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# Solution of scaling quantum networks

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We show that all scaling quantum graphs are explicitly integrable, i.e. any one of their spectral eigenvalues $E_n$ is computable analytically, explicitly, and individually for any given $n$. This is surprising, since quantum graphs are excellent models of quantum chaos [see, e.g., T. Kottos and H. Schanz, Physica E **9**, 523 (2001)].

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Graphs are networks of bonds and vertices. Figure 1 shows two examples: a three-bond four-vertex star graph (Fig. 1a) and a three-bond four-vertex linear graph (Fig. 1b). A quantum particle moving on the graph turns the graph into a quantum graph [1]. If the quantum particle moves freely on the graph, subjected only to flux conservation at its vertices, we call it a *standard quantum graph*. This is the type of quantum graph most frequently studied in the literature [1–5]. A larger class of quantum graphs, including the standard quantum graphs, are *dressed quantum graphs* [6]. A dressed quantum graph has potentials on its bonds and $\delta$-functions on its vertices. The potentials on its bonds are essentially arbitrary as long as they do not introduce turning points on the bonds. But even this case can be dealt with trivially by re-defining the topology of the graph.

An important subset of dressed quantum graphs are *scaling quantum graphs* [7–9]. In this case the graph bonds are dressed with scaling potentials and the graph vertices are dressed with scaling $\delta$-functions. A scaling potential is one whose strength $V_0$ scales with the energy $E$ of the quantum particle according to $V_0 = \lambda E$, where $\lambda$ is a constant. The strength of a scaling $\delta$-function scales with $k = \sqrt{E}$. Scaling potentials and $\delta$-functions are a natural choice to consider. On the one hand they frequently occur in physical systems [10–15], on the other hand they are mathematically convenient, since they allow studying a quantum

system without causing phase-space metamorphoses [16] in the underlying classical system.

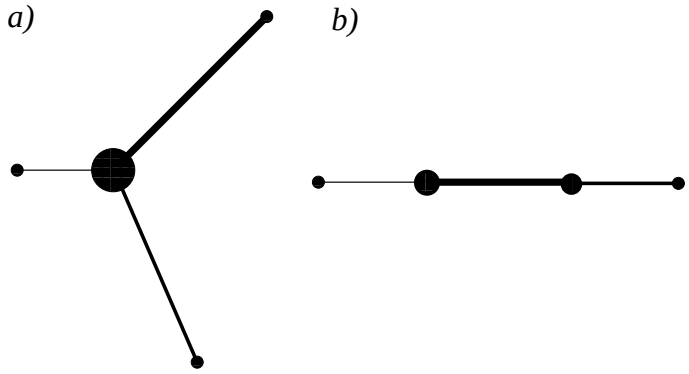


FIG. 1. (a) Dressed three-bond star graph and (b) dressed four-vertex chain graph. Different potential strengths on the bonds are indicated by different thickness of the bonds. Different vertex strengths are indicated by different dot sizes representing the vertices.

It has been pointed out before [17] that this is the most natural way of studying quantum systems, in particular quantum chaos [18,19]. Since quantum graphs are popular and successful models of quantum chaos [1–5], it may come as a surprise that the energy spectrum $E_n$, $n = 1, 2, \ldots$ of all scaling quantum graphs is explicitly and analytically solvable in the form $E_n = \ldots$, involving only known quantities on the right-hand side. In many cases the solutions can be stated in closed analytical form.

The spectral function $g^{(0)}(k)$ of a general scaling, dressed quantum graph is of the form [7]

$$g^{(0)}(k) = \cos(S_0 k - \pi\gamma_0) - \sum_{j=1}^{N} a_j^{(0)} \cos(S_j k - \pi\gamma_j), \tag{1}$$

where $S_0 > 0$ is the total reduced action length of the graph [7,8], $0 < S_j < S_0$ are certain combinations of the reduced bond actions [7,8], $N$ is the number of action combinations in (1), $\gamma_0$, $\gamma_j$ are constant phases and $a_j^{(0)}$ are constant amplitudes. The spectrum $E_n$ of the quantum graph is obtained by solving the spectral equation

$$g^{(0)}(k_n^{(0)}) = 0, \quad n = 1, 2, \ldots \tag{2}$$

via $E_n = (k_n^{(0)})^2$. For the purposes of this paper we are only interested in the positive solutions of (2), and obtain a well-defined counting index $n$ by defining $k_1^{(0)}$ to be the first positive root of (2). As a first step toward the solution of the general problem, it was shown in [7–9] that (2) can be solved explicitly in the form $k_n^{(0)} = \ldots$ if the regularity condition

$$\sum_{j=1}^{N} |a_j^{(0)}| < 1 \tag{3}$$

is fulfilled. In order to substantiate our claim that (2) is solvable explicitly for *all* scaling quantum graphs, we have to show that (2) is solvable explicitly even if (3) is not fulfilled.

