Title
Propositional Quantification and Comparison in Modal Logic

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Publication Date
2021

Peer reviewed|Thesis/dissertation

Propositional Quantification and Comparison in Modal Logic

by

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A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Logic and the Methodology of Science

in the

Graduate Division

of the

University of California, Berkeley

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Spring 2021

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Abstract

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We make the following contributions to modal logics with propositional quantifiers and modal logics with comparative operators in this dissertation:

- We define a general notion of normal modal logics with propositional quantifiers. We call them normal Π-logics. Then, as was done by Scrogg's theorem on extensions of the modal logic $\mathsf{S5}$, we study in general the normal Π-logics extending $\mathsf{S5}$. We show that they are all complete with respect to their algebraic semantics based on complete simple monadic algebras. We also show that the lattice formed by these logics is isomorphic to the lattice of the open sets of the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the natural order topology. Further, we show how to determine the computability of normal Π-logics extending $\mathsf{S5}\Pi$. A corollary is that they can be of arbitrarily high Turing-degree.

- Regarding the normal Π-logics extending the modal logic $\mathsf{KD45}$, we identify two important axioms: Immod : $\Box\forall p(\Box p \to p)$ and $\mathsf{4}^{\forall}$: $\forall p\Box\varphi \to \Box\forall p\Box\varphi$. We argue that when $\Box$ is interpreted as the belief operator, we should not accept Immod in the logic of belief while $\mathsf{4}^{\forall}$ is desirable in settings with full introspection. Then we provide algebraic semantics based on complete well-connected pseudo monadic algebras for $\Box$ and $\forall p$ and show that with respect to these algebras, normal Π-logics extending $\mathsf{KD45}$ and $\mathsf{4}^{\forall}$ with a finite list of formulas are complete. We also give a general completeness theorem for atomic complete well-connected pseudo monadic algebras and a sufficient condition for the decidability of logics obtained by classes of these algebras, atomic or not. A special case of these general theorems is that the normal Π-logic of serial, transitive, and Euclidean Kripke frames, that is, the Kripke frames validating $\mathsf{KD45}$, is axiomatized by $\mathsf{KD45}$, $\mathsf{4}^{\forall}$, Immod, and $\exists p(p \wedge \forall q(q \to \Box(p \to q)))$ together with the usual axioms and rules for propositional quantifiers, and this logic is decidable.

Other than completeness and decidability, we also show that $\mathsf{4}^{\forall}$ is not in the smallest normal Π-logic extending $\mathsf{KD45}$, using a countermodel based on a possible-world frame with propositional contingency, and that $\mathsf{4}^{\forall}$ is valid in any complete Boolean algebra expansion validating $\mathsf{KD45}$.

- For modal logics with comparative operators, we first provide an axiomatization of the logic of comparing the cardinality of sets, as defined by Cantor. The main technical contribution is the observation that in a purely comparative language, we can define finiteness well enough so that an axiomatization can be done by combining the logic of comparative cardinality for finite sets and the logic of comparative cardinality for infinite sets. Note that these two logics are very different: the former contains the axiom of qualitative additivity: $|A| \geq |B|$ iff $|A \setminus B| \geq |B \setminus A|$ but not the axiom of absorption: if $|A| \geq |B|$ and $|A| \geq |C|$ then $|A| \geq |B \cup C|$, while the latter does the opposite.

- Then we consider modal logics for comparative imprecise probability. In these logics, comparisons are made according to a set of probability measures and can be intuitively read as either "at least as likely as" (symbolized by $\succsim$) or "more likely than" (symbolized by $\succ$). We first disambiguate two interpretations of "more likely than" based on a set of probability measures and show that the stronger interpretation is not definable from "at least as likely as" while the weaker sense is. Then, we go on to axiomatize the logic of imprecise probability in a sequence of languages obtained by adding to the language with just "at least as likely as", one by one, the comparative operator $\succ$ for "more likely than" (in the stronger sense), a unary operator $\Diamond$ for "possibly", and a binary operator $\langle\varphi\rangle\psi$ for "possibly φ, and after learning the truth of φ, ψ". We also comment on the expressivity of these languages and the decidability of the logics in these languages. In particular, we show that many distinctive features of the imprecise probability approach of representing the doxastic states of agents, such as the problem of dilation, are observable at this purely comparative level. Finally, we add a pair of operators $I_p^+\varphi$ and $I_p^-\varphi$, intuitively read, respectively, as "after learning the existence of an actually true new proposition, now named by p, φ", and "after learning the existence of an actually false new proposition, now named by p, φ". We show that this pair of operators allow us to formalize a common kind of information dynamics and will boost the expressivity of the language to a quantitative level.

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Acknowledgments

My first and foremost thanks go to Wesley Holliday for his incredible guidance and support throughout my time at Berkeley. It's impossible to recount all the things I benefited from him and all the ways I'm influenced by him, so I'll just state a simple fact: this dissertation grew out of two of his projects, and most of my other researches done at Berkeley are also in collaboration with him (but of course the mistakes are due to me). Many thanks are also due to Matthew Harrison-Trainer and Thomas Icard who collaborated on parts of this dissertation, and my other dissertation committee members: Shamik Dasgupta, Paolo Mancosu, Dana Scott, and Seth Yalcin. Given my slight social phobia, their agreeing to be on the committee already alleviated a lot of anxiety in me, but I am also very lucky to have learned so many things from them and obtained their feedback on my research. I also thank Lara Buchak, who was on my qualification exam committee for the social choice and judgment aggregation topic and was also on my dissertation committee before she moved to Princeton. I'm grateful also to Tom Scanlon, Ted Slaman, and George Bergman for their classes that helped me a lot in learning logic. Outside Berkeley, I'm forever indebted to Yanjing Wang, who initiated my journey in logic, and Johan van Benthem, who lead me to see even more about logic from very early on and also provided valuable feedback on parts of this dissertation. I also thank the audiences at BLAST 2018, AiML 2018, the 2019 ASL Annual Meeting, and Modal Logic workshop 2019 at Peking University, especially Fengkui Ju, for discussions after the presentations of parts of this dissertation, and the referees for the published versions of the chapters.

Special thanks are to those who simply make life better, sometimes with intellectual conversations and sometimes by just hanging around. Benny Siskind and David Casey have been great office mates, and the general logic group people with Student Logic Colloquium talks and logic beers have been a delight. Thanks for organizing them, Nick Ramsey, Patrick Lutz, and Guillaume Massas, and big thanks to everyone who's been there! For the holiday parties at John Steel's place, I sense that many more people besides John are involved in the organization, and I want to thank them all for making the parties happen. I'll also never forget the hours I spent in Wes's office with Guillaume Massas, Kentaro Yamamoto, James Walsh, and Ced Zhang, and the logic circle gatherings organized by Wes and Thomas; if only there could be more of them! Reid Dale organized stimulating and low-pressure courses from which I learned a lot, and the Formal Epistemology Reading Course organized by Sven Neth and Mikayla Kelley is simply a gem. The visits of Arc Kocurek and Melissa Fusco have also been a source of warmth and happiness. Jixin Liu's one-year visit gave me a much-needed twist in my somewhat monotonous life, and I really enjoyed it.

My partner Shuoqi has been there throughout the six years, and I'm very fortunate to share the totality of my life with her. Much of my life is made meaningful because of her.

My parents have always been curious about my writings and what is this 'logic' that made me spend six years at Berkeley. Sadly, my papers are all in a language completely foreign to them. It's certainly time to change this, but really, it will take a lot more to repay their constant patience, unconditional trust, and the kindest caring.

Chapter 1

Introduction

This dissertation considers logics in which we can quantify over or compare propositions. These operations are an essential part of natural languages. For example, in English we easily find the following sentences:

- “You lied about everything.”
- “She knows more than I do.”
- “That there is no typo in this dissertation is no more likely than two plus two being five.”

The first two English sentences quantify over propositions salient in contexts, and the last sentence compares two propositions, that there is no typo in this dissertation and that $2 + 2 = 5$, in terms of likelihood. Despite their frequent uses in natural language, a logical study—especially a model-theoretical study of them—requires a departure from the “classical” setting in which only the truth values of propositions are modeled. After all, we are not merely quantifying or comparing truth values when we quantify or compare propositions. To get a sense of this, note that adding propositional quantifiers to the classical propositional logic results in only an addition of computational complexity and no gain in expressivity: any formula with propositional quantifiers can easily be translated to an equivalent formula in classical propositional logic without propositional quantifiers, though typically of an exponentially longer length. Hence, the booming of the logical studies on propositional quantifiers and operators comparing propositions came together with the maturation of the logical studies of modalities, in which an account of propositions as sets of possible worlds surfaced.

We will treat propositional quantifiers inside modal logics and see how, with propositional quantifiers, new questions about the philosophical interpretation and mathematical properties of modalities arise. We will also treat propositional comparisons as modalities/propositional operators. Chapter 2 and Chapter 3 of this dissertation deal with propositional quantifiers, and Chapter 4 and Chapter 5 are on propositional comparisons. In the next two sections, we introduce the background and main results of these chapters. These chapters are previously published, so in the last section of this introduction, we note the changes we made when incorporating them in this dissertation.

1.1 Modal Logics with Propositional Quantifiers

Compared to other topics of modal logic, propositional quantifiers have been less investigated. Their introduction into modal logic was very early, however. In Kripke's now landmark paper [114] where he provided a possible-world-based semantics for $\mathsf{S5}$ with first-order quantifiers, he also mentioned that propositional quantifiers can easily be added as well. Indeed, once we arrive at possible-world semantics, a natural candidate for representing propositions themselves becomes obvious: sets of possible worlds (or equivalently functions from all possible worlds to $\{0, 1\}$).

The most prominent effect of having propositional quantifiers that quantify over all these sets of worlds is the gain in expressivity. Other things being equal, a more expressive language is better than a less expressive one, since we can capture more valid reasoning in the more expressive language, and often the reason we devise a more expressive language is that we see an intuitively valid reasoning pattern that is not captured in existing languages: first-order logic versus propositional logic is a case in point. However, expressivity very often complicates the logic. The complication can be either intuitive/cognitive in that more complicated axioms must be used or measurable in mathematically precise ways such as computational complexity and computability. Again, an example of this is the progression from propositional logic to first-order logic and then to second-order logic. With the increase in expressivity, we see that the complexity/computability also rises from NP-complete to recursively enumerable and then to something beyond even the analytical hierarchy and hence without any recursive axiomatization.

As an example of the extra expressivity offered by propositional quantifiers, consider the English sentence "there is some truth she doesn't know." In the standard Kripke semantics, the knowledge of an agent is represented by a set of possible worlds; intuitively, this is the set of possible worlds that are compatible with the knowledge of the agent. Since in different worlds, the agent may know different things, we use a binary relation R on possible worlds, so that the set of worlds related to a given world represents the knowledge of the agent in that given world. Also, we take as primitives a countably infinite set $\mathsf{Prop} = \{\mathsf{p}_0, \mathsf{p}_1, \mathsf{p}_2, \cdots\}$ of atomic propositional variables. Then a Kripke model is a triple $\langle W, R, V\rangle$ where W is a non-empty set interpreted as a set of possible words, R is a binary relation interpreted as above, and V is a valuation function assigning to each $p \in \mathsf{Prop}$ a subset $V(p)$ of W, intuitively the set of worlds where p is true. On the language side, with $\Box$ representing "she knows that", the basic propositional modal language $\mathcal{L}$ is given by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

with $p \in \mathsf{Prop}$. The usual abbreviations, including $\Diamond$ as $\neg\Box\neg$, apply. The truth of formulas in a model $\mathcal{M} = \langle W, R, V \rangle$ is relativized to each possible world in the model and is defined recursively as follows:

$$\begin{array}{lcl} \mathcal{M}, w \vDash p & \iff & w \in V(p) \\ \mathcal{M}, w \vDash \neg\varphi & \iff & \mathcal{M}, w \nvDash \varphi \\ \mathcal{M}, w \vDash (\varphi \wedge \psi) & \iff & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\ \mathcal{M}, w \vDash \Box\varphi & \iff & \forall w' \in R(w), \mathcal{M}, w' \vDash \varphi. \end{array}$$

The last clause is equivalent to: $\mathcal{M}, w \vDash \Box\varphi$ iff $R(w) \subseteq [\![\varphi]\!]^{\mathcal{M}}$, where $[\![\varphi]\!]^{\mathcal{M}} = \{w \in \mathcal{M} \mid \mathcal{M}, w \vDash \varphi\}$ and $w \in \mathcal{M}$ means that $w \in W$ of $\mathcal{M}$. Intuitively, $[\![\varphi]\!]^{\mathcal{M}}$ the set of worlds in $\mathcal{M}$ where φ is true, which can also be regarded as the proposition expressed by φ. We say that a formula φ is *valid* on a model $\mathcal{M}$ if φ is true at every world in $\mathcal{M}$.

A well known result on the limit of the expressive power of $\mathcal{L}$ is that the truth of the formulas in $\mathcal{L}$ is invariant under bisimulation [19]. In particular, consider the following two models:

$$\begin{aligned} \mathcal{M}_1 &= \langle W_1, R_1, V_1 \rangle \\ W_1 &= \{0\} \\ R_1 &= \{\langle 0, 0 \rangle\} \\ V_1(p) &= \begin{cases} W_1 & \text{if } p = \mathtt{p}_1 \\ \varnothing & \text{if } p \neq \mathtt{p}_1 \end{cases} \end{aligned} \qquad \begin{aligned} \mathcal{M}_2 &= \langle W_2, R_2, V_2 \rangle \\ W_2 &= \{1, 2\} \\ R_2 &= W_2 \times W_2 \\ V_2(p) &= \begin{cases} W_2 & \text{if } p = \mathtt{p}_1 \\ \varnothing & \text{if } p \neq \mathtt{p}_1. \end{cases} \end{aligned}$$

The two models can be pictured as follows:

$\mathcal{M}_1$ $\qquad$ $\mathcal{M}_2$.

