3.10}$$

The next result concerns resonances of $\mathscr{V}_0 + \mathscr{V}_1$ near $\lambda_0$, assuming that $|\mathscr{V}_1|_{\mathscr{H}^{-2}}$ is small. It can be seen as a local characteristic equation for scattering resonances.

**Lemma 1.3.3.** *Under the notations of (1.3.10), there exist $r_0, \delta_0$ and $C$ all positive such that if $|\mathscr{V}_1|_{\mathscr{H}^{-2}} \leq \delta_0$, then:*

*(i) For any $\kappa \geq 0$,*

$$\lambda \in \mathbb{D}(0, R) \ \Rightarrow \ \sum_{k=\kappa}^{\infty} \left| \left\langle (\mathscr{V}_1 L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho)^k \mathscr{V}_1 f, g \right\rangle \right| \leq C |\mathscr{V}_1|_{\mathscr{H}^{-1}}^{\kappa+1}. \tag{1.3.11}$$

*(ii) The potential $\mathscr{V}_0 + \mathscr{V}_1$ has a unique resonance $\lambda_1$ in $\mathbb{D}(\lambda_0, r_0)$.*

*(iii) If $\varphi : \mathbb{D}(\lambda_0, r_0) \to \mathbb{C}$ is the holomorphic function given by*

$$\varphi(\lambda) = \lambda - \lambda_0 + i \sum_{k=0}^{\infty} (-1)^k \left\langle (\mathscr{V}_1 L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho)^k \mathscr{V}_1 f, \overline{g} \right\rangle \tag{1.3.12}$$

*then $\varphi$ has a unique zero $\lambda_2$ in $\mathbb{D}(\lambda_0, r_0)$. In addition, $\lambda_2 = \lambda_1$.*

**Remark 1.3.1.** *This result is related to [GW05, Theorem 4.1], which asserts that $\lambda_1$ has an expansion that depends analytically on $\mathscr{V}_1$. The proof relies on analytic Fredholm theory instead of the implicit function theorem. The novelty of the result is the characterization of resonances as zeroes of $\varphi$. It can be seen as a local characteristic equation for resonances.*

*Proof.* Start with $(i)$. Let $r_0 > 0$ such that $\mathscr{V}_0$ has no resonances but $\lambda_0$ in $\mathbb{D}(\lambda_0, r_0)$. Below we work with $\lambda \in \mathbb{D}(\lambda_0, r_0)$. We first check that $\rho L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho$ maps $H^{-1}$ to $H^1$. We have:

$$\rho L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho = \oint_{\partial\mathbb{D}(\lambda_0, r_0)} \frac{\rho R_{\mathscr{V}_0}(\mu)\rho}{\mu - \lambda} d\mu. \tag{1.3.13}$$

This comes from the Cauchy formula applied to $L_{\mathscr{V}_0}^{\lambda_0}(\lambda) = R_{\mathscr{V}_0}(\lambda) - i\frac{f\otimes g}{\lambda - \lambda_0}$ and the identity

$$\oint_{\partial\mathbb{D}(\lambda_0, r_0)} \frac{if \otimes g}{(\mu - \lambda)(\mu - \lambda_0)} d\mu = \frac{1}{\lambda_0 - \lambda} \oint_{\lambda_0} \frac{if \otimes g}{\mu - \lambda_0} d\mu - \frac{1}{\lambda_0 - \lambda} \oint_{\lambda_0} \frac{if \otimes g}{\mu - \lambda} d\mu = 0.$$

When $\mu \in \partial\mathbb{D}(\lambda_0, r_0)$, the operator $\rho R_{\mathscr{V}_0}(\mu)\rho$ maps $L^2$ to $H^2$, see Theorem 7. For any $u \in L^2, v \in H^{-2}$, the pairing $\langle \rho R_{\mathscr{V}_0}(\mu)\rho u, v\rangle$ is a holomorphic function on $\mathbb{D}(\lambda_0, r_0) \setminus \lambda_0$ – in particular,

$$\sup_{\mu \in \partial\mathbb{D}(\lambda_0, r_0)} |\langle \rho R_{\mathscr{V}_0}(\mu)\rho u, v\rangle| < \infty$$

The Banach–Steinhauss principle implies that the family $R_{\mathscr{V}_0}(\mu)$ is bounded from $L^2$ to $H^2$ uniformly for $\mu \in \mathbb{D}(\lambda_0, r_0)$. Since (1.3.13) expresses $\rho L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho$ as the contour integral of a uniformly bounded $L^2 \to H^2$ family of operators function over a circle, the operator $\rho L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho$ is itself bounded from $L^2$ to $H^2$. Using a duality argument, it also maps $H^{-2}$ to $L^2$, and (by interpolation) $H^{-1}$ to $H^1$. This shows that for $\lambda \in \mathbb{D}(\lambda_0, r_0)$ the operator $B(\lambda) = \langle D\rangle\rho L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho\langle D\rangle$ is bounded on $L^2$. As operators on $L^2$,

$$\langle D\rangle^{-1}(\mathscr{V}_1 L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho)^k \mathscr{V}_1\langle D\rangle^{-1} = \left(\langle D\rangle^{-1}\mathscr{V}_1\langle D\rangle^{-1}B(\lambda)\right)^k \langle D\rangle^{-1}\mathscr{V}_1\langle D\rangle^{-1}. \tag{1.3.14}$$

Since $f$ and $g$ are locally in $H^1$, we can use (1.3.14) to obtain

$$\begin{aligned}\left|\left\langle(\mathscr{V}_1 L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho)^k \mathscr{V}_1 f, g\right\rangle\right| &= \left|\left\langle\langle D\rangle^{-1}(\mathscr{V}_1 L_{\mathscr{V}_0}^{\lambda_0}(\lambda)\rho)^k \mathscr{V}_1\langle D\rangle^{-1}\cdot\langle D\rangle\rho f, \langle D\rangle\rho g\right\rangle\right| \\ &\leq C^{k+1}|\langle D\rangle^{-1}\mathscr{V}_1\langle D\rangle^{-1}|^{k+1}|\rho f|_{H^1}|\rho g|_{H^1} \leq C^{k+1}|\mathscr{V}_1|_{\mathscr{H}^{-1}}^{k+1}.\end{aligned}$$

If $|\mathscr{V}_1|_{\mathscr{H}^{-2}} \leq \delta_0^2$ with $\delta_0 \leq 1/(2C)$, then $|\mathscr{V}_1|_{\mathscr{H}^{-1}} \leq 1/(2C)$ and we can sum the above inequality over $k$. This yields the bound (1.3.11), thus part $(i)$.

Part $(ii)$ is an immediate consequence of Lemma 1.3.2 – possibly after reducing the value of $\delta_0$. We now prove part $(iii)$. We first show that if $|\mathscr{V}_1|_{\mathscr{H}^{-1}}$ is sufficiently small, the function $\varphi$ defined by (1.3.12) has a unique zero in $\mathbb{D}(\lambda_0, r_0)$. If $\varphi_0(\lambda) = \lambda - \lambda_0$ and $\delta_0$ is sufficiently small compared to $r_0$,

$$\sup_{\partial\mathbb{D}(\lambda_0} |\varphi - \varphi_0| \leq C\delta_0 < r_0 = \inf_{\partial\mathbb{D}(\lambda_0, r_0)} |\varphi_0|$$

hence Rouché's theorem applies and shows that $\varphi$ and $\varphi_0$ have the same number of zeros in $\mathbb{D}(\lambda_0, r_0)$ – i.e. exactly one, denoted by $\lambda_2$. We now investigate the relation between the resonance $\lambda_1$ of $\mathscr{V}_0 + \mathscr{V}_1$ and the zero $\lambda_2$ of $\varphi$. We will need a relative Lippman-Schwinger formula:

$$R_{\mathscr{V}_0+\mathscr{V}_1}(\lambda) = R_{\mathscr{V}_0}(\lambda)\left(\mathrm{Id} + \mathscr{V}_1 R_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}\left(\mathrm{Id} - \mathscr{V}_1 R_{\mathscr{V}_0}(\lambda)(1-\rho)\right) \tag{1.3.15}$$

To show (1.3.15), we observe that when $\operatorname{Im}\lambda \gg 1$, we can write

$$\mathscr{V}_1 R_{\mathscr{V}_0}(\lambda)\rho = \mathscr{V}_1 R_0(\lambda)(\mathrm{Id} + \mathscr{V}_0 R_0(\lambda))^{-1}\rho. \tag{1.3.16}$$

The operator $-\Delta_{\mathbb{R}^d}$ has absolutely continuous spectrum equal to $[0, \infty)$. Hence the operator $R_0(\lambda)$ satisfies $|R_0(\lambda)|_{\mathscr{B}} \leq |\lambda|^{-1}$ when $\operatorname{Im}\lambda \geq 1$. In particular, for $\operatorname{Im}\lambda \geq 1$, $\mathrm{Id} + \mathscr{V}_0 R_0(\lambda)$ is invertible by a Neumann series and the norm of its inverse is smaller than 2. The bound on $|R_0(\lambda)|_{\mathscr{B}}$ and (1.3.16) imply that for $\operatorname{Im}\lambda$ large enough, $\mathscr{V}_1 R_0(\lambda)\rho$ is bounded on $L^2$ with norm smaller than $1/2$. We deduce that $\mathrm{Id} + \mathscr{V}_1 R_{\mathscr{V}_0}(\lambda)\rho$ is invertible by a Neumann series;

we use this representation of the inverse to verify (1.3.15) when $\operatorname{Im}\lambda$ is large:

$$\begin{aligned}
&R_{\mathscr{V}_0}(\lambda)\left(\mathrm{Id}+\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}\left(\mathrm{Id}-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)(1-\rho)\right)\\
&= R_{\mathscr{V}_0}(\lambda)\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^k\left(\mathrm{Id}-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)(1-\rho)\right)\\
&= R_{\mathscr{V}_0}(\lambda)\left(\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^k-\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^k\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)(1-\rho)\right)\\
&= R_{\mathscr{V}_0}(\lambda)\left(\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^k+\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda))^{k+1}(1-\rho)\right)\\
&= R_{\mathscr{V}_0}(\lambda)\left(\mathrm{Id}+\sum_{k=0}^{\infty}(-\mathscr{V}_1R_{\mathscr{V}_0}(\lambda))^{k+1}\right)=R_{\mathscr{V}_0}(\lambda)(\mathrm{Id}+\mathscr{V}_1R_{\mathscr{V}_0}(\lambda))^{-1}=R_{\mathscr{V}_0+\mathscr{V}_1}(\lambda).
\end{aligned}$$

This identity extends meromorphically for all $\lambda\in\mathbb{C}$. We only need to check that $(\mathrm{Id}+\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^{-1}$ maps $L^2_{\mathrm{comp}}$ to itself: this is immediate for $\operatorname{Im}\lambda\gg1$ thanks to the Neumann series representation; and it extends to all $\lambda$ by the unique continuation principle. This implies (1.3.15).

Assuming that $r_0$ is sufficiently small, $\lambda_0$ is the unique resonance of $\mathscr{V}_0$ on the disk $\mathbb{D}(\lambda_0,r_0)$. The identity (1.3.15) implies that resonances of $\mathscr{V}_0+\mathscr{V}_1$ in the punctured disk $\mathbb{D}(\lambda_0,r_0)\setminus\lambda_0$ are the poles of

$$(\mathrm{Id}+\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^{-1}=\left(\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho+i\frac{\mathscr{V}_1f\otimes g\rho}{\lambda-\lambda_0}\right)^{-1}.$$

When $|\mathscr{V}_1|_{\mathscr{H}^{-2}}$ sufficiently small and $\lambda\in\mathbb{D}(\lambda_0,r_0)\setminus\lambda_0$, the operator $\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho$ is invertible by a Neumann series. Indeed, since $\rho L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho$ maps $L^2$ to $H^2$ and $H^{-2}$ to $L^2$,

$$|(\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho)^2|_{\mathscr{B}}\leq|\mathscr{V}_1|_\infty|\rho L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho|_{H^{-2}\to L^2}|\mathscr{V}_1|_{H^2\to H^{-2}}|\rho L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho|_{L^2\to H^2}\leq C|\mathscr{V}_1|_{\mathscr{H}^{-2}}<1.$$

Therefore, we can write

$$(\mathrm{Id}+\mathscr{V}_1R_{\mathscr{V}_0}(\lambda)\rho)^{-1}=\left(\mathrm{Id}+i\frac{\left(\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}\mathscr{V}_1f\otimes g\rho}{\lambda-\lambda_0}\right)^{-1}\left(\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}.$$

Hence, $\lambda$ is a resonance of $\mathscr{V}_0+\mathscr{V}_1$ in the disk $\mathbb{D}(\lambda_0,r_0)\setminus\lambda_0$ if and only if

$$\mathrm{Id}+i\frac{\left(\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}\mathscr{V}_1f\otimes f\rho}{\lambda-\lambda_0}\text{ is not invertible.}$$

This operator is the sum of the identity with a rank one projector, hence it is not invertible if and only if

$$1+\frac{i}{\lambda-\lambda_0}\mathrm{Tr}\left(\left(\mathrm{Id}+\mathscr{V}_1L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}\mathscr{V}_1f\otimes g\rho\right)=0.$$

Using the Neumann series representation of $\left(\mathrm{Id} + \mathscr{V}_1 L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho\right)^{-1}$, we obtain the characteristic equation

$$\lambda - \lambda_0 + i\sum_{k=0}^{\infty}(-1)^k \left\langle (\mathscr{V}_1 L^{\lambda_0}_{\mathscr{V}_0}(\lambda)\rho)^k \mathscr{V}_1 f, \overline{g}\right\rangle = 0$$

which is exactly the equation $\varphi(\lambda) = 0$ on $\mathbb{D}(\lambda_0, r_0) \setminus \lambda_0$. Thus $\lambda_1 \neq \lambda_0$ implies $\lambda_1 = \lambda_2$. To conclude, we show that we cannot have $\lambda_1 = \lambda_0$ and $\lambda_2 \neq \lambda_0$. Otherwise, we could reverse the above argument – that showed that $\lambda_1$ is a zero of $\varphi$ – to deduce that $\lambda_2$ is a resonance of $\mathscr{V}_0 + \mathscr{V}_1$. But this is a contradiction, because according to $(ii)$ the unique resonance of $\mathscr{V}_0 + \mathscr{V}_1$ on $\mathbb{D}(\lambda_0, r_0)$ is $\lambda_1$, itself equal to $\lambda_0$. □

## 1.4 Probabilistic tools

Until the end of this chapter, we consider given a sequence $\{u_j\}_{j\in\mathbb{Z}^d}$ of independent identically distributed random variables, with

$$\mathbb{E}(u_j) = 0, \quad \mathbb{E}(u_j^2) = 1, \quad u_j \in L^\infty.$$

Unless specified otherwise, all the sums below are realized over indices in $[-N, N]^d$. We fix $q, q_0 \in C_0^\infty(\mathbb{R}^d, \mathbb{C})$ and we define $V = q_0 + V_\#$, where $V_\#$ is the random potential

$$V_\#(x) \overset{\text{def}}{=} \sum_j u_j q(Nx - j) \quad N \gg 1.$$

The potential $V_N$ has support contained in a fixed compact set. Indeed, if $E_j$ denotes the support of $q(N\cdot -j)$, then $E_j$ is contained in a ball of radius $C/N$, centered at $j/N$. It follows that

$$\operatorname{supp}(V_N) \subset \operatorname{supp}(q_0) \cup \bigcup_{j\in[-N,N]^d} E_j \subset \operatorname{supp}(q_0) \cup [-C-1, C+1]^d. \tag{1.4.1}$$

Moreover, $V_N$ is bounded almost surely independently of $N$ or of the value of $\{u_j\}$. Indeed, since $E_j$ of $q(N\cdot -j)$ is contained in a ball of radius $C/N$, centered at $j/N$, any singleton of $\mathbb{R}^d$ intersects with at most $C^d$ sets $E_j$. As the $u_j$ are i.i.d. and bounded almost surely, the estimate

$$|V_N(x)| \leq |q_0|_\infty + |q|_\infty \sum_j |u_j| \mathbb{1}_{E_j} \leq |q_0|_\infty + C^d |q|_\infty |u_j|_\infty < \infty \tag{1.4.2}$$

holds independently of $N$ and of $\{u_j\}$.

In this section, we use the deterministic lemma of §1.3 to compare the resonances of $V_N$ with the resonances of $q_0$. This requires estimates on the $\mathscr{H}^{-s}$-norms of $V_\#$, and precise asymptotic of the leading terms in the expansion (1.3.12) in the limit $N \to \infty$.

### 1.4.1 Hanson–Wright inequality

Let $\alpha = (\alpha_{j\ell})$ be a matrix with complex entries. We denote by $|\alpha|_{\mathrm{HS}}$ its Hilbert–Schmidt norm: $|\alpha|_{\mathrm{HS}}^2 = \sum_{j,\ell} |\alpha_{j\ell}|^2$. We recall here the following lemma, which follows immediately from the Hanson–Wright inequality [HW71]:

**Lemma 1.4.1.** *There exist $c, C > 0$ depending only on the distribution of the $u_j$'s such that the following holds. For any $\alpha = (\alpha_{j\ell})_{j,\ell}$, the expected value of $\sum_{j,\ell} \alpha_{j\ell} u_j u_\ell$ is $\mathrm{Tr}(\alpha)$. In addition, for any $t > 0$ with $t^2 \geq 2|\mathrm{Tr}(\alpha)|$,*

$$\mathbb{P}\left( \left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell \right| \geq t^2 \right) \leq C \exp\left( -ct^2/|\alpha|_{\mathrm{HS}} \right). \tag{1.4.3}$$

*Proof.* For $\ell \neq j$, $\mathbb{E}(u_j u_\ell) = \mathbb{E}(u_j)\mathbb{E}(u_\ell) = 0$, hence

$$\mathbb{E}\left( \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell \right) = \sum_{j,\ell} \alpha_{j\ell} \mathbb{E}(u_j u_\ell) = \sum_j \alpha_{jj} \mathbb{E}(u_j^2) = \mathrm{Tr}(\alpha).$$

This proves the statement about the expected value. To show (1.4.3), we first observe that if $t^2 \geq 2|\mathrm{Tr}(\alpha)|$,

$$\left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell \right| \geq t^2 \Rightarrow \left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell - \mathrm{Tr}(\alpha) \right| \geq t^2 - |\mathrm{Tr}(\alpha)| \geq \frac{t^2}{2}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\left( \left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell \right| \geq t^2 \right) &\leq \mathbb{P}\left( \left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell - \mathrm{Tr}(\alpha) \right| \geq \frac{t^2}{2} \right) \\ &\leq \exp\left( -c \min\left( \frac{t^2}{|\alpha|}, \frac{t^4}{|\alpha|_{\mathrm{HS}}^2} \right) \right). \end{aligned}$$

In the above we applied the Hanson–Wright inequality [HW71] – for the general version needed here and an elementary proof see Rudelson–Vershynin [RV13]. The constant $c$ depends only on the distribution of the $u_j$'s and $|\alpha|$ is the operator norm of $\alpha : \mathbb{C}^{N^d} \to \mathbb{C}^{N^d}$, when $\mathbb{C}^{N^d}$ is provided with its Euclidean norm. If in addition we assume that $t^2 \geq |\alpha|_{\mathrm{HS}}$, we can use $|\alpha| \leq |\alpha|_{\mathrm{HS}}$ and $\frac{t^2}{|\alpha|_{\mathrm{HS}}} \geq 1$ to get

$$\min\left( \frac{t^2}{|\alpha|}, \frac{t^4}{|\alpha|_{\mathrm{HS}}^2} \right) \geq \min\left( \frac{t^2}{|\alpha|_{\mathrm{HS}}}, \frac{t^4}{|\alpha|_{\mathrm{HS}}^2} \right) = \frac{t^2}{|\alpha|_{\mathrm{HS}}}.$$

This implies (1.4.3) in the case $t^2 \geq 2|\mathrm{Tr}(\alpha)|$ and $t^2 \geq |\alpha|_{\mathrm{HS}}$. We remove the assumption $t^2 \geq |\alpha|_{\mathrm{HS}}$ by observing that the opposite case implies $\frac{t^2}{|\alpha|_{\mathrm{HS}}} \leq 1$, which leads to

$$\mathbb{P}\left( \left| \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell \right| \geq t^2 \right) \leq 1 \leq e^c e^{-ct^2/|\alpha|_{\mathrm{HS}}}.$$

It suffices to set $C = e^c$ to end the proof. ∎

### 1.4.2 Central limit theorem

To estimate the linear term that appear in the expansion (1.3.12), we will need the following version of the central limit theorem.

**Lemma 1.4.2.** *Let $\varphi \in C^\infty(\mathbb{R}^d, \mathbb{C})$, not identically vanishing on $[-1,1]^d$ and $\Sigma[\varphi]$ be the $2 \times 2$ matrix defined in (1.1.7). Then,*

$$\frac{1}{N^{d/2}\int_{\mathbb{R}^d} q(x)dx} \sum_{j\in[-N,N]^d} u_j \int_{\mathbb{R}^d} q(x)\varphi\left(\frac{x+j}{N}\right)dx \xrightarrow{\mathrm{d}} \mathscr{N}(0, \Sigma[\varphi]) \tag{1.4.4}$$

*where the LHS of (1.4.4) is seen as a two dimensional vector.*

*Proof.* Write $\varphi = \varphi_1 + i\varphi_2$. We first assume that $\Sigma[\varphi]$ is not degenerate. This is equivalent to $\varphi_1$ and $\varphi_2$ linearly independent over $\mathbb{R}$. Let $\sigma^1_{j,N}, \sigma^2_{j,N}$ be the two real numbers defined by

$$\sigma^1_{j,N} + i\sigma^2_{j,N} = \frac{1}{\int_{\mathbb{R}^d} q(x)dx}\int_{\mathbb{R}^d} q(x)\varphi\left(\frac{x+j}{N}\right)dx.$$

In order to show (1.4.4), it suffices to study the convergence in distribution to a Gaussian of

$$\frac{1}{N^{d/2}}\sum_{j\in[-N,N]^d} u_j(s\sigma^1_{j,N} + t\sigma^2_{j,N}), \quad (s,t) \neq (0,0)$$

because of the Cramér–Wold theorem [Bi95, Theorem 29.4]. We apply the central limit theorem in its version due to Lyapounov, see [Bi95, Theorem 27.3]. We remark that Lyapounov's condition:

$$\limsup_{N\to\infty} \frac{1}{N^{3d/2}} \sum_{j\in[-N,N]^d} (s\sigma^1_{j,N} + t\sigma^2_{j,N})^3 = 0$$

is immediately satisfied because $s\sigma^1_{j,N} + t\sigma^2_{j,N} = O(1)$. Hence, we deduce that

$$\frac{1}{N^{d/2}}\sum_{j\in[-N,N]^d} u_j(s\sigma^1_{j,N} + t\sigma^2_{j,N}) \to \mathscr{N}(0, \sigma(s,t)^2),$$

$$\sigma(s,t)^2 = \lim_{N\to\infty}\frac{1}{N^d}\sum_{j\in[-N,N]^d}(s\sigma^1_{j,N} + t\sigma^2_{j,N})^2$$

and it remains to compute $\sigma(s,t)^2$ and check that it is not vanishing. Since $\varphi$ is smooth and $q$ has compact support, a Taylor expansion and a Riemann series argument shows

$$\sigma^k_{j,N} = \varphi_k\left(\frac{j}{N}\right) + O(N^{-1}), \quad \sigma(s,t)^2 = \int_{[-1,1]^d} (s\varphi_1 + t\varphi_2)^2(x)dx.$$

Since $\varphi_1$ and $\varphi_2$ are linearly independent, $\sigma(s,t) \neq 0$ as long as $(s,t) \neq (0,0)$. One recognize the distribution of $sX + tY$, where $(X, Y)$ is a Gaussian vector with mean 0 and covariance $\Sigma[\varphi]$. This proves the lemma when $\Sigma[\varphi]$ is non degenerate.

We now prove the lemma when $\Sigma[\varphi]$ is degenerate. The determinant of $\Sigma[\varphi]$ vanishes; this yields the case of equality in the Cauchy–Schwarz inequality. Hence, we can assume $\varphi_1 = \alpha\varphi_2$ for some $\alpha \in \mathbb{R} \setminus 0$ – the case $\varphi_2 = \alpha\varphi_1$ is treated similarly. We now study the convergence of

$$\frac{1}{N^{d/2}(1+i\alpha)\int_{\mathbb{R}^d} q(x)dx} \sum_{j\in[-N,N]^d} u_j \int_{\mathbb{R}^d} q(x)\varphi\left(\frac{x+j}{N}\right) dx$$
$$= \frac{1}{N^{d/2}\int_{\mathbb{R}^d} q(x)dx} \sum_{j\in[-N,N]^d} u_j \int_{\mathbb{R}^d} q(x)\varphi_1\left(\frac{x+j}{N}\right) dx.$$

Again, we let $\tilde{\sigma}^1_{j,N}, \tilde{\sigma}^2_{j,N}$ be the real numbers such that

$$\tilde{\sigma}^1_{j,N} + i\tilde{\sigma}^2_{j,N} = \frac{1}{\int_{\mathbb{R}^d} q(x)dx} \int_{\mathbb{R}^d} q(x)\varphi_1\left(\frac{x+j}{N}\right) dx.$$

We now apply Lyapounov's central limit theorem to study the convergence in distribution of

$$\frac{1}{N^{d/2}} \sum_{j\in[-N,N]^d} u_j(s\tilde{\sigma}^1_{j,N} + t\tilde{\sigma}^2_{j,N}).$$

As in the case $\Sigma[\varphi]$ non-degenerate, we first check Lyapounov's condition – which is obviously verified because $\tilde{\sigma}^k_{j,N} = O(1)$. In fact, we even have

$$\tilde{\sigma}^1_{j,N} = \varphi_1\left(\frac{j}{N}\right) + O(N^{-1}), \quad \tilde{\sigma}^2_{j,N} = O(N^{-1}),$$

and as previously, an evaluation by a Riemann sum yields

$$\frac{1}{N^{d/2}} \sum_{j\in[-N,N]^d} u_j(s\tilde{\sigma}^1_{j,N} + t\tilde{\sigma}^2_{j,N}) \to \mathscr{N}(0, \tilde{\sigma}(s,t)^2),$$
$$\tilde{\sigma}(s,t)^2 = \limsup_{N\to\infty} \frac{1}{N^d} \sum_{j\in[-N,N]^d} (s\tilde{\sigma}^1_{j,N} + t\tilde{\sigma}^2_{j,N})^2 = s^2 \int_{[-1,1]^d} \varphi_1(x)^2 dx.$$

If $(X, Y)$ are independent random variables with distributions $\mathscr{N}\left(0, \int_{[-1,1]^d} \varphi_1(x)^2 dx\right)$ and $\delta_0$, respectively, the random variable $sX+tY$ has distribution $\mathscr{N}(0, \tilde{\sigma}(s,t)^2)$. An application of the Cramér–Wold theorem [Bi95, Theorem 29.4] concludes the proof. □

## 1.5 Large N estimates on expansion terms

### 1.5.1 Bounds on $\mathscr{H}^{-s}$-norms

Recall that $m$ is the order of vanishing of $\hat{q}(\xi)$ at $\xi = 0$ and that $\gamma = \min(7/4, d/2 + m)$. The lemma below states that with high probability, $V_\#$ is highly oscillatory – specifically, it is small when measured in $\mathscr{H}^{-2}$. This will allow us to apply the deterministic Lemma 1.3.2.

**Lemma 1.5.1.** *There exist $C_0, c_0 > 0$ such that for any $N$,*

$$\mathbb{P}(|V_\#|_{\mathscr{H}^{-2}} \geq N^{-\gamma/2}) \leq C_0 e^{-c_0 N^\gamma}$$

*Proof.* We start first with $d \leq 3$. In this case $2 > d/2$, therefore Lemma 1.3.1 implies that $|V_\#|_{\mathscr{H}^{-2}} \leq C|V_\#|_{H^{-2}}$. Hence

$$\mathbb{P}(|V_\#|_{\mathscr{H}^{-2}} \geq N^{-\gamma/2}) \leq \mathbb{P}(|V_\#|^2_{H^{-2}} \geq C^{-2}N^{-\gamma}).$$

The advantage of the $H^2$-norm over the $\mathscr{H}^{-2}$ is its bilinear character. This will allow us to apply the Hanson–Wright inequality in the version of Lemma 1.4.1. We observe that

$$|V_\#|^2_{H^{-2}} = \int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^2} \left| \int_{\mathbb{R}^d} e^{-ix\xi} \sum_j u_j q(Nx - j) dx \right|^2 d\xi$$

$$= \sum_{j,\ell} u_j u_\ell \int_{\mathbb{R}^d} \frac{1}{(1+|\xi|^2)^2} \int_{\mathbb{R}^d} e^{-ix\xi} q(Nx-j)dx \int_{\mathbb{R}^d} e^{-ix\xi} q(Ny-j)dyd\xi$$

Substitutions $x \mapsto \frac{x+j}{N}$, $y \mapsto \frac{y+\ell}{N}$, $\xi \mapsto N\xi$ yield

$$|V_\#|^2_{H^{-2}} = \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell, \quad \alpha_{j\ell} \overset{\text{def}}{=} \frac{1}{N^d} \int_{\mathbb{R}^d} \frac{e^{i\xi(j-\ell)}|\hat{q}(\xi)|^2}{(1+N^2|\xi|^2)^2} d\xi. \tag{1.5.1}$$

We note that $\mathbb{E}(u_j) = 0$, and we recall that $\mathbb{E}(u_j^2) = 1$. Hence, $\mathrm{Tr}(\alpha) = N^d \alpha_{00}$. If $\hat{q}$ vanishes at order $m$, we obtain

$$\mathrm{Tr}(\alpha) = \int_{\mathbb{R}^d} \frac{|\hat{q}(\xi)|^2}{(1+N^2|\xi|^2)^2} d\xi = \int_{r=0}^\infty \int_{\mathbb{S}^{d-1}} \frac{|\hat{q}(r\theta)|^2 r^{d-1}}{(1+N^2r^2)^2} dr d\sigma(\theta)$$

$$\leq C \int_{r=0}^1 \frac{r^{2m+d-1}dr}{(1+N^2r^2)^2} + O(N^{-4}) = O(N^{-2m-d}) \int_{r=0}^N \frac{r^{2m+d-1}dr}{(1+r^2)^2} + O(N^{-4}) = O(N^{-2\gamma}).$$

In particular, $\mathrm{Tr}(\alpha) = O(N^{-2\gamma})$. Since $|\alpha_{j\ell}| \leq |\alpha_{00}|$, the same computation shows $|\alpha|^2_{\mathrm{HS}} \leq N^{2d}|\alpha_{00}|^2 = O(N^{-4\gamma})$. Lemma 1.4.1 implies that for $N$ large enough,

$$\mathbb{P}\left(|V_\#|^2_{H^{-2}} \geq t^2\right) \leq Ce^{-cN^\gamma}. \tag{1.5.2}$$

We remove the assumption on $N$ by increasing the value of $C$ in (1.5.2).

We now deal with $d \geq 5$. In this case, $\gamma = 7/4$. Fix $s > d/2$, and apply Lemma 1.3.1: $|V_\#|_{\mathscr{H}^{-2}} \leq C|V_\#|_{H^{-s}}^{2/s}$. Therefore,

$$\mathbb{P}\left(|V_\#|_{\mathscr{H}^{-2}} \geq N^{-\gamma/2}\right) \leq \mathbb{P}\left(|V_\#|_{H^{-s}}^{2/s} \geq C^{-1}N^{-\gamma/2}\right) = \mathbb{P}\left(|V_\#|_{H^{-s}}^2 \geq C^{-s}N^{-s\gamma/2}\right). \quad (1.5.3)$$

We now compute $|V_\#|_{H^{-s}}^2$: as in (1.5.1),

$$|V_\#|_{H^{-s}}^2 = \sum \alpha_{j\ell} u_j u_\ell, \quad \alpha_{j\ell} \overset{\text{def}}{=} \frac{1}{N^d} \int_{\mathbb{R}^d} \frac{e^{i\xi(j-\ell)}|\hat{q}(\xi)|^2}{(1+N^2|\xi|^2)^s} d\xi.$$

We observe that $\text{Tr}(\alpha) = N^d\alpha_{00}$, and

$$\begin{aligned} N^d\alpha_{00} &= \int_{\mathbb{R}^d} \frac{|\hat{q}(\xi)|^2}{(1+N^2|\xi|^2)^s} d\xi = \int_{r=0}^{\infty} \int_{\mathbb{S}^{d-1}} \frac{|\hat{q}(r\theta)|^2 r^{d-1} dr d\sigma(\theta)}{(1+N^2r^2)^s} \\ &\leq C \int_{r=0}^{1} \frac{r^{d-1}dr}{(1+N^2r^2)^s} + O(N^{-2s}) \sim N^{-d} + N^{-2s}. \end{aligned}$$

Since $s > d/2$, we obtain $\text{Tr}(\alpha) = O(N^{-d})$. Similarly, $|\alpha|_{\text{HS}}^2 = O(N^{-2d})$. Lemma 1.4.1 shows that for any $t > 0$ with $t^s \geq O(N^{-d})$,

$$\mathbb{P}\left(|V_\#|_{H^{-s}}^2 \geq C^{-s}t^s\right) \leq Ce^{-ct^sN^d}. \quad (1.5.4)$$

Fix now $s = d/\gamma > d/2$. We have $t^sN^d = (t^2N^{2\gamma})^{d/(2\gamma)}$. If we take $t = N^{-\gamma/2}$, then (1.5.4) implies for $N$ sufficiently large

$$\mathbb{P}\left(|V_\#|_{H^{-s}}^2 \geq C^{-s}N^{-s\gamma/2}\right) \leq Ce^{-cN^\gamma}.$$

Again, we can get rid of the assumption on $N$ by increasing the value of $C$. The conclusion follows now from (1.5.3). □

Because of (1.3.11), we also need to estimate $|V_\#|_{\mathscr{H}^{-1}}$:

**Lemma 1.5.2.** *There exist $C, c > 0$ such that with probability $1 - Ce^{-cN^{1/4}}$,*

$$\begin{aligned} |V_\#|_{\mathscr{H}^{-1}} &\leq N^{-3/8} \quad \text{if } d = 1 \text{ and } \int_{\mathbb{R}} q(x)dx \neq 0, \\ |V_\#|_{\mathscr{H}^{-1}} &\leq N^{-7/8} \quad \text{if } d \geq 3 \text{ or } d = 1 \text{ and } \int_{\mathbb{R}} q(x)dx = 0. \end{aligned} \quad (1.5.5)$$

*Proof.* The proof is similar to that of Lemma 1.5.1. We start with the case $d = 1$. In this case, $|V_\#|_{\mathscr{H}^{-1}} \leq C|V_\#|_{H^{-1}}$. As in (1.5.1),

$$|V_\#|_{\mathscr{H}^{-1}}^2 = \sum_{j,\ell} \alpha_{j\ell} u_j u_\ell, \qquad \alpha_{j\ell} \overset{\text{def}}{=} \frac{1}{N} \int_{\mathbb{R}} \frac{e^{i\xi(j-\ell)}|\hat{q}(\xi)|^2}{1+N^2|\xi|^2} d\xi.$$

We observe that $\mathrm{Tr}(\alpha) = O(N^{-1})$. Similarly, $|\alpha|_{\mathrm{HS}}^2 = O(N^{-2})$. The Hanson–Wright inequality implies

$$\mathbb{P}(|V_\#|_{\mathscr{H}^{-1}} \geq N^{-3/8}) = O(e^{-cN^{1/4}}).$$

If in addition $\int_{\mathbb{R}} q(x)dx = 0$ then we can split the integral defining $\alpha_{00}$ in low and high frequency parts – as in the proof of Lemma 1.5.1 – and obtain $\alpha_{00} = O(N^{-3})$, $\mathrm{Tr}(\alpha) = O(N^{-2})$ and $|\alpha|_{\mathrm{HS}} = O(N^{-4})$. Hence,

$$d = 1, \int_{\mathbb{R}} q(x)dx = 0 \ \Rightarrow \ \mathbb{P}(|V_\#|_{\mathscr{H}^{-1}} \geq N^{-7/8}) = O(e^{-cN^{1/4}}).$$

We continue the proof for $d \geq 3$. Because of Lemma 1.3.1, for $s > d/2$, $|V_\#|_{\mathscr{H}^{-1}} \leq C|V_\#|_{H^{-s}}^{1/s}$. Apply (1.5.4) (which is also valid for any $d$) to $t = N^{-8/5}$ obtain

$$\mathbb{P}(|V_\#|_{H^{-s}} \geq N^{-7/8}) = O(e^{-cN^{d-7s/4}}).$$

Since $d \geq 3$, there exists $s > d/2$ such that $d - 7s/4 = 1/4$. The lemma follows. □

### 1.5.2 Resonances as zeroes of a random holomorphic function

Thanks to (1.3.11) and of §1.5.1, we can write (with high probability) resonances of $V_N$ near simple resonances of $q_0$ as the zeroes of a random holomorphic function that takes the form $\sum_{k=0}^\infty a_k(V_\#, \lambda)$. In this section we estimate $a_1(V_\#, \lambda)$ and $a_2(V_\#, \lambda)$. The remaining terms $a_k(V_\#, \lambda)$, $k \geq 3$, will be seen to be negligible compared to $a_1(V_\#, \lambda) + a_2(V_\#, \lambda)$ – thanks to (1.5.5). Below, $\lambda_0$ denotes a simple resonance of $q_0$: there exist resonant/coresonant states $f, g \in C^\infty(\mathbb{R}^d)$ and $L_{q_0}^{\lambda_0}(\lambda)$ a family of operators that is holomorphic near $\lambda_0$, such that

$$R_{q_0}(\lambda) = L_{q_0}^{\lambda_0}(\lambda) + i\frac{f \otimes g}{\lambda - \lambda_0}.$$

An efficient application of Lemma 1.3.3 requires to estimate the first two terms which appear in (1.3.12): $\langle V_\# f, \overline{g}\rangle$ and $\langle V_\# L_{q_0}^{\lambda_0}(\lambda) V_\# f, \overline{g}\rangle$.

**Lemma 1.5.3.** *If $\int_{\mathbb{R}^d} q(x)dx \neq 0$,*

$$\begin{gathered} \frac{N^{d/2}}{\int_{\mathbb{R}^d} q(x)dx}\langle V_\# f, \overline{g}\rangle \xrightarrow{\mathrm{d}} \mathscr{N}(0, \Sigma[fg]), \ \text{as } N \to \infty, \\ \mathbb{P}(N^{d/2}|\langle V_\# f, g\rangle| \geq N^{1/4}) = O(e^{-cN^{1/2}}). \end{gathered} \tag{1.5.6}$$

*Proof.* We have:

$$\langle V_\# f, \overline{g}\rangle = \sum_j u_j \int_{\mathbb{R}^d} q(Nx - j)f(x)g(x)dx = \frac{1}{N^d}\sum_j u_j \int_{\mathbb{R}^d} q(x)(fg)\left(\frac{x+j}{N}\right)dx. \tag{1.5.7}$$

If $fg$ is not identically vanishing on $[-1,1]^d$, then Lemma 1.4.2 applies and yields (1.5.6). It remains to show the non-vanishing condition. If $fg$ vanishes on $[-1,1]^d$, either $f$ or $g$ vanishes on an open set $\Omega \subset [-1,1]^d$. Assume that $f$ vanishes on $\Omega$ and define $E = \mathbb{R}^d \setminus \operatorname{supp}(f)$. This is an open subset of $\mathbb{R}^d$ containing $\Omega$ and we show that it is also closed using the unique continuation principle. Since $f$ is a resonant state,

$$-\Delta f = (-q_0 + \lambda^2) f.$$

Therefore, for any $x_0 \in \operatorname{adh}(E)$ and any $x \in B(x_0, 1)$,

$$|\Delta f(x)| = |(-q_0(x) + \lambda^2) f(x) - (-q_0(x_0) + \lambda^2) f(x_0)| \leq (|q_0|_\infty + |\lambda|^2) \sup_{B(x_0,1)} |\nabla f|.$$

In addition, since $f$ is smooth and vanishes at infinite order at $x_0$, $f(x) = O(|x - x_0|^N)$ for any $N$. [Hö07, Theorem 17.2.6] applies and shows that $f$ vanishes on $B(x_0, 1)$. Hence $E$ is closed and $f \equiv 0$, which is not possible. If now $g$ vanishes on $\Omega$, then the same argument using that $g$ is a coresonant state:

$$-\Delta g = (-\overline{q_0} + \overline{\lambda}^2) g$$

implies $g \equiv 0$, which is not possible either. This shows the convergence in distribution of $\langle V_\# f, g\rangle$.