Before we turn our attention to the general case, we introduce our methods with the help of a simple example. Let us consider a scaling quantum graph derived from the three-bond star graph shown in Fig. 1a by putting the scaling potentials $V_l(E) = \lambda_l E$, $0 < \lambda_l < 1$ on its three bonds of length $L_l$, $l = 1, 2, 3$, require the "Kirchhoff-type" [5] flux conservation condition $\sum_{l=1}^{3} d\psi_l/dx_l = 0$ at its central vertex ($\psi_l$ is the quantum wave function on bond number $l$ of the graph and $x_l$ is the coordinate on bond number $l$) and require Dirichlet boundary conditions on its three dead-end vertices. The spectral equation is of the form (1) with $N = 3$, $\gamma_1 = \gamma_2 = \gamma_3 = 0$ and

$$S_0 = \alpha_1 + \alpha_2 + \alpha_3, \quad S_1 = -\alpha_1 + \alpha_2 + \alpha_3, \quad S_2 = \alpha_1 - \alpha_2 + \alpha_3, \quad S_3 = \alpha_1 + \alpha_2 - \alpha_3, \tag{4}$$

$$a_1^{(0)} = \frac{-\beta_1 + \beta_2 + \beta_3}{\beta_1 + \beta_2 + \beta_3}, \quad a_2^{(0)} = \frac{\beta_1 - \beta_2 + \beta_3}{\beta_1 + \beta_2 + \beta_3}, \quad a_3^{(0)} = \frac{\beta_1 + \beta_2 - \beta_3}{\beta_1 + \beta_2 + \beta_3}, \tag{5}$$

where

$$\alpha_l = \beta_l L_l, \quad \beta_l = \sqrt{1 - \lambda_l}, \quad l = 1, 2, 3. \tag{6}$$

The amplitudes in (5) do not fulfill the regularity condition (3). In some cases $\sum_{j=1}^{3} |a_j^{(0)}| = 1$ (for instance for $a_j^{(0)} > 0$, $j = 1, 2, 3$), and in many cases $\sum_{j=1}^{3} |a_j^{(0)}| > 1$, which strongly violates the regularity condition (3). Since the methods and techniques presented in [7–9] for obtaining the spectrum of a graph explicitly depend crucially on (3), it seems that completely

different methods have to be developed for general graphs, such as the three-bond star graph of Fig. 1a, which do not fulfill (3). There is, however, a way to reduce (1) to a form that allows to bring the powerful theory of regular quantum graphs [7–9] to bear. In order to motivate and to illustrate this method, let us study the case $\alpha_1 = 1$, $\alpha_2 = 7$, $\alpha_3 = 11$, $\beta_1 = 1/10$, $\beta_2 = 1/5$, $\beta_3 = 1/2$. In this case $a_1^{(0)} = 3/4$, $a_2^{(0)} = 1/2$, $a_3^{(0)} = -1/4$, $S_0 = 19$, $S_1 = 17$, $S_2 = 5$, $S_3 = -3$ and the spectral equation is given by

$$g^{(0)}(k) = \cos(19k) - \frac{3}{4}\cos(17k) - \frac{1}{2}\cos(5k) + \frac{1}{4}\cos(3k). \tag{7}$$

Since $|a_1^{(0)}| + |a_2^{(0)}| + |a_3^{(0)}| = 3/2 > 1$, this quantum graph is certainly not regular. But let us look at the first derivative of (7). Dividing by $S_0$, we obtain

$$g^{(1)}(k) = \cos[S_0 k + \pi/2] - \sum_{j=1}^{3} a_j^{(1)} \cos[S_j k + \pi/2] =$$

$$-\sin(19k) + \frac{51}{76}\sin(17k) + \frac{5}{38}\sin(5k) - \frac{3}{76}\sin(3k). \tag{8}$$

This time we have $\sum_{j=1}^{3} |a_j^{(1)}| = 16/19 < 1$ and therefore, since (8) is precisely of the form (1) and satisfies (3), it can be solved explicitly using the methods of [7–9]. In particular it was shown in [7–9] that root number $n$ of a spectral equation that satisfies (3), such as (8), is found in the root interval $[\hat{k}_{n-1}, \hat{k}_n]$, where $\hat{k}_n$ are the root separators [7–9]. It was also shown in [7–9] that the location of the root separators is entirely controlled by the local extrema of the trigonometric function with the largest action argument. Thus, in our case, the root separators of (8) are given by $\hat{k}_n = (2n+1)\pi/38$. Since according to [7–9] root number $n$ and only root number $n$ is located in the interval $[\hat{k}_{n-1}, \hat{k}_n]$, we can now compute all roots of (8) explicitly and individually according to

$$k_n^{(1)} = \int_{\hat{k}_{n-1}}^{\hat{k}_n} k \left| \frac{dg^{(1)}(k)}{dk} \right| \delta(g^{(1)}(k))\, dk. \tag{9}$$

