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Author
Piltner, Reinhard

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TREFFTZ-TYPE BOUNDARY ELEMENTS FOR THE EVALUATION OF SYMMETRIC COEFFICIENT MATRICES

BY

R. PILTNER

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DEPARTMENT OF CIVIL ENGINEERING
UNIVERSITY OF CALIFORNIA
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Trefftz-type boundary elements for the evaluation of symmetric coefficient matrices

R. Piltner
Department of Civil Engineering, SEMM, University of California at Berkeley, Berkeley, CA 94720, U.S.A.

Abstract. The classical Trefftz-method can be generalized such that different types of finite elements and boundary elements are obtained. In a Trefftz-type approach we utilize functions which a priori satisfy the governing differential equations. In this paper the systematic construction of singular Trefftz-trial functions for elasticity problems is discussed. For convenience a list of solution representations and particular solutions is given which did not appear together elsewhere. The Trefftz-trial functions with singular expressions on the boundary are constructed such that the physical components (stresses, strains, displacements) remain finite in the solution domain and on the boundary. The unknown coefficients of the linearly independent Trefftz-trial functions for the physical components can be obtained by using a variational formulation. The symmetric coefficient matrix in the discussed procedure can be obtained from the evaluation of boundary integrals. As an application of the proposed boundary element algorithm, the symmetric stiffness matrices of subdomains (finite element domains) are calculated. For the numerical example the solution domain is decomposed into triangular subdomains so that a standard finite element program could be used to assemble the system of equations. The chosen example is meant as a simple test for the proposed algorithm and should not be understood as a proposal for a new triangular finite element. Using the proposed boundary element techniques, symmetric stiffness matrices for irregular shaped subdomains (finite elements) can be derived. However, in order to use the method in a finite element package for the coupling of irregular shaped subdomains some program modifications will be necessary.

1. Introduction

The research in boundary integral equations and boundary element methods covers by now many areas and different approaches, see e.g. Annigeri and Tseng (1990), Antes (1984, 1988), Banerjee and Butterfield (1981), Banerjee and Watson (1986), Beskos

(1987), Brebbia and Dominguez (1989), Brebbia and Connor (1989), Crouch and Starfield (1983), Cruse (1969, 1988a, 1988b), Geis (1989), Hartmann (1989), Jaswon and Symm (1977), Kobayashi and Nishimura (1992), Tanaka and Cruse (1988), Deb and Banerjee (1990), Heise (1978), Hsiao, Kopp and Wendland (1984), Hunt and Isaacs (1981), Lachat and Watson (1976), Oliveira (1968), Piltner and Taylor (1989, 1990a, 1990b, 1990c), Rizzo (1967,1990), Wendland (1985a, 1985b). Since we are particularly interested in obtaining symmetric coefficient matrices and convenient ways of coupling the boundary element method with finite elements, it might be helpful for future research to look at the subject under the aspect of a Trefftz-type approach.

The concept of using a set of linearly independent solution functions of a differential equation under consideration was introduced by Trefftz (1926). Trefftz chose harmonic polynomials for the example of the Laplace equation and calculated the unknown coefficients of his trial functions by using a variational formulation. Since the Trefftz-trial functions satisfy a priori the governing differential equations, the variational formulation can be rewritten in a form which contains only boundary integrals.

Since computers were not available during Trefftz's time, it took a long time until the basic ideas of Trefftz were used for numerical purposes and further developments. Some of the early numerical applications of the Trefftz method dealt with the coupling of boundary solution procedures with finite elements, see e.g Stein (1973), Ruoff (1973), Zienkiewics, Kelly and Bettess (1977).

Further research led to different techniques of obtaining symmetric finite element matrices by using functions in the sense of Trefftz and evaluating integrals along the finite element edges, see e.g. Jirousek (1978, 1985, 1986), Piltner (1982, 1985a, 1985b, 1985c), Zienkiewics and Taylor (1989). Examples for those finite elements which utilize variational formulations involving only boundary integrals are elements which can contain cracks, holes or sharp corners. These types of elements can be obtained from a mixed variational formulation. Often a Hellinger-Reissner variational formulation is used for such finite elements. Finite elements based on a Hellinger-Reissner formulation and stress fields satisfying the equilibrium equations are often called hybrid elements. For hybrid elements we use two independently assumed fields in the variational formulation. From the literature on hybrid elements especially the pioneering work of Pian (1964, 1983, 1984, 1988) has to be emphasized.

In order to describe those finite elements for which the trial functions for the stresses satisfy not only the equilibrium equations but also the compatibility equations, the term hybrid Trefftz elements was proposed by Jirousek and Guex (1986) and used thereafter (Jirousek 1987; Jirousek and Venkatesh 1989; Piltner 1992b). The name "Trefftz" indicates that the trial functions satisfy all governing differential equations. Taking this point of view one could classify, for example, the hybrid crack elements of Tong, Pian and Lasry (1973) and Piltner (1985a) as hybrid Trefftz elements.

The techniques used for hybrid Trefftz elements can also be utilized for the derivation of symmetric boundary element techniques. Whereas in finite element applications we usually use nonsingular trial functions (except for elements which include a stress singularity, as for example in crack elements), in boundary elements we utilize singular functions. For the proposed boundary element algorithm of this paper, suitable singular functions with poles on the boundary are constructed.

For plane problems a detailed derivation of stress and displacement formulations in complex form is given in the books of Muskhelishvili (1953) and Lekhnitskii (1963, 1968). Since for plane elasticity problems complex representations for displacements and stresses have been used with success for numerical applications (Piltner 1982, 1985a, 1985b, 1985c), one of the research topics of the author became the derivation of solution representations for three-dimensional elasticity problems and for thick elastic plates, in extension and under bending (Piltner 1987, 1988a, 1988b, 1989, 1992a). For all elasticity problems considered, the displacements and stresses can be given in terms of arbitrary complex valued functions. The functions are arbitrary in the sense that with any choice of the complex functions the governing differential equations are automatically satisfied. The displacements of the solution representations satisfy the Navier-equations, and the stresses satisfy the equilibrium equations. Moreover, the stresses satisfy the compatibility equations and can be obtained from the displacements $\mathbf{u}$ through the constitutive equation $\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{D}\mathbf{u}$, where $\mathbf{D}$ is a differential operator matrix.

In brief, the Trefftz-type boundary element concept can be characterized as follows:

- The trial functions satisfy the differential equations.
- The trial functions contain singular functions with poles on the boundary.
- The trial functions depend on unknowns on the boundary.

$$\text{displacements:}\quad \mathbf{u} = \mathbf{U}^{*}\mathbf{c} + \mathbf{u}_p \quad \text{in } \Omega \text{ and on } \Gamma \tag{1.1}$$

stresses: $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{*}\mathbf{c} + \boldsymbol{\sigma}_p$ in Ω and on Γ (1.2)

tractions: $\mathbf{T} = \mathbf{T}^{*}\mathbf{c} + \mathbf{T}_p$ on Γ (1.3)

- **The trial functions containing singular expressions have to be constructed such that all displacements and stresses remain finite in Ω and on Γ.**

The displacements, stresses, and tractions are decomposed into a homogeneous and a particular solution part. Only the homogeneous solution part, which satisfy the homogeneous differential equations, contains unknown parameters collected in a vector **c**. $\mathbf{U}^{*}$, $\boldsymbol{\sigma}^{*}$ and $\mathbf{T}^{*}$ contain singular functions.

The solution terms $\mathbf{u}_p$, $\boldsymbol{\sigma}_p$, $\mathbf{T}_p$ contain solutions of the inhomogeneous differential equations. They can also contain solutions for particular load cases and involve only quantities which are already known. In a variational formulation, the particular solution terms will enable us to get rid of the domain integrals which contain body force distributions.

The vector **c**, involving unknown parameters on the boundary only, can be calculated with the following methods:

(i) point collocation

(ii) variational formulation, containing only boundary integrals

(iii) least square method, for the minimization of the difference between prescribed boundary values and approximated values on the boundary.

In method (i), we satisfy the boundary conditions at selected points on the boundary and get a nonsymmetric coefficient matrix. In methods (ii) and (iii) the boundary conditions are satisfied in an integral sense and symmetric coefficient matrices are obtained.

Since the main objective of this paper is to present solution representations as a tool for a systematic construction of singular Trefftz-type trial functions for displacements and stresses, the following section is devoted to the representation of these physical quantities in elasticity.

2 Overview about solution representations for elasticity problems and some selected particular solutions

In the following a list of homogeneous solutions will be given in complex form. For some load cases of practical interest particular solutions are given as well. The indices h and p for "homogeneous" and "particular" are omitted since the section headings indicate already the type of solution.

2.1 Plane stress and plane strain in isotropic bodies

2.1.1 Homogeneous solution

$$
\begin{aligned}
2\mu u &= \mathrm{Re}[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}] \\
2\mu v &= \mathrm{Im}[\kappa\Phi(z) - z\overline{\Phi'(z)} - \overline{\Psi(z)}] \\
\sigma_{xx} &= \mathrm{Re}[2\Phi'(z) - \bar{z}\Phi''(z) - \Psi'(z)] \\
\sigma_{yy} &= \mathrm{Re}[2\Phi'(z) + \bar{z}\Phi''(z) + \Psi'(z)] \\
\tau_{xy} &= \mathrm{Im}[\bar{z}\Phi''(z) + \Psi'(z)]
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
& z = x + iy, \\
& 2\mu = E/(1+\nu) \\
& \kappa = \begin{cases} (3-\nu)/(1+\nu) & \text{for plane stress} \\ (3-4\nu) & \text{for plane strain} \end{cases}
\end{aligned}
\tag{2.2}
$$

2.1.2 Particular solution for constant body forces $\bar{f}_x$, $\bar{f}_y$

$$
2\mu u = \frac{1}{2} a\left(1 - \frac{\lambda}{2(\lambda+\mu)}\right)\bar{f}_x\, x^2 - \frac{\lambda}{2(\lambda+\mu)}\, b\bar{f}_y\, xy + \left(\frac{\lambda}{4(\lambda+\mu)}\, a - (1+a)\right)\bar{f}_x\, y^2
$$

$$
2\mu v = \frac{1}{2} b\left(1 - \frac{\lambda}{2(\lambda+\mu)}\right)\bar{f}_y\, y^2 - \frac{\lambda}{2(\lambda+\mu)}\, a\bar{f}_x\, xy + \left(\frac{\lambda}{4(\lambda+\mu)}\, b - (1+b)\right)\bar{f}_y\, x^2
$$

$$
\sigma_{xx} = a\,\bar{f}_x\, x, \qquad \sigma_{yy} = b\,\bar{f}_y\, y, \qquad \tau_{xy} = -(1+b)\bar{f}_y\, x - (1+a)\bar{f}_x\, y \tag{2.3}
$$

where

$$\frac{\lambda}{2(\lambda+\mu)} = \begin{cases} \nu/(1+\nu) & \text{for plane stress} \\ \nu & \text{for plane strain} \end{cases} \tag{2.4}$$

and a and b can be chosen arbitrarily, for example as a = −1, b = −1 or as a = 0, b = 0.