To show the large deviation estimate, we first write

$$|\langle V_\# f, \overline{g}\rangle|^2 = \sum_{j,\ell} u_j u_\ell \alpha_j \overline{\alpha_\ell}, \quad \alpha_j \overset{\text{def}}{=} \frac{1}{N^d} \int_{\mathbb{R}^d} q(x)(fg)\left(\frac{x+j}{N}\right) dx. \tag{1.5.8}$$

Since $f$ and $g$ are bounded, $\alpha_j = O(N^{-d})$, $\sum_j |\alpha_j|^2 = O(N^{-d})$, $|\alpha_j \alpha_\ell|_{\text{HS}}^2 = O(N^{-2d})$. We can then apply the Hanson–Wright inequality to obtain $\mathbb{P}(|\langle V_\# f, g\rangle| \geq N^{1/4-d/2}) = O(e^{-cN^{1/2}})$, as claimed. ☐

**Lemma 1.5.4.** *Assume that $d = 1$ and $\int_{\mathbb{R}} q(x)dx = \int_{\mathbb{R}} xq(x)dx = 0$, or that $d = 3$ and $\int_{\mathbb{R}^3} q(x)dx = 0$. Then,*

$$\mathbb{P}(|\langle V_\# f, \overline{g}\rangle| \geq N^{-9/4}) = O(e^{-cN^{1/2}}).$$

*Proof.* If $d = 1$ and $\int_{\mathbb{R}} q(x)dx = \int_{\mathbb{R}} xq(x)dx = 0$, we can find $Q \in C_0^\infty(\mathbb{R}, \mathbb{C})$ such that $Q'' = q$. We use the bilinear expression (1.5.8) for $|\langle V_\# f, g\rangle|^2$. Thanks to a double integration by parts, we see that $\alpha_j = O(N^{-3})$:

$$\alpha_j = \frac{1}{N} \int_{\mathbb{R}} Q''(x)(fg)\left(\frac{x+j}{N}\right) dx = \frac{1}{N^3} \int_{\mathbb{R}} Q(x)(fg)''\left(\frac{x+j}{N}\right) dx.$$

It follows that $\sum_j |\alpha_j|^2 = O(N^{-5})$ and $|\alpha_j \alpha_\ell|_{\text{HS}} = O(N^{-5})$. Hence, the Hanson–Wright inequality yields $\mathbb{P}(|\langle V_\# f, \overline{g}\rangle| \geq N^{-9/4}) \leq Ce^{-cN^{1/2}}$.

If $d = 3$ and $\int_{\mathbb{R}} q(x)dx = 0$, we use again (1.5.8). Since $f$ and $g$ are smooth,

$$(fg)\left(\frac{x+j}{N}\right) = (fg)\left(\frac{j}{N}\right) + O(N^{-1}) \tag{1.5.9}$$

uniformly for $j \in [-N, N]^3$ and $x \in \operatorname{supp}(q)$. Using that $\int_{\mathbb{R}^3} q(x)dx = 0$, we deduce

$$\alpha_j = \frac{1}{N^3}\int_{\mathbb{R}^3} q(x)(fg)\left(\frac{x+j}{N}\right)dx = O(N^{-4}).$$

In particular, $|\alpha_j\alpha_\ell|^2_{\mathrm{HS}} = |\alpha_j|^4_{\ell^2}$. Using that $\int_{\mathbb{R}^3} q(x)dx = 0$ and (1.5.9), we see that $\alpha_j = O(N^{-4})$. Therefore, $\sum_j |\alpha_j|^2 = O(N^{-5})$ and $|\alpha_j\alpha_\ell|^2_{\mathrm{HS}} = O(N^{-10})$ and we conclude as above. □

**Lemma 1.5.5.** *Assume that $d = 1$, $\int_{\mathbb{R}} q(x)dx = 0$ and $\int_{\mathbb{R}} xq(x)dx \neq 0$. As $N \to +\infty$,*

$$\begin{gathered}(fg)' \not\equiv 0 \text{ on } [-1,1] \quad \Rightarrow \quad \frac{N^{3/2}}{\int_{\mathbb{R}} xq(x)dx}\langle V_\# f, \overline{g}\rangle \xrightarrow{\mathrm{d}} \mathscr{N}(0, \Sigma[(fg)']), \\ (fg)' \equiv 0 \text{ on } [-1,1] \quad \Rightarrow \quad \langle V_\# f, \overline{g}\rangle = O(N^{-3}).\end{gathered} \tag{1.5.10}$$

**Remark 1.5.1.** *In practice, we can have $(fg)' \equiv 0$ on $[-1, 1]$: for instance if $q_0 = 0$ and $\lambda_0 = 0$, then $f$ and $g$ are constant functions – see the discussion following Corollary 5.*

*Proof.* As in (1.5.7),

$$\langle V_\# f, \overline{g}\rangle = \frac{1}{N}\sum_j u_j \int_{\mathbb{R}} q(x)(fg)\left(\frac{x+j}{N}\right)dx.$$

Since $\int_{\mathbb{R}^d} q(x)dx = 0$, there exists a unique $Q \in C_0^\infty(\mathbb{R})$ such that $Q' = q$. Integrating by parts in the above yields

$$\langle V_\# f, \overline{g}\rangle = -\frac{1}{N^2}\sum_j u_j \int_{\mathbb{R}} Q(x)(fg)'\left(\frac{x+j}{N}\right)dx.$$

If $(fg)'$ is not identically vanishing on $[-1, 1]$, then the first implication of (1.5.10) follows from $\int_{\mathbb{R}} xq(x)dx = \int_{\mathbb{R}} Q(x)dx$ and Lemma 1.4.2.

If now $(fg)'$ vanishes on $[-1, 1]$ then

$$\langle V_\# f, \overline{g}\rangle = -\frac{1}{N^2}\sum_j u_j \int_{|x+j|\geq N} Q(x)(fg)'\left(\frac{x+j}{N}\right)dx. \tag{1.5.11}$$

Let $M > 0$ such that $\operatorname{supp}(Q) \subset [-M, M]$. The indices $j$ such that $Q$ does not vanish identically on the set $\{|x + j| \geq N\}$ must satisfy $|x + j| \geq N$ for some $|x| \leq M$, in particular

$|j| \geq N - M$. This happens for at most $2M$ values of the $j$'s, that must remain at fixed distance from $\pm N$. For such $j$'s,

$$(fg)'\left(\frac{x+j}{N}\right) = (fg)'\left(\frac{x+j \mp N}{N} \pm 1\right) = (fg)'(\pm 1) + O(N^{-1}) = O(N^{-1}),$$

uniformly for $x \in \text{supp}(Q)$. Hence, the sum (1.5.11) is realized over only finitely many $j$, and each term is of order $O(N^{-1})$. This leads to $\langle V_\# f, \overline{g}\rangle = O(N^{-3})$. □

**Lemma 1.5.6.** *Assume that $d \geq 3$ or that $d = 1$ and $\int_{\mathbb{R}} q(x)dx = 0$. Define*

$$L \overset{\text{def}}{=} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\hat{q}(\xi)\hat{q}(-\xi)}{|\xi|^2} d\xi \cdot \int_{[-1,1]^d} f(x)g(x)dx \tag{1.5.12}$$

*Then, there exists $r_0 > 0$ such that for any $\lambda \in \mathbb{D}(\lambda_0, r_0)$,*

$$\mathbb{P}\left(\left|N^2\langle L_{q_0}^{\lambda_0}(\lambda)V_\# f, \overline{V_\# g}\rangle - L\right| \geq 2t\right) \leq C\exp\left(-\frac{ctN^{1/2}}{\ln(N)}\right).$$

*Proof.* In the steps 1 to 7 below, we assume that $\lambda_0 \neq 0$ if $d = 1$. In the step 8, we deal with the case $\lambda = 0$ and $d = 1$ – which requires a special (though simpler) treatment.

**Step 1.** Fix $r_0 > 0$ such that $q_0$ has no resonance on $\mathbb{D}(\lambda_0, r_0) \setminus \lambda_0$. In order to estimate $\langle L_{q_0}^{\lambda_0}(\lambda)V_\# f, \overline{V_\# g}\rangle$, we first use (1.3.13):

$$L_{q_0}^{\lambda_0}(\lambda) = \frac{1}{2\pi i}\oint_{\lambda_0} \frac{R_{q_0}(\mu)}{\mu - \lambda} d\mu$$

(the contour integral is realized over $\partial\mathbb{D}(\lambda_0, r_0)$). We combine the identity

$$R_{q_0}(\mu) = R_0(\mu)\left(\text{Id} + q_0 R_0(\mu)\rho\right)^{-1}\left(\text{Id} - q_0 R_0(\mu)(1-\rho)\right)$$

with the fact $(1-\rho)V_\# = 0$ to obtain $R_{q_0}(\mu)V_\# = R_0(\mu)V_\# + A(\mu)$, where

$$A(\mu) = -R_0(\mu)\left(\text{Id} + q_0 R_0(\mu)\rho\right)^{-1} q_0 R_0(\mu)V_\#.$$

It follows that

$$\begin{aligned}\langle L_{q_0}^{\lambda_0}(\lambda)V_\# f, \overline{V_\# g}\rangle &= \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\langle R_0(\mu)V_\# f, \overline{V_\# g}\rangle}{\mu - \lambda} d\mu - \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\langle A(\mu)f, \overline{V_\# g}\rangle}{\mu - \lambda} d\mu \\ &= \langle R_0(\lambda)V_\# f, \overline{V_\# g}\rangle - \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\langle A(\mu)f, \overline{V_\# g}\rangle}{\mu - \lambda} d\mu.\end{aligned} \tag{1.5.13}$$

**Step 2.** Since resonances of $q_0$ form a discrete set, $q_0$ has no resonances on a sufficiently small punctured neighborhood $U$ of $\lambda$. Therefore, the operator $\text{Id} + q_0 R_0(\mu)\rho$ is invertible on

$U$. Its inverses form an analytic family of operators on $L^2$, hence by the Banach–Steinhauss theorem their operator norms are uniformly bounded on compact subsets of $U$. In addition,

$$(\mathrm{Id} + q_0 R_0(\mu)\rho)^{-1} q_0 = \rho(\mathrm{Id} + q_0 R_0(\mu)\rho)^{-1} q_0.$$

This can be established by first expanding $(\mathrm{Id} + q_0 R_0(\mu)\rho)^{-1}$ as a Neumann series for $\mathrm{Im}\,\mu \gg 1$ (with the same argument as needed for (1.2.9)), then by meromorphic continuation and unique continuation principle for $\mu \in \mathbb{C} \setminus \mathrm{Res}(q_0)$. Therefore,

$$\left| \frac{1}{2\pi i} \oint_{\lambda_0} \frac{\langle A(\mu) f, \overline{V_\# g} \rangle}{\mu - \lambda} d\mu \right| \leq C \sup_{\mu \in \partial \mathbb{D}(\lambda, r)} |\rho R_0(\mu) V_\# f| |\rho R_0(\mu)^* V_\# g|. \tag{1.5.14}$$

Since $R_0(\mu)$ and its adjoint $R_0(\mu)^* = R_0(-\overline{\mu})$ map $H^{-2}$ to $L^2$, since $f$ and $g$ are smooth functions, the right hand side of (1.5.14) is controlled by $C|V_\#|^2_{\mathscr{H}^{-2}}$. By Lemma 1.5.1,

$$\mathbb{P}\left( N^2 \left| \frac{1}{2\pi i} \oint_{\lambda_0} \frac{\langle A(\mu) f, \overline{V_\# g} \rangle}{\mu - \lambda} d\mu \right| \geq t \right) \leq \mathbb{P}(C N^2 |V_\#|^2_{\mathscr{H}^{-2}} \geq t) \leq C e^{-ctN^{2\gamma - 1}} \leq C e^{-ctN}.$$

We used that the under the assumptions of Lemma 1.5.6, $\gamma \geq 3/2$. This estimate and (1.5.13) imply

$$\mathbb{P}\left( \left| N^2 \langle L_{q_0}^{\lambda_0}(\lambda) V_\# f, \overline{V_\# g} \rangle - L \right| \geq 2t \right) \leq \mathbb{P}\left( \left| N^2 \langle R_0(\lambda) V_\# f, \overline{V_\# g} \rangle - L \right| \geq t \right) + O(e^{-ctN}).$$

Hence, with high probability and for $\lambda \in \mathbb{D}(\lambda_0, r_0)$, the terms $N^2 \langle L_{q_0}^{\lambda_0}(\lambda) V_\# f, \overline{V_\# g} \rangle$ and $\langle R_0(\lambda) V_\# f, \overline{V_\# g} \rangle$ are comparable. In contrast with $N^2 \langle L_{q_0}^{\lambda_0}(\lambda) V_\# f, \overline{V_\# g} \rangle$, the function $\lambda \mapsto N^2 \langle R_0(\lambda) V_\# f, \overline{V_\# g} \rangle$ is meromorphic on the domain of holomorphy $X_d$ of $R_0(\lambda)$:

$$X_d = \mathbb{C} \text{ if } d \geq 3, \quad X_d = \mathbb{C} \setminus 0 \text{ if } d = 1.$$

We write

$$\langle R_0(\lambda) V_\# f, \overline{V_\# g} \rangle = \sum_{j,\ell} u_j u_\ell \alpha_{j\ell}(\lambda), \qquad \alpha_{j\ell}(\lambda) \overset{\text{def}}{=} \langle R_0(\lambda) q(N \cdot - j) f, \overline{q(N \cdot - \ell) g} \rangle.$$

In the steps 3, 4 and 5 below, we estimate the terms $\alpha_{j\ell}(\lambda)$ for $\mathrm{Im}\,\lambda \geq 1$, $\mathrm{Im}\,\lambda \leq -1$ and $|\mathrm{Im}\,\lambda| \leq 1$, respectively.

**Step 3.** We assume that $\mathrm{Im}\,\lambda \geq 1$. In this case, the operator $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$ is a Fourier multiplier with symbol $(|\xi|^2 - \lambda^2)^{-1}$. Using the Plancherel's identity and the substitutions $x \mapsto (x + j)/N$, $y \mapsto (y + \ell)/N$ and $\xi \mapsto N\xi$, we obtain

$$\begin{gathered} \alpha_{j\ell}(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{|\xi|^2 - \lambda^2} \int_{\mathbb{R}^d} e^{-i\xi x} q(Nx - j) f(x) dx \int_{\mathbb{R}^d} e^{i\xi y} q(Ny - \ell) g(y) dy d\xi \\ = \frac{1}{(2\pi N)^d} \int_{\mathbb{R}^d} \frac{e^{i\xi(\ell - j)} \zeta_{j\ell}(\xi)}{N^2 |\xi|^2 - \lambda^2} d\xi, \\ \zeta_{j\ell}(\xi) \overset{\text{def}}{=} \int_{\mathbb{R}^d} e^{-i\xi x} q(x) f\left( \frac{x + j}{N} \right) dx \int_{\mathbb{R}^d} e^{i\xi y} q(y) g\left( \frac{y + \ell}{N} \right) dy. \end{gathered}$$

The function $\zeta_{j\ell}$ is the product of Fourier transforms of functions whose derivatives are all bounded uniformly in $x, j, \ell, N$. It follows that

$$\zeta_{j\ell}(\xi) = O(\langle \xi \rangle^{-\infty}), \qquad \partial_\xi \zeta_{j\ell}(\xi) = O(\langle \xi \rangle^{-\infty}). \tag{1.5.15}$$

If in addition $d = 1$, then $\int_{\mathbb{R}} q(x)dx = \hat{q}(0) = 0$ according to the assumptions of the statement. Therefore, we can use the expansion (1.5.9) to obtain

$$d = 1 \ \Rightarrow \ \int_{\mathbb{R}} e^{-ix\xi} q(x) u\left(\frac{x+j}{N}\right) dx = u\left(\frac{j}{N}\right) \hat{q}(\xi) + O(N^{-1}) = O(|\xi| + N^{-1}),$$

uniformly for $\xi$ in bounded sets, $N \geq 1$ and $j \in [-N, N]$. Using that a similar estimate is available when $u$ is replaced by $v$ and $j$ is replaced by $\ell$, we get

$$d = 1 \ \Rightarrow \ |\zeta_{j\ell}(\xi)| = O(|\xi|^2 + N^{-2}) \qquad |\partial_\xi \zeta_{j\ell}(\xi)| = O(|\xi| + N^{-1}). \tag{1.5.16}$$

We now estimate $\alpha_{jj}(\lambda)$. When $d \geq 3$, we split the integral over $\xi$ near 0 and $\xi$ away from 0 and we use the bounds (1.5.15) on $\zeta_{jj}(\xi)$:

$$\begin{aligned}
|\alpha_{jj}(\lambda)| &\leq \frac{1}{(2\pi N)^d} \int_{\mathbb{R}^d} \frac{|\zeta_{jj}(\xi)|}{|N^2|\xi|^2 - \lambda^2|} d\xi \\
&\leq \frac{C}{N^d} \int_{|\xi| \leq 1} \frac{|\zeta_{jj}(\xi)|}{|N^2|\xi|^2 - \lambda^2|} d\xi + \frac{1}{N^d} \int_{|\xi| \geq 1} \frac{\langle \xi \rangle^{-2d}}{|N^2|\xi|^2 - \lambda^2|} d\xi \\
&\leq \frac{C}{N^d} \int_{r=0}^{1} \frac{\sup_{\omega \in \mathbb{S}^{d-1}} |\zeta_{jj}(r\omega)|}{|N^2 r^2 - \lambda^2|} r^{d-1} dr + \frac{C}{N^d} \int_{r=1}^{\infty} \frac{\langle r \rangle^{-d-1}}{|N^2 r^2 - \lambda^2|} dr \\
&\leq \frac{C}{N^{2d}} \int_{r=0}^{N} \frac{\sup_{\omega \in \mathbb{S}^{d-1}} |\zeta_{jj}(r\omega/N)|}{|r^2 - \lambda^2|} r^{d-1} dr + \frac{C}{N^d} \int_{r=1}^{\infty} \frac{\langle r \rangle^{-d-1}}{|N^2 r^2 - \lambda^2|} dr \\
&\leq \frac{C\langle \lambda \rangle}{N^{2d}} \int_{r=0}^{N} \frac{\sup_{\omega \in \mathbb{S}^{d-1}} |\zeta_{jj}(r\omega/N)|}{r^2 + 1} r^{d-1} dr + \frac{C\langle \lambda \rangle}{N^{d+2}} \int_{r=1}^{\infty} \langle r \rangle^{-d-1} dr.
\end{aligned} \tag{1.5.17}$$

In the last line, we used the bound

$$\operatorname{Im} \lambda \geq 1 \quad \Rightarrow \quad \sup_{r \geq 0} \frac{r^2 + 1}{|r^2 - \lambda^2|} \leq C\langle \lambda \rangle. \tag{1.5.18}$$

The second integral in the last line of (1.5.17) is finite and the contribution of the corresponding term is $O(\langle \lambda \rangle N^{-d-2})$. In dimension $d \geq 3$, we control the first integral by observing that the function $r \mapsto r^{d-1} \langle r \rangle^{-2}$ grows like $r^{d-3}$, which is not integrable. We deduce from (1.5.15) that

$$d \geq 3 \ \Rightarrow \ \frac{C\langle \lambda \rangle}{N^{2d}} \int_{r=0}^{N} \frac{\sup_{\omega \in \mathbb{S}^{d-1}} |\zeta_{jj}(r\omega/N)|}{r^2 + 1} r^{d-1} dr \leq \frac{C\langle \lambda \rangle}{N^{d+2}}. \tag{1.5.19}$$

When $d = 1$, the estimate (1.5.16) implies

$$\frac{C\langle\lambda\rangle}{N^2}\int_{r=0}^{N}\frac{\sup_{\omega\in\pm 1}|\zeta_{jj}(r\omega/N)|}{r^2+1}dr \le \frac{C\langle\lambda\rangle}{N^2}\int_{r=0}^{N}\frac{r^2N^{-2}+N^{-2}}{r^2+1}dr \le \frac{C\langle\lambda\rangle}{N^3} = \frac{C}{N^{1+2}}.$$

This proves the validity of (1.5.19) for all $d \ge 1$. Combining the above estimates together, we obtain that for any odd $d$, uniformly in $\lambda$ with $\operatorname{Im}\lambda \ge 1$, $N$ and $j \in [-N,N]^d$,

$$|\alpha_{jj}(\lambda)| \le \frac{C\langle\lambda\rangle}{N^{d+2}}.$$

We now estimate $\alpha_{j\ell}(\lambda)$, for $j \neq \ell$. We first integrate by parts in $\xi$, using the identity $(j-\ell)D_\xi e^{i\xi(\ell-j)} = |j-\ell|^2 e^{i\xi(\ell-j)}$:

$$\begin{aligned}\alpha_{j\ell}(\lambda) &= \frac{1}{(2\pi N)^d}\int_{\mathbb{R}^d}\frac{(j-\ell)D_\xi e^{i\xi(\ell-j)}}{|j-\ell|^2}\cdot\frac{\zeta_{j\ell}(\xi)}{N^2|\xi|^2-\lambda^2}d\xi \\ &= \frac{1}{(2\pi N)^d|j-\ell|^2}\int_{\mathbb{R}^d}e^{i\xi(\ell-j)}\left(\frac{(j-\ell)D_\xi\zeta_{j\ell}(\xi)}{N^2|\xi|^2-\lambda^2} - \frac{2N^2(j-\ell)\xi}{i(N^2|\xi|^2-\lambda^2)^2}\zeta_{j\ell}(\xi)\right)d\xi.\end{aligned} \tag{1.5.20}$$

We argue as in the case $j = \ell$ to bound the first term. This yields

$$\begin{aligned}&\left|\frac{1}{(2\pi N)^d|j-\ell|^2}\int_{\mathbb{R}^d}e^{i\xi(\ell-j)}\frac{(j-\ell)D_\xi\zeta_{j\ell}(\xi)}{N^2|\xi|^2-\lambda^2}d\xi\right| \\ &\le \frac{C\langle\lambda\rangle}{N^{2d}|j-\ell|}\int_{r=0}^{N}\frac{\sup_{\omega\in\mathbb{S}^{d-1}}|D_\xi\zeta_{j\ell}(r\omega/N)|}{r^2+1}r^{d-1}dr + \frac{C\langle\lambda\rangle}{N^{d+2}|j-\ell|}\int_{r=1}^{\infty}\langle r\rangle^{-d-1}dr.\end{aligned}$$

The second integral contributes to $O(\langle\lambda\rangle N^{-d-2}|j-\ell|^{-1})$. So does the first, when $d \ge 3$. When $d = 1$, we use (1.5.16) to obtain

$$\frac{C\langle\lambda\rangle}{N^2|j-\ell|}\int_{r=0}^{N}\frac{\sup_{\omega\in\{\pm1\}}|D_\xi\zeta_{j\ell}(r\omega/N)|}{r^2+1}dr \le \frac{C\langle\lambda\rangle}{N^2|j-\ell|}\int_{r=0}^{N}\frac{N^{-1}r+N^{-1}}{r^2+1}dr \le \frac{C\langle\lambda\rangle\ln(N)}{N^3|j-\ell|}.$$

In order to control the second term in the second line of (1.5.20), we use techniques similar to the case $j = \ell$ and obtain

$$\begin{aligned}&\left|\frac{C}{N^d|j-\ell|^2}\int_{\mathbb{R}^d}e^{i\xi(\ell-j)}\frac{2N^2(j-\ell)\xi}{i(N^2|\xi|^2-\lambda^2)^2}\zeta_{j\ell}(\xi)d\xi\right| \\ &\le \frac{C\langle\lambda\rangle^2}{N^{2d-1}|j-\ell|}\int_{r=0}^{N}\frac{r^d\sup_{\omega\in\mathbb{S}^{d-1}}|\zeta_{j\ell}(r\omega/N)|}{(1+r^2)^2}dr + \frac{C\langle\lambda\rangle^2}{N^{d+2}|j-\ell|}\int_{r=1}^{\infty}\langle r\rangle^{-d-1}dr\end{aligned}$$

The second integral is $O(\langle\lambda\rangle^2 N^{-d-2}|j-\ell|^{-1})$. In dimension $d \ge 5$, given that the function $r^d\langle r\rangle^{-4}$ is not integrable, the first integral can be controlled by $O(\langle\lambda\rangle^2 N^{-d-2}|j-\ell|^{-1})$. For the same reason, in dimension 3, it can be controlled by $O(\langle\lambda\rangle^2 N^{-5}\ln(N)|j-\ell|^{-1})$. In dimension 1, we use (1.5.16) to obtain

$$\frac{C\langle\lambda\rangle^2}{N|j-\ell|}\int_{r=0}^{N}\frac{r\sup_{\omega\in\{\pm1\}}|\zeta_{j\ell}(r\omega/N)|}{(1+r^2)^2}dr \le \frac{C\langle\lambda\rangle^2}{N^3|j-\ell|}\int_{r=0}^{N}\frac{r}{1+r^2}dr \le \frac{C\langle\lambda\rangle^2\ln(N)}{N^3|j-\ell|}.$$

Grouping all these estimates together, we obtain that for any odd $d$, uniformly in $\lambda$ with $\operatorname{Im}\lambda \geq 1$, $N$ and $j \neq \ell \in [-N,N]^d$,

$$|\alpha_{j\ell}(\lambda)| \leq \frac{C\langle\lambda\rangle^2 \ln(N)}{|j-\ell| N^{d+2}}. \tag{1.5.21}$$

**Step 4.** We show here estimates for $\operatorname{Im}\lambda \leq -1$. In such cases, $\operatorname{Im}(-\lambda) \geq 1$, and we can write

$$\alpha_{j\ell}(\lambda) = \alpha_{j\ell}(-\lambda) + \langle (R_0(\lambda) - R_0(-\lambda)) q(N\cdot -j) f, \overline{q(N\cdot-\ell)g}\rangle.$$

The same arguments as in [Dr15, Lemma 4.9] shows that $\rho(R_0(\lambda)-R_0(-\lambda))\rho$ maps $L^2$ to $H^{2d}$, with norm controlled by $C|\lambda|^{3d-2}e^{C|\operatorname{Im}\lambda|}$ – when $\operatorname{Im}\lambda \leq -1$. By duality, it also maps $H^{-2d}$ to $L^2$ (with same norm) and by interpolation, $H^{-d}$ to $H^d$ (with same norm). It follows that

$$\begin{aligned}&\left|\left\langle (R_0(\lambda) - R_0(-\lambda)) q(N\cdot -j) f, \overline{q(N\cdot-\ell)g}\right\rangle\right| \\ &\leq C|\lambda|^{3d-2}e^{C|\operatorname{Im}\lambda|}|q(N\cdot-j)f|_{H^{-d}}|q(N\cdot-\ell)g|_{H^{-d}}.\end{aligned}$$

Since $H^d$ is an algebra, the dual bound $|f_1f_2|_{H^{-d}} \leq C|f_1|_{H^d}|f_2|_{H^{-d}}$ holds for arbitrary $f_1, f_2 \in C_0^\infty(\mathbb{R}^d,\mathbb{C})$. Since $f$ and $g$ are smooth and $q$ has compact support,

$$|q(N\cdot-j)f|_{H^{-d}}|q(N\cdot-\ell)g|_{H^{-d}} \leq C|q(N\cdot)|_{H^{-d}}^2.$$

A computation shows

$$|q(N\cdot)|_{H^{-d}}^2 = \frac{1}{(2\pi N)^d}\int_{\mathbb{R}^d}\frac{|\hat{q}(\xi)|^2}{(1+N^2\xi^2)^{2d}}d\xi \leq \frac{C}{N^{2d}}\int_0^\infty \frac{\sup_{\omega\in\mathbb{S}^{d-1}}|\hat{q}(r\omega)|^2}{(1+N^2r^2)^{2d}}r^{d-1}dr.$$

Again, splitting this integral for $r$ near 0 and $r$ away from 0, and using $\hat{q}(0)=0$ in dimension 1, shows that it is $|q(N\cdot)|_{H^{-d}}^2 = O(N^{-d-3})$. Therefore,

$$|\alpha_{j\ell}(\lambda)| \leq |\alpha_{j\ell}(-\lambda)| + \frac{C|\lambda|^{3d-2}e^{C|\operatorname{Im}\lambda|}}{N^{d+3}}. \tag{1.5.22}$$

In particular, this bound combined with (1.5.21) shows that for $|\operatorname{Im}\lambda| = 1$,

$$|\alpha_{jj}(\lambda)| \leq \frac{C\langle\lambda\rangle^{3d-2}}{N^{d+2}}, \quad |\alpha_{j\ell}(\lambda)| \leq \frac{C\langle\lambda\rangle^2\ln(N)}{|j-\ell|N^{d+2}} + \frac{C|\lambda|^{3d-2}}{N^{d+3}}. \tag{1.5.23}$$

**Step 5.** We now use the three lines theorem to estimate locally $\alpha_{j\ell}(\lambda)$, for any $\lambda \in X_d$. Because of Steps 3 and 4, it suffices to focus on the strip $|\operatorname{Im}\lambda| \leq 1$. For $d \geq 3$, the function $\alpha_{j\ell}(\lambda)$ is bounded in this strip:

$$|\alpha_{j\ell}(\lambda)| = |\langle R_0(\lambda)q(N\cdot-j)f, \overline{q(N\cdot-\ell)g}\rangle| \leq Ce^{C|\operatorname{Im}\lambda|}|q|_\infty^2 \cdot \sup_{x,y\in\operatorname{supp}(q)^2}|u(x)||v(y)|.$$

The function $(\lambda+2i)^{-3d}\alpha_{j\ell}(\lambda)$ is also bounded in the strip $|\operatorname{Im}\lambda| \leq 1$, and (1.5.23) estimates it on the edge of this strip. The three lines theorem imply that the bound (1.5.23) holds uniformly inside the strip $|\operatorname{Im}\lambda| \leq 1$.

When $d = 1$, we remove the pole of $R_0(\lambda)$ at 0 by considering the function $\lambda\alpha_{j\ell}(\lambda)$ instead. The same arguments as in the case $d \geq 3$ extends (1.5.23) to $|\operatorname{Im}\lambda| \leq 1$, $\lambda \neq 0$. In particular, for any $\lambda \in X_d$, there exists $C > 0$ such that uniformly in $j, \ell, N$,

$$|\alpha_{jj}(\lambda)| \leq \frac{C}{N^{d+2}}, \quad |\alpha_{j\ell}(\lambda)| \leq \frac{C\ln(N)}{|j-\ell|N^{d+2}} + \frac{C}{N^{d+3}}.$$

**Step 6.** We now estimate the HS-norm of $\alpha_{j\ell}(\lambda)$ for $\lambda$ in compact subsets of $X_d$, so that we can apply later the Hanson–Wright inequality. According to Step 5, it suffices to estimate the sum

$$\sum_j \frac{1}{N^{2d+4}} + \sum_{j\neq\ell} \frac{\ln(N)^2}{N^{4+2d}|j-\ell|^2} + \sum_{j\neq\ell} \frac{1}{N^{2d+6}}.$$

Given $m$, the number of sites $j, \ell$ such that $|j-\ell| = m$ is controlled by $N^{2d-1}$. Therefore,

$$\sum_j \frac{1}{N^{2d+4}} + \sum_{j\neq\ell} \frac{\ln(N)^2}{N^{4+2d}|j-\ell|^2} + \sum_{j\neq\ell} \frac{1}{N^{2d+6}}$$
$$\leq \frac{N^d}{N^{2d+4}} + \frac{CN^{2d-1}\ln(N)^2}{N^{4+2d}} \sum_{m=1}^{2N} \frac{1}{m^2} + \frac{N^{2d}}{N^{2d+6}} \leq \frac{C\ln(N)^2}{N^5}.$$

It follows that $|\alpha(\lambda)|_{\mathrm{HS}} = O(N^{-5/2}\ln(N))$. By the Hanson–Wright inequality (in its original version), for $\lambda$ in compact subsets of $X_d$,

$$\mathbb{P}\left(N^2 \left|\langle R_0(\lambda)V_\# f, V_\# g\rangle - \sum_j \alpha_{jj}(\lambda)\right| \geq t\right) \leq C\exp\left(-\frac{ctN^{1/2}}{\ln(N)}\right). \tag{1.5.24}$$

**Step 7.** To conclude we estimate $\sum_j \alpha_{jj}(\lambda)$. For $\operatorname{Im}\lambda \geq 1$, we have

$$\sum_j \alpha_{jj}(\lambda) = \frac{1}{(2\pi N)^d} \sum_j \int_{\mathbb{R}^d} \frac{\zeta_{jj}(\xi)}{N^2|\xi|^2 - \lambda^2} d\xi.$$

The Taylor expansion (1.5.9) shows that uniformly in $j$,

$$\zeta_{jj}(\xi) = \hat{q}(\xi)\hat{q}(-\xi)\cdot(fg)\left(\frac{j}{n}\right) + O(N^{-1}\langle\xi\rangle^{-2d}).$$

It follows that

$$\sum_j \alpha_{jj}(\lambda) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\hat{q}(\xi)\hat{q}(-\xi)}{N^2|\xi|^2 - \lambda^2} d\xi \cdot \frac{1}{N^d} \sum_j (fg)\left(\frac{j}{n}\right) + O(N^{-1}) \int_{\mathbb{R}^d} \frac{\langle\xi\rangle^{-2d}}{|N^2|\xi|^2 - \lambda^2|} d\xi.$$

We recognize a Riemann sum with step $N^{-1}$ on the right hand side. Since the function $fg$ is smooth, this Riemann sum is equal to $\int_{[-1,1]^d}(fg)(x)dx$ modulo $O(N^{-1})$. In addition, we can use (1.5.18) to control the second term, and obtain

$$\sum_j \alpha_{jj}(\lambda) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{N^2|\xi|^2-\lambda^2}d\xi\cdot\int_{[-1,1]^d}(fg)(x)dx + O(N^{-3}\langle\lambda\rangle).$$

To obtain an asymptotic of the right hand side, we observe that the function $\hat{q}(\xi)\hat{q}(-\xi)/|\xi|^2$ is integrable (because $\hat{q}(0)=0$ when $d=1$ by assumption), and

$$\left|\int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{N^2|\xi|^2-\lambda^2}d\xi - \int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{N^2|\xi|^2}d\xi\right| \leq \int_{\mathbb{R}^d}\frac{|\hat{q}(\xi)\hat{q}(-\xi)||\lambda|^2}{N^2|\xi|^2\cdot|N^2|\xi|^2-\lambda^2|}d\xi.$$

We apply (1.5.18) to control the LHS:

$$\int_{\mathbb{R}^d}\frac{|\hat{q}(\xi)\hat{q}(-\xi)||\lambda|^2}{N^2|\xi|^2\cdot|N^2|\xi|^2-\lambda^2|}d\xi \leq \int_{\mathbb{R}^d}\frac{|\hat{q}(\xi)\hat{q}(-\xi)|\langle\lambda\rangle^3}{N^2|\xi|^2\cdot(N^2|\xi|^2+1)}d\xi.$$

As in Step 3, we can split this integral in a part near $\xi=0$ and a part away from $\xi=0$. Using that $\hat{q}$ has fast decay (and $\hat{q}(0)=0$ when $d=1$), an upper bound is $O(N^{-3}\langle\lambda\rangle^3)$, therefore

$$\sum_j \alpha_{jj}(\lambda) = \frac{1}{(2\pi)^d N^2}\int_{\mathbb{R}^d}\frac{\hat{q}(\xi)\hat{q}(-\xi)}{|\xi|^2}d\xi\cdot\int_{[-1,1]^d}(fg)(x)dx + O(N^{-3}\langle\lambda\rangle^3) = \frac{L}{N^2} + O(N^{-3}\langle\lambda\rangle^3),$$

where $L$ was defined in (1.5.12). Thanks to (1.5.22), this estimate holds also for $\operatorname{Im}\lambda \leq -1$, and by the same arguments as in Step 5, for any $\lambda \in X_d$. Now, (1.5.24) yields that for $\lambda$ in compact subsets of $X_d$,

$$\mathbb{P}\left(\left|N^2\langle R_0(\lambda)V_\# f, \overline{V_\# g}\rangle - L\right| \geq t\right) \leq C\exp\left(-\frac{ctN^{1/2}}{\ln(N)}\right).$$

**Step 8.** Here we deal with the case $d=1$ and $\lambda_0=0$. Step 1 goes through and yields

$$\langle L_{q_0}^0(\lambda)V_\# f, \overline{V_\# g}\rangle = \frac{1}{2\pi i}\oint_0 \frac{\langle R_0(\mu)V_\# f, \overline{V_\# g}\rangle}{\mu-\lambda}d\mu - \frac{1}{2\pi i}\oint_0\frac{\langle A(\mu)f, \overline{V_\# g}\rangle}{\mu-\lambda}d\mu.$$

Since $R_0(\mu)$ has a simple pole at 0, we obtain

$$\langle L_{q_0}^0(\lambda)V_\# f, \overline{V_\# g}\rangle = \langle L_0^0(0)V_\# f, \overline{V_\# g}\rangle - \frac{1}{2\pi i}\oint_0\frac{\langle A(\mu)f, \overline{V_\# g}\rangle}{\mu-\lambda}d\mu.$$

The same method as in Step 2 above shows that

$$\mathbb{P}\left(N^2\left|\frac{1}{2\pi i}\oint_{\lambda_0}\frac{\langle A(\mu)f, \overline{V_\# g}\rangle}{\mu-\lambda}d\mu\right| \geq t\right) \leq Ce^{-ctN}.$$

We now need to estimate $N^2\langle L_0^0(\lambda)V_\# f, \overline{V_\# g}\rangle$. The kernel of $L_0^0(\lambda)$ is explicitly given by $(x,y) \mapsto K(\lambda, |x-y|)$, where $K(\lambda, r) = \frac{i}{2\lambda}(e^{i\lambda r} - 1)$ – see for instance (1.2.2). Therefore, we can write

$$\langle L_0^0(\lambda)V_\# f, \overline{V_\# g}\rangle = \sum_{j,\ell} \beta_{j\ell} u_j u_\ell, \quad \beta_{j\ell}(\lambda) \overset{\text{def}}{=} \int_{\mathbb{R}^2} K(\lambda, |x-y|) q(Nx-j) q(Ny-\ell) f(x) g(y) dxdy.$$

We estimate the term $\beta_{j\ell}$. For this purpose, we define

$$\gamma[\widetilde{K}, q_1, q_2, \tilde{u}, \tilde{\omega}] = \frac{1}{N^2}\int_{\mathbb{R}^2} \widetilde{K}\left(\lambda, \left|\frac{x+j}{N} - \frac{y+\ell}{N}\right|\right) q_1(x) q_2(y) \tilde{u}\left(\frac{x+j}{N}\right) \tilde{\omega}\left(\frac{y+\ell}{N}\right) dxdy,$$

where $q_1$, $q_2$ are smooth with compact support, $\tilde{u}$, $\tilde{\omega}$ are $C^\infty$ functions on $\mathbb{R}$, and $\widetilde{K} : \mathbb{C}\times\mathbb{R} \to \mathbb{C}$ is holomorphic in the first variable and locally bounded in the second. In this context, it is clear that $\gamma[\widetilde{K}, q_1, q_2, \tilde{u}, \tilde{\omega}]$ is $O(N^{-2})$. We observe that $\beta_{j\ell} = \gamma[f, q, q, u, v]$. Let $Q$ be the compactly supported antiderivative of $q$. Using that $Q' = q$, we can integrate by parts $\gamma[f, q, q, u, v]$ in $x$, and get:

$$\beta_{j\ell} = -\frac{\gamma[\text{sgn}\cdot\partial_2 f, Q, q, u, v] + \gamma[f, Q, q, u', v]}{N}$$

The function $\text{sgn}\cdot\partial_2 f$ is locally bounded but it has a discontinuity at 0 in the second variable. Its derivative is the distribution $2\delta_0 \cdot \partial_2 f$. Therefore, an integration by parts in $y$ generates a boundary term:

$$\beta_{j\ell} = -\frac{2}{N^3}\int_{\mathbb{R}} \partial_2 K(\lambda, 0) Q(x) Q(x+j-\ell) f\left(\frac{x+j}{N}\right) g\left(\frac{y+\ell}{N}\right) dx$$
$$+\frac{\gamma[\partial_2^2 f, Q, Q, u, v] + \gamma[\text{sgn}\cdot\partial_2 f, Q, Q, u, v'] + \gamma[\text{sgn}\cdot\partial_2 f, Q, Q, u', v] + \gamma[f, Q, Q, u', v']}{N^2}$$

Because of $\gamma[\widetilde{K}, q_1, q_2, \tilde{u}, \tilde{\omega}] = O(N^{-2})$, of $\partial_2 K(\lambda, 0) = -\frac{1}{2}$ and of (1.5.9),

$$\beta_{j\ell} = \frac{1}{N^3}\int_{\mathbb{R}} Q(x) Q(x+j-\ell) dx \cdot f\left(\frac{j}{N}\right) g\left(\frac{\ell}{N}\right) + O(N^{-4}). \tag{1.5.25}$$

When $|j-\ell|$ is large enough, $Q$ and $Q(\cdot + j - \ell)$ have disjoint support. Therefore, the leading term in (1.5.25) vanishes unless $j-\ell$ is smaller than a constant independent of $N$. This yields the estimate:

$$|\beta|_{\text{HS}}^2 = O(N)O(N^{-6}) + O(N^2)O(N^{-8}) = O(N^{-5}).$$

We conclude by estimating $\sum_j \beta_{jj}$: recognizing a Riemann sum,

$$\sum_j \beta_{jj} = \frac{1}{N^2}\int_{\mathbb{R}} Q(x)^2 dx \cdot \frac{1}{N}\sum_j f\left(\frac{j}{N}\right) g\left(\frac{j}{N}\right) + O(N^{-3})$$
$$= \frac{1}{N^2}\int_{\mathbb{R}} Q(x)^2 dx \cdot \int_{-1}^1 (fg)(x) dx + O(N^{-3}).$$

We conclude that

$$\mathbb{P}(|N^2\langle L^0_{q_0}(\lambda)V_\# f, \overline{V_\# g}\rangle - L| \geq t) \leq Ce^{-ctN^{1/2}}.$$

Since $\hat{Q}(\xi) = \hat{q}(\xi)/\xi$, the lemma for $\lambda = 0$ and $d = 1$ follows. □

## 1.6 Proofs of the theorems

We conclude this chapter with the proof of the theorems.