It is easy to verify that $\mathcal{M}_1, 0$ and $\mathcal{M}_2, 1$ are bisimilar. Hence, for every $\varphi \in \mathcal{L}$, $\mathcal{M}_1, 0 \vDash \varphi$ iff $\mathcal{M}_2, 1 \vDash \varphi$. In other words, $\mathcal{L}$ cannot distinguish the two situations. However, in $\mathcal{M}_1$, we cannot find a subset X of worlds that contains the world 0 and yet is not known at 0 by the agent represented there in the sense that $R(0) \subseteq X$. In $\mathcal{M}_2$, this is easy: $\{1\}$ contains 1 but $R(1) = \{1, 2\} \nsubseteq \{1\}$. This means that we cannot find a formula in $\mathcal{L}$ that captures the meaning of "there is some truth she doesn't know." To be more precise, define the language $\mathcal{L}\Pi$ with propositional quantifiers by the grammar

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p \varphi$$

with $p \in \mathsf{Prop}$. As usual, $\exists p$ abbreviates $\neg\forall p\neg$. Then, the semantic clause for $\forall p\varphi$ is

$$\mathcal{M}, w \vDash \forall p\varphi \iff \forall X \subseteq W, \mathcal{M}[X/p], w \vDash \varphi.$$

Here $\mathcal{M}[X/p]$ is the result of replacing the valuation function V in $\mathcal{M}$ by $V[X/p]$, which is defined by letting $V[X/p](p) = X$ and $V[X/p](q) = V(q)$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Then, the above observation is that there is no formula in $\mathcal{L}$ that is equivalent to $\exists p(p \wedge \neg\Box p)$.

For modal logics with propositional quantifiers, we need to be reminded that modal operators come in different flavors, and hence for different intended interpretations of $\Box$, different restrictions apply to the models. In the above example where $\Box$ is interpreted as knowledge, a minimal constraint to be put on R is reflexivity, since one cannot know a falsity. Kit Fine [54, 55] is the first to systematically study modal logics with propositional quantifiers in different model classes. His observations include:

- the set of formulas valid on all Kripke models in which R is an equivalence relation is axiomatizable and in fact decidable;
- the set of formulas valid on all Kripke models in which R is a transitive relation is not axiomatizable;
- the set of formulas valid on all Kripke models in which R is a symmetric relation is not axiomatizable;
- the set of formulas valid on all Kripke models is not axiomatizable.

Later literature [102] improved Fine's result to the following theorem:

Theorem 1.1.1. *The set of formulas valid on any class of Kripke models containing all those in which R is a directed preorder*[1] *is recursively equivalent to full second-order logic.*

These results show clearly the cost of the expressivity brought by propositional quantifiers.

Early literature also realized that Kripke models with the semantics requiring that $\forall p$ to test every set of worlds as a possible valuation of p are not the most general setting for $\mathcal{L}\Pi$. It is easy to see that the following formulas are valid on all Kripke models:

$$\exists p(p \wedge \forall q(q \to \Box(p \to q))), \tag{At}$$
$$\forall p\Box\varphi \to \Box\forall p\varphi. \tag{Bc}$$

The validity of At is due to the existence of singletons, and the validity of the Bc comes from exchanging the order of two universal quantifiers. Fine uses models with a distinguished algebra of sets of possible worlds for $\forall p$ to quantify over to refute At. And to refute Bc, Bull in [23] and Fine in [55] noted that a variable domain model can be used, while Gabbay in [65] showed that neighborhood semantics can be used instead.

[1] R is a directed preorder iff R is reflexive, transitive, and has the Church-Rosser property: for all x, y_1, y_2, if xRy_1 and xRy_2, then there is z with y_1Rz and y_2Rz.

To be more precise on the variations of the semantics for $\forall p$ mentioned above, let us formally introduce the most general possible-world-based semantics for $\Box$ and $\forall p$. In this semantics, we start with a neighborhood model and add to it a domain function that assigns to each world a family of subsets of worlds for $\forall p$ to quantify over at that world.

Definition 1.1.2. A *neighborhood model* $\mathcal{M}$ is a tuple $\langle W, N, P, V\rangle$ where

- W is a non-empty set;
- N is a neighborhood function from W to $\wp(\wp(W))$;
- P is a domain function from W to $\wp(\wp(W))$;
- V is a valuation function from Prop to $\wp(W)$.

Truth for formulas in $\mathcal{L}\Pi$ on $\mathcal{M}$ is defined relative to its worlds by:

$$\begin{array}{lll}
\mathcal{M}, w \vDash p & \iff & w \in V(p) \\
\mathcal{M}, w \vDash \neg\varphi & \iff & \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) & \iff & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \Box\varphi & \iff & [\![\varphi]\!]^{\mathcal{M}} \in N(w) \\
\mathcal{M}, w \vDash \forall p\varphi & \iff & \forall X \in P(w), \mathcal{M}[X/p], w \vDash \varphi.
\end{array}$$

Validity is defined as usual: φ is valid on $\mathcal{M}$ if $[\![\varphi]\!]^{\mathcal{M}} = W$.

A neighborhood model $\mathcal{M} = \langle W, N, P, V\rangle$

- is *relational* if N is representable by a binary relation R is the sense that for each $w \in W$, $N(w) = \{X \subseteq W \mid R(w) \subseteq X\}$,
- is *constant-domain* if P is a constant function, and
- is *full* if for each $w \in W$, $P(w) = \wp(W)$.

We will call relational neighborhood models just relational models.

Observe that Kripke models can be regarded as full relational models. The interpretation of $\forall p$ with Kripke domains is also called "the primary interpretation of $\forall p$" in [23, 113], but we will call it the standard Kripke semantics. Also, as mentioned above, the refutation of At can be achieved by using a constant domain function P such that $P(w)$ is an atomless field of sets of W, and the refutation of Bc can be achieved by either using a neighborhood function N such that $N(w)$ is not closed under arbitrary intersection, or by using a domain function P that is not constant.

A major part of the later literature continued on the theme of expressivity and non-axiomatizability for modal logics with propositional quantifiers. On special classes of Kripke models, besides Kaminski's above result in [102], another well-studied one is the class of

Kripke models with two equivalence relations: [3, 115, 64]. Other special case studies (also including relevance and intuitionistic logics) of the expressivity and non-axiomatizability of modal logics with propositional quantifiers include [113, 112, 111, 71, 62]. Among them, [62] engages with non-constant-domain models. For the expressivity of $\mathcal{L}\Pi$ on all Kripke models, [29] provided the following important results:

- Every formula in $\mathcal{L}\Pi$ is equivalent to one in prenex form: $Q_1p_1Q_2p_2\cdots Q_np_n\chi$ where $Q_i \in \{\forall, \exists\}$ and χ is quantifier free.

- For any $\varphi \in \mathcal{L}\Pi$ with its modal depth being $md(\varphi)$, $\mathcal{M}, w \vDash \varphi$ iff $\mathcal{M}^{\leq md(\varphi)}, w \vDash \varphi$, where $\mathcal{M}_w^{\leq md(\varphi)}$ is the restriction of $\mathcal{M}$ to worlds accessible from w using R in $\mathcal{M}$ in no more than $md(\varphi)$ steps.

- With respect to all Kripke models or all finite Kripke models: $\varphi \in \mathcal{L}\Pi$ is invariant under bisimulation iff it is equivalent to a formula in $\mathcal{L}$.

- With respect to all transitive Kripke models: every formula of the modal μ-calculus is equivalent to a formula in $\mathcal{L}\Pi$, and a formula in $\mathcal{L}\Pi$ is bisimulation invariant iff it is equivalent to a formula of the modal μ-calculus.

- If a formula φ in $\mathcal{L}\Pi$ is equivalent to a formula of the first-order correspondence language (with a binary predicate interpreted by R and a unary predicate for each $p \in \mathsf{Prop}$ interpreted by $V(p)$), φ is also equivalent to a formula of the first-order correspondence language with one free variable where every quantifier is bounded by R.

- A class $\mathbf{K}$ of pointed Kripke models is definable by a formula in $\mathcal{L}\Pi$ iff $\mathbf{K}$ is definable by a formula of monadic second-order logic and has finite degree in the sense that there is $d \in \mathbb{N}$ such that $\mathcal{M}, w \in \mathbf{K}$ iff $\mathcal{M}_w^{\leq d}, w \in \mathbf{K}$.

Relatedly, [116] and [117] discussed the quantifier alternation hierarchy for $\mathcal{L}\Pi$ made available by the prenex form theorem. There is also a cluster of more applied work on propositional quantifiers: [11, 9, 10, 12, 52, 8, 61]. Note that [11] claimed essentially an axiomatization result for validities of a multi-modal $\mathcal{L}\Pi$ on the class of Kripke frames with multiple equivalence relations. This is in contradiction with [3, 115, 64]. The results in [10, 12] are in fact quite general, with results on expressivity, axiomatization, and non-axiomatizability, and the authors there mentioned that the aforementioned logic for two equivalence relations is not axiomatizable, though they did not mention the early work [3, 115] on this topic. It should be noted that their axiomatizations are all with respect to classes of constant-domain relational models that include non-full models. Other axiomatization results include [160], which provided an axiomatization for a very special class of Kripke models based on trees, and [110], which allows non-full models.

Our contributions to this literature are marked by three features that are largely absent in the existing literature. First, we aim to provide general studies encompassing more than just a handful of logics. While it is true that in a given applied setting we may focus on only a few logics that make the most sense, the power of the mathematical theory of modal logic lies in having a unified theory for all the existing and to-be-discovered logics. A nice example is the Sahlqvist canonicity theorem, a theorem that provides completeness for most normal propositional modal logics for discussions in epistemology and metaphysics. Second, we focus on full models while still aiming for positive results whenever possible. The literature has not seen a new axiomatizability/decidability result for full models for a long time. As we mentioned, almost all new axiomatizability/decidability results after Fine's [55] depend on using non-full models (with the exception of [160] mentioned above). The requirement on fullness is from what seems to us the most natural reading of $\forall p$: "no matter what p means". While non-full models provide coherent semantics for $\forall p$, they either do not afford the same intuitive reading or are hard to construct, and it is significantly harder to judge the intuitive validity of more complicated formulas involving $\forall p$ that can only quantify over a rather small set of propositions.

Finally, and perhaps most importantly, we will use algebraic semantics extensively. Algebraic semantics for $\forall p$ was mentioned already by Bull in [23], and a more explicit proposal can be found in a later non-technical paper [76]. It should be noted that algebraic semantics predate Kripke's possible-world based semantics [74], and as [76] argues, they form a very natural setting for propositional quantifiers. The basic idea is to take propositions directly as the primary building blocks of a model. Propositions should form a Boolean algebra under classical logic and every formula should express a proposition in a model. The compositional semantic clause for $\forall p\varphi$ comes from the reading of $\forall p$ we mentioned: no matter what p means. As we vary the meaning of p, φ mean express different propositions, and $\forall p\varphi$ as we read it should be the "conjunction" of these propositions. In the theory of Boolean algebra, such a "conjunction" is defined by greatest lower bound, or *meet* in short. The formal definition is as follows.

Definition 1.1.3. A *Boolean algebra expansion* (BAE in short) $\mathcal{B}$ is a tuple $\langle B, \Box\rangle$ where B is a Boolean algebra and $\Box$ is a function from B to B. We use '$\neg$' and '$\wedge$' also for the complementation and meet in Boolean algebras, and $\top$ and $\bot$ for the top and bottom element of B. A BAE is called *complete* if its Boolean algebra part is complete in the sense that every set of elements of the Boolean algebra has a meet in it.

Functions $V : \mathsf{Prop} \to B$ are called valuation functions on $\mathcal{B}$, and when $\mathcal{B}$ is complete, they can be extended to $\widehat{V} : \mathcal{L}\Pi \to B$ by

$$\begin{aligned}
\widehat{V}(p) &= V(p)\\
\widehat{V}(\neg\varphi) &= \neg\widehat{V}(\varphi)\\
\widehat{V}(\varphi\wedge\psi) &= \widehat{V}(\varphi)\wedge\widehat{V}(\psi)\\
\widehat{V}(\Box\varphi) &= \Box(\widehat{V}(\varphi))\\
\widehat{V}(\forall p\varphi) &= \bigwedge\{\widehat{V[a/p]}(\varphi) \mid a\in B\}.
\end{aligned}$$

A formula φ is *valid* on $\mathcal{B}$ if $\widehat{V}(\varphi) = \top$ for all valuations on $\mathcal{B}$.

The algebraic semantics above generalize full neighborhood models by the following observation: for any full neighborhood model $\mathcal{M} = \langle W, N, P, V \rangle$, form the complete (indeed complete and atomic) BAE $\langle \wp(W), \Box \rangle$ where $\Box(X) = \{w \in W \mid X \in N(w)\}$ for all $X \in \wp(W)$. Then, for any $\varphi \in \mathcal{L}\Pi$, $\widehat{V}(\varphi) = [\![\varphi]\!]^{\mathcal{M}}$.