In [20] we show that because of the hermiticity of the spectral eigenvalue problem on quantum graphs the locations of the local extrema of $g^{(0)}(k)$ are separators for the roots of $g^{(0)}(k)$. The location of the local extrema of $g^{(0)}(k)$, however, are given by the zeros of $g^{(1)}(k)$, which, up

to constants, is the derivative of $g^{(0)}(k)$. Therefore, using the roots $k_n^{(1)}$, explicitly computed in (9), as the root separators of (7), we obtain, again explicitly and individually,

$$k_n^{(0)} = \int_{k_{n-1}^{(1)}}^{k_n^{(1)}} k \left| \frac{dg^{(0)}(k)}{dk} \right| \delta(g^{(0)}(k))\, dk. \tag{10}$$

This solves the task of computing the spectrum of our example of the three-bond dressed star graph explicitly.

In general, given a spectral equation (1) which does not fulfill (3), we generate a chain of derivative spectral equations $g^{(m)}(k)$, where $g^{(m)}(k)$ is the $m$'th derivative of (1) divided by $S_0^m$, explicitly given by

$$g^{(m)}(k) = \cos(S_0 k - \pi\gamma_0 + m\pi/2) - \sum_{j=1}^{N} a_j^{(m)} \cos(S_j k - \pi\gamma_j + m\pi/2), \tag{11}$$

where $a_j^{(m)} = a_j^{(0)}(S_j/S_0)^m$. Since $S_0 < S_j$, there always exists an $M$ such that the amplitudes $a_j^{(m)}$ satisfy the regularity condition (3), i.e., $\sum_{j=1}^{N} |a_j^{(M)}| < 1$. Therefore, according to [7–9], root separators $\hat{k}_n^{(M)}$ exist on the level $M$ and the roots $k_n^{(M)}$ of $g^{(M)}(k) = 0$ are explicitly computable via

$$k_n^{(M)} = \int_{\hat{k}_{n-1}^{(M)}}^{\hat{k}_n^{(M)}} k \left| \frac{dg^{(M)}(k)}{dk} \right| \delta(g^{(M)}(k))\, dk. \tag{12}$$

Since we now know the roots on the level $M$, we can go one step backwards to level $M-1$. According to a root-counting argument [20] based on the Weyl formula [18,19], the root separators $\hat{k}_n^{(M-1)}$ on the level $M-1$ are the locations of the local extrema of $g^{(M-1)}(k)$, which are given explicitly by the roots $k_n^{(M)}$, which we know. Therefore, $\hat{k}_n^{(M-1)} = k_n^{(M)}$ and the roots of $g^{(M-1)}(k) = 0$ can now be computed explicitly, according to

$$k_n^{(M-1)} = \int_{k_{n-1}^{(M)}}^{k_n^{(M)}} k\, S_0 \left| g^{(M)}(k) \right| \delta(g^{(M-1)}(k))\, dk. \tag{13}$$

Steps (12) and (13) define a recursive procedure,

$$k_n^{(m-1)} = \int_{k_{n-1}^{(m)}}^{k_n^{(m)}} k\, S_0 \left| g^{(m)}(k) \right| \delta(g^{(m-1)}(k))\, dk, \quad m = M, M-1, \ldots, 2, \tag{14}$$

which can be followed until the level 0 is reached and the roots $k_n^{(0)}$, i.e. the spectrum of the quantum graph, is known explicitly.

It is important to notice that (12) – (14) are not just formal solutions. They yield $k_n^{(m)}$, $m = 0, \ldots, M$ explicitly, by quadratures. Thus (12) – (14) constitute explicit solutions of the problem, very much in the spirit of the definition of explicit solutions by quadratures in the theory of differential equations [21].

Several special cases require discussion. If the regularity condition (3) is fulfilled, a root $k_n$ lies *strictly inside* of the interval $[\hat{k}_{n-1}, \hat{k}_n]$. However, if (3) is not fulfilled, it is possible that a root $k_n^{(m)}$ coincides with one of its separators $\hat{k}_{n-1}^{(m)}$ or $\hat{k}_n^{(m)}$. This is, e.g., the case in our star-graph example above, where $k_{18}^{(0)} = \pi$ is a root *and* a root separator of (7). In the parameter space of $\alpha$'s and $\beta$'s cases like this are extremely rare (nongeneric). But even if such a case occurs, it does not present a problem for our theory. At the contrary, it saves one integration step since it can always be checked before performing the integration in (14), whether one of the separators $k_{n-1}^{(m)}$ or $k_n^{(m)}$ is a root of $g^{(m-1)}(k)$. If so, the result $k_{n-1}^{(m-1)} = k_{n-1}^{(m)}$, or $k_{n-1}^{(m-1)} = k_n^{(m)}$, respectively, is known in advance, without actually performing the integration.