*Proof of Theorem 1 and of Corollary 2.* Assume that $q_0$ has no resonances on $\partial\mathbb{D}(0, R)$. According to Lemma 1.3.2, it suffices to show that after removing a set of probability $O(e^{-cN^\gamma})$, $|V_\#|_{\mathscr{H}^{-2}} \leq c' N^{-\gamma/2}$, for some $c'$ sufficiently large. This follows from Lemma 1.5.1. This concludes the proof of Theorem 1.

We now show Corollary 2. Fix $R > 0$ such that $q_0$ has no resonances on $\partial\mathbb{D}(0, R)$. Introduce the event:

$$A_N \overset{\text{def}}{=} \left\{ \ V^\omega_N \text{ does not satisfy (1.1.4)} \right\}.$$

We know that $\mathbb{P}(A_N) \leq Ce^{-cN^\gamma}$. In particular, $\sum_{N=0}^\infty \mathbb{P}(A_N) < \infty$, and the Borel–Cantelli lemma implies that $A_N$ happens only finitely many times. Therefore, $\mathbb{P}$-almost surely, there exists $N_0$ such that for every $N \geq N_0$, (1.1.4) is realized. It suffices to take a countable sequence $R \to \infty$ to conclude. □

*Proof of Theorem 3.* The proof of this theorem follows from Lemma 1.5.1 and arguments from [Dr15, Dr16b]. Recall that $X_d = \mathbb{C}$ if $d \geq 3$ and $X_1 = \mathbb{C} \setminus 0$. According to the formula

$$R_{V_N}(\lambda) = R_0(\lambda)(\mathrm{Id} + V_\# R_0(\lambda)\rho)^{-1}(\mathrm{Id} - V_\# R_0(\lambda)(1-\rho)),$$

the resonances of $V_N = V_\#$ in $X_d$ are exactly the poles of $(\mathrm{Id} + V_\# R_0(\lambda)\rho)^{-1}$. In addition, Lemma 1.2.1 implies

$$|(V_\# R_0(\lambda)\rho)^2|_{\mathscr{B}} \leq |V_\#|_\infty |\rho R_0(\lambda)\rho|_{H^{-2}\to L^2} |V_\#|_{\mathscr{H}^{-2}} |\rho R_0(\lambda)\rho|_{L^2\to H^2} \leq \frac{Ce^{c(\mathrm{Im}\,\lambda)_-}}{d-1+|\lambda|^2}|V_\#|_{\mathscr{H}^{-2}}.$$

If the RHS is bounded by 1/2, then the operator $\mathrm{Id} + V_\# R_0(\lambda)\rho$ is invertible and $\lambda$ is not a resonance. According to Lemma 1.5.1, $|V_\#|_{\mathscr{H}^{-2}} \leq N^{-\gamma}$ with probability $1 - Ce^{-cN^\gamma}$. Therefore, if $\mathrm{Im}\,\lambda \geq (\ln(2C) - \gamma\ln(N))/c$ then with same probability,

$$\frac{Ce^{c(\mathrm{Im}\,\lambda)_-}}{d-1+|\lambda|^2}|V_\#|_{\mathscr{H}^{-2}} < 1/2.$$

Since Theorem 3 says something only for large values of $N$, and since the only possible resonance near 0 was localized in Theorem 1, the proof is over. □

*Proof of Theorem 4.* Fix $\lambda_0 \in \mathrm{Res}(q_0)$, of multiplicity 1, with resonant states $f$ and $g$. Recall the estimate (1.5.5): with probability $1 - O(e^{-cN^{1/4}})$,

$$\begin{aligned} |V_\#|_{\mathscr{H}^{-1}} &\leq N^{-3/8} \quad \text{if } d = 1 \text{ and } \int_{\mathbb{R}} q(x)dx \neq 0, \\ |V_\#|_{\mathscr{H}^{-1}} &\leq N^{-7/8} \quad \text{if } d \geq 3 \text{ or } d = 1 \text{ and } \int_{\mathbb{R}} q(x)dx = 0. \end{aligned} \tag{1.6.1}$$

Let us define the following event:

$$B_N = \{|V_\#|_{\mathscr{H}^{-2}} \geq N^{-\gamma/2}\} \cup \{V_\# \text{ does not satisfy (1.6.1)}\}.$$

Because of Lemma 1.5.1 and , $\mathbb{P}(B_N) = O(e^{-cN^{1/4}})$. Let $\delta_0, r_0$ be given by Lemma 1.3.3. As in the proof of Theorem 1, we know that for $N$ sufficiently large, $V_N$ has a single resonance $\lambda_N \in \mathbb{D}(\lambda_0, r_0)$, for the event $\Omega \setminus B_N$. We extend $\lambda_N$ to all of $\Omega$ by setting $\lambda_N|_{B_N} = \lambda_0$. This definition shows that the random variable $\lambda_N$ is a resonance of $V_N$ with probability $1 - \mathbb{P}(B_N) = 1 - O(e^{-cN^{1/4}})$. Moreover, Lemma 1.3.3 implies that $\lambda_N$ satisfies the equation

$$\lambda_N - \lambda_0 = \mathbb{1}_{\Omega \setminus B_N} \sum_{k=0}^{\infty} (-1)^{k+1} \left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g} \right\rangle. \tag{1.6.2}$$

as long as $N$ is sufficiently large. It remains to study the speed of convergence of $\lambda_N$, in the Cases I, II and III.

**Case I.** In this case, $d = 1$ or 3 and $\int_{\mathbb{R}^d} q(x)dx \neq 0$, hence $\gamma = d/2$. We have:

$$\begin{aligned} \frac{N^{d/2}}{i}(\lambda_N - \lambda_0) &= \mathbb{1}_{\Omega \setminus B_N} N^{d/2} \sum_{k=0}^{\infty} (-1)^{k+1} \left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g} \right\rangle \\ &= -N^{d/2}\langle V_\# f, \overline{g}\rangle + \mathbb{1}_{B_N} N^{d/2}\langle V_\# f, \overline{g}\rangle + \mathbb{1}_{\Omega \setminus B_N} N^{d/2} \sum_{k=1}^{\infty} (-1)^{k+1} \left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g} \right\rangle. \end{aligned} \tag{1.6.3}$$

We show that the first and second terms in the second line of (1.6.3) converge in $L^1$ to 0. Using that $\mathbb{P}(B_N)$ decay exponentially and that $\langle V_\# f, g\rangle = O(1)$,

$$\mathbb{E}(\mathbb{1}_{B_N} N^{d/2}|\langle V_\# f, \overline{g}\rangle|) = O(N^{d/2} e^{-cN^{1/4}}) \to 0.$$

In addition, on $\Omega \setminus B_N$, $N^{d/2}|V_\#|^2_{\mathscr{H}^{-1}} \leq N^{-1/4}$ by (1.5.5) (recall that $d = 1$ or 3). The estimate (1.3.11) and the definition of $B_N$ yields

$$\left| \mathbb{1}_{\Omega \setminus B_N} N^{d/2} \sum_{k=1}^{\infty} (-1)^{k+1} \left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g} \right\rangle \right| \leq C \mathbb{1}_{\Omega \setminus B_N} N^{d/2} |V_\#|^2_{\mathscr{H}^{-1}}) = O(N^{-1/4}).$$

We deduce that

$$\frac{N^{d/2}}{i}(\lambda_N - \lambda_0) \xrightarrow{\mathrm{d}} N^{d/2}\langle V_\# f, \overline{g}\rangle.$$

– see for instance [Bi95, Theorem 25.4]. Lemma 1.5.3 shows that $N^{d/2}\langle V_\# f, \overline{g}\rangle$ converges to a Gaussian and this concludes the proof of Theorem 4 in Case I.

**Case II.** In this case, $d = 1$ and $\int_{\mathbb{R}} q(x)dx = 0$, $\int_{\mathbb{R}} xq(x)dx \neq 0$ and $(f\cdot g)' \not\equiv 0$ on $[-1,1]$. We use (1.6.3) with a factor $N^{3/2}$ instead of $N^{1/2}$: $\frac{N^{3/2}}{i}(\lambda_N - \lambda_0) =$

$$-N^{3/2}\langle V_\# f, \overline{g}\rangle + \mathbb{1}_{B_N} N^{3/2}\langle V_\# f, \overline{g}\rangle + \mathbb{1}_{\Omega\setminus B_N} N^{3/2}\sum_{k=1}^{\infty}(-1)^{k+1}\left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g}\right\rangle.$$

The first term converges to a Gaussian according to Lemma 1.5.5. The second term converges in $L^1$ to 0 because $\mathbb{P}(B_N)$ is exponentially small. The third term is $O(N^{-1/4})$ because it is bounded by $N^{3/2}|V_\#|^2_{\mathscr{H}^{-1}}$ – see (1.3.11) – itself being $O(N^{-1/4})$ for events in $\Omega\setminus B_N$ – see (1.6.1) and the definition of $B_N$. An application of [Bi95, Theorem 25.4] as in Case I allows us to conclude.

**Case III.** Thanks to (1.6.2), we can write

$$\begin{aligned}\frac{N^2}{i}(\lambda_N - \lambda_0) = -\mathbb{1}_{\Omega\setminus B_N} N^2\langle V_\# f, \overline{g}\rangle + \mathbb{1}_{\Omega\setminus B_N} N^2\left\langle V_\# L_{q_0}^{\lambda_0}(\lambda_N) V_\# f, \overline{g}\right\rangle \\ +\mathbb{1}_{\Omega\setminus B_N} N^2\sum_{k=2}^{\infty}(-1)^{k+1}\left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g}\right\rangle.\end{aligned} \tag{1.6.4}$$

We now evaluate the probability that the RHS of (1.6.4) is significantly different from $L$ – defined in Lemma 1.5.6. Since Case III is satisfied,

$$\mathbb{P}(|N^2\langle V_\# f, \overline{g}\rangle| \geq N^{-1/4}) = O(e^{-cN^{1/2}}). \tag{1.6.5}$$

This comes from Lemma 1.5.3 when $d \geq 5$; Lemma 1.5.4 when $d = 3$ and $\int_{\mathbb{R}^3} q(x)dx = 0$ or $d = 1$ and $\int_{\mathbb{R}} q(x)dx = \int_{\mathbb{R}} xq(x)dx = 0$; and Lemma 1.5.5 when $\int_{\mathbb{R}} q(x)dx = 0$ and $(fg)' = 0$ on $[-1,1]$. According to Lemma 1.5.6 and $\mathbb{P}(B_N) = O(e^{-cN^{1/4}})$,

$$\begin{aligned}&\mathbb{P}(|\mathbb{1}_{\Omega\setminus B_N} N^2\left\langle V_\# L_{q_0}^{\lambda_0}(\lambda_N) V_\# f, \overline{g}\right\rangle - L| \geq N^{-1/5}) \\ &\leq \mathbb{P}(2|N^2\left\langle V_\# L_{q_0}^{\lambda_0}(\lambda_N) V_\# f, \overline{g}\right\rangle - L| \geq N^{-1/5}) + \mathbb{P}(B_N) = O(e^{-cN^{1/4}}).\end{aligned} \tag{1.6.6}$$

Since Case III is satisfied, then $m \geq 1$ when $d = 1$. Hence, $|V_\#|_{\mathscr{H}^{-1}} = O(N^{-7/8})$ on $\Omega\setminus B_N$ – see (1.5.5). Thanks to (1.3.11),

$$\left|\mathbb{1}_{\Omega\setminus B_N} N^2\sum_{k=2}^{\infty}(-1)^{k+1}\left\langle (V_\# L_{q_0}^{\lambda_0}(\lambda_N)\rho)^k V_\# f, \overline{g}\right\rangle\right| \leq C\mathbb{1}_{\Omega\setminus B_N} N^2|V_\#|^3_{\mathscr{H}^{-1}} = O(N^{-1/4}). \tag{1.6.7}$$

Combining (1.6.4), (1.6.5), (1.6.6) and (1.6.7), we obtain

$$\mathbb{P}(|N^2(\lambda_N - \lambda_0) - iL| \geq N^{-1/5}) = O(e^{-cN^{1/4}}).$$

In particular,

$$\sum_{N=1}^{\infty} \mathbb{P}(|N^2(\lambda_N - \lambda_0) - iL| \geq N^{-1/5}) < \infty.$$

This implies by the Borel–Cantelli lemma that for each elementary event, $|N^2(\lambda_N - \lambda_0) - L| \geq N^{-1/5}$ for only finitely many $N$. In particular, $N^2(\lambda_N - \lambda_0) \xrightarrow{\mathbb{P}\text{-a.s.}} iL$ as claimed. □

*Proof of Corollary 5.* Assume that $q_0$ is real-valued and $\lambda_0 \in \mathrm{Res}(q_0) \cap i\mathbb{R}$. Let $\lambda_N$ be the random variable constructed in the proof of Theorem 4. Then $\lambda_N$ is purely imaginary. Indeed, $\lambda_N|_{B_N} = \lambda_0 \in \mathbb{R}$; and on $\Omega_N \setminus B_N$, $\lambda_N$ is the unique resonance of $V_N$ in $\mathbb{D}(\lambda_0, r_0)$, in particular it is purely imaginary – otherwise $-\overline{\lambda_N}$ would be another resonance of $V_N$ in the disk $\mathbb{D}(\lambda_0, r_0)$.

The convergence results follows now from Theorem 4, from the identity $g = \overline{f}$ and from the convergence mapping theorem [Bi95, Theorem 25.7]. For instance, in Case I,

$$\frac{N^{d/2}(\lambda_N - \lambda_0)}{i \int_{\mathbb{R}^d} q(x)dx} = \mathrm{Re}\left(\frac{N^{d/2}(\lambda_N - \lambda_0)}{i \int_{\mathbb{R}^d} q(x)dx}\right) = \pi\left(\frac{N^{d/2}(\lambda_N - \lambda_0)}{i \int_{\mathbb{R}^d} q(x)dx}\right) \xrightarrow{\mathrm{d}} \mathscr{N}\left(0, \sigma^2\right).$$

where a complex number $x + iy$ is seen as a vector $(x, y)$; $\pi(x + iy) = \pi(x, y) = x$; and $\sigma^2 = \int_{[-1,1]^d} |f(x)|^4 dx$. The statement about eigenvalues follows from the remark preceding Lemma 1.2.2: in the context of real-valued potentials, eigenvalues corresponds exactly to resonances on the half-line $i(0, \infty)$. □

## 1.7 Necessity of the assumptions of Theorem 1

We show on a simple explicit example that the conclusion of Theorem 1,

$$N \text{ sufficiently large } \Rightarrow \ \mathrm{Res}(V_N) \cap \mathbb{D}(0, R) \subset \bigcup_{\lambda \in \mathrm{Res}(q_0)} \mathbb{D}\left(\lambda, N^{-\frac{\gamma}{2m_\lambda}}\right) \tag{1.7.1}$$

cannot hold with probability 1.

We fix $d = 1$, $q_0 \equiv 0$, $q \in C_0^\infty(\mathbb{R}, \mathbb{R})$ with $\int_{\mathbb{R}} q(x)dx = 1$ and $u_j$ independent Bernouilli random variables ($\mathbb{P}(u_j = 1) = \mathbb{P}(u_j = -1) = 1/2$). We observe that

$$\mathbb{P}(\{u_j = 1 \ \forall j \in [-N, N]^3\}) = 2^{-N} > 0,$$

and the potential corresponding to this event is $\widetilde{V}_N(x) \stackrel{\text{def}}{=} \sum_j q(Nx - j)$. The weak limit of $\widetilde{V}_N$ as $N \to +\infty$ is $\mathbb{1}_{[-1,1]}$, and the convergence is in fact strong in $H^{-2}$. Indeed, if $\varphi \in C_0^\infty(\mathbb{R}, \mathbb{C})$

then using a representation of $\int_{[-1,1]} \varphi(x)dx$ as a Riemann sum modulo $O(N^{-1}|\varphi'|_\infty)$,

$$\langle \widetilde{V}_N, \varphi\rangle - \langle \mathbb{1}_{[-1,1]}, \varphi\rangle = \sum_j \int_{\mathbb{R}} q(Nx-j)\varphi(x)dx - \int_{[-1,1]} \varphi(x)dx$$
$$= \frac{1}{N}\left( \sum_j \int_{\mathbb{R}} q(x)\varphi\left(\frac{x+j}{N}\right)dx - \sum_j \int_{\mathbb{R}} q(x)\varphi\left(\frac{j}{N}\right)dx \right) + O(N^{-1}|\varphi'|_\infty)$$
$$= \frac{1}{N}\sum_j \int_{\mathbb{R}} q(x)\left(\varphi\left(\frac{x+j}{N}\right) - \varphi\left(\frac{j}{N}\right)\right)dx + O(N^{-1}|\varphi'|_\infty).$$

This is fully estimated by $O(N^{-1}|\varphi'|_\infty)$. Hence, $\widetilde{V}_N - \mathbb{1}_{[-1,1]}$ converges to 0 for the topology of distributions of order 1 on $\mathbb{R}$. In dimension one, functions that are locally in $C^1$ are locally in $H^2$. Therefore $\widetilde{V}_N - \mathbb{1}_{[-1,1]}$ also converges to 0 for the topology of bounded linear functionals on $H^2$, i.e. for the $H^{-2}$ norm-topology, as claimed.

According to Lemma 1.3.2, resonances of $\widetilde{V}_N$ converge to resonances of $\mathbb{1}_{[-1,1]}$ – uniformly on compact sets. On the other hand, the potential $\mathbb{1}_{[-1,1]}$ has infinitely many resonances – see for instance [DZ16d, Theorem 2.14] – while $q_0$ has a single resonance. This shows that (1.7.1) cannot hold for every $R > 0$ when $\widetilde{V}_N$ replaces $V_N$, in particular Theorem 1 holds with probability at most $1 - 2^{-N}$.

One can construct similar examples in higher dimensions, with the conclusion that Theorem 1 cannot hold with probability greater than $1 - 2^{-N^d}$. The argument requires Smith–Zworski [SZ16] instead of [DZ16d]: in odd dimension $d \geq 3$, bounded compactly supported real-valued potentials have at least one resonance.

# Chapter 2

# Stochastic stability of Pollicott–Ruelle resonances

## 2.1 Introduction

The trajectories of individual particles submitted to a chaotic evolution is in general too difficult to predict. This is why the asymptotic of chaotic dynamical systems are studied via statistical correlations. Specifically, if $\mathscr{M}$ is (say) a compact Riemannian manifold and $\Phi_t$ is a continuous dynamical system on $\mathscr{M}$ which preserves the Riemannian measure, the correlation associated to two smooth functions $f$ and $g$ on $\mathscr{M}$ is

$$\langle f, \Phi_t^* g\rangle \overset{\text{def}}{=} \int_{\mathscr{M}} f(x) g(\Phi_t(x)) d\mu(x).$$

The dynamical system is called mixing if asymptotically, $f$ and $\Phi_t^* g$ behave like uncorrelated random variables:

$$\lim_{t\to\infty} \langle f, \Phi_t^* g\rangle = \int_{\mathscr{M}} f d\mu \cdot \int_{\mathscr{M}} g d\mu. \tag{2.1.1}$$

This is a form of chaos because knowing the initial state of the system tells you nothing about the state of the system at large times. A stronger form of chaos consists of requiring that the convergence (2.1.1) occurs exponentially fast. This is formally satisfied by a universal class of dynamical systems that are called Anosov (or Axiom A) in the mathematical literature. Trajectories of Anosov flows are either expanding or contracting – they cannot be expanding at a time then contracting at a later time. The goal of this chapter is to study the stochastic stability of spectral quantities, called Pollicott–Ruelle resonances, associated to certain Anosov flows. This form of stability should allow to observe physical manifestations of these resonances in nature.

### 2.1.1 Results

We will focus on the most fundamental example of Anosov flows, which is induced by chaotic geodesic flows. We consider a smooth compact Riemannian manifold $\mathbb{M}$ with negative sectional curvatures and cosphere bundle $S^*\mathbb{M}$. The generator of the geodesic flow $H_1 \in TS^*\mathbb{M}$ is an Anosov vector field, as was first noted by Anosov [An67]. On suitable spaces, $P_0 = \frac{1}{i}H_1$ has a discrete spectrum with eigenvalues called Pollicott–Ruelle resonances, denoted by $\mathrm{Res}(P_0)$. These complex numbers appear as exponential decay rates in expansions of classical correlations – see Tsuji [Ts10] and Nonnemacher–Zworski [NZ15]. We refer to §2.2.2 for precise definitions.

Several authors introduced recently a stochastic process on $S^*\mathbb{M}$ that is a natural perturbation of the geodesic equation – see Franchi–Le Jan [FL07], Grothaus–Stilgenbauer [GS13], Angst–Bailleul–Tardif [ABT15] and Li [Li16]. It is called in [ABT15] kinetic Brownian motion. In contrast with the Langevin process [La08], kinetic Brownian motion models diffusive phenomena with finite speed of propagation.

We concentrate on analytic and spectral aspects of this stochastic process. Our main object of study is the infinitesimal generator of kinetic Brownian motion. It is equal to $H_1 + \varepsilon\Delta_{\mathbb{S}}$, where $\Delta_{\mathbb{S}} \geq 0$ is the vertical spherical Laplacian – see §2.3.1. We investigate the convergence of the $L^2$-spectrum $\Sigma(P_\varepsilon)$ of $P_\varepsilon = \frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}})$, as $\varepsilon$ goes to $0^+$. Although the $L^2$-spectrum of $P_0$ is absolutely continuous and equal to $\mathbb{R}$, we have:

**Theorem 9.** *The set of accumulation points of $\Sigma(P_\varepsilon)$ as $\varepsilon \to 0^+$ is equal to* $\mathrm{Res}(P_0)$.

A finer statement is Theorem 13 below. It states that the spectral projections of $P_\varepsilon$ depend smoothly on $\varepsilon$; and that if all Pollicott–Ruelle resonances of $P_0$ have simple multiplicity, the $L^2$-eigenvalues of $P_\varepsilon$ admit a full expansion in powers of $\varepsilon$. Remark 2.6.1 analyzes the convergence as $\varepsilon \to 0^-$. In a previous version [Dr16a] of this work, we proved Theorem 9 when $\mathbb{M}$ is an orientable surface.

### Motivation and outline of proof

Dyatlov–Zworski [DZ15] showed that the Pollicott–Ruelle resonances of an Anosov vector field $X$ on a Riemannian manifold are the limits as $\varepsilon \to 0^+$ of the $L^2$-eigenvalues of $\frac{1}{i}(X+\varepsilon\Delta)$. From the point of view of partial differential equations, this realizes resonances as viscosity limits. From the point of view of probability theory, this indicates stochastic stability of Pollicott–Ruelle resonances, because the operator $\frac{1}{i}X + i\varepsilon\Delta$ generates the stochastic differential equation

$$\partial_t \Phi_t = -X(\Phi_t) - \sqrt{2\varepsilon}B(t), \qquad \Phi_0 = \mathrm{Id}_{\mathscr{M}}, \tag{2.1.2}$$

where $B(t)$ is a Brownian motion on $\mathscr{M}$. Their approach also shows that the $L^2$-eigenvalues of $\frac{1}{i}X + i\varepsilon\Delta$ converge to complex conjugates of Pollicott–Ruelle resonances as $\varepsilon \to 0^-$. This fact also holds here, see Remark 2.6.1.

The geodesic flow on the cosphere bundle $S^*\mathbb{M}$ of a Riemannian manifold $\mathbb{M}$ is a fundamental example of Anosov flow. If $X$ denotes the generator of the geodesic flow, (2.1.2) is a random perturbation of the geodesic equation. The perturbative term in (2.1.2) acts on both momenta and positions. As was first modeled by Langevin's equation [La08], a physical random perturbation created by collisions should only act on the momentum variables. A generalization of Langevin's equation to cotangent bundles $T^*\mathbb{M}$ was studied in Jørgensen [Jø78], Soloveitchik [So95] and Kolokoltsov [Ko00].

In this chapter, we remain on the cosphere bundle $S^*\mathbb{M}$ and we consider *kinetic Brownian motion*. This stochastic process is a random perturbation in the momentum random of the geodesic equation on $S^*\mathbb{M}$. It models diffusions with constant speed of propagation, and has generator $H_1 + \varepsilon\Delta_{\mathbb{S}}$. Kinetic Brownian motion was first introduced in Franchi–Le Jan [FL07], as an extension of Langevin's equation in general relativity: it models the relativistic motion of random particles, whose speed has to be bounded by the speed of light. Grothaus–Stilgenbauer [GS13] extended the construction to cosphere bundles of Riemannian manifolds, with applications to industry. Li [Li16] showed the first perturbative results in the small-and-large white force limit (respectively, $\varepsilon \to 0$ and $\varepsilon \to \infty$). Angst–Bailleul–Tardif [ABT15] improved upon Li's result and derived asymptotic in the context of rotationally invariant manifolds. We refer to §2.3.1 for precise definitions.

Dolgopyat–Liverani [DL11] studied another perturbation of the geodesic equation. They considered the geodesic motion of particles, coupled with an interaction of size $\varepsilon$. When the initial data is random and $\varepsilon$ goes to 0, they showed that a suitable rescaling of the energy at time $t$ solves an explicit stochastic differential equation. Bernadin et al. [BHLLO11] obtained a formal expansion of the heat conductivity for systems of weakly coupled random particles. Conceptually, both results can be seen as a step towards deriving macroscopic equations from principles of microscopic dynamics.

This chapter aims to generalize the main result of Dyatlov–Zworski [DZ15] to kinetic Brownian motion. In contrast with [DZ15], the operator $P_\varepsilon = \frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}})$ is hypoelliptic instead of being elliptic. An earlier version [Dr16a] contains a proof of Theorem 9 when $\mathbb{M}$ is an orientable surface. It can be seen as an introduction to the present chapter. The technical details are simpler there, because in that case $\Delta_{\mathbb{S}} = -V^2$, with $V$ the generator of the circle action on the fibers of $S^*\mathbb{M}$.

The lack of ellipticity of $P_\varepsilon$ creates serious new difficulties that we overcome by showing that the operator $P_\varepsilon$ is maximally hypoelliptic in the regime $\varepsilon \to 0$, see Theorem 10. For technical reasons, we will lift $P_\varepsilon$ to an operator $\widetilde{P}_\varepsilon$ acting on functions on the orthonormal coframe bundle of $\mathbb{M}$. The proof continues with the Lebeau [Le07], where the maximal hypoellipticity of Bismut's hypoelliptic Laplacian [Bi05] is shown. Lebeau ingeniously uses certain commutation relations to reduce his study to the case of the model operator $x_1^2 D_{x'}^2 + D_{x_1}$, microlocally near $(0, x', 0, \xi')$, $\xi' \neq 0$. In our approach, we bypass the microlocal reduction and we work directly with $P_\varepsilon$. We replace Lebeau's main step with a positive commutator argument. This yields a maximal hypoellipticity result for $\widetilde{P}_\varepsilon$, that descends to an estimate for $P_\varepsilon$. Lifting geometric equations to the orthonormal frame bundle has been an efficient technique in probability theory, starting with the pioneering constructions of stochastic processes on manifolds by Elworthy [El82]. It was used in both Li [Li16] and Angst–Bailleul–Tardif [ABT15] to show asymptotic results for kinetic Brownian motion.

The remainder of the proof of Theorem 9 is similar to [DZ15]. We will decompose the operator $P_\varepsilon$ in two parts $P_\varepsilon^\sharp + P_\varepsilon^\flat$. The first part acts on momentum frequencies greater than $\varepsilon^{-1}$, and the maximal hypoelliptic estimate will take care of it. For the second part, we will use the anisotropic Sobolev spaces designed in Faure–Sjöstrand [FS11] in a modified form due to Dyatlov–Zworski [DZ16a]. Their construction relies on Melrose's propagation estimate at radial points [Me94], in the improved version of [DZ16a, Propositions 2.6-2.7]. For the original version of anisotropic spaces used in Anosov dynamics, see Baladi [Ba05], Liverani [Li05], Gouëzel–Liverani [GL06] and Baladi–Tsuji [BaTs07]. We also mention Vasy [Va13] for application of similar anisotropic Sobolev spaces in the context of asymptotically hyperbolic manifolds and general relativity.

The operator $P_\varepsilon$ can be realized as the restriction of the hypoelliptic Laplacian of Bismut [Bi05] to the cosphere bundle. This connection provides another motivation for the study of $P_\varepsilon$. Li [Li16] and Angst–Bailleul–Tardif [ABT15] showed that kinetic Brownian motion interpolates between geodesic trajectories as $\varepsilon \to 0$ and Brownian motion on $\mathbb{M}$ as $\varepsilon \to \infty$ (after projection and rescaling). This dramatically echoes Bismut–Lebeau's motivation to study the hypoelliptic Laplacian, obtained in [BL08] as an operator interpolating between the generator of the geodesic flow and the Laplacian on $\mathbb{M}$ (after rescaling and projection). For the corresponding interpretation in probability theory, see Bismut [Bi15]. Improving upon work of Bismut [Bi11], Shen [Sh16] recently obtained far-reaching applications of the hypoelliptic Laplacian, including a proof of Fried's conjecture [Fr95] for maximally symmetric spaces.

Baudoin–Tardif [BaTa16] showed exponential convergence of the heat operator $e^{-itP_\varepsilon}$ to equilibrium: there exists $\nu_\varepsilon > 0$ such that for every $u \in S^\infty(S^*\mathbb{M})$,

$$\left| e^{-itP_\varepsilon} u - \int_{S^*\mathbb{M}} u \right| \leq C e^{-\nu_\varepsilon t} \left| u - \int_{S^*\mathbb{M}} u \right|.$$

Because of the connection of $P_\varepsilon$ with the Laplacian on $\mathbb{M}$, Baudoin and Tardif expected that the optimal value of $\nu_\varepsilon$ converges as $\varepsilon \to \infty$ to the first eigenvalue of the *non-negative* Laplacian on $\mathbb{M}$. Though the explicit value of $\nu_\varepsilon$ derived there converges to 0 as $\varepsilon \to \infty$. When $\mathbb{M}$ is negatively curved, we conjecture that the optimal value of $\nu_\varepsilon$ converges as $\varepsilon \to 0$ to the largest imaginary parts of Pollicott–Ruelle resonances of $\frac{1}{i} H_1$.

When $\mathbb{M}$ is not negatively curved, we can still study the accumulation points of the $L^2$-eigenvalues of $P_\varepsilon$ as $\varepsilon \to 0$. Already in the case of the 2-torus, the behavior of this spectrum is quite mysterious. See (in a slightly different context) [DZ15, Figure 3] and the discussion following it, originating from Galtsev–Shafarevich [GS06]. The general case is far from being understood. Recently, Dyatlov–Zworski [DZ16c] showed a deep connection between Pollicott–Ruelle resonances and topology: the order of vanishing of the Ruelle zeta function at 0 determines the genus of a negatively curved surface. We believe that the spectrum of $P_\varepsilon$ relates closed geodesics and topology, even when $\mathbb{M}$ is not negatively curved. The maximal hypoelliptic estimate (2.4.2) holds with no restrictions on the sign of the curvature. However, the methods of §2.6 are strictly restricted to the negative curvature case.

In order to facilitate the lecture, we provide in §2.2 an introduction to microlocal analysis in the context of hyperbolic dynamical systems. Our presentation is axiomatic and relies on approaches of Faure–Sjoštrand [FS11] and Dyatlov–Zworski [DZ16a]. In §2.3, we recall the theory of kinetic Brownian motion. In particular, we show that its lift to the orthonormal frame bundle satisfies many convenient identities.

## 2.2 Dynamical systems and phase-space analysis

### 2.2.1 Anosov flows and Pollicott–Ruelle resonances

We start with an overview of Anosov diffeomorphisms. These are hyperbolic dynamical systems which present a very chaotic behavior: they contract in certain directions while expanding in others; this intuitively generates a high degree of disorder. This feature, called the Anosov property, is rigorously stated in (2.2.2)-(2.2.3) below. The first systematic study of such flows goes back to Anosov [An67]. He showed that a fundamental example is given by geodesic flow on negatively curved manifolds. In addition, he proved that Anosov flows enjoy a universal property: they are mixing – which means that the state of the system at a large time is mainly independent from the initial state. Specifically, if $\Phi_t : \mathscr{M} \Rightarrow \mathscr{M}$ has the Anosov property, then:

$$\forall \mu \ \Phi_t - \text{invariant measure}, \ \forall f, g \in C^\infty(\mathscr{M}), \ \int_{\mathscr{M}} f \cdot g \circ \Phi_t d\mu = \int_{\mathscr{M}} f d\mu \int_{\mathscr{M}} g d\mu. \quad (2.2.1)$$

The convergence of (2.2.1) occurs in fact exponentially for a large class of Anosov flows – see below.

Rigorously speaking, if $\mathscr{M}$ is a smooth manifold and $X$ is a vector field on $\mathscr{M}$, the flow $e^{tX} : \mathscr{M} \rightarrow \mathscr{M}$ is Anosov if and only if at each point $x \in \mathscr{M}$, there exists a splitting of $T_x\mathscr{M}$ as

$$T_x\mathscr{M} = E_u(x) \oplus \mathbb{R} \cdot X(x) \oplus E_s(x), \quad (2.2.2)$$

with the following properties:

- $E_u(x)$ and $E_s(x)$ are invariant under the flow $e^{tX}$:

$$de^{tX}(x)(E_u(x)) \subset E_u(e^{tX}(x)), \qquad de^{tX}(x)(E_s(x)) \subset E_s(e^{tX}(x)).$$

- $E_u(x)$ and $E_s(x)$ contract in the past and in the future, respectively: if $|\cdot|$ is induced by a metric on $\mathscr{M}$, there exists $\nu > 0$ such that

$$\forall v \in E_u(x), \ |de^{tX}(x)v| \leq e^{-\nu|t|}|v|, \ t < 0, \qquad \forall v \in E_s(x), \ |de^{tX}(x)v| \leq e^{-\nu|t|}|v|, \ t > 0. \quad (2.2.3)$$

The sets $E_u(x)$ and $E_s(x)$ are called the unstable and stable subspaces, respectively. We will say that $X$ is Anosov if $e^{tX}$ is Anosov.

We recall that a contact structure on a manifold $\mathscr{M}$ is a smooth one-form $\alpha$ with the following property: for any vector field $V$,

$$i_V\alpha = 0,\ i_V d\alpha = 0\ \Rightarrow V = 0$$

($i_V$ denotes the exterior multiplication by $V$). This implies that the dimension of the manifold of $\mathscr{M}$ is odd: $\dim(\mathscr{M}) = 2k - 1$; and that $\alpha \wedge (d\alpha)^{k-1}$ is a smooth volume form. A contact form $\alpha$ always admits a Reeb vector field, i.e. a smooth vector field $Y$ such that $\alpha(Y) = 1$ and $i_Y d\alpha = 0$. The volume form $\alpha \wedge (d\alpha)^{k-1}$ is invariant under the Reeb vector field, because

$$\mathscr{L}_Y\alpha = i_Y d\alpha + di_Y\alpha = 0, \quad \mathscr{L}_Y d\alpha = d\mathscr{L}_Y\alpha = 0.$$

(we used Cartan's magical formula). In particular, the vector field $Y$ – seen as a differential operator – is antiselfadjoint on $L^2(\mathscr{M}, \mu)$. Its spectrum is equal absolutely continuous, equal to $i\mathbb{R}$.

On the other hand, it was observed relatively recently that Anosov vector fields enjoy a surprising universal property: their spectrum on specifically designed Sobolev spaces (or more generally, Banach spaces) is discrete – this offered a modern definition of Pollicott–Ruelle resonances. This is in striking contrast with their $L^2$-spectrum. Such anisotropic spaces find their origin in work of Blank–Keller–Liverani [BKL02], Gouëzel–Liverani [GL06] and Baladi–Tsuji [BaTa16]. We will use an approach due to Faure–Sjöstrand [FS11] and Dyatlov–Zworski [DZ16a], see 2.5.1.