Our first contribution is an analogue of the celebrated Scrogg's theorem [143]. Scrogg's theorem establishes a general completeness theorem for all modal logics extending $\mathsf{S5}$, and we establish similarly a general completeness theorem for all normal Π-logics extending $\mathsf{S5\Pi}$. Here, a normal Π-logic is simply a set of formulas in $\mathcal{L}\Pi$ containing certain axioms and closed under certain rules that are most clearly valid for reasoning with a normal modality and the propositional quantifier $\forall p$ that is meant to quantify over all propositions. The modal part of them is as usual the K axiom: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and the necessitation rule: from φ derive $\Box\varphi$. The quantificational part is essentially the same as the first-order quantificational axioms and rules, and we call them Π-principles:

- Dist : $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$,
- Inst : $\forall p\varphi \to \varphi[\psi/p]$, where ψ is substitutable for p in φ and $\varphi[\psi/p]$ is the result of replacing all free occurrences of p in φ by ψ,
- Vacu : $\varphi \to \forall p\varphi$, if p is not free in φ,
- Univ : from φ derive $\forall p\varphi$.

Under this general definition, $\mathsf{S5\Pi}$ is defined as the smallest normal Π-logic that contains the modal logic $\mathsf{S5}$ in $\mathcal{L}$. In Chapter 2, we provide the following results on normal Π-logics extending $\mathsf{S5\Pi}$:

- All normal Π-logics extending $\mathsf{S5\Pi}$ are complete with respect to their complete simple S5 algebras. These are just complete BAEs $\langle B, \Box \rangle$ where $\Box(b)$ is the top element of B if b also is, and is the bottom element of B otherwise. We can view them as slight generalization of Kripke models with a universal relation (dropping atomicity).
- The normal Π-logics extending $\mathsf{S5\Pi}$ forms a lattice under the subset relation. We show that this lattice is isomorphic to the lattice, ordered by subset relation again, of the open sets of the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the natural order topology.
- We also show how to determine the computability of normal Π-logics extending $\mathsf{S5\Pi}$. A corollary is that they can be of arbitrarily high Turing-degree.

We will also mention that there are non-normal Π-logics extending $\mathsf{S5\Pi}$. Here a general Π-logic is defined like normal Π-logics without requiring it to be closed under the necessitation rule.

Our next contribution turns to KD45, a logic properly weaker than S5 and with a significant practical interpretation: the logic of belief. The expressivity gained with the introduction of propositional quantifiers now bears foundational importance: we find reasoning principles reflecting foundational properties of belief expressible in $\mathcal{L}\Pi$ but not in $\mathcal{L}$. We pay special attention to what we call the principle of immodesty: "I believe that everything I believe is true." This principle is easily formalizable in $\mathcal{L}\Pi$ by

$$\Box\forall p(\Box p \to p) \qquad \mathsf{Immod}$$

where $\Box$ is interpreted by belief. We argue that a *logic* of belief should not treat Immod as a theorem (though of course the logic can and should determine the consequences of Immod) and observe that to invalidate Immod while maintaining the validity of KD45, a departure from the standard Kripke models is in order. Indeed, any relational neighborhood model, even allowing variable domains, will have difficulty invalidating Immod if we would like the binary relation interpreting the belief modality $\Box$ to have the nice first-order properties corresponding to the axioms of KD45, namely seriality, transitivity, and Euclidicity (Euclidicity is to be blamed here). On the other hand, algebraic semantics can separate Immod from KD45 easily. Essentially, we will identify a class of complete BAEs called complete well-connected pseudo-monadic BAEs. These are BAEs of the form $\langle B, \Box\rangle$ where there is a proper filter $\mathcal{F}_{\mathcal{B}}$ on B such that $\Box b$ is the top element of B if $b \in \mathcal{F}$ and is the bottom element of B otherwise. This $\mathcal{F}$ can be regarded as the set of all propositions believed by the agent modeled in this BAE, and that it is a proper filter correspond to the requirement that one's belief should be non-contradictory and closed under logical deduction.

While natural-looking, these well-connected BAEs validate a strengthening of the introspection axiom 4:

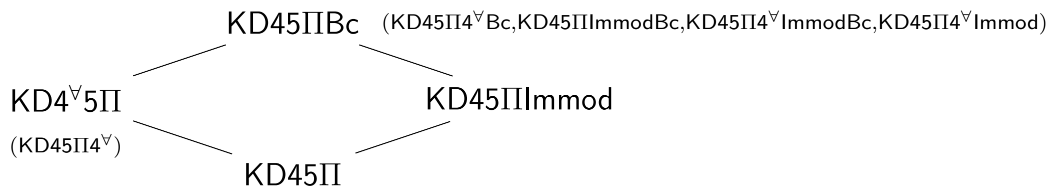
$$\forall p\Box\varphi \to \Box\forall p\Box\varphi. \qquad \mathsf{4}^{\forall}$$

We will argue that $\mathsf{4}^{\forall}$, unlike Immod, can be regarded as a logical principle for belief, though it is a proper strengthening of $\mathsf{4} : \Box\varphi \to \Box\Box\varphi$. In fact, we have the following theorem regarding extending $\mathsf{KD45}\Pi$, the smallest normal Π-logic containing KD45, by one or more of Immod, $\mathsf{4}^{\forall}$, and Bc. Here we follow the convention that when putting names of axioms together with Π, we mean the smallest normal Π-logic containing these axioms.

Theorem 1.1.4. *First,* $\mathsf{4} \in \mathsf{KD45}\Pi\mathsf{4}^{\forall}$. *Also,*

- $\mathsf{Immod} \notin \mathsf{KD4}^{\forall}\mathsf{5}\Pi$,
- $\mathsf{4}^{\forall} \notin \mathsf{KD45}\Pi\mathsf{Immod}$, *and*
- $\mathsf{KD45}\Pi\mathsf{Bc} = \mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{Immod}$.

Consequently, there are exactly 4 logics generated by adding some or none of $\mathsf{4}^{\forall}$, Immod, *and* Bc *to* $\mathsf{KD45}\Pi$*:* $\mathsf{KD45}\Pi$, $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, $\mathsf{KD45}\Pi\mathsf{Immod}$, $\mathsf{KD45}\Pi\mathsf{Bc}$. *They are ordered by inclusion as follows:*



Interestingly, algebraic semantics cannot be used to show that $\mathsf{4}^\forall \notin \mathsf{KD45}\Pi$, and we will do this using variable-domain relational models. Indeed, any complete BAE that validates $\mathsf{KD45}$ also validates $\mathsf{4}^\forall$. This raises a natural though underdefined question: what accounts for the difference between $\mathsf{S5}$ and $\mathsf{KD45}$ that the logic (validities) of complete BAEs validating $\mathsf{S5}$ is $\mathsf{S5}\Pi$ while the logic of complete BAEs validating $\mathsf{KD45}$ is not $\mathsf{KD45}\Pi$? While we could attribute this difference to the derivability of Bc in $\mathsf{S5}\Pi$ and eventually to the axiom $\mathsf{B} : \varphi \to \Box\Diamond\varphi$ that is sufficient for the derivation of Bc, it is not at all clear if this gets to the heart of the issue: it seems rather difficult to show that the logic of all complete BAEs validating KB is precisely $\mathsf{KB}\Pi$. We leave questions along these lines for future work.

The positive results we offer are general completeness for normal Π-logics generated by adding one (or a finite list by conjunction) axiom to $\mathsf{KD4}^\forall\mathsf{5}\Pi$ and a general criterion for determining the decidability of the logic of a class of complete well-connected pseudo-monadic BAEs. But to avoid unnecessary details for this introduction, we list only some important special cases:

- The logic, in the language $\mathcal{L}\Pi$, of all complete well-connected pseudo-monadic algebras is $\mathsf{KD4}^\forall\mathsf{5}\Pi$.

- The logic, in the language $\mathcal{L}\Pi$, of all complete well-connected pseudo-monadic algebras $\mathcal{B}$ such that $\mathcal{F}_\mathcal{B}$ is a principal filter (has a minimal element) is $\mathsf{KD4}^\forall\mathsf{5}\Pi\mathsf{Immod}$.

- The logic, in the language $\mathcal{L}\Pi$, of all atomic complete well-connected pseudo-monadic algebras is $\mathsf{KD4}^\forall\mathsf{5}\Pi\mathsf{At}$.

- The logic, in the language $\mathcal{L}\Pi$, of all atomic complete well-connected pseudo-monadic algebras $\mathcal{B}$ with $\mathcal{F}_\mathcal{B}$ being principal is $\mathsf{KD4}^\forall\mathsf{5}\Pi\mathsf{AtImmod}$.

- By the duality between atomic and complete BAEs validating $\mathsf{KD45}$ and Kripke frames validating $\mathsf{KD45}$, the logic, in the language $\mathcal{L}\Pi$, of all Kripke models with a serial, transitive, and Euclidean relation is $\mathsf{KD4}^\forall\mathsf{5}\Pi\mathsf{AtImmod}$.

- All of the above logics are decidable.

1.2 Modal Logics for Comparing Propositions

While every binary sentential operator can be regarded in a very general sense as comparing propositions, we are most interested in operators that compare propositions according to their "size", broadly construed. Such comparisons cannot be truth-functional since the relative "size" of propositions cannot be deduced from whether they are true or not at a particular world.

Such comparisons can be viewed as generalizing the usual black-or-white judgments about necessity in different flavors to graded judgments on "how necessary" they are without committing to a fixed scale of "grades". For example, other than fully believing or disbelieving propositions, we often have more fine-grained belief attitudes such as by how much do we believe that a proposition is true. These attitudes are important in decision making under uncertainty, since we often encounter actions whose consequences depend on propositions that we neither fully believe nor fully disbelieve. A standard way to represent such graded beliefs is through a Kolmogorov probability function which assigns a real number in $[0, 1]$ to every proposition and in doing so satisfies certain axioms. But this approach assumes that $[0, 1]$ (or more generally a closed interval $[0, r]$ of real numbers) is the right scale of grades to use, and a number of concerns have been raised in this regard:

- Real numbers, in general, are beyond human cognitive capacities. This concern has a very long tradition ([20, 103, 106, 75, 149]), and Suppes in [149] expressed it as follows: "almost everyone who has thought about the problems of measuring beliefs in the tradition of subjective probability or Bayesian statistical procedures concedes some uneasiness with the problem of always asking for the next decimal of accuracy in the prior estimation of a probability" (p. 160).

- Real numbers are sometimes also not enough when we have infinitely many propositions. For example, we cannot have a Kolmogorov probability function P on a uncountable set W defined on all singletons such that for all $w \in W$, $P(\{w\}) > 0$. This means that there must be possible worlds that are no more likely than the contradiction, if rely only on the function P to decide relative likelihood. This is the famous problem of regularity [120, 157, 86, 136, 90, 48].

- Also, even for idealized agents without representational and computational constraints, it has been argued that there are situations where even an idealized agent cannot rationally uniquely determine a Kolmogorov probability function to account for the right epistemic attitudes [28, 121, 101, 105].

While a purely comparative approach is not the only way to address the above three concerns—sets of hyperreal-valued probability functions also work—the comparative approach is uniquely parsimonious and general: technically, it is only providing a language that every representation of graded propositional attitude should be able to "speak". For example, a Kolmogorov probability function P "speaks" the comparative language in the most obvious way: a proposition A is at least as likely as a proposition B (both represented by sets of possible worlds) according to P iff $P(A) \geq P(B)$. Since this comparative language is foundational to all representations, the following natural questions are immediate:

- Given a representation method of graded propositional attitudes, what is its logic in a comparative language, namely, its *comparative logic*? The comparative logics, given the foundational role of comparative languages, should reflect the core theoretical commitments of the representation method in discussion.
- Given multiple representation methods, how do their comparative logics compare?
- Given a logic in a comparative language, is it the comparative logic of a reasonable representation method?

While these are not the only questions we will answer (in particular, we will also compare comparative languages, not just logics in a comparative language), these questions are the main motivation for our work.

We will offer many languages for propositional comparison in this dissertation, but the main distinction is whether iterated comparisons are allowed. In Chapter 3, we use a language that does not allow iteration on comparisons. This is defined by the following 2-layered grammar

$$\begin{aligned} t &::= a \mid t^c \mid (t \cap t) \\ \varphi &::= |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi) \end{aligned}$$

where a is in a countably infinite set Φ. Here the first layer defining t generates what we call *set terms* and the second layer defines the usual *formulas*. While set terms can easily be interpreted as denoting propositions, in Chapter 3, we remain purely mathematical and only interpret them as denoting subsets of some given set X, with $\cdot^c$ and $\cap$ standing for complement with respect to X and intersection instead of negation and conjunction. This is because we are interested in solving a purely mathematical problem: what is the comparative logic for comparing the cardinality of sets as Cantor defines it? The difficulty in this question lies in the fact that finite sets and infinite sets obey very different laws of cardinality comparison. For a special case, let A, B, and C be non-empty sets such that (1) $A = B \cup C$, (2) $B \cap C = \varnothing$, and (3) B and C have the same cardinality: there is a bijection between them. Then if A is finite, A is strictly larger than B and C in terms of cardinality, but if A is infinite, A then has the same cardinality as B and C do. More generally, finite sets obey what we call the axiom of qualitative additivity

$$|s| \geq |t| \leftrightarrow |s \cap t^c| \geq |t \cap s^c| \tag{1.1}$$

while failing the axiom of absorption

$$(|s| \geq |t| \wedge |s| \geq |t'|) \to |s| \geq |t \cup t'|, \tag{1.2}$$

but infinite sets do the opposite: they obey the axiom of absorption but fail the axiom of qualitative additivity. In Chapter 4, we will see how we can almost define finiteness in this purely comparative language and use that to axiomatize the logic of cardinality comparison.