2.2 Plane stress and plane strain in anisotropic bodies

2.2.1 Homogeneous solution

$$\begin{aligned} u &= 2\,\mathrm{Re}[p_1\Phi_1(z_1) + p_2\Phi_2(z_2)], \\ v &= 2\,\mathrm{Re}[q_1\Phi_1(z_1) + q_2\Phi_2(z_2)], \\ \sigma_{xx} &= 2\,\mathrm{Re}[\mu_1^2\Phi_1'(z_1) + \mu_2^2\Phi_2'(z_2)], \\ \sigma_{yy} &= 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)], \\ \tau_{xy} &= -2\,\mathrm{Re}[\mu_1\Phi_1'(z_1) + \mu_2\Phi_2'(z_2)]. \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} z_1 &= x + \mu_1 y = (x + \alpha y) + i\beta y, \\ z_2 &= x + \mu_2 y = (x + \gamma y) + i\delta y, \\ p_1 &= b_{11}\mu_1^2 + b_{12} - b_{16}\mu_1, \\ p_2 &= b_{11}\mu_2^2 + b_{12} - b_{16}\mu_2, \\ q_1 &= b_{12}\mu_1 + \frac{b_{22}}{\mu_1} - b_{26}, \\ q_2 &= b_{12}\mu_2 + \frac{b_{22}}{\mu_2} - b_{26}, \\ b_{ij} &= \begin{cases} a_{ij} & \text{for plane stress} \\ a_{ij} - \dfrac{a_{i3}a_{j3}}{a_{33}} \quad (i,j = 1,2,6) & \text{for plane strain.} \end{cases} \end{aligned} \tag{2.6}$$

a_{ij} are the elastic compliance coefficients, and μ_1, μ_2, $\bar{\mu}_1$, $\bar{\mu}_2$ are the solutions of the characteristic equation

$$b_{11}\mu_k^4 - 2b_{16}\mu_k^3 + (2b_{12} + b_{66})\mu_k^2 - 2b_{26}\mu_k + b_{22} = 0. \tag{2.7}$$

2.2.2 Particular solution for constant body forces $\bar{f}_x$, $\bar{f}_y$

$$u = -\frac{1}{2}\bar{f}_x b_{11} x^2 - \bar{f}_y b_{12} xy + \frac{1}{2}(\bar{f}_x b_{12} - \bar{f}_y b_{26}) y^2,$$

$$v = \frac{1}{2}(\bar{f}_y b_{12} - \bar{f}_x b_{16}) x^2 - \bar{f}_x b_{12} xy - \frac{1}{2}\bar{f}_y b_{22} y^2,$$

$$\sigma_{xx} = -\bar{f}_x x, \qquad (2.8)$$

$$\sigma_{yy} = -\bar{f}_y y,$$

$$\tau_{xy} = 0.$$

2.3 Bending of isotropic thin plates (Kirchhoff-plate theory)

Moments and shear forces:

$$M_{xx} = -D(w_{xx} + \nu w_{yy})$$

$$M_{yy} = -D(w_{yy} + \nu w_{xx})$$

$$M_{xy} = D(1 - \nu) w_{xy} \qquad (2.9)$$

$$Q_x = -D\frac{\partial}{\partial x}\Delta w$$

$$Q_y = -D\frac{\partial}{\partial y}\Delta w$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} \qquad (2.10)$$

2.3.1 Homogeneous solution

$$w(x,y) = \mathrm{Re}[\bar{z}\Phi(z) + \Psi(z)]$$

$$w_x = \mathrm{Re}[\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi'(z)}]$$

$$w_y = \mathrm{Im}[\Phi(z) + z\overline{\Phi'(z)} + \overline{\Psi'(z)}]$$

$$w_{xx} = \mathrm{Re}[\Phi'(z) + \overline{\Phi'(z)} + z\overline{\Phi''(z)} + \overline{\Psi''(z)}]$$

$$w_{yy} = \mathrm{Re}[\Phi'(z) + \overline{\Phi'(z)} - z\overline{\Phi''(z)} - \overline{\Psi''(z)}]$$

$$w_{xy} = \mathrm{Im}[z\overline{\Phi''(z)} + \overline{\Psi''(z)}] \qquad (2.11)$$

$$\Delta w = 4\mathrm{Re}[\Phi'(z)]$$

$$\frac{\partial}{\partial x}\Delta w = 4\mathrm{Re}[\Phi''(z)]$$

$$\frac{\partial}{\partial y}\Delta w = -4\mathrm{Im}[\Phi''(z)]$$

where

$$z = x + iy. \tag{2.12}$$

2.3.2 Particular solution for a constant normal load $p(x,y)=q_0$

$$w = \frac{q_0}{48D}(x^4 + y^4), \quad w_x = \frac{q_0}{12D}x^3, \quad w_y = \frac{q_0}{12D}y^3$$

$$w_{xx} = \frac{q_0}{4D}x^2, \quad w_{yy} = \frac{q_0}{4D}y^2, \quad w_{xy} = 0 \tag{2.13}$$

$$\Delta w = \frac{q_0}{4D}(x^2 + y^2), \quad \frac{\partial}{\partial x}\Delta w = \frac{q_0}{2D}x, \quad \frac{\partial}{\partial y}\Delta w = \frac{q_0}{2D}y$$

2.4 Bending of anisotropic thin plates

Moments and shear forces:

$$M_{xx} = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$M_{yy} = -\left(D_{12}\frac{\partial^2 w}{\partial x^2} + D_{22}\frac{\partial^2 w}{\partial y^2} + 2D_{26}\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$M_{xy} = -\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x \partial y}\right), \tag{2.14}$$

$$Q_x = -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + 3D_{16}\frac{\partial^3 w}{\partial x^2 \partial y} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x \partial y^2} + D_{26}\frac{\partial^3 w}{\partial y^3}\right],$$

$$Q_y = -\left[D_{16}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x^2 \partial y} + 3D_{26}\frac{\partial^3 w}{\partial x \partial y^2} + D_{22}\frac{\partial^3 w}{\partial y^3}\right],$$

where

$$D_{ij} = c_{ij}\frac{h^3}{12}. \tag{2.15}$$

and c_{ij} are the elastic constants.

2.4.1 Homogeneous solution

$$
\begin{aligned}
w &= 2\mathrm{Re}[\Phi_1(z_1) + \Phi_2(z_2)],\\
M_{xx} &= -\mathrm{Re}[p_1\Phi_1''(z_1) + p_2\Phi_2''(z_2)],\\
M_{yy} &= -\mathrm{Re}[q_1\Phi_1''(z_1) + q_2\Phi_2''(z_2)],\\
M_{xy} &= -\mathrm{Re}[r_1\Phi_1''(z_1) + r_2\Phi_2''(z_2)],\\
Q_x &= -\mathrm{Re}[\mu_1 s_1\Phi_1'''(z_1) + \mu_2 s_2\Phi_2'''(z_2)],\\
Q_y &= \mathrm{Re}[s_1\Phi_1'''(z_1) + s_2\Phi_2'''(z_2)],
\end{aligned}
\tag{2.16}
$$

where

$$
\begin{aligned}
z_1 &= x + \mu_1 y = (x + \alpha y) + i\beta y,\\
z_2 &= x + \mu_2 y = (x + \gamma y) + i\delta y,\\
p_1 &= D_{11} + D_{12}\mu_1^2 + 2D_{16}\mu_1, \quad p_2 = D_{11} + D_{12}\mu_2^2 + 2D_{16}\mu_2,\\
q_1 &= D_{12} + D_{22}\mu_1^2 + 2D_{26}\mu_1, \quad q_2 = D_{12} + D_{22}\mu_2^2 + 2D_{26}\mu_2,\\
r_1 &= D_{16} + D_{26}\mu_1^2 + 2D_{66}\mu_1, \quad r_2 = D_{16} + D_{26}\mu_2^2 + 2D_{66}\mu_2,\\
s_1 &= \frac{D_{11}}{\mu_1} + 3D_{16} + (D_{12} + 2D_{66})\mu_1 + D_{26}\mu_1^2,\\
s_2 &= \frac{D_{11}}{\mu_2} + 3D_{16} + (D_{12} + 2D_{66})\mu_2 + D_{26}\mu_2^2,\\
s_1 - r_1 &= \frac{p_1}{\mu_1}, \quad s_2 - r_2 = \frac{p_2}{\mu_2}, \quad s_1 + r_1 = -q_1\mu_1, \quad s_2 + r_2 = -q_2\mu_2.
\end{aligned}
\tag{2.17}
$$

and $\mu_1, \mu_2, \bar{\mu}_1, \bar{\mu}_2$ are the solutions of the characteristic equation

$$
D_{22}\mu_k^4 + 4D_{26}\mu_k^3 + 2(D_{12} + 2D_{66})\mu_k^2 + 4D_{16}\mu_k + D_{11} = 0. \tag{2.18}
$$

2.4.2 Particular solution for a constant normal load $p(x,y)=q_0$

$$
\begin{aligned}
w &= \frac{q_0(x^4 + y^4)}{24(D_{11} + D_{22})}, \quad w_x = \frac{q_0 x^3}{6(D_{11} + D_{22})}, \quad w_y = \frac{q_0 y^3}{6(D_{11} + D_{22})},\\
w_{xx} &= \frac{q_0 x^2}{2(D_{11} + D_{22})}, \quad w_{yy} = \frac{q_0 y^2}{2(D_{11} + D_{22})}, \quad w_{xy} = 0\\
w_{xxx} &= \frac{q_0 x}{D_{11} + D_{22}}, \quad w_{yyy} = \frac{q_0 y}{D_{11} + D_{22}}, \quad w_{xxy} = 0, \quad w_{xyy} = 0.
\end{aligned}
\tag{2.19}
$$

