Title
Weighted Projective Hypersurfaces with Extreme Invariants

Author
Esser, Louis

Publication Date
2023

Peer reviewed|Thesis/dissertation

UNIVERSITY OF CALIFORNIA
Los Angeles

Weighted Projective Hypersurfaces with Extreme Invariants

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Louis Franklin Esser

2023

ABSTRACT OF THE DISSERTATION

Weighted Projective Hypersurfaces with Extreme Invariants

by

Louis Franklin Esser
Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2023
Professor Burt Totaro, Chair

The goal of this dissertation is to study weighted projective hypersurfaces and their application to optimization problems in algebraic geometry. First, we generalize and strengthen several well-known results on the automorphisms of hypersurfaces due to Grothendieck-Lefschetz and Matsumura-Monsky to the weighted setting. Then, we construct special examples of weighted projective hypersurfaces with extreme properties. These are used to prove strong asymptotics on certain invariants from birational geometry as dimension increases. In particular, we show that the minimum volume of smooth varieties of general type approaches zero doubly exponentially with dimension; we also show that the index of mildly singular Calabi-Yau varieties can grow doubly exponentially with dimension. For several classes of varieties, we conjecture the optimal bounds on volume or index in every dimension; these conjectures are supported by low-dimensional evidence.

The dissertation of Louis Franklin Esser is approved.

Paul Balmer

Raphaël Alexis Rouquier

Alexander Sergee Merkurjev

Burt Totaro, Committee Chair

University of California, Los Angeles

2023

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGMENTS

First, I would like to thank my advisor, Burt Totaro, for his continual support and generosity of time over the last several years. I wasn't sure which field I would be studying in graduate school when I arrived at UCLA, and Burt did a great deal to help me find my path.

I also owe thanks to my friends here in the department who helped to make the UCLA math graduate student community so welcoming and engaging. There are too many of you to name, but I'd like to acknowledge Will Baker, David Hemminger, Ben Spitz, Talon Stark, and Dominic Yang in particular for their friendship, advice, and mathematical help.

So much of who I am, I owe to my family. Mom, Dad, Char, and Kendra: I've always been able to count on your love and support even across thousands of miles, and for that I'm deeply grateful. Finally, thank you to Trisha for being the greatest partner one could ask for; I couldn't have done any of it without you.

Note: Chapters 2, 3, and 4 are based on material from [Ess23], [ETW23], and [ETW22], respectively. The latter two works are joint with Burt Totaro and Chengxi Wang. I was also partially supported by NSF grant DMS-2054553 and UCLA's Dissertation Year Fellowship.

VITA

2018	B.A. (Mathematics, Physics), University of Pennsylvania.
2020	M.A. (Mathematics), University of California, Los Angeles.

CHAPTER 1

Introduction

1.1 Overview

A principal aim of algebraic geometry is the classification of algebraic varieties, in particular complex projective manifolds. These are separated into classes roughly by the sign of the curvature. The last few decades have seen spectacular advances in proving the "boundedness" of different classes of varieties in a fixed dimension, including work of Hacon, M[c]Kernan, Xu, Birkar, and many others (for example, [HM06, HMX13, JL20, Bir19, Bir20, Xu19]). These results are significant since they mean that all objects in the class of dimension n are parameterized by a space of finite size, so it's possible in principle to classify and study them all simultaneously.

However, these type of results typically show only the existence of a bound for some invariant among varieties of class $\mathcal{A}$ of dimension n. Any explicit classification of varieties of class $\mathcal{A}$ would require knowing such bounds exactly. Therefore, we aim to determine exact bounds on invariants for varieties of class $\mathcal{A}$ and fixed dimension n. This dissertation will describe results in this direction for varieties of several different classes. These illuminate a surprising common principle: "the complexity of algebraic geometry increases doubly exponentially with dimension."

The key idea is to find weighted projective hypersurfaces with extreme behavior. These objects serve as an excellent testing ground for conjectures in algebraic geometry. This is because many of their properties are determined combinatorially from the weights and degree, but they are nevertheless also flexible enough to use in a wide range of applications; indeed, we'll see below that there is ample evidence that they are in fact varied enough to attain the extreme values of many important invariants in algebraic geometry.

Moreover, weighted projective hypersurfaces are interesting objects to study in their own right. For instance, a large body of existing literature examines the properties of automorphisms of hypersurfaces in ordinary projective space, but considerably less is known about automorphisms in the weighted setting. Another objective of this dissertation is to investigate to what extent known results generalize to hypersurfaces in weighted projective space.

This dissertation is based on three projects: [Ess23], [ETW23], and [ETW22]. The first two chapters are focused on studying weighted projective hypersurfaces themselves, while the latter two, written in collaboration with Burt Totaro and Chengxi Wang, treat applications to optimization problems in algebraic geometry. The remainder of this chapter discusses the necessary preliminaries on the objects of study, many of which are required for all three of the later chapters. Chapter 2 discusses automorphisms groups of weighted projective hypersurfaces and is adapted from [Ess23]. Chapter 3 deals with varieties of small volume and is based on [ETW23]. Finally, Chapter 4 examines Calabi-Yau varieties with extreme invariants and is based on [ETW22].

1.2 Singularities of the Minimal Model Program

We'll frequently use terminology relating to singularities of the minimal model program, such as terminal, canonical, or Kawamata log terminal (klt). Some standard introductions to these notions are [Reib, KM98]. We briefly summarize the relevant properties we'll make use of here.

A *pair* (X, D) is a normal variety X with an effective $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q}$-Cartier. For a proper birational morphism $\mu\colon X' \to X$ with X' normal and an irreducible divisor $E \subset X'$, the *log discrepancy* of E with respect to (X, D), written $a_E(X, D)$, is the coefficient of E in the $\mathbb{Q}$-divisor $K_{X'} + E - \mu^*(K_X + D)$. The *center* of E in X is the image $\mu(E)$. The log discrepancy and the center depend only on the valuation defined by E on the function field of X; that allows us to identify some irreducible divisors on different birational models of X.

Various classes of singularities of the minimal model program are defined using the log discrepancy. We say that the pair (X, D) is *canonical* (resp. *terminal*) if the log discrepancy $a_E(X, D) \geq 1$ (resp. $a_E(X, D) > 1$) for every exceptional divisor E over (X, D). Most of the time, we'll use these notions for varieties rather than pairs, where we take $D = 0$.

A pair (X, D) is *log canonical* (resp. *Kawamata log terminal*) if the log discrepancy $a_E(X, D) \geq 0$ (resp. $a_E(X, D) > 0$) for *all* divisors E over (X, D). In particular, if (X, D) is log canonical, the coefficients of the prime divisors D_i in D are at most 1. We abbreviate log canonical by "lc" and Kawamata log terminal by "klt".

For a pair (X, D) and a point x of the scheme X, the *minimal log discrepancy* of (X, D) at x is the infimum

$$\mathrm{mld}_x(X, D) \coloneqq \inf\{a_E(X, D)\colon\ \mathrm{center}_X(E) = \overline{x}\},$$

and the (global) *minimal log discrepancy* of (X, D) is

$$\mathrm{mld}(X, D) \coloneqq \inf_{x \in X} \mathrm{mld}_x(X, D).$$

Furthermore, when (X, D) is lc (meaning that its mld is nonnegative), the mld can be computed using the finitely many irreducible divisors that appear on a log resolution of (X, D) (and so it is rational) [Kol97, Definition 7.1]. The condition that (X, D) is klt is equivalent to $\mathrm{mld}(X, D) > 0$. Finite quotient singularities are klt [Kol97, Corollary 2.43], so all quasismooth weighted projective hypersurfaces and their quotients by finite groups will have klt singularities (see Section 1.3 below). This makes klt singularities a natural class to study using these objects. In Chapter 4, for example, we'll investigate the question of how small the mld of a Calabi-Yau pair (X, D) with standard coefficients can be while still remaining positive (so that the pair is klt).

Varieties with canonical singularities will be of particular importance in Chapter 3, where we aim to construct smooth varieties of general type of small volume, or many vanishing plurigenera. To do this, we construct instead varieties X with canonical singularities such that K_X is ample. A resolution of singularities $\tilde{X}$ of X will then be smooth of general type and satisfy

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(mK_{\tilde{X}})) \cong H^0(X, \mathcal{O}_X(mK_X))$$

for all $m \geq 0$ [Kol97, Theorem 1.15]. Any invariant defined in terms of the plurigenera of X, such as $\mathrm{vol}(K_X)$, will then have the same value on $\tilde{X}$. Terminal singularities are the smallest class of singularities necessary for running the minimal model program on smooth varieties, so we spend time constructing terminal varieties with extreme properties as well.

1.3 Preliminaries on Weighted Projective Spaces

We'll work over the complex numbers throughout, unless otherwise noted. However, many statements would remain true in any characteristic. For a more thorough introduction to the basics of weighted projective varieties, see [Ian00].

Given a collection of positive integers $a_0, \ldots, a_N$, the *weighted projective space* $Y = \mathbb{P}(a_0, \ldots, a_N)$ is the quotient variety $(\mathbb{A}^{N+1} \setminus \{0\})/\mathbb{G}_m$, where the multiplicative group $\mathbb{G}_m$ acts by $t(x_0, \ldots, x_N) = (t^{a_0}x_0, \ldots, t^{a_N}x_N)$. It is also useful to describe weighted projective space in terms of the Proj construction as follows: $\mathbb{P}(a_0, \ldots, a_N) = \mathrm{Proj}(S)$, where $S = \mathbb{C}[x_0, \ldots, x_N]$ is the graded polynomial ring with variables of the given weights.

We say that Y is *well-formed* when $\gcd(a_0, \ldots, \widehat{a_j}, \ldots, a_N) = 1$ for each j [Ian00, Definition 6.9]. We always assume that Y is well-formed. (In other words, the analogous quotient stack $[(\mathbb{A}^{N+1} - 0)/\mathbb{G}_m]$ has trivial stabilizer group in codimension 1.) For well-formed Y, the canonical divisor of Y is given by $K_Y = \mathcal{O}(-a_0 - \cdots - a_N)$. Here $\mathcal{O}(d)$ is the sheaf on Y associated to the dth graded piece of S. It is a reflexive sheaf of rank 1; it is in particular a line bundle if and only if d is a multiple of every weight a_j. Though $\mathcal{O}(1)$ may not correspond to a Cartier divisor, it does correspond to some Weil divisor on Y. We say that a Weil divisor (with integral coefficients) is *ample* if some multiple of it is an ample Cartier divisor. For example, $\mathcal{O}(1)$ is ample. It makes sense to talk about the *volume*, or top self-intersection number, of an ample Weil divisor. The volume of $\mathcal{O}(1)$ is $1/(a_0 \cdots a_N)$.

We write $\mathbb{P}(a_0^{(b_0)}, \dots, a_r^{(b_r)})$ for the weighted projective space with the weight a_i repeated b_i times.

Weighted projective spaces have only cyclic quotient singularities. As remarked in Section 1.2, such singularities are always klt. To determine whether they are in addition canonical or terminal, we use the Reid-Tai criterion [Reib, Theorem 4.11]. For a an integer and b a positive integer, consider $a \bmod b$ as an integer in the set $\{0, \dots, b-1\}$.

Theorem 1.3.1. *For a positive integer r, let the group μ_r of rth roots of unity act on affine space by $\zeta(t_1, \dots, t_s) = (\zeta^{b_1} t_1, \dots, \zeta^{b_s} t_s)$. The quotient $\mathbb{A}^s/\mu_r$ is said to be a cyclic quotient singularity of type $\frac{1}{r}(b_1, \dots, b_s)$. Assume that this description is well-formed in the sense that $\gcd(r, b_1, \dots, \widehat{b_j}, \dots, b_s) = 1$ for all $j = 1, \dots, s$. Then the quotient singularity is canonical (resp. terminal) if and only if*

$$\sum_{j=1}^{s} i b_j \bmod r \geq r$$

(resp. $> r$) for all $i = 1, \dots, r-1$.

Each weighted projective space $Y = \mathbb{P}(a_0, \dots, a_N)$ is a toric variety, with an action of the torus $T = (\mathbb{G}_m)^{N+1}/\mathbb{G}_m \cong (\mathbb{G}_m)^N$ by scaling the variables. Since the locus where Y is canonical or terminal is open as well as T-invariant, we have the following:

Lemma 1.3.2. *Let Y be a weighted projective space. If Y is canonical (resp. terminal) at a point q, then Y is also canonical (resp. terminal) at all points p such that q is in the closure of the T-orbit of p.*

Because of this lemma, to prove that Y is canonical (or terminal), it suffices to check only the coordinate points $[0 : \cdots : 0 : 1 : 0 : \cdots : 0]$. Moreover, if Y is canonical at every coordinate point besides p, then $Y \setminus p$ is canonical. Nevertheless, we'll sometimes need to know the singularities of Y away from the coordinate points, in particular on some other stratum U_I of the torus action, where $I \subset \{0, \dots, N\}$ is the set of indices of nonzero coordinates on this stratum. Therefore, we state how to identify the quotient singularities at any point of Y (elaborating on the statements in [Ian00, section 5.15]).

Proposition 1.3.3. *Let $Y = \mathbb{P}(a_0, \ldots, a_N)$ be a weighted projective space and $I \subset \{0, ..., N\}$ a nonempty subset with size $|I| = k + 1$. Let $r = \gcd(a_i : i \in I)$. If p is a point on U_I, then p has a neighborhood analytically isomorphic to a quotient singularity of type*

$$\frac{1}{r}(a_i : i \notin I) \times \mathbb{A}^k.$$

The automorphism group of Y is easily determined using its structure as Proj of a graded ring S. We also use the notation S_m to denote the mth graded piece of S. Any graded automorphism of S induces an automorphism of weighted projective space; let $\mathrm{Aut}(S)$ denote the group of graded automorphisms. In fact, every automorphism of $\mathbb{P}$ comes from $\mathrm{Aut}(S)$. This is proven in [Amr89, Section 8], but for completeness, we provide a proof here:

Lemma 1.3.4. *Let $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_N)$ be a well-formed weighted projective space and $S := \mathbb{C}[x_0, \ldots, x_N]$ the graded polynomial ring with the weight of x_i equal to a_i. Then the natural map $\mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P})$ is surjective. The kernel is isomorphic to $\mathbb{C}^*$, where the isomorphism associates to any $t \in \mathbb{C}^*$ the automorphism mapping $x_i \mapsto t^{a_i} x_i$ for each i.*

Proof. Suppose that $f : \mathbb{P} \to \mathbb{P}$ is an automorphism. Pullback by f yields an isomorphism of class groups $f^* : \mathrm{Cl}(\mathbb{P}) \to \mathrm{Cl}(\mathbb{P})$. Since $\mathrm{Cl}(\mathbb{P}) \cong \mathbb{Z}$ and ampleness of classes must be preserved, $f^*\mathcal{O}(1) = \mathcal{O}(1)$. It follows that for every m we have an isomorphism $f^* : H^0(\mathbb{P}, \mathcal{O}(m)) \to H^0(\mathbb{P}, \mathcal{O}(m))$ and that these isomorphisms are compatible with tensor product. Since $H^0(\mathbb{P}, \mathcal{O}(m))$ is the mth graded piece of S, these assemble to a graded isomorphism $f^* : S \to S$, which induces the original f. It's straightforward to check that only maps of the form $x_i \mapsto t^{a_i} x_i$ for all i are in the kernel, and that every $t \neq 1$ gives a non-identity map $S \to S$ by well-formedness. □

We'll need the following fact about automorphisms of graded polynomial rings in Sections 2.3 and 2.4. We say that a group $G \subset \mathrm{Aut}(S)$ is *diagonalizable* if, after conjugating G by an automorphism of S, each element of g sends each variable x_i to some scalar multiple of itself. Any such automorphism also sends an arbitrary monomial to a scalar multiple of itself.

Lemma 1.3.5. *Let* $S = \mathbb{C}[x_0, \ldots, x_N]$ *be a weighted polynomial ring with weights* $a_0, \ldots, a_N$ *and* $\mathrm{Aut}(S)$ *the group of graded automorphisms. If* $G \subset \mathrm{Aut}(S)$ *is a finite abelian group, then* G *is diagonalizable.*

This is, of course, a generalization of the well-known fact that any finite abelian group in $\mathrm{GL}_{N+1}(\mathbb{C})$ is diagonalizable.

Proof. The group $\mathrm{Aut}(S)$ embeds naturally in $\bigoplus_{b \in \mathcal{B}} \mathrm{GL}(S_b)$, where $\mathcal{B} = \{b : b = a_i \text{ for some } i\}$ is the set of integers that appear as a weight of S. (In particular, $\mathrm{Aut}(S)$ is a linear algebraic group.) To diagonalize a finite abelian group $G \subset \mathrm{Aut}(S)$, we'll focus on the representation on each of these pieces S_b in turn. Of course, within each S_b, we can diagonalize G using some change of coordinates in S_b, but we must prove that we can do this simultaneously for all b using conjugation by an element γ in $\mathrm{Aut}(S)$.

To do this, we put the integers in B in increasing order and construct γ inductively. For the smallest integer $b_0 \in \mathcal{B}$, there is a basis of S_{b_0} given by $\{x_i : a_i = b_0\}$. Define γ on the smallest weight variables in such a way that the representation of G on S_{b_0} becomes diagonal in the given basis (this is no problem because all monomials of degree b_0 are generators of S). By the inductive hypothesis, we now assume G acts by multiplication by a scalar on all variables x_i of weight $a_i < b$ and consider the action on S_b. Write a basis for S_b beginning with the variables of weight b, followed by the other monomials of degree b. By the inductive hypothesis, we've already constructed a γ so that the representation of G after changing coordinates in only the smaller variables is of the form

$$\begin{pmatrix} A(g) & 0 \\ B(g) & D(g) \end{pmatrix}.$$

Here $D(g)$ is a diagonal sum of characters of G, because each $g \in G$ acts on monomials in the smaller weight variables by scalar multiplication. Since the entire space S_b must also be a direct sum of one-dimensional characters of G, we can find a change of coordinates affecting only coordinates in the first part of the basis (variables of weight b) so that the representation becomes diagonal. This finishes the definition of the automorphism γ on variables of weight up to b. By induction, the proof is complete. □

Next, we'll consider subvarieties of weighted projective space. A closed subvariety X of the weighted projective space Y is *quasismooth* if its affine cone in $\mathbb{A}^{N+1}$ is smooth away from the origin. We say that X is *well-formed* if Y is well-formed and the codimension of the intersection $X \cap Y_{\text{sing}}$ in X is at least 2. We'll be primarily considering the case where X is a hypersurface. In this situation, as long as the degree d is not equal to any of the weights (this assumption will always hold in our examples), every quasismooth hypersurface of dimension at least 3 is well-formed [Ian00, Theorem 6.17]. The following criterion, which works in characteristic zero, determines whether a general hypersurface is quasismooth [Ian00, Theorem 8.1]:

Proposition 1.3.6. *A general hypersurface X of degree d in the weighted projective space $\mathbb{P}(a_0, \ldots, a_N)$ is quasismooth if and only if one of the following properties holds:*

1. *$a_i = d$ for some i, or*
2. *for each nonempty subset I of $\{0, \ldots, N\}$, either*
   (*a*) *d is an $\mathbb{N}$-linear combination of the weights a_i for $i \in I$, or*
   (*b*) *there are at least $|I|$ numbers $j \notin I$ such that $d - a_j$ is an $\mathbb{N}$-linear combination of the numbers a_i with $i \in I$.*

Here, "$\mathbb{N}$-linear combination" means a linear combination with nonnegative integer coefficients. We'll frequently use the following version of the proposition applied to singleton sets $I = \{i\}$:

Proposition 1.3.7. *Suppose $X = \{f = 0\} \subset \mathbb{P}$ is a quasismooth hypersurface of degree d in $\mathbb{P}$. Then for each $i = 0, \ldots, N$, there exists a monomial of degree d with nonzero coefficient in f having the form either (a) x_i^k, or (b) $x_i^k x_j$ for some $j \neq i$.*

Proof. If such a monomial does not exist for some i, then the function f and all its derivatives would vanish at the coordinate point $e_i \in \mathbb{A}^{N+1}$, contradicting the smoothness of the punctured affine cone C_X^*. □

A key property of well-formed quasismooth hypersurfaces X is that the adjunction formula holds, so that $K_X = \mathcal{O}_X(d - a_0 - \cdots - a_N)$ [Ian00, section 6.14]. Also, the volume of the ample Weil divisor $\mathcal{O}_X(1)$ is $d/(a_0 \cdots a_N)$.

Given a fixed embedding of a weighted projective hypersurface $X \subset \mathbb{P}$, we'll define the subgroup $\mathrm{Lin}(X) \subset \mathrm{Aut}(X)$ of *linear automorphisms* to consist of those automorphisms of X which extend to $\mathbb{P}$ (or, equivalently, to some graded automorphism of S). We retain the terminology "linear" by analogy with ordinary projective space, but it's important to note that the images of the variables x_i under a graded automorphism of S need not be linear as polynomials. For example, if $S = \mathbb{C}[x_0, x_1, x_2]$ and x_0, x_1, x_2 have weights $4, 3,$ and 1, respectively, we could define an element of $\mathrm{Aut}(S)$ by $x_0 \mapsto x_0 + x_1x_2 - x_2^4$, $x_1 \mapsto 2x_1 + x_2^3$, $x_2 \mapsto -x_2$.

We now show that all quasismooth hypersurfaces of a given degree have the same singularities, étale-locally (and hence up to local analytic isomorphism). The argument shows more generally that for any smooth proper family of complex Deligne-Mumford stacks, their coarse moduli spaces have locally constant singularities. It follows, for example, that if one quasismooth hypersurface of some degree d is canonical (resp. terminal), then every quasismooth hypersurface of degree d is canonical (resp. terminal).

Lemma 1.3.8. *Let Y be a weighted projective space over $\mathbb{C}$, and let d be a positive integer. Suppose that there is a quasismooth hypersurface of degree d in Y. Then all quasismooth hypersurfaces of degree d have the same singularities, étale-locally.*

Proof. Let $X \to B$ be the family of all quasismooth hypersurfaces of degree d in Y. Thus B is a Zariski open subset of the projective space of all hypersurfaces of degree d. Let $\pi \colon W \to B$ be the corresponding family of affine cones minus their origins. Then X is the quotient variety W/H, where $H := \mathbb{G}_m$ acts on W with finite stabilizer groups. By definition of quasismoothness, W is smooth over B.

Let w be a complex point of W, and let x be its image in X and b its image in B. Then the stabilizer group of w in H is the subgroup μ_n of nth roots of unity, for some positive integer n. By Alper-Hall-Rydh's relative version of Luna's étale slice theorem, using that $W \to B$ is smooth, there is a vector bundle V over an étale neighborhood U of b with an action of μ_n on V such that $W \to B$ is étale-locally isomorphic near w to $(H \times V)/\mu_n \to U$, compatibly with the action of H [AHR19, Theorem 20.4(4)]. Therefore, $X = W/H$ is étale-locally isomorphic over B to V/μ_n, a family of vector spaces divided by μ_n.

The point is that representations of μ_n are locally constant, up to isomorphism. So the singularity of the hypersurface X_b at x is étale-locally isomorphic to the singularity of some point on every nearby hypersurface X_c. Conversely, suppose we have a sequence of points x_i in X such that their images c_i in B approach b, and such that the points x_i all have isomorphic singularities in their fibers X_{c_i}. By properness of $X \to B$, we can assume after passing to a subsequence that the points x_i approach some point x in X. But then the local triviality above implies that the singularity of the points x_i in their fibers also occurs at some point in X_b (not necessarily at x). Thus the set of singularities that occur on X_b is locally constant as a function of b in B. Since B is connected, all the hypersurfaces X_b have the same singularities. □

The condition in Proposition 1.3.6 for quasismoothness is always satisfied when all weights of the weighted projective space Y divide the degree d. When this holds, the sheaf $\mathcal{O}_Y(d)$ is a line bundle and is basepoint-free. In this case, if Y is canonical (resp. terminal), then so is a general hypersurface X of degree d, by Kollár's Bertini theorem [Kol97, Proposition 7.7]. (This uses again that we are in characteristic zero.) More generally, still assuming that d is a multiple of all weights, X is canonical (resp. terminal) if Y has the same property outside the coordinate points, since X misses those points. However, we'll often have to deal with examples where not all weights divide d, so that $\mathcal{O}_Y(d)$ is not basepoint-free. In those cases, we use the following result to determine the type of singularities a quasismooth hypersurface has. It is a generalization of results of Iano-Fletcher for surfaces and threefolds [Ian00, Theorems 13.1 and 14.4]:

Proposition 1.3.9. *Let $X_d \subset \mathbb{P}(a_0, \ldots, a_N)$ be a quasismooth hypersurface of degree d. Let p be a point in the stratum U_I, $|I| = k+1$, and let $r = \gcd(a_i : i \in I)$.*

1. *If U_I is not in the base locus of $\mathcal{O}_X(d)$, then a neighborhood of p in X is analytically isomorphic to a quotient singularity of type $\frac{1}{r}(a_i : i \notin I) \times \mathbb{A}^{k-1}$, where $j \in I$.*

2. *If U_I is in the base locus of $\mathcal{O}_X(d)$, then d is not an $\mathbb{N}$-linear combination of a_i with $i \in I$, so there exists $j \notin I$ such that r divides $d - a_j$ by quasismoothness. In this case, a neighborhood of p in X is analytically isomorphic to a quotient singularity of type $\frac{1}{r}(a_i : i \notin I, i \neq j) \times \mathbb{A}^k$.*

The simplest application of the second part of the proposition is to the case where we only have one weight a_0 that doesn't divide d. Then the only I to which (2) applies is $I = \{0\}$. In this situation, the coordinate point of a_0 is a basepoint of $\mathcal{O}_X(d)$, and so a quasismooth X of this degree in Y has a singularity of type

$$\frac{1}{a_0}(a_1, \ldots, \widehat{a_j}, \ldots, a_N)$$

at that point.

Proof. By Lemma 1.3.8, we can assume that the hypersurface X is general in its degree. We'll focus on calculating the singularities of a general X on U_I; this determines the singularities of every quasismooth hypersurface by the lemma. Note that the quotient singularity type in both (1) and (2) does not depend on which index we choose satisfying the given condition.

By Proposition 1.3.3, the weighted projective space Y has a singularity of type $\frac{1}{r}(a_i : i \notin I) \times \mathbb{A}^k$ at p. The idea is that the hypersurface X is locally an affine slice of this singularity, and hence has the same type except possibly with one weight removed. In case (1), the general X is transverse to the stratum U_I, so locally we can take X to be $\{x_j = 0\}$ for x_j some coordinate of $\mathbb{A}^k$, meaning that $j \in I$. This proves part (1).

Next, we need to check singularities on the base locus. The base locus of $\mathcal{O}_X(d)$ is then the union of the U_I corresponding to those I with d not a linear combination of weights in I. Indeed, for a point $p \in U_I$, any homogeneous f of degree d vanishes at p because there is no monomial of degree d containing only variables x_i, $i \in I$. Conversely, if $p \notin U_I$ for any I of this sort, there is some monomial of degree d not vanishing at p, and it is not in the base locus.

Since X is quasismooth, there is a set J as in 1.3.6(2). For each $j \in J$, there is a monomial $g(x_I)x_j$ of degree d, where $g(x_I)$ is a monomial in $\{x_i : i \in I\}$. This implies that for f general of degree d, $\frac{\partial f}{\partial x_j}$ is a polynomial that doesn't vanish identically on U_I. If it doesn't vanish at p, the inverse function theorem implies that we can take the remaining variables as coordinates of X, so the local picture is the same as the quotient singularity above with a_j removed, just as Proposition 1.3.3 states.

However, $\frac{\partial f}{\partial x_j}$ may vanish on a codimension 1 subset of U_I when f is general. This is why we need $|I|$ different indices $j \in J$ to choose from. The key fact is the following:

Lemma 1.3.10. *Let $\mathbb{P}$ be any weighted projective space and consider a linear system L on $\mathbb{P}$ consisting of all linear combinations of a set of monomials. Then the base locus of L (if nonempty) is a union of coordinate linear subspaces of $\mathbb{P}$.*

Proof. The linear system is invariant under the torus $T \cong (\mathbb{G}_m)^N$ acting on $\mathbb{P}$, and so its zero set in $\mathbb{P}$ is T-invariant as well as closed. □

To apply this, consider the set S of the monomials $g(x_I)$ of degree $d - a_j$ that appear in $\frac{\partial f}{\partial x_j}$. Restricting to V_I, the partial derivative becomes an arbitrary linear combination of monomials in S. As f varies over all polynomials of degree d, the corresponding linear system of restrictions of $\frac{\partial f}{\partial x_j}$ has no base points on U_I (if they exist anywhere on V_I, the closure $\overline{U_I}$ of U_I, they are in smaller strata).

Now, take a general f and consider all $\frac{\partial f}{\partial x_j}$ for $j \in J$ restricted to V_I. The dimension of V_I is $|I| - 1$, while $|J| = |I|$, so either the $\frac{\partial f}{\partial x_j}$ don't have a common vanishing point on V_I, or their common vanishing set lies on $V_I \setminus U_I$ (i.e. on the coordinate planes of V_I). This completes the proof, because f will have some partial derivative in J nonvanishing at any $p \in U_I$. □

We next prove two useful restrictions on the singularities of weighted projective hypersurfaces. The first result (Corollary 1.3.12), due to Artebani-Comparin-Guilbot [ACG16, Proposition 2.12], says that we get canonical singularities very easily in the Calabi-Yau case. We start with the well-known (cf. [Rei80, Remark 1.8]):

Lemma 1.3.11. *Let X be a klt variety with the property that K_X is a Cartier divisor. Then X has canonical singularities.*

Proof. Let $f : W \to X$ be a resolution of singularities. Then we have

$$K_W = f^* K_X + \sum_j b_j E_j,$$

where the E_j are the exceptional divisors of f. Since X is klt, each coefficient b_j is greater than -1. Because K_X is Cartier, each b_j is an integer. Therefore, $b_j \geq 0$ for all j and X has canonical singularities. □

In the context of weighted projective spaces, this leads to the following corollaries.

Corollary 1.3.12. *Let $X \subset \mathbb{P}(a_0, \ldots, a_N)$ be a quasismooth hypersurface of degree $\sum_j a_j$, so that $K_X = \mathcal{O}_X$. Then X has canonical singularities.*

Proof. Since X is quasismooth, it has only quotient singularities and is therefore klt. Moreover, $K_X = \mathcal{O}_X(d - \sum_i a_i) = \mathcal{O}_X$ is Cartier, so Lemma 1.3.11 applies. □

Corollary 1.3.13. *Let $Y = \mathbb{P}(a_0, \ldots, a_N)$ be a well-formed weighted projective space with the property that each weight a_j divides the sum $\sum_j a_j$. Then Y has canonical singularities.*

Proof. Y has only quotient singularities, so it is klt. Furthermore, $K_Y = \mathcal{O}_Y\left(-\sum_j a_j\right)$ is a line bundle because the sum is a multiple of each weight. □

Another related trick may be used on certain subsets of weights.

Lemma 1.3.14. *Let $\frac{1}{r}(b_1, \ldots, b_s)$ be a well-formed quotient singularity with the property that some nonempty subset $I \subset \{b_1, ..., b_s\}$ has sum congruent to $0 \bmod r$ and $\gcd(I \cup \{r\}) = 1$. Then the singularity is canonical.*

Proof. Since the singularity is well-formed, we may apply the Reid-Tai criterion. Let $1 \le i < r$ be an integer and consider

$$\sum_{j=1}^{s} ib_j \bmod r = \sum_{j\in I} ib_j \bmod r + \sum_{j\notin I} ib_j \bmod r.$$

The first sum on the right-hand side must be a multiple of r because the sum of weights in I is a multiple of r. Moreover, it cannot be zero because then each ib_j would be a multiple of r. This would imply that all b_j, $j \in I$ share a common factor with r, a contradiction. Therefore, the right-hand side is at least r and the singularity is canonical. □

The second point is that we need not distinguish between canonical and terminal under some circumstances. For each point x on a complex variety X with K_X $\mathbb{Q}$-Cartier, some neighborhood N of x has an "index-1 cover" $Y \to N$ (unique up to isomorphism étale-locally near x), which makes the canonical divisor Cartier [Reib, Section 3.6]. We use the following result of Reid's [Reia, Proposition 3.1(II)]:

Theorem 1.3.15. *A canonical singularity whose index-1 cover is terminal must be terminal.*

The case we need is that a canonical singularity whose index-1 cover is smooth must be terminal. This case can also be checked directly from the Reid-Tai criterion for quotient singularities.

Corollary 1.3.16. *Let X be a well-formed, quasismooth subvariety of a weighted projective space such that $K_X = \mathcal{O}_X(1)$ or $K_X = \mathcal{O}_X(-1)$. If X is canonical, then it is terminal.*

Proof. Let X be a well-formed, quasismooth subvariety of a weighted projective space Y. Let U be the affine cone over X minus the origin; then U is smooth, and X is the quotient of U by $\mathbb{G}_m$. The action of $\mathbb{G}_m$ on U is proper, and it is free in codimension 1 by the well-formedness of X. Let x be a complex point of X, and u a lift of this point to U. The stabilizer subgroup of u is $\mu_r \subset \mathbb{G}_m$ for some positive integer r. By Luna's étale slice theorem, an étale neighborhood N of x in X is the quotient of a smooth variety M by μ_r, and the action is free in codimension 1. So the local class group of X at x is cyclic of order

r, generated by $\mathcal{O}_X(1)$, using that the corresponding line bundle on U is the trivial bundle with $\mathbb{G}_m$ acting by scalars.

Suppose now that K_X is $\mathcal{O}_X(1)$ or $\mathcal{O}_X(-1)$. Then K_X generates the local class group of X at x, and so the index-1 cover of X at x is precisely $M \to N$. Here M is smooth. By Theorem 1.3.15, if X is canonical, then it is terminal. □

CHAPTER 2

Automorphisms of Weighted Projective Hypersurfaces

2.1 Introduction

Hypersurfaces in projective space are among the most well-studied varieties. In particular, a great deal is known about their automorphism groups, due to landmark theorems of Grothendieck-Lefschetz [Gro], Matsumura-Monsky [MM64], and others. In this chapter, we generalize and strengthen several of these results to hypersurfaces in any weighted projective space over $\mathbb{C}$. We'll use the notation $X_d \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ for a weighted projective hypersurface X of degree d and dimension n with the indicated weights.