In Chapter 5, we consider comparative likelihood. Here, we use a language allowing iterated comparisons. With a single primitive comparison operator $\succsim$ intuitively read as "at least as likely as", we have the basic language $\mathcal{L}(\succsim)$ defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi)$$

with $p \in \mathsf{Prop}$. While iterations are allowed in this language, logically we do not need to pay much attention to them as we will assume the validity of the following introspection axioms:

$$(\varphi \succsim \psi) \to \big((\varphi \succsim \psi) \succsim \top\big);$$
$$\neg(\varphi \succsim \psi) \to \big(\neg(\varphi \succsim \psi) \succsim \top\big).$$

These axioms are akin to the axioms 4 and 5 for the unary belief operator. With these axioms, it can be shown that every formula is logically equivalent to a formula without iterated comparisons.

As we mentioned, $\mathcal{L}(\succsim)$, being a simple language, can be used to compare different approaches to uncertainty representation. The dominant approach is of course the single-probability-measure approach. To contrast with what will follow, we call this the precise probability approach. The comparative logic of this approach is well understood [109, 142, 145]. To define the comparative logic of precise probability, we first turn probability spaces into models for $\mathcal{L}(\succsim)$ as follows:

Definition 1.2.1. A *precise probability model* (SP model in short) is a tuple $\langle W, \mathcal{F}, \mu, V\rangle$ where W is a non-empty set, $\mathcal{F}$ is a field of sets on W, μ is a finitely additive probability function defined on $\mathcal{F}$, and V is a valuation function from Prop to $\mathcal{F}$. In other words, an SP model is simply a probability space plus a valuation function from Prop to events.

Truth for formulas of $\mathcal{L}(\succsim)$ on any SP model $\mathcal{M} = \langle W, \mathcal{F}, \mu, V\rangle$ is defined recursively as follows, where $[\![\varphi]\!]^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \vDash \varphi\}$ as usual:

$$\begin{array}{lcl}
\mathcal{M}, w \vDash p & \iff & w \in V(p) \\
\mathcal{M}, w \vDash \neg\varphi & \iff & \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) & \iff & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash (\varphi \succsim \psi) & \iff & \mu([\![\varphi]\!]^{\mathcal{M}}) \geq \mu([\![\psi]\!]^{\mathcal{M}}) \;.
\end{array}$$

For this definition to work, it must be that for any $\varphi \in \mathcal{L}(\succsim)$, $[\![\varphi]\!]^{\mathcal{M}} \in \mathcal{F}$, but this is not hard to show once we observe that $[\![\varphi \succsim \psi]\!]^{\mathcal{M}}$ is either $\varnothing$ or W. As usual, a formula is valid on a model if it is true at every world in the model, and is valid on a class of models if it is valid on every model of the class.

The comparative logic of the precise probability approach is then the set of formulas valid on all SP models. This set is axiomatized by the following axioms and rules:

- instances of propositional tautologies and the rule of *modus ponens*

- the necessitation rule for $\succsim$: from φ derive $\varphi \succsim \top$

(**A0**) $(\varphi \succsim \psi) \vee (\psi \succsim \psi)$;

(**A1**) $\varphi \succsim \bot$;

(**A2**) $\varphi \succsim \varphi$;[2]

(**A3**) $\neg(\bot \succsim \top)$;

(**A4**) $\big((\varphi_1, \ldots, \varphi_n, \varphi') \equiv (\psi_1, \ldots, \psi_n, \psi') \succsim \top\big) \to \big(\bigwedge_{i=1}^{n}(\varphi_i \succsim \psi_i)) \to (\psi' \succsim \varphi')\big)$;

(**A5**) $(\varphi \succsim \psi) \to \big((\varphi \succsim \psi) \succsim \top\big)$;

(**A6**) $\neg(\varphi \succsim \psi) \to \big(\neg(\varphi \succsim \psi) \succsim \top\big)$.

The antecedent $(\varphi_1, \ldots, \varphi_n, \varphi') \equiv (\psi_1, \ldots, \psi_n, \psi') \succsim \top$ of the axiom (**A4**) uses an abbreviated symbol $\equiv$. Its formal definition can be found on page 148, and intuitively, the antecedent expresses the idea that with probability 1, the number of true formulas (counting repetition) in the sequence $(\varphi_1, \ldots, \varphi_n)$ is the same as the number of true formulas (counting repetition) in the sequence $(\psi_1, \ldots, \psi_n)$. It is not hard to see that the truth of this at any/all world(s) in a model $\mathcal{M}$ guarantees that $\sum_{i=1}^{n} \mu([\![\varphi_i]\!]^{\mathcal{M}}) = \sum_{i=1}^{n} \mu([\![\psi_i]\!])^{\mathcal{M}}$. The truth of the consequent of (**A4**) then follows. It should be noted that the qualitative additivity axiom, $(\varphi \succsim \psi) \leftrightarrow ((\varphi \wedge \neg\psi) \succsim (\psi \wedge \neg\psi))$ formulated in $\mathcal{L}(\succsim)$, is a special case of (**A4**) with the help of propositional tautologies, (**A2**), and the rules. This means that comparing likelihood is closer to comparing the sizes of finite sets than to comparing the sizes of infinite sets.

This axiomatization also highlights a distinctive feature of the precise probability approach: it validates the comparability axiom (**A0**). This feature has been treated as a decisive reason to reject the precise probability approach. As Keynes [103] expressed it a century ago:

> Is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on some occasions none of these alternatives hold, and that it will be an arbitrary matter to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our minds, or even that we should balance them. (p. 30)

A prominent deviation from the precise probability approach rejecting (**A0**) is the so-called imprecise probability approach. The deviation is simple: instead of using a single probability function, the imprecise probability approach allows a non-empty set of probability functions. How does an agent compare likelihoods of propositions according to the imprecise probability approach? The agent will judge a proposition A to be at least as likely as a proposition B iff all probability functions used to represent the doxastic state of the agent say that A is as

[2] Axiom (**A2**) is redundant, but we include (**A2**) to match [1].

likely as B. It is now clear how imprecise probability rejects (**A0**): the set representing an agent may contain one probability function saying that A is more likely than B and another probability function saying that B is more likely than B, and taking both into account, neither is A at least as likely as B, nor is B at least as likely as A.

It would be nice to compare precisely how the comparative logic of imprecise probability differs from the comparative logic of precise probability. For this, we set up the imprecise probability semantics for $\mathcal{L}(\succsim)$ as follows:

Definition 1.2.2. An *imprecise probability model* (IP model in short) is a tuple $\langle W, \mathcal{F}, \mathcal{P}, V\rangle$ where W is a non-empty set, $\mathcal{F}$ is a field of sets on W, $\mathcal{P}$ is a non-empty set of finitely additive probability functions defined on $\mathcal{F}$, and V a valuation function from Prop to $\mathcal{F}$. When $\mathcal{P}$ is a singleton, we also call it an SP model.

Truth for formulas of $\mathcal{L}(\succsim)$ on any IP model $\mathcal{M} = \langle W, \mathcal{F}, \mathcal{P}, V\rangle$ is defined recursively as follows, where $[\![\varphi]\!]^{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \vDash \varphi\}$ as usual:

$$\begin{aligned}
\mathcal{M}, w \vDash p &\iff w \in V(p)\\
\mathcal{M}, w \vDash \neg\varphi &\iff \mathcal{M}, w \nvDash \varphi\\
\mathcal{M}, w \vDash (\varphi \wedge \psi) &\iff \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi\\
\mathcal{M}, w \vDash (\varphi \succsim \psi) &\iff \forall \mu \in \mathcal{P}, \mu([\![\varphi]\!]^{\mathcal{M}}) \geq \mu([\![\psi]\!]^{\mathcal{M}}).
\end{aligned}$$

Validity on SP models and SP model classes is defined as usual.

The logic of IP models in $\mathcal{L}(\succsim)$ is axiomatized not long ago in [2], and the axiomatization can be achieved by the following modifications to the above axiomatization for the logic of SP models:

- drop the comparability axiom (**A0**) and
- replace (**A4**) with the following strengthening of it, which we call (**A4'**):

$$\Big((\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k\,\text{times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k\,\text{times}}) \succsim \top\Big) \to \Big(\bigwedge_{i=1}^{n}(\varphi_i \succsim \psi_i)) \to (\psi' \succsim \varphi')\Big).$$

That the strengthening is required is shown in [83]. This axiomatization shows that imprecise probability, at least in the language $\mathcal{L}(\succsim)$, is not very far away from precise probability: the only conceptual change is the invalidation of the comparability axiom.

It is important to note that, while the change in axioms looks simple, the elimination of the comparability axiom opens a sea of reasoning that was previously vacuous. The most notable perhaps is the dilation phenomena in imprecise probability where an agent apparently loses knowledge upon learning a proposition. Such phenomena can be partially represented in $\mathcal{L}(\succsim)$. Consider the following formula where $\varphi \approx \psi$ abbreviates $(\varphi \succsim \psi) \wedge (\psi \succsim \varphi)$:

$$(p \approx \neg p) \wedge \neg(q \succsim \neg q) \wedge \neg(\neg q \succsim q). \qquad \textsf{Dilation}$$

This formula says that p is equally as likely as $\neg p$, and q and $\neg q$ are incomparable. This formula is inconsistent in the logic of SP models, but is consistent in the logic of IP models. Say an agent is correctly described by this formula. Then, what happens if the agent learns that $p \leftrightarrow q$? While we do not have an operator for learning a proposition in $\mathcal{L}(\succsim)$, this can be approximated by comparing propositions in conjunction with the learned proposition. Indeed, IP models predict, and the axioms of the logic of IP models derives, the following formula:

$$\neg\big((p \wedge (p \leftrightarrow q)) \succsim (\neg p \wedge (p \leftrightarrow q))\big) \wedge \neg\big((\neg p \wedge (p \leftrightarrow q)) \succsim (p \wedge (p \leftrightarrow q))\big).$$

Also, learning the negation of $p \leftrightarrow q$ has the same effect—the following formula is also a logical consequence of Dilation:

$$\neg\big((p \wedge \neg(p \leftrightarrow q)) \succsim (\neg p \wedge \neg(p \leftrightarrow q))\big) \wedge \neg\big((\neg p \wedge \neg(p \leftrightarrow q)) \succsim (p \wedge \neg(p \leftrightarrow q))\big).$$

It has been argued that this is a big problem for the imprecise probability approach, since, according to this argument, if an agent sees that either way (learning either $p \leftrightarrow q$ or $\neg(p \leftrightarrow q)$) she will believe that p and $\neg p$ are incomparable, she should believe that now, but in Dilation, she believes that p and $\neg p$ are comparable and equally likely (see [22]).

In Chapter 5, we will formally extend the language $\mathcal{L}(\succsim)$ to include a learning operator and formalize many concrete examples with probabilistic reasoning. But before the introduction of the learning operator, we have two more operators to consider. The first is the strict comparison operator "more likely than". The addition of this as a primitive is not necessary with the precise probability approach, since if there is only one probability function, for A to be more likely than B ($\mu(A) > \mu(B)$) is simply for B to be not at least as likely as A ($\mu(B) \not\geq \mu(A)$). However, with multiple probability functions, there is a strong sense of "more likely than" and a weaker sense. For A to be more likely than B in the weaker sense is for A to be at least as likely as B and for there to be a probability function judging A to be more likely than B. This weaker sense of "more like than" is definable by "at least as likely as": just write $(\varphi \succsim \psi) \wedge \neg(\psi \succsim \varphi)$ when we need to say that "φ is more likely than ψ in the weaker sense". However, a stronger sense of "more likely than", defined by requiring unanimous judgment of "more likely than" from the representing probability functions is not definable with "at least as likely as". Hence, we will add a symbol $\succ$ formalizing this stronger notion of "more likely than" and discuss the logic of IP models in the resulting language $\mathcal{L}(\succsim, \succ)$. After this, we go on to add an operator $\Diamond$ formalizing the operator "possibly". While many semantic choices are possible, the semantics for $\Diamond$ we use is that for $\Diamond\varphi$ to be true, we need one probability function such that when evaluated with only that probability function, that probability function judges that φ has a non-zero probability. In particular, $\Diamond\varphi$ is neither equivalent to $\neg\bot \succsim \varphi$ nor to $\varphi \succ \bot$, since we would like $\Diamond(p \succ \neg p) \wedge \Diamond\neg(\neg p \succ p)$ to be consistent, reflecting the observation that "it's possible that raining is more likely than not, but it's also possible that raining is no more likely than not raining" does not sound marked. The $\Diamond$ operator offers another boost in expressivity, and we will axiomatize the logic of IP models in the language with $\Diamond$ added as well.