### Spectral gaps

In the context of contact Anosov flows, a remarkable application of the existence of a discrete spectrum is the exponential decay of correlations: the convergence (2.2.1) occurs exponentially rapidly. The proof of this fact uses that there are only finitely many Pollicott–Ruelle resonances (i.e. eigenvalues of $\frac{1}{i}X$) in the strip

$$\{\lambda : \operatorname{Im}\lambda \geq -\delta\}$$

for some $\delta$ sufficiently small. This property is called a spectral gap. One can then derive a rigorous resonance expansion for correlations in a strip of size $\delta$. This is closely related to the resonance expansion for scattered waves (Theorem 8). Since the existence of a spectral gap governs whether correlations decay exponentially, it is a subject of intense study. In the context of Anosov flows $e^{tX}$ on manifolds, a spectral gap is known when $X$ is the generator of

- a contact Anosov flow on a smooth compact manifold, see Dolgopyat [Do98], Tsuji [Ts10] and Nonnemacher–Zworski [NZ15];
- the geodesic flow on a convex, co-compact hyperbolic surface under a pressure condition, see Patterson [Pa76], Sullivan [Su79];

- the geodesic flow on a convex, co-compact hyperbolic manifold under a relaxed pressure condition, see Patterson [Na05] for surfaces and Stoyanov [St11], Dyatlov–Zahl [DZ16] and Dyatlov–Jin [DJ17] for higher-dimensional manifolds;
- the geodesic flow on a convex, co-compact hyperbolic surface without any pressure condition, see Dyatlov–Bourgain [BD17].

**Geodesic flow on negatively curved manifolds**

We give here a particularly important example of a contact Anosov flow, induced by the geodesic flow on negatively curved manifolds. If $(\mathbb{M}, g)$ is a smooth compact Riemannian manifold, its cotangent bundle $T^*\mathbb{M}$ admits a canonical symplectic structure, i.e. a 2-form $\omega \in \Gamma(\Lambda^2 T^*\mathbb{M})$ that is non-degenerate. The Hamiltonian vector field of a function $f \in C^\infty(T^*\mathbb{M}, R)$ is the vector field on $T^*\mathbb{M}$ that is uniquely defined by

$$\forall v \in \Gamma(T(T^*\mathbb{M})),\ \omega(H_f, V) = -dp(V).$$

This is alternatively written as $i_{H_f}\omega = -dp$. If $p(x,\xi) = |\xi|_g^2$, the vector field $H_p$ is tangent to the sphere bundle $\mathscr{M} = S^*\mathbb{M}$. It generates the geodesic flow in the following sense: the geodesic $\gamma : \mathbb{R} \to \mathbb{M}$ starting at $(x_0, v_0) \in S\mathbb{M}$ is the projection onto $\mathbb{M}$ of the curve $t \mapsto e^{tH_p}(x_0, \xi_0)$, where $(x_0, \xi_0) \in S^*\mathbb{M}$ is the dual element to $(x_0, v_0) \in SM$. If $\mathbb{M}$ is negatively curved, the flow $e^{tH_p}$ on $\mathscr{M} = S^*\mathbb{M}$ is a fundamental example of an Anosov diffeomorphism as Anosov [An67] first noticed.

This flow is contact. The contact form is the restriction of the canonical one-form on $T^*\mathbb{M}$ (a smooth section of $T^*(T^*\mathbb{M})$, also called the Liouville one-form) to $\mathscr{M} = S^*\mathbb{M}$. One can check that $H_p$ defined above is the Reeb vector field of $\alpha$, i.e.

$$\alpha(H_p) = 1, \quad i_{H_p} d\alpha = 0.$$

In §2.3.1 and below, we will write $H_p = H_1$, and $|\cdot|$ for the $L^2$-norm on functions on $\mathscr{M} = S^*\mathbb{M}$ induced by the volume form $\alpha \wedge (d\alpha)^{k-1}$.

### 2.2.2 Anosov flows and microlocal analysis

The material here is mostly taken from [DZ16a, §2.1] and [DZ16d, Appendix E.5.2].

When $\mathbb{M}$ has negative curvature, $H_1$ generates an Anosov flow on $S^*\mathbb{M}$: there exists a decomposition of $TS^*\mathbb{M}$, invariant under the geodesic flow $e^{tH_1}$, of the form

$$T_x S^*\mathbb{M} = E_0(x) \oplus E_u(x) \oplus E_s(x),$$

where $E_0(x) = \mathbb{R} \cdot H_1(x)$ and $E_u(x), E_s(x)$ satisfy:

$$\begin{aligned} v \in E_u(x) &\ \Rightarrow\ |de^{tH_1}(x)v| \le Ce^{ct}|v|, \quad t < 0, \\ v \in E_s(x) &\ \Rightarrow\ |de^{tH_1}(x)v| \le Ce^{-ct}|v|, \quad t > 0. \end{aligned}$$

For $(x,\xi) \in TS^*\mathbb{M}$, let $\sigma_{H_1}(x,\xi) = \langle \xi, H_1(x)\rangle$ – a smooth function on $TS^*\mathbb{M}$. The Hamiltonian vector field $H_{\sigma_{H_1}}$ of $\sigma_{H_1}$ generates the flow $\exp(tH_{\sigma_{H_1}})$ given by

$$\exp(tH_{\sigma_{H_1}})(x,\xi) = \left(e^{tH_1}(x), (de^{tH_1}(x)^{-1})^*\xi\right).$$

Since $\sigma_{H_1}(x,\xi) = \frac{1}{i}\langle \xi, H_1(x)\rangle$ is homogeneous of degree 1 in $\xi$, $\exp(tH_{\sigma_{H_1}})$ extends to a map $\overline{T}^* S^*\mathbb{M} \to \overline{T}^* S^*\mathbb{M}$, see [DZ16a, Proposition E.5]. A radial sink (with respect to $H_{\sigma_{H_1}}$) is a $\exp(tH_{\sigma_{H_1}})$-invariant closed conic set $L \subset T^*S^*\mathbb{M} \setminus 0$ with a conical neighborhood $U$ satisfying

$$\begin{aligned} t \to +\infty \ &\Rightarrow \ \mathrm{d}(\kappa(\exp(tH_{\sigma_{H_1}})(U)), \kappa(L)) \to 0, \\ (x,\xi) \in U \ &\Rightarrow \ |\pi_\xi \exp(tH_{\sigma_{H_1}})(x,\xi)| \geq C^{-1}e^{ct}|\xi|. \end{aligned} \tag{2.2.4}$$

Here $\pi_\xi(x,\xi) = \xi$. A radial source is defined by reversing the flow direction in (2.2.4).

The decomposition $T_xS^*\mathbb{M} = E_u(x) \oplus E_0(x) \oplus E_s(x)$ induces a dual decomposition $T_x^*S^*\mathbb{M} = E_s^*(x) \oplus E_0^*(x) \oplus E_u^*(x)$. *Note that in this notation, $E_s^*(x)$ is the dual of $E_u(x)$ and $E_u^*(x)$ is the dual of $E_s^*(x)$.* The stable and unstable foliations of Anosov flows are related to the radial source and sinks as follows: $E_s^* \setminus 0$ is a radial source and $E_u^* \setminus 0$ is a radial sink, see [DZ16a, §2.3].

As mentioned above, Pollicott–Ruelle resonances are dynamical quantities associated to $\mathbb{M}$, that quantify the decay of classical correlations, see [Ts10, Corollary 1.2] and [NZ15, Corollary 5]. These numbers can also be realized as eigenvalues of $\frac{1}{i}H_1$ on specifically designed Sobolev spaces. They are the poles of the meromorphic continuation of the Fredholm family of operators $(P_0-\lambda)^{-1} = (\frac{1}{i}H_1-\lambda)^{-1} : C^\infty \to \mathscr{D}'$, where $\mathscr{D}'$ is the set of distributions on $S^*\mathbb{M}$. The poles of $(P_0-\lambda)^{-1}$ have finite rank; the multiplicity of a pole $\lambda_0 \in \mathbb{C}$ is $\mathrm{rank}(\Pi_{\lambda_0})$, where

$$\Pi_{\lambda_0} \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\partial\mathbb{D}(\lambda_0,r_0)} (P_0-\lambda)^{-1} d\lambda \tag{2.2.5}$$

and $r_0$ is small enough so that $\lambda_0$ is the unique pole of $(P_0-\lambda)^{-1}$ on $\mathbb{D}(\lambda_0,r_0)$. In order to investigate further the residues of $(P_0-\lambda)^{-1}$, we recall that one can associate to each $u \in \mathscr{D}'$ a conical set $\mathrm{WF}(u)$, called the classical wavefront set, which measures in phase space where $u$ is not smooth. We refer to [GS94, §7] for precise definitions. For $\Gamma \subset T^*S^*\mathbb{M}$ a conical set, let $\mathscr{D}'_\Gamma$ be the set of distributions with classical wavefront set contained in $\Gamma$.

**Lemma 2.2.1.** *If $\lambda_0$ is a simple Pollicott–Ruelle resonance of $P_0 = \frac{1}{i}H_1$, there exist $u \in \mathscr{D}'_{E_u^*}, v \in \mathscr{D}'_{E_s^*}$ and a holomorphic family of operators $A(\lambda)$ defined near $\lambda_0$, with*

$$(P_0-\lambda)^{-1} = \frac{u\otimes v}{\lambda-\lambda_0} + A(\lambda).$$

*Proof.* According to [DZ16a, Proposition 3.3], the operator $\Pi_{\lambda_0}$ defined in (2.2.5) is equal to $u\otimes v$, where $\mathrm{WF}(u) \subset E_u^*$, $\mathrm{WF}(v) \subset E_s^*$; and there exist $J > 0$ and a family of operators $A(\lambda) : C^\infty \to \mathscr{D}'$ holomorphic near $\lambda_0$ such that

$$(P_0-\lambda)^{-1} = A(\lambda) + \sum_{j=1}^{J} \frac{(P_0-\lambda_0)^{j-1}\Pi_{\lambda_0}}{(\lambda-\lambda_0)^j}. \tag{2.2.6}$$

By the same argument as in the proof of [DZ16d, Theorem 2.4] the operator $P_0 - \lambda_0$ maps $\mathrm{Range}(\Pi_{\lambda_0})$ to itself and $(P_0 - \lambda_0)|_{\mathrm{Range}(\Pi_{\lambda_0})}$ is nilpotent. Since $\mathrm{Range}(\Pi_{\lambda_0})$ has dimension 1, $(P_0 - \lambda)|_{\mathrm{Range}(\Pi_{\lambda_0})}$ is equal to 0 and the index $J$ in (2.2.6) is equal to 1. □

In [DZ16a] the meromorphic continuation of $(P_0 - \lambda)^{-1}$ is realized via analytic Fredholm theory. Therefore, Pollicott–Ruelle resonances of $P_0$ are identified with the roots of a suitable Fredholm determinant, see [DZ15, Proposition 3.2].

### 2.2.3 Semiclassical analysis

We describe here a refinement of microlocal analysis, called semiclassical analysis. This theory not only captures the lack of regularity of distributions at a point and a direction, but also the oscillations at scale $h^{-1}$, where $h$ is small. For the sake of simplicity, our presentation is axiomatic.

#### Symbolic calculus

Symbols are smooth functions on $T^*\mathscr{M}$ that satisfy decay conditions sufficient to associate $h$-canonical operator – this process is called quantization. Specifically, a symbol of order $m \in \mathbb{R}$ on $\mathscr{M}$ is a smooth function $a : T^*\mathscr{M} \times (0,1] \to\to \mathbb{C}$ that satisfies

$$\forall \alpha, \beta, \ \exists C_{\alpha\beta} > 0, \ \forall 0 < h < 1, \ \sup_{(x,\xi)\in T^*\mathscr{M}} \langle \xi \rangle_g^{m-|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)| \leq C_{\alpha\beta}.$$

In the above, $g$ is any dual metric on $\mathscr{M}$ (i.e. definitive positive section of $T\mathscr{M} \otimes T\mathscr{M}$). The operators $\partial_x, \partial_\xi$ in the above are taken using local coordinates. The dependence in $h$ of $a$ is often dismissed: we write $a(x,\xi)$ for $a(x,\xi,h)$. Semiclassical pseudodifferential operators of order $m$ are operators on $C^\infty(\mathscr{M})$ that are realized as $h$-quantization of symbols in $S^m$. They form an algebra, denoted $\Psi_h^m$.

Conversely, to each $A \in \Psi_h^m$ we can associate a unique symbol equivalence class $\sigma(A) \in S^m/hS^{m-1}$ – called the principal symbol of $A$. The map

$$\sigma_m : A \in \Psi_h^m \to \sigma_m(A) \in S^m/hS^{m-1}$$

induces a morphism of $\mathbb{C}^*$-algebra, whose kernel is given by operators in $\Psi_h^{m-1}$. In particular, the principal symbol of differential operators on $\mathscr{M}$ is an invariantly defined class of equivalence in $S^{m-1}$. Some concrete applications are described below:

- If $A \in \Psi_h^m$, then $A^* \in \Psi_h^m$, and $\sigma(A^*) = \overline{\sigma(A)} \in S^m/hS^{m-1}$.
- If $A \in \Psi_h^m$ and $B \in \Psi_h^n$, $A \circ B \in \Psi_h^{m+n}$ and has principal symbol $\sigma_m(A)\sigma_n(B) \in S_h^{m+n}/hS_h^{m+n-1}$.

- If $A \in \Psi_h^m$ is such that $\sigma_m(A)$ has a non-vanishing represent $a \in S^m$, then we can construct $B \in \Psi_h^{-m}$ such that $AB - \mathrm{Id} \in h\Psi_h^{-1}$ and $BA - \mathrm{Id} \in h\Psi_h^{-1}$. Indeed, one can check that $a^{-1} \in S^{-m}$ and define $B = \mathrm{Op}_h(a^{-1}) \in \Psi_h^{-m}$. The operator $AB - \mathrm{Id}$ lies in $\Psi_h^0$, and $\sigma_0(AB - \mathrm{Id}) = 0 \in S^0/hS^{-1}$. Hence, $h^{-1}(AB - \mathrm{Id}) \in \ker(\sigma_0) = \Psi_h^{-1}$, as claimed.

- If $A \in \Psi_h^m$ and $B \in \Psi_h^n$, $[A, B] = AB - BA \in h\Psi_h^{m+n-1}$: indeed, $\sigma_m(AB - BA) = 0 \in S^{m+n}/hS^{m+n-1}$ which implies that $h^{-1}[A, B] \in \Psi_h^{m+n-1}$. In addition, the principal symbol of $[A, B] \in \Psi_h^{m+n-1}$ can be computed using Poisson brackets: if $a, b$ are representant of $\sigma_m(A)$, $\sigma_n(B)$ respectively, then a representant of $\sigma_{m+n-1}([A, B])$ is

$$\frac{h}{i}\{a, b\} = \frac{h}{i}H_a b \quad \text{– here } H_a \text{ is the Hamiltonian vector field of } a \in C^\infty(T^*\mathscr{M}).$$

  This formula is very important because it connects quantum dynamics - in the Heisenberg picture, the evolution of $B$ under $A$ is $e^{tA}Be^{-tA} = A + t[A, B] + O(t^2)$ – with classical dynamics – which evolves according to the Hamiltonian flow.

When there is no possible confusion, we will write $\sigma(A)$ instead of $\sigma_m(A)$ for the principle symbol of an operator $A \in \Psi_h^m$. We will also write $\sigma(A) = a \mod hS^{m-1}$ if $a$ is a representant of $\sigma(A)$. For instance, if $X$ is a smooth vector field on $\mathscr{M}$, then $\frac{h}{i}X \in \Psi_h^1$ and

$$\sigma\left(\frac{h}{i}X\right)(x, \xi) = \langle \xi, X(x) \rangle \mod hS^0.$$

Similarly, if $\Delta \geq 0$ denotes the nonnegative Laplace–Beltrami operator on $(\mathscr{M}, g)$, then $h^2\Delta \in \Psi_h^2$ and

$$\sigma(h^2\Delta) = |\xi|_g^2 \mod hS^1.$$

**Elliptic set and wavefront set**

The elliptic and wavefront sets of a pseudodifferential operator $A$ on $\mathscr{M}$ are two invariantly defined objects, that measure where $A$ is microlocally invertible and microlocally significant, respectively. They are defined here as subsets of the radial compactification $\overline{T}^*\mathscr{M}$ of $T^*\mathscr{M}$, which this is a smooth manifold with interior $T^*\mathscr{M}$ and boundary $S^*\mathscr{M}$ associated with the map

$$\kappa : (x, \xi) \in T^*\mathscr{M} \setminus 0 \to \left(x, \frac{\xi}{|\xi|_g}\right) \in \partial S^*\mathscr{M} \equiv \overline{T}^*\mathscr{M}. \tag{2.2.7}$$

In the above $g$ is any dual metric on $\mathscr{M}$. We will most importantly care about the topology of $\mathscr{M}$; a basis of open sets is concretely given by the open sets in $T^*\mathscr{M}$, together with the sets $\kappa^{-1}(U) \setminus K$, where $U$ is an open set in $S^*\mathscr{M} \equiv \partial\overline{T}^*\mathscr{M}$ and $K$ is a compact subset of $T^*\mathscr{M}$.

Given an operator $A \in \Psi_h^m$, a point $(x_0, \xi_0) \in \overline{T}^*\mathscr{M}$ belongs to the elliptic set of $A$ – denoted $\mathrm{Ell}_h(A)$ – if there exists a neighborhood $U$ of $(x_0, \xi_0)$ in $\overline{T}^*\mathscr{M}$ such that

$$\sigma(A) = a \mod hS^{m-1} \Rightarrow \inf_{(x,\xi)\in T^*\mathscr{M}\cap U, h\in(0,1)} |\langle\xi\rangle^{-m} a(x,\xi)| > 0. \tag{2.2.8}$$

(one can easily verify that this definition is independent of the choice of $a$ nor of the dual metric on $\mathscr{M}$). Hence, $A$ is elliptic at $(x_0, \xi_0) \in T^*\mathscr{M}$ if and only if $a(x_0, \xi_0) > 0$, and (roughly speaking) elliptic at $(x_0, \xi_0) \in \partial\overline{T}^*\mathscr{M}$ if and only if $a(x_0, \xi)$ grows like $\langle\xi\rangle^m$ in the direction $\xi_0$. The characterization (2.2.8) shows that the elliptic set is an open subset of $\overline{T}^*\mathscr{M}$.

We define the semiclassical wavefront set of $A$ – denoted $\mathrm{WF}_h(A)$ – by its complement: a point $(x_0, \xi_0) \notin \mathrm{WF}_h(A)$ if and only if there exists $U$ neighborhood of $(x_0, \xi_0)$ in $\overline{T}^*\mathscr{M}$ such that

$$\sigma(A) = a \mod hS^{m-1} \Rightarrow \forall m' \in \mathbb{R}, \sup_{(x,\xi)\in T^*\mathscr{M}\cap U, h\in(0,1)} |\langle\xi\rangle^{m'} a(x,\xi)| < \infty.$$

Roughly speaking, a point $(x_0, \xi_0)$ belongs to $\mathrm{WF}_h(A)$ if $A$ is not semiclassically negligible at $(x_0, \xi_0)$. The wavefront set of an operator $A$ is a closed subset of $\overline{T}^*\mathscr{M}$.

### Semiclassical estimates

The wavefront set and the elliptic sets of an operator $A$ carry a lot of essential information about $A$. This is most importantly seen in semiclassical estimates. These can be seen as general black-box estimates that require only dynamical information on the behavior of the elliptic and wavefront sets under the classical flow. They are often stated in semiclassical Sobolev spaces $H_h^s = \Lambda_{-s} L^2(\mathscr{M})$, where $\Lambda_s = (\mathrm{Id} + h^2\Delta)^{s/2} \in \Psi_h^s$. Roughly speaking, they quantize (in a non trivial way) facts available in the classical case (that are sometimes trivial).

We start with the microlocal partition lemma, which quantize the following classical statement: let $f, f_1, ..., f_n$ be complex-valued functions on $\overline{T}^*\mathscr{M}$ such that

$$\mathrm{supp}(f) \subset \bigcup_{i=1}^n \{x \in \overline{T}^*\mathscr{M} : |f_i(x)| > 0\}$$

Then there exists $C > 0$ such that

$$\forall x \in \overline{T}^*\mathscr{M}, \quad |f(x)| \leq C \sum_{i=1}^n |f_i(x)|.$$

**Lemma 2.2.2.** *Assume that $A, A_1, ..., A_n$ are pseudodifferential operators in $\Psi_h^0$ such that*

$$\mathrm{WF}_h(A) \subset \bigcup_{i=1}^n \mathrm{Ell}_h(A).$$

*Then for any $N, s$, there exists a constant $C$ such that*

$$u \in H_h^s \;\Rightarrow\; |Au|_{H_h^s} \leq C_N \sum_{i=1}^n |A_i u|_{H_h^s} + O(h^\infty)|u|_{H_h^{-N}}$$

This estimates allows us to separate possible difficulties. It shows that to control $u$ microlocally on $W_1 \cup W_2$, it suffices to control $u$ microlocally on $W_1$ and on $W_2$. The proof relies on the following estimate – which can also be seen as a corollary:

**Lemma 2.2.3.** *Let $A \in \Psi_h^0$, $B \in \Psi_h^0$ such that $\mathrm{WF}_h(A) \subset \mathrm{Ell}_h(B)$. Then, for any $N, s$, there exists a constant $C$ such that*

$$u \in H_h^s \;\Rightarrow\; |Au|_{H_h^s} \leq C|Bu|_{H_h^s} + O(h^{-N})|u|_{H_h^{-N}}.$$

A standard proof of this lemma constructs a parametrix for the operator $B$ on $\mathrm{WF}_h(A)$, i.e. an operator $Q \in \Psi_h^0$ such that $\mathrm{WF}_h(BQ - \mathrm{Id}) \cap \mathrm{WF}_h(A) = \mathrm{WF}_h(QB - \mathrm{Id}) \cap \mathrm{WF}_h(A) = \emptyset$. Under this condition,

$$A(\mathrm{Id} - QB) \in h^\infty \Psi_h^{-\infty} \;\Rightarrow\; Au = QBu + h^\infty \Psi_h^{-\infty} u \;\Rightarrow\; |Au|_{H_h^s} \leq C|Bu|_{H_h^s} + O(h^{-N})|u|_{H_h^{-N}}.$$

The construction of the parametrix is realized via an iterative scheme. The first step quantizes $\sigma(B)^{-1} \in S^0/hS^{-1}$ on the conical set $\mathrm{WF}_h(A)$, hence it relies strongly on the assumption $\mathrm{WF}_h(A) \subset \mathrm{Ell}_h(B)$.

A more sophisticated inequality is due to Gårding. It quantizes in a rather subtle way the inequality $a \geq 0$.

**Lemma 2.2.4.** *Let $A \in \Psi_h^0$ with principal symbol $a \mod hS^{-1}$, such that $\mathrm{Re}(a) \geq 0$. Then there exists $C > 0$ such that*

$$\langle Au, u \rangle_{H_h^s} \geq -Ch|u|^2_{H_h^{s-1}}.$$

These estimates are static in the sense that they do not require any assumption about the classical dynamics. The next inequality, due to Duistermaat–Hörmander, is called the semiclassical propagation estimate.

**Lemma 2.2.5.** *Let $P \in \Psi_h^1$ have principal symbol $p - iq \mod hS^0$, where $p$ is real valued, independent of $h$, and homogeneous for $\xi$ sufficiently large; and $q \geq 0$. Assume that $A, B, B_1 \in \Psi_h^0$ are such that*

$$e^{-TH_p}(\mathrm{WF}_h(A)) \subset \mathrm{Ell}_h(B), \quad \forall t \in [0,T],\; e^{-tH_p}(\mathrm{WF}_h(A)) \subset \mathrm{Ell}_h(B_1).$$

*Then for any $N, s$, there exists $C > 0$ such that*

$$u \in H_h^s \;\Rightarrow\; |Au|_{H_h^s} \leq C|Bu|_{H_h^s} + Ch^{-1}|B_1 Pu|_{H_h^s} + O(h^N)|u|_{H^{-N}}. \tag{2.2.9}$$

We briefly explain the proof of the estimate when $q = 0$ and $Pu = 0$. It relies on the construction of a function $f$ that satisfies $f > 0$ on $\mathrm{supp}(a)$, $f \geq 0$ everywhere and $H_p f \leq -cf$ – here $\sigma(A) = a \mod hS^{-1}$. In particular, $a \leq Cf|_{\mathrm{supp}(a)}$ (because $f > 0$ on $\mathrm{supp}(a)$) and $f|_{\mathrm{supp}(a)} \leq Cb$ (because $f$ decreases along the flow lines of $H_p$ and $e^{-TH_p}(\mathrm{WF}_h(A)) \subset \mathrm{Ell}_h(B)$). Hence, $a \leq Cb$ which yields (2.2.9) after a few additional details.

This brief explanation shows that the constant $C$ in (2.2.9) does not depend uniformly on $T$: indeed, if $f > 0$ on $\mathrm{supp}(a)$ and $H_p f \leq -cf$, then the argument shows $f|_{\mathrm{supp}(a)} \leq C_T b$ with $C_T$ a priori growing exponentially with $T$. Therefore, the proof cannot be adapted to the situation $T \to \infty$. Further dynamical assumptions are needed.

Let $X$ be a vector field on $\mathscr{M}$, with (representant of its) principal symbol $\sigma_X(x,\xi) = \langle \xi, X(x) \rangle$. Since $\sigma_X$ is homogeneous of degree 1 in $\xi$, the Hamiltonian flow $\exp(tH_{\sigma_X}) : T^*\mathscr{M} \to T^*\mathscr{M}$ naturally extends as a flow $\overline{T}^*\mathscr{M} \to \overline{T}^*\mathscr{M}$. A radial sink (with respect to $H_{\sigma_X}$) is a closed conic set $L \subset T^*\mathscr{M} \setminus 0$ that is invariant under $\exp(tH_{\sigma_X})$ and such that there exists a conical neighborhood $U$ of $L$ in $T^*\mathscr{M}$ with:

$$\begin{aligned} t \to +\infty \ &\Rightarrow \ \mathrm{d}(\kappa(\exp(tH_{\sigma_X})(U)), \kappa(L)) \to 0, \\ (x,\xi) \in U \ &\Rightarrow \ |\pi_\xi \exp(tH_{\sigma_X})(x,\xi)|_g \geq C^{-1} e^{ct} |\xi|_g. \end{aligned} \tag{2.2.10}$$

Here $\pi_\xi(x,\xi) = \xi$ and $g$ is a dual metric on $T^*\mathscr{M}$. A radial source is defined by reversing the flow direction in (2.2.10). Intuitively, a radial source (respectively sink) is a set where flow lines of $H_{\sigma_X}$ tend to concentrate in the far past (respectively future). Their definition depend on the long-times behavior of the flow. They appear naturally in relation with Anosov flows: if $X$ is an Anosov vector field on $\mathscr{M}$, then the invariant splitting in unstable, invariant and stable directions

$$T_x\mathscr{M} = E_u(x) \oplus \mathbb{R} \cdot X(x) \oplus E_s(x)$$

induces a dual splitting

$$T_x^*\mathscr{M} = E_s(x)^* \oplus (\mathbb{R} \cdot X(x))^* \oplus E_s^*(x), \qquad E_s^*(x) = E_u(x), \quad E_u^*(s) = E_s^*(x).$$

In this notation, $\overline{E_s^*(x)}$ (the closure realized for the topology of $\overline{T}^*\mathscr{M}$) is a radial source and $\overline{E_u^*(x)}$ is a radial sink. These sets also appear in relation with interior and horizons of black-holes in general relativity, see [Va13, Zw16] but this will not be developed further here.

Radial sources and sinks provide a convenient framework to state estimates similar to Lemma 2.2.5 in the limit $T \to \infty$. They were introduced in work of Melrose [Me94] and developed further in Vasy [Va13] and Dyatlov–Zworski [DZ16a].

**Lemma 2.2.6.** *Let $P \in \Psi_h^1$ have principal symbol $p - iq \mod hS^0$, where $p$ is real valued, independent of $h$, and homogeneous for $\xi$ sufficiently large; and $q \geq 0$. Let $L$ be a radial source with respect to $H_p$. Then, there exists $s_0 > 0$ such that for any $B_1 \in \psi_h^0$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0$ elliptic on $\kappa(L)$ with*

$$\forall N, s \geq s_0, \ u \in H_h^s, \quad |Au|_{H_h^s} \leq Ch^{-1} |B_1 Pu|_{H_h^s} + O(h^\infty)|u|_{H_h^{-N}}.$$

This estimates shows that one can control $u$ microlocally near $\kappa(L)$ provided that $Pu$ is sufficiently regular. Since radial sources and radial sinks are dual to one another, we naturally expect a similar estimate near radial sinks:

**Lemma 2.2.7.** *Let $P \in \Psi_h^1$ have principal symbol $p - iq \mod hS^0$, where $p$ is real valued, independent of $h$, and homogeneous for $\xi$ sufficiently large; and $q \geq 0$. Let $L$ be a radial sink with respect to $H_p$. Then, there exists $s_0 < 0$ such that for any $B_1 \in \psi_h^0$ elliptic on $\kappa(L)$, there exists $A \in \Psi_h^0$ elliptic on $\kappa(L)$ and $B \in \Psi_h^0$ with $\mathrm{WF}_h(B) \subset \mathrm{Ell}_h(B_1) \setminus \kappa(L)$, such that*

$$\forall N, s \leq s_0,\ u \in H_h^s,\quad |Au|_{H_h^s} \leq |Bu|_{H_h^s} + Ch^{-1}|B_1 Pu|_{H_h^s} + O(h^\infty)|u|_{H_h^{-N}}.$$

**The class $S = S^{0,0}$**

In §2.4.3 only, we will need a slightly more exotic class of symbols, in the context of the flat based space $\mathbb{R}^n, n > 0$. This class, first introduced by Hörmander as $S^{0,0}$ and denoted by $S$ here contains all functions on $T^*\mathbb{R}^n$ such that

$$\forall \alpha, \beta,\ \exists C_{\alpha\beta} > 0, \forall 0 < h < 1, \sup_{(x,\xi) \in T^*\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta}.$$

The quantization of symbols in $S$ forms an algebra $\Psi_h$. This algebra is *not* invariant under change of variables. In the class $\Psi_h$, the remainders in the composition formula are smaller than the leading part, but they are not more smoothing – in contrast with $\Psi_h^0$. We will use this class exclusively in §2.4.3. Our basic reference for such operators is [Zw12, Chapter 4].

## 2.3 Random perturbations of the geodesic equation

### 2.3.1 The operator $P_\varepsilon$

For every $z \in \mathbb{M}$, the fiber $T_z^*\mathbb{M}$ is a Euclidean space and we can see $S_x^*\mathbb{M}$ as a Riemannian submanifold of $T_z^*\mathbb{M}$ – provided with the induced metric. The non-negative Laplacian on $S_z^*\mathbb{M}$ is a differential operator $\Delta_{\mathbb{S}}(z) : C^\infty(S_x^*\mathbb{M}) \to C^\infty(S_x^*\mathbb{M})$. Varying $z$ we obtain a differential operator $\Delta_{\mathbb{S}} : C^\infty(S_x^*\mathbb{M}) \to C^\infty(S_x^*\mathbb{M})$ called the spherical vertical Laplacian. Similarly we can define a spherical vertical gradient operator $\nabla_{\mathbb{S}} : C^\infty(S^*\mathbb{M}) \to C^\infty(TS^*\mathbb{M})$ by first freezing $x$ and taking $\nabla_{\mathbb{S}}(x)$ the gradient operator on the Riemannian manifold $S_z^*\mathbb{M}$, then by varying $z$.

For every $z \in \mathbb{M}$, the fiber $T_z^*\mathbb{M}$ admits a Euclidean structure and the fiber $S_z^*\mathbb{M}$, provided with the induced metric, is a Riemannian submanifold of $T_z^*\mathbb{M}$. The *non-negative* Laplacian on $S_z^*\mathbb{M}$ is a differential operator $\Delta_{\mathbb{S}}(z) : C^\infty(S_z^*\mathbb{M}) \to C^\infty(S_z^*\mathbb{M})$. Varying $z$ we obtain a differential operator $\Delta_{\mathbb{S}} : C^\infty(S_z^*\mathbb{M}) \to C^\infty(S_z^*\mathbb{M})$ called the spherical vertical Laplacian. Similarly there is a spherical vertical gradient operator $\nabla_{\mathbb{S}} : C^\infty(S^*\mathbb{M}) \to C^\infty(TS^*\mathbb{M})$, defined on each fiber $S_z^*\mathbb{M}$ as the standard gradient. We will see below that $\Delta_{\mathbb{S}}$ is selfadjoint with respect to the Liouville measure on $S^*\mathbb{M}$.

Let $P_\varepsilon$ be the operator

$$P_\varepsilon \stackrel{\text{def}}{=} \frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}}) = \frac{1}{i}H_1 - i\varepsilon\Delta_{\mathbb{S}},$$

with $L^2$-domain $D(P_\varepsilon) \stackrel{\text{def}}{=} \{u \in L^2 : \ P_\varepsilon u \in L^2\}$ – here $P_\varepsilon u$ is seen as a distribution. Angst–Bailleul–Tardif [ABT15] call $P_\varepsilon$ the generator of *kinetic Brownian motion*. In §2.3.2 below we compute certain Lie brackets, showing that $P_\varepsilon$ satisfies Hörmander's condition [Hö67] for hypoellipticity. The Rothschild–Stein theory of hypoelliptic operators [RS76, §18] yields the subelliptic estimate (2.3.13): there exists a constant $c_\varepsilon > 0$ such that $|u|_{H^{2/3}} \leq c_\varepsilon(|P_\varepsilon u| + |u|)$. A significant part of this chapter, §2.4, studies the behavior of $c_\varepsilon$ as $\varepsilon \to 0$ when $H^{2/3}$ is replaced by its semiclassical version $H_\varepsilon^{2/3}$.

This chapter studies the accumulation points as $\varepsilon \to 0$ of the $L^2$-eigenvalues of $P_\varepsilon$ when $\mathbb{M}$ has negative curvature and is provided with the Liouville measure. For the sake of completeness, we prove here the following standard result:

**Lemma 2.3.1.** *The operator $P_\varepsilon$ has discrete $L^2$-spectrum.*

*Proof.* This is a general statement that uses that $P_\varepsilon$ is subelliptic and $\mathrm{Im}(P_\varepsilon) \leq 0$. We first show that the operator $P_\varepsilon - \lambda$ is injective for $\mathrm{Im}\,\lambda > 0$. We have

$$\begin{aligned} \mathrm{Im}(\langle (P_\varepsilon - \lambda)u, u\rangle) &= \mathrm{Im}\left(\langle -i(H_1 + \varepsilon\Delta_{\mathbb{S}} - \lambda)u, u\rangle\right) = \mathrm{Im}(-i\varepsilon|\nabla_{\mathbb{S}} u|^2) - \mathrm{Im}\,\lambda|u|^2 \\ &= -\varepsilon|\nabla_{\mathbb{S}} u|^2 - \mathrm{Im}\,\lambda|u|^2 \leq -\mathrm{Im}\,\lambda|u|^2. \end{aligned} \tag{2.3.1}$$

In particular $P_\varepsilon - \lambda$ is injective on $L^2$ as long as $\mathrm{Im}\,\lambda > 0$.

We next show that $P_\varepsilon - \lambda$ is surjective for $\mathrm{Im}\,\lambda > 0$. The graph of $P_\varepsilon - \lambda$ is closed in $L^2 \times L^2$: indeed assume that $u_n \in D(P_\varepsilon)$ satisfies $f_n = (P_\varepsilon - \lambda)u_n \to f \in L^2$ and $u_n \to u \in L^2$. Then $(P_\varepsilon - \lambda)u_n \to (P_\varepsilon - \lambda)u$ in the sense of distribution, which shows that $(P_\varepsilon - \lambda)u = f$ and hence $P_\varepsilon - \lambda$ has a closed graph. An estimate similar to (2.3.1) shows that

$$\begin{aligned} \mathrm{Im}(\langle (P_\varepsilon - \lambda)^* u, u\rangle) &= \mathrm{Im}\left(\langle i(-H_1 + \varepsilon\Delta_{\mathbb{S}} - \overline{\lambda})u, u\rangle\right) = \mathrm{Im}(i\varepsilon|\nabla_{\mathbb{S}} u|^2) - \mathrm{Im}\,\overline{\lambda}|u|^2 \\ &= \varepsilon|\nabla_{\mathbb{S}} u|^2 + \mathrm{Im}\,\lambda|u|^2 \geq \mathrm{Im}\,\lambda|u|^2. \end{aligned}$$

Hence, $(P_\varepsilon - \lambda)^*$ is also injective. Since $P_\varepsilon - \lambda$ has a closed graph,

$$\{0\} = \ker((P_\varepsilon - \lambda)^*) = \overline{R(P_\varepsilon - \lambda)}^{\perp}.$$

This shows that the range of $P_\varepsilon - \lambda$ is dense in $L^2$. It remains to prove that this range is also closed. If $f_n = (P_\varepsilon - \lambda)u_n \to f \in L^2$ then (2.3.1) shows that $u_n$ remains bounded in $L^2$. Hence possibly after passing to a subsequence it converges weakly to some $u$ in $L^2$. In the sense of distribution we must have $(P_\varepsilon - \lambda)u = f$. This shows that the range of $P_\varepsilon - \lambda$ is closed and dense in $L^2$, hence it is $L^2$.

We deduce that the operator $P_\varepsilon - \lambda$ is invertible when $\mathrm{Im}\,\lambda > 0$. We now show show that the operator $P_\varepsilon - \lambda$ is Fredholm, i.e. that both its kernel and cokernel are finite-dimensional.

We recall the subelliptic estimate of [RS76]: there exists a constant $C_\varepsilon$ such that for every $u \in D(P_\varepsilon)$,

$$|u|_{H^{2/3}} \leq C(|P_\varepsilon u| + |u|).$$

Every element $u$ lying in the kernel of $P_\varepsilon - \lambda$ satisfies $|u|_{H^{2/3}} \leq (C + |\lambda|)|u|$. This shows that the unit ball of $\ker(P_\varepsilon - \lambda)$ is contained in a bounded subset of $H^{2/3}$, hence it is compact in $L^2$, thus finite-dimensional. This proves that $\ker(P_\varepsilon - \lambda)$ is finite-dimensional. The same argument shows that $\ker((P_\varepsilon - \lambda)^*)$ is finite dimensional.

Hence $P_\varepsilon - \lambda$ is an analytic family of Fredholm operators with index 0. Analytic Fredholm theory (see for instance [DZ16d, Appendix C]) show that their resolvent $(P_\varepsilon - \lambda)^{-1} : L^2 \to L^2$ initially defined for $\operatorname{Im} \lambda > 0$ meromorphically continues to the whole complex plane. The set of (discrete) poles forms the $L^2$-spectrum of $P_\varepsilon - \lambda$, which proves the lemma. □

### 2.3.2 Operators on frame bundles

This section reviews Cartan's lifting process from the cosphere bundle $S^*\mathbb{M}$ to the bundle of orthonormal frames $O^*\mathbb{M}$. Angst–Bailleul–Tardif [ABT15] and Li [Li16] previously used it to show asymptotic of kinetic Brownian motion in the limits $\varepsilon \to 0, \infty$. We mention that when $\mathbb{M}$ is an orientable surface, $O^*\mathbb{M} \equiv S^*\mathbb{M} \times \{\pm 1\}$ and this lifting process is unnecessary. This simplifies the technical aspects in the earlier version [Dr16a] of this work.

#### Horizontal and vertical vector fields

The space of frames at $z \in \mathbb{M}$ – denoted $\mathscr{F}_z^*\mathbb{M}$ – is the vector space of linear maps $\zeta : \mathbb{R}^d \to T_z^*\mathbb{M}$. At this point $\zeta$ is not required to be orthogonal nor an invertible. The space $\mathscr{F}_z^*\mathbb{M}$ is a Euclidean when provided with the scalar product $(\zeta, \zeta') \mapsto \operatorname{Tr}(\zeta^*\zeta')$. Varying the base point $z$ we obtain a vector bundle $\mathscr{F}^*\mathbb{M}$ over $\mathbb{M}$ which admits a Riemannian structure.