The outline of the chapter is as follows: in Section 2.2, we prove that all automorphisms of well-formed quasismooth weighted projective hypersurfaces $X \subset \mathbb{P}$ extend to the ambient weighted projective space if $\dim(X) \geq 3$ or $\dim(X) = 2$ and X has non-trivial canonical class (Theorem 2.2.1). This generalizes the same statement for hypersurfaces in $\mathbb{P}^{n+1}$, which is a consequence of the Grothendieck-Lefschetz theorem. Some partial results in this direction have appeared in works of Przyjalkowski and Shramov [PS19, PS21]. Recall that an automorphism of X that extends to $\mathbb{P}$ is called *linear*, and that we use the notation $\mathrm{Lin}(X) \subset \mathrm{Aut}(X)$ to denote the subgroup of linear automorphisms.

We prove several results on the size of the linear automorphism group $\mathrm{Lin}(X)$ of a well-formed quasismooth X in Section 2.3. In particular, we give an exact characterization of when this group is finite in terms of the degree and weights of X (Theorem 2.3.1). When it is finite, we prove that there is an upper bound

$$|\mathrm{Lin}(X)| \leq C_n \frac{d^{n+1}}{a_0 \cdots a_{n+1}},$$

where C_n depends only on the dimension n (Theorem 2.3.3). We give a procedure for effectively calculating a value C_n for which the inequality above holds using the Jordan constants of general linear groups over $\mathbb{C}$. We also compute the optimal value for C_1.

Finally, in Section 2.4, we consider automorphisms of a very general quasismooth hypersurface X with a given degree and weights. We prove that when $d \geq 5\max\{a_0, \ldots, a_{n+1}\}$, the group $\mathrm{Lin}(X)$ is contained in the center of $\mathrm{Aut}(\mathbb{P})$ (Theorem 2.4.1). When $\mathbb{P} = \mathbb{P}^{n+1}$, the center of $\mathrm{Aut}(\mathbb{P}^{n+1}) = \mathrm{PGL}_{n+2}$ is trivial, so we recover the usual statement that $\mathrm{Lin}(X) = 1$ for X very general. However, this stronger statement is not always true for other choices of weights and degree. We give several examples illustrating new phenomena that arise for generic automorphisms of weighted projective hypersurfaces that don't occur in ordinary projective space.

2.2 Linearity of Hypersurface Automorphisms

Let X be a smooth hypersurface in $\mathbb{P}^{n+1}$. Then, automorphisms of X extend to $\mathbb{P}^{n+1}$ whenever $n \geq 1$ and $d \geq 3$ unless $(n, d) = (1, 3)$ or $(2, 4)$. When $n \geq 3$, this is a consequence of the Grothendieck-Lefschetz theorem [Gro, Exposé XII, Corollaire 3.6], which holds even for arbitrarily singular hypersurfaces in projective space. For $n = 2$, $d \neq 4$, this is a theorem due to Matsumura and Monsky [MM64]; when $n = 1, d \geq 4$, it is due to Chang [Cha78]. In this section, we prove a generalization to hypersurfaces in weighted projective space. We will deduce this statement from the local version of Grothendieck's Lefschetz theorems, but some care is needed. One complication is that a hypersurface in weighted projective space need not be a Cartier divisor (this is usually assumed even for variants of the global Lefschetz theorems that deal with singular varieties, e.g. [RS06]).

Theorem 2.2.1. *Let $X \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ and $X' \subset \mathbb{P}(a'_0, \ldots, a'_{n+1})$ be two complex weighted projective hypersurfaces of weighted degrees d and d', respectively. Suppose further that X and X' are well-formed and quasismooth, neither is a linear cone, and one of the following holds:*

1. *$n \geq 3$; or*

2. $n = 2$ *and* $a_0 + a_1 + a_2 + a_3 \neq d$.

Then, if $g : X' \to X$ *is an isomorphism, we have* $d = d'$*, the* a_i *coincide with the* a_i' *up to rearrangement, and* g *is induced by an automorphism of* $\mathbb{P}(a_0, \ldots, a_{n+1})$.

Remark 2.2.2. 1. The assumption of well-formedness is necessary (note that a quasismooth hypersurface in a well-formed weighted projective space of dimension at least 3 is automatically itself well-formed [Ian00, Theorem 6.17]). For example, any hypersurface $X_4 \subset \mathbb{P}^4(2,2,2,2,2)$ is isomorphic as a variety to a hypersurface $X_2 \subset \mathbb{P}^4(1,1,1,1,1)$ with the same equation.

2. A hypersurface is a linear cone if the equation f contains a linear term x_i for some i. In this case, the hypersurface $f = 0$ is isomorphic to a weighted projective space of smaller dimension, and the theorem above fails rather trivially: for example, $\{x_0 = 0\} \subset \mathbb{P}^4(1,1,1,1,1)$ and $\{x_0 = 0\} \subset \mathbb{P}^4(2,1,1,1,1)$ are both isomorphic to $\mathbb{P}^3$. We avoid the linear cone case to exclude this pathology.

3. The part of the theorem which states that the degree and weights coincide for any two embeddings of the same weighted projective hypersurface was shown in the greater generality of quasismooth weighted complete intersections of dimension at least 3 by Przyjalkowski and Shramov [PS19, Proposition 1.5]. They also proved in another paper that the action of any *linearly reductive* algebraic group Γ on a quasismooth weighted complete intersection of dimension at least 3 or of dimension 2 with nontrivial canonical class comes from an action of Γ on the ambient weighted projective space [PS21, Theorem 1.3]. We remove the reductivity assumption, answering [PS21, Question 1.5] for hypersurfaces. Some results on the class groups of weighted complete intersections of dimension at least 3 that we'll use below date back further to [Mor75, BR86].

Proof. Let $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$ be the graded polynomial ring in variables x_i of degrees a_i, and define S' in the same way, so that $\mathbb{P} := \mathbb{P}(a_0, \ldots, a_{n+1}) = \operatorname{Proj}(S)$ and $\mathbb{P}' := \mathbb{P}(a_0', \ldots, a_{n+1}') = \operatorname{Proj}(S')$. If f and f' are the homogeneous polynomials of weighted degrees d, d' defining X and X', respectively, then $X = \mathrm{Proj}(S/(f))$ and $X' = \mathrm{Proj}(S'/(f'))$. We'll first show that the isomorphism $g : X' \to X$ comes from an isomorphism of graded rings $g^* : S/(f) \to S'/(f')$. This is non-trivial since not every isomorphism of two Proj's comes from an underlying isomorphism of the graded rings. We use the notation $\mathcal{O}_{\mathbb{P}}(m)$ and $\mathcal{O}_X(m)$ for the sheaves coming from each respective Proj construction and $i : X \to \mathbb{P}$ for the closed immersion coming from the quotient of graded rings $S \to S/(f)$.

Lemma 2.2.3. *In the setting of Theorem 2.2.1, for each $m \geq 0$, the natural map $i^*\mathcal{O}_{\mathbb{P}}(m) \to \mathcal{O}_X(m)$ of sheaves is an isomorphism. Furthermore, the sheaf $\mathcal{O}_X(m)$ is reflexive on X.*

Proof. We may check these conditions on each affine open chart $D^+_X(x_j) \subset X, j = 0, \ldots, n+1$ where x_j doesn't vanish. (Here we note that each x_j is still nonzero in the quotient $S/(f)$ because X is not a linear cone.) These cover $X = \mathrm{Proj}(S/(f))$ and the corresponding collection of opens $D^+_{\mathbb{P}}(x_j)$ also covers $\mathbb{P}$. Use the notation $S_{(\alpha)}$ for $\alpha \in S$ a homogeneous element of positive degree to mean the degree zero part of the localization S_α; similarly, when M is a graded S-module, $M_{(\alpha)}$ means the degree zero part of the localization. We then have $D^+_{\mathbb{P}}(x_j) = \mathrm{Spec}(S_{(x_j)})$, $D^+_X(x_j) = \mathrm{Spec}((S/(f))_{(x_j)})$, and that $\mathcal{O}_{\mathbb{P}}(m)|_{D^+_{\mathbb{P}}(x_j)}$ is the quasicoherent sheaf associated to the $S_{(x_j)}$-module $(S(m))_{(x_j)}$. The claimed isomorphism amounts to checking

$$(S(m))_{(x_j)} \otimes_{S_{(x_j)}} (S/f)_{(x_j)} \cong ((S/f)(m))_{(x_j)}.$$

This follows from the commutativity of localizations and tensor product, as well as keeping track of the degrees on each side. To show that $\mathcal{O}_X(m)$ is reflexive, we use the following criterion [Har80, Proposition 1.6]: a coherent sheaf $\mathcal{F}$ on a normal, integral scheme X is reflexive iff 1) it is torsion-free, and 2) for every open set $U \subset X$ and closed $Y \subset U$ of codimension at least 2, the restriction $\mathcal{F}(U) \to \mathcal{F}(U \setminus Y)$ is a bijection.

It's clear that $\mathcal{O}_X(m)$ is torsion-free because $S/(f)$ is (otherwise, the affine cone would be non-reduced, contradicting quasismoothness). We can check the second property on the affine charts $D^+_X(x_j)$. On this chart, X is a quotient of the variety $X' \subset \mathbb{A}^{n+1}_{x_0,\ldots,\hat{x}_j,\ldots,x_{n+1}}$

by μ_{a_j}, where X' is cut out by the equation $f(x_0, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_{n+1}) = 0$. By well-formedness of X, this action is free away from a codimension 2 subset $Z \subset X'$. Therefore, given $s \in \mathcal{F}(U \setminus Y)$, the restriction of s to $U \setminus (Y \cup Z)$ lifts to a regular function on the preimage of this set in X'. By quasismoothness, X' is smooth, and in particular normal, so this function extends to all of the preimage of U and remains homogeneous of the same weight for the μ_{a_j}-action. Hence s extends to U, completing the proof. □

We require the reflexivity of $\mathcal{O}_X(m)$ in order to view it as a member of the class group $\mathrm{Cl}(X)$ below.

For each integer m, we claim that the following sequence of sheaves on $\mathbb{P}$ is exact:

$$0 \to \mathcal{O}_{\mathbb{P}}(m-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}}(m) \to \mathcal{O}_X(m) \to 0. \tag{2.1}$$

Here the second map is multiplication by f. Indeed, the same sequence with $\mathcal{O}_X(m)$ replaced by $\mathcal{O}_{\mathbb{P}}(m)|_X$ is manifestly exact because it corresponds to the exact sequence of modules $0 \to S(m-d)) \xrightarrow{\cdot f} S(m) \to (S/(f))(m) \to 0$. By Lemma 2.2.3, $\mathcal{O}_{\mathbb{P}}(m)|_X \cong \mathcal{O}_X(m)$, so (2.1) is also exact.

By [Dol82, Section 1.4], $H^1(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m-d)) = 0$ for all m because the dimension of the weighted projective space $\mathbb{P}$ is at least 2. This is the analog of projective normality for weighted projective hypersurfaces. Therefore, we arrive at the exact sequence of global sections

$$0 \to H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m-d)) \xrightarrow{\cdot f} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)) \to H^0(X, \mathcal{O}_X(m)) \to 0.$$

Since $S_m \cong \mathcal{O}_{\mathbb{P}}(m)$ and the image of the first map is fS_{m-d}, we may identify $H^0(X, \mathcal{O}_X(m))$ with the mth graded piece of $S/(f)$. Therefore,

$$S/(f) = \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(m)),$$

and similarly with $S'/(f')$. Thus, it will be enough to find maps $H^0(X, \mathcal{O}_X(m)) \to H^0(X', \mathcal{O}_{X'}(m))$ for each m to build our desired map $S/(f) \to S'/(f')$.

Since X is quasismooth, the punctured affine cone $C_X^* \subset \mathbb{A}^{n+2} \setminus \{0\}$ is smooth. The hypersurface X is a quotient of C_X^* by the action of $G \coloneqq \mathbb{G}_{\mathrm{m}}$ on $\mathbb{A}^{n+2} \setminus \{0\}$; we'll denote the quotient morphism by $q : C_X^* \to X$. Using the assumption of well-formedness of X, $\mathrm{Sing}(X)$ is codimension at least 2 in X. $\mathrm{Sing}(X)$ is also the image of the locus in C_X^* where the G-action has nontrivial stabilizers.

Thus, we have $\mathrm{Cl}(X) \cong \mathrm{Cl}(X_{\mathrm{sm}})$, where $X_{\mathrm{sm}} = X \setminus \mathrm{Sing}(X)$ is the smooth locus. Because X_{sm} is smooth, $\mathrm{Cl}(X_{\mathrm{sm}}) = \mathrm{Pic}(X_{\mathrm{sm}})$. Further, $\mathrm{Pic}(X_{\mathrm{sm}})$ is isomorphic to the group $\mathrm{Pic}^G(q^{-1}(X_{\mathrm{sm}}))$ of G-equivariant vector bundles on $q^{-1}(X_{\mathrm{sm}})$, since $q^{-1}(X_{\mathrm{sm}}) \to X_{\mathrm{sm}}$ is the quotient by a free group action. Finally, $q^{-1}(X_{\mathrm{sm}})$ is codimension at least 2 in C_X^*, so G-equivariant line bundles on X_{sm} extend to all of C_X^*. In summary,

$$\mathrm{Cl}(X) \cong \mathrm{Pic}(X_{\mathrm{sm}}) \cong \mathrm{Pic}^G(q^{-1}(X_{\mathrm{sm}})) \cong \mathrm{Pic}^G(C_X^*).$$

The isomorphism $g : X' \to X$ induces an isomorphism $X'_{\mathrm{sm}} \to X_{\mathrm{sm}}$, which also gives a pullback map $\mathrm{Pic}(X_{\mathrm{sm}}) \xrightarrow{\cong} \mathrm{Pic}(X'_{\mathrm{sm}})$. We'll identify $\mathrm{Cl}(X)$ and $\mathrm{Pic}^G(C_X^*)$ below without further comment and use the same notation $\mathcal{O}_X(m)$ for the corresponding elements in either of these groups.

Proposition 2.2.4. *Suppose that X and X' satisfy the conditions of Theorem 2.2.1. Then an isomorphism $g : X' \to X$ induces an isomorphism $\mathrm{Cl}(X) \xrightarrow{\cong} \mathrm{Cl}(X')$ which maps $\mathcal{O}_X(1)$ to $\mathcal{O}_{X'}(1)$.*

Proof. We argue differently depending on the dimension of X. When $\dim(X) \geq 3$, we claim that $\mathrm{Pic}^G(C_X^*) \cong \mathbb{Z} \cdot \mathcal{O}_X(1)$ (and analogously for X').

Indeed, if we forget the G-equivariant structure, all line bundles on the smooth variety C_X^* are trivial when $\dim(X) \geq 3$. This is because the local ring $\mathcal{O}_{C_X,0}$ is a complete intersection ring of dimension at least 4, regular outside its maximal ideal, so it is a parafactorial ring [Gro, Exposé XI, Théorème 3.13]. It follows that $\mathrm{Pic}(C_X) \to \mathrm{Pic}(C_X^*)$ is an isomorphism; furthermore, $\mathrm{Pic}(C_X)$ is trivial [Dol82, Section 3.2.2]. From $\mathrm{Pic}(C_X^*) = 0$, we deduce that the group of G-equivariant line bundles on C_X^* is naturally isomorphic to the character group of G, namely $\mathbb{Z}$ [Bri18, Lemma 4.1.7]. It's straightforward to check that the linearization of the trivial bundle by the character $t \mapsto t^m$ coincides with the G-equivariant bundle $\mathcal{O}_X(m)$. Therefore, $\mathrm{Cl}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$. Since X and X' are isomorphic, $\dim(X') \geq 3$ also, so identical reasoning shows $\mathrm{Cl}(X') = \mathbb{Z} \cdot \mathcal{O}_{X'}(1)$.

The pullback of an ample divisor by an isomorphism is still ample, so the isomorphism $\mathrm{Cl}(X) \xrightarrow{\cong} \mathrm{Cl}(X')$ must send $\mathcal{O}_X(1)$ to $\mathcal{O}_{X'}(1)$. This proves the proposition when $n \geq 3$.

If $\dim(X) = 2$, we use a different argument, since we may not have $\mathrm{Cl}(X) \cong \mathbb{Z}$ in this case. The canonical class of X is $K_X = \mathcal{O}_X(r)$, where $r = d - \sum_i a_i \neq 0$ by assumption. We also have $K_{X'} = \mathcal{O}_X(r')$ with $r = d' - \sum_i a_i'$ having the same sign as r, depending on whether X and X' have ample or anti-ample canonical class. The canonical class is preserved by isomorphism, so pullback sends $\mathcal{O}_X(r)$ to $\mathcal{O}_{X'}(r')$. We'll prove that $r = r'$ and moreover that $\mathcal{O}_X(1)$ maps to $\mathcal{O}_{X'}(1)$.

Lemma 2.2.5. *Let V be a connected scheme of finite type over $\mathbb{C}$. If $\pi_1(V) = 1$, then $\mathrm{Pic}(V)$ is torsion-free.*

Proof. For any positive integer ℓ, we have the following Kummer exact sequence of sheaves of abelian groups on V in the étale topology:

$$1 \to \mu_\ell \to \mathbb{G}_{\mathrm{m},V} \xrightarrow{x \mapsto x^\ell} \mathbb{G}_{\mathrm{m},V} \to 1.$$

The associated long exact sequence in cohomology gives

$$\cdots \to H^1(V, \mu_\ell) \to \mathrm{Pic}(V) \xrightarrow{L \mapsto L^\ell} \mathrm{Pic}(V) \to \cdots.$$

Since V is connected, $H_1(V, \mathbb{Z})$ is the abelianization of $\pi_1(V)$; hence $H_1(V, \mathbb{Z}) = 0$. The universal coefficient theorem then gives that $H^1(V, \mu_\ell) \cong H^1(V, \mathbb{Z}/\ell) = 0$, so that the ℓth tensor power map on $\mathrm{Pic}(V)$ is injective. Since this holds for any positive integer ℓ, $\mathrm{Pic}(V)$ is torsion-free. □

Now we'll conclude the proof of Proposition 2.2.4. By [Dol82, Section 3.2.2], the fundamental group of the punctured affine cone C_X^* vanishes whenever X is a quasismooth hypersurface of dimension at least 2. Therefore, the (non-equivariant) Picard group $\mathrm{Pic}(C_X^*)$ is torsion-free. Next, we use the exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Pic}^G(C_X^*) \to \mathrm{Pic}(C_X^*),$$

where the first map sends $1 \mapsto \mathcal{O}_X(1)$ and the second map forgets the linearization. Exactness of this sequence follows from [Bri18, Theorem 4.2.2] plus the observation that $\mathbb{Z} \to \mathrm{Pic}^G(C_X^*)$ is injective by ampleness of $\mathcal{O}_X(1)$ on X. It follows that $\mathrm{Pic}^G(C_X^*)$ is torsion-free, since any torsion element must map to zero in $\mathrm{Pic}(C_X^*)$, and hence be in the image of $\mathbb{Z} \to \mathrm{Pic}^G(C_X^*)$. In addition, we see that $\mathcal{O}_X(k) \in \mathrm{Pic}^G(C_X^*)$ is not divisible by any integer other than the factors of k, because otherwise the cokernel of the first map would contain torsion elements.

This implies $r = r'$ above. Indeed, if $g^*(\mathcal{O}_X(r)) \cong \mathcal{O}_{X'}(r')$, then the image $L \in \mathrm{Pic}^G(C_{X'}^*)$ of $\mathcal{O}_X(1)$ satisfies $L^r = \mathcal{O}_{X'}(r')$. Since $\mathcal{O}_{X'}(r')$ is divisible by r, r is a factor of r' by the above. Arguing symmetrically with the inverse isomorphism gives $r = r'$. In particular, we then have that the image of $\mathcal{O}_X(1)$ differs from $\mathcal{O}_{X'}(1)$ by a torsion element of order r. Because we also saw that $\mathrm{Pic}^G(C_X^*)$ is torsion free, this proves that $g^*(\mathcal{O}_X(1)) \cong \mathcal{O}_{X'}(1)$. □

Finally, we're ready to complete the proof of Theorem 2.2.1. We've now seen that in the assumptions of the theorem, pullback by the isomorphism g sends $\mathcal{O}_X(m)$ to $\mathcal{O}_{X'}(m)$ for any $m \in \mathbb{Z}$. Therefore, g induces isomorphisms

$$g^* : H^0(X, \mathcal{O}_X(m)) \to H^0(X', \mathcal{O}_{X'}(m)).$$

These maps respect the tensor product of sections, so we may assemble them into an isomorphism of graded rings $g^* : S/(f) \to S'/(f')$. In particular, g^* gives an isomorphism of the maximal irrelevant ideal $\mathfrak{m}$ of elements of positive degree in $S/(f)$ with $\mathfrak{m}' \subset S'/(f')$. The number of generators of $\mathfrak{m}$ and their degrees (up to reordering) coincide with those of $\mathfrak{m}'$. This is because if m is the smallest positive index with the graded piece $\mathfrak{m}_m$ nonempty, the number of generators of degree m equals $\dim(\mathfrak{m}_m)$, and then we can factor out by these generators and repeat inductively.

Since X is quasismooth and not a linear cone, we have that d, the degree of the smallest relation among the x_i, is strictly greater than all weights a_i. It follows that $x_0, \ldots, x_{n+1}$ are a minimal system of $n+1$ generators for the homogeneous ideal $\mathfrak{m}$. Similarly, $x'_0, \ldots, x'_{n+1}$ generate $\mathfrak{m}'$, which is an isomorphic to $\mathfrak{m}$, so this set of generators must also be minimal and the collection $a_0, \ldots, a_{n+1}$ must be the same as $a'_0, \ldots, a'_{n+1}$, up to reordering. Dimension counting yields that the relations are then also in the same degree, so we have $d = d'$.

Lastly, we observe that the isomorphism $g^*\colon S/(f) \to S'/(f')$ is induced by an isomorphism $S \to S'$. Indeed, for $m < d$, the mth graded piece of S is isomorphic to that of $S/(f)$, so g^* gives isomorphisms $S_m \to S'_m$. All the generators of S are contained in S_m for $m < d$ so this gives rise to a homomorphism $S \to S'$. The inverse of g similarly gives a morphism $S' \to S$ whose composition with the previous map is the identity on generators, and hence on all of S. Therefore, the homomorphism $S \to S'$ is an isomorphism. By our work above, both $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S')$ are isomorphic to $\mathbb{P}(a_0, \ldots, a_{n+1})$, so the isomorphism $S \to S'$ gives an automorphism of this weighted projective space inducing the original isomorphism $g\colon X' \to X$ of hypersurfaces, as claimed. □

This theorem fails if we weaken the hypotheses on dimension. As mentioned above, two smooth plane curves C and C' in $\mathbb{P}^2$ of degree at least 4 are isomorphic if and only if they differ by an isomorphism of the projective plane [Cha78]. However, the situation for weighted projective curves is considerably more complicated: there exist many curves of genus at least 2 which can be embedded as well-formed quasismooth hypersurfaces in different weighted projective spaces.

Example 2.2.6 (Hyperelliptic Curves)**.** Let C be a hyperelliptic curve of genus g, $p : C \to \mathbb{P}^1$ a $2:1$ cover of $\mathbb{P}^1$, and $P \in C$ a ramification point of the cover. Then C is isomorphic to $\mathrm{Proj}(R(C, P))$, where

$$R(C, P) := \bigoplus_{k=0}^{\infty} H^0(C, kP).$$

This ring has generators x, y, and z in degrees 1, 2, and $2g+1$, respectively, and a single relation in degree $4g+2$. It is possible to choose the generator z so that the relation has the form $f(x, y, z) \coloneqq z^2 - h(x^2, y) = 0$, where h is a polynomial of degree $2g+1$. This embeds C as a quasismooth hypersurface of degree $4g+2$ in $\mathbb{P}^2(2g+1, 2, 1)$. However, if we use $R(2C, P) = R(C, K_C)$ instead, we have another embedding of the same curve as a degree $2g+2$ hypersurface in $\mathbb{P}^2(g+1, 1, 1)$.

Example 2.2.7. There are also non-hyperelliptic curves exhibiting similar behavior. For C a smooth non-hyperelliptic curve of genus 3, we have the canonical embedding of C in $\mathbb{P}^2 = \mathbb{P}^2(1, 1, 1)$ as a degree 4 plane curve. Suppose further that C has the property that there is a line $\ell \subset \mathbb{P}^2$ tangent to C at a point P with multiplicity 4. Then we have $4P \sim K_C$. One can show that ring $R(C, P)$ has generators in degrees 1, 3, and 4, giving an embedding $C \subset \mathbb{P}^2(1, 3, 4)$ as a hypersurface of degree 12. There are many similar examples for curves of higher genus.

When $\dim X = 2$ but $\sum_i a_i = d$, Theorem 2.2.1 also fails. This is because the resulting hypersurfaces are K3 surfaces in this case, which frequently have infinite automorphism group. An example of Fano and Severi of a quartic surface in $\mathbb{P}^3$ with infinite automorphism group is described in the proof of Theorem 4 in [MM64], for instance. However, we'll see below in Theorem 2.3.1 that the linear automorphism groups of weighted projective surfaces with $\sum_i a_i = d$ are always finite. Hence, some automorphisms of these surfaces aren't linear.

2.3 Bounds on Linear Automorphism Groups

Recall that the linear automorphism group $\mathrm{Lin}(X) \subset \mathrm{Aut}(X)$ of a hypersurface $X \subset \mathbb{P}$ is the subgroup of automorphisms that extend to $\mathbb{P}$. As long as the degree of X exceeds all weights of $\mathbb{P}$, the only automorphism of $\mathbb{P}$ that fixes X pointwise is the identity (use the same argument on extending morphisms of graded rings as above), so in this case we may consider $\mathrm{Lin}(X)$ as a subgroup of both $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\mathbb{P})$. Theorem 2.2.1 shows that $\mathrm{Lin}(X) = \mathrm{Aut}(X)$ whenever $\dim(X) \geq 3$ or $\dim(X) = 2$ and $K_X \ncong \mathcal{O}_X$.

In this section, we prove several results on the size of $\mathrm{Lin}(X)$. These will imply the same results for $\mathrm{Aut}(X)$ in most dimensions in light of the last section. First, we give a criterion in terms of the degree d and the weights $a_0, \ldots, a_{n+1}$ which determines whether or not $\mathrm{Lin}(X)$ is finite.

Theorem 2.3.1. *Let $X \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ be a well-formed, quasismooth weighted projective hypersurface of degree d, where $n \geq 1$. Then $\mathrm{Lin}(X)$ is finite if and only if one of the following two conditions holds:*

1. *$d > 2\max\{a_0, \ldots, a_{n+1}\}$; or*
2. *$d = 2\max\{a_0, \ldots, a_{n+1}\}$ and only a single weight achieves the maximum.*

Further, if neither (1) nor (2) holds (so that $\mathrm{Lin}(X)$ is infinite), then X is rational.

Remark 2.3.2. This generalizes a theorem of Matsumura and Monsky [MM64, Theorem 1], which shows that the linear automorphism group of a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ over an algebraically closed field k is finite if $n \geq 2$ and $d \geq 3$. For $k = \mathbb{C}$, this result was known in some form at least as far back as 1880, when it appeared in a work of Jordan [Jor80] (see also [OS78, Section 6] for further historical remarks on this theorem). Note also that if $d < 3$, X is a hyperplane or a quadric. These always have infinite linear automorphism groups and are rational for $n \geq 1$.

In the special case when X is a *smooth* (rather than just quasismooth) well-formed weighted projective hypersurface, $\dim(X) \geq 3$ and $K_X \not\cong 0$, it was proven in [PS19, Corollary 1.4] that $\mathrm{Aut}(X)$ is finite unless X is isomorphic to either $\mathbb{P}^n$ or a quadric hypersurface of $\mathbb{P}^{n+1}$. Note that X is never smooth unless the weights are pairwise relatively prime and all divide the degree. However, the result of [PS19] also applies to smooth weighted complete intersections with these properties; their methods are rather different than ours.

Proof. First, assume that one of the two conditions on d in Theorem 2.3.1 holds. We'll show that $\mathrm{Lin}(X)$ is finite, using generally the same approach as in [MM64]. By Lemma 1.3.4, any automorphism of $\mathbb{P}$ comes from a graded automorphism of $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$, so we perform most of our analysis on the level of graded ring automorphisms. For any graded homomorphism $S \to S$, the image of each generator x_i is contained in the finite-dimensional vector space S_{a_i}. Thus, the endomorphism monoid of S is isomorphic to $\mathbb{A}^N$ as a variety, where $N := \sum_i \dim(S_{a_i})$. The linear algebraic group $\mathrm{Aut}(S)$ is an open subvariety of $\mathbb{A}^N$. This is a generalization of the fact that $\mathrm{GL}_{n+2}(\mathbb{C})$ is an open subvariety of $\mathbb{A}^{(n+2)^2}$.

We saw in Lemma 1.3.4 that $\mathrm{Aut}(S)$ acts on $\mathbb{P}(a_0, \dots, a_{n+1})$ with kernel isomorphic to $\mathbb{C}^*$, where $t \in \mathbb{C}^*$ acts as $t \cdot x_i = t^{a_i} x_i$ for each i. Let $G \subset \mathrm{Aut}(S)$ be the subgroup of elements mapping the polynomial f defining X to a multiple of itself. Then $\mathrm{Lin}(X) = G/\mathbb{C}^*$. Since $\mathrm{Lin}(X)$ is an algebraic group, if it has trivial Lie algebra, then it must be finite. We'll show that the Lie algebra $\mathfrak{g}$ of G equals that of the subgroup $\ker(\mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P})) \cong \mathbb{C}^*$; this implies that the Lie algebra of the quotient $\mathrm{Lin}(X) = G/\mathbb{C}^*$ is trivial, as required.

The tangent space to $\mathrm{Aut}(S)$ at the identity is naturally isomorphic to $S_{a_0} \oplus \cdots \oplus S_{a_{n+1}}$, where an element $z := (z_0, \dots, z_{n+1})$ corresponds to the infinitesimal automorphism $x_i \mapsto x_i + \epsilon z_i$. Our aim is to show that if $z \in \mathfrak{g}$, then in fact z is a multiple of $(a_0 x_0, \dots, a_{n+1} x_{n+1})$, which is the derivative of the function $\mathbb{C}^* \to G$ given by $t \mapsto (x_i \mapsto t^{a_i} x_i)$ at $t = 1$.

Every z in the Lie algebra of $\mathrm{Aut}(S)$ defines a derivation $D_z : S \to S$ by the formula $h \mapsto \frac{d}{d\epsilon} h(x + \epsilon z)|_{\epsilon=0}$. The fact that $z \in \mathfrak{g}$ means that $D_z(f) = cf$ for some constant c. But we may express $D_z(f)$ in terms of partial derivatives $f_i := \frac{\partial f}{\partial x_i}$ as:

$$D_Z(f) = \sum_i f_i z_i.$$

Therefore, we have

$$\sum_i f_i z_i = cf = \sum_i \frac{c a_i x_i}{d} f_i.$$

In this equation, the last equality follows from the following weighted variant of Euler's formula for homogeneous polynomials. Namely, for f homogeneous of weighted degree d in variables x_i with weights a_i:

$$\sum_i a_i x_i f_i = df.$$

Rearranging the equation above now gives

$$\sum_i \left(z_i - \frac{ca_ix_i}{d}\right) f_i = 0. \tag{2.2}$$

Since X is a quasismooth hypersurface, its punctured affine cone in $\mathbb{A}^{n+2} \setminus \{0\}$ is smooth, so that the only common zero of the partial derivatives $f_0, \ldots, f_{n+1}$ is at the origin $x = 0$ in $\mathbb{A}^{n+2}$. The ring S is Cohen-Macaulay, each f_i is a homogeneous element of positive degree in a graded ring, and these $n+2$ polynomials cut out the subvariety $\{0\}$ of codimension $n+2$ in $\mathbb{A}^{n+2}$. Therefore, it follows that any permutation of the f_i form a regular sequence in the ring S.

A particular consequence of that fact is that f_i is not a zero divisor in the ring S/I_i, where $I_i \coloneqq (f_0, \ldots, \hat{f}_i, \ldots, f_{n+1})$. The equation (2.2) then implies that $z_i - ca_ix_i/d \in I_i$ for each i. That element is homogeneous of weighted degree a_i, so we can guarantee that it is zero if every nonzero polynomial in I_i has degree greater than a_i. Each partial derivative f_j has degree $d - a_j$, so we can conclude that $z_i - ca_ix_i/d = 0$ if $d - a_j$ is greater than a_i for all $j \neq i$. Either of the two conditions in the theorem guarantees that this criterion is met for all i. Therefore, if we assume one of these conditions, then $z_i = ca_ix_i/d = (c/d)a_ix_i$ for all i, as required.

Next, we'll show the converse: if either $d < 2\max\{a_0, \ldots, a_{n+1}\}$ or $d = 2\max\{a_0, \ldots, a_{n+1}\}$ and there are multiple weights equal to the maximum, then $\mathrm{Lin}(X)$ is infinite. Furthermore, we'll prove that X is rational. It's helpful to consider a few distinct cases. Throughout, we'll assume the weights are arranged in decreasing order, so in particular $a_0 = \max\{a_0, \ldots, a_{n+1}\}$. We note that either assumption on degree above guarantees that $d < a_0 + \cdots + a_{n+1}$ so that the hypersurface is Fano.

If $d = \max\{a_0, \ldots, a_{n+1}\} = a_0$, then X is a linear cone, and hence isomorphic to $\mathbb{P}(a_1, \ldots, a_{n+1})$, which is rational. We may also assume that the equation of X is $x_0 + f(x_1, \ldots, x_n) = 0$, where f is homogenenous of degree a_0. Under the automorphism $x_0 \mapsto x_0 - f$ of $\mathbb{P}$, this becomes $x_0 = 0$. Every automorphism of $\{x_0 = 0\} = \mathbb{P}(a_1, \ldots, a_{n+1})$ extends to $\mathbb{P}(a_0, \ldots, a_{n+1})$ and this group is infinite since $n \geq 1$.

Now suppose $d < \max\{a_0, \ldots, a_{n+1}\} < 2d$. In order for X to be quasismooth, its equation must contain a monomial x_0x_l (for some $l \neq 0$) with nonzero coefficient by Proposition 1.3.7. By a transformation of x_l, we may assume that x_0x_l is the only term involving x_0. The equation then looks like

$$x_0x_l + x_lf(x_1, \ldots, x_{n+1}) + g(x_1, \ldots, \hat{x}_l, \ldots, x_n),$$

where f is homogeneous of degree $d - a_l = a_0$ and may include x_l, while g is homogeneous of degree d and consists of terms not containing x_0 or x_l. Composing with $x_0 \mapsto x_0 - f$ then eliminates the middle term. After these transformations, it's clear that X contains an infinite group of automorphisms which for each $t \in \mathbb{C}^*$ maps $x_0 \mapsto tx_0$, $x_l \mapsto \frac{1}{t}x_l$, and fixes all other variables. To show that X is rational, note that the g term above is nonzero since X is quasismooth. On the open set $x_l \neq 0$, we may isolate $x_0 = -g/x_l$, so that the projection forgetting x_0 is a birational map to the rational toric variety $\mathbb{P}(a_1, \ldots, a_{n+1})$.