After the introduction of $\Diamond$, we introduce the learning operator $\langle\,\rangle$. It turns out that the learning operator does not add expressivity, and an axiomatization based on logically reducing formulas with $\langle\,\rangle$ to formulas without $\langle\,\rangle$ is given. Finally, we consider adding a pair of operators, I_p^+ and I_p^-, that introduce a new true proposition or a new false proposition, respectively. The precise semantics of them will be slightly more complicated, but they are meant to formalize a very common kind of information dynamics: besides learning the truth of propositions, we also learn the existence of propositions. For example, we may be introduced by doctors to the existence of an important gland in our body. Without knowing the existence of the gland, we cannot even from beliefs, probabilistic or not, about it, and once introduced to it, we are forced to form some opinions about it. We often find it hard to form precise opinions, even only probabilistic ones, about propositions with little background information, but the imprecise probability approach solves this problem elegantly: just take all probability functions defined on old and new propositions that are compatible with the existing probability functions defined only on the old propositions. We will formalize this idea of belief change under the introduction of new propositions and express it using I_p^+ and I_p^-. It turns out that with these new operators, the expressivity of the language becomes quantitative, and we have to leave the full axiomatization of the logic of IP models in this language for future work.

1.3 Comparison with the Published Versions

Chapter 2 was published as [40] in AiML 2018 with the title "On the Logics with Propositional Quantifiers Extending $\mathsf{S5}\Pi$". Other than some minor editorial changes, there is one major change: Theorem 5.4 in the published version is strengthened to Theorem 2.5.4 in this dissertation where instead of just the sets of the form $\{1\} \times (X \cup \{\infty\})$, we now show that any closed set in $\mathcal{S}$ is Turing equivalent to its logic.

Chapter 3 was published as [39] in Journal of Philosophical Logic with the title "On the Logic of Belief and Propositional Quantification". Other than the minor editorial changes, we made the following major changes:

- An analysis of propositionally contingent models, culminating in Proposition 3.2.15 and Theorem 3.2.16 showing that $\mathsf{4}^\forall$ is not derivable in $\mathsf{KD45}\Pi$, is newly added.
- We added proofs for item 3 and 4 in Proposition 4.2 in the published version (now Proposition 3.4.7).
- In the published version, the extra axiom needed for atomic complete well-connected pseudo-monadic algebras is $\mathsf{z} \to \mathsf{g}$. While it looks short, it uses the two extra propositional constants not seen in $\mathcal{L}\Pi$. While they can be defined axiomatically in $\mathcal{L}\Pi$, the definition of g is rather long. In Chapter 3, we show in Proposition 3.5.8 that the simple At: $\exists q \forall p(p \to \Box(q \to p))$ (with $\Box$ replaced by $\mathtt{B}$ for the suggestive notation in Chapter 3) is in fact logically equivalent to $\mathsf{z} \to \mathsf{g}$. Recall that At defines atomicity

on complete simple S5 algebras. So At, while not defining atomicity on complete well-connected pseudo-monadic algebras (or equivalently complete proper filter algebras to be defined in Chapter 3), is nevertheless enough for axiomatizing atomicity.

Chapter 4 was published as [41] in The Journal of Symbolic Logic with the title "The Logic of Comparative Cardinality" and coauthors Matthew Harrison-Trainer and Wesley Holliday. There is a major editorial change: we make the appendix in the published version a normal section (Section 4.3). Also, we largely simplified the definition of Fin (compare Definition 4.7.6 from Chapter 4 to the same in the published version).

Chapter 5 was published as [42] in International Journal of Approximate Reasoning with the title "Logics of imprecise comparative probability" and coauthors Wesley Holliday and Thomas Icard. Besides the minor editorial changes made, we fix a mistake in the published version: the logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ requires the instances of the theorems in $\mathsf{IP}(\succsim, \succ)$ as axioms (see Definition 5.5.4). We also offer a simple derivation of the $\mathsf{4}$ axiom $\Box\varphi \to \Box\Box\varphi$ in $\mathsf{IP}(\succsim, \succ, \Diamond)$ after the Definition 5.5.4.

Chapter 2

Logics with Propositional Quantifiers Extending $\mathsf{S5}\Pi$

2.1 Introduction

In this chapter, we study the modal logics with propositional quantifiers extending the well-studied modal logic $\mathsf{S5}$. Modal logics with propositional quantifiers have been of considerable interest to many modal logicians since their appearances in Fine's dissertation [54] and an early paper by Bull [23]. However, much of the interest is devoted to a few particular systems (e.g., [115, 112, 9, 11, 17]) and the expressive power under Kripke semantics (e.g., [29, 117, 116, 102, 71, 3, 10]), and there is an obvious lack of general study of classes of such logics. An exemplary early general study of propositional modal logics is found in Scroggs's famous 1951 paper [143], and it is our intention here to extend it to modal logics with propositional quantifiers.

To this end, we must first define, in general, what is a modal logic with propositional quantifiers. Since we consider here only logics with one modal operator, the language $\mathcal{L}\Pi$ defined below suffices.

Definition 2.1.1. Let $\mathcal{L}\Pi$ be the language with the following grammar

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \forall p\varphi$$

where $p \in \mathsf{Prop}$, a countably infinite set of propositional *variables*.[1] Other Boolean connectives, $\bot$, and $\Diamond$ are defined as usual.

As is common in the general study of modal logics, we take a modal logic with propositional quantifiers to be a set of formulas satisfying certain closure conditions, which represent the necessary axioms and rules for connectives with fixed meaning. There are many readings of the propositionally quantified sentence $\forall p\varphi$, which result in different axioms and semantics (see [55] for example), but here we take the most straightforward reading: "no matter what proposition p expresses, φ." From a purely logical point of view, this reading should warrant the following widely accepted principles, which we call the Π-*principles*:

- All instances of the universal distribution axiom schema: $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$.
- All instances of the universal instantiation axiom schema: $\forall p\varphi \to \varphi^p_\psi$ where ψ is substitutable for p in φ, and φ^p_ψ is the result of this substitution.
- All instances of the vacuous quantification axiom schema: $\varphi \to \forall p\varphi$ where p is not free in φ.
- Universalization rule: if φ is derivable, then $\forall p\varphi$ is derivable.

Then the modal logics with propositional quantifiers, which we call Π-logics in accordance with [23] and most recently [91], can now be defined.

[1] In contrast, $\top$ is a propositional *constant*. Later we will have another propositional constant.

Definition 2.1.2. A Π-*logic* is a set Λ of formulas in $\mathcal{L}\Pi$ such that Λ contains all instances of propositional tautologies and axioms in the Π-principles, and is closed under *modus ponens* and the only rule, universalization, in the Π-principles.

A *normal* Π-logic Λ is a Π-logic that contains the K axiom and is further closed under necessitation: if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$.

For any normal modal logic L in the usual basic modal language, let $\mathsf{L}\Pi$ be the smallest (in terms of inclusion) normal Π-logic containing L.

Then, for example, $\mathsf{S5}\Pi$ is the smallest normal Π-logic extending $\mathsf{S5}$, and $\mathsf{K}\Pi$ is the smallest normal Π-logic extending K, which is just the smallest normal Π-logic.

Following Scroggs, we address the following questions in this chapter regarding the Π-logics extending $\mathsf{S5}\Pi$, the set of which we call $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$.

General completeness of logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ It is well known that $\mathsf{S5}\Pi$ is incomplete with respect to its Kripke frames where every set of possible worlds counts as a proposition. This was observed by Fine already in [54] and is in stark contrast to the situation without propositional quantifiers: as is shown by Scroggs, all modal logics in the basic modal language extending $\mathsf{S5}$ are complete with respect to their finite Kripke frames with a totally connected relation. However, Scroggs's proof is algebraic in spirit, and indeed, an algebraic semantics for $\mathcal{L}\Pi$ based on modal algebras is more natural for the normal Π-logics, given our straightforward reading of $\forall p\varphi$. Algebraically, $\forall p\varphi$ is interpreted as the meet (greatest lower bound) of all possible semantic values of φ when we only vary the valuation of p. In short, $\forall p\varphi$ expresses an arbitrary meet. Dually, $\exists p\varphi$ expresses an arbitrary join. For this to work, however, we need the modal algebras to be complete in the sense that for any set of elements in the algebra, the meet and join of this set exist. We will show that all logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ are complete with respect to their complete simple S5 algebras, which are just slight generalization of totally connected Kripke frames (dropping atomicity).

The lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ Using the general completeness result, the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ can be reduced to the lattice structure of classes of algebras defined by logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$. We will show from this that the logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ correspond to the closed sets of a Stone space $\mathcal{S}$, which is homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the natural order topology. Then the lattice $\langle \mathsf{Next}\Pi(\mathsf{S5}\Pi), \subseteq\rangle$ is isomorphic to the lattice of the open sets of $\mathcal{S}$ ordered by inclusion.

The computability of logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ From the correspondence between the logics and the closed sets, we also obtain that there are logics in $\mathsf{Next}\Pi(\mathsf{S5}\Pi)$ of arbitrarily high Turing-degree. While it is known that many natural modal logics with propositional quantifiers are of very high complexity [55, 102], this shows that we may still need to face the problem even above $\mathsf{S5}$.

The non-normal Π-logics extending $\mathsf{S5\Pi}$ We will also show that there are many non-normal Π-logics extending $\mathsf{S5\Pi}$, contrary to the situation in the basic modal language, where all modal logics extending $\mathsf{S5}$ are normal. However, we leave a complete study of the non-normal Π-logics extending $\mathsf{S5\Pi}$ to future work.

The plan to address these questions is as follows. In § 2.2, we present the semantics for $\mathcal{L}\Pi$ and collect the necessary results already appearing in [54] and more recently in [91]. In § 2.3, we show that, in terms of validity or theoremhood, every formula in $\mathcal{L}\Pi$ is equivalent to a Boolean combination of a few simple formulas. This serves as a good preparation for § 2.4, where we construct a topological space $\mathcal{S}$ based on all complete simple S5 algebras, which encodes what classes of algebras are definable in terms of validity by $\mathcal{L}\Pi$. Crucially, $\mathcal{S}$ is a Stone space. In § 2.5, we prove all the main results, which make essential use of the fact that $\mathcal{S}$ is a Stone space and, in particular, that $\mathcal{S}$ is compact. This allows us to prove completeness without using the usual Lindenbaum algebra and quotient construction, though we need to rely on the already proven completeness of $\mathsf{S5\Pi}$. Finally, we conclude with related open problems in § 2.6.

2.2 Preliminaries

Recall that a modal algebra is a pair $\langle B, \Box \rangle$ where B is a Boolean algebra and $\Box$ is a unary operator on B satisfying $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$ for any $a, b \in B$. In most cases, we will conflate the notation of an algebra and its carrier set, and we will take $\neg, \wedge, \vee, \Box$ to be the complement, meet, join, and modal operators in modal algebra, despite that they are also in our formal language $\mathcal{L}\Pi$. The usual abbreviations also apply to operations on modal algebras, including $\Diamond a := \neg\Box\neg a$ for all $a \in B$. When confusion may arise, we will use $\neg_B, \wedge_B, \vee_B, \Box_B$ for the operators in a modal algebra B. A modal algebra B is *complete* when its Boolean part is a complete Boolean algebra. Then the semantics for $\mathcal{L}\Pi$ can be defined as follows.

Definition 2.2.1. For any modal algebra B, a *valuation* V on B is a function from Prop to B. When B is complete, any such valuation can then be extended to an $\mathcal{L}\Pi$-*valuation* $\widehat{V}$ from $\mathcal{L}\Pi$ to B defined recursively by:

1. $\widehat{V}(p) = V(p)$ for all $p \in \mathsf{Prop}$;
2. $\widehat{V}(\top) = 1$; $\widehat{V}(\neg\varphi) = \neg\widehat{V}(\varphi)$; $\widehat{V}(\varphi \wedge \psi) = \widehat{V}(\varphi) \wedge \widehat{V}(\psi)$; $\widehat{V}(\Box\varphi) = \Box\widehat{V}(\varphi)$;
3. $\widehat{V}(\forall p\varphi) = \bigwedge\{\widehat{V'}(\varphi) \mid V' : \mathsf{Prop} \to B, V' \sim_p V\}$, where we define $V' \sim_p V$ by $V'(q) = V(q)$ for any $q \in \mathsf{Prop} \setminus \{p\}$.

A formula $\varphi \in \mathcal{L}\Pi$ is *valid* on a complete modal algebra B, written as $B \vDash \varphi$, if for all valuations V on B, $\widehat{V}(\varphi) = 1$.

Since we are only interested in Π-logics extending $\mathsf{S5\Pi}$, we only need modal algebras validating $\mathsf{S5}$. In fact, we only need a very special class of such modal algebras called simple S5 algebras.

Definition 2.2.2. A *simple S5 algebra* is pair $\langle B, \Box\rangle$ where B is a non-trivial Boolean algebra and $\Box$ is the unary function on B defined for $a \in B$ by

$$\Box a = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{Then } \Diamond a = \begin{cases} 1 & \text{if } a \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote the class of all simple S5 algebras by $\mathsf{sS5A}$ and the class of all complete simple S5 algebras by $\mathsf{csS5A}$.

Modal algebras validating $\mathsf{S5}$ are also known as *monadic algebra* (see [77, 78]). However, in the context of monadic algebras, $\Diamond$ and $\Box$ operators are usually denoted by $\exists$ and $\forall$, which we need for propositional quantifiers. We also remark that our simple S5 algebras are indeed simple in its general algebraic sense: they have no non-trivial congruence relation. In [47] (see p.367), these algebras are also referred to as *Henle algebras*.