2.5 Three-dimensional plate representations

2.5.1 Homogeneous bending solution involving powers of z

$$2\mu u = -z\,\mathrm{Re}[\,\Phi + \zeta\,\overline{\Phi'} + \overline{\chi'}\,] - \frac{1}{1-\nu}\,[\,h^2 z - 2(2-\nu)\frac{z^3}{3}\,]\,\mathrm{Re}[\,\Phi''\,],$$

$$2\mu v = -z\,\mathrm{Im}[\,\Phi + \zeta\,\overline{\Phi'} + \overline{\chi'}\,] + \frac{1}{1-\nu}\,[\,h^2 z - 2(2-\nu)\frac{z^3}{3}\,]\,\mathrm{Im}[\,\Phi''\,],$$

$$2\mu w = \mathrm{Re}[\,\bar{\zeta}\,\Phi + \chi\,] + \frac{2\nu}{1-\nu} z^2\,\mathrm{Re}[\,\Phi'\,],$$

$$\sigma_{xx} = -\frac{1}{1-\nu} z\,\mathrm{Re}[\,2(1+\nu)\Phi' + (1-\nu)(\,\zeta\overline{\Phi''} + \overline{\chi''}\,)\,]$$

$$- \frac{1}{1-\nu}\,[\,h^2 z - 2(2-\nu)\frac{z^3}{3}\,]\,\mathrm{Re}[\,\Phi'''\,],$$

$$\sigma_{yy} = -\frac{1}{1-\nu} z\,\mathrm{Re}[\,2(1+\nu)\Phi' - (1-\nu)(\,\zeta\overline{\Phi''} + \overline{\chi''}\,)\,] \tag{2.20}$$

$$+ \frac{1}{1-\nu}\,[\,h^2 z - 2(2-\nu)\frac{z^3}{3}\,]\,\mathrm{Re}[\,\Phi'''\,],$$

$$\sigma_{zz} = 0,$$

$$\tau_{xy} = -z\,\mathrm{Im}[\,\zeta\,\overline{\Phi''} + \overline{\chi''}\,] + \frac{1}{1-\nu}\,[\,h^2 z - 2(2-\nu)\frac{z^3}{3}\,]\,\mathrm{Im}[\,\Phi'''\,],$$

$$\tau_{xz} = \frac{2}{1-\nu}\,[\,z^2 - \frac{h^2}{4}\,]\,\mathrm{Re}[\,\Phi''\,],$$

$$\tau_{yz} = -\frac{2}{1-\nu}\,[\,z^2 - \frac{h^2}{4}\,]\,\mathrm{Im}[\,\Phi''\,],$$

where

$$\zeta = x + iy, \qquad \Phi = \Phi(\zeta), \qquad \chi = \chi(\zeta). \tag{2.21}$$

2.5.2 Homogeneous membrane solution involving powers of z

$$2\mu u = \mathrm{Re}[\,\frac{3-\nu}{1+\nu}\Phi - \zeta\overline{\Phi'} - \overline{\chi'}\,] - 2\,\frac{\nu}{1+\nu}[\,\frac{h^2}{12} - z^2\,]\,\mathrm{Re}[\Phi''],$$

$$2\mu v = \mathrm{Im}[\,\frac{3-\nu}{1+\nu}\Phi - \zeta\overline{\Phi'} - \overline{\chi'}\,] + 2\,\frac{\nu}{1+\nu}[\,\frac{h^2}{12} - z^2\,]\,\mathrm{Im}[\Phi''],$$

$$2\mu w = -4z\,\frac{\nu}{1+\nu}\,\mathrm{Re}[\Phi'], \tag{2.22}$$

$$\sigma_{xx} = \mathrm{Re}[\ \Phi' + \overline{\Phi'} - \zeta\overline{\Phi''} - \overline{\chi''}\ ] - 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Re}[\Phi'''],$$

$$\sigma_{yy} = \mathrm{Re}[\ \Phi' + \overline{\Phi'} + \zeta\overline{\Phi''} + \overline{\chi''}\ ] - 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Re}[\Phi'''],$$

$$\tau_{xy} = -\mathrm{Im}[\ \zeta\overline{\Phi''} + \overline{\chi''}\ ] + 2\frac{\nu}{1+\nu}[\ \frac{h^2}{12} - z^2\ ]\ \mathrm{Im}[\Phi'''],$$

$$\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0.$$

2.5.3 Homogeneous bending solution involving trigonometric functions of z

$$2\mu u = \frac{\partial g_n}{\partial y} \sin \omega_n z,$$

$$2\mu v = -\frac{\partial g_n}{\partial x} \sin \omega_n z,$$

$$2\mu w = 0,$$

$$\sigma_{xx} = \frac{\partial^2 g_n}{\partial x \partial y} \sin \omega_n z,$$

$$\sigma_{yy} = -\frac{\partial^2 g_n}{\partial x \partial y} \sin \omega_n z, \tag{2.23}$$

$$\sigma_{zz} = 0,$$

$$\tau_{xy} = \frac{1}{2}[\ \frac{\partial^2 g_n}{\partial y^2} - \frac{\partial^2 g_n}{\partial x^2}\ ] \sin \omega_n z,$$

$$\tau_{xz} = \frac{1}{2}\omega_n \frac{\partial g_n}{\partial y} \cos \omega_n z,$$

$$\tau_{yz} = -\frac{1}{2}\omega_n \frac{\partial g_n}{\partial x} \cos \omega_n z,$$

where $g_n(x,y)$ has to satisfy

$$\Delta g_n - \omega_n^2\, g_n = 0 \tag{2.24}$$

and $\omega_n = n\pi/h \quad (n=1,3,5,....)$.

Complex representation of $g_n(x,y)$ in a star-shaped domain (Vekua 1967):

$$g_n(x,y) = \mathrm{Re}[\Phi_n(\zeta)] - \int_0^1 \mathrm{Re}[\Phi_n(t\zeta)]\frac{\partial}{\partial t} I_0(\omega_n r\sqrt{1-t}\ )\ dt \tag{2.25}$$

or

$$g_n(x,y) = \mathrm{Re}[\Phi_n(0)]\ I_0(\omega_n r) + \int_0^1 \mathrm{Re}[\zeta\Phi_n'(t\zeta)]\ I_0(\omega_n r\sqrt{1-t}\ )\ dt, \tag{2.26}$$

where

I_0 = modified Bessel function of the first kind.

2.5.4 Homogeneous membrane solution involving trigonometric functions of z

Take relationships (2.23) and write $\hat{g}_m$, λ_m, $-\cos\lambda_m z$, $\sin\lambda_m z$ instead of g_n, ω_n, $\sin\omega_n z$, and $\cos\omega_n z$, respectively, where $\lambda_m = m\pi/h$ $(m=0,2,4,....)$.

2.5.5 Homogeneous bending and membrane solution parts involving hyperbolic functions of z

$$
\begin{aligned}
2\mu u &= -G_x\,[\, a\,\hat{z}\cosh q\hat{z} + b\,\hat{z}\sinh q\hat{z} + c\cosh q\hat{z} + d\sinh q\hat{z}\,],\\
2\mu v &= -G_y\,[\, a\,\hat{z}\cosh q\hat{z} + b\,\hat{z}\sinh q\hat{z} + c\cosh q\hat{z} + d\sinh q\hat{z}\,],\\
2\mu w &= G\,\Big[\, a\Big\{(3-4\nu)\cosh q\hat{z} - q\hat{z}\sinh q\hat{z}\Big\} + b\Big\{(3-4\nu)\sinh q\hat{z} - q\hat{z}\cosh q\hat{z}\Big\}\\
&\qquad - c\,q\sinh q\hat{z} - d\,q\cosh q\hat{z}\,\Big],\\
\sigma_{xx} &= -G_{xx}\,[\, a\,\hat{z}\cosh q\hat{z} + b\,\hat{z}\sinh q\hat{z} + c\cosh q\hat{z} + d\sinh q\hat{z}\,]\\
&\qquad + G\,2\nu q\,[\, a\sinh q\hat{z} + b\cosh q\hat{z}\,],\\
\sigma_{yy} &= -G_{yy}\,[\, a\,\hat{z}\cosh q\hat{z} + b\,\hat{z}\sinh q\hat{z} + c\cosh q\hat{z} + d\sinh q\hat{z}\,]\\
&\qquad + G\,2\nu q\,[\, a\sinh q\hat{z} + b\cosh q\hat{z}\,], \qquad\qquad (2.27)\\
\sigma_{zz} &= G\,q\,\Big[\, a\Big\{-q\hat{z}\cosh q\hat{z} + 2(1-\nu)\sinh q\hat{z}\Big\} + b\Big\{-q\hat{z}\sinh q\hat{z} + 2(1-\nu)\cosh q\hat{z}\Big\}\\
&\qquad - c\,q\cosh q\hat{z} - d\,q\sinh q\hat{z}\,\Big],\\
\tau_{xy} &= -G_{xy}\,[\, a\,\hat{z}\cosh q\hat{z} + b\,\hat{z}\sinh q\hat{z} + c\cosh q\hat{z} + d\sinh q\hat{z}\,],\\
\tau_{xz} &= G_x\,\Big[\, a\Big\{(1-2\nu)\cosh q\hat{z} - q\hat{z}\sinh q\hat{z}\Big\} + b\Big\{(1-2\nu)\sinh q\hat{z} - q\hat{z}\cosh q\hat{z}\Big\}\\
&\qquad - c\,q\sinh q\hat{z} - d\,q\cosh q\hat{z}\,\Big],\\
\tau_{yz} &= G_y\,\Big[\, a\Big\{(1-2\nu)\cosh q\hat{z} - q\hat{z}\sinh q\hat{z}\Big\} + b\Big\{(1-2\nu)\sinh q\hat{z} - q\hat{z}\cosh q\hat{z}\Big\}\\
&\qquad - c\,q\sinh q\hat{z} - d\,q\cosh q\hat{z}\,\Big],
\end{aligned}
$$

where G(x,y) has to satisfy

$$\Delta G + q^2 G = 0 \tag{2.28}$$

and

$$\hat{z} = z + h/2. \tag{2.29}$$

For the bending solution use:

$$qh - \sinh qh = 0 \tag{2.30}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \begin{bmatrix} 1 \\ \dfrac{1 - \cosh qh}{qh} \\ \dfrac{1-2\nu}{q} \\ \dfrac{2(1-\nu)}{q} \; \dfrac{1 - \cosh qh}{qh} \end{bmatrix}. \tag{2.31}$$

For the membrane solution use:

$$qh + \sinh qh = 0 \tag{2.32}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A \begin{bmatrix} 1 \\ \dfrac{1 + \cosh qh}{qh} \\ \dfrac{1-2\nu}{q} \\ \dfrac{2(1-\nu)}{q} \; \dfrac{1 + \cosh qh}{qh} \end{bmatrix}. \tag{2.33}$$

For complex values of q, A and G take the real parts of the right-hand sides of (2.27).

Complex representation of G(x,y):

$$\begin{aligned} G(x,y) = {} & \Phi(\zeta) - \int_0^{\zeta} \Phi(t) \frac{\partial}{\partial t} J_0(q\sqrt{\bar{\zeta}(\zeta-t)}\,)\, dt \\ & + \Phi^*(\bar{\zeta}) - \int_0^{\bar{\zeta}} \Phi^*(\tau) \frac{\partial}{\partial \tau} J_0(q\sqrt{\zeta(\bar{\zeta}-\tau)}\,)\, d\tau, \end{aligned} \tag{2.34}$$

where

$\zeta = x + iy, \quad \bar{\zeta} = x - iy,$

J_0 = Bessel function of the first kind.