For $(z_0, \zeta_0) \in \mathscr{F}^*\mathbb{M}$, a vector $X_0 \in T_{z_0,\zeta_0}\mathscr{F}^*\mathbb{M}$ is said to be vertical if $X_0$ is tangent to the fiber $\mathscr{F}_{z_0}^*\mathbb{M}$. A smooth vector field $X \in T\mathscr{F}^*\mathbb{M}$ is vertical if $X(z_0, \zeta_0)$ is vertical for all $(z_0, \zeta_0) \in \mathscr{F}^*\mathbb{M}$. A curve $t \mapsto (z_t, \zeta_t) \in \mathscr{F}^*\mathbb{M}$ is said to be horizontal if for all $e \in \mathbb{R}^d$, $\zeta_t(e)$ (which belongs to $T_{z_t}\mathbb{M}$) is parallel along $z_t$ with respect to the Levi–Civita connection. A vector $X_0 \in T_{z_0,\zeta_0}\mathscr{F}^*\mathbb{M}$ is horizontal if there exists a horizontal curve $(z_t, \zeta_t)$ with $\partial_t(z_t, \zeta_t)(0) = X_0$; a smooth vector field $X \in T\mathscr{F}^*\mathbb{M}$ is horizontal if $X(z, \zeta)$ is horizontal for every $(z, \zeta) \in \mathscr{F}^*\mathbb{M}$.

The bundle of orthonormal frames $O^*\mathbb{M}$ is the subbundle of $\mathscr{F}^*\mathbb{M}$ with fibers formed of orthogonal maps $\zeta : \mathbb{R}^d \to T_z^*\mathbb{M}$. Since parallel transport preserves angles, the Levi–Civita derivative of an orthogonal frame along a curve is still an orthogonal frame. Vertical and horizontal vector fields in $TO^*\mathbb{M}$ are defined similarly as before. We also observe that $O^*\mathbb{M}$ is a bundle over $S^*\mathbb{M}$, provided with the projection $\pi_{\mathbb{S}} : (z, \zeta) \mapsto (z, \zeta(e_1))$, where $e_1 = (1, 0, ..., 0) \in \mathbb{R}^d$.

Geodesics on $\mathbb{M}$ are identified with integral curves of the vector field $H_1$ defined in §2.3.1; the geodesic flow is then $\exp(tH_1)$. The vector field $H_1$ lifts to a horizontal vector field $\widetilde{H}_1$ on $O^*\mathbb{M}$ defined as follows. Fix $(z_0, \zeta_0) \in O^*\mathbb{M}$ and let $(z_0, \zeta_0^1) = (z_0, \zeta_0(e_1))$ be its projection on

$S^*\mathbb{M}$; let $(z_t, \zeta_t^1) = \exp(tH_1)(z_0, \zeta_0(e_1))$ be the geodesic starting at $(z_0, \zeta_0^1)$. Parallel transport of $\zeta_0$ along $z_t$ yields a flow $(z_t, \zeta_t) = \Phi_t(z_0, \zeta_0)$ on $\mathscr{F}^*\mathbb{M}$. Since the parallel transport preserves angles this flow actually takes values in $O^*\mathbb{M}$. As $(z_t, \zeta_t^1)$ is a geodesic, $\zeta_t^1 = \zeta_t(e_1)$ is the parallel transport of $\zeta_0^1$ along $z_t$ hence $\zeta_t(e_1) = \zeta_t^1$. This shows that $(z_t, \zeta_t)$ is a lift of $(z_t, \zeta_t^1)$ to the orthogonal frame bundle. The vector field $\widetilde{H}_1 \in TO^*\mathbb{M}$ is the generator of $\Phi_t$:

$$\widetilde{H}_1(z_0, \zeta_0) \overset{\text{def}}{=} \left.\frac{d}{dt}\right|_{t=0} \Phi_t(z_0, \zeta_0).$$

The integral curves of $\widetilde{H}_1$ are horizontal, which shows that $\widetilde{H}_1$ is horizontal.

Let $E_{k\ell}$ be the matrix $E_{k\ell} \overset{\text{def}}{=} (\delta_{ki}\delta_{j\ell})_{ij}$ and $A_{k\ell}$ be the anti-symmetric matrix $A_{k\ell} \overset{\text{def}}{=} E_{k\ell} - E_{\ell k}$. The matrix $e^{tA_{k\ell}}$ is orthogonal and $V_{k\ell}$ is the vector field on $O^*\mathbb{M}$ given by

$$V_{k\ell}(z, \zeta) = \left.\frac{d}{dt}\right|_{t=0} \left(z, \zeta \circ e^{tA_{k\ell}}\right).$$

Since the projection of $(z, \zeta \circ e^{tA_{k\ell}})$ on $\mathbb{M}$ does not depend on $t$ the vector fiels $V_{k\ell}$ are vertical. The brackets of $\widetilde{H}_1$ with $V_{1k}$ define new vector fields on $O^*\mathbb{M}$: $\widetilde{H}_k \overset{\text{def}}{=} [\widetilde{H}_1, V_{1k}]$.

### Expression in coordinates

A system of coordinates $z_m \in \mathbb{R}^d$ on $\mathbb{M}$ lifts canonically to a system of coordinates $(z_m, \zeta_j^1)$ on $T^*\mathbb{M}$. If $(z, \zeta) \in \mathscr{F}^*\mathbb{M}$ then $\zeta(e_i) \in T_z^*\mathbb{M}$ and we denote by $\zeta_j^i$ its coordinates. This defines a system of coordinates on $\mathscr{F}^*\mathbb{M}$.

Unless precised otherwise, all the sums appearing below are run through indices from 1 to $d$. Let $(z, \zeta) \in O^*\mathbb{M} \subset \mathscr{F}^*\mathbb{M}$ with coordinates $(z_m, \zeta_i^j)$. Then

$$\zeta \circ e^{tA_{k\ell}}(e_i) = \zeta + t\zeta(A_{k\ell}e_i) + O(t^2) = \zeta + t\delta_{i\ell}\zeta(e_k) - t\delta_{ik}\zeta(e_\ell) + O(t^2).$$

Hence $\zeta \circ e^{tA_{k\ell}}$ has coordinates $\zeta_j^i + t\delta_{i\ell}\zeta_j^k - t\delta_{ik}\zeta_j^\ell + O(t^2)$ and

$$V_{k\ell} = \sum_{i,j} \left(\delta_{i\ell}\zeta_j^k - \delta_{ik}\zeta_j^\ell\right) \frac{\partial}{\partial \zeta_j^i} = \sum_j \zeta_j^k \frac{\partial}{\partial \zeta_j^\ell} - \zeta_j^\ell \frac{\partial}{\partial \zeta_j^k}. \tag{2.3.2}$$

Geodesic trajectories $(z, \zeta^1) \in T^*\mathbb{M}$ satisfy the equation

$$\dot{z}_m = \zeta_m^1, \quad \dot{\zeta}_m^1 = \sum_{i,j} \Gamma_{ij}^m(z)\zeta_i^1\zeta_j^1$$

while covectors $\eta \in T^*\mathbb{M}$ that are parallely transported along $(z, \zeta^1)$ satisfy

$$\dot{\eta}_m = -\sum_{i,j} \Gamma_{ij}^m \zeta_j^1 \eta_i.$$

This yields the coordinate expression of $\widetilde{H}_1$, $\widetilde{H}_m$:

$$\widetilde{H}_1 = \sum_i \zeta_i^1 \frac{\partial}{\partial z_i} - \sum_{i,j,k,\ell} \Gamma_{ij}^\ell \zeta_i^1 \zeta_j^k \frac{\partial}{\partial \zeta_\ell^k}, \qquad \widetilde{H}_m = \sum_i \zeta_i^m \frac{\partial}{\partial z_i} - \sum_{i,j,k,\ell} \Gamma_{ij}^\ell \zeta_i^m \zeta_j^k \frac{\partial}{\partial \zeta_\ell^k}.$$

### Some differential operators

Recall that $\Delta_{\mathbb{S}}$ is the operator defined in §2.3.1 and let $\Delta_{\mathbb{M}}$ the *non-negative* Laplacian operator of $\mathbb{M}$. The operator $\Delta \overset{\text{def}}{=} \Delta_{\mathbb{M}} + \Delta_{\mathbb{S}}$ is an elliptic operator acting on $C^\infty(S^*\mathbb{M})$.

The operators $\Delta_O^V, \Delta_O^H$ acting on $C^\infty(O^*\mathbb{M})$ are defined by $\Delta_O^V \overset{\text{def}}{=} -\sum_{i,j} V_{ij}^2$, $\Delta_O^H \overset{\text{def}}{=} -\sum_i \widetilde{H}_i^2$. The operator $\Delta_O \overset{\text{def}}{=} \Delta_O^H + \Delta_O^V$ is an elliptic operator on $O^*\mathbb{M}$. Let $\pi_{\mathbb{S}} : (z,\zeta) \in O^*\mathbb{M} \mapsto (z, \zeta(e_1)) \in S^*\mathbb{M}$ be the bundle projection of $O^*\mathbb{M}$ to $S^*\mathbb{M}$. It lifts the operators $\Delta_O^V, \Delta_O^H, \widetilde{H}_1$ as follows:

$$\Delta_O^V \pi_{\mathbb{S}}^* = \pi_{\mathbb{S}}^* \Delta_{\mathbb{S}}, \qquad \Delta_O^H \pi_{\mathbb{S}}^* = \pi_{\mathbb{S}}^* \Delta_{\mathbb{M}}, \qquad \pi_{\mathbb{S}}^* H_1 = \widetilde{H}_1 \pi_{\mathbb{S}}^*. \tag{2.3.3}$$

*Proof of* (2.3.3). In order to prove the first identity of (2.3.3) it is enough to show that for every $z \in \mathbb{M}$, $\pi_{\mathbb{S}}(z)^* \Delta_{\mathbb{S}}(z) = -\pi_{\mathbb{S}}(z)^* \sum_{i,j} V_{ij}^2(z)$, where $\pi_{\mathbb{S}}(z)$ is the canonical projection $\zeta \in O_z^*\mathbb{M} \to \zeta(e_1) \in S_z^*\mathbb{M}$ and $V_{ij}(z) = V_{ij}|_{C^\infty(O_z^*\mathbb{M})}$. Normal coordinates centered at $z$ on $\mathbb{M}$ induce coordinates $\zeta_i^1$ on $T_z^*\mathbb{M}$ (and $\zeta_i^j$ on $\mathscr{F}_z^*\mathbb{M}$). In these coordinates the Euclidean metric on $T_z^*\mathbb{M}$ takes the form $\sum_i (d\zeta_i^1)^2$; hence they provide an isometric identification of $S_z^*\mathbb{M}$ with $\mathbb{S}^{d-1}$, $O_z^*\mathbb{M}$ with $O(d)$, and $\mathscr{F}_z^*\mathbb{M}$ with $\mathbb{R}^{d\times d}$. Therefore, it suffices to show that if $\pi_{\mathbb{S}^{d-1}} : O(d) \to \mathbb{S}^{d-1}$ is the canonical projection, if $\Delta_{\mathbb{S}^{d-1}}$ and $\Delta_{O(d)}$ are respectively the Laplacians on $\mathbb{S}^{d-1}$ and $O(d)$ (with respect to the metric induce by the Euclidean structure of $\mathbb{R}^{d\times d}$), then

$$\Delta_{O(d)} \pi_{\mathbb{S}^{d-1}}^* = \pi_{\mathbb{S}^{d-1}}^* \Delta_{\mathbb{S}^{d-1}}. \tag{2.3.4}$$

This identity should be available in the literature, though we have found no reference. We prove it below.

Since $\mathbb{S}^{d-1} \subset \mathbb{R}^d \subset \mathbb{R}^{d\times d}$, $\Delta_{\mathbb{S}^{d-1}}$ can be written as $-\sum_j X_j^2$, where the $X_j$ are the projections of $\partial_{\zeta_j^1}$ on $\mathbb{S}^{d-1}$ – see [Hs02, Theorem 3.1.4]. In coordinates,

$$X_j = \partial_{\zeta_j^1} - \sum_k \zeta_j^1 \zeta_j^k \partial_{\zeta_j^k}. \tag{2.3.5}$$

A direct computation combining (2.3.5) with $\sum_j (\zeta_j^1)^2 = 1$ on $\mathbb{S}^{d-1}$ shows that if $u$ is a function on $\mathbb{R}^{d\times d}$ depending only on $(\zeta_1^1, ..., \zeta_d^1)$,

$$\Delta_{\mathbb{S}^{d-1}} u|_{\mathbb{S}^{d-1}} = -\sum_j \frac{\partial^2 u}{\partial {\zeta_j^1}^2} + \sum_{j,k} \zeta_j^1 \zeta_k^1 \frac{\partial^2 u}{\partial \zeta_k^1 \partial \zeta_j^1} + (d-1) \sum_k \zeta_k^1 \frac{\partial u}{\partial \zeta_k^1}.$$

We similarly compute $\Delta_{O(d)} u|_{O(d)}$. Using (2.3.2) and that $u$ depends only on $(\zeta_1^1, ..., \zeta_d^1)$,

$$\Delta_{O(d)} u|_{O(d)} = -\sum_{k,\ell} \left( \sum_j \zeta_j^k \frac{\partial}{\partial \zeta_j^\ell} - \zeta_j^\ell \frac{\partial}{\partial \zeta_j^k} \right)^2 u = \sum_{i>1} \sum_{j,k} \left( \zeta_j^1 \frac{\partial}{\partial \zeta_k^i} - \zeta_j^i \frac{\partial}{\partial \zeta_j^1} \right) \zeta_k^i \frac{\partial u}{\partial \zeta_k^1}$$

$$= -\sum_{i>1} \sum_{j,k} \zeta_j^i \zeta_k^i \frac{\partial^2 u}{\partial \zeta_k^1 \partial \zeta_j^1} + \sum_{i>1} \sum_{j,k} \zeta_j^1 \delta_{jk} \frac{\partial u}{\partial \zeta_k^1} = -\sum_{i>1} \sum_{j,k} \zeta_j^i \zeta_k^i \frac{\partial^2 u}{\partial \zeta_k^1 \partial \zeta_j^1} + (d-1) \sum_j \zeta_j^1 \frac{\partial u}{\partial \zeta_j^1}.$$

Because of these formula, proving (2.3.4) amounts to show that for $\zeta \in O(d)$,

$$\sum_j \frac{\partial^2 u}{\partial {\zeta_j^1}^2} = \sum_{i,j,k} \zeta_j^i \zeta_k^i \frac{\partial^2 u}{\partial \zeta_k^1 \partial \zeta_j^1}. \tag{2.3.6}$$

Since $\zeta \in O(d)$, $\zeta^* \in O(d)$ which implies that $\sum_i \zeta_j^i \zeta_k^i = \delta_{jk}$. This relation shows that (2.3.6) holds on $O(d)$, which proves (2.3.4) and the first identity of (2.3.3).

The second identity in (2.3.3) is [Hs02, Proposition 3.1.2].

If $(z_t, \zeta_t) = \exp(t\widetilde{H}_1)(z_0, \zeta_0)$ with $(z_0, \zeta_0) \in O^*\mathbb{M}$ then $\pi_{\mathbb{S}}(z_t, \zeta_t)$ is the geodesic starting at $\pi_{\mathbb{S}}(z_0, \zeta_0)$: $\pi_{\mathbb{S}}(z_t, \zeta_t) = \exp(tH_1)\pi_{\mathbb{S}}(z_0, \zeta_0)$. The identity $\pi_{\mathbb{S}}^* H_1 = \widetilde{H}_1 \pi_{\mathbb{S}}^*$ follows. □

We define $\widetilde{P}_\varepsilon \overset{\text{def}}{=} \frac{1}{i}(\widetilde{H}_1 + \varepsilon \Delta_O^V)$. Because of (2.3.3), the operator $\widetilde{P}_\varepsilon$ is the lift of $P_\varepsilon$ to the orthogonal coframe bundle: $\widetilde{P}_\varepsilon \pi_{\mathbb{S}}^* = \pi_{\mathbb{S}}^* P_\varepsilon$.

### Commutation identities

A computation using (2.3.2) yields the commutation relation

$$[V_{k\ell}, V_{mn}] = \delta_{\ell m} V_{kn} + \delta_{nk} V_{\ell m} + \delta_{km} V_{n\ell} + \delta_{\ell n} V_{mk}. \tag{2.3.7}$$

We next study the commutation relations between the $V_{k\ell}$ and $\widetilde{H}_m$. Fix $z \in \mathbb{M}$ together with normal coordinates centered at $z$. In particular, $\Gamma_{ij}^\ell(z) = 0$ and

$$[V_{k\ell}, \widetilde{H}_m](z) = \sum_{i,j} [\zeta_j^k \partial_{\zeta_j^\ell} - \zeta_j^\ell \partial_{\zeta_j^k}, \zeta_i^m \partial_{z_i}] = \sum_i \delta_{\ell m} \zeta_i^k \partial_{z_i} - \delta_{km} \zeta_i^\ell \partial_{z_i} = \delta_{\ell m} \widetilde{H}_k(z) - \delta_{km} \widetilde{H}_\ell(z).$$

Since $z$ was arbitrary, this shows that

$$[V_{k\ell}, \widetilde{H}_m] = \delta_{\ell m} \widetilde{H}_k - \delta_{km} \widetilde{H}_\ell. \tag{2.3.8}$$

We conclude this section by proving that the operators $\Delta_O^V, \Delta_O^H$ enjoy some important commutation properties:

$$[\Delta_O^V, V_{mn}] = 0, \qquad [\Delta_O^H, \Delta_O^V] = 0, \qquad [\Delta_{\mathbb{M}}, \Delta_{\mathbb{S}}] = 0. \tag{2.3.9}$$

*Proof of* (2.3.9). We start with the first identity. By (2.3.7), $[V_{mn}, \Delta_O^V]$

$$= \sum_{k,\ell} (\delta_{\ell m} V_{kn} + \delta_{nk} V_{\ell m} + \delta_{km} V_{n\ell} + \delta_{\ell n} V_{mk}) V_{k\ell} + V_{k\ell} (\delta_{\ell m} V_{kn} + \delta_{nk} V_{\ell m} + \delta_{km} V_{n\ell} + \delta_{\ell n} V_{mk})$$
$$= \sum_k V_{kn}(V_{km} + V_{mk}) + (V_{km} + V_{mk}) V_{kn} - \sum_\ell V_{\ell n}(V_{\ell m} + V_{m\ell}) + (V_{\ell m} + V_{m\ell}) V_{k\ell} = 0,$$

where we used that $V_{ij} + V_{ji} = 0$.

For the second identity, we first observe that (2.3.8) implies

$$\begin{aligned}[V_{k\ell}, \Delta_O^H] &= -\sum_m \left(\delta_{\ell m}\widetilde{H}_k - \delta_{km}\widetilde{H}_\ell\right)\widetilde{H}_m + \widetilde{H}_m\left(\delta_{\ell m}\widetilde{H}_k - \delta_{km}\widetilde{H}_\ell\right) \\ &= -\widetilde{H}_k\widetilde{H}_\ell + \widetilde{H}_\ell\widetilde{H}_k - \widetilde{H}_\ell\widetilde{H}_k + \widetilde{H}_k\widetilde{H}_\ell = 0.\end{aligned}$$

Therefore $\Delta_O^H$ commutes with the $V_{k\ell}$ and a fortiori with $\Delta_O^V$.

The third identity is equivalent to $\pi_{\mathbb{S}}^*[\Delta_{\mathbb{M}}, \Delta_{\mathbb{S}}] = 0$. This is automatically satisfied since $[\Delta_O^H, \Delta_O^V] = 0$ and $\pi_{\mathbb{S}}^*$ intertwines $\Delta_{\mathbb{M}}$ with $\Delta_O^H$ and $\Delta_{\mathbb{S}}$ with $\Delta_O^V$ – see (2.3.3). □

**Sobolev equivalence**

Recall that $\mu$ is the Liouville measure on $S^*\mathbb{M}$, that $\pi_{\mathbb{S}}$ denotes the bundle projection $O^*\mathbb{M} \to S^*\mathbb{M}$ and that $\pi_{\mathbb{S}}$ intertwines $\Delta_O$ with $\Delta$ – see (2.3.3). Let $\mu_O$ be a measure on $O^*\mathbb{M}$ with

$$v \in C^\infty(S^*\mathbb{M}) \ \Rightarrow \ \int_{S^*\mathbb{M}} v d\mu = \int_{O^*\mathbb{M}} \pi_{\mathbb{S}}^* v d\mu_O. \tag{2.3.10}$$

Let $\Lambda_s = (\mathrm{Id} + \varepsilon^2\Delta)^{s/2}$, $\widetilde{\Lambda}_s = (\mathrm{Id} + \varepsilon^2\Delta_O)^{s/2}$. We *define* the semiclassical Sobolev space $H_\varepsilon^s$ on $S^*\mathbb{M}$ (resp. $\widetilde{H}_\varepsilon^s$ on $O^*\mathbb{M}$) by $H_\varepsilon^s = \Lambda_{-s}L^2$ (resp. $\widetilde{\Lambda}_{-s}L^2$) with the corresponding norm with respect to $\mu$ (resp. $\mu_O$). The identity (2.3.10) implies

$$|\pi_{\mathbb{S}}^* u|_{\widetilde{H}_\varepsilon^s}^2 = \int_{O^*\mathbb{M}} \left|\widetilde{\Lambda}_s \pi_{\mathbb{S}}^* u\right|^2 d\mu_O = \int_{S^*\mathbb{M}} |\Lambda_s u|^2 \mu = |u|_{H_\varepsilon^s}^2. \tag{2.3.11}$$

The commutation relation (2.3.8) shows that the vector fields $V_{k\ell}, [V_{1m}, \widetilde{H}_1]$ span the whole tangent bundle $TO^*\mathbb{M}$. The operator $\widetilde{P}_\varepsilon$ satisfies Hörmander's condition [Hö67] for hypoellipicity, with only one commutator needed. The Rothschild–Stein theory [RS76, §18] shows that there exists a constant $C_\varepsilon > 0$ such that

$$v \in C^\infty(O^*\mathbb{M}) \ \Rightarrow \ |v|_{\widetilde{H}_\varepsilon^{2/3}} \leq C_\varepsilon(|\widetilde{P}_\varepsilon v| + |v|). \tag{2.3.12}$$

Thanks to (2.3.11), this subelliptic estimate for $\widetilde{P}_\varepsilon$ transfers to a subelliptic estimate on $P_\varepsilon$: it suffices to plug $v = \pi_{\mathbb{S}}^* u$ in (2.3.12) to obtain

$$u \in C^\infty(S^*\mathbb{M}) \ \Rightarrow \ |u|_{H_\varepsilon^{2/3}} \leq C_\varepsilon(|P_\varepsilon u| + |u|). \tag{2.3.13}$$

**Spherical vertical Laplacian as a sum of squares**

We will need the following result: there exist $n > 0$ and $X_1, ..., X_n$ smooth vector fields on $S^*\mathbb{M}$ such that

$$\Delta_{\mathbb{S}} = -\sum_{j=1}^n X_j^2, \qquad \mathrm{div}(X_j) = 0. \tag{2.3.14}$$

Indeed, Nash's theorem shows there exist $n > 0$ and an isometric embedding $\iota : \mathbb{M} \hookrightarrow \mathbb{R}^n$. The manifold $S^*\mathbb{M}$ can be seen as a submanifold of $T^*\mathbb{R}^n$ thanks to the embedding

$$\left(z, \zeta^1\right) \mapsto \left(\iota(z),\ (d\iota(z)^*)^{-1} \cdot \zeta^1\right),$$

which in addition preserves the bundle structure. Let $X_1, ..., X_n$ be the orthogonal projections of $\partial_{n+1}, ..., \partial_{2n}$ on $S^*\mathbb{M}$. Following the proof of [Hs02, Theorem 3.1.4], the $X_j$'s are divergence-free vector fields hence (2.3.14) holds.

## 2.4 Maximal hypoelliptic estimates

### 2.4.1 Statement of the result

Recall that the operator $P_\varepsilon$ is given by $\frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}})$, that the semiclassical Sobolev spaces $H^s_\varepsilon$ were defined in §2.3.2, and that there exist $X_1, ..., X_n \in TS^*\mathbb{M}$ such that such that $\Delta_{\mathbb{S}} = -\sum_{j=1}^n X_j^2$. Here we prove an estimate for $P_\varepsilon$ similar to [RS76, Theorem 18], but uniform in the *semiclassical regime* $\varepsilon \to 0$. Let $\rho_1, \rho_2$ be two smooth functions satisfying

$$\mathrm{supp}(\rho_1, \rho_2) \subset \mathbb{R} \setminus 0, \quad 1 - \rho_1,\ 1 - \rho_2 \in C_0^\infty(\mathbb{R}, [0, 1]), \quad \rho_2 = 1 \text{ on } \mathrm{supp}(\rho_1). \tag{2.4.1}$$

**Theorem 10.** *Let $R > 0$ and $\rho_1, \rho_2$ two functions satisfying (2.4.1). For any $N > 0$, there exists $C_{N,R} > 0$ such that for every $|\lambda| \leq R$, $u \in C^\infty(S^*\mathbb{M})$, and $0 < \varepsilon < 1$,*

$$\begin{aligned} \varepsilon^{2/3}|\rho_1(\varepsilon^2\Delta)u|_{H_\varepsilon^{2/3}} + \varepsilon^{1/3}\sum_{j=1}^n |\varepsilon X_j \rho_1(\varepsilon^2\Delta)u|_{H_\varepsilon^{1/3}} + |\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}u| \\ \leq C_{N,R}|\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u| + O(\varepsilon^N)|u|. \end{aligned} \tag{2.4.2}$$

This Theorem applies to any smooth compact Riemannian manifold $\mathbb{M}$, *with no restriction on the sign of its sectional curvatures, and with no change in the proof.*

The paper [RS76] shows that for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\rho_1(\varepsilon^2\Delta)u|_{H_\varepsilon^{2/3}} \leq C_\varepsilon(|\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u| + |u|).$$

Theorem 10 shows that $C_\varepsilon = O(\varepsilon^{-2/3})$. Because of related estimates in [DSZ04] and [Le07, §3] we believe that this upper bound is optimal. This is the subject of a work in progress of Smith [Sm16].

We proved Theorem 10 in [Dr16a], when $\mathbb{M}$ is an orientable surface. In this case, $\Delta_{\mathbb{S}} = -V^2$ where $V \in TS^*\mathbb{M}$ generates the circle action on the fibers of $S^*\mathbb{M}$. Thus, $\Delta_{\mathbb{S}}$ is a sum of squares of vector fields that commute with $\Delta_{\mathbb{S}}$, a fact used in a crucial manner in the proof of [Dr16a, Proposition 3.1]. This no longer holds when $d \geq 3$ or $\mathbb{M}$ is not orientable. In order to apply nevertheless the main idea of [Dr16a] we observe that $\Delta_O^V$ – the lift of $\Delta_{\mathbb{S}}$

to the orthonormal coframe bundle $O^*\mathbb{M}$ – is the sum of squares of vector fields which all commute with $\Delta_O^V$:

$$\Delta_O^V = -\sum_{i,j} V_{ij}^2, \qquad [\Delta_O^V, V_{ij}] = 0, \tag{2.4.3}$$

see §2.3.2-2.3.2. The operator $P_\varepsilon = \frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}})$ on $C^\infty(S^*\mathbb{M})$ lifts to $\widetilde{P}_\varepsilon = \frac{1}{i}(\widetilde{H}_1 + \varepsilon\Delta_O^V)$ on $C^\infty(O^*\mathbb{M})$. Because of (2.4.3), we can modify the techniques of [Dr16a] to apply them to the operator $\widetilde{P}_\varepsilon$. This will yield estimates for functions on $O^*\mathbb{M}$, which we will descend to function on $S^*\mathbb{M}$.

We will use semiclassical analysis to show Theorem 10. To conform with standard notations, we define

$$h \overset{\text{def}}{=} \varepsilon, \qquad P \overset{\text{def}}{=} ihP_h = h^2\Delta_{\mathbb{S}} + hH_1, \qquad \widetilde{P} \overset{\text{def}}{=} ih\widetilde{P}_h = h^2\Delta_O^V + h\widetilde{H}_1,$$

*for use in §2.4.2-2.4.3 only.* We see $h$ as a small parameter and $P$ as a $h$-semiclassical operator in $\Psi_h^2$. As in [Dr16a], we base our investigation on ideas of Lebeau [Le07], where a subelliptic estimate for the Bismutian is shown, for $\varepsilon = 1$. The strategy starts to differ when Lebeau uses a microlocal reduction to a toy model. Instead, we continue to work with $P_\varepsilon$ and we replace the microlocal reduction by a positive commutator estimate. This avoids to use semiclassical Fourier integral operators.

### 2.4.2 Reduction to a subelliptic estimate

The first lemma shows that Theorem 10 is a consequence of a subelliptic estimate.

**Lemma 2.4.1.** *Let $\mathscr{S}_1, ..., \mathscr{S}_q, \mathscr{T} \subset \Psi_h^1$ be a collection of selfadjoint semiclassical operators on $S^*\mathbb{M}$ or $O^*\mathbb{M}$ and $\mathscr{P} \overset{\text{def}}{=} \sum_{j=1}^q \mathscr{S}_j^2 + i\mathscr{T}$. There exist $C, h_0 > 0$ such that*

$$0 < h < h_0 \ \Rightarrow \ |\mathscr{T}u| + \left|\sum_{j=1}^q \mathscr{S}_j^2 u\right| + \sum_{j=1}^q h^{1/3}|\mathscr{S}_j u|_{H_h^{1/3}} \le C|\mathscr{P}u| + O(h^{2/3})|u|_{H_h^{2/3}}.$$

*Proof.* We prove the result only in the case of $S^*\mathbb{M}$; the proof is identical when considering operators on $O^*\mathbb{M}$. We first show the estimate

$$|\mathscr{S}_j u|_{H_h^{1/3}}^2 \le C|\mathscr{P}u||u|_{H_h^{2/3}} + O(h)|u|_{H_h^{2/3}}^2. \tag{2.4.4}$$

Recall that $\Delta \overset{\text{def}}{=} \Delta_{\mathbb{M}} + \Delta_{\mathbb{S}}$, where $\Delta_{\mathbb{M}}$ is the *non-negative* standard Laplacian on $\mathbb{M}$ (lifted to $S^*\mathbb{M}$) and $\Delta_{\mathbb{S}}$ is the spherical Laplacian on $S^*\mathbb{M}$. The $H_h^s$-norm was defined in §2.3.2 by $|u|_{H_h^s} \overset{\text{def}}{=} |\Lambda_s u|$, where $\Lambda_s = (\mathrm{Id} + h^2\Delta)^{s/2}$. Thus,

$$|\mathscr{S}_j u|_{H_h^{1/3}}^2 = |\Lambda_{1/3}\mathscr{S}_j u|^2 \le 2|\mathscr{S}_j\Lambda_{1/3}u|^2 + 2|[\Lambda_{1/3}, \mathscr{S}_j]u|^2 \le 2|\mathscr{S}_j\Lambda_{1/3}u|^2 + O(h^2)|u|_{H_h^{1/3}}^2 \tag{2.4.5}$$

because $[\Lambda_{1/3}, \mathscr{S}_j] \in h\Psi_h^{1/3}$. Next we study $|\mathscr{S}_j\Lambda_{1/3}u|$: using $\sum_{j=1}^q \mathscr{S}_j^2 = \mathrm{Re}(\mathscr{P})$,

$$\begin{aligned}|\mathscr{S}_j\Lambda_{1/3}u|^2 = \langle \mathscr{S}_j^2\Lambda_{1/3}u, \Lambda_{1/3}u\rangle \leq \langle \sum_{j=1}^q \mathscr{S}_j^2\Lambda_{1/3}u, \Lambda_{1/3}u\rangle \\ \leq \mathrm{Re}(\langle \mathscr{P}\Lambda_{1/3}u, \Lambda_{1/3}u\rangle) = \mathrm{Re}(\langle \mathscr{P}u, \Lambda_{2/3}u\rangle) + \mathrm{Re}(\langle [\mathscr{P}, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle).\end{aligned} \tag{2.4.6}$$

We can estimate $\langle \mathscr{P}u, \Lambda_{2/3}u\rangle$ by $|\mathscr{P}u||u|_{H_h^{2/3}}$. The identity $\mathscr{P} = \sum_{j=1}^q \mathscr{S}_j^2 + i\mathscr{T}$ yields

$$\mathrm{Re}(\langle [\mathscr{P}, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle) = \sum_{j=1}^q \mathrm{Re}(\langle [\mathscr{S}_j^2, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle) + \mathrm{Re}(\langle [i\mathscr{T}, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle).$$

The operator $[i\mathscr{T}, \Lambda_{1/3}]$ belongs to $h\Psi_h^{1/3}$ therefore $|\langle [i\mathscr{T}, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle| = O(h)|u|^2_{H_h^{1/3}}$. Using the relation $[\mathscr{S}_j^2, \Lambda_{1/3}] = \mathscr{S}_j[\mathscr{S}_j, \Lambda_{1/3}] + [\mathscr{S}_j, \Lambda_{1/3}]\mathscr{S}_j$ and the fact that $[\mathscr{S}_j, \Lambda_{1/3}]$ is anti-selfadjoint we obtain

$$\begin{aligned}\langle [\mathscr{S}_j^2, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle = -\langle \mathscr{S}_ju, [\mathscr{S}_j, \Lambda_{1/3}]\Lambda_{1/3}u\rangle + \langle [\mathscr{S}_j, \Lambda_{1/3}]u, \mathscr{S}_j\Lambda_{1/3}u\rangle \\ = -\langle \mathscr{S}_ju, [\mathscr{S}_j, \Lambda_{1/3}]\Lambda_{1/3}u\rangle + \langle \Lambda_{1/3}[\mathscr{S}_j, \Lambda_{1/3}]u, \mathscr{S}_ju\rangle + \langle [\mathscr{S}_j, \Lambda_{1/3}]u, [\mathscr{S}_j, \Lambda_{1/3}]u\rangle.\end{aligned}$$

The operators $\Lambda_{1/3}[\mathscr{S}_j, \Lambda_{1/3}]$ and $[\mathscr{S}_j, \Lambda_{1/3}]$ belong to $h\Psi_h^{2/3}$ and $h\Psi_h^{1/3}$, respectively. Moreover $\mathscr{S}_j^2 \leq \mathrm{Re}(\mathscr{P})$, hence $|\mathscr{S}_ju| \leq |\mathscr{P}u|^{1/2}|u|^{1/2}$. It follows that

$$\begin{aligned}|\langle [\mathscr{S}_j^2, \Lambda_{1/3}]u, \Lambda_{1/3}u\rangle| \leq |\mathscr{S}_ju||\Lambda_{1/3}[\mathscr{S}_j, \Lambda_{1/3}]u| + |[\mathscr{S}_j, \Lambda_{1/3}]u|^2 \\ \leq O(h)|\mathscr{P}u|^{1/2}|u|^{1/2}|u|_{H_h^{2/3}} + O(h^2)|u|^2_{H_h^{1/3}}.\end{aligned}$$

Gluing this estimate with (2.4.5), (2.4.6), we get the bound

$$\begin{aligned}|\mathscr{S}_ju|^2_{H_h^{1/3}} \leq C|\mathscr{P}u||u|_{H_h^{2/3}} + O(h)|u|^2_{H_h^{1/3}} + O(h)|\mathscr{P}u|^{1/2}|u|^{1/2}|u|_{H_h^{2/3}} \\ \leq C|\mathscr{P}u||u|_{H_h^{2/3}} + O(h)|u|^2_{H_h^{2/3}} + O(h)|\mathscr{P}u|^{1/2}|u|^{3/2}_{H_h^{2/3}} \leq C|\mathscr{P}u||u|_{H_h^{2/3}} + O(h)|u|^2_{H_h^{2/3}}.\end{aligned}$$

In the last inequality we used $ab \leq a^2 + b^2$ with $a = |\mathscr{P}u|^{1/2}|u|^{1/2}_{H_h^{2/3}}$ and $b = h|u|_{H_h^{2/3}}$. This proves (2.4.4). We observe that (2.4.4) gives the estimate on $|\mathscr{S}_ju|_{H_h^{1/3}}$ provided by the lemma:

$$h^{2/3}|\mathscr{S}_ju|^2 \leq Ch^{2/3}|\mathscr{P}u||u|_{H_h^{2/3}} + O(h^{5/3})|u|^2_{H_h^{2/3}} \leq C|Pu|^2 + O(h^{4/3})|u|^2_{H_h^{2/3}}. \tag{2.4.7}$$

Next we observe that

$$|\mathscr{P}u|^2 = \left|\sum_{j=1}^q \mathscr{S}_j^2u\right|^2 + |\mathscr{T}u|^2 + \sum_{j=1}^q \langle [\mathscr{S}_j^2, i\mathscr{T}]u, u\rangle.$$

To conclude the proof of the lemma it suffices to control the commutators $\langle[\mathscr{S}_j^2, i\mathscr{T}]u, u\rangle$. We have

$$\langle[\mathscr{S}_j^2, i\mathscr{T}]u, u\rangle = \langle[\mathscr{S}_j, i\mathscr{T}]u, \mathscr{S}_j u\rangle + \langle\mathscr{S}_j u, [\mathscr{S}_j, i\mathscr{T}]u\rangle = 2\operatorname{Re}(\langle\mathscr{S}_j u, [\mathscr{S}_j, i\mathscr{T}]u\rangle).$$

By interpolation, $|\langle[\mathscr{S}_j^2, i\mathscr{T}]u, u\rangle| \leq |\mathscr{S}_j u|_{H_h^{1/3}} |[\mathscr{S}_j, i\mathscr{T}]u|_{H_h^{-1/3}}$. Since $[\mathscr{S}_j, i\mathscr{T}] \in h\Psi_h^1$ it is bounded from $H_h^{2/3}$ to $H_h^{-1/3}$ with norm $O(h)$. By (2.4.4),

$$|\langle[\mathscr{S}_j^2, i\mathscr{T}]u, u\rangle| \leq Ch\left(|\mathscr{P}u|^{1/2}|u|^{1/2}_{H_h^{2/3}} + h^{1/2}|u|_{H_h^{2/3}}\right)|u|_{H_h^{2/3}}.$$

Hence we obtain

$$\begin{aligned}\left|\sum_{j=1}^{q}\mathscr{S}_j^2 u\right|^2 + |\mathscr{T}u|^2 &\leq C|\mathscr{P}u|^2 + O(h)|\mathscr{P}u|^{1/2}|u|^{3/2}_{H_h^{2/3}} + O(h^{3/2})|u|^2_{H_h^{2/3}} \\ &\leq C|\mathscr{P}u|^2 + O(h^{4/3})|u|^2_{H_h^{2/3}}.\end{aligned} \tag{2.4.8}$$

In the second line we used Young's inequality: $ab \leq a^4 + b^{4/3}$ with $a = |\mathscr{P}u|^{1/2}$, $b = h|u|^{3/2}_{H_h^{2/3}}$. The estimates (2.4.7) and (2.4.8) are enough to conclude the proof. □

Roughly speaking, this lemma reduces the proof of (2.4.1) to an estimate of the form

$$u \in C^\infty(S^*\mathbb{M}) \;\Rightarrow\; h^{2/3}|\rho_1(h^2\Delta)u|_{H_h^{2/3}} \leq C|\rho_2(h^2\Delta)Pu| + O(h^\infty)|u|. \tag{2.4.9}$$

Because of the reasons detailed above, we will work with the lift of $P$ to $O^*\mathbb{M}$ rather than directly with $P$. We will show the estimate

$$v \in C^\infty(O^*\mathbb{M}) \;\Rightarrow\; h^{2/3}|\rho_1(h^2\Delta_O)v|_{H_h^{2/3}} \leq C|\rho_2(h^2\Delta_O)\widetilde{P}v| + O(h^\infty)|v|. \tag{2.4.10}$$

To see that (2.4.10) implies (2.4.9) we plug $v = \pi_{\mathbb{S}}^* u$ in (2.4.10), then we use the identity (2.3.3) between $\widetilde{P}$ and $P$, $\Delta$ and $\Delta_O$, and finally the relation (2.3.11) between Sobolev spaces on $S^*\mathbb{M}$ and $O^*\mathbb{M}$. The bound (2.4.10) will be implied by microlocal estimates on $\widetilde{P}$:

**Proposition 2.4.2.** *For every $(x_0, \xi_0) \in \overline{T}^* O^*\mathbb{M}\backslash 0$ there exists an open neighborhood $W_{x_0,\xi_0}$ of $(x_0, \xi_0)$ in $\overline{T}^* O^*\mathbb{M}\backslash 0$ with the following property. For every $A \in \Psi_h^0$ with $\operatorname{WF}_h(A) \subset W_{x_0,\xi_0}$, there exists $B$ with $\operatorname{WF}_h(B) \subset W_{x_0,\xi_0}$ such that*

$$v \in C^\infty(O^*\mathbb{M}) \;\Rightarrow\; h^{2/3}|Av|_{\widetilde{H}_h^{2/3}} \leq C|\widetilde{P}Bv| + O(h)|v|_{\widetilde{H}_h^{3/5}}.$$

*Proof of Theorem 10 assuming Proposition 2.4.2.* It suffices to prove the Theorem when $h$ is sufficiently small. We first fix $N, R > 0$ and $\rho_1, \rho_2$ two functions satisfying (2.4.1). Recall that we can write $P = -h^2\sum_{j=1}^n X_j^2 + hH_1$, where $\frac{h}{i}X_j$, $\frac{h}{i}H_1$ are selfadjoint semiclassical operators in $\Psi_h^1$.