To formulate completeness with respect to $\mathsf{csS5A}$, it is natural to use the following Galois connection:

Definition 2.2.3. For any class $\mathsf{C} \subseteq \mathsf{csS5A}$, define $\mathrm{Log}(\mathsf{C}) = \{\varphi \in \mathcal{L}\Pi \mid \forall B \in \mathsf{C}, B \vDash \varphi\}$. We also write $\mathrm{Log}(\{B\})$ as simply $\mathrm{Log}(B)$ for any $B \in \mathsf{csS5A}$. Conversely, for any set of formulas $\Gamma \subseteq \mathcal{L}\Pi$, define $\mathrm{Alg}(\Gamma) = \{B \in \mathsf{csS5A} \mid \forall \varphi \in \Gamma, B \vDash \varphi\}$. Similarly, $\mathrm{Alg}(\varphi)$ abbreviates $\mathrm{Alg}(\{\varphi\})$.

This finishes the semantics for $\mathcal{L}\Pi$, and now we march into expanding $\mathcal{L}\Pi$, as Fine did in [54], to $\mathcal{L}\Pi\mathrm{Mg}$. This is instrumental for formulating the quantifier elimination on which completeness for $\mathsf{S5\Pi}$ alone in [54, 91] depends, and all our new results will also need it. In the following, let $\mathbb{N}_+$ be the set of positive natural numbers, and $\mathbb{N}^\infty$ be the set of natural numbers plus an infinite element ∞. Also, we will use $\mathbb{N}^\infty_+$, which has ∞ but not 0.

Definition 2.2.4. ([54]) Define $\mathcal{L}\Pi\mathrm{Mg}$ by extending the grammar for $\mathcal{L}\Pi$ with a propositional *constant* g (not in Prop) and countably many new unary operators $\{\mathrm{M}_i \mid i \in \mathbb{N}_+\}$. Then, define $\mathcal{L}\mathrm{Mg}$ as the quantifier free fragment of $\mathcal{L}\Pi\mathrm{Mg}$, which has the following grammar:

$$\varphi ::= p \mid \top \mid \mathrm{g} \mid \Box\varphi \mid \mathrm{M}_i\varphi \mid \neg\varphi \mid (\varphi \wedge \varphi)$$

with $p \in \mathsf{Prop}$.

For future convenience, we refer to the elements in $\mathsf{Prop} \cup \{\top, g\}$ in general as propositional *letters*, and we define $\mathrm{md}(\varphi)$ to be the *modal depth* of φ defined as usual, with M_i's and $\Box$ all treated as modal operators, $\mathsf{Free}(\varphi)$ to be the set of free propositional variables in φ, and the *quantificational depth* of φ to be the maximal length of any chain of nested quantifers in φ, analogous to the usual definition in first-order logics.

Let us also define as in [54] for every $\alpha \in \mathcal{L}\Pi\mathrm{Mg}$ an important formula $\mathrm{atom}(\alpha)$:

$$\mathrm{atom}(\alpha) := \Diamond\alpha \wedge \forall q(\Box(q \to \alpha) \vee \Box(q \to \neg\alpha)) \tag{2.1}$$

where $q \in \mathsf{Prop}$ does not occur in α. To fix this choice, we assume that there is an enumeration of Prop fixed from the outset. Then whenever we need fresh propositional letters in a definition, the definition picks out the first available propositional variable.

Here g is intended to express the proposition that some atomic proposition is true, and $\mathrm{M}_i\varphi$ the proposition that φ is entailed by at least i many atomic propositions. Hence, g should be evaluated to the join of the atoms in a modal algebra. But this requires that the join exists. Let us call a modal algebra *separable* if the join of its atoms exists. Then we can give the semantics for $\mathcal{L}\mathrm{Mg}$ and $\mathcal{L}\Pi\mathrm{Mg}$ on appropriate modal algebras.

Definition 2.2.5. For any separable modal algebra B, define g (or g_B when ambiguity arises) as the join of all atoms of B, and M_i an operator on B as follows:

$$M_i a = \begin{cases} 1 & \text{if there are at least } i \text{ distinct atoms below } a \\ 0 & \text{otherwise} \end{cases}$$

for $i \in \mathbb{N}_+$.

Then, any valuation V on a separable B can be recursively extended to an $\mathcal{L}\mathrm{Mg}$-*valuation* $\widehat{V}$ from $\mathcal{L}\mathrm{Mg}$ to B by the same clauses for Boolean connectives and $\Box$ as in Definition 2.2.1, plus the following two clauses:

1. $\widehat{V}(\mathrm{g}) = g_B$
2. $\widehat{V}(\mathrm{M}_i\varphi) = M_i\widehat{V}(\varphi)$.

If B is furthermore complete, define the $\mathcal{L}\Pi\mathrm{Mg}$-*valuation* extending $\widehat{V}$ by combining the clauses above and in Definition 2.2.1. It is not hard to see that the $\mathcal{L}\mathrm{Mg}$-valuation, $\mathcal{L}\Pi$-valuation, and $\mathcal{L}\Pi\mathrm{Mg}$-valuation extending V are compatible. Hence by $\widehat{V}$, we always mean the defined valuation with the maximal domain. This will be either an $\mathcal{L}\mathrm{Mg}$-valuation or an $\mathcal{L}\Pi\mathrm{Mg}$-valuation, depending on whether the codomain of V is merely separable or is complete. We also extend the definition of validity and also the Alg operator to formulas in $\mathcal{L}\Pi\mathrm{Mg}$ in the obvious way.

Regarding $\mathrm{atom}(\alpha)$, it is intended to express the proposition that α expresses an atomic proposition. Its definition does not always achieve this intended meaning, but assuming that it is interpreted on complete simple S5 algebras, this definition indeed singles out atoms in simple S5 algebras. The proof of this can be found in [91].

Lemma 2.2.6. *For any $\alpha \in \mathcal{L}\Pi\mathrm{Mg}$, any complete simple S5 algebra B, and any valuation V on B, we have*

$$\widehat{V}(\mathrm{atom}(\alpha)) = \begin{cases} 1 & \textit{if } \widehat{V}(\alpha) \textit{ is an atom in } B \\ 0 & \textit{otherwise.} \end{cases}$$

Now to the logics in the language $\mathcal{L}\Pi\mathrm{Mg}$. They are obtained by adding two axiom schemata to define the new operators M_i's and g by formulas in $\mathcal{L}\Pi$.

Definition 2.2.7. For any normal Π-logic Λ, define $\Lambda\mathsf{Mg}$ as the smallest normal Π-logic (with formula variables in the schemata and rules of the definition now ranging over $\mathcal{L}\Pi\mathrm{Mg}$) that contains the following two axiom schemata for each $n \in \mathbb{N}_+$:

$$\mathrm{M}_n\varphi \leftrightarrow \exists q_1 \cdots \exists q_n(\bigwedge_{1\leq i<j\leq n} \Box(q_i \to \neg q_j) \wedge \bigwedge_{1\leq i\leq n} (\mathrm{atom}(q_i) \wedge \Box(q_i \to \varphi))) \qquad (\mathsf{M})$$

$$\mathrm{g} \leftrightarrow \exists q(q \wedge \mathrm{atom}(q)) \qquad (\mathsf{g})$$

where $q_1, \cdots q_n \in \mathsf{Prop}$ do not occur in φ, and $q \in \mathsf{Prop}$.

With the help of Lemma 2.2.6, it is not hard to directly observe that both (M) and (g) are sound. In fact, we have the following theorem, mostly by Fine and Holliday, on which our new results depend.

Lemma 2.2.8. $\mathsf{S5\Pi Mg}$ *is a conservative extension of* $\mathsf{S5\Pi}$*. Namely,* $\mathsf{S5\Pi Mg} \cap \mathcal{L}\Pi = \mathsf{S5\Pi}$*. Also,* $\mathsf{S5\Pi Mg}$ *is sound and complete with respect to* $\mathsf{csS5A}$*.*

Proof. For any $\varphi \in \mathcal{L}\Pi$, if $\varphi \in \mathsf{S5\Pi Mg}$, then we can replace all M_i's and g in its derivation by their definitions in Definition 2.2.7. The resulting derivation is in $\mathsf{S5\Pi}$. Hence $\varphi \in \mathsf{S5\Pi}$. This shows that $\mathsf{S5\Pi Mg} \cap \mathcal{L}\Pi \subseteq \mathsf{S5\Pi}$. The other direction is trivial. This is observed and first used by Fine in [54].

It is first shown algebraically in [91] that $\mathsf{S5\Pi Mg}$ is sound and that $\mathsf{S5\Pi}$ is sound and complete with respect to $\mathsf{csS5A}$. Now for any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$ that is valid in $\mathsf{csS5A}$, we can first replace all M_i's and g in φ by their definitions to obtain ψ. Then $\varphi \leftrightarrow \psi \in \mathsf{S5\Pi Mg}$. We know that $\mathsf{S5\Pi Mg}$ is sound on $\mathsf{csS5A}$. So ψ is valid. Then, since $\psi \in \mathcal{L}\Pi$, $\psi \in \mathsf{S5\Pi}$, which means $\psi \in \mathsf{S5\Pi Mg}$. By *modus ponens*, $\varphi \in \mathsf{S5\Pi Mg}$. $\Box$

The proof of the completeness of $\mathsf{S5\Pi}$ in [91] relies on a fairly intricate quantifier elimination in $\mathsf{S5\Pi Mg}$ found first by Fine in [54], which says that for any $\varphi \in \mathcal{L}\Pi$, there is a formula $\psi \in \mathcal{L}\mathrm{Mg}$ such that $\varphi \leftrightarrow \psi \in \mathsf{S5\Pi Mg}$. We will also make use of this technical result. In fact, ψ can be chosen from a much smaller fragment of $\mathcal{L}\mathrm{Mg}$. Following Fine, we call them model descriptions and define them now.

Definition 2.2.9. For any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$, first define the following abbreviations:

$$\mathrm{Q}_0 := \neg\mathrm{M}_1\varphi; \quad \mathrm{Q}_i\varphi := \mathrm{M}_i\varphi \wedge \mathrm{M}_{i+1}\varphi, i \in \mathbb{N}_+; \quad \mathrm{N}\varphi := \Diamond(\neg\mathrm{g} \wedge \varphi).$$

For any finite subset $P \subseteq \mathsf{Prop}$, a *state description* s over P is a conjunction of literals from P in which every $p \in P$ occurs exactly once. We follow the convention that an empty conjunction is $\top$ and an empty disjunction is $\bot$. Let 2^P be the set of all state descriptions over P. Then, a *model description of degree* n *over* P is a conjunction of:

1. either g or $\neg$g;
2. a state description $a \in 2^P$;
3. for each $s \in 2^P$, either $\mathrm{M}_n s$ or some $\mathrm{Q}_i s$ for some $i < n$;
4. for each $s \in 2^P$, either $\mathrm{N}s$ or $\mathrm{N}\neg s$.

Lemma 2.2.10 ([54], § 4.2)**.** *For any $\varphi \in \mathcal{L}\Pi$, there exists a $qf(\varphi) \in \mathcal{L}\mathrm{Mg}$ such that $\varphi \leftrightarrow qf(\varphi) \in \mathsf{S5\Pi Mg}$. Moreover, $qf(\varphi)$ is a disjunction of model descriptions over $\mathsf{Free}(\varphi)$ of degree 2^n where n is the quantification degree of φ.*

For the construction of qf, the reader can also see the appendix of [91].

2.3 Semantical and syntactical reduction

In this section, we show that for any $\varphi \in \mathcal{L}\Pi$, any complete simple S5 algebra B, and any $\Lambda \in \mathsf{Next}\Pi(\mathsf{S5\Pi})$, we can construct a formula, which we call $basic(\varphi)$, such that:

- $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$;
- $B \vDash \varphi$ iff $B \vDash basic(\varphi)$;
- $basic(\varphi)$ is a Boolean combination of $\Diamond\neg g$ and $\mathrm{M}_i\top$ for $i \in \mathbb{N}_+$.

To facilitate the proof, let us first define a number of useful fragments of $\mathcal{L}\Pi$.

Definition 2.3.1. Recall that $\mathcal{L}\mathrm{Mg}$ is the quantifier free fragment of $\mathcal{L}\Pi\mathrm{Mg}$. Now, define the following propositional-variable-free fragments of $\mathcal{L}\mathrm{Mg}$ where i ranges over elements in $\mathbb{N}_+$:

$$\begin{aligned}
\mathcal{S}\mathrm{Mg} \ni \varphi &::= \top \mid \mathrm{g} \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi \mid \mathrm{M}_i\varphi \\
\mathcal{S}_{\leq 1}\mathrm{Mg} \ni \varphi &::= \top \mid \mathrm{g} \mid \Diamond\neg\mathrm{g} \mid \mathrm{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi) \\
\mathcal{S}\mathrm{Basic} \ni \varphi &::= \top \mid \Diamond\neg\mathrm{g} \mid \mathrm{M}_i\top \mid \neg\varphi \mid (\varphi \wedge \varphi)\ .
\end{aligned}$$

The $\mathcal{S}$ instead of $\mathcal{L}$ in their names means "$\mathcal{S}$entence." It is not hard to see that $\mathcal{S}\mathrm{Mg}$ collects all propositional-variable-free formulas in $\mathcal{L}\mathrm{Mg}$ and that $\mathcal{S}_{\leq 1}\mathrm{Mg}$ collects some formulas with modal depth at most 1 in $\mathcal{S}\mathrm{Mg}$, which are enough for our purposes.