Finally, suppose $d = 2\max\{a_0, \ldots, a_{n+1}\}$, but that both a_0 and a_1 equal $\frac{d}{2}$. A similar series of reductions to the equation of f works here: we can change variables so that the quadratic in x_0 and x_1 equals x_0x_1 and eliminate any other terms involving x_0 and x_1. The equation $x_0x_1 + f(x_2, \ldots, x_{n+1}) = 0$ has the same infinite family $x_0 \mapsto tx_0$, $x_1 \mapsto \frac{1}{t}x_1$ of automorphisms, and the projection forgetting the first coordinate is again a birational map. This completes the proof. □

Next, we consider bounds on the size of the linear automorphism groups of hypersurfaces when they are finite. Some results in this direction are known for hypersurfaces of degree d in ordinary projective space $\mathbb{P}^{n+1}$, which we know have finite linear automorphism groups when $d \geq 3$. An unpublished work of Bott and Tate from around 1961 showed that there is a bound on the size of the $\mathrm{Lin}(X)$ in terms of d and n (see [OS78] for an exposition of these ideas). Around twenty years later, Howard and Sommese [HS81] showed that there is a constant k_n for every dimension n such that $|\mathrm{Lin}(X)| < k_nd^{n+1}$, for any $d \geq 3$. We'll prove an even stronger theorem in the setting of weighted projective hypersurfaces.

Theorem 2.3.3. *For every positive integer n, there exists a constant C_n depending only on n with* *the following property: for any well-formed, quasismooth hypersurface $X \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ of degree d and dimension n, if $\operatorname{Lin}(X)$ is finite, then*

$$|\operatorname{Lin}(X)| \leq C_n \frac{d^{n+1}}{a_0 \cdots a_{n+1}}. \tag{2.3}$$

In particular, the same constant C_n works for hypersurfaces in *any* weighted projective space of dimension n. The comments following the proof of Lemma 2.3.5 describe an explicit procedure for effectively producing a constant C_n for which the theorem holds. We'll need the following definitions during the proof.

Definition 2.3.4. Let $\mathcal{G}$ be a group. We say that $\mathcal{G}$ has the *Jordan property* if there exists a constant $J(\mathcal{G})$ such that for every finite subgroup $H \subset \mathcal{G}$, there exists a normal abelian subgroup $A \subset H$ with index $[H : A] \leq J(\mathcal{G})$. The minimum $J(\mathcal{G})$ with this property is called the *Jordan constant* of $\mathcal{G}$. The *weak Jordan constant* $\bar{J}(\mathcal{G})$ of $\mathcal{G}$ is the minimum constant such that every finite $H \subset \mathcal{G}$ has a (not necessarily normal) abelian subgroup $A \subset H$ with $[H : A] \leq \bar{J}(\mathcal{G})$.

An 1878 result by Jordan [Jor78] shows that $\mathrm{GL}_N(\mathbb{C})$ has the Jordan property for all N (for a modern exposition of his original proof and subsequent developments, see [Bre12]). The explicit values of the Jordan constants $J_N := J(\mathrm{GL}_N(\mathbb{C}))$ were not computed until much later: Collins [Col07] calculated all the J_N and in particular showed that when $N \geq 71$, $J_N = (N+1)!$; this index is achieved by the N-dimensional standard representation of the symmetric group S_{N+1}. His proof relies on the classification of finite simple groups.

Weak Jordan constants have not been as well studied, but it follows from a theorem of A. Chermak and A. Delgado that for any group with the Jordan property, $\bar{J}(\mathcal{G}) \leq J(\mathcal{G}) \leq \bar{J}(\mathcal{G})^2$ (see [Isa08, Theorem 1.41] and [PS17, Remark 1.2.2]). The precise values of $\bar{J}_N := \bar{J}(\mathrm{GL}_N(\mathbb{C}))$ for small N are computed in [PS17], but to the author's knowledge there has been no complete calculation of all the $\bar{J}_N$.

Proof of Theorem 2.3.3. We'll prove the theorem in the following three steps:

Step 1: Show that $\mathrm{Lin}(X)$ is the image of a finite group of graded ring automorphisms which fixes the function f defining X and has order $d|\mathrm{Lin}(X)|$.

Step 2: Find a uniform bound C_n on the weak Jordan constants $\bar{J}(\mathrm{Aut}(S))$ of the graded automorphism groups of weighted polynomial rings S in $n+2$ variables.

Step 3: Show that the order of an abelian group of graded ring automorphisms fixing f is at most $d^{n+2}/(a_0 \cdots a_{n+1})$.

Step 1: Suppose that $G := \mathrm{Lin}(X)$ is a finite group, for a quasismooth hypersurface X of degree d in $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_{n+1})$. Let $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$ be the weighted polynomial ring with weights a_i, f be the polynomial defining the hypersurface X, and π be the natural quotient homomorphism $\pi : \mathrm{Aut}(S) \to \mathrm{Aut}(\mathbb{P})$. If $g \in \pi^{-1}(G) \subset \mathrm{Aut}(S)$, the induced automorphism of $\mathbb{P}$ preserves f, so $g \cdot f = cf$ for some constant $c \in \mathbb{C}$. Define H to be the subgroup of $\pi^{-1}(G)$ of elements g which satisfy the stronger condition $g \cdot f = f$. We claim that $\pi|_H : H \to G$ is a surjective homomorphism with kernel of order d, so that in particular H is a finite group with $|H| = d|G|$.

It's clear that $\pi|_H : H \to G$ is surjective because we can compose any automorphism in $\pi^{-1}(G)$ with an element of $\ker(\pi) \cong \mathbb{C}^*$ (see Lemma 1.3.4) to scale the factor c to 1. An element of the kernel of $\pi|_H$ is a $t \in \mathbb{C}^*$ with $f(t^{a_0}x_0, \ldots, t^{a_{n+1}}x_{n+1}) = t^d f = f$, so t is a dth root of unity. This proves $|H| = d|G|$. To bound the order of G, we can therefore analyze the group $H \subset \mathrm{Aut}(S)$ instead.

Step 2: Next, we reduce to only considering abelian groups by computing the weak Jordan constant for the group $\mathrm{Aut}(S)$ of graded automorphisms. For any weighted polynomial ring S, $\mathrm{Aut}(S)$ is a linear algebraic group. This implies that $\mathrm{Aut}(S)$ has the Jordan property. However, even for a fixed number of variables, $\mathrm{Aut}(S)$ can have arbitrarily large dimension as an algebraic group: for example, if $S = \mathbb{C}[x_0, x_1, x_2]$ with weights a, 1, and 1, respectively, then $\dim(\mathrm{Aut}(S)) = \dim(S_1) + \dim(S_1) + \dim(S_a) = a + 6$. Despite this, we'll prove that the Jordan constant of $\mathrm{Aut}(S)$ is uniformly bounded among polynomial rings S with a fixed number of variables.

Following the notation used in Lemma 1.3.5, we let $\mathcal{B} := \{b : b = a_i \text{ for some } i\}$ be the set of positive integers that occur as a weight of the polynomial ring S. For each $b \in \mathcal{B}$, N_b is the number of weights equal to b. Recall that $\bar{J}_N := \bar{J}(\mathrm{GL}_N(\mathbb{C}))$.

Lemma 2.3.5. *Let $S = \mathbb{C}[x_0, \dots, x_{n+1}]$ be a weighted polynomial ring with weights $a_0, \dots, a_{n+1}$. Then $\bar{J}(\mathrm{Aut}(S)) = \prod_{b \in \mathcal{B}} \bar{J}_{N_b}$. In particular, for any integer n, there is a uniform upper bound C_n on the weak Jordan constants of all groups $\mathrm{Aut}(S)$ where S has $n+2$ variables.*

Proof. Inside each graded piece S_b with $b \in \mathcal{B}$, there is a subspace V_b of dimension N_b spanned by the variables of weight b, and a complementary subspace W_b spanned by the remaining monomials of weighted degree b. The direct sum $\bigoplus \mathrm{GL}(V_b)$ embeds as a subgroup of $\mathrm{Aut}(S)$, consisting of all automorphisms that don't "mix" variables of different weights. We'll show that any finite group $G \subset \mathrm{Aut}(S)$ is conjugate to a subgroup of $\bigoplus \mathrm{GL}(V_b) \subset \mathrm{Aut}(S)$. To do this, we'll construct the necessary coordinate change inside each S_b.

Since finite groups are linearly reductive in characteristic zero, the representation of G on S_b splits into a direct sum of irreducible representations. In particular, since W_b is G-invariant, we can find a complementary G-invariant subspace V_b' inside S_b. Define the change of coordinates on variables of weight b in such a way that the span of the variables of weights b becomes V_b'. We can construct this change of coordinates independently within each S_b and arrive at an automorphism of the entire graded ring S. By construction, elements $g \in G$ don't mix variables of different weights in the new coordinates.

This proves that any finite group G that appears as a subgroup of $\mathrm{Aut}(S)$ also embeds in $\bigoplus \mathrm{GL}(V_b)$. Therefore,

$$\bar{J}(\mathrm{Aut}(S)) = \bar{J}\left(\bigoplus_{b \in \mathcal{B}} \mathrm{GL}(W_b)\right) = \prod_{b \in \mathcal{B}} \bar{J}_{N_b}.$$

Here the last equality comes from the general fact that $\bar{J}(G_1 \times G_2) = \bar{J}(G_1)\bar{J}(G_2)$ (this is one convenient property of weak Jordan constants that doesn't hold for regular Jordan constants). Because there are $n+2$ weights total, we have $\sum_{b \in \mathcal{B}} N_b = n+2$. There are only finitely many possibilities for the collection of positive integers N_b for a fixed n, so there is a uniform upper bound C_n on the weak Jordan constant of $\mathrm{Aut}(S)$ depending only on n. $\square$

We may explicitly compute a value for C_n for a particular n by considering all partitions of $n+2$ as a sum of positive integers N_b and multiplying the corresponding values of J_{N_b} computed by Collins [Col07] (since $\bar{J}_N \leq J_N$).

Step 3: Using Lemma 2.3.5, we now only need to bound the order of abelian subgroups of automorphisms $A \subset \mathrm{Aut}(S)$, where S has $n+2$ variables. Suppose that $A \subset H$ is an abelian subgroup of smallest index and assume we've changed coordinates on S so that the action of A is diagonal, using Lemma 1.3.5.

Now suppose that f is a sum of s monomials with nonzero coefficients and write it as

$$f = \sum_{i=1}^{s} K_i \prod_{j=0}^{n+1} x_j^{m_{ij}},$$

where each K_i is nonzero by assumption. Package the exponents m_{ij} into an $s \times (n+2)$ matrix M. Each row corresponds to a monomial in f. We can use Proposition 1.3.7 to pick a distinguished collection of $n+2$ monomials in f: indeed, for each i, select a monomial of the form $x_i^{b_i}$ or $x_i^{b_i} x_j$, $j \neq i$ which has a nonzero coefficient in f. Take only the $n+2$ rows of M corresponding to these, and assemble them into a square $(n+2) \times (n+2)$ minor B of M in such a way that the monomial associated to x_i goes in the ith row.

Lemma 2.3.6. *The matrix B constructed above is invertible and has determinant satisfying*

$$0 < \det(B) \leq \frac{d^{n+2}}{a_0 \cdots a_{n+1}}.$$

Proof. We first note the following properties of B: first, every entry b_i on the main diagonal is a positive integer satisfying $2 \leq b_i \leq d/a_i$. (The lower bound is by the criterion in Theorem 2.3.1, while the upper bound is because $x_i^{b_i}$ or $x_i^{b_i} x_j$ is a monomial of weighted degree d.) Second, each row of B contains at most one nonzero element off the main diagonal; if it does, this element must be a 1.

We can begin to compute the determinant by expanding along any rows or columns that have only one nonzero entry, namely the diagonal entry. At each such step, the diagonal entry b_i is a positive integer which is at most d/a_i, where i is the index of the row in question. After removing the ith row and column, the resulting minor always has the same properties as B, so it suffices to prove the inequality in the lemma with one copy of d in the numerator and the a_i in the denominator removed. Continuing in this way, we may assume B has exactly one off-diagonal 1 in each row and column. Up to a permutation of the indices, we can now further assume that B is block diagonal with blocks of the form

$$\begin{pmatrix} b_0 & 1 & 0 & \cdots & 0 \\ 0 & b_1 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{r-1} & 1 \\ 1 & 0 & \cdots & 0 & b_r \end{pmatrix}.$$

It now suffices to prove the lemma in the case that B is a single block of the form above (so $r = n+1$). It's straightforward to compute that this "loop matrix" has determinant $b_0 \cdots b_{n+1} + (-1)^{n+1} \neq 0$, so it is invertible (here we use $b_i \geq 2$). As for the bound on the determinant, it automatically holds when n is even since $b_0 \cdots b_{n+1} - 1 < b_0 \cdots b_{n+1}$ and each $b_i < \frac{d}{a_i}$. When n is odd, use the series of equations $b_0 = (d - a_1)/a_0, b_1 = (d - a_2)/a_1, \ldots, b_{n+1} = (d - a_0)/a_{n+1}$ to compute

$$\begin{aligned} \det(B) = b_0 \cdots b_{n+1} + 1 &= \frac{(d-a_0)(d-a_1)\cdots(d-a_{n+1})}{a_0 \cdots a_{n+1}} + 1 \\ &= \frac{d^{n+2} - d^{n+1}s_1 + d^n s_2 - \cdots + d s_{n+1}}{a_0 \cdots a_{n+1}} < \frac{d^{n+2}}{a_0 \cdots a_{n+1}}. \end{aligned}$$

Here $s_l \coloneqq s_l(a_0, \ldots, a_{n+1})$ is the degree l elementary symmetric polynomial in the weights $a_0, \ldots, a_{n+1}$. In the last equality, the final term $-s_{n+2}/(a_0 \cdots a_{n+1}) = -1$ in the expansion cancelled with the 1. Since all weights are smaller than d, the terms $d^{n+2-l}s_l$ decrease in magnitude as l increases. This justifies the last inequality. □

With these properties of B in hand, we return to the proof of the theorem. We'll show that for any diagonal automorphism $x_j \mapsto c_j x_j$ in A, the scalars c_j satisfy $|c_j| = 1$. Indeed, since this automorphism preserves f, it preserves each monomial individually, and

$$K_i \prod_j (c_j x_j)^{m_{ij}} = K_i \prod_j x_j^{m_{ij}},$$

for each $i = 1, \ldots, s$. Therefore, $\prod_j c_j^{m_{ij}} = 1$. Taking logarithms of the $|c_j|$, this means that $(\ln |c_0|, \ldots, \ln |c_{n+1}|) \in \ker M$. But $\ker M \subset \ker B = \{0\}$ since B is invertible, so $|c_j| = 1$ for each j. Therefore, any element of A can be represented as an $(n+2)$-tuple $(\theta_0, \ldots, \theta_{n+1})$ of elements of $\mathbb{Q}/\mathbb{Z}$, where $c_j = e^{2\pi i \theta_j}$.

The condition that $(\theta_0, \ldots, \theta_{n+1})$ preserves f can be expressed as $M(\theta_0, \ldots, \theta_{n+1})^\intercal \in \mathbb{Z}^s$. We can obtain an upper bound for the order of A by considering the weaker condition $B(\theta_0, \ldots, \theta_{n+1})^\intercal \in \mathbb{Z}^{n+2}$ instead (this says that at least the $n+2$ selected monomials in f are preserved by the automorphism). The number of distinct solutions for $(\theta_0, \ldots, \theta_{n+1})$ modulo $\mathbb{Z}^{n+2}$ to this latter equation is the index of $\mathbb{Z}^{n+2}$ in the superlattice spanned by $B^{-1}e_0, \ldots, B^{-1}e_{n+1}$, where $e_0, \ldots, e_{n+1}$ are the standard basis vectors in $\mathbb{Z}^{n+2}$. This index equals $\det(B)$, so

$$|A| \leq |\det(B)| \leq \frac{d^{n+2}}{a_0 \cdots a_{n+1}}$$

by Lemma 2.3.6. The original group H which contained A as a smallest index abelian subgroup therefore has order

$$|H| \leq \bar{J}(\mathrm{Aut}(S)) \frac{d^{n+2}}{a_0 \cdots a_{n+1}} \leq C_n \frac{d^{n+2}}{a_0 \cdots a_{n+1}}.$$

Finally, we have the desired bound

$$|\operatorname{Lin}(X)| = |G| = \frac{|H|}{d} \leq C_n \frac{d^{n+1}}{a_0 \cdots a_{n+1}}.$$

□

Example 2.3.7 (Fermat Hypersurfaces)**.** For any positive integer n and degree $d \geq 3$, the

Fermat hypersurface of dimension n and degree d in $\mathbb{P}^{n+1}$ is $X := \{x_0^d + \cdots + x_{n+1}^d = 0\} \subset \mathbb{P}^{n+1}$. Then $\operatorname{Lin}(X)$ contains a copy of the symmetric group S_{n+2} acting by permutation of the variables x_i and the diagonal automorphisms given by multiplying each x_i by an arbitrary dth root of unity (modulo the scalar transformations). Therefore, $|\operatorname{Lin}(X)| \geq (n+2)!d^{n+1}$.

In fact, a computation of Shioda [Shi87] shows that $|\operatorname{Lin}(X)| = (n+2)!d^{n+1}$ when X is defined over an algebraically closed field of characteristic zero (see also [Kon02] for another proof and some generalizations of this result). Note that a Fermat hypersurface can have extra automorphisms in positive characteristic [Kon02, Section 1].

This example shows that the order of growth with respect to the degree d in the estimate Theorem 2.3.3 is optimal and that we must have $C_n \geq (n+2)!$ for all n. A natural question is whether we may actually take $C_n = (n+2)!$ for all n. We'll show that this is nearly true for $n = 1$, but not quite.

Proposition 2.3.8. *Let X be a well-formed quasismooth weighted projective curve of degree d in $\mathbb{P}(a,b,c)$ with finite linear automorphism group. Then*

$$|\operatorname{Lin}(X)| \leq \frac{6d^2}{abc},$$

unless $a = b = c = 1$ and X is projectively equivalent to one of the following two plane curves in $\mathbb{P}^2 = \mathbb{P}(1,1,1)$:

1. *The Klein quartic $xy^3 + yz^3 + zx^3 = 0$, with automorphism group isomorphic to $\mathrm{PSL}_2(\mathbb{F}_7)$ of order 168;*
2. *The Wiman sextic $10x^3y^3 + 9x^5z + 9y^5z - 45x^2y^2z^2 - 135xyz^4 + 27z^6 = 0$, with automorphism group isomorphic to A_6 of order 360.*

Proof. In order for a weighted projective curve to be well-formed, we must have that the weights a, b and c are pairwise relatively prime, and that each weight divides d. Indeed, if some weight does not divide d, then the intersection $X \cap \mathbb{P}_{\text{sing}}$ contains the corresponding coordinate point, which has codimension 1 in X. There are three cases to consider, depending on which of a, b, or c coincide.

Suppose first that a, b, and c are all distinct. Then, any finite subgroup of $\mathrm{Aut}(\mathbb{P}(a,b,c))$ is abelian. This follows from Lemma 2.3.5, which shows that $\bar{J}(\mathrm{Aut}(S)) = 1$ in this case. Abelian subgroups of $\mathrm{Aut}(S)$ fixing the defining polynomial f of X have order at most $d^3/(abc)$ by **Step 3** of the proof of Theorem 2.3.3; hence by **Step 1**, $|\mathrm{Lin}(X)| \leq d^2/(abc)$.

Now suppose that $b = c$, but that a is distinct from the other two weights. Since $\mathbb{P}(a,b,c)$ is well-formed, we must have $b = c = 1$. The weak Jordan constant of $\mathrm{Aut}(S)$ is $\bar{J}_1\bar{J}_2 = \bar{J}_2 = 12$ in this case [PS17, Section 2.2]. Suppose our hypersurface is given by $X = \{f = 0\}$, where f has weighted degree d. In order for $|\mathrm{Lin}(X)|$ to exceed $6d^2/(abc) = 6d^2/a$, we would have to have (after conjugation) a finite subgroup G of $\mathrm{GL}_1(\mathbb{C}) \oplus \mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(S)$ fixing f of order exceeding $6d^3/a$. The maximal possible order of an abelian subgroup preserving f is d^3/a, so we require our hypothetical G to have no abelian subgroup of index less than or equal to 6. The image of G under the projection

$$\mathrm{GL}_1(\mathbb{C}) \oplus \mathrm{GL}_2(\mathbb{C}) \xrightarrow{p_2} \mathrm{GL}_2(\mathbb{C})$$

would also have no abelian subgroup of index less than or equal to 6. All finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ are central extensions of cyclic groups, dihedral groups, A_4, S_4, or A_5. Of these, only A_5 has the required property: the largest abelian subgroup has index 12. Therefore, the image of G in $\mathrm{PGL}_2(\mathbb{C})$ is isomorphic to A_5. It follows that $p_2(G)$ is a central extension of A_5 in $\mathrm{GL}_2(\mathbb{C})$. Since X is quasismooth, the polynomial f is of the form

$$f = x^{d/a} + x^{d/a-1}g_a(y,z) + x^{d/a-2}g_{2a}(y,z) + \cdots + g_d(y,z),$$

for some polynomials $g_a, g_{2a}, \ldots, g_d$ of the indicated degrees in y and z. Here $g_d(y,z)$ must be nonzero since f is irreducible. Each of the terms must be individually preserved by the action of G because that action is block diagonal; in particular, $g_d(y,z)$ is an invariant polynomial under the action of $p_2(G)$. But this means that the intersection of $p_2(G)$ with the center of $\mathrm{GL}_2(\mathbb{C})$ has order at most d (primitive roots of unity of higher degree could not preserve this polynomial), so $|p_2(G)| \leq |A_5|d = 60d$. Similarly, $|\ker(p_2) \cap G| \leq d/a$, so

$$|G| \leq \frac{60d^2}{a}.$$

The combination of the inequalities $|G| \leq 60d^2/a$ and $|G| > 6d^3/a$ means that $d < 10$. However, we've already seen that $g_d(y, z)$ is a polynomial invariant of the action of the binary icosahedral group in $\mathrm{GL}_2(\mathbb{C})$. The homogeneous generators for that invariant ring have degrees 12, 20, and 30 by a result of Klein [Kle84], contradicting the bound on the degree. It follows that no weighted projective curve of this form has more than $6d^2/a$ linear automorphisms.

The last possibility is that $a = b = c$, so that X is a smooth plane curve. In this case, the problem of finding the largest possible automorphism groups in different degrees is well studied. For $d \geq 4$, recall that all plane curve automorphisms are linear [Cha78]. Klein [Kle79] computed the linear automorphism group of the quartic curve in Proposition 2.3.8; it has the largest possible automorphism group of any curve of genus $g = 3$ by the Hurwitz bound $|\mathrm{Aut}(X)| < 84(g-1)$. Wiman [Wim96] first computed that the sextic in the proposition has automorphism group A_6. Later work showed that the Wiman sextic is the unique degree 6 curve with largest automorphism group up to projective equivalence [DIK00] and that the Fermat curve has the same property for various other $d \leq 20$ [KMP99, KMP01]. Finally, Harui [Har19, Theorem 2.5] proved that the two curves listed in Proposition 2.3.8 are the only ones with $|\mathrm{Lin}(X)| > 6d^2$ for *any* degree d. This proves the proposition. □

This classification shows that we may take $C_1 = \frac{21}{2}$ in Theorem 2.3.3. The author is unaware of any counterexamples to the theorem with $C_n = (n+2)!$ for $n \geq 2$. By analogy with Collins' computations of Jordan constants, we might expect that unusual behavior such as in the $n = 1$ case occurs only for small n.

Question 2.3.9. Does Theorem 2.3.3 hold with $C_n = (n+2)!$ for $n \geq 2$? In particular, does the Fermat hypersurface have the largest automorphism group of any smooth hypersurface of degree d in $\mathbb{P}^{n+1}$ when $n \geq 2$ and $d \geq 3$?

Many partial results in this direction are known for smooth hypersurfaces in ordinary projective space. For instance, we have a fairly complete picture of the possible *orders* of automorphisms that can occur [GL13, Zhe22]. The possible automorphism groups of smooth cubic surfaces over an algebraically closed field of arbitrary characteristic were classified by Dolgachev and Duncan [DD19]. Moreover, the linear automorphism groups of smooth cubic threefolds and smooth quintic threefolds were classified by works of Wei and Yu [WY20] and Oguiso and Yu [OY19], respectively. The Fermat cubic fourfold is also known to have the largest possible automorphism group by a result of Laza and Zheng [LZ22]. In summary, the second part of Question 2.3.9 is known to have an affirmative answer at least for the following pairs (n, d): $(2, 3), (3, 3), (3, 5)$, and $(4, 3)$.

2.4 Automorphisms of a Very General Hypersurface

Another result of Matsumura and Monsky is that the automorphism group of a *very general* hypersurface in $\mathbb{P}^{n+1}$ with $n \geq 2$ and degree $d \geq 3$ is trivial [MM64, Theorem 5]. One might hope that the same result holds for weighted projective hypersurfaces whenever the conditions of Theorem 2.3.1 are met. This turns out to be false, but we have the following result under a slightly stronger assumption on the degree.

Theorem 2.4.1. *Suppose that there exists a hypersurface $X \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ of degree d which is quasismooth and well-formed, where $n \geq 1$ and $d \geq 5 \max\{a_0, \ldots, a_{n+1}\}$. Then for a very general such X, $\mathrm{Lin}(X)$ is contained in the center of $\mathrm{Aut}(\mathbb{P})$ and is toric. In particular, $\mathrm{Lin}(X)$ is abelian.*

Proof. Fix a very general hypersurface $X = \{f = 0\}$ with the given weights and degree. Any element of $\mathrm{Lin}(X)$ comes from an automorphism $\alpha : S \to S$ of the graded ring $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$. The fact that α descends to X means that $\alpha(f) = cf$ for some constant $c \in \mathbb{C}$.

The conditions of Theorem 2.4.1 on the weights and degree are strictly stronger than those of Theorem 2.3.1, so we know that $\mathrm{Lin}(X)$ is finite. In particular, the automorphism α has finite order. It follows from Lemma 1.3.5 that after conjugating by some automorphism of the graded ring S, α becomes diagonal, i.e., maps each x_i to a scalar multiple of itself. Let $\gamma : S \to S$ be such an automorphism that brings α into diagonal form. Define $\beta \coloneqq \gamma\alpha\gamma^{-1}$ and $g \coloneqq \gamma(f)$, so that $\beta(g) = cg$. For each i, let c_i be the scalar such that $\beta(x_i) = c_i x_i$. Next, let $G \coloneqq \mathrm{Aut}(S)$ and $H \coloneqq C_G(\beta)$ be the centralizer of the element β in G.

We will show that unless $G = H$, that is, unless β is actually contained in the center of $\mathrm{Aut}(S)$, the fact that $\{g = 0\}$ has automorphism β forces more than $\dim(G) - \dim(H)$ monomials of degree d in the polynomial g to vanish. This would contradict the assumption that f was originally chosen to be very general, since the space of degree d polynomials with β as an automorphism would have codimension greater than $\dim(G/H)$. The homogeneous space G/H, in turn, is isomorphic to the orbit of β under conjugation. This is the same idea used in Matsumura and Monsky's proof [MM64, Theorem 5] of the analogous fact for hypersurfaces in $\mathbb{P}^n$. (In that paper, they considered both diagonalizable and unipotent automorphisms; since we are working over $\mathbb{C}$ instead of an arbitrary algebraically closed field, we only need to consider the former type.)

We've already seen that the dimension of $G = \mathrm{Aut}(S)$ is $\dim(G) = \dim(S_{a_0}) + \cdots + \dim(S_{a_{n+1}})$. To compute the dimension of the centralizer H, it suffices to compute the dimension of its Lie algebra; we may do this by seeing which infinitesimal transformations commute with β. Indeed, let $\sigma : x \mapsto x + \epsilon z$ be such a transformation, where $z = (z_0, \ldots, z_{n+1})$ is an $(n+2)$-tuple of homogeneous polynomials with $z_i \in S_{a_i}$. We have that

$$\sigma\beta(x_i) = \sigma(c_i x_i) = c_i\sigma(x_i) = c_i(x_i + \epsilon z_i),$$

while

$$\beta\sigma(x_i) = \beta(x_i + \epsilon z_i) = c_i x_i + \epsilon\beta(z_i).$$

Comparing these two equations, we have that β and σ commute if and only if $\beta(z_i) = c_i z_i$; that is, if and only if each monomial in z_i is multiplied by c_i when applying β. Therefore, $\dim(H)$ is equal to the cardinality of the set of ordered 2-tuples (i, y) such that $i \in \{0, \ldots, n+1\}$ and y is a monomial of degree a_i with $\beta(y) = c_i y$. We may also describe the dimension of the entire group $G = \mathrm{Aut}(S)$ in a similar way: it is just the size of the set of *all* 2-tuples

(i, y) with y a monomial of degree a_i. Therefore, $\dim(G/H) = |\Gamma|$, where Γ is the set

$$\Gamma \coloneqq \{(i, y) : \beta(y) \neq c_i y\}.$$

If Γ is empty, then $G = H$, which is what we want. Assuming that it is nonempty, we will now exhibit a vanishing monomial in g for each $(i, y) \in \Gamma$, plus exactly one extra. This would show that the number of vanishing monomials is greater than $\dim(G/H)$, as required.

We'll begin by finding one vanishing monomial for each $(i, y) \in \Gamma$. Since X is quasismooth, Proposition 1.3.7 guarantees that we may choose a monomial of degree d of one of the following two forms: x_i^k or $x_i^k x_j$, $i \neq j$, for some positive integer k. By default, we'll always choose the form x_i^k when a_i actually divides d. By the assumption on degree, we have $k \geq 5$ in either case (if $k < 5$ and $ka_i + a_j = d$ for some i, j, then we must have $k = 4$, $a_i = a_j$ by the assumption on degree, so $5a_i = d$ and we can choose x_i^5 instead).

If a_i divides d, consider the pair of monomials $\{x_i^{k-1}y, x_i^{k-2}y^2\}$. Since $(i, y) \in \Gamma$, $\beta(y) = c_y y$ where $c_y \neq c_i$. But then, our two monomials are multiplied by $c_i^{k-1}c_y$ and $c_i^{k-2}c_y^2$, respectively, under β. These constants cannot be equal or else we would have $c_y = c_i$. If both monomials had nonzero coefficients in g, that would contradict the fact that $\beta(g) = cg$. Therefore, at least one monomial of the pair vanishes in g. The same reasoning works for the pair $\{x_i^{k-1}yx_j, x_i^{k-2}y^2x_j\}$ in the event that we began with $x_i^k x_j$ of degree d instead.

This argument exhibits exactly $|\Gamma| = \dim(G/H)$ distinct vanishing monomials in the polynomial g. They are all distinct because any two pairs of monomials chosen above are disjoint. This follows from the fact that we can recover the pair (i, y) uniquely from either monomial of the pair. This works as follows: given a monomial x^I belonging to the pair we created from $(i, y) \in \Gamma$, find an index i' such that: (1) the corresponding weight $a_{i'}$ is maximal among all variables appearing in x^I with exponent at least 2, and (2) the exponent of $x_{i'}$ is itself maximal among variables with indices satisfying the first condition. Examining the forms of the monomials we chose above, one can show that since $k - 2 > 2$, the index i' identified by this procedure must be unique and equal to i. If we have two elements (i, y_1) and (i, y_2) in Γ with $y_1 \neq y_2$, it's clear that the chosen pairs of monomials are disjoint. Thus, y is also uniquely determined.

The final step is to find just one extra monomial in g that vanishes. To do this, we'll make a slight modification to the list of pairs above, without breaking the disjointness property of the previous paragraph. Since Γ is nonempty, we can fix a particular element $(i, y) \in \Gamma$. Depending on the properties of i and y, we find two vanishing monomials associated to (i, y) rather than just one as follows:

- If we chose x_i^k with degree d (here $k \geq 5$) and y is not equal to some other variable $x_{i'}$, replace the pair $\{x_i^{k-1}y, x_i^{k-2}y^2\}$ with the two pairs $\{x_i^k, x_i^{k-1}y\}$ and $\{x_i^{k-2}y^2, x_i^{k-3}y^3\}$. We can do the same modification when we have $x_i^k x_j$ of degree d (and $k \geq 5$, y not linear) instead. None of the new monomials we've introduced can repeat among the ones we previously found for other elements in Γ. We may now find two vanishing monomials for (i, y) instead of one.

- If x_i^k has degree d with $k \geq 5$ and $y = x_{i'}$, then we still replace the pair $\{x_i^{k-1}x_{i'}, x_i^{k-2}x_{i'}^2\}$ with the two pairs $\{x_i^k, x_i^{k-1}x_{i'}\}$ and $\{x_i^{k-2}x_{i'}^2, x_i^{k-3}x_{i'}^3\}$. However, the latter pair overlaps with the one we found for $(i', x_i) \in \Gamma$ in the special case that $k = 5$. To remedy the issue in this one case, also replace the pair $\{x_i x_{i'}^4, x_i^2 x_{i'}^3\}$ associated to (i', x_i) with $\{x_{i'}^5, x_i x_{i'}^4\}$. As before, the process would be the same if we had started with $x_i^k x_j$ of degree d in the beginning; no repeats are introduced.

By contradiction, we've now shown that $G = H$ so that β is in the center of $\mathrm{Aut}(S)$. This means that $\alpha = \gamma^{-1}\beta\gamma = \beta$, and more generally that α is diagonal in any choice of coordinates. The induced automorphism of $\mathbb{P}(a_0, \ldots, a_{n+1})$ is therefore always toric, as claimed. □

For hypersurfaces satisfying the condition $d \geq 5 \max\{a_0, \ldots, a_{n+1}\}$ in Theorem 2.4.1, the stronger statement that $\mathrm{Lin}(X) = \{1\}$ for X very general is not always true.

Example 2.4.2. Consider the family of hypersurfaces of degree 180 in $\mathbb{P}^3(36, 31, 30, 25)$. The general X in this family is quasismooth and well-formed. Furthermore, the weights and degree satisfy the hypothesis of Theorem 2.4.1. However, since the only monomial of degree 180 involving the variable x_0 of weight 36 is x_0^5, any quasismooth X has a non-trivial automorphism of order 5 given by $x_0 \mapsto \zeta x_0$ for ζ a primitive fifth root of unity. As predicted by the theorem, this automorphism is in the center of $\mathrm{Aut}(\mathbb{P})$.