2.5.6 Particular homogeneous solution for a constant normal load on the upper plate face

$$2\mu u = \frac{p\,x}{4h^3}\left[(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2+y^2)z + \frac{2\nu h^3}{1+\nu} \right],$$

$$2\mu v = \frac{p\,y}{4h^3}\left[(2-\nu)(4z^3 - 3h^2 z) - 3(1-\nu)(x^2+y^2)z + \frac{2\nu h^3}{1+\nu} \right],$$

$$2\mu w = \frac{p}{16h^3}\left[-8(1+\nu)z^4 + 24\nu(x^2+y^2)z^2 + 12h^2(1+\nu)z^2 \right.$$

$$\left. + 3(1-\nu)(x^2+y^2)^2 - 6h^2\nu(x^2+y^2) - \frac{8h^3}{1+\nu} z \right],$$

$$\sigma_{xx} = \frac{p}{4h^3}\left[4(2+\nu)z^3 - (9+3\nu)x^2 z - (3+9\nu)y^2 z - 3h^2(2+\nu)z \right], \qquad (2.35)$$

$$\sigma_{yy} = \frac{p}{4h^3}\left[4(2+\nu)z^3 - (3+9\nu)x^2 z - (9+3\nu)y^2 z - 3h^2(2+\nu)z \right],$$

$$\sigma_{zz} = -\frac{p}{2h^3}(z+h)(2z-h)^2,$$

$$\tau_{xy} = -\frac{3p}{2h^3}(1-\nu)xyz,$$

$$\tau_{xz} = \frac{3p}{4h^3}x(2z-h)(2z+h),$$

$$\tau_{yz} = \frac{3p}{4h^3}y(2z-h)(2z+h).$$

2.5.7 Particular solution for a constant body force $\bar{f}_z$

$$2\mu u = \frac{\bar{f}_z}{4h^2}\left[4(2-\nu)z^3 - 3(1-\nu)(x^2+y^2)z - 2h^2(1-\nu)z \right]x,$$

$$2\mu v = \frac{\bar{f}_z}{4h^2}\left[4(2-\nu)z^3 - 3(1-\nu)(x^2+y^2)z - 2h^2(1-\nu)z \right]y,$$

$$2\mu w = \frac{\bar{f}_z}{16h^2}\left[-8(1+\nu)z^4 + 24\nu(x^2+y^2)z^2 + 4h^2(1+\nu)z^2 \right.$$

$$\left. - 2h^2(1+\nu)(x^2+y^2) + 3(1-\nu)(x^2+y^2)^2 \right], \qquad (2.36)$$

$$\sigma_{xx} = \frac{\bar{f}_z}{4h^2}\,[\,4(2+\nu)z^3 - (9+3\nu)x^2z - (3+9\nu)y^2z - h^2(2+\nu)z\,],$$

$$\sigma_{yy} = \frac{\bar{f}_z}{4h^2}\,[\,4(2+\nu)z^3 - (3+9\nu)x^2z - (9+3\nu)y^2z - h^2(2+\nu)z\,],$$

$$\sigma_{zz} = -\frac{\bar{f}_z}{2h^2}\,z(2z-h)(2z+h),$$

$$\tau_{xy} = -\frac{3\bar{f}_z}{2h^2}\,(1-\nu)xyz,$$

$$\tau_{xz} = \frac{3\bar{f}_z}{4h^2}\,x(2z-h)(2z+h)$$

$$\tau_{yz} = \frac{3\bar{f}_z}{4h^2}\,y(2z-h)(2z+h)$$

2.6 General three-dimensional elasticity solution representations

In Piltner (1987, 1988a, 1989) a 3-dimensional elasticity solution representation in terms of six arbitrary complex valued functions was introduced. Using the following three complex variables

$$\begin{aligned} \zeta_1 &= ix + b_1(t)y + c_1(t)z \\ \zeta_2 &= a_2(t)x + iy + c_2(t)z \\ \zeta_3 &= a_3(t)x + b_3(t)y + iz \end{aligned} \tag{2.37}$$

the displacements can be written in terms of $\Phi_k(\zeta_k)$, $\Psi_k(\zeta_k)$ (where k=1,2,3):

$$2\mu u = \int \left\{ \begin{aligned} &\mathrm{Im}[\Psi_1 - 2ix\Phi_1' + 2(3-4\nu)\Phi_1] \\ &a_2\,\mathrm{Re}[-\Psi_2 + 2iy\Phi_2'\,] \\ &a_3\,\mathrm{Re}[-\Psi_3 + 2iz\Phi_3'\,] \end{aligned} \right\} dt$$

$$2\mu v = \int \left\{ \begin{aligned} &b_1\,\mathrm{Re}[-\Psi_1 + 2ix\Phi_1'\,] \\ &+ \mathrm{Im}[\,\Psi_2 - 2iy\Phi_2' + 2(3-4\nu)\Phi_2] \\ &+ b_3\,\mathrm{Re}[-\Psi_3 + 2iz\Phi_3'\,] \end{aligned} \right\} dt \tag{2.38}$$

$$2\mu w = \int \Big\{ c_1 \,\mathrm{Re}[-\Psi_1 + 2ix\Phi_1'] + c_2 \,\mathrm{Re}[-\Psi_2 + 2iy\Phi_2'] + \mathrm{Im}[\Psi_3 - 2iz\Phi_3' + 2(3-4\nu)\Phi_3]\Big\}\, dt$$

where $a_2(t)$, $a_3(t)$, $b_1(t)$, $b_3(t)$, $c_1(t)$, $c_2(t)$ are parameter functions satisfying

$$\begin{aligned} b_1^2(t) + c_1^2(t) &= 1 \\ a_2^2(t) + c_2^2(t) &= 1 \\ a_3^2(t) + b_3^2(t) &= 1. \end{aligned} \qquad (2.39)$$

Important examples for the choice of parameter functions are $a_3(t) = \cos t$, $b_3(t) = \sin t$.

An alternative to the representation (2.38) involving three different complex variables is a representation of the displacement field which needs only one of the complex variables from (2.37), say ζ_3:

$$\begin{aligned} 2\mu u &= \int \Big\{ a_3 \,\mathrm{Re}[-\hat\Psi_1 + 2ix\hat\Phi_1'] + 2(3-4\nu)\,\mathrm{Im}[\hat\Phi_1] + a_3 \,\mathrm{Re}[-\hat\Psi_2 + 2iy\hat\Phi_2'] + a_3 \,\mathrm{Re}[-\hat\Psi_3 + 2iz\hat\Phi_3'] \Big\}\, dt \\ 2\mu v &= \int \Big\{ b_3 \,\mathrm{Re}[-\hat\Psi_1 + 2ix\hat\Phi_1'] + b_3 \,\mathrm{Re}[-\hat\Psi_2 + 2iy\hat\Phi_2'] + 2(3-4\nu)\,\mathrm{Im}[\hat\Phi_2] + b_3 \,\mathrm{Re}[-\hat\Psi_3 + 2iz\hat\Phi_3'] \Big\}\, dt \qquad (2.40) \\ 2\mu w &= \int \Big\{ \mathrm{Im}[\hat\Psi_1 - 2ix\hat\Phi_1'] + \mathrm{Im}[\hat\Psi_2 - 2iy\hat\Phi_2'] + \mathrm{Im}[\hat\Psi_3 - 2iz\hat\Phi_3'] + 2(3-4\nu)\,\mathrm{Im}[\hat\Phi_3] \Big\}\, dt \end{aligned}$$

where $\hat\Phi_k = \hat\Phi_k(\zeta_3)$, $\hat\Psi_k = \hat\Psi_k(\zeta_3)$ and k=1,2,3.

An illustrative example for the use of the 3-dimensional solution representation in the case of singular functions is given in Piltner (1989). The derivation of the 3-dimensional plate representations in section 2.5 can be found in Piltner (1992a).

All solution representations involve "arbitrary" complex valued functions. Once those arbitrary complex functions ($\Phi(z)$, $\Psi(z)$, $\Phi_n(z)$ etc.) are chosen we get the real trial functions for the displacements and stresses. Very helpful for the systematic construction of the real Trefftz-trial functions are symbolic manipulation codes like MACSYMA and MATHEMATICA. The complex representations are very suitable for a use in symbolic manipulation programs which can also convert the results into FORTRAN-statements. This means that we have a convenient way of filling major parts of the subroutines for the boundary element algorithm without worrying about typing errors.

3 The systematic construction of singular Trefftz-trial functions based on a boundary element discretization

The representations of stresses and displacements given in section 2 can be used to obtain systematically linearly independent trial functions for the physical quantities. For a Trefftz-type finite element we usually choose the complex functions as complex power series whereas for a boundary element algorithm we choose singular functions for Φ, X, Ψ, etc. Those complex functions which depend on a variable involving two real space coordinates (e.g. $x + iy$) can be chosen in the form of a Cauchy integral

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \tag{3.1}$$

where the function f(z) stands for the complex functions Φ, Ψ, X, etc. Γ is the boundary of the domain. After discretizing the boundary Γ into N straight line boundary elements with the boundary portions Γ_j (j=1,2,...,N) (Figure 1). and choosing basic functions $\hat{f}^j$ for every element we use a discretized form for the Cauchy type integrals:

$$f(z) = \frac{1}{2\pi i} \sum_{j} \int_{\Gamma_j} \frac{\hat{f}^j(\zeta)}{\zeta - z} d\zeta, \tag{3.2}$$

In the following notation, the boundary element algorithm will be explained for the example of the plane strain/stress solution representation (2.1). In this representation we have the two functions $\Phi(z)$ and $\Psi(z)$ for which we choose the following discretizations:

$$\Phi(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\hat{\Phi}^j(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \sum_{j=1}^{N} \Phi^j(z),$$
$$\Psi(z) = \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{\Gamma_j} \frac{\hat{\Psi}^j(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \sum_{j=1}^{N} \Psi^j(z), \tag{3.3}$$

where $z = x + iy$ and ζ is a complex boundary coordinate. Since the boundary values of Φ and Ψ are not explicitly given in the plate boundary value problem, we assume for both functions distributions on the boundary Γ.