For any $\varphi \in \mathcal{L}\Pi$, we will construct $basic(\varphi)$ as the following with u and $comp$ to be defined:

$$basic(\varphi) = comp(\Box qf(u(\varphi))).$$

First, $u(\varphi)$ is the universal closure of φ, which is $\forall p_1 \forall p_2 \cdots \forall p_n \varphi$ where $p_1, p_2, \cdots, p_n$ enumerate the free propositional variables in φ. And recall that qf returns the result of quantifier elimination. Since $u(\varphi)$ has no free propositional variable, according to Lemma 2.2.10,

$qf(u(\varphi)) \in \mathcal{S}\mathrm{Mg}$ is a disjunction of model descriptions of some finite degree over $\varnothing$. From Definition 2.2.9, we can see that all model descriptions of degree n over $\varnothing$ are of the form

$$\pm g \wedge \mathrm{M}_i\top \wedge \neg\mathrm{M}_{i+1}\top \wedge \pm\Diamond\neg\mathrm{g} \quad \text{or} \quad \pm\, g \wedge \mathrm{M}_n\top \wedge \pm\Diamond\neg\mathrm{g}$$

where $i < n$, $\pm$ stands for $\neg$ or nothing, and $\mathrm{M}_0\top$ for $\top$. In short, $qf(u(\varphi))$ is a Boolean combination of g, $\mathrm{M}_i\top$ for $i \in \mathbb{N}_+$, and $\Diamond\neg g$, and hence is in $\mathcal{S}_{\leq 1}\mathrm{Mg}$.

Now we construct *comp* as a function that simplifies a boxed modal description over $\varnothing$ to a formula in $\mathcal{S}$Basic in a provably equivalent way.

Lemma 2.3.2. *For any ψ a disjunction of model descriptions of degree n over $\varnothing$, there is a formula $comp(\Box\psi) \in \mathcal{S}\mathrm{Basic}$ such that $\Box\psi \leftrightarrow comp(\Box\psi) \in \mathsf{S5\Pi Mg}$.*

Proof. Let *pos* be the number of model descriptions in ψ where g appears positively and *neg* the number of model descriptions in ψ where g appears negatively. For any $1 \leq i \leq pos$, let α_i be the result of deleting the conjunct g in the ith model description in ψ where g appears positively, and similarly define β_i for $1 \leq i \leq neg$, where we need to delete the $\neg$g conjunct.

Let $\alpha = \bigvee_{1\leq i\leq pos} \alpha_i$ and $\beta = \bigvee_{1\leq i\leq neg} \beta_i$, which are now Boolean combinations of $\mathrm{M}_i\top$'s and $\Diamond\neg$g. Then obviously $\psi \leftrightarrow ((\mathrm{g}\wedge\alpha)\vee(\neg\mathrm{g}\wedge\beta))$ and $\Box\psi \leftrightarrow \Box((\mathrm{g}\wedge\alpha)\vee(\neg\mathrm{g}\wedge\beta))$ are in $\mathsf{S5\Pi Mg}$ using propositional tautologies and normality. Let us write for any $\varphi_1, \varphi_2 \in \mathcal{L}\Pi\mathrm{Mg}$, $\varphi_1 \equiv_{\mathsf{S5\Pi Mg}} \varphi_2$ iff $\varphi_1 \leftrightarrow \varphi_2 \in \mathsf{S5\Pi Mg}$. Then we have

$$\begin{aligned}
\Box\psi &\equiv_{\mathsf{S5\Pi Mg}} \Box((\mathrm{g}\wedge\alpha)\vee(\neg\mathrm{g}\wedge\beta)) && (2.2)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box((\mathrm{g}\vee\neg\mathrm{g})\wedge(\mathrm{g}\vee\beta)\wedge(\neg\mathrm{g}\vee\alpha)\wedge(\alpha\vee\beta)) && (2.3)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box(\mathrm{g}\vee\neg\mathrm{g})\wedge\Box(\mathrm{g}\vee\beta)\wedge\Box(\neg\mathrm{g}\vee\alpha)\wedge\Box(\alpha\vee\beta) && (2.4)\\
&\equiv_{\mathsf{S5\Pi Mg}} \Box(\mathrm{g}\vee\Box\beta)\wedge\Box(\neg\mathrm{g}\vee\Box\alpha)\wedge\Box(\Box\alpha\vee\Box\beta) && (2.5)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\Box\mathrm{g}\vee\Box\beta)\wedge(\Box\neg\mathrm{g}\vee\Box\alpha)\wedge(\Box\alpha\vee\Box\beta) && (2.6)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\Box\mathrm{g}\vee\beta)\wedge(\Box\neg\mathrm{g}\vee\alpha)\wedge(\alpha\vee\beta) && (2.7)\\
&\equiv_{\mathsf{S5\Pi Mg}} (\neg\Diamond\neg\mathrm{g}\vee\beta)\wedge(\neg\mathrm{M}_1\top\vee\alpha)\wedge(\alpha\vee\beta). && (2.8)
\end{aligned}$$

In the above chain of provable equivalences, (5), (7), and (8) require more explanation. Note that in $\mathsf{S5\Pi Mg}$, we have $\Box\forall p\varphi \equiv_{\mathsf{S5\Pi Mg}} \forall p\Box\varphi$, and dually, $\Diamond\exists p\varphi \equiv_{\mathsf{S5\Pi Mg}} \exists p\Diamond\varphi$. With all the axioms in $\mathsf{S5}$, and also the axiom (M) defined in Definition 2.2.7, $\mathrm{M}_i\top \equiv_{\mathsf{S5\Pi Mg}} \Box\mathrm{M}_i\top$. Then, $\alpha \equiv_{\mathsf{S5\Pi Mg}} \Box\alpha$ and $\beta \equiv_{\mathsf{S5\Pi Mg}} \Box\beta$, since α and β are Boolean combinations of $\mathrm{M}_i\top$ and $\Diamond\neg g$. Thus we have (5) and (7). For (8), some manipulation of axioms (g) and (M) gives us $\Box\neg g \equiv_{\mathsf{S5\Pi Mg}} \neg\mathrm{M}_1\top$. The rest of the equivalences are standard $\mathsf{S5}$ reasoning. We can then define $basic(\Box\psi)$ to be the right hand side in (8), which is provably equivalent to $\Box\psi$ and is in $\mathcal{S}$Basic. $\Box$

Now we show that for any $\varphi \in \mathcal{L}\Pi$, $basic(\varphi)$ has the required properties.

Lemma 2.3.3. *For any $\varphi \in \mathcal{L}\Pi$ and $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$, $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$.*

Proof. Since $\Lambda\mathsf{Mg}$ is a conservative extension of Λ (by Lemma 2.2.8), $\varphi \in \Lambda$ iff $\varphi \in \Lambda\mathsf{Mg}$. Using universalization and also universal instantiation, $\varphi \in \Lambda\mathsf{Mg}$ iff $u(\varphi) \in \Lambda\mathsf{Mg}$. Since $\Lambda \in \mathsf{Next}\Pi(\mathsf{S5\Pi})$, $\Lambda\mathsf{Mg}$ extends $\mathsf{S5\Pi Mg}$. Together with the quantifier elimination result in Lemma 2.2.10, $qf(u(\varphi)) \leftrightarrow u(\varphi) \in \Lambda\mathsf{Mg}$. Thus $u(\varphi) \in \Lambda\mathsf{Mg}$ iff $qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. By necessitation and also the T axiom derivable in $\mathsf{S5}$, $qf(u(\varphi)) \in \Lambda\mathsf{Mg}$ iff $\Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Finally, due to Lemma 2.3.2 and the fact that $qf(u(\varphi))$ is indeed a disjunction of model descriptions of some finite degree over $\varnothing$, we have $basic(\varphi) = comp(\Box qf(u(\varphi))) \leftrightarrow \Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Thus $basic(\varphi) \in \Lambda\mathsf{Mg}$ iff $\Box qf(u(\varphi)) \in \Lambda\mathsf{Mg}$. Connecting all the equivalences, $\varphi \in \Lambda$ iff $basic(\varphi) \in \Lambda\mathsf{Mg}$. □

On the semantical side, we first make an easy observation.

Lemma 2.3.4. *For any complete (resp. separable) modal algebra B and any $\varphi \in \mathcal{L}\Pi\mathrm{Mg}$ (resp. $\mathcal{L}\mathrm{Mg}$) such that $\mathsf{Free}(\varphi) = \varnothing$, for any two valuations V and V' on B, $\widehat{V}(\varphi) = \widehat{V'}(\varphi)$.*

Due to this observation, we define for any separable modal algebra B, a fixed trivial valuation V_B which maps every $p \in \mathsf{Prop}$ to 1_B. Then $B \vDash \varphi$ iff $\widehat{V_B}(\varphi) = 1_B$ for any $\varphi \in \mathcal{L}\mathrm{Mg}$ (or $\mathcal{L}\Pi\mathrm{Mg}$ when B is complete) such that $\mathsf{Free}(\varphi) = \varnothing$.

Then we can prove the semantical requirement for *basic*.

Lemma 2.3.5. *For any $\varphi \in \mathcal{L}\Pi$ and any complete simple S5 algebra B, $B \vDash \varphi$ iff $B \vDash basic(\varphi)$.*

Proof. First consider the following chain of equivalences:

$B \vDash \varphi$ iff $B \vDash u(\varphi)$	by the definition of validity
iff $\widehat{V_B}(u(\varphi)) = 1$	by the definition of validity
iff $\widehat{V_B}(qf(u(\varphi))) = 1$	B validates $\mathsf{S5\Pi Mg}$, and $u(\varphi) \leftrightarrow qf(u(\varphi)) \in \mathsf{S5\Pi Mg}$
iff $\widehat{V_B}(\Box(qf(u(\varphi)))) = 1$	$\Box 1 = 1$ and $\Box a \neq 1$ if $a \neq 1$
iff $\widehat{V_B}(comp(\Box(qf(u(\varphi))))) = 1$	B validates $\mathsf{S5\Pi Mg}$, and Lemma 2.3.2

Note that $basic(\varphi)$ must have no free propositional variable. This is because $basic(\varphi) = comp(\Box qf(u(\varphi)))$, $\mathsf{Free}(u(\varphi)) = \varnothing$, and neither qf nor $comp$ introduces new free variables. Then by the observation in the previous lemma, $B \vDash basic(\varphi)$ iff $\widehat{V_B}(basic(\varphi)) = 1$. Connecting all the equivalences, $B \vDash \varphi$ iff $B \vDash basic(\varphi)$. □

2.4 Types and type space

In the last section, we have shown that many formulas are equivalent in terms of validity or theoremhood. In this section, we do the same to the algebras: many algebras are equivalent in terms of the formulas in $\mathcal{L}\Pi$ they validate. This equivalence relation can in fact bring csS5A from a class to a countable set, which we will call the type space. Then, to study the classes of algebras definable by formulas in $\mathcal{L}\Pi$, we can just study the sets of types of the algebras in those classes. This in turn gives us a topology on the type space. Now we start with the definition of the types, which is in fact a much simplified version of the famous Tarski invariant for Boolean algebras (see § 5.5 in [31]), due to the completenss of the algebras we are interested in.

Definition 2.4.1. For any complete simple S5 algebra B, its *type* $t(B)$ is a pair $\langle t_0(B), t_1(B)\rangle$ where

$$t_0(B) = \begin{cases} 1 & \text{if } g \neq 1 \\ 0 & \text{if } g = 1, \end{cases} \quad t_1(B) = \begin{cases} i \in \mathbb{N} & \text{if } B \text{ has exactly } i \text{ atoms} \\ \infty & \text{if } B \text{ has infinitely many atoms.} \end{cases} \tag{2.9}$$

Recall that g of B is the join of its atoms. Hence, t_0 says whether this algebra contains an atomless part, and t_1 counts the atoms it has. Let S be the set of all types of complete simple S5 algebras, the type space.

Proposition 2.4.2. $S = (\{0,1\} \times \mathbb{N}^\infty) \backslash \{\langle 0,0\rangle\}$.

Proof. Clearly, $S \subseteq \{0,1\} \times \mathbb{N}^\infty$ as any type is a pair $\langle t_0, t_1\rangle$ where the first component can only be 0 or 1 and the second component can only be a natural number of ∞. Also, if the type of a complete simple S5 algebra A is $\langle 0,0\rangle$, then A has no atom and also no atomless part, which means A is trivial and thus not a complete simple S5 algebra in our Definition 2.2.5. So we have shown the inclusion from left to right. Now to the other direction. The right-hand-side can be decomposed into three parts:

- $\langle 0,n\rangle$ for $n \in \mathbb{N}^\infty_+$. Types of this form can be realized by the Boolean algebra B_n of the powerset of a set of n elements, supplemented with $\Box$ defined as in Definition 2.2.5. When $n = \infty$, we can use the powerset of $\mathbb{N}$.

- $\langle 1,0\rangle$. To realize this type, take the countable free Boolean algebra B. It is well known that B is atomless, but not complete. However, we can take the MacNeille completion B^+ of B, a complete Boolean algebra (unique up to isomorphism) such that B embeds into and that every element of B^+ is a join of images of elements of B. For a construction of this B^+, see [73], Chap. 25. Then B^+ is complete and atomless, as if there is an atom, it must be the image of an atom of B, but B is atomless. Now turn B^+ to a simple S5 algebra by defining $\Box$ as in Definition 2.2.5. Then B^+ has type $\langle 1,0\rangle$.