In a similar way, one can construct examples where d is arbitrarily large relative to the maximum of the weights, but non-trivial automorphisms still exist for any quasismooth X. By having multiple "isolated" weights, generic automorphism groups can be made to contain any abelian group.

Further, in the range where the conditions of Theorem 2.3.1 apply but those of Theorem 2.4.1 do not (i.e., when the degree satisfies $2\max\{a_0, \ldots, a_{n+1}\} \leq d < 5\max\{a_0, \ldots, a_{n+1}\}$), there are many examples of hypersurfaces with generic automorphisms outside the center of $\mathrm{Aut}(\mathbb{P})$. These show that Theorem 2.4.1 is close to optimal.

Example 2.4.3 (Hyperelliptic curves, revisited)**.** We saw above that hyperelliptic curves of genus g naturally embed via the canonical map as $X_{2g+2} \subset \mathbb{P}^2(g+1, 1, 1)$. Conversely, the general hypersurface of this degree is a hyperelliptic curve; up to a transformation of weighted projective space, a general equation becomes $x_0^2 + f(x_1, x_2) = 0$ and gives a double cover of $\mathbb{P}^1$. The hyperelliptic involution is given by $x_0 \mapsto -x_0$, which is a nontrivial automorphism of $\mathbb{P}^2(g+1, 1, 1)$ that descends to X. Further, this automorphism is not in the center of $\mathrm{Aut}(\mathbb{P})$.

Similar reasoning works for other families of hypersurfaces $X_d \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ in higher dimensions whenever $a_0 = d/2$; X always has the involution given by the double cover, and this involution is often not in the center of $\mathrm{Aut}(\mathbb{P})$.

Example 2.4.4. It's well known that any cubic plane curve has non-trivial linear automorphisms. Indeed, any smooth complex cubic plane curve $X_3 \subset \mathbb{P}^2(1, 1, 1) = \mathbb{P}^2$ is projectively equivalent to a curve in *Hesse normal form*

$$x^3 + y^3 + z^3 = 3Cxyz,$$

where $C \in \mathbb{C}$, $C^3 \neq 1$. This result dates back at least to the late 19th century [Web98, v.3, p.22]. The curve X defined by this equation has a linear automorphism group of order at least 18 (in fact, the order equals 18 except when C takes one of a handful of special values [BM17, Corollary 3.10]). For general C, this group is generated by permutations of x, y, z and the transformation $(x : y : z) \mapsto (x : \zeta y : \zeta^2 z)$ for ζ a primitive third root of unity. It acts transitively on the nine flex points of the curve, and is furthermore non-abelian. Since $\mathrm{Aut}(\mathbb{P}^2) = \mathrm{PGL}_3(\mathbb{C})$ is centerless, all non-trivial automorphisms in this group must be outside the center.

In the world of weighted projective hypersurfaces, we can bootstrap this example to any number of dimensions by looking at families such as $X_{15} \subset \mathbb{P}^4(5, 5, 5, 3, 3)$. The variables corresponding to weights of 3 and 5 never mix, so we can again change coordinates so that the equation f defining X assumes Hesse normal form in the first three variables. Then, the transformations above are still automorphisms of f, leaving the variables of weight 3 unchanged.

Example 2.4.5. Examples of generic non-central automorphisms with $d = 4 \max\{a_0, \ldots, a_{n+1}\}$ also exist because of the fact that any quartic "hypersurface" in $\mathbb{P}^1$, that is, a collection of four general points, has nontrival linear automorphism group. More precisely, let $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$ be four general points. Any automorphism of $\mathbb{P}^1$ must preserve the cross-ratio of these four points, and the stabilizer of the cross-ratio under the permutation action of S_4 on these points is the Klein four-group $K = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Further, since $\mathrm{PGL}_2(\mathbb{C})$ is three-transitive, there is an automorphism mapping p_1, p_2 and p_3 according to any permutation $\sigma \in K$; by cross-ratio considerations, it also acts as σ on the fourth. It follows that the subgroup of $\mathrm{PGL}_2(\mathbb{C})$ preserving this set of four points is isomorphic to $K \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

We can pick matrices in $\mathrm{GL}_2(\mathbb{C})$ which descend to these transformations and preserve the quartic equation in two variables defining the given set of four points. This construction allows us to find many positive-dimensional examples with non-central automorphisms. For instance, let $X_{20} \subset \mathbb{P}^3(5, 5, 4, 4)$ be very general. Then X has nontrivial automorphisms defined by the same linear transformations as above in the first two variables and the identity on the variables of weight 4.

CHAPTER 3

Small Volume

3.1 Introduction

For a smooth complex projective variety X of dimension n, the *volume* $\mathrm{vol}(X)$ measures the asymptotic growth of the plurigenera $h^0(X, \ell K_X)$, by

$$\mathrm{vol}(X) := \lim_{\ell \to \infty} h^0(X, \ell K_X)/(\ell^n/n!).$$

This is equal to the intersection number K_X^n if the canonical class K_X is ample or (more generally) nef. A variety is said to be of *general type* if its volume is positive. By a theorem of Hacon–M$^{\mathrm{c}}$Kernan, Takayama, and Tsuji, for each positive integer n there is a constant r_n such that the pluricanonical linear system $|\ell K_X|$ gives a birational embedding of X into projective space for every $\ell \geq r_n$ and every smooth projective n-fold X of general type [HM06, Tak06, Tsu07]. This implies a positive lower bound $1/(r_n)^n$ on volume for all n-folds of general type. However, the asymptotics of these bounds for n large remain mysterious.

Following a long tradition in algebraic geometry [Ian00, BPT13, BK16], we look for examples of low volume among weighted projective hypersurfaces X with canonical singularities and ample canonical class. A resolution of singularities of X will then be a smooth projective variety of general type (see Section 1.2). Weighted projective hypersurfaces exhibit a huge range of behavior, and finding good examples is not easy. In version 1 of [ETT22] on the arXiv, Tao, Totaro, Wang, and the author found examples of n-folds of general type with volume less than $1/e^{n^{3/2}}$, also showing that the bound r_n must grow at least as $e^{n^{1/2}}$. In this chapter, adapted from [ETW23], we considerably improve those results.

Theorem 3.1.1. *1. For every integer $n \geq 3$, there is a smooth complex projective variety X of general type with dimension n and volume less than $1/2^{2^{n/2}}$. It is possible to choose X with geometric genus $p_g = h^0(X, K_X)$ positive.*

2. For every integer $n \geq 2$, there is a smooth complex projective variety X of general type with dimension n such that the linear system $|\ell K_X|$ does not give a birational embedding for any $\ell < 2^{2^{(n-2)/2}}$.

3. For every integer $n \geq 2$, there is a smooth complex projective variety X of general type with dimension n such that $H^0(X, \ell K_X) = 0$ for $1 \leq \ell < 2^{2^{(n-4)/2}}$.

Prior to version 1 of [ETT22], the best examples in high dimensions were by Ballico, Pignatelli, and Tasin. They gave smooth n-folds of general type with volume about $1/n^n$, and with about $n/3$ vanishing plurigenera [BPT13, Theorems 1 and 2]. Theorem 3.1.1 is a big advance. In particular, the constants r_n grow at least doubly exponentially with dimension. Moreover, it is expected that the optimal bound is indeed doubly exponential. Indeed, in the more general situation of klt pairs of general type with standard coefficients, Kollár has proposed a conjecturally optimal example [Kol94], [HMX13, Introduction]:

$$(X, \Delta) = \left(\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_{n+1}-1}{s_{n+1}}H_{n+1}\right),$$

where $H_0, \ldots, H_{n+1}$ are $n+2$ general hyperplanes in $\mathbb{P}^n$ and the sequence s_m is Sylvester's sequence, defined recursively by $s_0 = 2$ and $s_m = s_{m-1}(s_{m-1} - 1) + 1$ for $m \geq 1$. (To say that the pair (X, Δ) is of general type with standard coefficients means that $K_X + \Delta$ is big and all coefficients of the $\mathbb{Q}$-divisor Δ are of the form $1 - 1/j$ with $j \in \mathbb{Z}_+$.)

The volume of $K_X + \Delta$ is $1/(s_{n+2} - 1)^n$, which is, crudely, about $1/2^{2^n}$. Totaro and Wang constructed klt varieties of general type (rather than pairs) for which the logarithm of the volume is asymptotic to the logarithm of the volume of Kollár's pair [TW21]. Later, Totaro found even more extreme examples of klt varieties, which are conjecturally optimal

[Tot22]. These examples have roughly the same asymptotics. We now construct *smooth* varieties of general type with volume around $1/2^{2^{n/2}}$. So the minimal volume under these assumptions should be (crudely) somewhere between $1/2^{2^{n/2}}$ and $1/2^{2^n}$.

We'll also consider the analogous problems for other classes of varieties. In total, we'll study 1) canonical Calabi-Yau varieties, 2) terminal Fano varieties, 3) smooth varieties of general type, and 4) terminal Calabi-Yau varieties. (Our constructions happen to become more complicated in this order.) For each class, we give examples a) of small volume, and b) with many vanishing spaces of sections. When X is of general type, these are the problems of finding a variety with small (canonical) volume and many vanishing plurigenera, as in Theorem 3.1.1. For X Fano, we'll consider $\mathrm{vol}(-K_X)$ and many vanishing spaces of sections $H^0(X, -\ell K_X)$, while for X Calabi-Yau, we'll look at $\mathrm{vol}(A)$ and the groups $H^0(X, \ell A)$ with A an ample Weil divisor on X.

For problems 1a), 2a), and 1b), we can conjecture *optimal* examples, with supporting evidence in low dimensions. The examples involve Sylvester's sequence s_m. In what follows, a projective variety is said to be Calabi-Yau if $K_X \sim_{\mathbb{Q}} 0$.

Conjecture 3.1.2 (Conjecture 3.2.2)**.** *For a positive integer n, let $d = (2s_n - 3)(s_n - 1)$. A general hypersurface X of degree d in $\mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-1}, s_n - 1, s_n - 2)$ is the canonical Calabi-Yau n-fold with ample Weil divisor $\mathcal{O}_X(1)$ of minimal volume.*

Conjecture 3.1.3 (Conjecture 3.2.7)**.** *For each integer $n \geq 2$, let $d = (2s_{n-1} - 3)(s_{n-1} - 1)$. Then a general hypersurface X of degree d in*

$$\mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-2}, s_{n-1} - 1, s_{n-1} - 2, 1)$$

is the terminal Fano n-fold of minimal anticanonical volume.

For problem 1b), our example is not optimal among all canonical Calabi-Yau varieties, but we expect it to be optimal among weighted projective hypersurfaces.

Conjecture 3.1.4 (Conjecture 3.2.4)**.** *For an integer $n \geq 2$, let $d = (s_{n-1} - 1)(3s_{n-1} - 4)^2$. Then* *a general hypersurface X of degree d in*

$$\mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-2}, (s_{n-1}-1)(3s_{n-1}-4), (s_{n-1}-1)(3s_{n-1}-5), 3s_{n-1}^2 - 9s_{n-1} + 7)$$

is the quasismooth canonical Calabi-Yau hypersurface of dimension n with ample Weil divisor $\mathcal{O}_X(1)$ with the largest number M having $H^0(X, \mathcal{O}_X(\ell)) = 0$ for $1 \le \ell < M$, among all such hypersurfaces. Namely, $M = 3s_{n-1}^2 - 9s_{n-1} + 7 > 2^{2^{n-1}}$.

We produce examples with doubly exponential behavior for all eight problems. The volumes of the conjecturally optimal examples 1a) and 2a) are (crudely) around $1/2^{2^n}$. Our other examples 3a) and 4a) have volume around $1/2^{2^{n/2}}$, and the optimal bound should be somewhere between that and $1/2^{2^n}$. The number of vanishing spaces of sections is around 2^{2^n} in example 1b) and around $2^{2^{n/2}}$ for 2b), 3b), and 4b).

Although our examples have mild singularities in terms of the minimal model program, note that such exotic behavior cannot occur for *smooth* Fano or Calabi-Yau varieties (or likewise for smooth projective varieties with ample canonical class, rather than of general type). This is a qualitative sense in which our examples are optimal. Indeed, every ample Weil divisor on a smooth projective variety is Cartier and hence has volume an integer. Also, the Ambro-Kawamata conjecture predicts that for every klt projective variety X and every ample *Cartier* divisor A with $A - K_X$ ample, $H^0(X, A)$ is not zero [Amb99, Kaw00, PST17].

Finally, we deduce from our examples that the constant a_n in a Noether-type inequality $\mathrm{vol}(X) \ge a_n p_g(X) - b_n$ for smooth projective n-folds of general type must be doubly exponentially small as a function of n (Theorem 3.3.1).

3.2 Proof of Main Results

In this section, we will find examples of hypersurfaces of various types with excellent asymptotics in high dimensions. For several of the problems, we conjecture that the examples we find are optimal. The examples we construct will all be based on Sylvester's sequence $s_0, s_1, s_2, \ldots$, defined by $s_0 = 2$ and $s_m = s_{m-1}(s_{m-1} - 1) + 1$ for $m \ge 1$. It follows

	1. Canonical CY	2. Terminal Fano	3. General type	4. Terminal CY
a) Volume	$1/(s_n)^{2n-1}$	$1/(s_{n-1})^{2n-3}$	$1/(s_{\lfloor (n+1)/2 \rfloor})^n$	$1/(s_{\lfloor n/2 \rfloor})^n$
b) First $H^0 \neq 0$	s_n	$s_{\lfloor (n-1)/2 \rfloor}$	$s_{\lfloor (n-1)/2 \rfloor}$	$(s_{\lfloor (n-4)/2 \rfloor})^{3/2}$

Table 3.1: Summary of our examples, approximately stated.

that $s_m = s_0 \cdots s_{m-1} + 1$, and hence that the integers in Sylvester's sequence are pairwise coprime. The first few values are $s_0 = 2$, $s_1 = 3$, $s_2 = 7$, $s_3 = 43$, $s_4 = 1807$.

The Sylvester sequence grows doubly exponentially; in fact, $s_m = \lfloor E^{2^{m+1}} + \frac{1}{2} \rfloor$ for a constant $E \doteq 1.264$ [GK08, equations 2.87 and 2.89]. We'll frequently make use of the estimate $s_n > 2^{2^{n-1}}$. Also, the sums of the reciprocals of the s_j converge to 1 more quickly than any other sequence of unit fractions [Sou05]. Namely, we have

$$\frac{1}{s_0} + \frac{1}{s_1} + \cdots + \frac{1}{s_{m-1}} = 1 - \frac{1}{s_m - 1} = 1 - \frac{1}{s_0 \cdots s_{m-1}}.$$

Table 3.1 summarizes the examples we construct for each of the 8 problems. The table shows (approximately) the volume or the first nonvanishing space of sections, in terms of Sylvester's sequence s_j. Note that s_{j+1} is roughly s_j^2. Our examples 1a) and 2a) are conjecturally optimal, and example 1b) is conjecturally optimal among quasismooth hypersurfaces.

3.2.1 Canonical Calabi-Yau Varieties

In light of Corollary 1.3.12, the problem of finding extreme behavior among weighted projective hypersurfaces is easiest for canonical Calabi-Yau varieties, where we need only check that a given example is well-formed and quasismooth. For each positive integer n, Birkar showed that there is a bound $M = M(n)$ such that for all ample Weil divisors A on klt Calabi-Yau n-folds, the linear system $|\ell A|$ defines a birational embedding into projective space for all $\ell \geq M$ [Bir20, Corollary 1.4]. This implies that there is a positive lower bound on volume in each dimension for problem 1a), and an upper bound on the number of vanishing H^0 groups in problem 1b). We'll first tackle 1a) by producing a Calabi-Yau hypersurface X for which the ample Weil divisor $\mathcal{O}_X(1)$ has small volume. We conjecture that this example is optimal, as discussed below.

Proposition 3.2.1. *For each positive integer n, let $d = (2s_n-3)(s_n-1)$. Then a general hypersurface X of degree d in the weighted projective space $Y = \mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-1}, s_n - 1, s_n - 2)$ is quasismooth and Calabi-Yau with canonical singularities. The ample Weil divisor $\mathcal{O}_X(1)$ has volume less than $1/2^{2^n}$ for $n \geq 2$.*

Proof. Since $s_n - 1 = s_0 \cdots s_{n-1}$, each weight is an integer and all but the last weight divide the degree. The last weight $s_n - 2$, however, divides $d-(s_n-1) = (2s_n-4)(s_n-1)$. Therefore, a general hypersurface X of degree d in Y is quasismooth, using Proposition 1.3.6. Because $s_n - 2$ and $s_n - 1$ are relatively prime, we need only check the well-formedness condition when one of these two weights is removed. However, the greatest common divisor of $d/s_0, \ldots d/s_{n-1}$ is $d/(s_0 \cdots s_{n-1}) = 2s_n - 3$. This is relatively prime to both $s_n - 2$ and $s_n - 1$, so the required condition holds. Furthermore, the sum of the weights is

$$d(1/s_0 + \cdots + 1/s_{n-1}) + (s_n - 1) + (s_n - 2) = d(1 - 1/(s_n - 1)) + (s_n - 1) + (s_n - 2)$$

$$= d - (2s_n - 3) + (s_n - 1) + (s_n - 2) = d,$$

so X is Calabi-Yau. Finally, the volume of the ample Weil divisor $\mathcal{O}_X(1)$ is

$$\begin{aligned} \operatorname{vol}(\mathcal{O}_X(1)) &= \frac{d}{(d/s_0)\cdots(d/s_{n-1})(s_n-1)(s_n-2)} = \frac{s_0\cdots s_{n-1}}{d^{n-1}(s_n-1)(s_n-2)} \\ &= \frac{1}{d^{n-1}(s_n-2)} = \frac{1}{(2s_n-3)^{n-1}(s_n-1)^{n-1}(s_n-2)}. \end{aligned}$$

As a crude estimate, this last volume is less than $1/(s_n)^{2n-1} < 1/1.264^{(2n-1)2^{n+1}}$, where 1.264 is less than $E \doteq 1.264$ above. Since $(2n-1)2^{n+1} = 2(2n-1)2^n$ and $1.264^3 > 2$, this last volume is less than $1/2^{2^n}$ when $n \geq 2$. □

For $n = 1$, this example gives $X_6 \subset \mathbb{P}^2(3, 2, 1)$ of volume 1, an elliptic curve E embedded into Proj of its section ring $R(E, \mathcal{O}(P))$, where P is the origin of E. When $n = 2$, we obtain the surface $X_{66} \subset \mathbb{P}^3(33, 22, 6, 5)$ for which $\mathcal{O}_X(1)$ has volume $1/330$, which is minimal among all canonical K3 surfaces with an ample Weil divisor [Bro07, Computation 1.1]. The next two examples are the threefold $X_{3486} \subset \mathbb{P}^4(1743, 1162, 498, 42, 41)$ with volume $1/498240036$ and the fourfold

$$X_{6521466} \subset \mathbb{P}^5(3260733, 2173822, 931638, 151662, 1806, 1805)$$

with volume approximately 2.0×10^{-24}. Both of these are known to have minimal volume among quasismooth Calabi-Yau hypersurfaces in their respective dimensions, by Brown and Kasprzyk (see [BK] for $n = 3$ and [BK16, Section 1.2] for $n = 4$). These results motivate our conjecture:

Conjecture 3.2.2. *Let n be a positive integer and X the n-fold in Proposition 3.2.1. Then $\mathcal{O}_X(1)$ has minimal volume among all ample Weil divisors on canonical n-folds X which are Calabi-Yau in the sense that $K_X \sim_{\mathbb{Q}} 0$.*

Next, we'll consider problem 1b), namely the requirement that $h^0(X, \ell A) = 0$ for some ample divisor A and for $1 \leq \ell < M$ with M large. Once again, we can conjecture an optimal hypersurface example using Sylvester's sequence:

Proposition 3.2.3. *For a positive integer n, let $d = (s_{n-1} - 1)(3s_{n-1} - 4)^2$. Then a general hypersurface X of degree d in the weighted projective space $Y = \mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-2}, (s_{n-1} - 1)(3s_{n-1} - 4), (s_{n-1} - 1)(3s_{n-1} - 5), 3s_{n-1}^2 - 9s_{n-1} + 7)$ is quasismooth and Calabi-Yau with canonical singularities. The spaces of sections $H^0(X, \mathcal{O}_X(\ell))$ vanish for $1 \leq \ell < M$, where $M = 3s_{n-1}^2 - 9s_{n-1} + 7 > 2^{2^{n-1}}$ if $n \geq 2$.*

Proof. All but the last two weights divide the degree. Let $a = (s_{n-1} - 1)(3s_{n-1} - 4)$, $b = (s_{n-1} - 1)(3s_{n-1} - 5)$, and $c = 3s_{n-1}^2 - 9s_{n-1} + 7$. First, we'll show that Y is well-formed. Observe that $\gcd(a, b, c) = 1$. This is because the greatest common factor of a and b is $s_{n-1} - 1$, while $c = (s_{n-1} - 1)(3s_{n-1} - 6) + 1$. Therefore, it remains to see that when one of these three weights is removed, the gcd of all other weights is still 1. We have $\gcd(d/s_0, \ldots, d/s_{n-2}) = (3s_{n-1} - 4)^2$, which is coprime to b. Finally, if b is removed, the gcd of all weights besides c is $3s_{n-1}-4$ and $c-(3s_{n-1}-4)(s_{n-1}-2)=s_{n-1}-1$. This last expression is coprime to $3s_{n-1}-4$, completing the argument.

We have $b=(d-a)/(3s_{n-1}-4)$ and $c=(d-b)/(3s_{n-1}-3)$. Thus X is quasismooth. Finally, taking the sum of all weights gives

$$d(1/s_0+\cdots+1/s_{n-2})+a+b+c=d(1-1/(s_{n-1}-1))+9s_{n-1}^2-24s_{n-1}+16$$

$$=d-(3s_{n-1}-4)^2+(3s_{n-1}-4)^2=d.$$

Therefore the hypersurface is Calabi-Yau and is canonical by Corollary 1.3.12. □

In low dimensions this example becomes $X_{50}\subset\mathbb{P}^3(25,10,8,7)$ for $n=2$, $X_{1734}\subset\mathbb{P}^4(867,578,102,96,91)$ for $n=3$, and

$$X_{656250}\subset\mathbb{P}^5(328125,218750,93750,5250,5208,5167)$$

for $n=4$. In each of these dimensions, X has the largest possible bottom weight for any quasismooth Calabi-Yau hypersurface (see [BK] and [BK16, Section 1.2]). In general, we make the following conjecture.

Conjecture 3.2.4. *Let n be a positive integer and X the n-fold in Proposition 3.2.3. Then X has the largest possible positive integer M such that $H^0(X,\mathcal{O}_X(\ell))$ vanishes for $1\leq\ell<M$, among all quasismooth Calabi-Yau hypersurfaces of dimension n.*

Note, however, that this value M is not optimal among all canonical Calabi-Yau varieties with an ample Weil divisor. We see this in dimension 2. Namely, Iano-Fletcher found that a general complete intersection $X_{24,30}\subset\mathbb{P}^4(15,12,10,9,8)$ of codimension 2 is a K3 surface with canonical singularities, and $h^0(X,\mathcal{O}_X(\ell))=0$ for $1\leq\ell<8$ [Ian00, Section 13.8]. In contrast, the maximum bottom weight in the hypersurface case is 7.

Putting together Propositions 3.2.1 and 3.2.3, we've now proven the following theorem.

Theorem 3.2.5. *1. For every integer $n\geq 2$, there is a canonical Calabi-Yau variety X of dimension n with an ample Weil divisor A such that $\mathrm{vol}(A)<1/2^{2^n}$.*

2. *For every integer $n \geq 2$, there is a canonical Calabi-Yau variety X of dimension n with an ample Weil divisor A such that $H^0(X, \ell A) = 0$ for $1 \leq \ell < 2^{2^{n-1}}$.*

3.2.2 Terminal Fano Varieties

We now turn to terminal Fano varieties. The class of klt Fano varieties of given dimension has no positive lower bound on volume [HMX14, Example 2.1.1]. But Birkar showed that there is a lower bound for ϵ-lc Fano varieties, hence for terminal Fano varieties [Bir19, Theorem 1.2]. The number of vanishing anti-plurigenera, meanwhile, is bounded among all klt Fano varieties of each dimension, by Birkar's theorem on boundedness of complements [Bir19, Theorem 1.1]. So problems 2a) and 2b) have some bound in each dimension. In the case of problem 2a), we can conjecture the optimal example, obtained by adding an extra weight of 1 to the Calabi-Yau example 1a). See the evidence below.

For problem 2b), we find a terminal Fano n-fold with $H^0(X, -mK_X) = 0$ for all $1 \leq \ell < M$ with M roughly $2^{2^{n/2}}$ (Theorem 3.2.9). In the broader class of klt Fano n-folds, Totaro and Wang gave examples with M roughly 2^{2^n} [TW21, Theorem 5.1].

Proposition 3.2.6. *For each integer $n \geq 2$, let $d = (2s_{n-1} - 3)(s_{n-1} - 1)$. Then a general hypersurface X of degree d in the weighted projective space $Y = \mathbb{P}^{n+1}(d/s_0, \ldots, d/s_{n-2}, s_{n-1} - 1, s_{n-1} - 2, 1)$ is a terminal Fano variety, and $\operatorname{vol}(-K_X) < 1/2^{2^n}$ when $n \geq 3$.*

Proof. The proof is nearly identical to that of Proposition 3.2.1 with n replaced by $n-1$. Here X is quasismooth with $K_X = \mathcal{O}_X(-1)$. Adding a weight 1 to the weights in Proposition 3.2.1 ensures by the Reid-Tai criterion that all canonical singularities become terminal (this is also a consequence of Corollary 1.3.16). We may still use $1/2^{2^n}$ as the volume bound because our estimate $1/1.264^{(2n-3)2^n}$ still becomes less than $1/2^{2^n}$ when $n \geq 3$. □

When $n = 2$, this example gives $X_6 \subset \mathbb{P}^3(3, 2, 1, 1)$ for $n = 2$, which is the natural embedding of a smooth del Pezzo surface of degree 1 via its anticanonical ring. Since terminal surfaces are smooth, its (anticanonical) volume of 1 is minimal. In dimension 3, we obtain $X_{66} \subset \mathbb{P}^4(33, 22, 6, 5, 1)$ of volume $1/330$, which is minimal among all terminal

Fano 3-folds, by J. Chen and M. Chen [CC08]. Finally, when $n = 4$, we obtain $X_{3486} \subset \mathbb{P}^5(1743, 1162, 498, 42, 41, 1)$, with volume $1/498240036$. This volume is minimal among all quasismooth terminal 4-fold hypersurfaces with $K_X = \mathcal{O}_X(-1)$, by Brown and Kasprzyk [BK16, Section 1.3]. These results justify the conjecture:

Conjecture 3.2.7. *For an integer $n \geq 2$, let X be the variety of dimension n in Proposition 3.2.6. Then X has minimal anticanonical volume among all terminal Fano n-folds.*

We can also find Fano hypersurfaces with many vanishing groups $H^0(X, -\ell K_X)$, though these examples are not optimal. (In dimension 3, Iano-Fletcher found the complete intersection 3-fold $X_{12,14} \subset \mathbb{P}^5(7, 6, 5, 4, 3, 2)$, a terminal variety with $K_X = \mathcal{O}_X(-1)$. This has $H^0(X, -K_X) = 0$, unlike any terminal Fano hypersurface of dimension 3 [Ian00, section 16.7].)

The example below is a slight modification of the example obtained from 1a) by repeating each weight twice.

Proposition 3.2.8.

1. *Let $n = 2m + 1$ be an odd integer at least 3, and let $d = (2s_m - 3)(s_m - 1)$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \dots, (d/s_{m-1})^{(2)}, 2(s_m - 1), s_m - 1, s_m - 2)$ is quasismooth and Fano with terminal singularities. Such an X satisfies $H^0(X, -\ell K_X) = 0$ for $1 \leq \ell < M$, where $M = s_m - 2 > 2^{2^{(n-3)/2}}$.*

2. *Let $n = 2m + 2$ be an even integer at least 6, and let $d = (2s_m - 3)(s_m - 1)$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \dots, (d/s_{m-2})^{(2)}, d/s_{m-1}, (d/2s_{m-1})^{(2)}, 2(s_m - 1), s_m - 1, s_m - 2)$ is quasismooth and Fano with terminal singularities. Such an X satisfies $H^0(X, -\ell K_X) = 0$ for $1 \leq \ell < M$ where $M = s_m - 2 > 2^{2^{(n-4)/2}}$.*

To get from the odd-dimensional case to the even-dimensional case for the same m, we split one copy of the weight d/s_{m-1} in two (it is even since it is divisible by $s_0 = 2$). We'll prove only (1), since (2) directly follows.

Proof. All weights but the last divide the degree, while $s_m - 2$ divides $2d - 2(s_m - 1) = (4s_m - 8)(s_m - 1)$. It follows that a general X is quasismooth. Since it contains the weights of the Calabi-Yau example of Proposition 3.2.1, the weighted projective space is well-formed. The sum of the weights is

$$2(d/s_0 + \cdots + d/s_{m-1}) + 2(s_m - 1) + s_m - 1 + s_m - 2 = 2d(1 - 1/(s_m - 1)) + 4s_m - 5 = 2d + 1.$$

Thus, X_{2d} is Fano with dimension $n = 2m + 1$.

We'll next show that X has canonical singularities. We already know that the weighted projective space $Y' := \mathbb{P}^{m+1}(d/s_0, \ldots, d/s_{m-1}, s_m - 1, s_m - 2)$ has canonical singularities, again by Proposition 3.2.1. Since the weights of Y' are a subset of those of Y, the Reid-Tai criterion is satisfied for each weight of Y which is also a weight of Y'. This includes the criterion for X at the coordinate point p for the weight $s_m - 2$, since only the weight $2(s_m - 1)$ not belonging to Y' is removed, by Proposition 1.3.9. This is the only basepoint of the sheaf $\mathcal{O}_Y(2d)$. In summary, we have that X and Y are both canonical away from the coordinate point with weight $2(s_m - 1)$. Because a general hypersurface misses this point, we may conclude that X is canonical. By Corollary 1.3.16, X is terminal. □

The following statement summarizes Propositions 3.2.6, 3.2.8, and the discussion above.

Theorem 3.2.9. 1. *For every integer $n \geq 3$, there is a terminal Fano variety X of dimension n with* $\mathrm{vol}(-K_X) < 1/2^{2^n}$.

2. *For every integer $n \geq 3$, there is a terminal Fano variety X of dimension n with the property that $H^0(X, -\ell K_X) = 0$ for $1 \leq \ell < 2^{2^{(n-4)/2}}$.*

3.2.3 Varieties of General Type

We'll next find examples for 3a) and 3b) to prove Theorem 3.1.1: that is, varieties of general type with small volume or many vanishing plurigenera. There is some bound for these problems in each dimension, by the results of Hacon-M$^{\text{c}}$Kernan, Takayama, and Tsuji mentioned in the introduction. (Note that the plurigenera of a variety with canonical singularities are equal to the plurigenera of any resolution of singularities. So it is a matter of choice whether to consider varieties with canonical singularities and ample canonical class or smooth projective varieties of general type.)

The constructions will be modifications of Calabi-Yau examples above. First is an example of small volume.

Lemma 3.2.10. *For each natural number m, let $d = s_{m+1} - 1$. Then the weighted projective space $Y' = \mathbb{P}^{m+1}(d/s_0, \ldots, d/s_m, 1)$ is well-formed and has canonical singularities.*

Proof. First, $\gcd(d/s_0, \ldots, d/s_m) = 1$, so the space is well-formed. Further, the sum of the weights is $d(1 - 1/(s_{m+1} - 1)) + 1 = d$. Since all weights divide this sum, $K_{Y'}$ is Cartier and Y' has canonical singularities by Corollary 1.3.13. □

Proposition 3.2.11. *Let $n = 2m+1$ be an odd positive integer, and let $d = s_{m+1} - 1$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_m)^{(2)}, 1)$ is quasismooth, with $K_X = \mathcal{O}_X(1)$, and has terminal singularities. When $n \geq 5$, $\operatorname{vol}(X) < 1/2^{2^{n/2}}$.*

Proof. As compared to the example in Lemma 3.2.10, all weights are repeated twice except for 1. All the required properties follow from the lemma; in particular $K_X = \mathcal{O}_X(1)$ because the sum of the weights is $2d - 1$. Since all weights are repeated from the example in the lemma, the Reid-Tai criterion automatically holds for each coordinate point and Y is canonical. Since the degree is a multiple of all weights, it follows that X is quasismooth and canonical. Since $K_X = \mathcal{O}_X(1)$, X is in fact terminal, by Corollary 1.3.16.

The volume of this example is

$$\operatorname{vol}(X) = \frac{2d}{(d/s_0)^2 \cdots (d/s_m)^2} = \frac{2d(s_0 \cdots s_m)^2}{d^{2m+2}} = \frac{2}{(s_{m+1} - 1)^{2m-1}} = \frac{2}{(s_{(n+1)/2} - 1)^{n-2}}.$$

Since $s_{(n+1)/2} > 2^{2^{(n-1)/2}}$, we have that when $n \geq 5$, the volume is smaller than $1/2^{2^{n/2}}$. □

Lemma 3.2.12. *For each natural number m, let $d = (s_m - 1)(2s_m - 1)$. Then the weighted projective space $Y' = \mathbb{P}^{m+2}(d/s_0, \ldots, d/s_{m-1}, (s_m - 1)^{(2)}, 1)$ is well-formed and has canonical singularities.*

Proof. For $m = 0$, $Y' = \mathbb{P}^2$ and the result is clear. For $m \geq 1$, $\gcd(d/s_0, \ldots, d/s_{m-1}) = 2s_m - 1$ is coprime to $s_m - 1$, and so Y' is well-formed. The sum of all but the last three weights is $d(1 - 1/(s_m - 1)) = d - (2s_m - 1)$, so the sum of all weights is d. Therefore, Y' is canonical by Corollary 1.3.13. □

Proposition 3.2.13. *Let $n = 2m + 2$ be an even integer at least 2, and let $d = (s_m - 1)(2s_m - 1)$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, 2(s_m - 1), (s_m - 1)^{(2)}, 1)$ is quasismooth with terminal singularities and has $K_X = \mathcal{O}_X(1)$. When $n \geq 4$, $\operatorname{vol}(X) < 1/2^{2^{n/2}}$.*

Proof. The weights of Y include those of Y' as a subset and each of them divides $2d$. Therefore, X is well-formed and quasismooth. The adjunction formula gives that $K_X = \mathcal{O}_X(1)$. The Reid-Tai criterion is again automatic at every weight besides $2(s_m - 1)$. Since a general hypersurface of degree d misses this point, we can conclude that X is canonical. By Corollary 1.3.16, X is in fact terminal.