In other special cases the roots of $g^{(m)}(k) = 0$ can be obtained in the form of explicit periodic orbit expansions [7,8]. In order to illustrate this, let us return to our example of the three-bond star graph. We notice that the spectral equation $g^{(1)}(k) = 0$ of the three-bond star graph looks the same as the spectral equation [7]

$$g^{(0)}_{4\mathrm{V-chain}}(k) = \sin(S_0 k) + r_2 \sin(S_1 k) + r_2 r_3 \sin(S_2 k) - r_3 \sin(S_3 k) = 0 \tag{15}$$

of the dressed four-vertex chain graph shown in Fig. 1b, where $r_2 = (\beta_1 - \beta_2)/(\beta_1 + \beta_2)$, $r_3 = (\beta_2 - \beta_3)/(\beta_2 + \beta_3)$ are the reflection coefficients at the vertices number 2 and 3 of the chain graph, and the actions $S_0, \ldots, S_3$ are the same as in (4). If we arrange for the bond actions of the chain graph to equal the bond actions of the three-bond star graph, and furthermore arrange for $a_1^{(1)} = -r_2$, $a_2^{(1)} = -r_2 r_3$, $a_3^{(1)} = r_3$, which is possible if the scaling constants of the three-bond star graph fulfill $\beta_1^2 - \beta_2^2 + \beta_3^2 = 0$, then $g^{(1)}(k)$ of the

three-bond star graph is the same as the spectral equation (15) of the associated four-vertex chain graph and the spectral points $k_n^{(1)}$ can be stated immediately and explicitly in the form of convergent, periodic orbit expansions [7,8], bypassing any integrations that would have been necessary according to the scheme defined in (14).

Although they presented the first examples of explicitly solvable quantum graphs, a major shortcoming of [7] and [8] is the fact that the theory presented in [7] and [8] is only applicable to *regular* quantum graphs, i.e. quantum graphs that fulfill the regularity condition (3). In this paper we showed that the restriction to regular quantum graphs is not necessary: all scaling quantum graphs can be solved explicitly. Nevertheless, the theory presented in [7–9] provides an indispensable foundation without which the present theory would not be possible.

A conceptual advance is the following. Frequently an operational definition of quantum chaos, or a quantum chaotic regime, is the "loss of quantum numbers". To illustrate, let us consider a Hamiltonian system with Hamiltonian $\hat{H} = \hat{H}_0 + \mu\hat{V}$, where $\hat{H}_0$ is an integrable Hamiltonian, $\mu$ is a real parameter and $\hat{V}$, with respect to and in conjunction with $\hat{H}_0$, is a nonintegrable perturbation. Many quantum systems, for instance the hydrogen atom in a strong magnetic field [10], can be described in this way. For $\mu = 0$ the system is integrable and possesses a complete set of quantum numbers

that can be obtained, at least approximately, using EBK quantization [18,19]. As the parameter $\mu$ increases, EBK quantization breaks down and the system makes a transition to quantum chaos. This explains the frequently employed practice of characterizing the onset of quantum chaos by a loss of quantum numbers, since the breakdown of the EBK quantization scheme implies the loss of quantum numbers. The results obtained in this paper, however, show that this is not necessarily a good way to characterize quantum chaos. Although not strictly chaotic in the classical limit (due to ray-splitting [11,12,22,23] the term *stochastic* may characterize the situation better), quantum graphs were shown by many authors [1–5] to be excellent models of quantum chaos. Yet, our results above show that a well-defined quantum number, the counting index $n$, still exists, and produces explicit energy levels in

exactly the same spirit as the EBK quantization scheme.

The iteration scheme (14) is perhaps the most interesting feature of our method of explicitly solving quantum graphs. We call the smallest $M$ that "regularizes" a given quantum graph (i.e. the amplitudes of $g^{(M)}$ fulfill (3)), the *order* of the quantum graph. For any given quantum graph its order is unique. Since the order $M$ of a quantum graph determines the length of the bootstrapping iteration scheme (14), it is possible that the order of a quantum graph is also an indication of the complexity of its spectrum. Quantum iterations similar to (14) were studied before [24] and were found to lead to sensitivity and chaos on the quantum level. This may explain the reason why certain quantum graphs are such good models of quantum chaos [1–5] and the order $M$ of the quantum graph may be an indication of how well a given quantum graph can be described in terms of the usual diagnostic tools of quantum chaos, such as, e.g., random matrix theory [18,19,25].

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