For every boundary element we choose fifth or seventh-order basis functions $\hat{\Phi}^j(\zeta)$ and $\hat{\Psi}^j(\zeta)$. Choosing, for example, fifth-order basis functions between nodes (j-1) and j, the

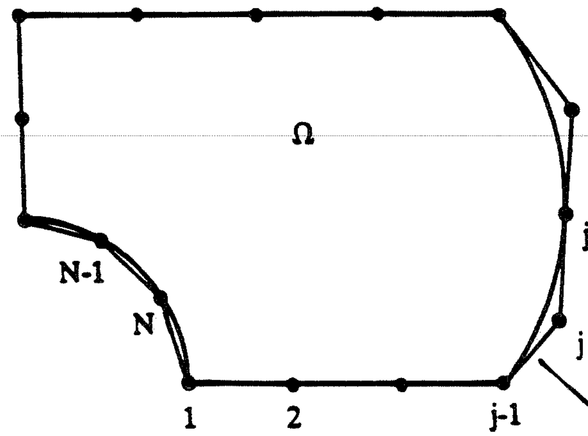

Figure 1: Boundary element discretization for a plane stress/strain problem

complex functions take the form

$$\hat{\Phi}^j(s) = N_1(s)\,\Phi_{j-1} + N_2(s)\,\Phi'_{j-1}(z_j - z_{j-1}) + N_3(s)\,\Phi''_{j-1}(z_j - z_{j-1})^2 + N_4(s)\,\Phi_j + N_5(s)\,\Phi'_j(z_j - z_{j-1}) + N_6(s)\,\Phi''_j(z_j - z_{j-1})^2 \tag{3.4}$$

and

$$\hat{\Psi}^j(s) = N_1(s)\,\Psi_{j-1} + N_2(s)\,\Psi'_{j-1}(z_j - z_{j-1}) + N_3(s)\,\Psi''_{j-1}(z_j - z_{j-1})^2 + N_4(s)\,\Psi_j + N_5(s)\,\Psi'_j(z_j - z_{j-1}) + N_6(s)\,\Psi''_j(z_j - z_{j-1})^2, \tag{3.5}$$

where $N_j(s)$ (j=1,2,...,6) are fifth-order shape functions of the variable

$$s = \frac{\zeta - z_{j-1}}{z_j - z_{j-1}}, \tag{3.6}$$

and Φ_j, Φ'_j, Φ''_j, Ψ_j, Ψ'_j, Ψ''_j are discrete complex node values at the point $z = z_j$. The shape functions N_j are given in the appendix. For straight line boundary elements the integration can be performed analytically. For every element with the boundary portion Γ_j, we get the boundary element contributions $\Phi^j(z)$ and $\Psi^j(z)$ from analytical integration.

After adding all contributions $\Phi^j(z)$ and $\Psi^j(z)$ from all boundary elements, we get approximation functions $\Phi(z)$, $\Psi(z)$, which, in the above example, have the form

$$\Phi(z) = \frac{1}{2\pi i}\Big[\ \Phi_1\,F_1(z) + \Phi'_1\,G_1(z) + \Phi''_1\,H_1(z) + \cdots + \Phi_j\,F_j(z) + \Phi'_j\,G_j(z) + \Phi''_j\,H_j(z) + \cdots + \Phi_N\,F_N(z) + \Phi'_N\,G_N(z) + \Phi''_N\,H_N(z)\ \Big], \tag{3.7}$$

$$\Psi(z) = \frac{1}{2\pi i}\Big[\quad \Psi_1\, F_1(z) + \Psi_1'\, G_1(z) + \Psi_1''\, H_1(z) + \cdots \tag{3.8}$$

$$\cdots + \Psi_j\, F_j(z) + \Psi_j'\, G_j(z) + \Psi_j''\, H_j(z) + \cdots$$

$$\cdots + \Psi_N\, F_N(z) + \Psi_N'\, G_N(z) + \Psi_N''\, H_N(z) \quad \Big].$$

The explicit expressions, for the approximation functions $\Phi(z)$ and $\Psi(z)$, can be taken from the appendix. In the appendix the approximation function is denoted by $f(z)$; so in order to get the functions for Φ and Ψ, only the notation has to be changed (i.e., change f into Φ and change f into Ψ, respectively).

For the example of fifth order basis functions we get for $F_j(z)$, $G_j(z)$ and $H_j(z)$ the following expressions:

$$F_j(z) = 3K + 9K^2 - 24K^3 + 12K^4 + 2N_4(K) \ln \frac{z_j - z}{z_{j-1} - z}$$

$$- 3L - 9L^2 + 24L^3 - 12L^4 + 2N_1(L) \ln \frac{z_{j+1} - z}{z_j - z}$$

$$G_j(z) = (z_j - z_{j-1})\left\{ -\frac{11}{30} - \frac{5}{6}K - 3K^2 + 11K^3 - 6K^4 + 2N_5(K) \ln \frac{z_j - z}{z_{j-1} - z} \right\} \tag{3.9}$$

$$+ (z_{j+1} - z_j)\left\{ \frac{4}{5} - \frac{13}{6}L - 6L^2 + 13L^3 - 6L^4 + 2N_2(L) \ln \frac{z_{j+1} - z}{z_j - z} \right\}$$

$$H_j(z) = (z_j - z_{j-1})^2\left\{ \frac{1}{30} + \frac{1}{12}K + \frac{1}{3}K^2 - \frac{3}{2}K^3 + K^4 + 2N_6(K) \ln \frac{z_j - z}{z_{j-1} - z} \right\}$$

$$+ (z_{j+1} - z_j)^2\left\{ \frac{1}{20} + \frac{1}{4}L - \frac{11}{6}L^2 + \frac{5}{2}L^3 - L^4 + 2N_3(L) \ln \frac{z_{j+1} - z}{z_j - z} \right\}$$

where

$$K = \frac{z - z_{j-1}}{z_j - z_{j-1}}, \qquad L = \frac{z - z_j}{z_{j+1} - z_j}. \tag{3.10}$$

Some of the representations for the displacements and stresses in section 2 contain the third-order complex derivative Φ'''. In this case we need seventh-order basis functions which include Φ_j''' as an unknown nodal value. For convenience, the results, for the case of seventh-order boundary element basis functions, are included in the appendix. The complex derivatives of the approximation functions are also given in the appendix.

It should be emphasized that although the approximation functions for Φ, Ψ, Φ', Ψ', Φ''

contain singular expressions, limit values for all points on the boundary can be calculated. This means that all physical quantities (stresses, strains, displacements) will remain finite everywhere in the solution domain and on the boundary.

For the functions in equation (3.7), for example, we will get the following limit values as we approach the nodal point $z = z_j$ on the boundary:

$$
\begin{aligned}
\lim_{z \to z_j} \Phi(z) = \frac{1}{4\pi i} \Bigg\{ \quad & \Phi_j \left[2 \ln \frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] \\
& + \Phi_j' \left[\frac{4}{5} \left\{ (z_j - z_{j-1}) + (z_{j+1} - z_j) \right\} \right] \\
& + \Phi_j'' \left[\frac{1}{20} \left\{ -(z_j - z_{j-1})^2 + (z_{j+1} - z_j)^2 \right\} \right] \\
& + \text{remaining terms} \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\lim_{z \to z_j} \Phi'(z) = \frac{1}{4\pi i} \Bigg\{ \quad & \Phi_j \left[-\frac{5}{z_j - z_{j-1}} - \frac{5}{z_{j+1} - z_j} \right] \\
& + \Phi_j' \left[2 \ln \frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] \\
& + \Phi_j'' \left[\frac{z_j - z_{j-1}}{4} + \frac{z_{j+1} - z_j}{4} \right] \\
& + \text{remaining terms} \Bigg\},
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
\lim_{z \to z_j} \Phi''(z) = \frac{1}{4\pi i} \Bigg\{ \quad & \Phi_j \left[\frac{20}{(z_j - z_{j-1})^2} - \frac{20}{(z_{j+1} - z_j)^2} \right] \\
& + \Phi_j' \left[-\frac{16}{z_j - z_{j-1}} - \frac{16}{z_{j+1} - z_j} \right] \\
& + \Phi_j'' \left[2 \ln \frac{z_j - z_{j+1}}{z_j - z_{j-1}} \right] \\
& + \text{remaining terms} \Bigg\}.
\end{aligned}
$$

Noting that we have

$$\Phi_j = \mathrm{Re}[\Phi_j] + i\,\mathrm{Im}[\Phi_j],$$

$$F_j = \mathrm{Re}[F_j] + i\,\mathrm{Im}[F_j], \tag{3.12}$$

and similar expressions for Φ_j', Φ_j'' and $G_j(z)$, $H_j(z)$, we get after substituting the approximation functions (3.7, 3.8) into the displacement representation (2.1), the real approximation functions for the displacements. In matrix notation we get $\mathbf{u} = \mathbf{U}^*\mathbf{c}$, where the vector $\mathbf{c}$ contains the real parameters

$$\mathrm{Re}[\Phi_j],\ \mathrm{Im}[\Phi_j],\ \mathrm{Re}[\Phi_j'],\ \mathrm{Im}[\Phi_j'],\ \mathrm{Re}[\Phi_j''],\ \mathrm{Im}[\Phi_j''],$$

and

$$\mathrm{Re}[\Psi_j],\ \mathrm{Im}[\Psi_j],\ \mathrm{Re}[\Psi_j'],\ \mathrm{Im}[\Psi_j'],\ \mathrm{Re}[\Psi_j''],\ \mathrm{Im}[\Psi_j''],$$

where j=1,2,...,N. Since both Φ and Ψ can represent rigid body translations, we need to eliminate the linearly dependent functions terms. This can be done by omitting one complex function term of $\Psi(z)$, using the complex coefficient Ψ_i, which can easily realized by choosing i=1. So at the first boundary element node, we set $\mathrm{Re}[\Psi_1] = \mathrm{Im}[\Psi_1] = 0$. In order to remove the rigid body motions for the plane strain/stress case we set $\mathrm{Re}[\Phi_1] = \mathrm{Im}[\Phi_1] = \mathrm{Im}[\Phi_1'] = 0$. Details about the computation of the complex logarithmic terms can be found in Piltner and Taylor (1990a).