The volume of this example is

$$\operatorname{vol}(X) = \frac{2d}{(d/s_0)^2 \cdots (d/s_{m-1})^2 (s_m - 1)^2 2(s_m - 1)} = \frac{2d(s_0 \cdots s_{m-1})^2}{2(s_m - 1)^{2m+3}(2s_m - 1)^{2m}}$$

$$= \frac{1}{(s_m - 1)^{2m}(2s_m - 1)^{2m-1}}.$$

This last expression is less than

$$\frac{1}{s_m^{4m-1}} = \frac{1}{s_{(n-2)/2}^{2n-5}}.$$

Since $s_{(n-2)/2} > 2^{2^{(n-4)/2}}$, this last expression is less than $2^{2^{n/2}}$ whenever $2n - 5 > 4$, i.e. for $n \geq 4$. □

We'll also write down some examples of general type with many vanishing plurigenera.

Proposition 3.2.14. *1. Let $n = 2m + 1$ be an odd integer at least 5, and let $d = (s_m - 1)(2s_m - 1)$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{2m+2}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, 2s_m - 2, s_{m-1}^2, (s_{m-1} - 1)^2)$ is quasismooth with* *terminal singularities and has* $K_X = \mathcal{O}_X(1)$. *Such an* X *satisfies* $H^0(X, \ell K_X) = 0$ *for* $1 \le \ell < M$ *where* $M = (s_{m-1} - 1)^2 \ge 2^{2^{(n-3)/2}}$.

2. *Let* $n = 2m + 2$ *be an even integer at least* 6, *and let* $d = (s_m - 1)(2s_m - 1)$. *Then a general hypersurface* X *of degree* $2d$ *in the weighted projective space* $Y = \mathbb{P}^{2m+3}((d/s_0)^{(2)}, \ldots, (d/s_{m-2})^{(2)}, d/s_{m-1}, (d/(2s_{m-1}))^{(2)}, 2s_m - 2, s_{m-1}^2, (s_{m-1}-1)^2)$ *is quasismooth with terminal singularities and has* $K_X = \mathcal{O}_X(1)$. *Such an* X *satisfies* $H^0(X, \ell K_X) = 0$ *for* $1 \le \ell < M$ *where* $M = (s_{m-1} - 1)^2 \ge 2^{2^{(n-4)/2}}$.

Proof. We'll first prove part (1). Suppose that p is a prime number dividing all but one of the weights. Since s_{m-1}^2 and $(s_{m-1} - 1)^2$ are relatively prime, one of these must be omitted, so that p divides $d/s_0, \ldots, d/s_{m-1}$, and $2s_m - 2$. However, $\gcd(d/s_0, \ldots, d/s_{m-1}) = 2s_m - 1$, which is coprime to $2s_m - 2$; so X is well-formed. To see that X is quasismooth, we verify Iano-Fletcher's criterion (Proposition 1.3.6). Neither s_{m-1}^2 nor $(s_{m-1}-1)^2$ divides d, but both divide $2d-(2s_m-2) = 2(s_m-1)(2s_m-2) = 4(s_m-1)^2 = 4s_{m-1}^2(s_{m-1}-1)^2$, so the condition is satisfied for one-element subsets. For the two-element subset $I = \{s_{m-1}^2, (s_{m-1} - 1)^2\}$, we claim that $2d$ is an $\mathbb{N}$-linear combination of these weights. Indeed, we have $2s_m - 1 = s_{m-1}^2 + (s_{m-1} - 1)^2$ so $2d = 2(s_m - 1)(s_{m-1}^2 + (s_{m-1} - 1)^2)$. Finally, the sum of the weights is

$$2d(1 - 1/(s_m - 1)) + 2s_m - 2 + s_{m-1}^2 + (s_{m-1} - 1)^2$$

$$= 2d - (4s_m - 2) + 2s_m - 2 + 2s_m - 1 = 2d - 1,$$

so that $K_X = \mathcal{O}_X(1)$. To show that X is terminal, it suffices to show that it is canonical, by Corollary 1.3.16. First, note that the sum $d/s_0 + \cdots + d/s_{m-1} + s_{m-1}^2 + (s_{m-1} - 1)^2$ is equal to d, and the last two weights are relatively prime. Since each d/s_j divides this sum, Lemma 1.3.14 shows that the singularity at each coordinate point with weight d/s_j is canonical. Using the torus action, this means that Y is canonical away from the weighted $\mathbb{P}^2_{x,y,z}$ with weights $a := 2s_m - 2$, $b := s_{m-1}^2$, and $c := (s_{m-1} - 1)^2$. Because b and c are coprime, Y is actually smooth at points on this weighted $\mathbb{P}^2$ outside of two $\mathbb{P}^1$s. We'll consider these in turn.

On the one-dimensional stratum given by $z = 0$, $x, y \neq 0$, Y has singularities of the form $\frac{1}{s_{m-1}}((d/s_0)^{(2)}, \dots, (d/s_{m-1})^{(2)}, c)$ because $\gcd(a, b) = s_{m-1}$ (here we use that $m \geq 2$, so s_{m-1} is odd). Note that the weight $d/s_{m-1} \equiv -1 \pmod{s_{m-1}}$ because it equals $(s_{m-1} - 1)(2s_m - 1) = (s_{m-1} - 1)(2s_{m-1}^2 - 2s_{m-1} + 1)$, where the first term has residue -1 and the second $+1$. Also, $c = (s_{m-1} - 1)^2 \equiv 1 \pmod{s_{m-1}}$. Therefore, the sum of these two weights is divisible by s_{m-1}, while c is coprime to s_{m-1}. Lemma 1.3.14 again applies to show that the singularity is canonical.

Next, on the one-dimensional stratum given by $y = 0$, $x, z \neq 0$, Y has singularities of the form $\frac{1}{2(s_{m-1}-1)}((d/s_0)^{(2)}, \dots, (d/s_{m-1})^{(2)}, b)$ because $\gcd(a, c) = 2(s_{m-1} - 1)$. We'll use a similar method to the above to analyze this singularity. First, the sum of the first $m - 1$ weights $d/s_0 + \cdots + d/s_{m-2}$ (with each only repeated once) is

$$d(1 - 1/(s_{m-1} - 1)) = (s_m - 1)(2s_m - 1) - (s_{m-1})(2s_m - 1)$$
$$= (s_m - 1 - s_{m-1})(2s_m - 1) = (s_{m-1}^2 - 2s_{m-1})(2s_{m-1}^2 - 2s_{m-1} + 1).$$

The second factor is congruent to $1 \bmod 2(s_{m-1} - 1)$. The first is equal to $(s_{m-1} - 1)^2 - 1$, which is congruent to $-1 \bmod 2(s_{m-1} - 1)$ because $s_{m-1} - 1$ is even. The product has residue $-1 \bmod 2(s_{m-1} - 1)$. Similarly, $b = s_{m-1}^2 \equiv 1 \pmod{2(s_{m-1} - 1)}$. Therefore, we've found a subset of the weights with sum divisible by $2(s_{m-1} - 1)$, to which b is coprime. Therefore, the singularity is canonical.

At this point, we've shown that Y, and hence X, is canonical away from the three coordinate points with weights a, b, and c. The weight a divides $2d$, so the general hypersurface X misses the corresponding coordinate point and we needn't check the singularity of Y there. We'll conclude by looking at the singularities at the basepoints $[0 : \cdots : 0 : 1 : 0]$ and $[0 : \cdots : 0 : 0 : 1]$ of $|\mathcal{O}_Y(2d)|$ (corresponding to weights b and c, respectively). At $[0 : \cdots : 0 : 1 : 0]$, X has a singularity of type $\frac{1}{b}((d/s_0)^{(2)}, \dots, (d/s_{m-1})^{(2)}, c)$, where a is omitted. We claim that the sum of weights $d/s_0 + d/s_0 + d/s_{m-1} + c$ is divisible by b. Indeed,

$$d/s_0 + d/s_0 = d = s_{m-1}(s_{m-1} - 1)(2s_{m-1}^2 - 2s_{m-1} + 1) \equiv -s_{m-1} \pmod{b},$$

while $d/s_{m-1} = (s_{m-1}-1)(2s_{m-1}^2 - 2s_{m-1}+1) \equiv 3s_{m-1}-1 \pmod{b}$ and $c = (s_{m-1}-1)^2 \equiv -2s_{m-1}+1 \pmod{b}$. The sum of these residues is $0 \bmod b$. Since $\gcd(d/s_0, d/s_0, d/s_{m-1}, c, b) = 1$, this shows that the singularity is canonical.

Finally, consider the point $[0 : \cdots : 0 : 0 : 1]$. The singularity there is $\frac{1}{c}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, b)$. The sum of the first $m-1$ weights (once each), $d/s_0 + \cdots + d/s_{m-2}$, is $(s_{m-1}^2 - 2s_{m-1})(2s_{m-1}^2 - 2s_{m-1}+1)$ as calculated above. The first factor is $-1 \pmod{c}$ and the second is $-2s_{m-1}+1 \pmod{c}$, so the product is $2s_{m-1}-1 \pmod{c}$. But $b = s_{m-1}^2 \equiv -2s_{m-1}+1 \pmod{c}$, so the sum of these weights is divisible by c. As before, their gcd with c is 1, so Lemma 1.3.14 applies. This completes the proof that X is canonical.

The hypersurface in part (2) is identical except that we've split one copy of d/s_{m-1} into two copies of $d/(2s_{m-1})$. The same proof works for this case, where we may replace sums involving d/s_{m-1} with $d/(2s_{m-1}) + d/(2s_{m-1})$ where necessary. □

Proof of Theorem 3.1.1. First, consider part (1). For any dimension $n \geq 4$, let W be a resolution of singularities of the hypersurface X appearing in either Proposition 3.2.11 or 3.2.13 of the appropriate dimension. Since X is terminal, we have $\mathrm{vol}(W) = \mathrm{vol}(X) < 1/2^{2^{n/2}}$, as required. In both the examples, the last weight is 1, so the geometric genus p_g is positive (namely, 1). For $n = 3$, the example in Proposition 3.2.11 is $X_{12} \subset \mathbb{P}^4(3^{(2)}, 2^{(2)}, 1)$ with volume $1/3$, which is not good enough. Instead, we can use the 3-fold $X_{28} \subset \mathbb{P}^4(14, 5, 4, 3, 1)$, which is terminal with $K_X = \mathcal{O}_X(1)$ [BK]. This has positive geometric genus, and its volume $1/30$ is less than $1/2^{2^{n/2}}$, as we want.

To prove part (2), note that $|\ell K_W|$ does not give a birational embedding for the resolutions of examples X in Propositions 3.2.11 or 3.2.13 until ℓ is at least as large as the highest weight, since $K_X = \mathcal{O}_X(1)$. When $n = 2m+1$ is odd and $n \geq 5$, the highest weight in the example of Proposition 3.2.11 is $(s_{m+1}-1)/2 > 2^{2^{(n-2)/2}}$. When $n = 2m$ is even and $n \geq 4$, the highest weight in the example of Proposition 3.2.13 is $(s_m - 1)(2s_m - 1)/2 > 2^{2^{(n-2)/2}}$. Finally, in dimensions 2 and 3, the top weights of the examples $X_{10} \subset \mathbb{P}^3(5, 2, 1, 1)$ and $X_{28} \subset \mathbb{P}^4(14, 5, 4, 3, 1)$ have top weights 5 and 14, which are greater than $2^{2^{(2-2)/2}} = 2$ and $2^{2^{(3-2)/2}} \doteq 2.66$, respectively.

To prove part (3), let n be an integer at least 5, and let W be a resolution of singularities of the example in Proposition 3.2.14 of dimension n. Taking a resolution of singularities doesn't alter the plurigenera, and so $h^0(W, \ell K_W) = 0$ for $1 \leq \ell < 2^{2^{(n-4)/2}}$ as we want. For $2 \leq n \leq 4$, we just need to show that there is a smooth projective n-fold of general type with geometric genus p_g equal to zero. In dimension 2, various examples have been found, the first one due to Godeaux [BHP04, section VII.10]. It follows that such varieties exist in every dimension at least 2, since $p_g(X \times Y) = p_g(X)p_g(Y)$. □

3.2.4 Terminal Calabi-Yau Varieties

Just as in the canonical case, problems 4a) and 4b) have some bound in each dimension, by Birkar's results. To find terminal Calabi-Yau varieties of small volume, we can add an additional weight of 1 to the examples for problem 3a) from one dimension lower. The resulting examples have volume roughly $1/2^{2^{n/2}}$; compare our canonical Calabi-Yau examples, with volume roughly $1/2^{2^n}$. The proofs of the first two examples are identical to those of the previous section.

Proposition 3.2.15. *Let $n = 2m+2$ be an even integer at least 4, and let $d = s_{m+1} - 1$. Then a general hypersurface X of degree $2d$ in the weighted projective space $\mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_m)^{(2)}, 1^{(2)})$ is quasismooth and Calabi-Yau with terminal singularities. When $n \geq 6$, $\mathrm{vol}(\mathcal{O}_X(1)) < 1/2^{2^{n/2}}$.*

Proposition 3.2.16. *Let $n = 2m + 3$ be an odd integer at least 5, and let $d = (s_m - 1)(2s_m - 1)$. Then a general hypersurface X of degree $2d$ in the weighted projective space $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, 2(s_m - 1), (s_m - 1)^{(2)}, 1^{(2)})$ is quasismooth and Calabi-Yau with terminal singularities. When $n \geq 7$, $\mathrm{vol}(\mathcal{O}_X(1)) < 1/2^{2^{n/2}}$.*

Finally, we'll find an example for 4b), a terminal Calabi-Yau variety with an ample Weil divisor with many vanishing spaces of sections.

Proposition 3.2.17. *1. Let $n = 2m + 2$ be an even integer at least 8, and let $d = (s_m - 1)(4s_{m-2}^3 - 6s_{m-2}^2 + 5s_{m-2} - 2)$. Then a general hypersurface X of degree $2d$ in $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_{m-1})^{(2)}, s_{m-2}(2s_{m-1} - 1))^{(2)}, (2(s_{m-2} - 1)s_{m-1})^{(2)})$ is quasismooth* *and Calabi-Yau with terminal singularities. Such an X satisfies $H^0(X, \mathcal{O}_X(\ell)) = 0$ for $1 \le \ell < M$, where $M = 2(s_{m-2}-1)s_{m-1} > 2^{2^{(n-5)/2}}$.*

2. *Let $n = 2m+3$ be an odd integer at least 9, and let $d = (s_m - 1)(4s_{m-2}^3 - 6s_{m-2}^2 + 5s_{m-2} - 2)$. Then a general hypersurface X of degree $2d$ in $Y = \mathbb{P}^{n+1}((d/s_0)^{(2)}, \ldots, (d/s_{m-3})^{(2)}, d/s_{m-2}, (d/(2s_{m-2}))^{(2)}, (d/s_{m-1})^{(2)}, (s_{m-2}(2s_{m-1}-1))^{(2)}, (2(s_{m-2}-1)s_{m-1})^{(2)})$ is quasismooth and Calabi-Yau with terminal singularities. Such an X satisfies $H^0(X, \mathcal{O}_X(\ell)) = 0$ for $1 \le \ell < M$, where $M = 2(s_{m-2} - 1)s_{m-1} > 2^{2^{(n-6)/2}}$.*

Proof. We'll begin with part (1). Throughout the proof, we'll abbreviate the last weights as $a := s_{m-2}(2s_{m-1} - 1)$ and $b := 2(s_{m-2} - 1)s_{m-1}$. Since all weights are repeated twice, to show that X is well-formed, it suffices to see that a and b are coprime. This is true by the properties of Sylvester's sequence and the fact that $m \ge 3$, so that s_{m-2} is odd. All weights divide $2d$ except for a, which divides $2d - d/s_{m-1}$. Indeed, $2d - d/s_{m-1} = d(2s_{m-1} - 1)/s_{m-1}$ and s_{m-2} divides d (in particular, $s_m - 1$). There are two copies of a, but there are two copies of d/s_{m-1} as well; so the criterion for quasismoothness (Proposition 1.3.6) is satisfied.

The sum of all but the last four weights is $2d(1 - 1/(s_m - 1)) = 2d - 2(4s_{m-2}^3 - 6s_{m-2}^2 + 5s_{m-2} - 2)$, while

$$a + b = s_{m-2}(2s_{m-1} - 1) + 2(s_{m-2} - 1)s_{m-1} = 4s_{m-2}^3 - 6s_{m-2}^2 + 5s_{m-2} - 2,$$

so $K_X = \mathcal{O}_X$ and the hypersurface is Calabi-Yau.

To show that it is terminal, first note that the sum of each weight taken only once is d. Since each d/s_j divides d, the singularity

$$\frac{1}{d/s_j}(d/s_0, \ldots, \widehat{d/s_j}, \ldots, d/s_{m-1}, s_{m-2}(2s_{m-1}-1), 2(s_{m-2}-1)s_{m-1})$$

is canonical. Once we repeat each weight twice, the singularity is terminal, and so Y is terminal at the coordinate points of weight d/s_j.

We can't use this argument to get that Y is terminal at the coordinate points of weight b because b divides $2d$, but not d. However, we may use a different subset of weights. Let $w := 4s_{m-2}^3 - 6s_{m-2}^2 + 5s_{m-2} - 2$ so that $d = (s_m - 1)w = (s_{m-2} - 1)s_{m-1}s_{m-2}w$ and $d/s_{m-2} = (s_{m-2} - 1)s_{m-1}w$. Both d and d/s_{m-2} are multiples of $b/2$ by an odd number (both w and s_{m-2} are odd since $m \geq 3$), so $d - d/s_{m-2} \equiv 0 \pmod{b}$. But $d - d/s_{m-2}$ is the sum of each weight repeated once omitting d/s_{m-2}. This subset of weights appears twice in Y, so the singularities corresponding to weights b are terminal.

So far, we've proven that Y, and hence X, are terminal away from the base locus of $\mathcal{O}_Y(2d)$, which is the $\mathbb{P}^1$ corresponding to the two weights a. Regardless of which stratum of the $\mathbb{P}^1$ we consider, applying Proposition 1.3.9 leads to analyzing the singularity $\frac{1}{a}((d/s_0)^{(2)}, ..., (d/s_{m-2})^{(2)}, d/s_{m-1}, b^{(2)})$. (Looking at a zero-dimensional stratum would add another weight a, but this does not change the singularity type.)

Lemma 3.2.18. *For any integer $m \geq 3$, the quotient singularity*

$$\frac{1}{a}((d/s_0)^{(2)}, ..., (d/s_{m-2})^{(2)}, d/s_{m-1}, b^{(2)})$$

is terminal and Gorenstein.

This is the hardest case that we encounter. In this case, there is typically no nonempty proper subset of the weights whose sum is zero modulo a, which would be our most common approach to proving terminality.

Proof. The sum of the weights is zero modulo a, so the singularity is Gorenstein, and hence canonical (Lemma 1.3.11). Since a divides $2d - d/s_{m-1}$, it is also possible to write this quotient singularity as

$$\frac{1}{a}((d/s_0)^{(2)}, ..., (d/s_{m-2})^{(2)}, 2d, b^{(2)}).$$

Now, dividing the weight $2d$ into two copies of d gives

$$\frac{1}{a}((d/s_0)^{(2)}, ..., (d/s_{m-2})^{(2)}, d^{(2)}, b^{(2)}),$$

which is terminal Gorenstein. This is because a is odd, so taking a sum of only one copy of each weight gives $0 \bmod a$. Therefore, we need to show that combining the two weights d does not change the fact that our singularity is terminal. Suppose by way of contradiction that the original singularity is canonical but not terminal. Then for some i with $1 \leq i < a$,

$$2\left(ib \bmod a + \sum_{j=1}^{m-2} id/s_j \bmod a\right) + (2id) \bmod a = a.$$

For the same i, the fact that the singularity with $2d$ split is terminal means that

$$2\left(ib \bmod a + \sum_{j=1}^{m-2} id/s_j \bmod a\right) + 2(id \bmod a) = 2a.$$

The last expression must be at least $2a$ because the singularity is terminal and Gorenstein. It is at most $2a$ because combining the two d weights can lower the sum by no more than a. From now on, fix an i for which the above two equalities hold. Taking half of the second expression and setting $K = \sum_{j=1}^{m-2} id/s_j \bmod a + ib \bmod a$, we have that both $2K + (2id) \bmod a = a$ and $K + id \bmod a = a$. Using that a is odd, the first equation implies that $K < a/2$, and then the second equation gives that $id \bmod a > a/2$. In particular, $id \bmod a$ is nonzero. But $d = d/s_0 + \cdots + d/s_{m-1} + a + b$. Taking the smallest nonnegative residues modulo a of each term, we therefore have

$$id \bmod a \leq \sum_{j=1}^{m-1} id/s_j \bmod a + i(a+b) \bmod a.$$

(This expresses the fact we've used frequently that dividing a weight into multiple weights with the same sum can only increase the contribution to the Reid-Tai criterion.) The right-hand side may be rewritten as $ib \bmod a + \sum_{j=1}^{m-2} id/s_j \bmod a + (id/s_{m-1}) \bmod a = K + (id/s_{m-1}) \bmod a$, so in fact

$$id \bmod a \leq K + (id/s_{m-1}) \bmod a. \tag{3.1}$$

The two sides of (3.1) are congruent modulo a, so the only way $K < a/2$ and $id \bmod a > a/2$ can hold is if inequality 3.1 is an equality. We'll show that this can't happen.

For a start, consider the $id/s_0 = id/2$ term in K. We know that $id \bmod a$ is nonzero (because it is greater than $a/2$), so $id/2 \bmod a$ is as well. Moreover, $id/2 \bmod a$ is at least $(id \bmod a)/2$ and these can be equal only if 2 actually divides $id \bmod a$. Something similar happens with all the other terms: we must have $(id/s_j) \bmod a \geq (id \bmod a)/s_j$ for every j and an analogous statement for the term with $i(a+b) = id/(s_0 \cdots s_{m-1})$, namely $(id/(s_0 \cdots s_{m-1})) \bmod a \geq (id \bmod a)/(s_0 \cdots s_{m-1})$. In each case, the inequality is only an equality when $(id \bmod a)$ is divisible by the relevant s_j or $s_0 \cdots s_{m-1}$. Since

$$id \bmod a = (id \bmod a)(1/s_0 + \cdots + 1/s_{m-1} + 1/(s_0 \cdots s_{m-1})),$$

we must actually be in the equality case for each term. All this implies that $s_0 \cdots s_{m-1} = s_m - 1$ divides $id \bmod a$. However, $id \bmod a$ is a nonzero integer between 1 and a. Since $a = s_{m-2}(2s_{m-1} - 1) < s_m - 1$ when $m \geq 3$, this is a contradiction. □

This concludes the proof of part (1) of the proposition. The proof of (2) is nearly identical; we've split the weight d/s_{m-2} in two because two copies of d/s_{m-1} are required for quasismoothness. The bound on M comes from the fact that $b = 2(s_{m-2} - 1)s_{m-1} > s_{m-2}^3 > (2^{2^{m-3}})^3 > 2^{2^{m-(3/2)}}$. □

The examples from Propositions 3.2.15, 3.2.16, and 3.2.17 give the following theorem.

Theorem 3.2.19. 1. *For every integer $n \geq 6$, there is a terminal Calabi-Yau n-fold X with an ample Weil divisor A such that $\mathrm{vol}(A) < 1/2^{2^{n/2}}$.*

2. *For every integer $n \geq 8$, there is a terminal Calabi-Yau n-fold X with an ample Weil divisor A such that $H^0(X, \ell A) = 0$ for $1 \leq \ell < 2^{2^{(n-6)/2}}$.*

3.3 Noether-type inequalities

We now deduce from Propositions 3.2.11 and 3.2.13 that the constant a_n in a Noether-type inequality $\operatorname{vol}(X) \geq a_n p_g(X) - b_n$ (for n-folds X of general type) must be doubly exponentially small as a function of n.

Noether's inequality for surfaces of general type says that $\operatorname{vol}(X) \geq 2p_g - 4$, where the geometric genus p_g means $h^0(X, K_X)$. More generally, M. Chen and Z. Jiang showed that for every positive integer n there are positive constants a_n and b_n such that $\operatorname{vol}(X) \geq a_n p_g(X) - b_n$ for every smooth projective n-fold X of general type [CJ17, Corollary 5.1]. Strengthening earlier results, J. Chen, M. Chen, and C. Jiang recently proved a Noether inequality for 3-folds of general type, with optimal constants: we have $\operatorname{vol}(X) \geq (4/3)p_g(X) - 10/3$ if $p_g(X) \geq 11$ [CCJ20].

In high dimensions, no explicit constants in Noether's inequality are known, although they are related to lower bounds for the volume in lower dimensions. Using that relation, we now show that the constant a_n must tend rapidly to zero with n. Our argument uses the product of a given variety with curves of high genus, as suggested by J. Chen and C.-J. Lai [CL20, Example 1.4].

Theorem 3.3.1. *For every integer $n \geq 5$, there is a sequence of smooth complex projective n-folds of general type with $p_g \to \infty$ and $\operatorname{vol}/p_g < 1/2^{2^{n/2}}$.*

Proof. Let Z be the variety of dimension $n-1$ with $p_g(Z) > 0$ given by Propositions 3.2.11 and 3.2.13. For a smooth projective curve C of genus $g \geq 2$, we have $p_g(Z \times C) = g\, p_g(Z)$ and $\operatorname{vol}(Z \times C) = n(2g-2)\operatorname{vol}(Z)$. Therefore, taking a sequence of curves C with genera going to infinity, the n-folds $Z \times C$ have $p_g \to \infty$ and $\operatorname{vol}/p_g \to 2n\operatorname{vol}(Z)/p_g(Z) \leq 2n\operatorname{vol}(Z)$. For $n \geq 6$, this is less than $1/2^{2^{n/2}}$, as we want. For $n = 5$, the 4-fold Z is a resolution of $X_{20} \subset \mathbb{P}^5(5^{(2)}, 4, 2^{(2)}, 1)$, with volume $1/20$, which is not good enough. We can instead use the 4-fold $X_{64} \subset \mathbb{P}^5(19, 16, 11, 9, 7, 1)$, which is terminal with $K_X = \mathcal{O}_X(1)$ [BK]. This variety has $p_g > 0$ and volume $4/13167$, which is good enough to imply the theorem for $n = 5$. □

CHAPTER 4

Large Index

4.1 Introduction

We'll call a normal projective variety X *Calabi-Yau* if its canonical divisor K_X is $\mathbb{Q}$-linearly equivalent to zero. The smallest positive integer m with mK_X linearly equivalent to zero is called the *index* of X. A major conjecture on the classification of Calabi-Yau varieties predicts that, under suitable assumptions on singularities, the index is bounded in each dimension. More generally, the following conjecture predicts an index bound for Calabi-Yau pairs (X, D); such a pair consists of a normal projective variety X and an effective $\mathbb{Q}$-divisor D on X such that $K_X + D \sim_{\mathbb{Q}} 0$. Conjectures in this direction go back to Alexeev and M^{c}Kernan-Prokhorov, and Y. Xu gave a formulation close to what follows [Ale94], [MP04, Conjecture 3.8], [Xu19].

Conjecture 4.1.1 (Index Conjecture)**.** *Let n be a positive integer and $I \subset [0, 1]$ a set of rational numbers that satisfies the descending chain condition* (DCC)*. Then there is a positive integer $c(n, I)$ with the following property. Let (X, D) be a complex klt Calabi-Yau pair of dimension n such that the coefficients of D belong to I. Then $c(n, I)(K_X + D) \sim 0$.*

In dimension at most 3, the conjecture is true, and a similar boundedness statement holds even for slc Calabi-Yau pairs [Xu19, Theorem 1.13], [JL20, Corollary 1.6]. For klt Calabi-Yau pairs (X, D) with $D \neq 0$, the conjecture also holds in dimension 4 [Xu19, Theorem 1.14]. An important result in any dimension is that all the Calabi-Yau pairs in Conjecture 4.1.1 (with given dimension and coefficients in a given DCC set I) actually have coefficients only in some finite subset of I, by Hacon-M^{c}Kernan-Xu [HMX14, Theorem 1.5].

For several classes of Calabi-Yau varieties or pairs, we construct examples which we conjecture have the largest index in each dimension, with supporting evidence in low dimensions. In all cases, it's possible to express the examples as certain quotients of weighted projective hypersurfaces by finite groups. In our examples, the index grows doubly exponentially with dimension, and may be written in terms of Sylvester's sequence, similarly to the results of Chapter 3.

First, we produce terminal Calabi-Yau varieties with large index. The key idea is to apply mirror symmetry (Remark 4.3.8) to our Calabi-Yau varieties with an ample Weil divisor of small volume from Proposition 3.2.1.

Theorem 4.1.2 (Corollary 4.4.1)**.** *For each positive integer n, there is a complex terminal Calabi-Yau n-fold with index $(s_{n-1}-1)(2s_{n-1}-3)$. In particular, this is larger than $2^{2^{n-1}}$.*

This example should have the largest possible index among all canonical Calabi-Yau varieties of dimension n, and also among all terminal Calabi-Yau varieties of dimension n (Conjecture 4.4.2). For example, our construction gives a known Calabi-Yau 3-fold of index 66, the largest possible [MO98, Corollary 5], and a new terminal Calabi-Yau 4-fold of index 3486.

Next, we turn to klt Calabi-Yau pairs (X, D) with standard coefficients, meaning that all coefficients of D are of the form $1-\frac{1}{b}$ for positive integers b. Pairs with standard coefficients naturally arise as quotients by finite groups of varieties Y with $\mathbb{Q}$-Cartier canonical class, in the sense that $\pi\colon Y \to X$ has $K_Y = \pi^*(K_X + D)$. For Calabi-Yau pairs, the index can be somewhat bigger than for varieties. (In fact, the example in Theorem 4.1.2 is based on the same construction used to define the pair below.) Our example has a simple description as a pair on a weighted projective space; it can also be viewed as a hypersurface in an ill-formed fake weighted projective space.

Theorem 4.1.3 (Theorem 4.3.3)**.** *For each positive integer n, there is a complex klt Calabi-Yau pair of dimension n with standard coefficients that has index $(s_n-1)(2s_n-3)$. In particular, this is larger than 2^{2^n}.*

This should be the largest possible index among such pairs (Conjecture 4.3.4).

Jihao Liu recently constructed klt Calabi-Yau pairs with small minimal log discrepancy (mld) [Liu22, Remark 2.6]. We write out the details as Theorem 4.3.1. This is related to several examples by Kollár [Kol97, Example 8.16], [Kol94]. By definition, the klt property for a pair means that the mld is positive. It follows from Hacon-M^{c}Kernan-Xu's work that there is a positive lower bound for the mld's of all klt Calabi-Yau pairs with standard coefficients in a given dimension (Proposition 4.2.1). Liu's example is conjectured to achieve the minimum (Conjecture 4.3.2).

The pairs in Theorem 4.1.3 and Theorem 4.3.1 above may be viewed as quotients of certain Calabi-Yau weighted projective hypersurfaces. These are also notable for their extreme topological properties. We prove that the total rank of the orbifold cohomology of these hypersurfaces grows doubly exponentially with dimension.

Theorem 4.1.4 (Theorem 4.5.1(ii))**.** *For every positive integer n, there is a quasismooth Calabi-Yau hypersurface of dimension n whose sum of orbifold Betti numbers is*

$$H = 2(s_0 - 1) \cdots (s_n - 1).$$

In particular, this is larger than 2^{2^n} for $n > 1$.

This should be the largest possible sum of orbifold Betti numbers among all projective varieties with quotient singularities and trivial canonical class (Conjecture 4.5.3).

Finally, we conjecture the largest index for a klt Calabi-Yau variety (rather than a pair) in each dimension (Conjecture 4.7.10). We prove that conjecture in dimension 2: the largest index of a klt Calabi-Yau surface is 19 (Proposition 4.6.1). Also, the smallest mld of a klt Calabi-Yau surface is $\frac{1}{13}$.

4.2 Notation and Background

A *fake weighted projective space* means a projective toric variety whose divisor class group has rank 1, or equivalently the projective toric variety associated to a lattice simplex [Kas09]. Such a variety is the quotient of a weighted projective space by a finite abelian group.

Recall the following properties of Sylvester's sequence needed for our examples; a fuller explanation of the properties of Sylvester's sequence appeared at the beginning of Section 3.2. The sequence is defined by $s_0 = 2$ and $s_n = s_{n-1}(s_{n-1} - 1) + 1$ for $n \geq 1$. The first few terms are 2, 3, 7, 43, 1807. We have $s_n > 2^{2^{n-1}}$ for all n. Also, $s_n = s_0 \cdots s_{n-1} + 1$, and hence the numbers in the sequence are pairwise coprime. Finally, the key point for our applications is that the sum of the reciprocals tends very quickly to 1. Namely:

$$\frac{1}{s_0} + \frac{1}{s_1} + \cdots + \frac{1}{s_{n-1}} = 1 - \frac{1}{s_n - 1}.$$

Recall that a pair (X, D) is a normal variety X together with an effective $\mathbb{Q}$-divisor D such that $K_X + D$ is $\mathbb{Q} - Cartier$. By definition, a pair (X, D) is klt if and only if its mld is positive (the definitions of these terms were given in Section 1.2). Furthermore, in each dimension, there is a positive lower bound for the mld of every klt Calabi-Yau pair with standard coefficients. More generally:

Proposition 4.2.1. *Let n be a positive integer and $I \subset [0, 1]$ a DCC set. Then there is a positive number ϵ such that every klt Calabi-Yau pair (X, D) of dimension n with coefficients of D in I has mld at least ϵ.*

Proof. We repeat the argument of [CCH21, Lemma 3.13] for pairs rather than varieties. Suppose by way of contradiction that there is a sequence of klt Calabi-Yau pairs (X_i, D_i) of dimension n with coefficients in I such that the sequence $\epsilon_i := \mathrm{mld}(X_i, D_i)$ converges to 0. Replacing this with a subsequence, we may suppose that the sequence ϵ_i is decreasing and that each term is less than 1. For each positive integer i, define a new klt pair as follows: choose a point $x_i \in X_i$ with $\mathrm{mld}_{x_i}(X_i, D_i) = \epsilon_i$. If the closure Y of x_i has codimension 1 in X_i, then Y has coefficient $1 - \epsilon_i$ in D_i. In this case, let (X'_i, D'_i) be (X_i, D_i), i.e., leave the original pair unchanged. If the point $x_i \in X_i$ has codimension greater than 1, then choose an exceptional divisor E_i over X with log discrepancy ϵ_i. Since ϵ_i is less than 1, [Kol97, Corollary 1.39] gives that there is a projective birational morphism $\mu \colon X'_i \to X_i$ for which X'_i is $\mathbb{Q}$-factorial and E_i is the only exceptional divisor. Let $D'_i = \mu_*^{-1} D_i + (1 - \epsilon_i) E_i$. Then

(X_i', D_i') is a new klt pair which is still Calabi-Yau. Indeed,

$$K_{X_i'} + D_i' = \mu^*(K_{X_i} + D_i) \sim_{\mathbb{Q}} 0.$$

Thus, for every positive integer i, we've constructed a klt Calabi-Yau pair of dimension n that includes the coefficient $1 - \epsilon_i$. The coefficients of these pairs are in the union J of the set I with the numbers $1 - \epsilon_i$. Since J satisfies the descending chain condition, this conclusion contradicts Hacon-McKernan-Xu's result [HMX14, Theorem 1.5]. ☐

For a klt Calabi-Yau pair (X, D) with standard coefficients, let m be the index of (X, D). Then the (global) *index-1 cover* of (X, D) is a projective variety Y with canonical Gorenstein singularities such that the canonical class K_Y is linearly equivalent to zero [Kol97, Example 2.47, Corollary 2.51]. Here (X, D) is the quotient of Y by an action of the cyclic group μ_m such that μ_m acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$. (Explicitly, D has coefficient $1 - \frac{1}{b}$ on the image of an irreducible divisor on which the subgroup of μ_m that acts as the identity has order b.)