**Step 1.** By Lemma 2.4.1 applied to $P$ instead of $\mathscr{P}$ and $\rho_1(h^2\Delta)u$ instead of $u$,

$$|h^2\Delta_{\mathbb{S}}\rho_1(h^2\Delta)u| + h^{1/3}\sum_{j=1}^n |hX_j\rho_1(h^2\Delta)u|_{H_h^{1/3}} + h^{2/3}|\rho_1(h^2\Delta)u|_{H_h^{2/3}}$$
$$\leq C|P\rho_1(h^2\Delta)u| + O(h^{2/3})|\rho_1(h^2\Delta)u|_{H_h^{2/3}}.$$

Let $\tilde{\rho}_1 \in C_0^\infty$, be equal to 1 on $\operatorname{supp}(\rho_1)$ and 0 where $\rho_2 \neq 1$. Since $\Delta$ and $\Delta_{\mathbb{S}}$ commute, we have $P\rho_1(h^2\Delta) = \rho_1(h^2\Delta)(P-\lambda h) + \lambda h\rho_1(h^2\Delta) + [\frac{h}{i}H_1, \rho_1(h^2\Delta)]$. Both $\lambda h\rho_1(h^2\Delta)$ and $[\frac{h}{i}H_1, \rho_1(h^2\Delta)]$ have wavefront set contained in the elliptic set of $\tilde{\rho}_1(h^2\Delta)$. Therefore,

$$C|P\rho_1(h^2\Delta)u| + O(h^{2/3})|\rho_1(h^2\Delta)u|_{H_h^{2/3}}$$
$$\leq C|\rho_2(h^2\Delta)(P-\lambda h)u| + O(h^{2/3})|\tilde{\rho}_1(h^2\Delta)u|_{H_h^{2/3}} + O(h^\infty)|u|.$$

Hence the theorem follows from a bound on $h^{2/3}|\tilde{\rho}_1(h^2\Delta)u|_{H_h^{2/3}}$. After lifting to $O^*\mathbb{M}$ and using (2.3.3) and (2.3.11) it suffices to show that

$$v \in C^\infty(O^*\mathbb{M}) \;\Rightarrow\; h^{2/3}|\tilde{\rho}_1(h^2\Delta_O)v|_{\widetilde{H}_h^{2/3}} \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + O(h^N)|v|. \quad (2.4.11)$$

**Step 2.** Since $\mathrm{WF}_h(\tilde{\rho}_1(h^2\Delta))$ is a compact subset of $\overline{T}^*O^*\mathbb{M}\setminus 0$, there exists a finite collection of points $(x_1,\xi_1),...,(x_\nu,\xi_\nu) \in \overline{T}^*O^*\mathbb{M}$ and open sets $W_{x_1,\xi_1},...,W_{x_\nu,\xi_\nu}$ given by Proposition 2.4.2 such that

$$\mathrm{WF}_h(\tilde{\rho}_1(h^2\Delta)) \subset \bigcup_{k=1}^{\nu} W_{x_k,\xi_k}. \quad (2.4.12)$$

Let $\Psi^m_{h,k}$ be the set of operators in $\Psi^m_h$ with wavefront set contained in $W_{x_k,\xi_k}$. Using (2.4.12) and a microlocal partition of unity, we can construct operators $E_k \in \Psi^{2/3}_{h,k}$ with

$$v \in C^\infty(O^*\mathbb{M}) \;\Rightarrow\; |\tilde{\rho}_1(h^2\Delta_O)v|_{\widetilde{H}_h^{2/3}} \leq \sum_{k=1}^{\nu}|E_kv| + O(h^\infty)|v|. \quad (2.4.13)$$

Below we obtain bounds on the terms $|E_kv|$.

**Step 3.** Let $\delta = 1/15$ and $m \leq 2/3$. We first claim that for every $A \in \Psi^m_{h,k}$, there exist $B_1 \in \Psi^{m-2/3}_{h,k}$ and $A' \in \Psi^{m-\delta}_{h,k}$ with

$$h^{2/3}|Av| \leq C|\widetilde{P}B_1v| + O(h)|A'v| + O(h^\infty)|v|. \quad (2.4.14)$$

The operator $\Lambda_{-2/3}A\Lambda_{-m+2/3}$ belongs to $\Psi^0_{h,k}$. Proposition 2.4.2 gives an operator $B \in \Psi^0_{h,k}$ such that

$$h^{2/3}|\Lambda_{-2/3}A\Lambda_{-m+2/3}v|_{\widetilde{H}_h^{2/3}} \leq |\widetilde{P}Bv| + O(h)|v|_{\widetilde{H}_h^{3/5}}.$$

Pick $B' \in \Psi^0_{h,k}$ with $\mathrm{WF}_h(B'-\mathrm{Id}) \cap \mathrm{WF}_h(A) = \emptyset$ and replace $v$ by $\Lambda_{m-2/3}B'v$:

$$h^{2/3}|AB'v| \leq C|\widetilde{P}B\Lambda_{m-2/3}B'v| + O(h)|\Lambda_{m-2/3}B'v|_{\widetilde{H}_h^{3/5}}.$$

Since $h^{2/3}|A(\mathrm{Id}-B')v| = O(h^\infty)|v|$, (2.4.14) holds with $B_1 \overset{\mathrm{def}}{=} B\Lambda_{m-2/3}B' \in \Psi^{m-2/3}_{h,k}$ and $A' \overset{\mathrm{def}}{=} \Lambda_{3/5}\Lambda_{m-2/3}B' \in \Psi^{m-\delta}_{h,k}$.

**Step 4.** The goal is now to iterate (2.4.14). We first need a commutator-like estimate. For $B_1$ belongs to $\Psi^{m-2/3}_{h,k}$,

$$\begin{aligned}\widetilde{P}B_1 = B_1\widetilde{P} + [\widetilde{P}, B_1] &= B_1(\widetilde{P}-\lambda h) + 2h\sum_{k,\ell} V_{k\ell}[hV_{k\ell}, B_1] + h\Psi^{m-2/3}_{h,k} \\ &= B_1(\widetilde{P}-\lambda h) + 2h\sum_{k,\ell}\Lambda_{1/3}hV_{k\ell}\cdot\Psi^{m-1}_{h,k} + h\Psi^{m-2/3}_{h,k}.\end{aligned}$$

Hence there exist operators $B_2 \in \Psi^{m-1}_{h,k}$ and $C_0 \in \Psi^{m-2/3}_{h,k}$ such that

$$\begin{aligned}|\widetilde{P}B_1v| &\leq C|B_1(\widetilde{P}-\lambda h)v| + h\sum_{k,\ell}|hV_{k\ell}B_2v|_{\widetilde{H}_h^{1/3}} + h|C_0v| \\ &\leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + h^{4/3}|B_2v|_{\widetilde{H}_h^{2/3}} + h^{2/3}|\widetilde{P}B_2v| + h|C_0v| + O(h^\infty)|v|.\end{aligned}$$

In the second line we used Lemma 2.4.1 and the elliptic estimate. The slightly weaker bound holds: there exist $B_2 \in \Psi^{m-1}_{h,k}$ and $C_1 \in \Psi^{m-1/3}_{h,k}$ such that

$$|\widetilde{P}B_1v| \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + h^{2/3}|\widetilde{P}B_2v| + h|C_1v| + O(h^\infty)|v|. \tag{2.4.15}$$

Iterate (2.4.15) to obtain $B_N \in \Psi^{m-2/3-N/3}$ and $C_N \in \Psi^{m-1/3}_{h,k}$ such that

$$|\widetilde{P}B_1v| \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + h^{2N/3}|\widetilde{P}B_Nv| + h|C_Nv| + O(h^\infty)|v|.$$

For $N \geq 6$ the operator $\widetilde{P}B_N$ belongs to $\Psi^0_h$ and $|\widetilde{P}B_Nv| = O(|v|)$. It follows that for $N$ large enough,

$$|\widetilde{P}B_1v| \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + h|C_{2N}v| + O(h^N)|v|. \tag{2.4.16}$$

**Step 5.** The estimate (2.4.16) combined with (2.4.14) show that for every $A_1 \in \Psi^m_{h,k}$ there exists $A_2 \in \Psi^{m-\delta}_{h,k}$ with

$$h^{2/3}|A_1v| \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + h|A_2v| + O(h^N)|v|.$$

Here again we can iterate this inequality sufficiently many times to obtain

$$h^{2/3}|A_1v| \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + O(h^N)|v|. \tag{2.4.17}$$

Recall that $\tilde{\rho}_1$ is controlled by operators microlocalized inside $W_{x_k,\xi_k}$ thanks to (2.4.1). Apply (2.4.17) with $A_1 = E_k$, $k = 1, ..., \nu$ and sum over $k$ to get (2.4.11):

$$h^{2/3}|\tilde{\rho}_1(h^2\Delta_O)v|_{\widetilde{H}_h^{2/3}} \leq C|\rho_2(h^2\Delta_O)(\widetilde{P}-\lambda h)v| + O(h^N)|v|.$$

This ends the proof of the theorem. □

### 2.4.3 Proof of the subelliptic estimate

In this subsection we show Proposition 2.4.2. We fix $(x_0, \xi_0) \in \overline{T}^* O^*\mathbb{M} \setminus 0$. We distinguish three cases: whether $(x_0, \xi_0) \in \mathrm{Ell}_h(h^2\Delta_O^V)$ – in this case $\widetilde{P}$ is elliptic at $(x_0, \xi_0)$ – or $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_1)$ – in this case $\mathrm{Im}(\widetilde{P})$ is elliptic at $(x_0, \xi_0)$ – or $(x_0, \xi_0) \notin \mathrm{Ell}_h(h\widetilde{H}_1) \cup \mathrm{Ell}_h(h\widetilde{H}_1)$. The latter is the hardest; we will use that one of the commutators $[hV_{1\ell}, h\widetilde{H}_1]$ is elliptic at $(x_0, \xi_0)$.

*Proof of Proposition 2.4.2 in the case* $(x_0, \xi_0) \in \mathrm{Ell}_h(h^2\Delta_O^V)$. In this case $(x_0, \xi_0) \in \mathrm{Ell}_h(\widetilde{P})$. Let $W_{x_0,\xi_0}$ be an open neighborhood of $(x_0, \xi_0)$ contained in $\mathrm{Ell}_h(\widetilde{P})$, and $A \in \Psi_h^0$ with wavefront set contained in $W_{x_0,\xi_0}$. Let $B \in \Psi_h^0$ elliptic on $\mathrm{WF}_h(A)$ and with wavefront set contained in $W_{x_0,\xi_0}$. The operator $\widetilde{P}B$ is elliptic on the wavefront set of $A$. The elliptic estimate 2.2.3 shows that for $h$ small enough,

$$v \in C^\infty(O^*\mathbb{M}) \ \Rightarrow\ h^{2/3}|Av|_{\widetilde{H}_h^{2/3}} \leq C|\widetilde{P}Bv| + O(h^\infty)|v|.$$

This shows the proposition in this case. □

*Proof of Proposition 2.4.2 in the case* $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_1)$. Without loss of generalities, we can assume that $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_1) \setminus \mathrm{Ell}_h(h^2\Delta_O^V)$. In particular $V_{1\ell}$ is characteristic at $(x_0, \xi_0)$ for any $\ell$. Let $\sigma_{\widetilde{H}_m}, \sigma_{V_{k\ell}}$ be the principal symbols of $\frac{h}{i}\widetilde{H}_m, \frac{h}{i}V_{k\ell}$. We can find an open neighborhood $W_{x_0,\xi_0} \subset \mathrm{Ell}_h(h\widetilde{H}_1)$ of $(x_0, \xi_0)$ in $\overline{T}^* O^*\mathbb{M}$ such that on $W_{x_0,\xi_0} \cap T^*O^*\mathbb{M}$, $\sigma_{\widetilde{H}_1}^2 - 2\sigma_{V_{1\ell}}\sigma_{\widetilde{H}_\ell} \geq 0$. Let $A \in \Psi_h^0$ with wavefront set contained in $W_{x_0,\xi_0}$ and $B \in \Psi_h^0$ elliptic on $\mathrm{WF}_h(A)$, with wavefront set contained in $W_{x_0,\xi_0}$, and principal symbol $\sigma_B$. The operator $h\widetilde{H}_1 B$ is elliptic on $\mathrm{WF}_h(A)$ and 2.2.3 shows that

$$|Av|_{\widetilde{H}_h^{2/3}} \leq C|h\widetilde{H}_1 Bv| + O(h^\infty)|v|.$$

It remains to control $|h\widetilde{H}_1 Bv|$. Using that $\widetilde{P}$ is equal to $h^2\Delta_O^V + h\widetilde{H}_1$ with $\Delta_O^V$ selfadjoint and $\widetilde{H}_1$ anti-selfadjoint,

$$\begin{aligned} |\widetilde{P}Bv|^2 &= |h^2\Delta_O^V Bv|^2 + |h\widetilde{H}_1 Bv|^2 + \langle [h^2\Delta_O^V, h\widetilde{H}_1]Bv, Bv\rangle \\ &= |h^2\Delta_O^V Bv|^2 + |h\widetilde{H}_1 Bv|^2 + 2h\,\mathrm{Re}(\langle B^*\,\mathrm{Re}(h^2V_{1\ell}\widetilde{H}_\ell)Bv, v\rangle), \end{aligned} \tag{2.4.18}$$

where we used that $\mathrm{Re}([h^2\Delta_O^V, h\widetilde{H}_1]) = 2h\sum_\ell \mathrm{Re}(h^2V_{1\ell}\widetilde{H}_\ell)$. On $W_{x_0,\xi_0} \cap T^*O^*\mathbb{M}$, $\sigma_{\widetilde{H}_1}^2 - 2\sigma_{V_{1\ell}}\sigma_{\widetilde{H}_\ell} \geq 0$; hence $2|\sigma(B)|^2 i\sigma_{V_{1\ell}} i\sigma_{\widetilde{H}_\ell} \geq |\sigma_B|^2(i\sigma_{\widetilde{H}_1})^2$. The sharp Gårding inequality given by Lemma 2.2.4 shows that

$$2h\,\mathrm{Re}\langle B^*\,\mathrm{Re}(h^2V_{1\ell}\widetilde{H}_\ell)Bv, v\rangle \geq \langle B^* h^2\widetilde{H}_1^2 Bv, v\rangle - O(h)|v|^2_{\widetilde{H}_h^{1/2}} = -|h\widetilde{H}_1 Bv|^2 - O(h)|v|^2_{\widetilde{H}_h^{1/2}}.$$

Plug this inequality in (2.4.18) to obtain

$$v \in C^\infty(O^*\mathbb{M}) \ \Rightarrow\ |Av|^2_{\widetilde{H}_h^{2/3}} \leq C|h\widetilde{H}_1 Bv|^2 + O(h^\infty)|v|^2 \leq C|\widetilde{P}Bv|^2 + O(h^2)|v|^2_{\widetilde{H}_h^{1/2}}.$$

This shows the proposition in the case $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_1)$. □

The proof of Proposition 2.4.2 is substantially harder when $(x_0, \xi_0)$ does not belong to $\mathrm{Ell}_h(\Delta_O^V) \cup \mathrm{Ell}_h(h\widetilde{H}_1)$. In this case, $(x_0, \xi_0) \notin \mathrm{Ell}_h(hV_{k\ell})$ for any $k, \ell$. Since $\{V_{k\ell}, \widetilde{H}_m\}$ span $TO^*\mathbb{M}$, there exists $m$ such that $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_m)$. Our analysis near $(x_0, \xi_0)$ finds its origin in work of Lebeau [Le07]. It is best explained on the model operator

$$T_\varepsilon = \varepsilon \partial_{x_1} - (\varepsilon x_1 \partial_{x_2})^2$$

acting on $C_0^\infty(\mathbb{R}^2)$. We observe that at $(0,0,0,1) \in T^*\mathbb{R}^2$, both $\partial_{x_1}$ and $x_1\partial_{x_2}$ are characteristic; and $[\partial_{x_1}, x_1\partial_{x_2}] = \partial_{x_2}$ is elliptic. This situation is similar to that of $P_\varepsilon$: near $(x_0, \xi_0)$, both $\widetilde{H}_1$ and $hV_{k\ell}$ are characteristic; and $[V_{1m}, \widetilde{H}_1] = \widetilde{H}_m$ is elliptic.

We now explain how to get a subelliptic estimate for $T_\varepsilon$. Since $T_\varepsilon$ is invariant with respect to $x_2$-translations, it is natural to conjugate it with the Fourier transform in $x_2$, defining $\hat{T}_\varepsilon$ as

$$\hat{T}_\varepsilon \overset{\text{def}}{=} \mathscr{F}^{-1} T_\varepsilon \mathscr{F} = -\varepsilon \partial_{x_1} + (\varepsilon x_1 \xi_2)^2, \quad \mathscr{F}u(x_1, \xi_2) \overset{\text{def}}{=} \int_{\mathbb{R}} e^{-ix_2\xi_2} u(x_2, x_1) dx_2.$$

We work now with fixed $\xi_2$. Hence we can assume without loss of generalities that $\hat{T}_\varepsilon$ acts on (real-valued) functions $u \in C_0^\infty(\mathbb{R})$ in the single variable $x_1 \in \mathbb{R}$. For $t > 0$, consider $\Phi \in C^\infty(\mathbb{R})$ with the following property:

$$\forall x_1 \in \mathbb{R}, \quad \mathbb{1}_{[-2t,2t]} \geq (\Phi^2)'(x_1) \geq \mathbb{1}_{[-t,t]}(x_1). \tag{2.4.19}$$

We start by showing an estimate when $|x_1| \leq t$, using a positive commutator argument. Observe that

$$\begin{aligned} \mathrm{Re}(\langle \hat{T}_\varepsilon u, \Phi^2 u \rangle) &= \mathrm{Re}(\langle -\varepsilon u', \Phi^2 u \rangle) + \mathrm{Re}(\langle \varepsilon x_1 \xi_2)^2 u, \Phi^2 u \rangle) \\ &= -\varepsilon \int_{\mathbb{R}} \Phi^2(x_1)(u^2)'(x_1) dx_1 + \varepsilon^2 \xi_2^2 \int_{\mathbb{R}} (x_1 \Phi(x_1) u(x_1))^2 dx_1. \end{aligned}$$

Integrating by parts the first term in the RHS, observing that the second term is non-negative, and that $|\Phi^2|_\infty \leq 4t$ thanks to (2.4.19), we obtain:

$$4t|\hat{T}_\varepsilon u||u| \geq \varepsilon \int_{\mathbb{R}} (\Phi^2)'(x_1) u(x_1)^2 dx_1 \geq \varepsilon \int_{[-t,t]} u(x_1)^2 dx_1. \tag{2.4.20}$$

We now show an estimate for $|x_1| \geq t$:

$$\begin{aligned} \int_{|x_1| \geq t} u(x_1)^2 dx_1 &= \int_{|x_1| \geq t} \frac{(\varepsilon x_1 \xi_2 u(x_1))^2}{(\varepsilon x_1 \xi_2)^2} \\ &\leq \frac{1}{(\varepsilon t \xi_2)^2} \int_{\mathbb{R}} (\varepsilon x_1 \xi_2 u(x_1))^2 \leq \frac{1}{(\varepsilon t \xi_2)^2} \mathrm{Re}(\langle \hat{T}_\varepsilon u, u \rangle) \end{aligned} \tag{2.4.21}$$

Summing (2.4.20) with (2.4.21), we obtain

$$|u|^2 \leq \left( \frac{4t}{\varepsilon} + \frac{1}{\varepsilon t \xi_2} \right) |Pu||u|.$$

We optimize this bound by chosing $t = \varepsilon^{-1/3}\xi_2^{-2/3}$. This yields the estimate

$$|u| \leq 5\varepsilon^{-4/3}\xi_2^{-2/3}|\hat{T}_\varepsilon u| \ \Rightarrow\ \varepsilon^{2/3}|(\varepsilon D_{x_2})^{2/3}u| \leq 5|\hat{T}_\varepsilon u|.$$

This is a $H_\varepsilon^{2/3}$-estimate for $\hat{T}_\varepsilon$, with a entirely classical proof. We will adapt this very simple analysis to the case of $\widetilde{P}_\varepsilon$, using microlocal arguments instead. Roughly speaking, the condition "$x_1$ is small" will transfer to "$V_{1m}$ is very far from being elliptic"; the condition $(\Phi^2)' \geq \mathbb{1}_{[-t,t]}$ will be replaced by the positive argument $[\partial_{x_1}, \Phi^2] \geq \mathbb{1}_{[-t,t]}$, itself quantized by a sharp Gårding inequality; the condition "$x_1$ is not too small" will transfer to "$V_{1m}$ is not too characteristic".

*Proof of Proposition 2.4.2 in the case* $(x_0, \xi_0) \notin \mathrm{Ell}_h(\Delta_O^V) \cup \mathrm{Ell}_h(h\widetilde{H}_1)$. As explained above, there exists $m$ such that $(x_0, \xi_0) \in \mathrm{Ell}_h(h\widetilde{H}_m)$; and for $(x, \xi) \in T^*O^*\mathbb{M}$ in a neighborhood of $(x_0, \xi_0)$, $\sigma_{\widetilde{H}_m}(x, \xi) \neq 0$. Changing $V_{1m}$ to $-V_{1m}$ does not change $\widetilde{P}$; and under this change $\widetilde{H}_m = [V_{1m}, \widetilde{H}_1]$ becomes $-\widetilde{H}_m$. Hence we can assume without of generalities that $\sigma_{\widetilde{H}_m}(x, \xi) > 0$ for $(x, \xi) \in T^*O^*\mathbb{M}$ in a neighborhood of $(x_0, \xi_0)$.

We subdivide the proof it in 7 short steps. In the first step we localize the functions and operators involved in a small neighborhood of $x_0$, diffeomorphic to $\mathbb{R}^{d(d+1)/2}$. It allows us to use the class $\Psi_h$ introduced in §2.2.3 and to perform a second microlocalization in the steps 2 and 3. Step 4 is the main argument. Instead of using an energy estimate obtained after a microlocal reduction as in [Le07] we apply a positive commutator estimate. This allows us to control microlocally $u$ over certain small frequencies. In step 5 we use the spectral theorem to control microlocally $u$ over the remaining frequencies. In step 6 we combine the results of steps 4,5 to conclude the proof modulo an error term which is shown to be negligible in step 7.

**Step 1.** The first step in the proof is a localization process. We fix $W_{x_0,\xi_0}$ an open neighborhood of $(x_0, \xi_0)$ in $\overline{T}^*O^*\mathbb{M}$. We assume that $W_{x_0,\xi_0}$ is small enough, so that for all $(x, \xi) \in W_{x_0,\xi_0} \cap T^*O^*\mathbb{M}$, $\sigma_{\widetilde{H}_m}(x, \xi) > c|\xi|_g$, $c > 0$; and so that there exists a smooth diffeomorphism $\gamma : \mathscr{U} \subset \mathbb{R}_y^{d(d+1)/2-1} \times \mathbb{R}_\theta \to U \overset{\text{def}}{=} \{x \in O^*\mathbb{M} : \exists \xi, (x, \xi) \in W_{x_0,\xi_0}\}$ such that $d\gamma(\partial_\theta|_{\mathscr{U}}) = V_{1m}|_U$. Let $\Gamma : T^*\mathscr{U} \to T^*U$ be the symplectic lift of $\gamma$.

Let $\mathscr{V}_{k\ell} \overset{\text{def}}{=} \frac{1}{2}(d\gamma^{-1}V_{k\ell}|_U - (d\gamma^{-1}V_{k\ell}|_U)^*)$. This is an anti-selfadjoint differential operator on $\mathscr{U}$ which has the same principal symbol as $d\gamma^{-1}V_{k\ell}|_U$. In particular there exists a function $f_{k\ell} \in C^\infty(O^*\mathbb{M})$ such that

$$\mathscr{V}_{k\ell} = d\gamma^{-1}V_{k\ell}|_U + \gamma^* f_{k\ell}|_U. \tag{2.4.22}$$

Extend $\mathscr{V}_{k\ell}$ to an anti-selfadjoint differential operator of order 1 on $\mathbb{R}^{d(d+1)/2}$ with coefficients in $C_b^\infty(\mathbb{R}^{d(d+1)/2})$ – with $\mathscr{V}_{1m}$ specifically continued by $\partial_\theta$ – and define $\mathscr{L}_O^V \overset{\text{def}}{=} -\sum_{k,\ell} \mathscr{V}_{k\ell}^2$. Since $\Delta_O^V$ commutes with $V_{1m}$, $[\mathscr{L}_O^V, D_\theta]w = 0$ for each $w \in C^\infty(\mathbb{R}^{d(d+1)/2})$ supported on $\mathscr{U}$.

Similarly, we define $\mathscr{H}_1 \overset{\text{def}}{=} \frac{1}{2}(d\gamma^{-1}\widetilde{H}_1|_U - (d\gamma^{-1}\widetilde{H}_1|_U)^*)$, which is an anti-selfadjoint differential operator on $\mathscr{U}$. It satisfies

$$\mathscr{H}_1|_{\mathscr{U}} = d\gamma^{-1}\widetilde{H}_1|_U + \gamma^* f|_U, \tag{2.4.23}$$

for a certain function $f \in C^\infty(O^*\mathbb{M})$. It extends to an anti-selfadjoint differential operator of order 1 on $\mathbb{R}^{d(d+1)/2}$ with coefficients in $C_b^\infty(\mathbb{R}^{d(d+1)/2})$. We define $\mathscr{P} \overset{\text{def}}{=} h^2 \mathscr{L}_O^V + h\mathscr{H}_1$.

Let $A \in \Psi_h^0$ with $\mathrm{WF}_h(A) \subset W_{x_0,\xi_0}$ and $\psi \in C^\infty(O^*\mathbb{M})$ be equal to 1 on the set $\{x \in O^*\mathbb{M}, \exists \xi, (x,\xi) \in \mathrm{WF}_h(A)\}$ and 0 outside $U$. The function $1-\psi$ can be seen as a pseudodifferential operator in $\Psi_h^0$ with $\mathrm{WF}_h(1-\psi) \cap W_{x_0,\xi_0} = \emptyset$. In particular $A(1-\psi) \in h^\infty \Psi_h^{-\infty}$, $(1-\psi)A \in h^\infty \Psi_h^{-\infty}$ and to prove the proposition it suffices to show that

$$v \in C^\infty(O^*\mathbb{M}) \ \Rightarrow \ h^{2/3} |\psi A \psi^2 v|_{\widetilde{H}_h^{2/3}} \leq C|\widetilde{P}\psi A \psi^2 v| + O(h)|v|_{\widetilde{H}_h^{3/5}}. \tag{2.4.24}$$

We define $(\gamma^*)^{-1}$ (resp. $\gamma^*$) the operator defined on functions on $\mathscr{U}$ (resp. $O^*\mathbb{M}$) by

$$(\gamma^*)^{-1} w(x) = \begin{cases} w(\gamma^{-1}(x)) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases} \quad \left( \text{resp. } \gamma^* v(z) = \begin{cases} v(\gamma(z)) & \text{if } z \in \mathscr{U} \\ 0 & \text{otherwise} \end{cases} \right).$$

The function $\psi A \psi^2 u$ has support in $U$; the operator $\mathscr{A} \overset{\text{def}}{=} \gamma^* \psi A \psi (\gamma^*)^{-1}$ is a pseudodifferential operator in $\Psi_h^0$ on $\mathbb{R}^3$ with wavefront set in $\Gamma^{-1}(W_{x_0,\xi_0})$; and

$$|\psi A \psi^2 v|_{\widetilde{H}_h^{2/3}} \leq C|\gamma^* \psi A \psi^2 v|_{\widetilde{H}_h^{2/3}} = C|\mathscr{A} v|_{\widetilde{H}_h^{2/3}}, \quad w \overset{\text{def}}{=} \gamma^* \psi v. \tag{2.4.25}$$

Thanks to (2.4.22), (2.4.23),

$$\mathscr{P}\gamma^*\psi = -\gamma^* \sum_{k,\ell} h^2 (V_{k\ell} + f_{k\ell})^2 \psi + \gamma^* h(\widetilde{H}_1 + f)\psi = \gamma^* \widetilde{P} \psi - 2h\gamma^* \sum_{k,\ell} f_{k\ell} h V_{k\ell} \psi + h\gamma^* g \psi,$$

where $g \overset{\text{def}}{=} f - h\sum_{k,\ell} f_{k\ell}^2 + (V_{k\ell} f_{k\ell})$ belongs to $C^\infty(O^*\mathbb{M})$. It follows that $|\mathscr{P}\mathscr{A} v|$

$$\leq |\widetilde{P}\psi A \psi^2 v| + O(h) \sum_{k,\ell} |hV_{k\ell} \psi A \psi^2 v| + O(h)|v| \leq 2|\widetilde{P}\psi A \psi^2 u| + O(h)|v|. \tag{2.4.26}$$

In the last inequality we used that $\mathrm{Re}(\widetilde{P}) = h^2 \Delta_O^V = -\sum_{k,\ell} (hV_{k\ell})^2$ hence $|hV_{k\ell} v|^2 \leq |\widetilde{P} v||v|$. Finally we observe that since $w = \gamma^* \psi v$, $|w|_{\widetilde{H}_h^{3/5}} = |\gamma^* \psi v|_{\widetilde{H}_h^{3/5}} \leq C|v|_{\widetilde{H}_h^{3/5}}$. Thanks to (2.4.25) and (2.4.26) the bound (2.4.24) will follow from the estimate

$$w \in C_0^\infty(\mathbb{R}^{d(d+1)/2}), \ \mathrm{supp}(w) \subset \mathscr{U} \ \Rightarrow \ h^{2/3} |\mathscr{A} w|_{\widetilde{H}_h^{2/3}} \leq C|\mathscr{P}\mathscr{A} w| + O(h)|w|_{\widetilde{H}_h^{3/5}}. \tag{2.4.27}$$

We have reduced the estimate on $O^*\mathbb{M}$ to an estimate on $\mathbb{R}^{d(d+1)/2}$. In the following steps we prove (2.4.27).

**Step 2.** Let $\chi, \chi_0 \in C_0^\infty(\mathbb{R}^{d(d+1)/2})$ be two functions such that $\chi$ is supported away from 0, $\mathrm{WF}_h(\mathscr{A}) \cap \mathrm{WF}_h(\chi_0(hD)) = \emptyset$, and

$$1 = \sum_{j=0}^{\infty} \chi_j(\xi), \quad \chi_j(\xi) \overset{\text{def}}{=} \chi(2^{-j}\xi) \text{ for } j \geq 1.$$

Write a Littlewood-Paley decomposition of $\mathscr{A}$:

$$\mathscr{A} = \sum_{j=0}^{\infty} \mathscr{A}_j, \quad \mathscr{A}_j \overset{\text{def}}{=} \chi_j(hD)\mathscr{A}.$$

Given $a$ a symbol on $\mathbb{R}^{d(d+1)/2} \times \mathbb{R}^{d(d+1)/2}$ we denote by $\mathrm{Op}_h(a)$ the standard quantization of $a$ – see [Zw12, §4]. The following lemma studies the composition of a pseudodifferential operator with symbol in $S^m$ with a dyadic decomposition:

**Lemma 2.4.3.** *If $a \in S^m$, the operators $2^{-jm}\mathrm{Op}_h(a)\chi(2^{-j}hD)$ and $2^{-jm}\chi(2^{-j}hD)\mathrm{Op}_h(a)$ both belong to $\Psi_{2^{-j}h}$, with semiclassical symbol $a_j\chi + 2^{-j}h \cdot S$.*

*Proof.* We first note that if $a_j(x,\xi) \overset{\text{def}}{=} 2^{-jm}a(x, 2^j\xi)$ then

$$2^{-jm}\mathrm{Op}_h(a)\chi(2^{-j}hD) = \mathrm{Op}_{2^{-j}h}(a_j \# \chi) = \mathrm{Op}_{2^{-j}h}(a_j\chi).$$

It suffices to show that the $S$-seminorms of $a_j\chi$ are uniformly bounded in $j$. We have

$$\left|\partial_x^\alpha \partial_\xi^\beta a_j(x,\xi)\chi(\xi)\right| \leq C_{\alpha\beta} \sup_{2^j\xi \in \mathrm{supp}(\chi)} 2^{-jm+j|\beta|}\langle 2^j\xi\rangle^{m-|\beta|}. \tag{2.4.28}$$

Since $\mathrm{supp}(\chi)$ is a compact subset of $\mathbb{R}^3 \setminus 0$, the right hand side of (2.4.28) is uniformly bounded in $j$. This shows that $a_j \# \chi = a_j\chi \in S$, hence $2^{-jm}\mathrm{Op}_h(a)\chi(2^{-j}hD)$ belongs to $\Psi_{2^{-j}h}$ with symbol $a_j\chi$. The operator $2^{-jm}\chi(2^{-j}hD)\mathrm{Op}_h(a)$ is the adjoint of the operator $2^{-jm}\mathrm{Op}_h(a^*)\chi(2^{-j}hD)$, thus it also belongs to $\Psi_{2^{-j}h}$. By the composition formula for symbols of semiclassical operators, its semiclassical symbol is equal to $a_j\chi + 2^{-j}h \cdot S$. □

A direct application of this result shows that $\mathscr{A}_j$ belongs to $\Psi_{2^{-j}h}$. In addition, $\mathscr{A}_0 \in h^\infty\Psi_h^{-\infty}$, which implies immediately $|\mathscr{A}_0 w| \leq O(h)|w|_{\widetilde{H}_h^{3/5}}$. We obtain in the next steps estimates on $|\mathscr{A}_j w|$ for $j \geq 1$.

**Step 3.** We start with a simple result:

**Lemma 2.4.4.** *There exist functions $\Phi \in C_b^\infty(\mathbb{R})$ and $\phi \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 1$, $\phi \geq 0$ and $\phi^2 = (\Phi^2)'$.*

*Proof.* It is enough to construct $\phi$ with $\phi(0) > 0$ then to multiply $\Phi, \phi$ by a suitable multiplicative constant. Let $\Phi$ be a smooth non-decreasing function with

$$\Phi(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ e^{-(x+1)^{-1}} & \text{if } x \in [-1,0], \\ 1 & \text{if } x \geq 1. \end{cases}$$

If $\phi$ is the non-negative root of $(\Phi^2)'$ then $\phi$ has compact support and $\phi(0) > 0$. Since the $s \in [0,\infty) \mapsto \sqrt{s}$ is smooth everywhere but at 0, $\phi$ is smooth everywhere but possibly at $-1$. But

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq -1, \\ 2^{1/2}(x+1)^{-1}e^{-(x+1)^{-1}} & \text{if } x \in [-1,0], \end{cases}$$

which is smooth at $x = -1$. □

Let $\Phi, \phi$ be given by Lemma 2.4.4. Let $h_j \overset{\text{def}}{=} h^{2/3}2^{-j/3}$ and consider the operator $\Phi(h_j D_\theta)$. This operator belongs to $\Psi_{h_j}$ with semiclassical symbol $\Phi(\xi_\theta)$. Below we show an estimate on $|\mathscr{A}_j w|$, by splitting it into two parts, $|\phi(h_j D_\theta)w|$ and $|(\mathrm{Id} - \phi(h_j D_\theta))w|$.

**Step 4.** In order to estimate $|\phi(h_j D_\theta)w|$ we use a positive commutator argument and the sharp Gårding inequality. Observing that $\sigma_{\widetilde{H}_m}(x,\xi) > c|\xi|_g$ on $W_{x_0,\xi_0} \cap T^*O^*\mathbb{M}$, the principal symbol $\sigma_{\mathscr{H}_1}$ of $\frac{1}{i}\mathscr{H}_1$ satisfies

$$\{\xi_\theta, \sigma_{\mathscr{H}_1}\}(x,\xi) > c|\xi|_g \quad \text{on } \Gamma^{-1}(W_{x_0,\xi_0}) \cap T^*O^*\mathbb{M}.$$

Recall that $\mathscr{P} = h^2 \mathscr{L}_O^V + h\mathscr{H}_1$. Similarly to [Le07, Equation (2.47)],

$$\begin{aligned} &\mathrm{Re}(\langle \mathscr{P}\mathscr{A}_j w, \Phi(h_j D_\theta)^2 \mathscr{A}_j w\rangle) \\ &= \mathrm{Re}(\langle h^2 \Phi(h_j D_\theta)\mathscr{L}_O^V \mathscr{A}_j w, \Phi(h_j D_\theta)\mathscr{A}_j w\rangle) + \mathrm{Re}(\langle h\mathscr{H}_1 \mathscr{A}_j w, \Phi(h_j D_\theta)^2 \mathscr{A}_j w\rangle). \end{aligned}$$

We study the first term. We observe that $\mathscr{L}_O^V \mathscr{A}_j = \mathscr{L}_O^V \tilde{\chi}_j(hD) \cdot \mathscr{A}_j$. Lemma 2.4.3 shows that both the operators $2^{-2j}h^2\mathscr{L}_O^V\tilde{\chi}_j(hD)$ and $\mathscr{A}_j$ belong to $\Psi_{2^{-j}h}$. Since $2^{-j}h \le h_j = h^{2/3}2^{-j/3}$, they *a fortiori* belong to $\Psi_{h_j}$. In addition, $D_\theta$ and $\mathscr{L}_O^V$ commute on $\mathscr{U}$ and $\mathscr{A}$ has wavefront set contained in $T\mathscr{U}$. The asymptotic expansion formula for composition of pseudodifferential operators [Zw12, Theorem 4.14] show that

$$h^2\Phi(h_j D_\theta)\mathscr{L}_O^V \mathscr{A}_j = h^2 \mathscr{L}_O^V \Phi(h_j D_\theta)\mathscr{A}_j + h_j^\infty \Psi_{h_j}.$$

Using that $\mathscr{L}_V^O = -\sum_{k,\ell} \mathscr{V}_{k\ell}^2 \ge 0$ we get $\mathrm{Re}(\langle h^2\Phi(h_j D_\theta)\mathscr{L}_O^V \mathscr{A}_j w, \Phi(h_j D_\theta)\mathscr{A}_j w\rangle) =$

$$\langle h^2 \mathscr{L}_O^V \Phi(h_j D_\theta)\mathscr{A}_j w, \Phi(h_j D_\theta)\mathscr{A}_j w\rangle + O(h_j^\infty)|w| \ge O(h_j^\infty)|w|.$$

We next focus on the term $\mathrm{Re}(\langle h\mathscr{H}_1\mathscr{A}_j w, \Phi(h_j D_\theta)^2 \mathscr{A}_j w\rangle)$. Since $h\mathscr{H}_1$ is anti-selfadjoint, it is equal to $\mathrm{Re}(\langle [h\mathscr{H}_1, \Phi(h_j D_\theta)]\mathscr{A}_j w, \Phi(h_j D_\theta)\mathscr{A}_j w\rangle)$. The real part of $\Phi(h_j D_\theta)[h\mathscr{H}_1, \Phi(h_j D_\theta)]$ is equal to $\frac{1}{2}[h\mathscr{H}_1, \Phi(h_j D_\theta)^2]$. We obtain

$$\mathrm{Re}(\langle \mathscr{P}\mathscr{A}_j w, \Phi(h_j D_\theta)^2\mathscr{A}_j w\rangle) \ge \frac{1}{2}\langle \mathscr{A}_j^*[h\mathscr{H}_1, \Phi(h_j D_\theta)^2]\mathscr{A}_j w, w\rangle + O(h_j^\infty)|w|. \tag{2.4.29}$$

We now study the commutator term $\mathscr{E}_j \overset{\text{def}}{=} 2^{-j}\mathscr{A}_j^*[h\mathscr{H}_1, \Phi(h_j D_\theta)^2]\mathscr{A}_j$. We claim that it belongs to $\Psi_{h_j}$. To show this claim we fix $\tilde{\chi} \in C_0^\infty(\mathbb{R}^{d(d+1)/2} \setminus 0)$ equal to 1 near $\mathrm{supp}(\chi)$ and we write

$$\mathscr{E}_j = 2\,\mathrm{Re}\left((\mathscr{A}\tilde{\chi}_j(hD)) \cdot (2^{-j}\chi_j(hD)h\mathscr{H}_1) \cdot \Phi(h_j D_\theta)^2 \cdot \mathscr{A}_j\right).$$

By Lemma 2.4.3, the operators $\mathscr{A}\tilde{\chi}_j(hD)$, $2^{-j}\chi_j(hD)h\mathscr{H}_1$ and $\mathscr{A}_j$ belong to $\Psi_{2^{-j}h}$. Since $2^{-j}h \le h_j = h^{2/3}2^{-j/3}$, they also belong to $\Psi_{h_j}$. The operator $\Phi(h_j D_\theta)$ has symbol equal to $\Phi(\xi_\theta)$ in the $h_j$-quantization and the composition theorem for semiclassical operators shows that $\mathscr{E}_j \in \Psi_{h_j}$.