4 Variational Formulation

In order to obtain a symmetric form of a boundary element algorithm, we can use the following hybrid variational formulation:

$$\Pi_H^i = \int_V \left[\frac{1}{2}(\mathbf{u}^T \mathbf{D}^T)\,\mathbf{E}\,(\mathbf{Du}) - \mathbf{u}^T\bar{\mathbf{f}} \right] dV - \int_{S_T} \tilde{\mathbf{u}}^T\,\bar{\mathbf{T}}\,dS - \int_S \mathbf{T}^T(\mathbf{u}-\tilde{\mathbf{u}})\,dS, \qquad (4.1)$$

In the used notation $\bar{\mathbf{f}}$ is the body force vector, $\mathbf{D}$ is a differential operator matrix and $\mathbf{E}$ is the matrix of material coefficients. $\bar{\mathbf{T}}$ are prescribed tractions on the boundary portion S_T. The boundary S is decomposed into two portions according to $S = S_u + S_T$. The strains are calculated from $\boldsymbol{\varepsilon} = \mathbf{Du}$, and the stresses are obtained from $\boldsymbol{\sigma} = \mathbf{EDu}$. Using $\mathbf{n}$ as the matrix of direction cosines on the boundary the tractions can be written as

$$\mathbf{T} = \mathbf{nEDu}. \qquad (4.2)$$

The displacement field $\mathbf{u}$ is defined in the domain V and on the boundary S, whereas $\tilde{\mathbf{u}}$ is chosen on the boundary. Carrying out the variation in (4.1), and noting that $\delta\tilde{\mathbf{u}} = \bar{\mathbf{T}} = \mathbf{0}$ on S_u, we obtain the relationship

$$\delta\Pi_H^i = -\int_V \delta\mathbf{u}^T(\,\mathbf{D}^T\,\mathbf{E}\,\mathbf{Du} + \bar{\mathbf{f}}\,)\,dV - \int_S \delta\mathbf{T}^T(\mathbf{u}-\tilde{\mathbf{u}})\,dS + \int_S \delta\tilde{\mathbf{u}}^T(\,\mathbf{T}-\bar{\mathbf{T}})\,dS. \qquad (4.3)$$

For the discretization, we use

$$\mathbf{u} = \mathbf{U}^*\mathbf{c} + \mathbf{u}_p \qquad (4.4)$$

$$\tilde{\mathbf{u}} = \tilde{\mathbf{U}}\mathbf{q} \qquad (4.5)$$

where $\mathbf{U}^*$ contains the Trefftz-type functions with singular expressions. The construction of such functions was discussed in sections 2 and 3. We recall that, by construction, the displacement functions have the property that they lead to to stresses and strains for which we can calculate limit values for all points on the boundary. The vector $\mathbf{q}$ contains the nodal displacements on the boundary, whereas $\mathbf{c}$ is the vector of unknown coefficients in the Trefftz-trial functions.

Since $\mathbf{u}$ has the property of satisfying the differential equation

$$\mathbf{D}^T\,\mathbf{E}\,\mathbf{Du} + \bar{\mathbf{f}} = \mathbf{0} \qquad (4.6)$$

inside the solution domain, and all limit values on the boundary exist, we are left with the task of integrating terms along the boundary S. The discretization of (4.3) leads to the following form of equations

$$\begin{bmatrix} -\mathbf{H} & \mathbf{L} \\ \mathbf{L}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{c} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{bmatrix} \tag{4.7}$$

where **H** is a symmetric matrix. This system of equations can be reduced to

$$\mathbf{K}\mathbf{q} = \mathbf{p} \tag{4.8}$$

where

$$\mathbf{K} = \mathbf{L}^T \mathbf{H}^{-1} \mathbf{L} = \mathbf{K}^T \tag{4.9}$$

$$\mathbf{p} = \mathbf{L}^T \mathbf{H}^{-1} \mathbf{r}_1 + \mathbf{r}_2. \tag{4.10}$$

Hybrid variational formulations utilizing Trefftz-type trial functions for **u** have been successfully applied for the derivation of several finite elements (e.g., Jirousek and Guex 1986, Jirousek 1987, Jirousek and Vakatesh 1989, Piltner 1985, 1992b). Details of such discretizations can be found in the given references. The important difference of the approach in this paper to the one used for the Trefftz-type finite elements is that singular functions are used in the present approach, whereas nonsingular functions are used for the Trefftz-type finite elements. (Remark: Exceptions are finite elements containing a crack for which two singular functions are added to the nonsingular functions).

5 Numerical Results

Singular Trefftz-trial functions, as the ones described in section 3, have been used successfully with the point collocation techniques. Numerical results for plane strain/stress and plate bending problems have been reported in Piltner and Taylor (1989, 1990a, 1990b, 1990c).

For testing the discussed symmetric boundary element form, the stiffness matrices for triangular and quadrilateral domains under bending have been calculated. Choosing the same type of nodal degrees of freedom, as in a finite element, we can test the evaluation of symmetric finite element matrices from a discretization along the boundary. One of the examples was a triangular Kirchhoff plate bending element (Figure 2). On every edge of the triangular finite element, one boundary element was used for the discretization of the Cauchy integrals. Since the evaluation of the symmetric stiffness matrices is based on a variational formulation, we obtain finite element matrices for every triangular subdomain, and can use a standard finite element program to assemble the element stiffness matrices into a global stiffness matrix.

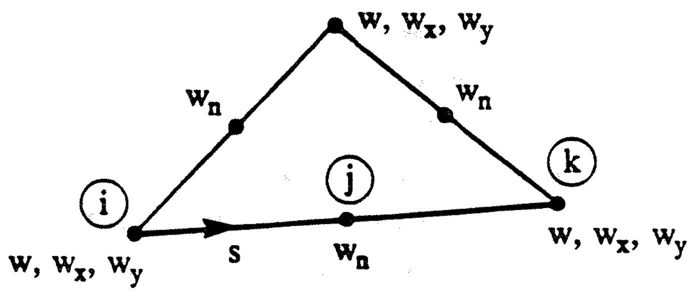

Figure 2: Triangular plate bending element

For this example, the complex basis functions for the Cauchy integrals are chosen as fifth-order functions. The approximation functions for Φ and Ψ, and the complex derivatives, can be taken from the results listed in the appendix.

For one quarter of a simply supported square plate under uniform load (length a=10, thickness h=0.1, $\nu = 0.3$, E=1, uniform load p=1), the results for the maximum deflection are listed in Table 1 and compared with the exact solution (Timoshenko 1940).

Table 1: Results obtained with a mesh of triangular plate bending elements based on a boundary element discretization along the finite element edges (one boundary element per edge).

number of finite elements	w (numerical)	w exact (Timoshenko)	error %
2	$4.338 * 10^5$	$4.43 * 10^5$	2.1
8	$4.406 * 10^5$	$4.43 * 10^5$	0.5
32	$4.419 * 10^5$	$4.43 * 10^5$	0.2

6 Concluding remarks

A systematic way of constructing singular functions for a variety of elasticity problems has been shown. After substituting the functions listed in the appendix into the solution representations, we can get limit values for stresses, strains, and displacements for all points on the boundary. A hybrid variational formulation with two independent fields can be used for the calculation of symmetric coefficient matrices. Once the discrete unknowns are computed, we can get, with the aid of the Trefftz-trial functions, the stresses and displacements inside the solution domain and on the boundary. It is hoped that some of the new solution representations in section 2 will be helpful for other aspects in boundary element techniques.

Appendix

A.1 Results for fifth-order complex basis functions

In the following, the fifth-order basis functions and the according shape functions and derivatives are listed. The results of the evaluated Cauchy integrals, for the approximation function f(z) and the complex derivatives f′(z) and f″(z), are given as well.

The fifth-order complex basis functions $\hat{f}^j$ and their complex derivatives can be written in the following form:

$$\begin{aligned}
\hat{f}^j(s) &= N_1(s)\, f_{j-1} + N_2(s)\, f'_{j-1}(z_j - z_{j-1}) + N_3(s)\, f''_{j-1}(z_j - z_{j-1})^2 + \\
&\quad N_4(s)\, f_j + N_5(s)\, f'_j(z_j - z_{j-1}) + N_6(s)\, f''_j(z_j - z_{j-1})^2, \\
\hat{f}^{j\prime}(s) &= \dot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1}) + \dot{N}_2(s)\, f'_{j-1} + \dot{N}_3(s)\, f''_{j-1}(z_j - z_{j-1}) + \\
&\quad \dot{N}_4(s)\, f_j/(z_j - z_{j-1}) + \dot{N}_5(s)\, f'_j + \dot{N}_6(s)\, f''_j(z_j - z_{j-1}), \qquad \text{(A1.1)} \\
\hat{f}^{j\prime\prime}(s) &= \ddot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1})^2 + \ddot{N}_2(s)\, f'_{j-1}/(z_j - z_{j-1}) + \ddot{N}_3(s)\, f''_{j-1} + \\
&\quad \ddot{N}_4(s)\, f_j/(z_j - z_{j-1})^2 + \ddot{N}_5(s)\, f'_j/(z_j - z_{j-1}) + \ddot{N}_6(s)\, f''_j,
\end{aligned}$$

where

$$s = \frac{\zeta - z_{j-1}}{z_j - z_{j-1}}, \qquad \text{(A1.2)}$$

$$\begin{aligned}
N_1(s) &= [\, 2 - 20s^3 + 30s^4 - 12s^5 \,]/2, \\
N_2(s) &= [\, 2s - 12s^3 + 16s^4 - 6s^5 \,]/2, \\
N_3(s) &= [\, s^2 - 3s^3 + 3s^4 - s^5 \,]/2, \qquad \text{(A1.3)} \\
N_4(s) &= [\, 20s^3 - 30s^4 + 12s^5 \,]/2, \\
N_5(s) &= [\, -8s^3 + 14s^4 - 6s^5 \,]/2, \\
N_6(s) &= [\, s^3 - 2s^4 + s^5 \,]/2,
\end{aligned}$$

$$\begin{aligned}
\dot{N}_1(s) &= [\, -60s^2 + 120s^3 - 60s^4 \,]/2, \\
\dot{N}_2(s) &= [\, 2 - 36s^2 + 64s^3 - 30s^4 \,]/2, \\
\dot{N}_3(s) &= [\, 2s - 9s^2 + 12s^3 - 5s^4 \,]/2, \qquad \text{(A1.4)} \\
\dot{N}_4(s) &= [\, 60s^2 - 120s^3 + 60s^4 \,]/2, \\
\dot{N}_5(s) &= [\, -24s^2 + 56s^3 - 30s^4 \,]/2, \\
\dot{N}_6(s) &= [\, 3s^2 - 8s^3 + 5s^4 \,]/2,
\end{aligned}$$

and

$$\ddot{N}_1(s) = [\; -120s + 360s^2 - 240s^3 \,]/2,$$
$$\ddot{N}_2(s) = [\; -72s + 192s^2 - 120s^3 \,]/2,$$
$$\ddot{N}_3(s) = [\,2 - 18s + 36s^2 - 20s^3 \,]/2, \qquad \text{(A1.5)}$$
$$\ddot{N}_4(s) = [\; 120s - 360s^2 + 240s^3 \,]/2,$$
$$\ddot{N}_5(s) = [\; -48s + 168s^2 - 120s^3 \,]/2,$$
$$\ddot{N}_6(s) = [\; 6s - 24s^2 + 20s^3 \,]/2.$$