Finally, we note the existence of equivariant terminalizations.

Proposition 4.2.2. *Let a finite group G act on a complex variety X. Then there is a G-equivariant projective birational morphism $\pi\colon Y \to X$ with Y terminal and K_Y nef over X. If X is canonical, then $K_Y = \pi^*(K_X)$.*

Proof. Using a canonical resolution procedure, there is a G-equivariant resolution of singularities $Z \to X$ [Kol07, section 3.4.1]. By the method of Birkar-Cascini-Hacon-McKernan, we can run a G-equivariant minimal model program for Z over X [BCH10], [Pro21, section 4.3]. That gives a G-equivariant birational contraction $Z \dashrightarrow Y$ over X (meaning that $Y \to X$ is a birational morphism) with Y terminal, Y projective over X, and K_Y nef over X. If X is canonical, then $K_Y = \pi^*(K_X) + \sum a_i E_i$ for some rational numbers $a_i \geq 0$. Since K_Y is nef over X, we have $a_i = 0$ for all i by the negativity lemma [KM98, Lemma 3.39]. ☐

Without the group action, we can also arrange that Y is $\mathbb{Q}$-factorial, but that is generally impossible in this equivariant setting. A variety Y produced by Proposition 4.2.2 is "$G\mathbb{Q}$-

factorial", meaning that every G-invariant Weil divisor is $\mathbb{Q}$-Cartier; but that does not imply that Y is $\mathbb{Q}$-factorial [Pro21, Example 1.1.1].

4.3 Calabi-Yau pairs with small mld or large index

In this section, we'll construct the pair used to prove Theorem 4.1.3. For completeness, we first present Theorem 4.3.1, due to Liu [Liu22, Remark 2.6].

Theorem 4.3.1. *For each positive integer n, there is a complex klt Calabi-Yau pair of dimension n with standard coefficients whose mld is $1/(s_{n+1}-1)$. In particular, this is less than $1/2^{2^n}$.*

Proof. Kollár constructed a klt pair (X, D) with standard coefficients such that $K_X + D$ is ample and $K_X + D$ has small volume, conjecturally the smallest in each dimension [Kol94], [HMX13, Introduction]. A small change gives Liu's Calabi-Yau pair. Namely, define a pair (X, D) by

$$(X, D) \coloneqq \left(\mathbb{P}^n, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \cdots + \frac{s_n - 1}{s_n}H_n + \frac{s_{n+1} - 2}{s_{n+1} - 1}H_{n+1}\right), \tag{4.1}$$

where $H_0, \ldots, H_{n+1}$ are $n + 2$ general hyperplanes in $\mathbb{P}^n$. (Here we have changed the last coefficient of Kollár's pair.) Then (X, D) is klt, it has standard coefficients, and $K_X + D$ is $\mathbb{Q}$-linearly equivalent to zero. It is clear from the last coefficient that (X, D) has mld $1/(s_{n+1} - 1)$, as we want. □

Conjecture 4.3.2. *Let (X, D) be the pair defined in (4.1). Then the mld $1/(s_{n+1} - 1)$ of (X, D) is the smallest possible among all klt Calabi-Yau pairs with standard coefficients of dimension n.*

In dimension 1, the example above is the klt Calabi-Yau pair $(\mathbb{P}^1, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{5}{6}H_2)$, with mld $\frac{1}{6}$, and in dimension 2 we have $(\mathbb{P}^2, \frac{1}{2}H_0 + \frac{2}{3}H_1 + \frac{6}{7}H_2 + \frac{41}{42}H_3)$, with mld $\frac{1}{42}$. Conjecture 4.3.2 is true in these cases; in dimension 2, we prove this as Proposition 4.6.8.

The klt Calabi-Yau pair (X, D) above has index $s_{n+1} - 1$, which is quite good (doubly exponential in n); but we now give a better example for that problem. Namely, we produce a klt Calabi-Yau pair with standard coefficients that has conjecturally maximal index. We first give a simple construction: the properties are easy to check, but the origin of the example is hidden. Our pair (X, D) has index 6 in dimension 1, 66 in dimension 2, and 3486 in dimension 3.

We found this example by applying mirror symmetry (Remark 4.3.8) to our canonical Calabi-Yau variety of conjecturally minimal volume from Proposition 3.2.1. With that approach, our pair arises as the quotient of a Calabi-Yau hypersurface in weighted projective space by the action of a cyclic group, or equivalently as a hypersurface in an ill-formed fake weighted projective space. We give those descriptions later in this section.

Theorem 4.3.3. *For an integer $n \geq 2$, let $d = 2s_n - 2$ and let X be the weighted projective space $\mathbb{P}^n(d^{(n-1)}, d-1, 1)$, with coordinates $y_1, \ldots, y_{n+1}$. For $1 \leq i \leq n$, let D_i be the divisor $\{y_i = 0\}$ on X. Let D_0 be the divisor $\{y_1 + \cdots + y_{n-1} + y_n y_{n+1} + y_{n+1}^d = 0\}$ on X. Let*

$$D = \frac{1}{2}D_0 + \frac{2}{3}D_1 + \cdots + \frac{s_{n-1}-1}{s_{n-1}}D_{n-1} + \frac{d-2}{d-1}D_n.$$

Then (X, D) is a klt Calabi-Yau pair of dimension n with standard coefficients and with index $(s_n - 1)(2s_n - 3)$.

For $n = 1$, let $X = \mathbb{P}^1$ with 3 distinct complex points p_1, p_2, p_3. Then $(X, \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3)$ is a klt Calabi-Yau pair with standard coefficients and with index $(s_1 - 1)(2s_1 - 3) = 6$.

Proof. The statement for $n = 1$ is straightforward, so assume that $n \geq 2$. Then X is a well-formed weighted projective space. We have $K_X = O_X(-(n-1)d - (d-1) - 1) = O_X(-nd)$. Also, we have linear equivalences $D_i \sim O_X(d)$ for $0 \leq i \leq n-1$, and $D_n \sim O_X(d-1)$. So

$K_X + D$ is $\mathbb{Q}$-linearly equivalent to $O_X(a)$, where

$$\begin{aligned}
a &= -nd + d - 2 + \sum_{i=0}^{n-1} d\left(1 - \frac{1}{s_i}\right) \\
&= -2 + d\left(1 - \sum_{i=0}^{n-1} \frac{1}{s_i}\right) \\
&= -2 + \frac{d}{s_n - 1} \\
&= 0.
\end{aligned}$$

So (X, D) is a Calabi-Yau pair. The coefficients of D are standard.

Since K_X and the divisors D_i are integral multiples of $O_X(1)$ in the divisor class group of X, the index of (X, D) is the least common multiple of the denominators of the coefficients of D. Since the numbers in Sylvester's sequence are pairwise coprime, this lcm is $\text{lcm}(s_n - 1, d - 1) = \text{lcm}(s_n - 1, 2s_n - 3) = (s_n - 1)(2s_n - 3)$, as we want.

It remains to show that (X, D) is klt. Since the klt property is preserved under taking quotients by finite groups which act freely in codimension 1 [Kol97, Corollary 2.43], it suffices to show this in the coordinate chart $y_i = 1$ for each i. Equivalently, it suffices to show that the affine cone $(\mathbb{A}^{n+1}, F)$ over (X, D) is klt outside the origin. Here F is a linear combination of irreducible divisors $F_0, \ldots, F_n$ with coefficients less than 1, and so it suffices to show that $F_0, \ldots, F_n$ are smooth and transverse outside the origin. This is clear for $F_1 = \{y_1 = 0\}, \ldots, F_n = \{y_n = 0\}$.

Next, $F_0 = \{y_1 + \cdots + y_{n-1} + y_n y_{n+1} + y_{n+1}^d = 0\}$ in $\mathbb{A}^{n+1}$. All intersections of subsets of $F_0, \ldots, F_n$ are smooth of the expected dimension except possibly for $F_0 \cap \cdots \cap F_{n-1}$ and $F_0 \cap \cdots \cap F_n$. Here $F_0 \cap \cdots \cap F_{n-1}$ (as a scheme) is the curve $y_n y_{n+1} + y_{n+1}^d = 0$ in $\mathbb{A}^2$, which is smooth outside the origin, as we want. And $F_0 \cap \cdots \cap F_n$ is the origin, as a set. So all intersections are transverse outside the origin in $\mathbb{A}^{n+1}$, and hence (X, D) is klt. □

This implies Theorem 4.1.3, using that $s_n > 2^{2^{n-1}}$ for all n. We conjecture that this is the example of largest index.

Conjecture 4.3.4. *Let (X, D) be the pair in Theorem 4.3.3. Then the index $(s_n - 1)(2s_n - 3)$ of (X, D) is the largest possible index among all klt Calabi-Yau pairs of dimension n with standard coefficients.*

Proposition 4.3.5. *Conjecture 4.3.4 is true in dimensions at most 2. Moreover, there is exactly one klt Calabi-Yau pair with standard coefficients that has index 6 in dimension 1 (up to isomorphism), and there are exactly two (related by a blow-up) of index 66 in dimension 2.*

Proof. This is elementary in dimension 1. (The index-1 cover of (X, D) is the unique elliptic curve over $\mathbb{C}$ with automorphism group of order 6.) So let (X, D) be a klt Calabi-Yau pair of dimension 2 with standard coefficients and index m. Let Y be the index-1 cover of (X, D) (section 4.2). Thus Y is a projective surface with canonical singularities such that K_Y is trivial. Here (X, D) is the quotient of Y by an action of the cyclic group μ_m which is purely non-symplectic, meaning that μ_m acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$.

Let Z be the minimal resolution of Y; then K_Z is trivial, and so Z is an abelian surface or a K3 surface. The action of μ_m lifts to Z by Proposition 4.2.2. Clearly the action of μ_m on Z is still purely non-symplectic. When Z is a K3 surface, Nikulin showed that $m \leq 66$, and Machida and Oguiso showed that for $m = 66$, Z and the action of μ_{66} are unique up to isomorphism and automorphisms of μ_{66} [MO98, Main Theorem 1]. When Z is an abelian surface, μ_m acts faithfully on $H^0(Z, K_Z) \subset H^2(Z, \mathbb{C}) = \Lambda^2 H^1(Z, \mathbb{C})$, so it acts faithfully on $H^1(Z, \mathbb{Z}) \cong \mathbb{Z}^4$, which implies that $m \leq 12$.

There remains the problem of classifying the contractions $Z \to Y$ of the K3 surface Z above. Machida and Oguiso show that Z has Picard rank 2, with cone of curves spanned by one (-2)-curve C and one elliptic curve with self-intersection zero [MO98, section 2]. It follows that C is the only (-2)-curve in Z. So the only canonical K3 surfaces with minimal resolution Z are Z itself and the surface Y with a node obtained by contracting C. So a klt Calabi-Yau pair with standard coefficients, dimension 2, and index 66 is isomorphic to Z/μ_{66} or to Y/μ_{66}. The surface X of Theorem 4.3.3 is the latter one, since its index-1 cover (Proposition 4.3.6) has a node. □

We now describe the example of Theorem 4.3.3 as a quotient of a hypersurface in weighted projective space.

Proposition 4.3.6. *For a positive integer n, let $d := 2s_n - 2 = 2s_0 \cdots s_{n-1}$. Then the hypersurface X' of degree d in the weighted projective space $Y' := \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ defined by the equation $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1}x_{n+1} + x_{n+1}^d = 0$ is quasismooth of dimension n, canonical, and has $K_{X'}$ linearly equivalent to zero.*

Proof. Since the weighted projective space Y' has two weights equal to 1, it is well-formed. One can verify that the hypersurface X' is quasismooth, meaning that its affine cone is smooth outside the origin in $\mathbb{A}^{n+2}$. The sum of the weights is $d/s_0 + \cdots + d/s_{n-1} + 2 = d(1 - 1/(s_n - 1)) + 2 = d$, and so $K_{X'}$ is linearly equivalent to zero. Since X' is quasismooth, it has quotient singularities, hence is klt. Since $K_{X'}$ is linearly equivalent to zero, it is Cartier. It follows that X' is canonical. □

Proposition 4.3.7. *With the same notation as above, let $m := (s_n - 1)(2s_n - 3)$ and let the group μ_m of mth roots of unity act on the weighted projective space Y' as follows. For any $\zeta \in \mu_m$,*

$$\zeta[x_0 : \cdots : x_{n+1}] = [\zeta^{d/(2s_0)}x_0 : \zeta^{d/(2s_1)}x_1 : \cdots : \zeta^{d/(2s_{n-1})}x_{n-1} : x_n : \zeta^{d/2}x_{n+1}]. \tag{4.2}$$

The hypersurface X' is invariant under this group action. The quotient of X' by μ_m is a klt Calabi-Yau pair (X, D) with standard coefficients and index m, isomorphic to the pair in Theorem 4.3.3.

We omit the proof, since this is just a different description of the same example.

Remark 4.3.8.

1. The index $m := (s_n - 1)(2s_n - 3) = (d - 1)d/2$ in this example coincides with the degree of the hypersurface which we conjecture is the canonical Calabi-Yau n-fold with an ample Weil divisor of minimum volume (see Conjecture 3.2.2). Indeed, the hypersurface defined in Proposition 4.3.6 is related to the small volume example by mirror symmetry. Specifically, the hypersurface X' in Proposition 4.3.6 above is the Berglund-Hübsch-Krawitz (BHK) mirror of the hypersurface

$$\widehat{X'_m} \subset \mathbb{P}(m/s_0, \ldots, m/s_{n-1}, s_n - 1, s_n - 2)$$

of degree m defined by the equation $x_0^2 + x_1^3 + \cdots + x_{n-1}^{s_n-1} + x_n^{d-1} + x_n x_{n+1}^d = 0$. That is, the equations for X' and $\widehat{X'}$, each with exactly $n+2$ monomials in $n+2$ variables, are the transposes of each other, in terms of the explicit BHK recipe for mirror symmetry [ABS14, Section 2.3].

2. Section 4.5 describes the extreme topological behavior of these Calabi-Yau hypersurfaces.

3. Another striking feature of Proposition 4.3.6 is that the weighted projective space $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ has the largest anticanonical volume of any canonical toric Fano $(n+1)$-fold [BKN, Corollary 1.3].

For possible future use, we give one last description of the klt Calabi-Yau pair (X, D) of Theorem 4.3.3, now as a hypersurface in an ill-formed fake weighted projective space. (Ill-formedness means that we have a toric pair, not just a toric variety.) It is convenient to use the language of toric Deligne-Mumford stacks [BCS05]. Just as there is a projective $\mathbb{Q}$-factorial toric variety associated to a complete simplicial fan in a lattice N, we can associate a toric Deligne-Mumford stack to the data of a *stacky fan* $\mathbf{\Sigma}$. A stacky fan $\mathbf{\Sigma}$ is a triple (N, Σ, β), where N is a finitely generated abelian group, Σ is a complete simplicial fan in $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ with r rays, and β is a set of r elements of N that span the rays of Σ. (They need not be the smallest lattice points on these rays.) In our case, $N \cong \mathbb{Z}^{n+1}$ is torsion-free, which means that the associated stack has trivial generic stabilizer. If the chosen point of a given ray ρ is a positive integer b times the smallest lattice point, then the associated toric stack has stabilizer group μ_b along the irreducible divisor D associated to ρ, and the associated pair has coefficient $(b-1)/b$ along D.

Define a stacky fan $\mathbf{\Sigma}$ using the following distinguished points generating rays of a simplicial fan in $N \cong \mathbb{Z}^{n+1}$, where $e_0, \ldots, e_n$ is a basis for N:

$$v_i = \begin{cases} s_i e_i, & i = 0, \ldots, n-1, \\ (d-1)e_n, & i = n, \\ -d(e_0 + \cdots + e_{n-1}) - (d-1)e_n, & i = n+1. \end{cases}$$

Most of these lattice points are not primitive on the respective rays, hence the stacky behavior. Note also that $(d/s_0)v_0 + \cdots (d/s_{n-1})v_{n-1} + v_n + v_{n+1} = 0$. Therefore, these vectors define the weighted projective space $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$ in the lattice spanned by $v_0, \ldots, v_{n+1}$. The resulting stack will thus be a quotient of this weighted projective space, namely the quotient in Proposition 4.3.7. The pair (X, D) is the pair associated to a hypersurface in this stack.

4.4 Terminal Calabi-Yau varieties of large index

Building on the klt Calabi-Yau pair of Theorem 4.3.3, we now construct terminal Calabi-Yau varieties with large index, conjecturally optimal.

Corollary 4.4.1. *Let n be an integer at least 2. Let X' be the Calabi-Yau hypersurface of dimension $n-1$ with an action of the cyclic group of order $m := (s_{n-1}-1)(2s_{n-1}-3)$ from Proposition 4.3.7. Let Z be a μ_m-equivariant terminalization of X' (Proposition 4.2.2). Let E be a smooth elliptic curve, and let $p \in E(\mathbb{C})$ be a point of order m. Let μ_m act on $Z \times E$ by*

$$\zeta(x, y) = (\zeta(x), y + p),$$

where ζ is a primitive mth root of unity. Then the quotient $S = (Z \times E)/\mu_m$ is a terminal Calabi-Yau n-fold that has index m.

Proof. The variety X' is canonical, by Proposition 4.3.6. So $K_Z = \pi^*(K_{X'})$ by Proposition 4.2.2, and hence Z is a terminal projective variety with K_Z trivial. Since E is a complex elliptic curve, it has a point p of order m. Translation by an non-identity point has no fixed points on E, so μ_m acts freely on E. It follows that μ_m acts freely on $Z \times E$, and so the quotient $S = (Z \times E)/\mu_m$ is a Calabi-Yau variety (rather than a pair). Since Z is terminal and μ_m acts freely on $Z \times E$, the quotient $S = (Z \times E)/\mu_m$ is also terminal.

It remains to see that S has index m. The group μ_m acts trivially on $H^0(E, K_E)$ since global sections of K_E are translation-invariant. Therefore, an element of μ_m acts trivially on $H^0(Z \times E, K_{Z\times E}) = H^0(Z, K_Z) \otimes_{\mathbb{C}} H^0(E, K_E)$ if and only if it acts trivially on $H^0(Z, K_Z)$. So the index of S is equal to the index of the quotient stack $[Z/\mu_m]$ (or the associated klt pair), namely m. □

We conjecture that this example is optimal.

Conjecture 4.4.2. *Let n be an integer at least 2 and let X be the Calabi-Yau variety of Corollary 4.4.1. Then the index $m = (s_{n-1} - 1)(2s_{n-1} - 3)$ of X is the largest possible index among all terminal Calabi-Yau varieties of dimension n, and also among all canonical Calabi-Yau varieties of dimension n.*

This conjecture holds in dimensions at most 3. Indeed, in dimension 2, terminal singularities are smooth, and the largest possible index of a smooth Calabi-Yau surface is $6 = (s_1 - 1)(2s_1 - 3)$, which occurs for a "bielliptic" surface $(W \times E)/\mu_6$ as above [Bea96, Corollary VIII.7]. Likewise, the largest possible index of a canonical or terminal Calabi-Yau 3-fold is $66 = (s_2 - 1)(2s_2 - 3)$ [MO98, Corollary 5].

For n at most 3, there is in fact a *smooth* Calabi-Yau n-fold with index $(s_{n-1}-1)(2s_{n-1}-3)$ [MO98, Corollary 5]. To see this in dimension 3, let X' be the canonical Calabi-Yau surface from Proposition 4.3.7 with an action of μ_{66}. Let $Z \to X'$ be a μ_{66}-equivariant terminalization of X' (Proposition 4.2.2). In dimension 2, Z is smooth and unique, known as the minimal resolution of X'. Also, $K_Z = \pi^*(K_{X'})$, and so K_Z is trivial (more precisely, Z is a smooth K3 surface). Then the quotient $(Z \times E)/\mu_{66}$ as in Corollary 4.4.1 is a smooth Calabi-Yau 3-fold of index 66. It would be interesting to construct smooth Calabi-Yau varieties of large index in high dimensions.

4.5 Calabi-Yau varieties with large Betti numbers

The Calabi-Yau hypersurfaces in section 4.3 are also notable for their topological properties. In dimension 3, the family of hypersurfaces $X'_{84} \subset \mathbb{P}^4(42, 28, 12, 1, 1)$ of Proposition 4.3.6 is an extreme in Kreuzer and Skarke's famous list of Calabi-Yau 3-folds [KS00, section 3]. (They found more than 400 million canonical 3-folds with trivial canonical class.) In particular, a crepant resolution of X' has the most negative known Euler characteristic, -960. (The special hypersurface in Proposition 4.3.6 with an action of μ_{3486} seems to be new, though.) The BHK mirror to the special hypersurface X' is the small-volume example

$$\widehat{X'_{3486}} \subset \mathbb{P}^4(1743, 1162, 498, 42, 41).$$

That has the largest known Euler characteristic among K-trivial 3-folds, 960 (for a crepant resolution).

A third hypersurface, $X'_{1806} \subset \mathbb{P}^4(903, 602, 258, 42, 1)$, has the same sum of Betti numbers (of a crepant resolution) as in the previous two examples, namely 1008. All three of these hypersurfaces are visible at the top of the graph of all pairs of Hodge numbers from Kreuzer and Skarke's list, shown in the figure below [Can13].

In this section, we'll show that these observations in dimension 3 fit into a pattern that holds for all dimensions n.

In higher dimensions, we do not know whether our weighted projective hypersurfaces have crepant resolutions. Instead, we'll consider Chen-Ruan's *orbifold cohomology* for these hypersurfaces [CR04]. For a projective variety with Gorenstein quotient singularities, orbifold cohomology (as a Hodge structure) agrees with the cohomology of a crepant resolution if one exists [Yas04, Corollary 1.5]. Assuming Gorenstein quotient singularities, the orbifold Hodge numbers also agree with Batyrev's *stringy Hodge numbers* [Yas04, Remark 1.4].

We'll make use of the following formula for orbifold Hodge numbers, building on Vafa's ideas (see [Bat20, (4.2)]). Given a quasismooth hypersurface X of degree d in

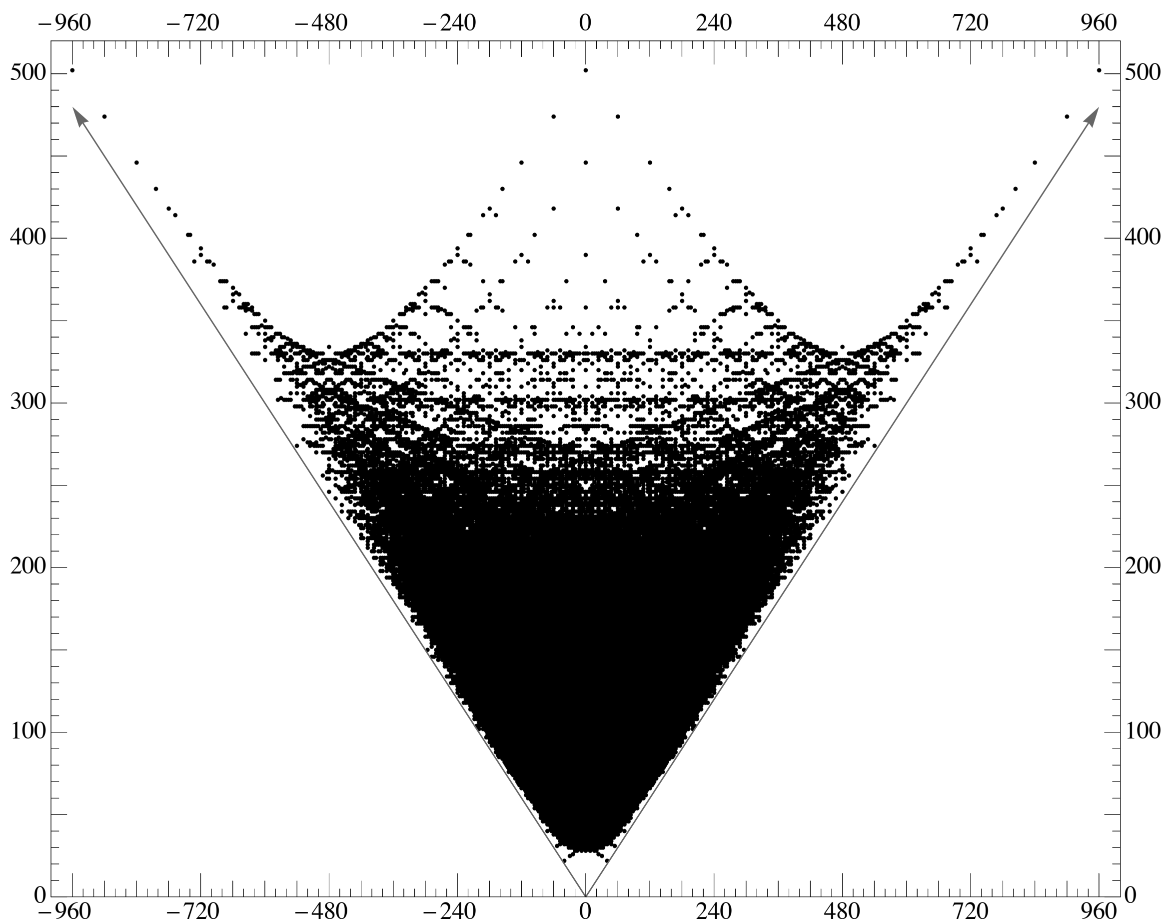

Figure 4.1: Hodge numbers of the Calabi-Yau 3-folds found by Kreuzer and Skarke; x-axis is $\chi = 2(h^{1,1} - h^{2,1})$, y-axis is $h^{1,1} + h^{2,1}$.

$\mathbb{P}(a_0, \ldots, a_{n+1})$, where $a_0 + \cdots + a_{n+1} = d$, we write: $q_i := a_i/d$ and $\tilde{\theta}_i(\ell)$ is the fractional part of $\ell a_i/d$. Then the orbifold Hodge number $h^{p,q}_{\text{orb}}$ of the mirror of X is the coefficient of $t^p \bar{t}^q$ in the expression

$$P(t, \bar{t}) = \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} \frac{1-(t\bar{t})^{1-q_i}}{1-(t\bar{t})^{q_i}} \prod_{\tilde{\theta}_i(\ell)\neq 0} (t\bar{t})^{\frac{1}{2}-q_i} \left(\frac{t}{\bar{t}}\right)^{\tilde{\theta}_i(\ell)-\frac{1}{2}} \right]. \tag{4.3}$$

In [Bat20], $P(t, \bar{t})$ is defined as the part of the above expression with integer exponents, but it will be useful to keep track of the parts with fractional exponents as well.

For each dimension n, there are three Calabi-Yau hypersurfaces which we expect to have the "most extreme cohomology":

$$\begin{aligned}
X_1^{(n)} &:= \{x_0^{s_0} + x_1^{s_1} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{d-1} x_{n+1} + x_{n+1}^d = 0\} \\
&\subset \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1) \text{ of degree } d = 2s_n - 2, \\
X_2^{(n)} &:= \{x_0^{s_0} + \cdots + x_n^{s_n} + x_{n+1}^d = 0\} \\
&\subset \mathbb{P}(d/s_0, \ldots, d/s_n, 1) \text{ of degree } d = s_{n+1} - 1, \text{ and} \\
X_3^{(n)} &:= \{x_0^{s_0} + x_1^{s_1} + \cdots + x_{n-1}^{s_{n-1}} + x_n^{2s_n-3} + x_n x_{n+1}^{2s_n-2} = 0\} \\
&\subset \mathbb{P}(d/s_0, \ldots, d/s_{n-1}, s_n - 1, s_n - 2) \text{ of degree } d = (s_n - 1)(2s_n - 3).
\end{aligned}$$

Note that any quasismooth hypersurface with the same degree and weights as one of the examples above would have identical cohomology, but we pick specific hypersurfaces to highlight the connection to previous sections. The first and third examples form a mirror pair and are the source of our large index and small volume results, respectively (see Remark 4.3.8). Note that the hypersurfaces of degree $d = s_{n+1} - 1$ in $\mathbb{P}(d/s_0, \ldots, d/s_n, 1)$ appeared in Lemma 3.2.10 as a building block of our more complicated small volume examples. The specific Fermat hypersurface $X_2^{(n)}$ is self-mirror and can be viewed as the source of our conjecturally optimal small mld example. Indeed, the pair (X, D) in 4.1 is the quotient of this hypersurface by an action of $\mu_{s_{n+1}-1}$.

Theorem 4.5.1. *Let n be a positive integer. Then the following properties hold:*

(*i*) $h^{p,q}_{\mathrm{orb}}(X^{(n)}_i) = 0$ *for* $i = 1, 2, 3$ *unless* $p = q$ *or* $p + q = n$.

(*ii*) *The sum of the orbifold Betti numbers of* $X^{(n)}_i$ *for* $i = 1, 2, 3$ *is*

$$H = 2(s_0 - 1) \cdots (s_n - 1).$$

This equals the orbifold Euler characteristic of X *if* n *is even.*

(*iii*) *Suppose that* n *is odd. Then the dimension of the middle orbifold cohomology group is*

$$\dim(H^n_{\mathrm{orb}}(X^{(n)}_i, \mathbb{Q})) = \begin{cases} (s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 4), & i = 1, \\ (s_0 - 1) \cdots (s_n - 1), & i = 2, \\ 2(s_0 - 1) \cdots (s_{n-1} - 1), & i = 3. \end{cases}$$

The orbifold Euler characteristic is

$$\chi_{\mathrm{orb}}(X^{(n)}_i) = \begin{cases} -(s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 6), & i = 1, \\ 0, & i = 2, \\ (s_0 - 1) \cdots (s_{n-1} - 1)(2s_n - 6), & i = 3. \end{cases}$$

Remark 4.5.2. The proof will actually show a slight generalization of (i): the stated vanishing of orbifold Hodge numbers holds for any quasismooth Calabi-Yau weighted hypersurface $X_d \subset \mathbb{P}(a_0, \ldots, a_{n+1})$ satisfying the conditions that a) each a_i divides d and b) the quotients d/a_i that are strictly smaller than d are pairwise coprime. Mirror symmetry preserves the vanishing, so it will also hold for mirrors of such hypersurfaces, e.g., $X^{(n)}_3$. However, it does not hold in general. A simple counterexample is the fourfold $X_{12} \subset \mathbb{P}(3^{(3)}, 1^{(3)})$, which has $h^{1,2} = 3$.

The sum $H = 1008$ of Betti numbers common to our three examples in dimension 3 is the largest known among all Calabi-Yau threefolds (a crepant resolution always exists in this case). The situation in dimension 3 motivates the following conjecture:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 11 & & 0 & \\ 1 & & 491 & & 491 & & 1 \\ & 0 & & 11 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(a) $X_1^{(3)}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 251 & & 0 & \\ 1 & & 251 & & 251 & & 1 \\ & 0 & & 251 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(b) $X_2^{(3)}$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 491 & & 0 & \\ 1 & & 11 & & 11 & & 1 \\ & 0 & & 491 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(c) $X_3^{(3)}$

Figure 4.2: The orbifold Hodge diamonds of three extreme Calabi-Yau 3-folds.

Conjecture 4.5.3. *In dimension n, the largest possible sum of orbifold Betti numbers for a projective variety with quotient singularities and trivial canonical class is*

$$H = 2(s_0 - 1)\cdots(s_n - 1).$$

For odd n, the smallest possible orbifold Euler characteristic is $-(s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 6)$, and the largest is $(s_0 - 1)\cdots(s_{n-1} - 1)(2s_n - 6)$.

The individual orbifold Hodge numbers are harder to compute, but the values for our examples in dimensions 3 and 4 are shown in Figure 4.2 and Figure 4.3, respectively.

Proof of Theorem 4.5.1. The properties of $X_1^{(n)}$ will imply those for $X_3^{(n)}$ by mirror symmetry [CR11, Theorem 4]; so we'll focus on the first two examples. For the moment, let X denote any quasismooth hypersurface with the property that all weights a_i divide the degree d.

First, we'll make the substitutions $t = x^d$, $\bar{t} = y^d$ to eliminate fractional exponents in (4.3) and define a polynomial

$$Q(x,y) := P(x^d, y^d) = \sum_{\ell=0}^{d-1}\left[\prod_{\tilde{\theta}_i(\ell)=0}\frac{1-(xy)^{d-a_i}}{1-(xy)^{a_i}}\prod_{\tilde{\theta}_i(\ell)\neq 0}(xy)^{\frac{d}{2}-a_i}\left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell)-\frac{d}{2}}\right].$$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 252 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 303148 & & 1213644 & & 303148 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 252 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(a) $X_1^{(4)}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 151700 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 151700 & & 1213644 & & 151700 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 151700 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(b) $X_2^{(4)}$

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 303148 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 252 & & 1213644 & & 252 & & 1 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 303148 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

(c) $X_3^{(4)}$

Figure 4.3: The orbifold Hodge diamonds of three extreme Calabi-Yau 4-folds.