The semiclassical symbols of $\mathscr{A}_j$ and $2^{-j}h\mathscr{H}_1$ are given modulo $O(h_j)S$ by

$$a(x, h^{1/3}2^{j/3}\xi)\chi(h^{1/3}2^{-2j/3}\xi), \qquad 2^{-2j/3}h^{1/3}\sigma_{\mathscr{H}_1},$$

where $a$ is the semiclassical symbol of $\mathscr{A}$ in the $h$-quantization. By the composition formula for symbols of semiclassical operators [Zw12, Theorem 4.14], the semiclassical symbol $\sigma_{\mathscr{E}_j}$ of $\mathscr{E}_j$ in the $h_j$-quantization is given modulo $O(h_j^2)S$ by

$$\begin{aligned}\chi(h^{1/3}2^{-2j/3}\xi)^2|a(x,h^{1/3}2^{j/3}\xi)|^2\cdot\frac{h_j}{i}\{\Phi(\xi_\theta)^2,2^{-2j/3}h^{1/3}\sigma_{\mathscr{H}_1}\}\\ =\chi(h^{1/3}2^{-2j/3}\xi)^2|a(x,h^{1/3}2^{j/3}\xi)|^2\phi(\xi_\theta)^2\cdot 2^{-j}h\{\xi_\theta,\sigma_{\mathscr{H}_1}\}.\end{aligned}\tag{2.4.30}$$

The wavefront set of $\mathscr{A}$ (hence the support of $a$) is contained in $\Gamma^{-1}(W_{x_0,\xi_0})$ itself contained in the conical set $\overline{\{\{\xi_\theta,\sigma_{\mathscr{H}_1}\}\geq c|\xi|\}}$, and $|\xi|\geq ch^{-1/3}2^{2j/3}$ whenever $\chi(h^{1/3}2^{-2j/3}\xi)\neq 0$. It follows that

$$\sigma_{\mathscr{E}_j}\geq\chi(h^{1/3}2^{-2j/3}\xi)^2|a(x,h^{1/3}2^{j/3}\xi)|^2\phi(\xi_\theta)^2\cdot ch^{2/3}2^{-j/3}.$$

The sharp Gårding inequality [Zw12, Theorem 4.32] implies

$$\langle\mathscr{E}_jw,w\rangle\geq ch^{2/3}2^{-j/3}|\phi(h_jD_\theta)\mathscr{A}_jw|^2+O(h_j^2)|w|^2.$$

Since $h_j=h^{2/3}2^{-j/3}$ and $\mathscr{E}_j=2^{-j}\mathscr{A}_j^*[h\mathscr{H}_1,\Phi(h_jD_\theta)^2]\mathscr{A}_j$ this yields

$$\langle\mathscr{A}_j^*[h\mathscr{H}_1,\Phi(h_jD_\theta)^2]\mathscr{A}_jw,w\rangle\geq ch^{2/3}2^{2j/3}|\phi(h_jD_\theta)\mathscr{A}_jw|^2+O(h^{4/3}2^{j/3})|w|^2.$$

Therefore we can come back to (2.4.29) and obtain

$$\mathrm{Re}(\langle\mathscr{P}\mathscr{A}_jw,\Phi(h_jD_\theta)^2\mathscr{A}_jw\rangle)\geq ch^{2/3}2^{2j/3}|\phi(h_jD_\theta)\mathscr{A}_jw|^2+O(h^{4/3}2^{j/3})|w|^2.$$

Since $\Phi$ is uniformly bounded, the operator $\Phi(h_jD_\theta)^2$ is bounded on $L^2$. This gives the estimate on $|\phi(h_jD_\theta)\mathscr{A}_jw|$:

$$h^{2/3}2^{2j/3}|\phi(h_jD_\theta)\mathscr{A}_jw|^2\leq C|\mathscr{P}\mathscr{A}_jw||\mathscr{A}_jw|+O(h^{4/3}2^{j/3})|w|^2.$$

**Step 5.** The estimate on $|(\mathrm{Id}-\phi(h_jD_\theta))w|$ follows from the spectral theorem. Since $\phi(0)=1$ there exists a smooth bounded function $\varphi$ such that $1-\phi(t)=t\varphi(t)$. The operator $\varphi(h_jD_\theta)$ is uniformly bounded on $L^2$ hence

$$h^{2/3}2^{2j/3}|(\mathrm{Id}-\phi(h_jD_\theta))\mathscr{A}_jw|^2=h^{2/3}2^{2j/3}|\varphi(h_jD_\theta)h_jD_\theta\mathscr{A}_jw|^2\leq C|hD_\theta\mathscr{A}_jw|^2.\tag{2.4.31}$$

We recall that $\partial_\theta=\mathscr{V}_{1m}$ and that $\mathscr{L}_O^V=-\sum_{k,\ell}\mathscr{V}_{k\ell}^2\geq-\mathscr{V}_{1m}^2$; hence

$$h^{2/3}2^{2j/3}|(\mathrm{Id}-\phi(h_jD_\theta))\mathscr{A}_jw|^2\leq C\langle\mathscr{L}_O^V\mathscr{A}_jw,\mathscr{A}_jw\rangle\leq C|\mathscr{P}\mathscr{A}_jw||\mathscr{A}_jw|.$$

**Step 6.** Combining the results of the steps 4 and 5, we obtain the estimate

$$h^{2/3}2^{2j/3}|\mathscr{A}_jw|^2\leq C|\mathscr{P}\mathscr{A}_jw||\mathscr{A}_jw|+O(h^{4/3}2^{j/3})|w|^2.$$

Let $\tilde\chi\in C_0^\infty(\mathbb{R}^{d(d+1)/2}\setminus 0)$ equal to 1 on $\mathrm{supp}(\chi)$. We apply the above estimate to $\tilde\chi_j(hD)w$ and we observe that both $\mathscr{A}_j$ and $\mathrm{Id}-\tilde\chi_j(hD)$ belong to $\Psi_{2^{-j}h}$ and that their symbols have disjoint supports; therefore $|\mathscr{A}_j(\mathrm{Id}-\tilde{\chi}_j(hD))w| = O(h^\infty 2^{-j\infty})|w|$ by the composition theorem. Similarly by Lemma 2.4.3, $2^{-2j}\mathscr{P}\mathscr{A}_j$ belongs to $\Psi_{2^{-j}h}$ and its symbol has disjoint support from the one of $\mathrm{Id}-\tilde{\chi}_j(hD)$; therefore $2^{-2j}|\mathscr{P}\mathscr{A}_j(\mathrm{Id}-\tilde{\chi}_j(hD))w| = O(h^\infty 2^{-j\infty})|w|$. It follows that

$$h^{2/3}2^{2j/3}|\mathscr{A}_j w|^2 \leq C|\mathscr{P}\mathscr{A}_j w||\mathscr{A}_j w| + O(h^{4/3}2^{j/3})|\tilde{\chi}_j(hD)w|^2 + O(h^\infty 2^{-j\infty})|w|^2.$$

The inequality $ab \leq a^2+b^2$ and the identity $\mathscr{A}_j = \chi_j(hD)\mathscr{A}$ shows that

$$\begin{aligned} h^{4/3}2^{4j/3}|\chi_j(hD)\mathscr{A}w|^2 &\leq C|\mathscr{P}\mathscr{A}_j w|^2 + O(h^2 2^j)|\tilde{\chi}_j(hD)w|^2 + O(h^\infty 2^{-j\infty})|w|^2 \\ &\leq C|\chi_j(hD)\mathscr{P}\mathscr{A}w|^2 + C|[\mathscr{P},\chi_j(hD)]\mathscr{A}w|^2 + O(h^2 2^j)|\tilde{\chi}_j(hD)w|^2 + O(2^{-j}h^2)|w|^2. \end{aligned} \tag{2.4.32}$$

**Step 7.** To conclude we show the commutator term $|[\mathscr{P},\chi_j(hD)]\mathscr{A}w|$ in the right hand side of (2.4.32) is negligible. Recall that $\mathscr{P} = -h^2\sum_{k,\ell}\mathscr{V}_{k\ell}^2 + h\mathscr{H}_1$ and write

$$[\mathscr{P},\chi_j(hD)] = [h\mathscr{H}_1,\chi_j(hD)] - \sum_{k,\ell} 2h\mathscr{V}_{k\ell}[h\mathscr{V}_{k\ell},\chi_j(hD)] + [h\mathscr{V}_{k\ell},[h\mathscr{V}_{k\ell},\chi_j(hD)]].$$

We first control the term $|[h\mathscr{H}_1,\chi_j(hD)]\mathscr{A}w|$. We can write

$$\begin{aligned} &2^{-j/2}[h\mathscr{H}_1,\chi_j(hD)]\langle hD\rangle^{-1/2} \\ &= 2^{-j}h\mathscr{H}_1\tilde{\chi}_j(hD)\cdot 2^{j/2}\chi_j(hD)\langle hD\rangle^{-1/2} - 2^{j/2}\chi_j(hD)\langle hD\rangle^{-1/2}\cdot 2^{-j}h\mathscr{H}_1\tilde{\chi}_j(hD). \end{aligned}$$

By Lemma 2.4.3, both $2^{-j}h\mathscr{H}_1\tilde{\chi}_j(hD)$ and $2^{j/2}\chi_j(hD)\langle hD\rangle^{-1/2}$ belong to $\Psi_{2^{-j}h}$. It follows that the operator $2^{-j/2}[h\mathscr{H}_1,\chi_j(hD)]\langle hD\rangle^{-1/2}$ belongs to $\Psi_{2^{-j}h}$. Its symbol in the $2^{-j}h$-quantization is given by the asymptotic formula and has vanishing leading term; therefore $2^{-j/2}[h\mathscr{H}_1,\chi_j(hD)]\langle hD\rangle^{-1/2}$ belongs to $2^{-j}h\Psi_{2^{-j}h}$. As such it is bounded on $L^2$ with norm $O(2^{-j}h)$. This yields

$$|[h\mathscr{H}_1,\chi_j(hD)]\mathscr{A}w| = O(2^{-j/2}h)|w|_{\widetilde{H}_h^{1/2}} = O(2^{-j/2}h)|w|_{\widetilde{H}_h^{3/5}}. \tag{2.4.33}$$

By arguments similar to the one needed to show (2.4.33), $[h\mathscr{V}_{k\ell},[h\mathscr{V}_{k\ell},\chi_j(hD)]]\langle hD\rangle^{-1/2}$ belongs to $2^{-j/2}h^2\Psi_{2^{-j}h}$ and

$$|[h\mathscr{V}_{k\ell},[h\mathscr{V}_{k\ell},\chi_j(hD)]]\mathscr{A}w| = O(2^{-j/2}h^2)|w|_{\widetilde{H}_h^{1/2}} = O(2^{-j/2}h^2)|w|_{\widetilde{H}_h^{3/5}}. \tag{2.4.34}$$

The term $h\mathscr{V}_{k\ell}[h\mathscr{V}_{k\ell},\chi_j(hD)]$ requires some extra work. Fix $j,k,\ell$ and define $B = [h\mathscr{V}_{k\ell},\chi_j(hD)]$. Then,

$$|h\mathscr{V}_{k\ell}B\mathscr{A}w|^2 \leq \mathrm{Re}(\langle\mathscr{P}B\mathscr{A}w, B\mathscr{A}w\rangle) = \mathrm{Re}(\langle\mathscr{P}\mathscr{A}w, B^*B\mathscr{A}w\rangle) + \mathrm{Re}(\langle[P,B]\mathscr{A}w, B\mathscr{A}w\rangle).$$

By the same arguments as needed to show (2.4.33), the operator $B^*B\langle hD\rangle^{-1/2}$ belongs to $2^{-j/2}h^2\Psi_{2^{-j}h}$. Hence $\mathrm{Re}(\langle\mathscr{P}\mathscr{A}w, B^*B\mathscr{A}w\rangle) = O(2^{-j/2}h^2)|\mathscr{P}\mathscr{A}w||w|_{\widetilde{H}_h^{1/2}}$. On the other hand the operator $\langle hD\rangle^{-3/5}[P,B]\langle hD\rangle^{-3/5}$ belongs to $2^{-j/5}h^2\Psi_{h_j}$ and this implies that $\mathrm{Re}(\langle [P,B]\mathscr{A}w, B\mathscr{A}w\rangle) = O(2^{-j/5}h^3)|w|^2_{\widetilde{H}_h^{3/5}}$. Combining all these estimates together we obtain that

$$|h\mathscr{V}_{k\ell}B\mathscr{A}w|^2 = O(2^{-j/5}h^3)|w|^2_{\widetilde{H}_h^{3/5}} + O(2^{-j/2}h^2)|\mathscr{P}\mathscr{A}w||w|_{\widetilde{H}_h^{3/5}}. \tag{2.4.35}$$

We plug (2.4.33), (2.4.34) and (2.4.35) in (2.4.32) to obtain the estimate

$$h^{4/3}2^{4j/3}|\chi_j(hD)\mathscr{A}w|^2 \leq$$
$$C|\chi_j(hD)\mathscr{P}\mathscr{A}w|^2 + O(h^2 2^j)|\tilde{\chi}_j(hD)w|^2 + O(2^{-j/5}h^3)|w|^2_{\widetilde{H}_h^{3/5}} + O(2^{-j/2}h^2)|\mathscr{P}\mathscr{A}w||w|_{\widetilde{H}_h^{3/5}}.$$

Summation over $j$ allows us to conclude thanks to [Zw12, Equation (9.3.29)]:

$$h^{4/3}|\mathscr{A}w|^2_{\widetilde{H}_h^{2/3}} \leq C|\mathscr{P}\mathscr{A}w|^2 + O(h^2)|w|^2_{\widetilde{H}_h^{1/2}} + O(h^3)|w|_{\widetilde{H}_h^{3/5}} + O(h^2)|\mathscr{P}\mathscr{A}w||w|_{\widetilde{H}_h^{3/5}}.$$

This implies (2.4.27), hence the proof is over. □

## 2.5 Subelliptic estimates in Anisotropic Sobolev spaces

### 2.5.1 Anisotropic Sobolev spaces

To define Pollicott–Ruelle resonances as eigenvalues we need to change the spaces on which $H_1$ acts. These spaces originally appeared as anisotropic Sobolev spaces in Baladi [Ba05], Liverani [Li05], Gouëzel–Liverani [GL06], Baladi–Tsuji [BaTs07]. We follow a microlocal approach due to Faure–Sjöstrand [FS11] in a version given by Dyatlov–Zworski [DZ16a]. It allows the use of PDE methods in the study of the Pollicott–Ruelle spectrum.

For $s, r \in \mathbb{R}$, let $G_{r,s}(h) \in \Psi_h^{0+}$ with principal symbol $\sigma_{G_{r,s}}$ given by

$$\sigma_{G_{r,s}}(x,\xi) \overset{\text{def}}{=} (sm(x,\xi) + r)\rho_0(|\xi|_g)\log(|\xi|_g), \tag{2.5.1}$$

where $\rho_0 \in C^\infty(\mathbb{R},[0,1])$ vanishes on $[-1,1]$ and is equal to 1 on $\mathbb{R}\setminus[-2,2]$ and $m \in C^\infty(T^*S^*\mathbb{M}\setminus 0, [-1,1])$ is homogeneous of degree 0 with

$$\begin{cases} m(x,\xi) = 1 & \text{near } E_s^* \\ m(x,\xi) = -1 & \text{near } E_u^* \end{cases} \quad \text{and } \{m, \sigma_{H_1}\} \geq 0.$$

The existence of $m$ is proved in [DZ16a, Lemma 3.1]. For every $s, r \geq 0$, the operator $e^{G_{r,s}(h)}$ belongs to $\Psi_h^{s+r+}$ and the semiclassical spaces of [DZ16a] are defined as $H_h^{r,s} \overset{\text{def}}{=} e^{-G_{r,s}(h)}L^2$.

In particular functions in $H_h^{r,s}$ are in $H_h^{r+s}$ microlocally near $E_s^*$ and in $H_h^{r-s}$ microlocally near $E_u^*$:

$$\begin{aligned} &A \in \Psi_h^0,\ \mathrm{WF}_h(A) \text{ sufficiently close to } E_s^* \ \Rightarrow\ |Au|_{H_h^{r+s}} \le C|u|_{H_h^{r,s}},\\ &A \in \Psi_h^0,\ \mathrm{WF}_h(A) \text{ sufficiently close to } E_u^* \ \Rightarrow\ |Au|_{H_h^{r-s}} \le C|u|_{H_h^{r,s}}. \end{aligned} \tag{2.5.2}$$

In addition if $r, s \in \mathbb{R}$ are fixed and $h > 0$ varies the spaces $H_h^{r,s}$ are equal and there exists a constant $C$ such that

$$C^{-1}h^{|s|+|r|}|u|_{H_1^{r,s}} \le |u|_{H_h^{r,s}} \le Ch^{-|s|-|r|}|u|_{H_1^{r,s}}. \tag{2.5.3}$$

### 2.5.2 High frequency estimate in $H_1^{r,s}$

The first result of this section extends the $L^2$-based hypoelliptic estimate of Theorem 10 to anisotropic Sobolev spaces:

**Proposition 2.5.1.** *For every $R, N \ge 0$ and $r, s \in \mathbb{R}$, $\rho_1, \rho_2$ satisfying (2.4.1), there exist $C_{R,N,r,s} > 0$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \le \varepsilon_0$ and $|\lambda| \le R$,*

$$|\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}u|_{H_1^{r,s}} \le C_{R,N,r,s}|\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u|_{H_1^{r,s}} + O(\varepsilon^N)|u|_{H_1^{r,s}}. \tag{2.5.4}$$

*Proof.* First observe that as in [DZ15, Equation (4.4)], if $B$ is a semiclassical pseudodifferential operator then

$$\mathrm{WF}_\varepsilon(B) \subset \overline{T}^* S^*\mathbb{M} \setminus 0 \ \Rightarrow\ (e^{G_{r,s}(1)} - e^{G_{r,s}(\varepsilon)})B \in \varepsilon^\infty\Psi_\varepsilon^{-\infty}.$$

Since $\rho_1(\varepsilon^2\Delta), \rho_2(\varepsilon^2\Delta)$ are microlocalized away from the zero section and because of (2.5.3), the proposition will follow from the bound

$$|\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}u|_{H_\varepsilon^{r,s}} \le C|\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u|_{H_\varepsilon^{r,s}} + O(\varepsilon^N)|u|_{H_\varepsilon^{r,s}}. \tag{2.5.5}$$

Below we conjugate the operators involved in (2.5.5) with $e^{G_{r,s}(\varepsilon)}$ and show a $L^2$-based estimate equivalent to (2.5.5).

For $A \in \Psi_\varepsilon^m$, let $[A]_{r,s}$ be the operator $e^{G_{r,s}(\varepsilon)}Ae^{-G_{r,s}(\varepsilon)}$. We have

$$[A]_{r,s} = A + [G_{r,s}(\varepsilon), A] + \varepsilon^2\Psi_\varepsilon^{m-2+}, \tag{2.5.6}$$

see the equation [DDZ14, (3.11)] and the discussion following it. For $\rho_1, \rho_2$ satisfying (2.4.1), let $\tilde\rho_1, \tilde\rho_2$ be smooth functions satisfying (2.4.1), with $\tilde\rho_1 = 1$ on $\mathrm{supp}(\rho_1)$ and $\tilde\rho_2 = 0$ on $\{\rho_2 \ne 1\}$. We use the identity (2.5.6) to prove that:

$$\left|\left(\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}} - [\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}]_{r,s}\right)v\right| \le C|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)v| + O(\varepsilon^N)|v|. \tag{2.5.7}$$

Since $\Delta_{\mathbb{S}} = -\sum_{j=1}^n X_j^2$, we have

$$\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}} - [\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}]_{r,s} \in \sum_{j=1}^n \varepsilon\Psi_\varepsilon^{0+}\cdot\varepsilon X_j + \varepsilon^2\Psi_\varepsilon^{0+}, \tag{2.5.8}$$

where the terms in $\Psi_\varepsilon^{0+}$ have wavefront sets contained in $\mathrm{WF}_\varepsilon(\rho_1(\varepsilon^2\Delta))$, itself contained in $\mathrm{Ell}_\varepsilon(\tilde\rho_1(\varepsilon^2\Delta))$. Thus,

$$\begin{gathered}\left|\left(\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}} - [(\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}]_{r,s}\right) v\right| \\ \leq O(\varepsilon)\sum_{j=1}^n |\tilde\rho_1(\varepsilon^2\Delta)\varepsilon X_j v|_{H_\varepsilon^{1/3}} + O(\varepsilon^2)|\tilde\rho_1(\varepsilon^2\Delta)v|_{H_\varepsilon^{2/3}} + O(\varepsilon^\infty)|v|.\end{gathered}$$

Theorem 10 applied with the pair $(\tilde\rho_1,\tilde\rho_2)$ estimates the right hand side by $C|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)v| + O(\varepsilon^N)|v|$. This gives (2.5.7).

Thanks to (2.5.7),

$$\begin{aligned}|[\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}]_{r,s}v| &\leq |\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}v| + C|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)v| + O(\varepsilon^N)|v| \\ &\leq C|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)v| + O(\varepsilon^N)|v|.\end{aligned} \tag{2.5.9}$$

In the second line we used Theorem 10 with the pair $(\rho_1,\tilde\rho_2)$. To show (2.5.5), it remains to control $|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)v|$ by $|[\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)]_{r,s}v|$. We will need the following lemma:

**Lemma 2.5.2.** *Let $m \leq 0$ and $B_0 \in \Psi_\varepsilon^m$ such that $\mathrm{WF}_\varepsilon(B_0) \subset \mathrm{Ell}_\varepsilon(\rho_2(\varepsilon^2\Delta))$. For every $N > 0$ there exists $B_1 \in \Psi_\varepsilon^{m-1/4}$ with $\mathrm{WF}_h(B_1) \subset \mathrm{Ell}_\varepsilon(\rho_2(\varepsilon^2\Delta))$ such that*

$$|B_0\varepsilon(P_\varepsilon-\lambda)v| \leq |B_0\varepsilon[P_\varepsilon-\lambda]_{r,s}v| + O(\varepsilon^{1/3})|B_1\varepsilon(P_\varepsilon-\lambda)v| + O(\varepsilon^N)|v|. \tag{2.5.10}$$

*Proof.* The idea is similar to the second part of the proof of Theorem 10. We have

$$\begin{aligned}B_0\varepsilon(P_\varepsilon-\lambda) &= B_0\varepsilon[P_\varepsilon-\lambda]_{r,s} + \varepsilon\sum_{j=1}^n \varepsilon X_j\cdot\Psi_\varepsilon^{m+} + \varepsilon\Psi_\varepsilon^{m+} \\ &= B_0\varepsilon[P_\varepsilon-\lambda]_{r,s} + \varepsilon\Lambda_{1/3}\cdot\sum_{j=1}^n \varepsilon X_j\cdot\Psi_\varepsilon^{m-1/4} + \varepsilon\Lambda_{2/3}\cdot\Psi_\varepsilon^{m-1/4}.\end{aligned} \tag{2.5.11}$$

Let $\tilde\rho_3,\tilde\rho_4 \in C^\infty(\mathbb{R}^3)$ satisfying (2.4.1) and such that $\mathrm{WF}_\varepsilon(B_0)\cap\mathrm{WF}_\varepsilon(\tilde\rho_3(\varepsilon^2\Delta)-\mathrm{Id}) = \emptyset$. Equivalently, $\tilde\rho_3(\varepsilon^2\Delta)B_0 = B_0 + \varepsilon^\infty\Psi_\varepsilon^{-\infty}$. We multiply both sides of (2.5.11) by $\tilde\rho_3(\varepsilon^2\Delta)$ to obtain $B_0\varepsilon(P_\varepsilon-\lambda) - B_0\varepsilon[P_\varepsilon-\lambda]_{r,s}$

$$= \varepsilon\Lambda_{1/3}\cdot\tilde\rho_3(\varepsilon^2\Delta)\sum_{j=1}^n \varepsilon X_j\cdot\varepsilon\Psi_\varepsilon^{m-1/4} + \varepsilon\Lambda_{2/3}\cdot\tilde\rho_3(\varepsilon^2\Delta)\cdot\Psi_\varepsilon^{m-1/4} + \varepsilon^\infty\Psi_\varepsilon^{-\infty}. \tag{2.5.12}$$

Thus there exist operators $\widetilde{B}_1^j \in \Psi_\varepsilon^{m-1/3+} \subset \Psi_\varepsilon^{m-1/4}$ and $\widetilde{B}_2^j \in \Psi^{m-2/3+} \subset \Psi_\varepsilon^{m-1/4}$ with wavefront sets contained in $\mathrm{WF}_\varepsilon(B_0)$ such that $|B_0\varepsilon(P_\varepsilon-\lambda)v - B_0\varepsilon[P_\varepsilon-\lambda]_{r,s}v|$

$$\leq \varepsilon\sum_{j=1}^n\sum_{k=1,2}|\tilde\rho_3(\varepsilon^2\Delta)\varepsilon X_j\widetilde{B}_k^j v|_{H_\varepsilon^{1/3}} + \varepsilon|\tilde\rho_3(\varepsilon^2\Delta)\widetilde{B}_k^j v|_{H_\varepsilon^{2/3}} + O(\varepsilon^\infty)|v|. \tag{2.5.13}$$

Theorem 10 applied to $(\tilde\rho_3, \tilde\rho_4)$ estimates the right hand side of (2.5.13):

$$|B_0\varepsilon(P_\varepsilon-\lambda)v| \leq |B_0\varepsilon[P_\varepsilon-\lambda]_{r,s}v| + O(\varepsilon^{1/3})\sum_{j,k}|\tilde\rho_4(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j v| + O(\varepsilon^N)|v|.$$

Since $\mathrm{WF}_\varepsilon(\widetilde{B}_k^j)\cap\mathrm{WF}_\varepsilon(\tilde\rho_3(\varepsilon^2\Delta)-\mathrm{Id})$ is empty and $\tilde\rho_4=1$ on $\mathrm{supp}(\tilde\rho_3)$, $\tilde\rho_4(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j = \varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j + \varepsilon^\infty\Psi_\varepsilon^{-\infty}$. It follows that

$$|B_0\varepsilon(P_\varepsilon-\lambda)v| \leq |B_0\varepsilon[P_\varepsilon-\lambda]_{r,s}v| + O(\varepsilon^{1/3})\sum_{j,k}|\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j v| + O(\varepsilon^N)|v|. \tag{2.5.14}$$

Fix $1\leq j\leq n$ and $1\leq k\leq 2$. We recall that $\widetilde{B}_k^j\in\Psi_\varepsilon^{m-1/4}$ with wavefront sets contained in $\mathrm{WF}_\varepsilon(B_0)$. Similarly to (2.5.12), we write

$$[\varepsilon(P_\varepsilon-\lambda),\widetilde{B}_k^j] = \varepsilon\Lambda_{1/3}\cdot\tilde\rho_3(\varepsilon^2\Delta)\sum_{\ell=1}^n\varepsilon X_\ell\widetilde{B}_{k,1}^{j,\ell} + \varepsilon\Lambda_{2/3}\cdot\tilde\rho_3(\varepsilon^2\Delta)\cdot\varepsilon\widetilde{B}_{k,2}^{j,\ell} + \varepsilon^\infty\Psi_\varepsilon^\infty,$$

for some operators $\widetilde{B}_{k,1}^{j,\ell}, \widetilde{B}_{k,2}^{j,\ell}\in\Psi_\varepsilon^{m-1/2}$ with wavefront sets contained in $\mathrm{WF}_h(B_0)$. And similarly to (2.5.14), we obtain the estimate

$$|\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j v| \leq |\widetilde{B}_k^j\varepsilon(P_\varepsilon-\lambda)v| + O(\varepsilon^{1/3})\sum_{\ell,k,k'}|\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_{k,k'}^{j,\ell}v| + O(\varepsilon^N)|v|. \tag{2.5.15}$$

We observe that the terms $O(\varepsilon^{1/3})|\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_{k,k'}^{j,\ell}v|$ above involve a factor $\varepsilon^{1/3}$ and an operator $\widetilde{B}_{k,k'}^{j,\ell}$ that is 1/4-smoother than $\widetilde{B}_k^j$. Since $\varepsilon(P_\varepsilon-\lambda)\cdot\Psi_\varepsilon^{-2}\subset\Psi_\varepsilon^0$, we can then iterate (2.5.15) sufficiently many times to get an operator $B_1\in\Psi_h^{m-1/4}$ with wavefront set contained in $\mathrm{WF}_h(B_0)$, such that

$$\sum_{j,k}|\varepsilon(P_\varepsilon-\lambda)\widetilde{B}_k^j v| \leq |B_1\varepsilon(P_\varepsilon-\lambda)v| + O(\varepsilon^N)|v|.$$

We combine this bound with (2.5.14) to conclude the proof. □

The right hand side of (2.5.10) involves the term $O(\varepsilon^{1/3})|B_1\varepsilon(P_\varepsilon-\lambda)v|$ which comes with the factor $\varepsilon^{1/3}$, and the operator $B_1$. This operator is 1/4-smoother than $B_0$. We can then iterate (2.5.10) sufficiently many times starting from $B_0=\tilde\rho_2(\varepsilon^2\Delta)\in\Psi_\varepsilon^0$ to obtain operators $B_1\in\Psi_\varepsilon^{-1/4}, ..., B_{3N}\in\Psi_\varepsilon^{-3N/4}$ with wavefront sets contained in $\mathrm{Ell}_\varepsilon(\rho_2(\varepsilon^2\Delta))$ and such that

$$|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)v| \leq \sum_{k=0}^{3N-1}\varepsilon^{k/3}|B_k\varepsilon[P_\varepsilon-\lambda]_{r,s}v| + O(\varepsilon^N)|\varepsilon(P_\varepsilon-\lambda)B_{3N}v|.$$

For $N$ large enough, $\varepsilon(P_\varepsilon-\lambda)B_{3N}\in\Psi_\varepsilon^0$ and $O(\varepsilon^N)|\varepsilon(P_\varepsilon-\lambda)B_{3N}v| = O(\varepsilon^N)|v|$. In addition the operator $[\rho_2(\varepsilon^2\Delta)]_{r,s}$ is elliptic on the wavefront set of the $B_k$ thus

$$|\tilde\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)v| \leq |[\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon-\lambda)]_{r,s}v| + O(\varepsilon^N)|v|.$$

Plug this estimate back in (2.5.9) to conclude the proof of the proposition. □

*Starting now we consider $R, N, r, s$ fixed, $\varepsilon_0$ given by Proposition 2.5.1 and $\varepsilon, h$ satisfying* $0 < \varepsilon \leq h \leq \varepsilon_0$. Fix $\rho_1, \rho_2$ satisfying (2.4.1), $\chi_1 \overset{\text{def}}{=} 1 - \rho_1$ and $\chi$ be equal to 1 near 0 and such that $\chi\rho_2 = 0$. Define $Q \overset{\text{def}}{=} \chi(h^2\Delta)$ and

$$\begin{aligned} P_\varepsilon(\lambda) &\overset{\text{def}}{=} h(P_\varepsilon - \lambda) - iQ = -ih\varepsilon\Delta_{\mathbb{S}} - ihH_1 - \lambda h - iQ, \\ \widetilde{P}_\varepsilon(\lambda) &\overset{\text{def}}{=} -ih\varepsilon\chi_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}} - ihH_1 - \lambda h - iQ. \end{aligned} \tag{2.5.16}$$

If $P_\varepsilon(\lambda)u \overset{\text{def}}{=} f$ then $\widetilde{P}_\varepsilon(\lambda)u = f + ih\varepsilon\rho_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}}u \overset{\text{def}}{=} F$. We use (2.5.3) to go from the space $H_h^{r,s}$ to the space $H_1^{r,s}$ and we bound $F$ by Proposition 2.5.1:

$$\begin{aligned} |F|_{H_h^{r,s}} &\leq |f|_{H_h^{r,s}} + h\varepsilon|\rho_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}}u|_{H_h^{r,s}} \leq |f|_{H_h^{r,s}} + h^{-|s|-|r|+1}\varepsilon^{-1}|\rho_1(\varepsilon^2\Delta)\varepsilon^2\Delta_{\mathbb{S}}u|_{H_1^{r,s}} \\ &\leq |f|_{H_1^{r,s}} + Ch^{-|s|-|r|+1}\varepsilon^{-1}|\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u|_{H_1^{r,s}} + O(h^{-|s|-|r|}\varepsilon^N)|u|_{H_1^{r,s}}. \end{aligned}$$

We note that $\rho_2(\varepsilon^2\Delta)Q = 0$ because $\varepsilon \leq h$, hence

$$h\varepsilon^{-1}\rho_2(\varepsilon^2\Delta)\varepsilon(P_\varepsilon - \lambda)u = \rho_2(\varepsilon^2\Delta)\left(h(P_\varepsilon - \lambda) - iQ\right)u = \rho_2(\varepsilon^2\Delta)P_\varepsilon(\lambda)u = \rho_2(\varepsilon^2\Delta)f.$$

It follows that

$$|F|_{H_h^{r,s}} \leq |f|_{H_h^{r,s}} + Ch^{-|s|-|r|}|\rho_2(\varepsilon^2\Delta)f|_{H_1^{r,s}} + O(h^{-|s|-|r|}\varepsilon^N)|u|_{H_1^{r,s}}. \tag{2.5.17}$$

The operator $\rho_2(\varepsilon^2\Delta)$ is bounded on $H_1^{r,s}$ since $\rho_2(\varepsilon^2\Delta) \in \Psi_\varepsilon^0 \subset \Psi_1^0$ and by (2.5.6),

$$e^{G_{r,s}(1)}\rho_2(\varepsilon^2\Delta)e^{-G_{r,s}(1)} = \rho_2(\varepsilon^2\Delta) + \Psi_1^{-1+} \in \Psi_1^0.$$

Therefore $|\rho_2(\varepsilon^2\Delta)f|_{H_1^{r,s}} \leq C|f|_{H_1^{r,s}}$; and $|f|_{H_1^{r,s}}$ is controlled by $h^{-|s|-|r|}|f|_{H_h^{r,s}}$ because of (2.5.3). The estimate (2.5.17) yields

$$|F|_{H_h^{r,s}} \leq Ch^{-2|s|-2|r|}|f|_{H_h^{r,s}} + O(h^{-2|s|-2|r|}\varepsilon^N)|u|_{H_h^{r,s}}.$$

Recalling that $f = P_\varepsilon(\lambda)u$ and $F = \widetilde{P}_\varepsilon(\lambda)u$ we obtain the main result of this section:

**Theorem 11.** *For every $R, N \geq 0$, and $r, s \in \mathbb{R}$ there exist $C_{R,N,r,s} > 0$ and $\varepsilon_0 > 0$ such that if $P_\varepsilon(\lambda)$ and $\widetilde{P}_\varepsilon(\lambda)$ are defined in* (2.5.16),

$$\begin{aligned} &\lambda \in \mathbb{D}(0, R),\ 0 < \varepsilon \leq h \leq \varepsilon_0 \\ \Rightarrow\ &|\widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r,s}} \leq C_{R,N,r,s}h^{-2|s|-2|r|}|P_\varepsilon(\lambda)u|_{H_h^{r,s}} + O(h^{-2|s|-2|r|}\varepsilon^N)|u|_{H_h^{r,s}}. \end{aligned}$$

## 2.6 Stochastic stability of Pollicott–Ruelle resonances

### 2.6.1 Invertibility of $P_\varepsilon(\lambda)$

Recall that $P_\varepsilon(\lambda)$ is given by $P_\varepsilon(\lambda) = h(P_\varepsilon - \lambda) - iQ$ on $H_h^{r,s}$, and let $D_h^{r,s}$ be its domain on $H_h^{r,s}$:

$$D_h^{r,s} \overset{\text{def}}{=} \{u \in H_h^{r,s},\ H_1u \in H_h^{r,s}, \Delta_{\mathbb{S}}u \in H_h^{r,s}\},$$

where $H_1 u, \Delta_{\mathbb{S}} u$ are first seen as distributions. We prove here that the operator $P_\varepsilon(\lambda)$ is invertible from $D_h^{r,s}$ to $H_h^{r,s}$, provided that $\lambda$ is in a compact set and that $h$ is small enough, $s$ is large enough.