The approximation function f(z) and the complex derivatives $f'(z)$, $f''(z)$ are obtained as:

$$f(z) = \frac{1}{4\pi i}\Bigg\{ \cdots + f_j\Bigg[\; 3K + 9K^2 - 24K^3 + 12K^4 + 2N_4(K)\ln\frac{z_j - z}{z_{j-1} - z}$$
$$- 3L - 9L^2 + 24L^3 - 12L^4 + 2N_1(L)\ln\frac{z_{j+1} - z}{z_j - z}\Bigg]$$
$$+ f_j'\Bigg[(z_j - z_{j-1})\Big\{ -\frac{11}{30} - \frac{5}{6}K - 3K^2 + 11K^3 - 6K^4 + 2N_5(K)\ln\frac{z_j - z}{z_{j-1} - z}\Big\}$$
$$+ (z_{j+1} - z_j)\Big\{ \frac{4}{5} - \frac{13}{6}L - 6L^2 + 13L^3 - 6L^4 + 2N_2(L)\ln\frac{z_{j+1} - z}{z_j - z}\Big\}\Bigg]$$
$$+ f_j''\Bigg[(z_j - z_{j-1})^2\Big\{ \frac{1}{30} + \frac{1}{12}K + \frac{1}{3}K^2 - \frac{3}{2}K^3 + K^4 + 2N_6(K)\ln\frac{z_j - z}{z_{j-1} - z}\Big\} +$$
$$+ (z_{j+1} - z_j)^2\Big\{ \frac{1}{20} + \frac{1}{4}L - \frac{11}{6}L^2 + \frac{5}{2}L^3 - L^4 + 2N_3(L)\ln\frac{z_{j+1} - z}{z_j - z}\Big\}\Bigg] + \cdots \Bigg\}$$

(A1.6)

$$f'(z) = \frac{1}{4\pi i}\Bigg\{ \cdots + f_j\Bigg[\frac{1}{z_j - z_{j-1}}\Big\{ 5 + 20K - 90K^2 + 60K^3 + 2\dot{N}_4(K)\ln\frac{z_j - z}{z_{j-1} - z}\Big\}$$
$$+ \frac{1}{z_{j+1} - z_j}\Big\{ -5 - 20L + 90L^2 - 60L^3 + 2\dot{N}_1(L)\ln\frac{z_{j+1} - z}{z_j - z}\Big\}\Bigg]$$

$$+ f_j' \left[\quad -5 - 6K + 41K^2 - 30K^3 + 2\dot{N}_5(K) \ln\frac{z_j - z}{z_{j-1} - z} \right.$$

$$\left. + \quad -14L + 49L^2 - 30L^3 + 2\dot{N}_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \right] \tag{A1.7}$$

$$+ f_j'' \left[(z_j - z_{j-1}) \left\{ \frac{1}{12} + \frac{2}{3}K - \frac{11}{2}K^2 + 5K^3 + 2\dot{N}_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} + \right.$$

$$\left. \left. + (z_{j+1} - z_j) \left\{ \frac{1}{4} - \frac{14}{3}L + \frac{19}{2}L^2 - 5L^3 + 2\dot{N}_3(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right] + \cdots \right\}$$

$$f''(z) = \frac{1}{4\pi i} \left\{ \cdots + f_j \left[\frac{1}{(z_j - z_{j-1})^2} \left\{ 20 - 240K + 240K^2 + 2\ddot{N}_4(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right. \right.$$

$$\left. + \frac{1}{(z_{j+1} - z_j)^2} \left\{ -20 + 240L - 240L^2 + 2\ddot{N}_1(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right]$$

$$+ f_j' \left[\frac{1}{z_j - z_{j-1}} \left\{ -4 + 108K - 120K^2 + 2\ddot{N}_5(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right.$$

$$\left. + \frac{1}{z_{j+1} - z_j} \left\{ -16 + 132L - 120L^2 + 2\ddot{N}_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right] \tag{A1.8}$$

$$+ f_j'' \left[\quad -6 - 14K + 20K^2 + 2\ddot{N}_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \right.$$

$$\left. \left. + 26L - 20L^2 + 2\ddot{N}_3(L) \ln\frac{z_{j+1} - z}{z_j - z} \right] + \cdots \right\},$$

where

$$K = \frac{z - z_{j-1}}{z_j - z_{j-1}}, \qquad L = \frac{z - z_j}{z_{j+1} - z_j}. \tag{A1.9}$$

A.2 Results for seventh-order complex basis functions

The seventh-order complex basis functions $\hat{f}^j$ and their derivatives can be written in the following form:

$$\hat{f}^j(s) = N_1(s)\, f_{j-1} + N_2(s)\, f'_{j-1}(z_j - z_{j-1}) + N_3(s)\, f''_{j-1}(z_j - z_{j-1})^2 + N_4(s)\, f'''_{j-1}(z_j - z_{j-1})^3 +$$
$$N_5(s)\, f_j + N_6(s)\, f'_j(z_j - z_{j-1}) + N_7(s)\, f''_j(z_j - z_{j-1})^2 + N_8(s)\, f'''_j(z_j - z_{j-1})^3,$$
$$\hat{f}^{j\prime}(s) = \dot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1}) + \dot{N}_2(s)\, f'_{j-1} + \dot{N}_3(s)\, f''_{j-1}(z_j - z_{j-1}) + \dot{N}_4(s)\, f'''_{j-1}(z_j - z_{j-1})^2 +$$
$$\dot{N}_5(s)\, f_j/(z_j - z_{j-1}) + \dot{N}_6(s)\, f'_j + \dot{N}_7(s)\, f''_j(z_j - z_{j-1}) + \dot{N}_8(s)\, f'''_j(z_j - z_{j-1})^2,$$

(A2.1)

$$\hat{f}^{j\prime\prime}(s) = \ddot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1})^2 + \ddot{N}_2(s)\, f'_{j-1}/(z_j - z_{j-1}) + \ddot{N}_3(s)\, f''_{j-1} + \ddot{N}_4(s)\, f'''_{j-1}(z_j - z_{j-1}) +$$
$$\ddot{N}_5(s)\, f_j/(z_j - z_{j-1})^2 + \ddot{N}_6(s)\, f'_j/(z_j - z_{j-1}) + \ddot{N}_7(s)\, f''_j + \ddot{N}_8(s)\, f'''_j(z_j - z_{j-1}),$$
$$\hat{f}^{j\prime\prime\prime}(s) = \dddot{N}_1(s)\, f_{j-1}/(z_j - z_{j-1})^3 + \dddot{N}_2(s)\, f'_{j-1}/(z_j - z_{j-1})^2 + \dddot{N}_3(s)\, f''_{j-1}/(z_j - z_{j-1}) + \dddot{N}_4(s)\, f'''_{j-1} +$$
$$\dddot{N}_5(s)\, f_j/(z_j - z_{j-1})^3 + \dddot{N}_6(s)\, f'_j/(z_j - z_{j-1})^2 + \dddot{N}_7(s)\, f''_j/(z_j - z_{j-1}) + \dddot{N}_8(s)\, f'''_j,$$

(A2.2)

where

$$N_1(s) = [\, 6 - 210s^4 + 504s^5 - 420s^6 + 120s^7 \,]/6,$$
$$N_2(s) = [\, 6s - 120s^4 + 270s^5 - 216s^6 + 60s^7 \,]/6,$$
$$N_3(s) = [\, 3s^2 - 30s^4 + 60s^5 - 45s^6 + 12s^7 \,]/6,$$
$$N_4(s) = [\, s^3 - 4s^4 + 6s^5 - 4s^6 + s^7 \,]/6,$$
$$N_5(s) = [\, 210s^4 - 504s^5 + 420s^6 - 120s^7 \,]/6,$$
$$N_6(s) = [\, -90s^4 + 234s^5 - 204s^6 + 60s^7 \,]/6,$$
$$N_7(s) = [\, 15s^4 - 42s^5 + 39s^6 - 12s^7 \,]/6,$$
$$N_8(s) = [\, - s^4 + 3s^5 - 3s^6 + s^7 \,]/6,$$

(A2.3)

and

$$\dot{N}_1(s) = [\ -840s^3 + 2520s^4 - 2520s^5 + 840s^6\ ]/6,$$
$$\dot{N}_2(s) = [\ 6 - 480s^3 + 1350s^4 - 1296s^5 + 420s^6\ ]/6,$$
$$\dot{N}_3(s) = [\ 6s - 120s^3 + 300s^4 - 270s^5 + 84s^6\ ]/6,$$
$$\dot{N}_4(s) = [\ 3s^2 - 16s^3 + 30s^4 - 24s^5 + 7s^6\ ]/6,$$
$$\dot{N}_5(s) = [\ 840s^3 - 2520s^4 + 2520s^5 - 840s^6\ ]/6, \qquad \text{(A2.4)}$$
$$\dot{N}_6(s) = [\ -360s^3 + 1170s^4 - 1224s^5 + 420s^6\ ]/6,$$
$$\dot{N}_7(s) = [\ 60s^3 - 210s^4 + 234s^5 - 84s^6\ ]/6,$$
$$\dot{N}_8(s) = [\ -4s^3 + 15s^4 - 18s^5 + 7s^6\ ]/6,$$

$$\ddot{N}_1(s) = [\ -2520s^2 + 10080s^3 - 12600s^4 + 5040s^5\ ]/6,$$
$$\ddot{N}_2(s) = [\ -1440s^2 + 5400s^3 - 6480s^4 + 2520s^5\ ]/6,$$
$$\ddot{N}_3(s) = [\ 6 - 360s^2 + 1200s^3 - 1350s^4 + 504s^5\ ]/6,$$
$$\ddot{N}_4(s) = [\ 6s - 48s^2 + 120s^3 - 120s^4 + 42s^5\ ]/6,$$
$$\ddot{N}_5(s) = [\ 2520s^2 - 10080s^3 + 12600s^4 - 5040s^5\ ]/6, \qquad \text{(A2.5)}$$
$$\ddot{N}_6(s) = [\ -1080s^2 + 4680s^3 - 6120s^4 + 2520s^5\ ]/6,$$
$$\ddot{N}_7(s) = [\ 180s^2 - 840s^3 + 1170s^4 - 504s^5\ ]/6,$$
$$\ddot{N}_8(s) = [\ -12s^2 + 60s^3 - 90s^4 + 42s^5\ ]/6,$$

$$\dddot{N}_1(s) = [\ -5040s + 30240s^2 - 50400s^3 + 25200s^4\ ]/6,$$
$$\dddot{N}_2(s) = [\ -2880s + 16200s^2 - 25920s^3 + 12600s^4\ ]/6,$$
$$\dddot{N}_3(s) = [\ -720s + 3600s^2 - 5400s^3 + 2520s^4\ ]/6,$$
$$\dddot{N}_4(s) = [\ 6 - 96s + 360s^2 - 480s^3 + 210s^4\ ]/6,$$

$$\dddot{N}_5(s) = [\quad 5040s - 30240s^2 + 50400s^3 - 25200s^4\,]/6, \tag{A2.6}$$

$$\dddot{N}_6(s) = [\quad -2160s + 14040s^2 - 24480s^3 + 12600s^4\,]/6,$$

$$\dddot{N}_7(s) = [\quad 360s - 2520s^2 + 4680s^3 - 2520s^4\,]/6,$$

$$\dddot{N}_8(s) = [\quad -24s + 180s^2 - 360s^3 + 210s^4\,]/6.$$

Substitution of (A2.1) into (3.2) and integration gives us the element contribution $f^j(z)$ in the form