Write

$$Q_\ell(x,y) \coloneqq \prod_{\tilde{\theta}_i(\ell)=0} \frac{1-(xy)^{d-a_i}}{1-(xy)^{a_i}} \prod_{\tilde{\theta}_i(\ell)\neq 0} (xy)^{\frac{d}{2}-a_i} \left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell)-\frac{d}{2}} \tag{4.4}$$

for the ℓth term in the sum.

Since each a_i divides d, we have that

$$\frac{1-(xy)^{d-a_i}}{1-(xy)^{a_i}} = 1 + (xy)^{a_i} + \cdots + (xy)^{d-2a_i}.$$

The orbifold Hodge numbers of X correspond to coefficients of monomials in $Q(x,y)$ of the form $x^a y^b$ with both a and b divisible by d. First, we claim that for any monomial $x^a y^b$ with a nonzero coefficient in $Q(x,y)$, d divides a if and only if d divides b. This is because $Q(x,y)$ is a polynomial in xy with the exception of the term $\left(\frac{x}{y}\right)^{d\tilde{\theta}_i(\ell)-\frac{d}{2}}$. The difference of the exponents of x and y in this expression is $2d\tilde{\theta}_i(\ell)-d$. But for any fixed ℓ, we have

$$\sum_{i,\tilde{\theta}_i(\ell)\neq 0} (2d\tilde{\theta}_i(\ell)-d) \equiv \sum_{i,d\nmid \ell a_i} (2\ell a_i - d) \equiv \ell \sum_i 2a_i \equiv 0 \bmod d.$$

Here the second-to-last congruence holds because adding the terms $2\ell a_i$ for which $d|\ell a_i$ does not change the value modulo d (removing extraneous d terms also, of course, does not change this value). Therefore, each summand $Q_\ell(x,y)$ is a polynomial in xy multiplied by a monomial with the powers of x and y differing by a multiple of d. This proves the claim. In particular, the sum of all orbifold Hodge numbers is the same as the sum of coefficients of powers of x^d in

$$q(x) \coloneqq Q(x,1) = \sum_{\ell=0}^{d-1} \left[\prod_{\tilde{\theta}_i(\ell)=0} (1 + x^{a_i} + \cdots + x^{d-2a_i}) \prod_{\tilde{\theta}_i(\ell)\neq 0} x^{d\tilde{\theta}_i(\ell)-a_i} \right].$$

To isolate powers of x^d, we can sum the values of this polynomial over dth roots of unity. More precisely, let H be the sum of Hodge numbers and ζ a primitive dth root of unity.

Then:

$$H = \frac{1}{d}\sum_{j=0}^{d-1} q(\zeta^j) = \sum_{j=0}^{d-1}\sum_{\ell=0}^{d-1}\left[\prod_{\tilde{\theta}_i(\ell)=0}(1+\zeta^{ja_i}+\cdots+\zeta^{j(d-2a_i)})\prod_{\tilde{\theta}_i(\ell)\neq 0}\zeta^{j(d\tilde{\theta}_i(\ell)-a_i)}\right].$$

We may simplify this further by noticing, as before, that the sum of all the $d\tilde{\theta}_i(\ell)$ is divisible by d for fixed ℓ. We can therefore drop the $d\tilde{\theta}_i(\ell)$ term in the last exponent. Further, since the sum of the a_i is d, we have

$$\prod_{\tilde{\theta}_i(\ell)\neq 0}\zeta^{j(-a_i)} = \prod_{\tilde{\theta}_i(\ell)=0}\zeta^{ja_i}.$$

After making this substitution, both products are now indexed over the weights a_i for which $\tilde{\theta}_i(\ell) = 0$ (i.e., for which $d|\ell a_i$). Thus, we may combine them into one:

$$H = \frac{1}{d}\sum_{j=0}^{d-1}\sum_{\ell=0}^{d-1}\prod_{i,d|\ell a_i}(\zeta^{ja_i}+\zeta^{j(2a_i)}+\cdots+\zeta^{j(d-a_i)}).$$

The quantity $\zeta^{ja_i}+\zeta^{j(2a_i)}+\cdots+\zeta^{j(d-a_i)}$ equals -1 if $d\nmid ja_i$ and equals $\frac{d}{a_i}-1$ if $d|ja_i$, since all terms will be equal to 1 in this case. We'll switch sums and write this as

$$H = \frac{1}{d}\sum_{\ell=0}^{d-1}\sum_{j=0}^{d-1}\prod_{i,d|\ell a_i}\begin{cases}\frac{d}{a_i}-1, & d|ja_i,\\ -1, & d\nmid ja_i\end{cases}.$$

Given a positive integer c, we'll use the notation

$$f_c(j) := \begin{cases} c-1, & c|j,\\ -1, & c\nmid j\end{cases}.$$

In this notation,

$$H = \frac{1}{d}\sum_{\ell=0}^{d-1}\sum_{j=0}^{d-1}\prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j). \tag{4.5}$$

The same reasoning shows that the contribution to H coming from $Q_\ell(x,y)$ is $\frac{1}{d}\sum_{j=0}^{d-1}\prod_{i,d|\ell a_i} f_{\frac{d}{a_i}}(j)$. The following lemma will help to evaluate such sums.

Lemma 4.5.4. *Let C be a nonempty finite set of pairwise coprime positive integers and d a positive integer divisible by every element of C. Then*

$$\sum_{j=0}^{d-1}\prod_{c\in C} f_c(j) = 0.$$

Proof. We'll prove the statement by induction on the size of the set C. Suppose for the base case that $C = \{c_1\}$. Then there are $\frac{d}{c_1}$ values of j divisible by c and $d - \frac{d}{c_1}$ which aren't. The expression in the lemma therefore equals

$$\frac{d}{c_1}(c_1 - 1) + \left(d - \frac{d}{c_1}\right)(-1) = d - \frac{d}{c_1} - d + \frac{d}{c_1} = 0.$$

Now, suppose that the statement is true for sets of size m and set $C = \{c_1, \ldots, c_m, c_{m+1}\}$. Let d be an integer divisible by each element of C. Then we can break up the expression into two pieces based on whether c_{m+1} divides j:

$$\sum_{j=0}^{d-1}\prod_{i=1}^{m+1} f_{c_i}(j) = \sum_{\substack{1\le j<d,\\ c_{m+1}|j}}\prod_{i=1}^{m+1} f_{c_i}(j) + \sum_{\substack{1\le j<d,\\ c_{m+1}\nmid j}}\prod_{i=1}^{m+1} f_{c_i}(j).$$

Next, we may factor the $i = m+1$ term out of each product to obtain:

$$(c_{m+1}-1)\sum_{\substack{1\le j<d,\\ c_{m+1}|j}}\prod_{i=1}^{m} f_{c_i}(j) + (-1)\sum_{\substack{1\le j<d,\\ c_{m+1}\nmid j}}\prod_{i=1}^{m} f_{c_i}(j) = c_{m+1}\sum_{\substack{1\le j<d,\\ c_{m+1}|j}}\prod_{i=1}^{m} f_{c_i}(j) - \sum_{j=0}^{d-1}\prod_{i=1}^{m} f_{c_i}(j).$$

In the sum over multiples of c_{m+1}, we can replace each j by j/c_{m+1} without changing the value of the other f_{c_i} because the c_i are pairwise coprime. Therefore, the last expression may be rewritten:

$$c_{m+1}\sum_{k=0}^{\frac{d}{c_{m+1}}-1}\prod_{i=1}^{m} f_{c_i}(k) - \sum_{j=0}^{d-1}\prod_{i=1}^{m} f_{c_i}(j).$$

Since both d and d/c_{m+1} are positive integers divisible by each element of the pairwise coprime set $\{c_1, \ldots, c_m\}$, the two sums above both equal zero by the inductive hypothesis. This completes the proof. □

Using this lemma, we prove the properties of Theorem 4.5.1. Let

$$S_\ell := \sum_{j=0}^{d-1} \prod_{i, d | \ell a_i} f_{\frac{d}{a_i}}(j)$$

be the ℓth term of the sum (4.5), so that $H = \frac{1}{d} \sum_{j=0}^{d-1} S_\ell$. Notice that the product is indexed by a set depending only on ℓ (and not j). The following lemma will show that S_ℓ is equal to zero for many values of ℓ in our examples.

Lemma 4.5.5. *Let X be a quasismooth Calabi-Yau hypersurface of degree d with weights $a_0, \ldots, a_{n+1}$ all dividing d. Suppose that the set $C := \{d/a_i : a_i \neq 1\}$ is pairwise coprime. Then in the notation above: if $\ell \neq 0$ and c divides ℓ for some $c \in C$, then $S_\ell = 0$.*

Proof. Let $c_i = d/a_i$ for every element $d/a_i \in C$. Assuming that $\ell \neq 0$, we have $d \nmid \ell$ so the sum S_ℓ may be written

$$S_\ell = \sum_{j=0}^{d-1} \prod_{i, d | \ell a_i} f_{\frac{d}{a_i}}(j) = \sum_{j=0}^{d-1} \prod_{c_i \in C, c_i | \ell} f_{c_i}(j).$$

Any subset of C is pairwise coprime and all elements of C divide d, so by Lemma 4.5.4, $S_\ell = 0$ whenever the product is nonempty. This occurs precisely when some c_i divides ℓ. □

Let X satisfy the condition of Lemma 4.5.5. (For instance, $X = X_2^{(n)}$ or $X_1^{(n)}$; in these cases, $C = \{s_0, \ldots, s_n\}$ or $C = \{s_0, \ldots, s_{n-1}\}$, respectively.) Then, S_ℓ is only nonzero if: a) $\ell = 0$, in which case the second product in $Q_\ell(x, y)$ is empty, or b) no s_i divides ℓ, in which case the first product in $Q_\ell(x, y)$ is empty. In case a), Q_0 is a polynomial in xy. In case b),

$Q_\ell(x,y)$ is a monomial $x^a y^b$ with

$$a+b = 2\sum_{i=0}^{n+1}\left(\frac{d}{2}-a_i\right) = 2\left(\frac{(n+2)d}{2}-d\right) = nd.$$

Therefore, the only orbifold Hodge numbers $h^{p,q}_{\text{orb}}$ of the mirror of X that could be nonzero satisfy $p=q$ or $p+q=n$. Taking the mirror just switches these two types of Hodge numbers, so the vanishing in Theorem 4.5.1(i) holds for both X and its mirror. This is enough to prove the first part of the theorem. We also note that, in odd dimensions, all contributions to the middle orbifold cohomology of X (i.e., to Hodge numbers $h^{p,p}$ of the mirror) come from $Q_0(x,y)$.

To prove parts (ii) and (iii) of Theorem 4.5.1, we'll consider $X_2^{(n)}$ and $X_1^{(n)}$ separately:

Proof for $X_2^{(n)}$: (hypersurface of degree $d = s_{n+1}-1$ in $\mathbb{P}(d/s_0,\ldots,d/s_n,1)$)

The sum of all orbifold Betti numbers (equivalently, the sum of all orbifold Hodge numbers) is $H = \frac{1}{d}(A+B)$, where $A := S_0$ and B is the sum of all S_ℓ such that $s_i \nmid \ell$, $i = 0,\ldots n$. First, we compute

$$\begin{aligned} A = S_0 &= \sum_{j=0}^{d-1} f_{s_0}(j)\cdots f_{s_n}(j) f_d(j) \\ &= (s_0-1)\cdots(s_n-1)(d-1) + (-1)\sum_{j=1}^{d-1} f_{s_0}(j)\cdots f_{s_n}(j). \end{aligned}$$

Here S_0 is divided into two pieces by explicitly evaluating the function f_d. By Lemma 4.5.4,

$$\sum_{j=1}^{d-1} f_{s_0}(j)\cdots f_{s_n}(j) = -f_{s_0}(0)\cdots f_{s_n}(0) = -(s_0-1)\cdots(s_n-1).$$

Therefore, A simplifies to $d(s_0-1)\cdots(s_n-1)$. The number of values ℓ summed in B is

$$d\left(1-\frac{1}{s_0}\right)\cdots\left(1-\frac{1}{s_n}\right) = (s_0-1)\cdots(s_n-1).$$

Since the product in S_ℓ is empty when no s_i divides ℓ, each S_ℓ term in B equals d; the total is $B = d(s_0-1)\cdots(s_n-1)$. Therefore, $H = \frac{1}{d}(A+B) = 2(s_0-1)\cdots(s_n-1)$, which proves property (ii) for this example. In odd dimensions, the middle orbifold cohomology has dimension $\frac{1}{d}A = (s_0-1)\cdots(s_n-1)$, which is half the total, as expected. Note that $X_2^{(n)}$ is self-mirror, so its orbifold Euler characteristic vanishes.

Proof for $X_1^{(n)}$: (hypersurface of degree $d = 2s_n - 2$ in $\mathbb{P}(d/s_0, \ldots, d/s_{n-1}, 1, 1)$)

The idea of the proof is similar in this case. Once again, let $A := S_0$ and B be the sum of the S_ℓ with $s_i \nmid \ell$, $i = 0, \ldots n-1$. The number of terms in B is

$$d\left(1 - \frac{1}{s_0}\right)\cdots\left(1 - \frac{1}{s_{n-1}}\right) = 2(s_0-1)\cdots(s_{n-1}-1)$$

and each of these satisfies $S_\ell = d$. So $B = 2d(s_0-1)\cdots(s_{n-1}-1)$. Similarly,

$$\begin{aligned} A = S_0 &= \sum_{j=0}^{d-1} f_{s_0}(j)\cdots f_{s_{n-1}}(j)(f_d(j))^2 \\ &= (s_0-1)\cdots(s_{n-1}-1)(d-1)^2 + (-1)^2 \sum_{j=1}^{d-1} f_{s_0}(j)\cdots f_{s_{n-1}}(j). \end{aligned}$$

Applying Lemma 4.5.4 again, the last sum equals $-f_{s_0}(0)\cdots f_{s_{n-1}}(0) = (s_0-1)\cdots(s_{n-1}-1)$. Thus,

$$A = (s_0-1)\cdots(s_{n-1}-1)d(d-2).$$

The total sum is

$$\begin{aligned} H &= \frac{1}{d}(A+B) = \frac{1}{d}(d(d-2)(s_0-1)\cdots(s_{n-1}-1) + 2d(s_0-1)\cdots(s_{n-1}-1)) \\ &= (s_0-1)\cdots(s_{n-1}-1)d = 2(s_0-1)\cdots(s_n-1). \end{aligned}$$

The dimension of the middle orbifold cohomology in odd dimensions is $\frac{1}{d}A = (d-2)(s_0-1)\cdots(s_{n-1}-1) = (s_0-1)\cdots(s_{n-1}-1)(2s_n-4)$. The third example is the mirror of the first, so properties (i) and (ii) also hold for $X_3^{(n)}$. The sums A and B switch under mirror symmetry, so the dimension of $H^n_{\text{orb}}(X_3^{(n)}, \mathbb{Q})$ is $\frac{1}{d}B = 2(s_0-1)\cdots(s_{n-1}-1)$. The orbifold Euler characteristics of $X_1^{(n)}$ and $X_3^{(n)}$ are $\frac{1}{d}(B-A)$ and $\frac{1}{d}(A-B)$, respectively, using the vanishing result (i). □

4.6 Klt Calabi-Yau surfaces with large index or small mld

We have conjectured the klt Calabi-Yau pairs with standard coefficients of largest index in each dimension, and also the terminal or canonical Calabi-Yau varieties of largest index. In section 4.7, we conjecture the answer to a natural intermediate problem: find the klt Calabi-Yau *varieties* of largest index. In this section, we prove that conjecture in dimension 2, finding the klt Calabi-Yau surfaces with the largest index. We also find the smallest minimal log discrepancy for klt Calabi-Yau surfaces. The main tool is Brandhorst-Hofmann's classification of finite group actions on K3 surfaces, extending many earlier results [BH21].

Proposition 4.6.1. *The largest index of any klt Calabi-Yau surface is* 19*. The smallest mld of any klt Calabi-Yau surface is* $\frac{1}{13}$.

For comparison, the largest index of a klt Calabi-Yau pair with standard coefficients of dimension 2 is 66 (Proposition 4.3.5), and the smallest mld of a klt Calabi-Yau pair with standard coefficients of dimension 2 is $\frac{1}{42}$ (Proposition 4.6.8, below).

Proof. For any klt Calabi-Yau surface X, let Y be the index-1 cover of X (section 4.2). Thus Y is a projective surface with canonical singularities such that K_Y is trivial. Writing m for the index of X, we have $X = Y/\mu_m$, where μ_m acts purely non-symplectically, meaning that it acts faithfully on $H^0(Y, K_Y) \cong \mathbb{C}$. Also, μ_m acts freely in codimension 1 on Y, because X is a Calabi-Yau surface rather than a pair.

Let $\pi\colon Z \to Y$ be the minimal resolution. Since $K_Z = \pi^*(K_Y)$, Z is a smooth projective surface with K_Z linearly equivalent to zero. By Proposition 4.2.2, the action of μ_m on Y lifts to an action on Z. By the classification of smooth projective surfaces with trivial canonical bundle, Z is a $K3$ surface or an abelian surface. If Z is an abelian surface, then m is at most 6 [Bla95, Theorem C], [Zha91, Theorem 4.1]. In that case, X has mld at least $\frac{1}{6}$.

Since we are trying to find the largest index or smallest mld among klt Calabi-Yau surfaces, we can assume from now on that Z is a smooth K3 surface, where more extreme values can occur. Note that Y may be a nontrivial contraction of Z. Let $W = Z/\mu_m$, and let (W, D) be the pair with standard coefficients associated to the μ_m-action on Z. Denote by $\sigma\colon W \to X$ the quotient of the morphism $\pi\colon Z \to Y$ by the action of μ_m, so that we have:

$$\begin{array}{ccc} Z & \xrightarrow{\pi} & Y \\ \downarrow & & \downarrow \\ W & \xrightarrow{\sigma} & X. \end{array}$$

Kondo found that the group μ_{19} acts purely non-symplectically on a certain smooth projective K3 surface Z [Kon92, section 7]. In fact, the K3 surface and the action are unique up to isomorphism and automorphisms of μ_{19} [Bra19, Theorem 1.1]. Also, μ_{19} acts freely in codimension 1 on Z [BH21, Table 37]. So $W = Z/\mu_{19}$ is a klt Calabi-Yau surface of index 19. In the spirit of this chapter, we can describe Z as the minimal resolution of the hypersurface $Y_{10} \subset \mathbb{P}^3(5, 3, 1, 1)$ defined by $0 = x_0^2 + x_1^3 x_3 + x_3^9 x_2 + x_2^7 x_1$. Here μ_{19} acts on Y by

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta^7 x_1 : \zeta^2 x_2 : x_3],$$

Y has an A_2 singularity, and Y/μ_{19} is also a klt Calabi-Yau surface of index 19.

In the other direction, Appendix B in [BH21] shows that for each purely non-symplectic action of a cyclic group of order greater than 19 on a smooth K3 surface, there is a smooth curve of positive genus that is fixed by a nontrivial subgroup. This curve cannot be contracted by $\pi\colon Z \to Y$ (in the notation above), and so μ_m cannot act freely in codimension 1 on Y. This contradicts that $X = Y/\mu_m$ is a klt Calabi-Yau surface (rather than a Calabi-Yau pair). So 19 is the largest index among all klt Calabi-Yau surfaces.

Next, we will find the smallest possible mld for a klt Calabi-Yau surface X. We know that X has index $m \leq 19$. As above, let Y be the index-1 cover of X, Z its minimal resolution, and (W, D) the quotient of Z by μ_m.

Lemma 4.6.2.

$$\mathrm{mld}(X) = \mathrm{mld}(W, D).$$

Proof. Consider the commutative square above. Since Y has canonical singularities, we have $K_Z = \pi^* K_Y$. Taking quotients by μ_m, we have that K_Y is the pullback of K_X and K_Z is the pullback of $K_W + D$. So $K_W + D$ and $\sigma^* K_X$ both pull back to K_Z on Z. Since they differ by an explicit $\mathbb{Q}$-divisor on W, it follows that $K_W + D = \sigma^* K_X$. Since the mld of X can be computed on a log resolution of (W, D), we conclude that $\mathrm{mld}(X) = \mathrm{mld}(W, D)$. ☐

Given a pair (X, D), we'll use the notation $\mathrm{mld}_{\geq 2}(X, D)$ to denote the minimal log discrepancy over all points $x \in X$ with codimension at least 2 (corresponding to exceptional divisors over X, rather than all divisors over X). We have the following general bound on discrepancies for quotients by finite groups.

Proposition 4.6.3. *Let (Z, F) be a quasi-projective terminal pair with a faithful action of a finite group G. Let $W = Z/G$, and let D be the $\mathbb{Q}$-divisor on W such that $K_Z + F$ is the pullback of $K_W + D$. Then*

$$\mathrm{mld}_{\geq 2}(W, D) > \frac{1}{|G|}.$$

If $Z = (Z, 0)$ is a terminal Gorenstein variety, then

$$\mathrm{mld}_{\geq 2}(W, D) \geq \frac{2}{|G|}.$$

We need to exclude points of codimension 1 in these inequalities. For example, if a cyclic group $G = \mu_m$ fixes an irreducible divisor S in a variety Z, then the image D_1 of S in W occurs in D with coefficient $(m-1)/m$, and so the generic point w of D_1 has $\mathrm{mld}_w(W, D) = 1/m = 1/|G|$.

Proof. By [Kol97, Corollary 2.43], we have the inequalities

$$\mathrm{mld}_{\geq 2}(W, D) \leq \mathrm{mld}_{\geq 2}(Z, F) \leq |G| \mathrm{mld}_{\geq 2}(W, D).$$

(The notation "$\mathrm{discrep}(W, D)$" of [Kol97] means the infimum of all discrepancies of exceptional divisors over X, which is the same as $\mathrm{mld}_{\geq 2}(W, D) - 1$.) By definition, the terminality of (Z, F) means that $\mathrm{mld}_{\geq 2}(Z, F) > 1$. It follows that

$$\mathrm{mld}_{\geq 2}(W, D) \geq \frac{1}{|G|}\mathrm{mld}_{\geq 2}(Z, F) > \frac{1}{|G|}.$$

If $Z = (Z, 0)$ is a terminal Gorenstein variety, then $|G|(K_W + D)$ is Cartier. It follows that the discrepancy of every divisor over W is a multiple of $1/|G|$, and so $\mathrm{mld}_{\geq 2}(W, D) \geq 2/|G|$. □

By [BH21, Table 35], there is a smooth K3 surface Z with a purely non-symplectic action of the cyclic group μ_{13}; the fixed locus consists of a smooth rational curve C and 9 other points. Since K_Z is trivial, the adjunction formula gives that C is a (-2)-curve. Let Y be the canonical K3 surface obtained by contracting C. Then the cyclic group μ_{13} acts freely in codimension 1 on Y. So $X := Y/\mu_{13}$ is a klt Calabi-Yau surface of index 13.

For an explicit example, start with the canonical K3 surface $S_{11} \subset \mathbb{P}^3(5, 3, 2, 1)$ defined by $0 = x_0^2x_3 + x_1^3x_2 + x_2^3x_0 + x_3^8x_1$, on which there is a purely non-symplectic action of μ_{13}:

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta^2 x_1 : \zeta^{-4}x_2 : x_3].$$

This action is free in codimension 1, and so S/μ_{13} is a klt Calabi-Yau surface of index 13. One checks that the minimal resolution Z of S contains a smooth rational curve C fixed by μ_{13} over the A_4 singularity $[1 : 0 : 0 : 0]$ of S, in agreement with [BH21, Table 35].

Lemma 4.6.4. *There is a klt Calabi-Yau surface of index* 13*. Every such surface has mld* $\frac{1}{13}$*.*

Proof. We have shown that there is a klt Calabi-Yau surface X of index 13. As throughout this section, let Y be the index-1 cover of X, Z the minimal resolution of Y, and $(W, D) = Z/\mu_{13}$. By Lemma 4.6.2, $\mathrm{mld}(X) = \mathrm{mld}(W, D)$. Here $13(K_W + D)$ is Cartier, and so $\mathrm{mld}(W, D)$ is a positive integer divided by 13. Also, D is $\frac{12}{13}$ times the image of the curve C above, and so (W, D) has log discrepancy $\frac{1}{13}$ at the generic point of this curve. Thus $\mathrm{mld}(W, D) = \frac{1}{13}$. □

Lemma 4.6.5. *The mld of a klt Calabi-Yau surface of index* 14 *is* $\frac{1}{7}$*.*

Remark 4.6.6. Although it is not needed for the proof of Proposition 4.6.1, we remark that there is a klt Calabi-Yau surface of index 14. Start with the canonical K3 surface $S_7 \subset \mathbb{P}^3(3,2,1,1)$ defined by $0 = x_0^2x_3 + x_1^3x_2 + x_1x_2^5 + x_3^7$, on which there is a purely non-symplectic action of μ_{14}:

$$\zeta[x_0 : x_1 : x_2 : x_3] = [\zeta x_0 : \zeta x_1 : \zeta^{-3}x_2 : \zeta^{-2}x_3].$$

The curve $C = \{0 = x_1 = x_3\} \cong \mathbb{P}^1(3,1)$ is contained in S and fixed by the subgroup μ_2. Here $C^2 = -4/3 < 0$, and so C can be contracted, by Artin's contraction theorem applied to the minimal resolution of S [Art62, Theorem 2.7]. This yields another canonical K3 surface Y on which μ_{14} acts freely in codimension 1. Then $X = Y/\mu_{14}$ is a klt Calabi-Yau surface of index 14.

Proof. (Lemma 4.6.5) Let X be a klt Calabi-Yau surface of index 14. In the notation above, Z must be a smooth K3 surface with a purely non-symplectic action of μ_{14}, and every curve in Z fixed by a nontrivial subgroup of μ_{14} must have genus zero. By [BH21, Table 15, 0.14.0.4], there is exactly one deformation type of smooth K3 surfaces Z with such an action, up to automorphisms of μ_{14}. The table shows that there is no curve in Z fixed by the whole group μ_{14}, but there are three smooth curves of genus zero fixed by the subgroup μ_2 and one smooth curve of genus zero fixed by μ_7.

Let $(W, D) = Z/\mu_{14}$. We have $\text{mld}(X) = \text{mld}(W, D)$ by Lemma 4.6.2. Here D has coefficients $6/7$ and $1/2$. So the mld of (W, D) is $1/7$ at one point of codimension 1 and greater than $1/7$ at every other point of codimension 1. Also, by Proposition 4.6.3, since (W, D) is the quotient of a smooth K3 surface by μ_{14}, the mld of (W, D) at any closed point is at least $2/14 = 1/7$. So the mld of X is $1/7$. □

Lemma 4.6.7. *For m equal to 17 or 19, the mld of a klt Calabi-Yau surface of index m is at least $\frac{2}{m}$.*

Proof. Let X be a klt Calabi-Yau surface of index m equal to 17 or 19. Let Y be the index-1 cover of X, Z the minimal resolution of Y, and $(W, D) = Z/\mu_m$. By [BH21, Appendix B], for m equal to 17 or 19, every purely non-symplectic action of μ_m on a smooth K3 surface is free in codimension 1; so $D = 0$. Then Lemma 4.6.2 gives that $\mathrm{mld}(X) = \mathrm{mld}(W)$. Since W is a normal variety (rather than a pair), $\mathrm{mld}(W)$ is equal to the minimum of its mld's at closed points, and these are at least $\frac{2}{m}$ by Proposition 4.6.3. □

We now show that among all klt Calabi-Yau surfaces X, the smallest mld is $1/13$. We know by Lemma 4.6.4 that $1/13$ does occur. Here X must be a quotient Y/μ_m as discussed above, in particular with μ_m acting freely in codimension 1. The mld of X is a positive integer divided by m. So if X has mld less than $1/13$, it must have index $m \geq 14$. Since the minimal resolution $Z \to Y$ contracts only smooth rational curves, every curve in Z fixed by a nontrivial subgroup of μ_m must have genus zero. By [BH21, Appendix B], if a smooth K3 surface Z has a purely non-symplectic action of the cyclic group μ_m with m at least 14 and not equal to 14, 17, or 19, then there is a curve of genus > 0 fixed by a nontrivial subgroup. So m must be 14, 17, or 19. Therefore, Lemmas 4.6.5 and 4.6.7 give that the smallest mld of a klt Calabi-Yau surface is $1/13$. Proposition 4.6.1 is proved. □

We conclude with the analogous bound for pairs.

Proposition 4.6.8. *The smallest mld of any klt Calabi-Yau pair with standard coefficients of dimension 2 is* $\frac{1}{42}$.

Proof. By Theorem 4.3.1, there is a klt Calabi-Yau pair with standard coefficients of dimension 2 that has mld $1/42$.

Suppose that there is a klt Calabi-Yau pair (X, E) with standard coefficients of dimension 2 that has mld less than $1/42$. Let m be the index of (X, E). The mld is a positive integer divided by m, and so we must have $m > 42$. By Proposition 4.3.5, we have $m \leq 66$. Let Y be the index-1 cover of (X, E), so that $(X, E) = Y/\mu_m$ with μ_m acting purely non-symplectically. Let Z be the minimal resolution of Y, and let $(W, D) = Z/\mu_m$. Here Z is a K3 surface. The proof of Lemma 4.6.2 shows that $\mathrm{mld}(X, E) = \mathrm{mld}(W, D)$.

By Brandhorst and Hofmann's classification of purely non-symplectic cyclic group actions on K3 surfaces, m must be 44, 48, 50, 54, or 66 [BH21, Appendix B]. In each case, the classification shows that the subspace of Z fixed by G has dimension zero. Equivalently, each curve in Z is fixed by a proper subgroup of G. So the coefficients of D have the form $(b-1)/b$ with $b \leq m/2$. Equivalently, the mld of (W, D) at each point of codimension 1 in X is at least $2/m$. By Proposition 4.6.3, the mld of (W, D) at each closed point is also at least $2/m$. So $\mathrm{mld}(W, D)$ is at least $2/m$, hence at least $1/33$, contradicting that it is less than $1/42$. □

4.7 Higher-dimensional klt Calabi-Yau varieties with large index

In this section, we find klt Calabi-Yau varieties in every dimension at least 2, which we conjecture have the largest possible index. In dimension 2, the construction reproduces the index 19 example mentioned in the proof of Proposition 4.6.1. Each of our varieties is constructed as a quotient of a weighted projective hypersurface by a cyclic group action. To define these actions, we'll use the techniques of Artebani-Boissière-Sarti [ABS14] to write down hypersurfaces with an action of a large cyclic group. We only manage to compute the index of our example exactly in dimensions at most 30. So, at least in low dimensions, we construct klt Calabi-Yau varieties of extremely large index.

First, we prove some general properties about hypersurfaces defined by loop potentials, some of which appear in [ABS14]. A *potential* is a sum of n monomials in n variables. A *loop potential* has the form

$$W := x_1^{b_1}x_2 + \cdots + x_{n-1}^{b_{n-1}}x_n + x_n^{b_n}x_1 \tag{4.6}$$

for some integers $b_i \geq 2$. Suppose that for some positive integer weights $a_1, \ldots, a_n$ on the variables, W is homogeneous of degree $d := \sum_i a_i$. Then the hypersurface $X := \{W = 0\}$ in $\mathbb{P}(a_1, \ldots, a_n)$ is quasismooth and Calabi-Yau. Writing W in the form

$$W = \sum_{i=1}^{n} \prod_{i=1}^{n} x_i^{a_{ij}},$$

define an associated matrix $A = (a_{ij})$:

$$A = \begin{pmatrix} b_1 & 1 & & & \\ & b_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & b_{n-1} & 1 \\ 1 & & & & b_n \end{pmatrix}.$$

Let $\mathrm{Aut}(W)$ be the group of diagonal automorphisms of $\mathbb{C}^n$ preserving the potential W. By the results of [ABS14], we may identify $\mathrm{Aut}(W)$ with $A^{-1}\mathbb{Z}^n/\mathbb{Z}^n$. That is, the columns of A^{-1} generate $\mathrm{Aut}(W)$ modulo $\mathbb{Z}^n$, where an element $(c_1, \dots, c_n) \in \mathrm{Aut}(W)$ acts on $\mathbb{C}^n$ by multiplication by the diagonal matrix $\mathrm{diag}(e^{2\pi i c_1}, \dots, e^{2\pi i c_n})$.

This action descends to an action on the weighted projective hypersurface X, giving a surjective homomorphism $\alpha\colon \mathrm{Aut}(W) \to \mathrm{Aut}_T(X)$ to the group of toric automorphisms of X.

Lemma 4.7.1 (cf. [ABS14, Proposition 2]). *The group* $\mathrm{Aut}(W)$ *has order* $\Gamma := \det(A) = (-1)^{n+1} + b_1 \cdots b_n$. *The last column of* A^{-1} *is*

$$v_n = \frac{1}{\Gamma}\begin{pmatrix} (-1)^{n-1} \\ (-1)^{n-2}b_1 \\ (-1)^{n-3}b_1 b_2 \\ \vdots \\ b_1 \cdots b_{n-1} \end{pmatrix}.$$

The other columns v_i *of* A^{-1} *satisfy the following relations modulo* $\mathbb{Z}^n$: $v_i = (-1)^{n-i} b_n b_{n-1} \cdots b_{i+1} v_n$, $i = 1, \dots, n-1$. *In particular,* $\mathrm{Aut}(W)$ *is cyclic with generator* $\varphi := v_n$ (*or any one of the other columns*). *The first row of* A^{-1} *is*

$$w_1 = \frac{1}{\Gamma}\begin{pmatrix} b_2 \cdots b_n & -b_3 \cdots b_n & b_4 \cdots b_n & \cdots & (-1)^{n-1} \end{pmatrix}.$$

The other rows w_i of A^{-1} satisfy the following relations modulo $\mathbb{Z}^n$: $w_i = (-1)^{i-1} b_1 b_2 \cdots b_{i-1}$, $i = 2, \ldots n$.

The kernel of $\alpha\colon \mathrm{Aut}(W) \to \mathrm{Aut}_T(W)$ is a subgroup $J_W \subset \mathrm{Aut}(W)$ of order d generated by the vector of charges $(q_1, \ldots, q_n)$, where each *charge* q_i is the sum of the entries of the ith row of A^{-1}. The charges have the property $q_i = a_i/d$. Lemma 4.7.1 implies that

$$q_1 = \frac{1}{\Gamma}\left((-1)^{n-1} + (-1)^{n-2} b_n + \cdots + b_2 \cdots b_n\right).$$

Modulo $\mathbb{Z}^n$, the other charges are multiples of this one, so that the degree d equals the denominator of q_1:

$$d = \frac{\Gamma}{\gcd(\Gamma, (-1)^{n-1} + (-1)^{n-2} b_n + \cdots + b_2 \cdots b_n)}. \tag{4.7}$$

The same reasoning shows that the degree d^T of the mirror hypersurface (which is defined by the potential W^T associated to the matrix A^T) is

$$d^\mathsf{T} = \frac{\Gamma}{\gcd(\Gamma, (-1)^{n-1} + (-1)^{n-2} b_1 + \cdots + b_1 \cdots b_{n-1})}. \tag{4.8}$$

Loop potentials are useful for constructing cyclic group actions which are free in codimension 1, in light of the following proposition.