**Theorem 12.** *Let $R > 0$ and $r \in \mathbb{R}$. There exists $s_0 > 0$ such that for every $s \geq s_0$, there exists $h_0 > 0$ with*

$$\varepsilon \leq h \leq h_0, \quad |\lambda| \leq R \quad \Rightarrow \quad P_\varepsilon(\lambda) : D_h^{r,s} \to H_h^{r,s} \text{ is invertible.}$$

A necessary step to prove this result is a bound of the form $|u|_{H_h^{r,s}} \leq C_h |P_\varepsilon(\lambda)|_{H_h^{r,s}}$. In view of Theorem 11 applied with $N = 2|s| + 2|r| + 1$ it suffices to show that $|u|_{H_h^{r,s}} \leq Ch^{-1}|\widetilde{P}_\varepsilon(\lambda)|_{H_h^{r,s}}$ where we recall that $\widetilde{P}_\varepsilon(\lambda)$ is given by

$$\widetilde{P}_\varepsilon(\lambda) = -ih\varepsilon\chi_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}} - ihH_1 - \lambda h - iQ.$$

See $\widetilde{P}_\varepsilon(\lambda)$ as a pseudodifferential operator in the semiclassical parameter $h$. Its semiclassical principal symbol is $p_\varepsilon - iq_\varepsilon$, where $p_\varepsilon = \sigma_{H_1}$ and

$$q_\varepsilon(x,\xi) = \chi_1\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}}(x,\xi) + \chi(|\xi|_g^2).$$

It is clear that $p_\varepsilon$ belongs to $S^1/hS^0$. We claim that $q_\varepsilon$ also belong to $S^1/hS^0$ or equivalently that

$$\chi_1\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}}(x,\xi) \in S^1/hS^0. \tag{2.6.1}$$

Recall that $\Delta_{\mathbb{S}} = -\sum_{j=1}^n X_j^2$, write $\sigma_{X_j}$ for the principal symbol of $\frac{h}{i}X_j$ and note that

$$q_\varepsilon(x,\xi) = \sum_{j=1}^n \sigma_{X_j}(x,\xi)\chi_1\left(|\xi'|_g^2\right)\sigma_{X_j}(x,\xi') + \chi(|\xi|_g^2), \qquad \xi' \overset{\text{def}}{=} \varepsilon h^{-1}\xi.$$

It suffices to show that each term in the above sum belongs to $S^1/hS_0$, thus that $(x,\xi) \mapsto \chi_1(|\xi'|_g^2)\sigma_{X_j}(x,\xi')$ belongs to $S^0/hS^{-1}$. When $|\alpha| + |\beta| > 0$,

$$\begin{aligned}\langle\xi\rangle^{|\beta|}\left|\partial_x^\alpha\partial_\xi^\beta\left(\chi_1(|\xi'|_g^2)\sigma_{X_j}(x,\xi')\right)\right| &= \langle\xi\rangle^{|\beta|}(\varepsilon h^{-1})^{|\beta|}|\partial_x^\alpha\partial_{\xi'}^\beta\chi_1(|\xi'|_g^2)\sigma_{X_j}(x,\xi')| \\ &\leq \langle\xi'\rangle^{|\beta|}|\partial_x^\alpha\partial_{\xi'}^\beta\chi_1(|\xi'|_g^2)\sigma_{X_j}(x,\xi')| \leq C_{\alpha\beta},\end{aligned}$$

where in the last inequality we used that $\chi'$ vanishes in a neighborhood of 0 and that $\chi_1(|\xi'|_g^2)\sigma_{X_j}(x,\xi')$ belongs to $S^0$ as a symbol in $\xi'$. Since for $\alpha = \beta = 0$ there is nothing to prove, we obtain (2.6.1) and $q_\varepsilon \in S^1/hS^0$.

Hence the operator $\widetilde{P}_\varepsilon(\lambda)$ belongs to $\Psi_h^1$. We next compute the principal symbol of the operator $[\widetilde{P}_\varepsilon(\lambda)]_{r,s} \overset{\text{def}}{=} e^{G_{r,s}(h)}P_\varepsilon(\lambda)e^{-G_{r,s}(h)}$. We write $p_{\varepsilon,r,s} - iq_{\varepsilon,r,s}$ for the principal symbol of

$[\widetilde{P}_\varepsilon(\lambda)]_{r,s}$, where $p_{\varepsilon,r,s}, q_{\varepsilon,r,s}$ are real-valued. The symbol $p_{\varepsilon,r,s}$ is given by:

$$\begin{aligned} p_{\varepsilon,r,s} &= \sigma_{H_1} - \left\{\sigma_{G_{r,s}}, \chi_1\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}}\right\} \\ &= \sigma_{H_1} - sh\left\{m, \chi_1\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}}\right\}\rho_0(|\xi|_g^2)\log|\xi|_g \mod hS^0. \end{aligned} \tag{2.6.2}$$

Here we used that $\sigma_{G_{r,s}} = \log(|\xi|_g)\rho_0(|\xi|_g^2)m \mod hS^{-1}$ by (2.5.1), and that $\{\sigma_{\Delta_{\mathbb{S}}}, |\xi|_g^2\} = 0$ because $\Delta_{\mathbb{S}}$ commutes with $\Delta$, see (2.3.9). Since $m$ is homogeneous of degree 0, we deduce from (2.6.1) and (2.6.2) that

$$p_{\varepsilon,r,s} = \sigma_{H_1} + sh\log|\xi|_g \cdot S^0 \mod hS^0. \tag{2.6.3}$$

Similarly the symbol $q_{\varepsilon,r,s}$ is given by:

$$\begin{aligned} q_{\varepsilon,r,s} &= Q(|\xi|_g^2) + \chi\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}} + h\{\sigma_{G_{r,s}}, \sigma_{H_1}\} \\ &= Q(|\xi|_g^2) + \chi\left(\frac{\varepsilon^2}{h^2}|\xi|_g^2\right)\frac{\varepsilon}{h}\sigma_{\Delta_{\mathbb{S}}} + sh\{m, \sigma_{H_1}\}\rho_0(|\xi|_g^2)\log|\xi|_g \mod hS^0, \end{aligned} \tag{2.6.4}$$

where we used that $h\rho_0 m\{\sigma_{H_1}, \log|\xi|_g\} \in hS^0$ and that $h\{\sigma_{H_1}, \rho_0(|\xi|_g^2)\}\log|\xi|_g \in hS^0$. We remark that since $\{m, \sigma_{H_1}\} \geq 0$, $q_{\varepsilon,r,s}$ is nonnegative when $s \geq 0$.

The key step to prove Theorem 12 is the following Proposition, whose proof is largely inspired from [DZ16a, Proposition 3.1] and [DZ15, Lemma 4.2]:

**Proposition 2.6.1.** *Let $R > 0$, $r \in \mathbb{R}$. There exists $s_0$ such that for $s \geq s_0$, there exist $h_0 > 0$ and $C_{R,r,s} > 0$ with*

$$0 < \varepsilon \leq h \leq h_0, \ |\lambda| \leq R \quad \Rightarrow \quad |u|_{H_h^{r,s}} \leq C_{R,r,s}h^{-1}|\widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r,s}}.$$

*Proof.* We define $v \overset{\text{def}}{=} e^{G_{r,s}(h)}u \in L^2$ and we recall that $[A]_{r,s} \overset{\text{def}}{=} e^{G_{r,s}(h)}Ae^{-G_{r,s}(h)}$ when $A \in \Psi_h^m$. Using a microlocal partition of unity it is sufficient to show the inequality

$$|[A]_{r,s}v| \leq Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|,$$

when $\mathrm{WF}_h(A)$ is supported in a small neighborhood of $(x_0, \xi_0) \in \overline{T}^* S^*\mathbb{M}$ in each of the following cases:

**Case I:** $(x_0, \xi_0) \in \mathrm{Ell}_h(Q)$**.** Since $\{m, \sigma_{H_1}\} \geq 0$ by construction of $m$, (2.6.4) shows that $q_{\varepsilon,r,s}(x_0, \xi_0) > 0$ when $s \geq 0$. In particular, $[\widetilde{P}_\varepsilon(\lambda)]_{r,s}$ is elliptic at $(x_0, \xi_0)$. By the elliptic estimate, $|A_{r,s}v| \leq C|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|$.

**Case II:** $(x_0, \xi_0) \in \kappa(E_s^*)$**.** Here $\kappa : T^*S^*\mathbb{M} \to \partial\overline{T}^* S^*\mathbb{M}$ is the projection map defined by (2.2.7). The operator $\widetilde{P}_\varepsilon(\lambda)$ has semiclassical principal symbol $p_\varepsilon - iq_\varepsilon$. We note that $q_\varepsilon \geq 0$ everywhere and that $p_\varepsilon = \sigma_{H_1}$ is homogeneous of degree 1 and independent of $h$. Hence we can apply the radial source estimate (Lemma 2.2.6). Fix $B_1 \in \Psi_h^0$ with wavefront set contained in the set $\{\rho_0 m = 1\}$ so that on $WF_h(B_1)$ the space $H_h^{r,s}$ and $H_h^{r+s}$ are microlocally equivalent, see (2.5.2). There exist $s_0 > 0$ and $U_0$ neighborhood of $\kappa(E_s^*)$ in $\overline{T}^* S^*\mathbb{M}$ such that

$$s \geq s_0, \quad \mathrm{WF}_h(A) \subset U_0 \ \Rightarrow \ |Au|_{H_h^{r+s}} \leq Ch^{-1}|B_1 \widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r+s}} + O(h^\infty)|u|_{H_h^{-|r|-s}}.$$

After possibly shrinking the size of $\mathrm{WF}_h(A)$ we can use that $H_h^{r,s}$ and $H_h^s$ are microlocally equivalent near $\mathrm{WF}_h(A)$, $\mathrm{WF}_h(B_1)$ to conclude that

$$|Au|_{H_h^{r,s}} \leq Ch^{-1}|\widetilde{P}_\varepsilon(\lambda)v|_{H_h^{r,s}} + O(h^\infty)|u|_{H_h^{-|r|-s}}.$$

Since $H_h^{r,s}$ embeds in $H_h^{-|r|-s}$, we deduce that for $v \overset{\text{def}}{=} e^{G_{r,s}(h)}u$,

$$|[A]_{r,s}v| \leq Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|.$$

**Case III:** $(x_0,\xi_0) \in \overline{T}^* S^*\mathbb{M}$**,** $(x_0,\xi_0) \notin \overline{E_0^*} \oplus \overline{E_u^*}$**.** In this case $(x_0,\xi_0)$ admits a neighborhood $U$ in $\overline{T}^* S^*\mathbb{M}$ such that

$$\mathrm{d}(\exp(-tH_{\sigma_{H_1}})(U), \kappa(E_s^*)) \to 0 \text{ as } t \to -\infty,$$

see [DZ16a, Equation (3.2)]. Hence for $T$ large enough, $\exp(-TH_{\sigma_{H_1}})(U) \subset U_0$ where $U_0$ is the open set defined in Case II. We recall that $p_{\varepsilon,r,s} - iq_{\varepsilon,r,s}$ is the principal symbol of $[\widetilde{P}_\varepsilon(\lambda)]_{r,s}$, and that $p_{\varepsilon,r,s} = \sigma_{H_1} + hS^{1/2}$ and $q_{\varepsilon,r,s} \geq 0$. Since $\sigma_{H_1}$ is homogeneous of degree 1 we can apply Lemma 2.2.5. It shows that if $B \in \Psi_h^0$ has wavefront set contained in $U_0$ then $|[A]_{r,s}v| \leq C|Bv| + Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|$. Combined with the result of Case II, we get

$$|[A]_{r,s}v| \leq Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|.$$

**Case IV:** $(x_0,\xi_0) \in E_u^* \setminus 0$**.** We recall that the lifted geodesic flow $\exp(-tH_{\sigma_{H_1}})(x_0,\xi_0)$ is equal to $\left(e^{-tH_1}(x_0), (de^{-tH_1}(x_0)^{-1})^*\xi_0\right)$. We observe that $\exp(-tH_{\sigma_{H_1}})(x_0,\xi_0)$ converges to the zero section as $t \to +\infty$: because of $\xi_0 \in E_u^*(x_0) = E_s(x_0)$ and of (2.2.4),

$$\left|(de^{-tH_1}(x_0)^{-1})^*\xi_0\right|_g = \left|(de^{tH_1}(e^{-tH_1}(x_0)))^*\xi_0\right|_g \leq Ce^{-ct}.$$

Since $\mathrm{Ell}_h(Q)$ contains the zero section, there exists $T > 0$ such that $\exp(-TH_{\sigma_{H_1}})(x_0,\xi_0)$ belongs to $\mathrm{Ell}_h(Q)$. We apply again Lemma 2.2.5: if $\mathrm{WF}_h(A)$ is supported sufficiently close to $E_u^*$, there exists $B \in \Psi_h^0$ with wavefront set contained in the elliptic set of $Q$ such that $|[A]_{r,s}v| \leq C|Bv| + Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|$. Together with Case I, it implies

$$|[A]_{r,s}v| \leq Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|.$$

**Case V:** $(x_0,\xi_0) \in \kappa(E_u^*)$**.** We recall that $q_\varepsilon \geq 0$ everywhere and that $p_\varepsilon = \sigma_{H_1}$ is homogeneous of degree 1 and independent of $h$. Hence we can apply the radial sink estimate

(Lemma 2.2.7). Fix $B_1 \in \Psi_h^0$ elliptic on $\kappa(E_u^*)$, such that $\mathrm{WF}_h(B_1) \cap \overline{E_0^*} = \emptyset$ and such that $\rho_0 m = -1$ on $\mathrm{WF}_h(B_1)$. Then (after possibly increasing the value of $s_0$ given in Case II) there exist a neighborhood $U_1$ of $(x_0, \xi_0)$ and $B \in \Psi_h^0$ with $\mathrm{WF}_h(B) \subset \mathrm{WF}_h(B_1) \setminus \kappa(E_u^*)$ such that if $\mathrm{WF}_h(A) \subset U_1$ and $s \geq s_0$,

$$|Au|_{H_h^{r-s}} \leq C|Bu|_{H_h^{r-s}} + Ch^{-1}|B_1 \widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r-s}} + O(h^\infty)|u|_{H_h^{-|r|-s}}. \tag{2.6.5}$$

Without loss of generality $U_1$ is small enough so that the spaces $H_h^{r-s}$, $H_h^{r,s}$ are microlocally equivalent on $\mathrm{WF}_h(A), \mathrm{WF}_h(B_1), \mathrm{WF}_h(B)$. Hence we can replace $H_h^{r-s}$ by $H_h^{r,s}$ in (2.6.5). In addition since $\mathrm{WF}_h(B_1)$ is supported away from $\kappa(E_0^*)$, it can be written as a finite sum of operators in $\Psi_h^0$ whose wavefront sets are supported near points $(x_0, \xi_0)$ satisfying Cases I-IV. Finally, since $H_h^{r,s}$ embeds in $H_h^{-s-|r|}$, the term $O(h^\infty)|u|_{H_h^{-|r|-s}}$ in the right hand side of (2.6.5) is bounded by $O(h^\infty)|u|_{H_h^{r,s}}$. It follows that

$$|Au|_{H_h^{r,s}} \leq Ch^{-1}|\widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r,s}} + O(h^\infty)|u|_{H_h^{r,s}}.$$

Since $v = e^{G_{r,s}(h)}u$ we deduce that

$$|[A]_{r,s}v| \leq Ch^{-1}|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|.$$

**Case VI:** $(x_0, \xi_0) \in \overline{E_0^*} \setminus \mathrm{Ell}_h(Q)$**.** In particular, $\xi_0 \neq 0$ and $\sigma_{H_1}(x_0, \xi_0) \neq 0$. By (2.6.3), we have $p_{\varepsilon,r,s} = \sigma_{H_1} + sh\log|\xi|_g \cdot S^0$. This shows that the operator $[\widetilde{P}_\varepsilon(\lambda)]_{r,s}$ is elliptic at $(x_0, \xi_0)$. Therefore if $A$ has wavefront set contained in a small neighborhood of $(x_0, \xi_0)$ the elliptic estimate shows that $|[A]_{r,s}v| \leq C|[\widetilde{P}_\varepsilon(\lambda)]_{r,s}v| + O(h^\infty)|v|$.

Since Cases I-VI cover the whole $\overline{T}^* S^*\mathbb{M}$ this ends the proof of the theorem. □

*Proof of Theorem 12.* It is very similar to the end of the proof of [DZ16a, Proposition 3.1]. Fix $R > 0$ and $r \in \mathbb{R}$. Proposition 2.6.1 shows that $|u|_{H_h^{r,s}} \leq C_R h^{-1}|\widetilde{P}_\varepsilon(\lambda)u|_{H_h^{r,s}}$ as long as $0 < \varepsilon \leq h \leq h_0$ and $s$ is large enough. Theorem 11 applied with $N = 2|s| + 2|r| + 1$ yields the estimate

$$|u|_{H_h^{r,s}} \leq C_R h^{-2r-2s-1}|P_\varepsilon(\lambda)u|_{H_h^{r,s}} + O(h)|u|_{H_h^{r,s}}.$$

After possibly decreasing the value of $h_0$ we can absorb the term $O(h)|u|_{H_h^{r,s}}$ by the left hand side. We get $|u|_{H_h^{r,s}} \leq C_R h^{-2r-2s-1}|P_\varepsilon(\lambda)u|_{H_h^{r,s}}$. This estimate implies that the operator $P_\varepsilon(\lambda) : D_h^{r,s} \to H_h^{r,s}$ is injective.

To show the surjectivity of $P_\varepsilon(\lambda)$ we first note that the range of $P_\varepsilon(\lambda)$ is closed in $H_h^{r,s}$. Indeed, let $u_j \in D_h^{r,s}$ such that $P_\varepsilon(\lambda)u_j$ converges in $H_h^{r,s}$. Then $u_j$ is a Cauchy sequence in $H_h^{r,s}$ and it converges to some $u \in H_h^{r,s}$. We must show that $u \in D_h^{r,s}$. The sequence $P_\varepsilon(\lambda)u_j$ is bounded in $H_h^{r,s}$ hence it converges weakly; it follows that $P_\varepsilon(\lambda)u \in H_h^{r,s}$. By Proposition 2.5.1, $\rho_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}}u \in H_h^{r,s}$. In addition for any $\varepsilon > 0$, $\chi_1(\varepsilon^2\Delta)\Delta_{\mathbb{S}}u \in C^\infty$. It follows that $\Delta_{\mathbb{S}}u \in H_h^{r,s}$ hence $H_1 u \in H_h^{r,s}$. Therefore $u$ belongs to the domain of $P_\varepsilon(\lambda)$ and the range of $P_\varepsilon(\lambda)$ is closed.

To conclude we show that the range of $P_\varepsilon(\lambda)$ is dense in $H_h^{r,s}$. The dual of $H_h^{r,s}$ is $H_h^{-r,-s}$. Thus it suffices to prove that if $f \in H_h^{-r,-s}$ is such that $\langle f, P_\varepsilon(\lambda)u\rangle = 0$ for every $u \in H_h^{r,s}$ then $f = 0$, or equivalently that $P_\varepsilon(\lambda)$ is injective. We have

$$P_\varepsilon(\lambda) = -ih\varepsilon\Delta_{\mathbb{S}} - ihH_1 - \lambda h - iQ, \quad -P_\varepsilon(-\overline{\lambda})^* = -ih\varepsilon\Delta_{\mathbb{S}} + ihH_1 - \lambda h - iQ.$$

Therefore $-P_\varepsilon(-\overline{\lambda})^*$ is equal to $P_\varepsilon(\lambda)$ except for $H_1$ which is replaced by $-H_1$. For the dynamics of $-H_1$, $E_u^*$ is a radial source and $E_s^*$ a radial sink. Moreover the imaginary part of $-P_\varepsilon(\lambda)^*$ is non-positive. The space $H_h^{-r,-s}$ has low regularity near $E_s^*$ (the radial sink for $-H_1$) since it is microlocally equivalent to $H_h^{-r-s}$ near $E_s^*$. Similarly $H_h^{-r,-s}$ has high regularity near $E_s^*$ (the radial source for $-H_1$) since it is microlocally equivalent to $H_h^{-r-s}$ near $E_s^*$. Hence the same analysis as in the proof of Proposition 2.6.1 can be applied to $-P_\varepsilon(\lambda)^*$. It shows that for $s$ large enough and $0 < \varepsilon \leq h$ small enough, $\lambda \in \mathbb{D}(0, R)$,

$$|f|_{H_h^{-r,-s}} \leq C_R h^{-2r-2s-1} |P_\varepsilon(-\overline{\lambda})^* f|_{H_h^{-r,-s}}.$$

This shows that $P_\varepsilon(\lambda)^*$ is injective. Hence the range of $P_\varepsilon(\lambda)$ is dense and $P_\varepsilon(\lambda)$ is surjective. This ends the proof of the theorem. □

### 2.6.2 Proof of Theorem 9

We conclude the chapter (and the thesis) with a more precise version of Theorem 9. A function $\varepsilon \in (0, \varepsilon_0) \mapsto f(\varepsilon)$ is said to be $C^1([0, \varepsilon_0))$ if $f$ is $C^1$ on $(0, \varepsilon_0)$ and $f'(\varepsilon)$ has a limit when $\varepsilon \to 0$. By induction we define the class $C^k([0, \varepsilon_0))$. In the following, we shall say that $f$ is smooth at 0 if for every $k > 0$, there exists $\varepsilon_k > 0$ such that $f \in C^k([0, \varepsilon_k))$. The set $\Sigma(P_\varepsilon)$ (resp. $\mathrm{Res}(P_0)$) is defined as the $L^2$-spectrum of $P_\varepsilon = \frac{1}{i}(H_1 + \varepsilon\Delta_{\mathbb{S}})$ (resp. Pollicott–Ruelle resonances of $P_0 = \frac{1}{i}H_1$), with inclusion according to multiplicity.

**Theorem 13.** *The set of accumulation points of $\Sigma(P_\varepsilon)$, as $\varepsilon \to 0^+$, is equal to* $\mathrm{Res}(P_0)$. *If $\lambda_0 \in \mathrm{Res}(P_0)$ has multiplicity $m$, there exist $r_0 > 0, \varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, there exists $\{\lambda_j(\varepsilon)\}_{j=1}^m$ such that $\Sigma(P_\varepsilon) \cap \mathbb{D}(\lambda_0, r_0) = \{\lambda_j(\varepsilon)\}_{j=1}^m$. Moreover,*

*(i) If $m = 1$, then $\varepsilon \in (0, \varepsilon_0) \mapsto \lambda_1(\varepsilon)$ is smooth at 0 and*

$$\lambda_1(\varepsilon) = \lambda_0 + i\varepsilon \int_{S^*\mathbb{M}} \langle \nabla_{\mathbb{S}} u, \nabla_{\mathbb{S}} v\rangle d\mu + O(\varepsilon^2), \tag{2.6.6}$$

*where $u, v$ are the left and right resonant states defined in Lemma 2.2.1.*

*(ii) The finite-rank operators*

$$\Pi_\varepsilon \stackrel{\text{def}}{=} \frac{1}{2\pi i} \oint_{\partial\mathbb{D}(\lambda_0, r_0)} (P_\varepsilon - \lambda)^{-1} d\lambda \ : \ C^\infty(S^*\mathbb{M}) \to \mathscr{D}'(S^*\mathbb{M}) \tag{2.6.7}$$

*form a smooth trace-class family of operators at $\varepsilon = 0$.*

**Remark 2.6.1.** *Theorem 13 shows that as $\varepsilon \to 0^-$, the spectrum of $P_{-\varepsilon}^*$ converges to complex conjugates of Pollicott–Ruelle resonances. Because of the identity $P_\varepsilon = P_{-\varepsilon}^*$, we deduce that the spectrum of $P_\varepsilon$ converges to complex conjugates of Pollicott–Ruelle resonances as $\varepsilon \to 0^-$.*

*Proof.* Fix $R > 0$ and $k_0$ a positive integer. For $1 \leq k \leq k_0$, let $r_k \overset{\text{def}}{=} 2^{k+1} - 2$. By Theorem 12 and [DZ16a, Proposition 3.4] there are $s_0, h_0 > 0$ such that for every $0 \leq \varepsilon \leq h_0$, $r \in [\![0, r_{k_0}]\!]$ and $\lambda \in \mathbb{D}(0, R)$ the operator

$$P_\varepsilon(\lambda) = -i\varepsilon h_0 \Delta_{\mathbb{S}} - ih_0 H_1 - \lambda h_0 - iQ$$

admits a right inverse on $\mathscr{H}^{-r} \overset{\text{def}}{=} H_{h_0}^{-r,s_0}$: there exists a bounded operator $P_\varepsilon(\lambda)^{-1} : \mathscr{H}^{-r} \to \mathscr{H}^{-r}$ with range contained in the domain of $P_\varepsilon(\lambda)$ such that $P_\varepsilon(\lambda) P_\varepsilon(\lambda)^{-1} = \mathrm{Id}_{\mathscr{H}^{-r}}$. We show below that for every $r \in [\![0, r_{k_0} - r_k]\!]$, the operator $P_\varepsilon(\lambda)^{-1} : \mathscr{H}^{-r} \to \mathscr{H}^{-r-r_k}$ is $C^k([0, h_0))$. We proceed by induction on $k$.

We start with $k = 1$. For every $r \in [\![0, r_{k_0}]\!] \cap [\![-2, r_{k_0} - 2]\!] = [\![0, r_{k_0} - r_1]\!]$, the operator $P_\varepsilon(\lambda)^{-1}$ maps $\mathscr{H}^{-r}$ to itself and $\mathscr{H}^{-r-2}$ to itself. This fact, together with the identity

$$P_\varepsilon(\lambda)^{-1} - P_{\varepsilon'}(\lambda)^{-1} = -i(\varepsilon - \varepsilon') P_{\varepsilon'}(\lambda)^{-1} h_0 \Delta_{\mathbb{S}} P_\varepsilon(\lambda)^{-1} \tag{2.6.8}$$

shows that $\varepsilon \in [0, h_0) \mapsto P_\varepsilon(\lambda)^{-1} : \mathscr{H}^{-r} \to \mathscr{H}^{-r-2}$ is differentiable (in particular continuous) with

$$\partial_\varepsilon P_\varepsilon(\lambda) = -i P_\varepsilon(\lambda)^{-1} h_0 \Delta_{\mathbb{S}} P_\varepsilon(\lambda)^{-1}. \tag{2.6.9}$$

The right hand side of (2.6.9) is continuous, hence $\varepsilon \in [0, h_0) \mapsto P_\varepsilon(\lambda)^{-1} : \mathscr{H}^{-r} \to \mathscr{H}^{-r-2}$ is $C^1([0, h_0))$.

Assume now that $k \leq k_0 - 1$ and that for every $r \in [\![0, r_{k_0} - r_k]\!]$, $P_\varepsilon(\lambda)^{-1} : \mathscr{H}^{-r} \to \mathscr{H}^{-r-r_k}$ is $C^k([0, h_0))$. The identity (2.6.9) shows that $\partial_\varepsilon P_\varepsilon(\lambda) : \mathscr{H}^{-r} \to \mathscr{H}^{-r-2r_k-2}$ is also $C^k([0, h_0))$ as long as $r \in [0, r_{k_0} - r_k]\!] \cap [-r_k - 2, r_{k_0} - 2r_k - 2]\!]$. Since $r_{k+1} = 2r_k - 2$, the operator $\partial_\varepsilon P_\varepsilon(\lambda) : \mathscr{H}^{-r} \to \mathscr{H}^{-r-r_{k+1}}$ is $C^k([0, h_0))$ as long as $r \in [0, r_{k_0} - r_{k+1}]\!]$. This implies that $P_\varepsilon(\lambda) : \mathscr{H}^{-r} \to \mathscr{H}^{-r-r_{k+1}}$ is $C^{k+1}([0, h_0))$ for the above range of $r$. This completes the induction process.

It follows that the operator $P_\varepsilon(\lambda)^{-1} : \mathscr{H}^0 \to \mathscr{H}^{-r_{k_0}}$ is $C^{k_0}([0, h_0))$. We recall that $Q$ is a smoothing operator. In particular, $Q$ maps $\mathscr{H}^{-r_{k_0}}$ to the Sobolev space $H^N$ for any $N$. It follows that $QP_\varepsilon(\lambda)$ is a trace-class operator with holomorphic dependence in $\lambda \in \mathbb{D}(0, R)$ and $C^{k_0}$ dependence in $\varepsilon \in [0, h_0)$. Since $k_0$ was arbitrary, $QP_\varepsilon(\lambda)$ is smooth at $\varepsilon = 0$. For $\varepsilon \in [0, h_0)$ and $\lambda \in \mathbb{D}(0, R)$, we define the Fredholm determinant

$$D_\varepsilon(\lambda) = \mathrm{Det}_{\mathscr{H}^0}(\mathrm{Id} + iQP_\varepsilon(\lambda)^{-1}),$$

which depends holomorphically in $\lambda$, and which is smooth at $\varepsilon = 0$.

The operator $h_0(P_\varepsilon - \lambda) = P_\varepsilon(\lambda) + iQ$ is Fredholm, because where $P_\varepsilon(\lambda)$ admits a right inverse on $\mathscr{H}^0$ and $Q$ is compact. Hence, the $\mathscr{H}^0$-spectrum of $P_\varepsilon$ in $\mathbb{D}(0, R)$ is discrete and equal to the zero set of $D_\varepsilon(\lambda)$. When $\varepsilon \neq 0$ the operator $P_\varepsilon$ is subelliptic. Consequently, $\mathscr{H}^0$-eigenvectors of $P_\varepsilon$ must belong to the (standard) Sobolev space $H^2$, thus to the domain of $P_\varepsilon$ on $L^2$. Conversely, $L^2$-eigenvectors of $P_\varepsilon$ must belong to the (standard) Sobolev space $H^{s_0}$, thus to $\mathscr{H}^0$. This shows that for $\varepsilon \neq 0$, the $L^2$-spectrum and $\mathscr{H}^0$-spectrum of $P_\varepsilon$ in $\mathbb{D}(0,R)$ are equal, and the $L^2$-eigenvalues of $P_\varepsilon$ in $\mathbb{D}(0,R)$ are exactly the zeroes of $D_\varepsilon(\lambda)$.

For $\varepsilon > 0$, $D_\varepsilon(\lambda)$ is a holomorphic function of $\lambda$ whose zero set is the $L^2$-spectrum of $P_\varepsilon$ in $\mathbb{D}(0,R)$, and the zero set of $D_0(\lambda)$ is the Pollicott–Ruelle spectrum of $P_0$ in $\mathbb{D}(0,R)$ – see [DZ16d, Proposition 3.2]. Since $D_\varepsilon(\lambda)$ is smooth at $\varepsilon = 0$, the first part of the theorem follows from an application of Hurwitz's theorem.

If $\lambda_0$ is a Pollicott–Ruelle resonance of $P_0$ and $\lambda_1(\varepsilon)$ is the unique eigenvalue of $P_\varepsilon$ converging to $\lambda_0$, the implicit function theorem shows that $\varepsilon \mapsto \lambda_1(\varepsilon)$ is smooth. We compute now the leading terms in the expansion (2.6.6), inspired by the method of [Dr15, §3.1]. Denote by $\mathrm{Res}(P_0)$ the set of Pollicott–Ruelle resonances of $P_0 = \frac{1}{i}H_1$ and fix $K$ be a compact subset of $\mathbb{D}(0,R)\backslash\mathrm{Res}(P_0)$. For every $\lambda \in K$, $D_0(\lambda) \neq 0$ and the operator $\mathrm{Id}+iQP_0(\lambda)^{-1} : \mathscr{H}^0 \to \mathscr{H}^0$ is invertible. Therefore, for every $0 < \varepsilon \leq h_0$ and $\lambda \in K$,

$$\mathrm{Id} + iQP_\varepsilon(\lambda)^{-1} = \left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right) \cdot \left(\mathrm{Id} + \left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} iQ\left(P_\varepsilon(\lambda)^{-1} - P_0(\lambda)^{-1}\right)\right).$$

Uniformly for $\lambda \in K$, the operator $\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1}$ is bounded on $\mathscr{H}^0$ and by (2.6.8), $Q\left(P_\varepsilon(\lambda)^{-1} - P_0(\lambda)^{-1}\right)$ has trace-class norm $O(\varepsilon)$. The identity (2.6.8) implies for $\lambda \in K$,

$$\begin{aligned}D_\varepsilon(\lambda) &= D_0(\lambda) \cdot \mathrm{Det}_{\mathscr{H}^0}\left(\mathrm{Id} + i\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} Q\left(P_\varepsilon(\lambda)^{-1} - P_0(\lambda)^{-1}\right)\right)\\ &= D_0(\lambda) \cdot \left(1 + \varepsilon h_0 \mathrm{Tr}_{\mathscr{H}^0}\left(\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}} P_\varepsilon(\lambda)^{-1}\right) + O(\varepsilon^2)\right).\end{aligned}$$

The operator $QP_0(\lambda)^{-1}\Delta_{\mathbb{S}}$ extends to a trace-class operator in $\mathscr{H}^0$. Because of the identity (2.6.8), we have uniformly for $\lambda \in K$,

$$\begin{aligned}h_0 \mathrm{Tr}_{\mathscr{H}^0}\left(\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}} P_\varepsilon(\lambda)^{-1}\right) &= f_1(\lambda) + O(\varepsilon),\\ f_1(\lambda) \overset{\mathrm{def}}{=} h_0 \mathrm{Tr}_{\mathscr{H}^0}\left(\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}} P_0(\lambda)^{-1}\right)&.\end{aligned}$$

It follows that uniformly for $\lambda \in K$,

$$D_\varepsilon(\lambda) = D_0(\lambda) \cdot \left(1 + f_1(\lambda)\varepsilon + O(\varepsilon^2)\right). \tag{2.6.10}$$

In (2.6.10), the function $D_\varepsilon$ is holomorphic on $\mathbb{D}(0,R)$ and $f_1(\lambda)$ is meromorphic in $\mathbb{D}(0,R)$, with poles in $\mathbb{D}(0,R) \cap \mathrm{Res}(P_0)$. Therefore we can apply [Dr15, Lemma 4.4] with $E = \mathbb{D}(0,R)$, $S_0 = \mathrm{Res}(P_0)$, $D_\varepsilon(\lambda)/D_0(\lambda) = 1 + f_1(\lambda)\varepsilon + O(\varepsilon^2)$ and $g(\lambda,\varepsilon) = D_0(\lambda)$ (strictly speaking, [Dr15, Lemma 4.4] is stated there with $E = \mathbb{C}$ or $\mathbb{C} \setminus 0$; but it also holds without change in the proof when $E = \mathbb{D}(0,R)$). It shows that (2.6.10) is valid uniformly for $\lambda \in \mathbb{D}(0,R) \setminus \mathrm{Res}(P_0)$ and that the function $D_0(\lambda)f_1(\lambda)$ is holomorphic on $\mathbb{D}(0,R)$.

Let $\lambda_0 \in \mathbb{D}(0,R)$ be a simple resonance of $\mathrm{Res}(P_0)$. We now work with $f_1(\lambda)$ for $\lambda$ in a small punctured disk $\mathscr{D} \setminus \lambda_0 \subset \mathbb{D}(0,R)$, so that $\lambda_0$ is the only resonance of $P_0$ in $\mathscr{D}$. We have

$$\begin{aligned}f_1(\lambda) &= h_0 \mathrm{Tr}_{\mathscr{H}^0}\left(\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}} P_0(\lambda)^{-1}\right)\\ &= h_0 \mathrm{Tr}_{\mathscr{H}^0}\left(P_0(\lambda)^{-1}\left(\mathrm{Id} + iQP_0(\lambda)^{-1}\right)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}}\right) = \mathrm{Tr}_{\mathscr{H}^0}\left((P_0 - \lambda)^{-1} QP_0(\lambda)^{-1}\Delta_{\mathbb{S}}\right).\end{aligned}$$

In the above we used the cyclicity of the trace and the identity

$$P_0(\lambda)^{-1}\left(\mathrm{Id}+iQP_0(\lambda)^{-1}\right)^{-1}=(P_0(\lambda)+iQ)^{-1}=h_0^{-1}(P_0-\lambda)^{-1}.$$

Because of (2.2.6) and since $P_0(\lambda)^{-1}$ is holomorphic near $\lambda_0$, we can write

$$(P_0-\lambda)^{-1}QP_0(\lambda)^{-1}\Delta_{\mathbb{S}}=(i(P_0-\lambda)^{-1}-h_0P_0(\lambda)^{-1})\Delta_{\mathbb{S}}=\frac{iu\otimes v\Delta_{\mathbb{S}}}{\lambda-\lambda_0}+B(\lambda), \qquad (2.6.11)$$

where $B(\lambda)$ denotes a holomorphic family of operators near $\lambda_0$. The right hand side of (2.6.11) is trace-class on $\mathscr{H}^0$ and the operator $u\otimes v\Delta_{\mathbb{S}}$ is of rank 1. Therefore $B(\lambda)$ is trace-class on $\mathscr{H}^0$ and $f_0(\lambda)\overset{\text{def}}{=}\mathrm{Tr}_{\mathscr{H}^0}(B(\lambda))$ is holomorphic. It follows that

$$f_1(\lambda)-f_0(\lambda)=\frac{i\mathrm{Tr}_{\mathscr{H}^0}\left(u\otimes v\Delta_{\mathbb{S}}\right)}{\lambda-\lambda_0}=\frac{i\mathrm{Tr}_{\mathscr{H}^0}\left(\Delta_{\mathbb{S}}u\otimes v\right)}{\lambda-\lambda_0}=\frac{i}{\lambda_0-\lambda}\int_{S^*\mathbb{M}}\langle\nabla_{\mathbb{S}}u,\nabla_{\mathbb{S}}v\rangle.$$

In the last equality we used that $\Delta_{\mathbb{S}}u$ and $v$ have wavefront sets contained in $E_u^*$ and $E_s^*$, respectively. Hence the trace of the operator $\Delta_{\mathbb{S}}u\otimes v$ is given by integrating the kernel $\Delta_{\mathbb{S}}u(x)v(y)$ along the diagonal $\{x=y\}$ according to [GS94, Proposition 7.6]. The operator $\nabla_{\mathbb{S}}$ was defined in §2.3.1 and the scalar product $\langle\cdot,\cdot\rangle$ is inherited from the Euclidean structure on the fibers of $T^*\mathbb{M}$.

Combining the above, we obtain that uniformly in $\varepsilon$ small enough and $\lambda\in\mathscr{D}\setminus\lambda_0$,

$$\begin{aligned}D_\varepsilon(\lambda)&=D_0(\lambda)-i\varepsilon\frac{D_0(\lambda)}{\lambda-\lambda_0}\int_{S^*\mathbb{M}}\langle\nabla_{\mathbb{S}}u,\nabla_{\mathbb{S}}v\rangle+\varepsilon D_0(\lambda)f_0(\lambda)+O(\varepsilon^2)\\&=D_0'(\lambda_0)\left(\lambda-\lambda_0-i\varepsilon\int_{S^*\mathbb{M}}\langle\nabla_{\mathbb{S}}u,\nabla_{\mathbb{S}}v\rangle+O(\varepsilon(\lambda-\lambda_0))+O(\varepsilon^2)\right).\end{aligned}$$

Recall that $\lambda_1(\varepsilon)$ is the unique eigenvalue of $P_\varepsilon$ near $\lambda_0$. In particular $D_\varepsilon(\lambda_1(\varepsilon))=0$. Since $\varepsilon\mapsto\lambda_1(\varepsilon)$ is smooth, $\lambda_1(\varepsilon)=\lambda_0+O(\varepsilon)$. This yields

$$\lambda_1(\varepsilon)=\lambda_0+i\varepsilon\int_{S^*\mathbb{M}}\langle\nabla_{\mathbb{S}}u,\nabla_{\mathbb{S}}v\rangle+O(\varepsilon^2).$$

This concludes the proof of $(i)$.

For $(ii)$, we fix $k_0>0$ and we recall that $P_\varepsilon(\lambda)^{-1}:\mathscr{H}^0\to\mathscr{H}^{-r_{k_0}}$ is $C^{k_0}([0,h_0))$. Since $h_0(P_\varepsilon-\lambda)=P_\varepsilon(\lambda)+Q$, where $Q$ is smoothing, the family $P_\varepsilon-\lambda:\mathscr{H}^{-r_{k_0}}\to\mathscr{H}^0$ is Fredholm with $C^{k_0}$ dependence in $\varepsilon$. Hence, $(P_\varepsilon-\lambda)^{-1}$ is a meromorphic family of operators with poles of finite rank, with $C^{k_0}$ dependence in $\varepsilon$. This shows that the family of operators $\varepsilon\to\Pi_\varepsilon:\mathscr{H}^0\to\mathscr{H}^{-r_{k_0}}$ given by (2.6.7) is $C^{k_0}([0,h_0))$. A fortiori, $\varepsilon\mapsto\Pi_\varepsilon:C^\infty(S^*\mathbb{M})\to\mathscr{D}'(S^*\mathbb{M})$ is also $C^{k_0}([0,h_0))$, hence smooth at $\varepsilon=0$. □

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