$$\begin{aligned}
f^j(z) = \frac{1}{12\pi i}\Bigg\{ & f_{j-1}\left[-\frac{319}{70} - 8K - 18K^2 - 68K^3 + 334K^4 - 360K^5 + 120K^6 + 6N_1(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})f'_{j-1}\left[\frac{18}{7} - \frac{57}{10}K - 12K^2 - 42K^3 + 182K^4 - 186K^5 + 60K^6 + 6N_2(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})^2 f''_{j-1}\left[\frac{3}{14} + K - \frac{77}{20}K^2 - 12K^3 + \frac{83}{2}K^4 - 39K^5 + 12K^6 + 6N_3(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})^3 f'''_{j-1}\left[\frac{1}{420}(4 + 14K + 84K^2 - 875K^3 + 1820K^4 - 1470K^5 + 420K^6) \right. \\
& \qquad \left. + 6N_4(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + f_j\left[\frac{319}{70} + 8K + 18K^2 + 68K^3 - 334K^4 + 360K^5 - 120K^6 + 6N_5(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})f'_j\left[-\frac{79}{70} - \frac{23}{10}K - 6K^2 - 26K^3 + 152K^4 - 174K^5 + 60K^6 + 2N_6(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})^2 f''_j\left[\frac{19}{140} + \frac{3}{10}K + \frac{17}{20}K^2 + 4K^3 - \frac{53}{2}K^4 + 33K^5 - 12K^6 + 2N_7(K)\ln\frac{z_j - z}{z_{j-1} - z} \right] \\
& + (z_j - z_{j-1})^3 f'''_j\left[\frac{1}{420}(-3 - 7K - 21K^2 - 105K^3 + 770K^4 + 1050K^5 + 420K^6) \right. \\
& \qquad \left. + 2N_8(K)\ln\frac{z_j - z}{z_{j-1} - z} \right]\Bigg\}.
\end{aligned} \tag{A2.7}$$

Adding neighboring element contributions we obtain for f(z) the representation

$$f(z) = \frac{1}{12\pi i}\Bigg\{ \cdots +$$

$$+ f_j \Bigg[8K + 18K^2 + 68K^3 - 334K^4 + 360K^5 - 120K^6 + 6N_5(K) \ln\frac{z_j - z}{z_{j-1} - z}$$

$$- 8L - 18L^2 - 68L^3 + 334L^4 - 360L^5 + 120L^6 + 6N_1(L) \ln\frac{z_{j+1} - z}{z_j - z} \Bigg]$$

$$+ f_j' \Bigg[(z_j - z_{j-1})\Big\{ -\frac{79}{70} - \frac{23}{10}K - 6K^2 - 26K^3 + 152K^4 - 174K^5 + 60K^6 + 6N_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \Big\}$$

$$+ (z_{j+1} - z_j)\Big\{ \frac{18}{7} - \frac{57}{10}L - 12L^2 - 42L^3 + 182L^4 - 186L^5 + 60L^6 + 6N_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \Big\} \Bigg]$$

$$+ f_j'' \Bigg[(z_j - z_{j-1})^2\Big\{ \frac{19}{140} + \frac{3}{10}K + \frac{17}{20}K^2 + 4K^3 - \frac{53}{2}K^4 + 33K^5 - 12K^6 + 6N_7(K) \ln\frac{z_j - z}{z_{j-1} - z} \Big\}$$

$$+ (z_{j+1} - z_j)^2\Big\{ \frac{3}{14} + L - \frac{77}{20}L^2 - 12L^3 + \frac{83}{2}L^4 - 39L^5 + 12L^6 + 6N_3(L) \ln\frac{z_{j+1} - z}{z_j - z} \Big\} \Bigg]$$

$$+ f_j''' \Bigg[(z_j - z_{j-1})^3\Big\{ \frac{1}{420}(-3 - 7K - 21K^2 - 105K^3 + 770K^4 - 1050K^5 + 420K^6)$$

$$+ 6N_8(K) \ln\frac{z_j - z}{z_{j-1} - z} \Big\} +$$

$$+ (z_{j+1} - z_j)^3\Big\{ \frac{1}{420}(4 + 14L + 84L^2 - 875L^3 + 1820L^4 - 1470L^5 + 420L^6)$$

$$+ 6N_4(L) \ln\frac{z_{j+1} - z}{z_j - z} \Big\} \Bigg] + \cdots \Bigg\}. \qquad \text{(A2.8)}$$

The complex derivatives $f'(z)$, $f''(z)$ and $f'''(z)$ are obtained as:

$$f'(z) = \frac{1}{12\pi i}\Bigg\{\cdots +$$

$$+ f_j\Bigg[\frac{1}{z_j - z_{j-1}}\left\{14 + 42K + 210K^2 - 1540K^3 + 2100K^4 - 840K^5 + 6\dot{N}_5(K)\ln\frac{z_j - z}{z_{j-1} - z}\right\}$$

$$+ \frac{1}{z_{j+1} - z_j}\left\{-14 - 42L - 210L^2 + 1540L^3 - 2100L^4 + 840L^5 + 6\dot{N}_1(L)\ln\frac{z_{j+1} - z}{z_j - z}\right\}\Bigg]$$

$$+ f_j'\Bigg[\quad -14 - 12K - 78K^2 + 698K^3 - 1014K^4 + 420K^5 + 6\dot{N}_6(K)\ln\frac{z_j - z}{z_{j-1} - z}$$

$$-30L - 132L^2 + 842L^3 - 1086L^4 + 420L^5 + 6\dot{N}_2(L)\ln\frac{z_{j+1} - z}{z_j - z}\Bigg]$$

$$+ f_j''\Bigg[(z_j - z_{j-1})\left\{\frac{3}{10} + \frac{17}{10}K + 12K^2 - 121K^3 + 192K^4 - 84K^5 + 6\dot{N}_7(K)\ln\frac{z_j - z}{z_{j-1} - z}\right\}$$

$$+ (z_{j+1} - z_j)\left\{1 - \frac{107}{10}L - 39L^2 + 193L^3 - 228L^4 + 84L^5 + 6\dot{N}_3(L)\ln\frac{z_{j+1} - z}{z_j - z}\right\}\Bigg]$$

$$+ f_j'''\Bigg[(z_j - z_{j-1})^2\Bigg\{\frac{1}{420}(-7 - 42K - 315K^2 + 3500K^3 - 6090K^4 + 2940K^5)$$

$$+ 6\dot{N}_8(K)\ln\frac{z_j - z}{z_{j-1} - z}\Bigg\} +$$

$$+ (z_{j+1} - z_j)^2\Bigg\{\frac{1}{420}(14 + 168L - 3045L^2 + 8540L^3 - 8610L^4 + 2940L^5)$$

$$+ 6\dot{N}_4(L)\ln\frac{z_{j+1} - z}{z_j - z}\Bigg\}\Bigg] + \cdots\Bigg\} \qquad \text{(A2.9)}$$

$$
\begin{aligned}
f''(z) = \frac{1}{12\pi i}\Bigg\{ \cdots +& \\
&+ f_j \left[\frac{1}{(z_j - z_{j-1})^2} \left\{ 42 + 420K - 5460K^2 + 10080K^3 - 5040K^4 + 6\ddot{N}_5(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right. \\
&\quad \left. + \frac{1}{(z_{j+1} - z_j)^2} \left\{ -42 - 420L + 5460L^2 - 10080L^3 + 5040L^4 + 6\ddot{N}_1(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right] \\
&+ f_j' \left[\frac{1}{z_j - z_{j-1}} \left\{ -6 - 150K + 2460K^2 - 4860K^3 + 2520K^4 + 6\ddot{N}_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right. \\
&\quad \left. + \frac{1}{z_{j+1} - z_j} \left\{ -36 - 270L + 3000L^2 - 5220L^3 + 2520L^4 + 6\ddot{N}_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right] \\
&+ f_j'' \left[-15 + 24K - 423K^2 + 918K^3 - 504K^4 + 6\ddot{N}_7(K) \ln\frac{z_j - z}{z_{j-1} - z} \right. \\
&\quad \left. - 84L + 693L^2 - 1098L^3 + 504L^4 + 6\ddot{N}_3(L) \ln\frac{z_{j+1} - z}{z_j - z} \right] \\
&+ f_j''' \left[(z_j - z_{j-1}) \left\{ -\frac{1}{10} - \frac{3}{2}K + 29K^2 - 69K^3 + 42K^4 + 6\ddot{N}_8(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} + \right. \\
&\quad \left. + (z_{j+1} - z_j) \left\{ \frac{2}{5} - \frac{35}{2}L + 74L^2 - 99L^3 + 42L^4 + 6\ddot{N}_4(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right] + \cdots \Bigg\},
\end{aligned}
$$

(A2.10)

$$
\begin{aligned}
f'''(z) = \frac{1}{12\pi i}\Bigg\{ \cdots +& \\
&+ f_j \left[\frac{1}{(z_j - z_{j-1})^3} \left\{ 420 - 13440K + 37800K^2 - 25200K^3 + 6\dddot{N}_5(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right. \\
&\quad \left. + \frac{1}{(z_{j+1} - z_j)^3} \left\{ -420 + 13440L - 37800L^2 + 25200L^3 + 6\dddot{N}_1(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right]
\end{aligned}
$$

$$+ f_j' \left[\frac{1}{(z_j - z_{j-1})^2} \left\{ -150 + 6000K - 18180K^2 + 12600K^3 + 6\dddot{N}_6(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} \right.$$

$$\left. + \frac{1}{(z_{j+1} - z_j)^2} \left\{ -270 + 7440L - 19620L^2 + 12600L^3 + 6\dddot{N}_2(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right]$$

$$+ f_j'' \left[\frac{1}{z_j - z_{j-1}} \left\{ 30 - 1020K + 3420K^2 - 2520K^3 + 6\ddot{N}_7(K) \ln\frac{z_j - z}{z_{j-1} - z} \right\} + \right.$$

$$\left. + \frac{1}{z_{j+1} - z_j} \left\{ -90 + 1740L - 4140L^2 + 2520L^3 + 6\ddot{N}_3(L) \ln\frac{z_{j+1} - z}{z_j - z} \right\} \right]$$

$$+ f_j''' \left[-25 + 70K - 255K^2 + 210K^3 + 6\ddot{N}_8(K) \ln\frac{z_j - z}{z_{j-1} - z} \right.$$

$$\left. + 190L - 375L^2 + 210L^3 + 6\ddot{N}_4(L) \ln\frac{z_{j+1} - z}{z_j - z} \right] + \cdots \Bigg\}.$$

(A2.11)

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