Proposition 4.7.2. *If a hypersurface X with $\dim(X) \geq 3$ is defined by a loop potential, the action of $\mathrm{Aut}_T(X)$ is free in codimension 1.*

Proof. First, we reduce to only checking the stabilizers of coordinate hyperplanes. Indeed, $\mathrm{Aut}_T(X)$ is a subgroup of the torus action on $\mathbb{P}(a_1, \ldots, a_n)$, so the only nontrivial stabilizers of the action of $\mathrm{Aut}_T(X)$ on weighted projective space occur on coordinate strata in the complement of the open torus. Further, we claim that the intersection of X with a toric stratum S is only codimension 1 in X if S is codimension 1 in $\mathbb{P}(a_1, \ldots, a_n)$. Indeed, if X contained S of codimension 2 in weighted projective space, then the potential $W = x_1^{b_1} x_2 + \cdots + x_{n-1}^{b_{n-1}} x_n + x_n^{b_n} x_1$ would become identically zero after setting some pair of variables x_i and x_j equal to zero. Since $\dim(X) \geq 3$, we have $n \geq 5$, and no choice of i and j makes the loop potential zero.

Thus, it's enough to compute the stabilizers of each $H_i \coloneqq X \cap \{x_i = 0\}$. Without loss of generality we'll just show the stabilizer of H_n is trivial. To this end, let $H \subset \mathrm{Aut}_T(W)$ be the subgroup fixing H_n and let $G = \alpha^{-1}(H)$. Then G is a cyclic subgroup of $\mathrm{Aut}(W)$. We want to show that in fact $G = J_W$.

For an element $g \coloneqq (c_1, \ldots, c_n)$ in $\mathrm{Aut}(W)$, the statement that the element $\alpha(g)$ fixes H_n is equivalent to the following statement in terms of coordinates on $\mathbb{C}^n$: for all $x_1, \ldots, x_{n-1} \in \mathbb{C}$, there exists a $t \in \mathbb{C}^*$ such that

$$g \cdot (x_1, \ldots, x_{n-1}, 0) = t \cdot (x_1, \ldots, x_{n-1}, 0),$$

where the $\mathbb{C}^*$ action is $t \cdot (y_1, \ldots, y_n) = (t^{a_1} y_1, \ldots, t^{a_n} y_n)$. Now, the action of $\mathrm{Aut}(W)$ is by diagonal matrices whose entries are unit complex numbers, so it's immediate that $|t| = 1$ and in fact that $t = e^{i\theta}$ for θ a rational multiple of π. In order to compute in terms of elements of $(\mathbb{Q}/\mathbb{Z})^n$, it will be convenient to express the element $t \in \mathbb{C}^*$ as $r(q_1, \ldots, q_n) \in (\mathbb{Q}/\mathbb{Z})^n$, where $(q_1, \ldots, q_n)$ is the vector of charges and r is some rational number. (Remember that we use the convention that each vector $(c_1, \ldots, c_n) \in (\mathbb{Q}/\mathbb{Z})^n$ corresponds to the action by the diagonal matrix $\mathrm{diag}(e^{2\pi i c_1}, \ldots, e^{2\pi i c_n})$).

There is a point of H_n with all coordinates $x_1, \ldots, x_{n-1}$ nonzero. Otherwise, since H_n is codimension 1 in X, it must contain some entire stratum $\{x_i = x_j = 0\}$, which we saw above cannot occur when $\dim(X) \geq 3$. Thus, if the element g fixes H_n, there is an $r \in \mathbb{Q}$ such that the following congruences hold:

$$rq_i = c_i \bmod \mathbb{Z}, 1 \leq i \leq n-1. \tag{4.9}$$

We already know that the vector of charges itself generates J_W, so if $r \in \mathbb{Z}$, $r(q_1, \ldots, q_n)$ will equal an element of J_W. We will derive a contradiction from the following assumption: there exists $g \in \mathrm{Aut}(W)$ and r *not* an integer such that the congruences (4.9) hold. This will show that $G = J_W$, as required.

First, reduce to the case that $r = 1/A$ for a positive integer $A \geq 2$. Indeed, we can always replace r and g by the same integer multiple of each and the congruences (4.9) will still hold; we can also add an integer to r and multiply g by the corresponding element of J_W. Using such adjustments, any rational number $r = B/A$ in lowest terms can be turned into $1/A$. Consider the first two coordinates c_1 and c_2 of g. We know that $c_2 \equiv -b_1 c_1 \mod \mathbb{Z}$, since this relation holds for the generator φ of the group $\mathrm{Aut}(W)$ and g is some multiple of φ. But $c_i \equiv r q_i \mod \mathbb{Z}$, so we have $r q_2 \equiv -b_1 r q_1 \mod \mathbb{Z}$. Therefore, $s := b_1 r q_1 + r q_2$ is an integer. Substituting $r = 1/A$ and using the definition of charges $q_i = a_i/d$, we have

$$b_1 a_1 + a_2 = Ads.$$

However, our loop potential is homogeneous of degree d and contains the monomial $x_1^{b_1} x_2$ so $b_1 a_1 + a_2 = d$. This would mean that $As = 1$, contradicting the assumption that $A \geq 2$ and s is an integer. This completes the proof. □

Next, we'll state a criterion for whether the induced action of $\mathrm{Aut}_T(X)$ on $H^0(X, K_X) \cong \mathbb{C}$ is faithful. This in turn implies that the quotient $X/\mathrm{Aut}_T(X)$ has index $|\mathrm{Aut}_T(X)| = \Gamma/d$. Let $\mathrm{SL}(W) \subset \mathrm{Aut}(W)$ be the subgroup with determinant 1. Since X is Calabi-Yau, we have $J_W \subset \mathrm{SL}(W)$. The image $\alpha(\mathrm{SL}(W)) \subset \mathrm{Aut}_T(X)$ is the kernel of the action on $H^0(X, K_X)$. By [ABS14, Corollary 1], $|\mathrm{SL}(W)| = \Gamma/d^{\mathsf{T}}$. Therefore, we have the following criterion:

Proposition 4.7.3. *The group $\mathrm{Aut}_T(X)$ acts faithfully on $H^0(X, K_X)$ if and only if $d \cdot d^{\mathsf{T}} = \Gamma$.*

Proof. The kernel of α is J_W, so the action is faithful if and only if $\mathrm{SL}(W) = J_W$. Since $J_W \subset \mathrm{SL}(W)$, this holds exactly when the two groups have the same order. But $|J_W| = d$, so this is equivalent to $d = \Gamma/d^{\mathsf{T}}$. □

Nearly identical arguments apply to any X with $\dim(X) \geq 3$ defined by a polynomial of the form $W = x_0^{b_0} + f_{\mathrm{loop}}$, where f_{loop} is of the form of (4.6). The corresponding matrix A is block diagonal, so there is a subgroup $G \cong \mathbb{Z}/\Gamma\mathbb{Z}$ of $\mathrm{Aut}(X)$ with index b_0 whose image in $\mathrm{Aut}_T(X)$ acts freely in codimension 1. The quotient of X by this cyclic group is therefore a klt Calabi-Yau variety (rather than a pair). We can check whether this action is purely non-symplectic using a criterion very similar to Proposition 4.7.3.

Next, we define several sequences of numbers based on Sylvester's sequence for each dimension n. These will be used to define the exponents and weights for our example of large index.

Definition 4.7.4. For $n = 2r+1$, $r \geq 1$ or $n = 2r$, $r \geq 1$, define integers $b_0, \ldots, b_{n-1}$ as follows. Set $b_i := s_i$ when $0 \leq i \leq r$, and define b_{r+i} by the inductive formula:

$$b_{r+i} := 1 + (b_{r+1-i} - 1)^2[1 + (b_r - 1)b_{r+1} + (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_{r+2-i} - 1)b_{r+3-i} \cdots b_{r+i-1}]$$

for $1 \leq i \leq r$ when $n = 2r+1$ or $1 \leq i \leq r-1$ when $n = 2r$.

Definition 4.7.5. For $n = 2r+1$, $r \geq 1$, define an integer d by:

$$d := (b_0 - 1)^2 b_1 \cdots b_{2r} + (b_1 - 1)[1 + (b_r - 1)b_{r+1} + (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_1 - 1)b_2 \cdots b_{2r}].$$

For $n = 2r$, $r \geq 1$, define an integer d by:

$$d := (b_1 - 1)^2 b_2 \cdots b_{2r-1} + (b_2 - 1)[1 + (b_r - 1)b_{r+1} + (b_{r-1} - 1)b_r b_{r+1} b_{r+2} + \cdots + (b_2 - 1)b_3 \cdots b_{2r-1}].$$

Definition 4.7.6. For $n = 2r+1$, $r \geq 1$, define $b_{2r+1} := \frac{d+1}{2}$.

For $n = 2r$, $r \geq 1$, define $b_{2r} := \frac{2d+1}{3}$.

Definition 4.7.7. For $n = 2r+1$, $r \geq 1$, define a_j for $0 \leq j \leq 2r+2$ by the following inductive formulas. They are positive integers.

$$
\begin{aligned}
&a_{2r+1} = a_{2r+2} := 1,\\
&a_{2r-i} := a_{2r-i+1}\left(\frac{1}{s_{i+1}}\right) + (s_{r+1}-1)\left(1-\frac{1}{s_{i+1}}\right)\left[a_{2r-i+1}\left(1-\frac{1}{s_{i+1}}\right) + \cdots\right.\\
&\left. + a_{2r}\left(1-\frac{1}{s_2}\right) + a_{2r+1}\left(1-\frac{1}{s_1}\right) + a_{2r+2}\left(1-\frac{1}{s_0}\right)\right] b_{r+1}\cdots b_{2r-i-1} \text{ for } 0 \le i \le r-1,\\
&a_i := d - a_{2r+1-i}b_{2r+1-i} \text{ for } 0 \le i \le r.
\end{aligned}
$$

For $n = 2r$, $r \geq 1$, define a_j for $0 \leq j \leq 2r+1$ by the inductive formulas:

$$
\begin{aligned}
&a_{2r} = a_{2r+1} := 1\\
&a_{2r-1-i} := a_{2r-i}\left(\frac{1}{s_{i+2}}\right) + (s_{r+1}-1)\left(1-\frac{1}{s_{i+2}}\right)\left[a_{2r-i}\left(1-\frac{1}{s_{i+2}}\right) + \cdots\right.\\
&\left. + a_{2r-1}\left(1-\frac{1}{s_3}\right) + a_{2r}\left(1-\frac{1}{s_2}\right) + a_{2r+1}\left(1-\frac{1}{s_1}\right)\right] b_{r+1}\cdots b_{2r-i-2} \text{ for } 1 \le i \le r-2,\\
&a_i := d - b_{2r+1-i}a_{2r+1-i} \text{ for } 0 \le i \le r,\\
&a_0 := \frac{d}{2}.
\end{aligned}
$$

Definition 4.7.8. For $n = 2r+1$, $r \geq 1$, we define an integer $b_{2r+2} := d - a_{r+1}$.

For $n = 2r$, $r \geq 1$, we define an integer $b_{2r+1} := d - a_{r+1}$.

Definition 4.7.9. For $n = 2r+1$, $r \geq 1$, define an integer m by:

$$
\begin{aligned}
m := {} & b_0\cdots b_{2r+1} - b_0\cdots b_r b_{r+2}\cdots b_{2r+1}\\
& + b_0\cdots b_{r-1}b_{r+2}\cdots b_{2r+1} - b_0\cdots b_{r-1}b_{r+3}\cdots b_{2r+1} + \cdots - b_0 + 1.
\end{aligned}
$$

For $n = 2r$, $r \geq 1$, define an integer m by:

$$
m := b_1\cdots b_{2r} - b_1\cdots b_r b_{r+2}\cdots b_{2r} + b_1\cdots b_{r-1}b_{r+2}\cdots b_{2r} - \cdots + 1.
$$

We were led to the choice of weights a_i and exponents b_i in Definition 4.7.7 and Definition 4.7.4 by an inductive procedure. First we chose the general form of the equation in (4.10) or (4.11) below. Since each monomial in the equation must have degree d, we get a relation involving d and some a_i and b_i. We use the relations given by all monomials and the relation that all the weights a_i sum to d to express d as $r+1$ linear combinations of $a_{r+i}, \ldots, a_{2r+2}$ for $1 \leq i \leq r+1$ and express each a_{r+i} as a linear combination of $a_{r+i+1}, \ldots, a_{2r+2}$ for $1 \leq i \leq r$. During each step, we choose one weight as big as possible which determines our choice of the corresponding exponent. In each step, we can increase i by one. This process yields the formulas above.

We now define our example. When $n = 2r+1$ for $r \geq 1$, let X_d be the hypersurface in $\mathbb{P}(a_0, a_1, \ldots, a_{2r+1}, a_{2r+2})$ of degree d given by

$$x_0^{b_0} x_{2r+2} + x_1^{b_1} x_{2r+1} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + x_{r+2}^{b_{r+2}} x_{r-1} + \cdots + x_{2r+1}^{b_{2r+1}} x_0 + x_{2r+2}^{b_{2r+2}} x_{r+1} = 0. \quad (4.10)$$

This is a loop potential, but with the variables written in a different order than in (4.6) above; one can check that it is homogeneous of weighted degree d. The group $G = \mathrm{Aut}_T(X)$ is cyclic of order m and acts freely in codimension 1 on X by Proposition 4.7.2. Similarly, when $n = 2r$ for $r \geq 1$, let X_d be the hypersurface in $\mathbb{P}(a_0, a_1, \ldots, a_{2r}, a_{2r+1})$ of degree d given by

$$x_0^{b_0} + x_1^{b_1} x_{2r+1} + x_2^{b_2} x_{2r} + \cdots + x_r^{b_r} x_{r+2} + x_{r+1}^{b_{r+1}} x_r + x_{r+2}^{b_{r+2}} x_{r-1} + \cdots + x_{2r}^{b_{2r}} x_1 + x_{2r+1}^{b_{2r+1}} x_{r+1} = 0. \quad (4.11)$$

This potential has the form $x_0^2 + f_{\text{loop}}$. There is a subgroup G of index 2 inside $\mathrm{Aut}_T(X)$ corresponding to f_{loop} which is cyclic of order m and which acts freely in codimension 1 on X (this does not follow from Proposition 4.7.2 when $n = 2$, but nevertheless is still true in this case).

We conjecture that the subgroup G acts faithfully on $H^0(X, K_X) \cong \mathbb{C}$, so that the index of X/G is in fact m. By Proposition 4.7.3, proving this conjecture would amount to a gcd calculation involving the exponents b_i. This is true in dimensions at most 30 by computer verification, and so we have klt Calabi-Yau varieties of extremely large index at least in low dimensions. The number m has $\log m$ asymptotic to $\log s_{n+1}$, and so this conjecture would imply that the index of our klt Calabi-Yau n-fold is comparable to that of the klt Calabi-Yau pair in Theorem 4.3.3, and much bigger than the index of our terminal Calabi-Yau variety in Corollary 4.4.1. More precisely, we conjecture:

Conjecture 4.7.10. *The quotient X/G above has index m. Moreover, in each dimension $n \geq 2$, this is the largest possible index for a klt Calabi-Yau variety.*

For $n = 2$, the hypersurface X is

$$\{x_0^2 + x_1^3 x_3 + x_2^7 x_1 + x_3^9 x_2 = 0\} \subset \mathbb{P}^3(5, 3, 1, 1),$$

with an action of the cyclic group G of order 19 that is free in codimension 1 and faithful on $H^0(X, K_X)$. So X/G is a klt Calabi-Yau surface of index 19, which is the largest possible, by Proposition 4.6.1. For $n = 3$, the hypersurface X is

$$\{x_0^2 x_4 + x_1^3 x_3 + x_2^5 x_1 + x_3^{19} x_0 + x_4^{32} x_2 = 0\} \subset \mathbb{P}^4(18, 12, 5, 1, 1).$$

The cyclic group of order 493 acts on X with quotient a klt Calabi-Yau 3-fold of index 493. That is well above the index 66 of the terminal Calabi-Yau 3-fold in Corollary 4.4.1.

For $n = 4$, the hypersurface X is

$$\{x_0^2 + x_1^3 x_5 + x_2^7 x_4 + x_3^{37} x_2 + x_4^{1583} x_1 + x_5^{2319} x_3 = 0\} \subset \mathbb{P}^5(1187, 791, 339, 55, 1, 1).$$

The cyclic group of order 1201495 acts on X with quotient a klt Calabi-Yau 4-fold of index 1201495. In dimensions 3 and 4, our example has the largest index among all quotients by toric automorphisms of quasismooth Calabi-Yau hypersurfaces defined by potentials. This was verified by computer search, using the databases of Calabi-Yau threefold and fourfold hypersurfaces in [BK].

[ABS14] M. Artebani, S. Boissière, and A. Sarti. "The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces." *J. Math. Pure Appl.*, **102**:758–781, 2014. 78, 99, 100, 103

[ACG16] M. Artebani, P. Comparin, and R. Guilbot. "Families of Calabi-Yau hypersurfaces in $\mathbb{Q}$-Fano toric varieties." *J. Math. Pures Appl.*, **106**:319–341, 2016. 13

[AHR19] J. Alper, J. Hall, and D. Rydh. "The étale local structure of algebraic stacks." Preprint at arXiv:1912.06162, 2019. 10

[Ale94] V. Alexeev. "Boundedness and K^2 for log surfaces." *Internat. J. Math.*, **5**:779–810, 1994. 68

[Amb99] F. Ambro. "Ladders on Fano varieties." *J. Math. Sci. (N.Y.)*, **94**:1126–1135, 1999. 49

[Amr89] A. Al Amrani. "Classes d'idéaux et groupe de Picard des fibrés projectifs tordus." *K-Theory*, **2**:559–578, 1989. 6

[Art62] M. Artin. "Some numerical criteria for contractability of curves on algebraic surfaces." *Amer. J. Math.*, **84**:485–496, 1962. 97

[Bat20] V. Batyrev. "On the stringy Hodge numbers of mirrors of quasi-smooth Calabi-Yau hypersurfaces." Preprint at arXiv:2006.15825, 2020. 81, 83

[BCH10] C. Birkar, P. Cascini, C. Hacon, and J. McKernan. "Existence of minimal models for varieties of log general type." *J. Amer. Math. Soc.*, **23**:405–468, 2010. 72

[BCS05] L. Borisov, L. Chen, and G. Smith. "The orbifold Chow ring of toric Deligne-Mumford stacks." *J. Amer. Math. Soc.*, **18**(1):193–215, 2005. 78

[Bea96] A. Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge Univ. Press, 1996. 80

[BH21] S. Brandhorst and T. Hofmann. "Finite subgroups of automorphisms of K3 surfaces." To appear in *Forum Math. Sigma*. Preprint at arXiv:2112.07715, 2021. 93, 94, 96, 97, 98

[BHP04] W. Barth, K. Hulek, C. Peters, and A. van de Ven. *Compact complex surfaces*. Springer, 2nd edition, 2004. 62

[Bir19] C. Birkar. "Anti-pluricanonical linear systems on Fano varieties." *Ann. of Math.*, **190**:345–463, 2019. 1, 54

[Bir20] C. Birkar. "Geometry of polarised varieties." *Pub. Math. IHES*, to appear. Preprint at arXiv:2006.11238, 2020. 1, 50

[BK] G. Brown and A. Kasprzyk. "Graded Ring Database." http://www.grdb.co.uk/. 52, 53, 61, 67, 107

[BK16] G. Brown and A. Kasprzyk. "Four-dimensional projective orbifold hypersurfaces." *Exp. Math*, **25**(2), 2016. 46, 52, 53, 55

[BKN] G. Balletti, A. Kasprzyk, and B. Nill. "On the maximum dual volume of a canonical Fano polytope." To appear in *Forum Math. Sigma*. Preprint at arXiv:1611.02455. 78

[Bla95] R. Blache. "The structure of l.c. surfaces of Kodaira dimension zero. I." *J. Alg. Geom.*, **4**:137–179, 1995. 93

[BM17] A. Bonifant and J. Milnor. "On real and complex cubic curves." *Enseign. Math*, **63**:21–61, 2017. 44

[BPT13] E. Ballico, R. Pignatelli, and L. Tasin. "Weighted hypersurfaces with either assigned volume or many vanishing plurigenera." *Comm. Algebra*, **41**:3745–3752, 2013. 46, 47

[BR86] M. Beltrametti and L. Robbiano. "Introduction to the theory of weighted projective spaces." *Exposition. Math*, **4**(2):111–162, 1986. 18

[Bra19] S. Brandhorst. "The classification of purely non-symplectic automorphisms of high order on K3 surfaces." *J. Algebra*, **533**:229–265, 2019. 94

[Bre12] E. Breuillard. "An exposition of Jordan's original proof of his theorem on finite subgroups of $\mathrm{GL}_n(\mathbb{C})$." `http://www.math.u-psud.fr/~breuilla/Jordan.pdf`, 2012. 30

[Bri18] M. Brion. "Linearization of algebraic group actions." *Adv. Lect. Math.*, **41**:291–340, 2018. 22, 23

[Bro07] G. Brown. "A database of polarized K3 surfaces." *Exp. Math*, **16**(1):1–20, 2007. 52

[Can13] P. Candelas. "Max Kreuzer's contributions to the study of Calabi-Yau manifolds." *Strings, gauge fields, and the geometry behind: the legacy of Maximilian Kreuzer, In: World Scientific, Singapore*, pp. 161–172, 2013. 81

[CC08] Jungkai A. Chen and Meng Chen. "An optimal boundedness on weak $\mathbb{Q}$-Fano 3-folds." *Adv. Math.*, **219**:2086–2104, 2008. 55

[CCH21] W. Chen, G. Di Cerbo, J. Han, C. Jiang, and R. Svaldi. "Birational boundedness of rationally connected Calabi-Yau 3-folds." *Adv. Math.*, **378**, 2021. 71

[CCJ20] Jungkai A. Chen, Meng Chen, and Chen Jiang. "The Noether inequality for algebraic threefolds (with an appendix by J Kollár)." *Duke Math. J.*, **169**:1603–1645, 2020. Addendum, *Duke Math. J.* **169**(2020) 2199-2204. 67

[Cha78] H. C. Chang. "On plane algebraic curves." *Chinese J. Math*, **6**(2):185–189, 1978. 17, 24, 38

[CJ17] Meng Chen and Zhi Jiang. "A reduction of canonical stability index of 4 and 5 dimensional projective varieties with large volume." *Ann. Inst. Fourier (Grenoble)*, **67**:2043–2082, 2017. 67

[CL20] Jungkai A. Chen and Ching-Jui Lai. "Varieties of general type with small volumes." *Internat. J. Math.*, **31**:2050005 (21 pages), 2020. 67

[Col07] M. Collins. "On Jordan's theorem for complex linear groups." *J. Group Theory*, **10**:411–423, 2007. 30, 33

[CR04] W. Chen and Y. Ruan. "A new cohomology theory of orbifold." *Comm. Math. Phys.*, **248**:1–31, 2004. 81

[CR11] A. Chiodo and Y. Ruan. "LG/CY correspondence: the state space isomorphism." *Adv. Math.*, **227**:2157–2188, 2011. 85

[DD19] I. Dolgachev and A. Duncan. "Automorphisms of cubic surfaces in positive characteristic." *Izv. Ross. Akad. Nauk Ser. Mat.*, **83**(3):15–92, 2019. 39

[DIK00] H. Doi, K. Idei, and H. Kaneta. "Uniqueness of the most symmetric non-singular plane sextics." *Osaka J. Math.*, **37**:667–687, 2000. 38

[Dol82] I. Dolgachev. "Weighted projective varieties." In *Group actions and vector fields (Vancouver, 1981), Lecture Notes in Mathematics 956*, pp. 34–71. 1982. 20, 21, 22

[Ess23] L. Esser. "Automorphisms of weighted projective hypersurfaces." Preprint at arXiv:2301.07872, 2023. viii, 2

[ETT22] L. Esser, T. Tao, B. Totaro, and C. Wang. "Optimal sine and sawtooth inequalities." *J. Fourier Anal. Appl.*, **28**(2), 2022. Paper No. 14, 21 pp. 46, 47

[ETW22] L. Esser, B. Totaro, and C. Wang. "Calabi-Yau varieties of large index." Preprint at arXiv:2209.04597, 2022. viii, 2

[ETW23] L. Esser, B. Totaro, and C. Wang. "Varieties of general type with doubly exponential asymptotics." *Trans. Amer. Math. Soc. Ser. B*, **10**:288–309, 2023. viii, 2, 47

[GK08] D. H. Greene and D. E. Knuth. *Mathematics for the analysis of algorithms*. Birkhäuser, 3rd edition, 2008. 50

[GL13] V. González-Aguilera and A. Liendo. "On the order of an automorphism of a smooth hypersurface." *Israel J. Math*, **197**(1):29–49, 2013. 39

[Gro] A. Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA 2*). Société Mathématique de France, volume 4, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d'un exposé de Michèle Raynaud [With an exposé by Michèle Raynaud]. With a preface and edited by Yves Laszlo. Revised reprint of the 1968 French original. 16, 17, 21

[Har80] R. Hartshorne. "Stable reflexive sheaves." *Math. Ann*, **254**(2):121–176, 1980. 19

[Har19] T. Harui. "Automorphism groups of plane curves." *Kodai Math. J*, **42**(2):308–331, 2019. 38

[HM06] C. Hacon and J. McKernan. "Boundedness of pluricanonical maps of varieties of general type." *Invent. Math.*, **166**:1–25, 2006. 1, 46

[HMX13] C. Hacon, J. McKernan, and C. Xu. "On the birational automorphisms of varieties of general type." *Ann. of Math.*, **177**:1077–1111, 2013. 1, 47, 73

[HMX14] C. Hacon, J. McKernan, and C. Xu. "ACC for log canonical thresholds." *Ann. of Math.*, **180**(2):523–571, 2014. 54, 68, 72

[HS81] A. Howard and A. Sommese. "On the orders of the automorphism groups of certain projective manifolds." *Manifolds and Lie groups*, pp. 145–158, 1981. 29

[Ian00] A. R. Iano-Fletcher. "Working with weighted complete intersections." *Explicit birational geometry of 3-folds, London Math. Soc. Lecture Notes Ser. Cambridge Univ. Press, Cambridge*, **281**:101–173, 2000. 4, 5, 8, 9, 10, 18, 46, 53, 55

[Isa08] I. M. Isaacs. *Finite group theory*, volume 92 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. 30

[JL20] C. Jiang and H. Liu. "Boundedness of log pluricanonical representations of log Calabi-Yau pairs in dimension 2." Preprint at arXiv:2002.11928, 2020. 1, 68

[Jor78] C. Jordan. "Mémoire sur les équations différentielles linéaires à intégrale algébrique." *J. Reine Angew. Math*, **84**:89–215, 1878. 30

[Jor80] C. Jordan. "Mémoire sur l'équivalence des formes." *J. Ecole Polytech*, **48**:112–150, 1880. 26

[Kas09] A. Kasprzyk. "Bounds on fake weighted projective space." *Kodai Math. J.*, **32**:197–208, 2009. 70

[Kaw00] Y. Kawamata. "On effective non-vanishing and base-point-freeness." *Asian J. Math.*, **4**:173–181, 2000. 49

[Kle79] F. Klein. "Ueber die Transformation siebenter Ordnung der elliptischen Functionen." *Math. Ann*, **14**:428–471, 1879. 38

[Kle84] F. Klein. *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade*. Teubner, 1884. 38

[KM98] J. Kollár and S. Mori. *Birational geometry of algebraic varieties.* Cambridge Univ. Press, Cambridge, 1998. 2, 72

[KMP99] H. Kaneta, S. Marcugini, and F. Pambianco. "The most symmetric non-singular plane curves of degree $n < 8$." *RIMS Kokyuroku*, **1109**:182–191, 1999. 38

[KMP01] H. Kaneta, S. Marcugini, and F. Pambianco. "The most symmetric non-singular plane curves of degree $n \leq 20$, I." *Geom. Dedicata*, **85**:317–334, 2001. 38

[Kol94] J. Kollár. "Log surfaces of general type: some conjectures." *Contemp. Math.*, **162**:261–275, 1994. 47, 70, 73

[Kol97] J. Kollár. *Singularities of pairs*, volume 62. American Mathematical Society, 1997. 3, 4, 10, 70, 71, 72, 75, 95

[Kol07] J. Kollár. *Lectures on resolution of singularities.* Princeton Univ. Press, Princeton, 2007. 72

[Kon92] S. Kondo. "Automorphisms of algebraic K3 surfaces which act trivially on Picard groups." *J. Math. Soc. Japan*, **44**:75–98, 1992. 94

[Kon02] A. Kontogeorgis. "Automorphisms of Fermat-like varieties." *Manuscripta Mathematica*, **107**:187–205, 2002. 36

[KS00] M. Kreuzer and H. Skarke. "Reflexive polyhedra, weights, and toric Calabi-Yau fibrations." *Adv. Theor. Math. Phys.*, **4**:1209–1230, 2000. 81

[Liu22] J. Liu. "Remark on complements on surfaces." Preprint at `arXiv:2208.09184`, 2022. 70, 73

[LZ22] R. Laza and Z. Zheng. "Automorphisms and periods of cubic fourfolds." *Math. Z*, **300**(2):1455–1507, 2022. 39

[MM64] H. Matsumura and P. Monsky. "On the automorphisms of hypersurfaces." *J. Math. Kyoto Univ*, **3**:347–361, 1963/1964. 16, 17, 25, 26, 39, 40

[MO98] N. Machida and K. Oguiso. "On K3 surfaces admitting finite non-symplectic group actions." *J. Math. Sci. Univ. Tokyo*, **5**(2):273–297, 1998. 69, 76, 80

[Mor75] S. Mori. "On a generalization of complete intersections." *J. Math. Kyoto Univ*, **15**(3):619–646, 1975. 18

[MP04] J. McKernan and Y. Prokhorov. "Threefold thresholds." *Manuscripta Math.*, **114**:281–304, 2004. 68

[OS78] P. Orlik and L. Solomon. "Singularities. II. Automorphisms of forms." *Math. Ann*, **231**(3):229–240, 1977/78. 26, 29

[OY19] K. Oguiso and X. Yu. "Automorphism groups of smooth quintic threefolds." *Asian J. Math*, **23**(2):201–256, 2019. 39

[Pro21] Y. Prokhorov. "Equivariant minimal model program." *Russian Math. Surveys*, **76**:461–542, 2021. 72, 73

[PS17] Y. Prokhorov and C. Shramov. "Jordan constant for Cremona group of rank 3." *Moscow Math. J*, **17**(3):457–509, 2017. 30, 37

[PS19] V. Przyjalkowski and C. Shramov. "Automorphisms of weighted complete intersections." *Proc. Steklov Inst. Math.*, **307**:198–209, 2019. 16, 18, 26

[PS21] V. Przyjalkowski and C. Shramov. "On automorphisms of quasi-smooth weighted complete intersections." *Sb. Math.*, **212**, 2021. 16, 18

[PST17] M. Pizzato, T. Sano, and L. Tasin. "Effective non-vanishing for Fano weighted complete intersections." *Algebra Number Theory*, **11**:2369–2395, 2017. 49

[Reia] M. Reid. "Minimal models of canonical threefolds." *Algebraic varieties and analytic varieties*, pp. 131–180. Advanced Studies in Pure Mathematics, 1, Mathematical Society of Japan, Tokyo (1983). 14

[Reib] M. Reid. "Young person's guide to canonical singularities." *Algebraic Geometry* (Bowduin, 1985), pp. 345–414. Proc. Symp. Pure Math. 46, Amer. Math. Soc. (1987). 2, 5, 14

[Rei80] M. Reid. "Canonical 3-folds." *Journées de géometrie algébrique d'Angers, Sijthoff & Noordhoff, Alphen aan den Rijn*, pp. 273–310, 1980. 13

[RS06] G. V. Ravindra and V. Srinivas. "The Grothendieck-Lefschetz theorem for normal projective varieties." *J. Algebraic Geom*, **15**:563–590, 2006. 17

[Shi87] T. Shioda. "Arithmetic and geometry of Fermat curves." *Algebraic Geometry Seminar, Singapore*, pp. 95–102, 1987. 36

[Sou05] K. Soundararajan. "Approximating 1 from below using n Egyptian fractions." Preprint at `arXiv:0502247`, 2005. 50

[Tak06] S. Takayama. "Pluricanonical systems on algebraic varieties of general type." *Invent. Math.*, **165**:551–587, 2006. 46

[Tot22] B. Totaro. "Klt varieties of conjecturally minimal volume." *International Mathematics Research Notices*, to appear. Preprint at arXiv:2210.11354, 2022. 48

[Tsu07] H. Tsuji. "Pluricanonical systems of projective varieties of general type. II." *Osaka J. Math.*, **44**:723–764, 2007. 46

[TW21] B. Totaro and C. Wang. "Klt varieties of general type with small volume." To appear in *Ann. Sc. Norm. Super. Pisa Cl. Sci*. Preprint at arXiv:2104.12200, 2021. 47, 54

[Web98] H. Weber. *Lehrbuch der Algebra*, volume 2. Braunschweig, F. Vieweg and Sohn (reprinted by Plenum Publ. Co., New-York, 1961), 1898. 44

[Wim96] A. Wiman. "Ueber eine einfache Gruppe von 360 ebenen Collineationen." *Math. Ann*, **47**:531–556, 1896. 38

[WY20] L. Wei and X. Yu. "Automorphism groups of smooth cubic threefolds." *J. Math. Soc. Japan*, **72**(4):1327–1343, 2020. 39

[Xu19] Y. Xu. "Some results about the index conjecture for log Calabi-Yau pairs." Preprint at arXiv:1905.00297, 2019. 1, 68

[Yas04] T. Yasuda. "Twisted jets, motivic measure and orbifold cohomology." *Compos. Math.*, **140**:396–422, 2004. 81

[Zha91] D.-Q. Zhang. "Logarithmic Enriques surfaces." *J. Math. Kyoto Univ*, **31**(2):419–466, 1991. 93

[Zhe22] Z. Zheng. "On abelian automorphism groups of hypersurfaces." *Israel J. Math*, **247**(1):479–498, 2022. 39