- $\langle 1,n\rangle$ for $n \in \mathbb{N}^\infty_+$. Consider the product of B^+ and B_n with $\Box$ again defined as above. It is not hard to see that it has an atomless part: g of this algebra is $\langle 1,0\rangle$. It also has n many atoms, listed by $\langle 0,a\rangle$ where a range over atoms in B_n. Thus this simple S5 algebra has type $\langle 1,n\rangle$.

Hence we realized all types in the right-hand-side. So the inclusion from right to left is also shown. □

Now we define the equivalences between algebras. Then we will show that types capture this equivalence relation.

Definition 2.4.3. For any two complete simple S5 algebras A, B, and any $\mathcal{L} \in \{\mathcal{L}\Pi, \mathcal{S}\text{Basic}\}$ we say $A \equiv_{\mathcal{L}} B$ if for any sentence $\varphi \in \mathcal{L}$, $A \vDash \varphi$ iff $B \vDash \varphi$.

Lemma 2.4.4. *For any two complete simple S5 algebras A, B, $A \equiv_{\mathcal{L}\Pi} B$ iff $A \equiv_{\mathcal{S}\text{Basic}} B$.*

Proof. Immediate from Lemma 2.3.5. □

Lemma 2.4.5. *For any two complete simple S5 algebras A, B, $t(A) = t(B)$ iff $A \equiv_{\mathcal{S}\text{Basic}} B$. Hence, together with Lemma 2.4.4, $t(A) = t(B)$ iff $A \equiv_{\mathcal{L}\Pi} B$.*

Proof. Recall that for all $\varphi \in \mathcal{S}\text{Basic}$ and any complete simple S5 algebra A, $A \vDash \varphi$ iff $\widehat{V_A}(\varphi) = 1$, because $\mathsf{Free}(\varphi) = \varnothing$. Also notice that for $\varphi = \Diamond\neg g$ or $\mathrm{M}_i\top$ for any $i \in \mathbb{N}_+$, $\widehat{V_A}(\varphi)$ is either 0 or 1. Since $\mathcal{S}\text{Basic}$ consists of all and only the Boolean combinations of these formulas, for any $\varphi \in \mathcal{S}\text{Basic}$, $\widehat{V_A}(\varphi) \in \{0, 1\}$. This means either $A \vDash \varphi$ or $A \vDash \neg\varphi$.

Now suppose $t(A) = t(B)$. Then, conflating 1_A and 1_B, and also 0_A and 0_B, we can easily verify that $\widehat{V_A}(\Diamond\neg \mathrm{g}) = \widehat{V_B}(\Diamond\neg \mathrm{g})$ and that for all $i \in \mathbb{N}_+$, $\widehat{V_A}(\mathrm{M}_i\top) = \widehat{V_B}(\mathrm{M}_i\top)$. Then a simple induction propagates these equalities to all $\varphi \in \mathcal{S}\text{Basic}$. Thus we see that if $t(A) = t(B)$, then $A \equiv_{\mathcal{S}\text{Basic}} B$.

On the other hand, if $t(A) \neq t(B)$, then there are two cases:

- $t_1(A) \neq t_1(B)$. In this case, $\Diamond\neg \mathrm{g}$ distinguishes the two algebras.
- $t_2(A) \neq t_2(B)$. Let n be the smaller number among them. Then $n \in \mathbb{N}$, $n + 1 \in \mathbb{N}_+$, and $\neg\mathrm{M}_{n+1}$ distinguishes the two algebras.

Hence, if $t(A) \neq t(B)$, then $A \not\equiv_{\mathcal{S}\text{Basic}} B$. □

Due to this lemma, the function t can be seen as the quotient map from $\mathsf{csS5A}$ to $\mathsf{csS5A}/{\equiv_{\mathcal{L}\Pi}}$. This means that the Galois connection between $\mathsf{csS5A}$ and $\mathcal{L}\Pi$ by Alg and Log can be reduced to the following Galois connection between S and $\mathcal{L}\Pi$.

Definition 2.4.6. For any type $s \in S$ and any $\varphi \in \mathcal{L}\Pi$, let us write $s \vDash \varphi$ just in case for any $A \in \mathsf{csS5A}$ such that $t(A) = s$, $A \vDash \varphi$.

Then define Type(Γ) for every $\Gamma \subseteq \mathcal{L}\Pi$ as $\{s \in S \mid \forall\varphi \in \Gamma, s \vDash \varphi\}$, with Type($\varphi$) again abbreviating Type($\{\varphi\}$). In the other direction, define Log(T) for any subset T of S as $\{\varphi \in \mathcal{L}\Pi \mid \forall s \in T, s \vDash \varphi\}$, with Log($s$) abbreviating Log($\{s\}$) for any $s \in S$ as well.

Then, we can collect the following easy but useful observations.

Lemma 2.4.7. *For any $\Gamma \subseteq \mathcal{L}\Pi$, $\varphi, \psi \in \mathcal{S}\text{Basic}$:*

- $\mathrm{Alg}(\Gamma) = t^{-1}(\mathrm{Type}(\Gamma))$, $t(\mathrm{Alg}(\Gamma)) = \mathrm{Type}(\Gamma)$, *and then* $\mathrm{Log}(\mathrm{Type}(\Gamma)) = \mathrm{Log}(\mathrm{Alg}(\Gamma))$;
- $\mathrm{Type}(\Gamma) = \bigcap\{\mathrm{Type}(\varphi) \mid \varphi \in \Gamma\}$;
- $\mathrm{Type}(\neg\varphi) = S \setminus \mathrm{Type}(\varphi)$, $\mathrm{Type}(\varphi \wedge \psi) = \mathrm{Type}(\varphi) \cap \mathrm{Type}(\psi)$.

Using this lemma, we can study the following topology that will be important to us for both general completeness and the lattice structure of $\mathsf{Next\Pi}(\mathsf{S5\Pi})$.

Definition 2.4.8. Let $\mathcal{S}$ be the topological space with the type space S as the underlying set and $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{L}\Pi\}$ as basic opens.

Lemma 2.4.9. *$\mathcal{S}$ is a Stone space, homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$ with the usual order topology.*

Proof. From Lemma 2.3.5, the basic opens of $\mathcal{S}$ are just sets in $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$. By the third bullet in Lemma 2.4.7, we know that $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$ is a field of sets on S. Thus $\mathcal{S}$ is zero-dimensional. To see that $\mathcal{S}$ is Hausdorff, take two different $s_1, s_2 \in S$. Recall that S is $t(\mathsf{csS5A})$. So we can find two complete simple S5 algebras B_1 and B_2 such that $t(B_1) = s_1$ and $t(B_2) = s_2$. Then, by Lemma 2.3.5, $B_1 \not\equiv_{\mathcal{S}\mathrm{Basic}} B_2$. So we can find a formula $\varphi \in \mathcal{S}\mathrm{Basic}$ such $\mathrm{Type}(\varphi)$ that separates B_1 and B_2. So $\mathcal{S}$ is Hausdorff.

To show that $\mathcal{S}$ is compact, we need a more detailed analysis of $\{\mathrm{Type}(\varphi) \mid \varphi \in \mathcal{S}\mathrm{Basic}\}$. First, note that $\mathrm{Type}(\Diamond\neg g) = \{1\} \times \mathbb{N}^\infty$, $\mathrm{Type}(\neg\Diamond\neg g) = \{0\} \times \mathbb{N}^\infty_+$, and they partition S into two parts. Let us name them by S_1 and S_0 respectively. Hence $\mathcal{S}$ is the disjoint union of $\mathcal{S}_1$ and $\mathcal{S}_0$ defined as the subspaces of S on S_1 and S_0 respectively. So we only need to show that they are both compact. On $\mathcal{S}_1$, the basic clopens are now Boolean combinations of $\mathrm{Type}(\mathrm{M}_i)$ for $i \in \mathbb{N}_+$ and $\mathrm{Type}(\Diamond\neg g)$, all restricted to S_1. But $\mathrm{Type}(\Diamond\neg g) = S_1$. Then the clopens are actually the field of sets on S_1 generated by $\{\mathrm{Type}(\mathrm{M}_n) \cap S_1 \mid n \in \mathbb{N}\} = \{\{\langle 1, i\rangle \mid n \leq i \leq \infty\} \mid n \in \mathbb{N}\}$. Hence it is not hard to see that $\mathcal{S}_1$ is just (homeomorphic to) the one-point compactification of the order topology on $\mathbb{N}$. The situation for $\mathcal{S}_0$ is almost the same, except that the space is on $\mathbb{N}^\infty_+$. But it is still homeomorphic to the one-point compactification of $\mathbb{N}$. $\square$

2.5 Main results

Now we are prepared to prove the main results regarding Π-logics extending $\mathsf{S5\Pi}$. Let us start with the general completeness.

Theorem 2.5.1. *For any $\Lambda \in \mathsf{Next\Pi}(\mathsf{S5\Pi})$, $\Lambda = \mathrm{Log}(\mathrm{Alg}(\Lambda))$.*

Proof. As is shown in Lemma 2.4.7, $\mathrm{Log}(\mathrm{Alg}(\Lambda)) = \mathrm{Log}(\mathrm{Type}(\Lambda))$. Also, it is trivial that $\Lambda \subseteq \mathrm{Log}(\mathrm{Type}(\Lambda))$. Hence we just need to show that, for any $\varphi \in \mathcal{L}\Pi$, if $\varphi \in \mathrm{Log}(\mathrm{Type}(\Lambda))$, then $\varphi \in \Lambda$.

Let us assume the antecedent. Then for any $s \in \text{Type}(\Lambda)$, $s \vDash \varphi$. In other words, $\text{Type}(\varphi) \supseteq \text{Type}(\Lambda)$. As we observed in Lemma 2.4.7, $\text{Type}(\Lambda) = \bigcap\{\text{Type}(\psi) \mid \psi \in \Lambda\}$. By Lemma 2.3.5, $\text{Type}(\psi) = \text{Type}(basic(\psi))$. Note also that $\psi \in \Lambda$ iff $basic(\psi) \in \Lambda\mathsf{Mg}$, which is shown in Lemma 2.3.3. Thus the set $\{\text{Type}(\psi) \mid \psi \in \Lambda\} \subseteq \{\text{Type}(\psi) \mid \psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}\}$. On the other hand, for any $\psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}$, using the axioms defining M_i and g, there is a $\psi' \in \mathcal{L}\Pi$ such that $\psi \leftrightarrow \psi' \in \Lambda\mathsf{Mg}$. This means that ψ' is in $\Lambda\mathsf{Mg}$, hence also in Λ, and that $\text{Type}(\psi) = \text{Type}(\psi')$, using Lemma 2.2.8. Hence $\{\text{Type}(\psi) \mid \psi \in \Lambda\} = \{\text{Type}(\psi) \mid \psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}\}$, which we now call F.

Now this is a filter of basic clopens in $\mathcal{S}$ for the following reasons.

- For any $X, Y \in F$, we can find $\alpha, \beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}$ such that $X = \text{Type}(\alpha)$ and $Y = \text{Type}(\beta)$. Now $\alpha \wedge \beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}$, since $\Lambda\mathsf{Mg}$ has all propositional tautologies and *modus ponens*. Hence $X \cap Y = \text{Type}(\alpha) \cap \text{Type}(\beta) = \text{Type}(\alpha \wedge \beta) \in F$.

- Recall that the basic clopens in $\mathcal{S}$ are just $\{\text{Type}(\beta) \mid \beta \in \mathcal{S}\text{Basic}\}$. For any $X \in F$ and any basic clopen Y such that $X \subseteq Y$, we first find $\alpha \in \Lambda\Pi\mathsf{Mg} \cap \mathcal{S}\text{Basic}$ and $\beta \in \mathcal{S}\text{Basic}$ such that $X = \text{Type}(\alpha)$ and $Y = \text{Type}(\beta)$. Then note that $\text{Type}(\alpha \to \beta) = (S \setminus X) \cup Y = S$, since $X \subseteq Y$. Then by the completeness of $\mathsf{S5\Pi Mg}$ (Lemma 2.2.8), $\alpha \to \beta \in \mathsf{S5\Pi Mg}$. Then by *modus ponens* in $\Lambda\mathsf{Mg}$, which extends $\mathsf{S5\Pi Mg}$ as Λ extends $\mathsf{S5}$, $\beta \in \Lambda\mathsf{Mg}$. Remember that $\beta \in \mathcal{S}\text{Basic}$. Hence $\beta \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}$ and $Y = \text{Type}(\beta) \in F$.

Thus we have $\text{Type}(\Lambda) = \cap F$, a filter of basic clopens in $\mathcal{S}$, and we assumed that $\text{Type}(\varphi) \supseteq \text{Type}(\Lambda)$. Take $basic(\varphi)$. We have $\text{Type}(basic(\varphi)) = \text{Type}(\varphi)$ and that it is basic clopen in $\mathcal{S}$. We have shown that $\mathcal{S}$ is a Stone space in Lemma 2.4.9. Hence by compactness, there is actually an element $Z \in F$ such that $\text{Type}(\varphi) \subseteq Z$. By the definition of F, we can find a $\psi \in \Lambda\mathsf{Mg} \cap \mathcal{S}\text{Basic}$ such that $Z = \text{Type}(\psi)$. Then $\text{Type}(\psi \to basic(\varphi)) = S$. By completeness again, we have $\psi \to basic(\varphi) \in \mathsf{S5\Pi Mg}$ and thus also $\Lambda\mathsf{Mg}$. Then, since ψ is taken in $\Lambda\mathsf{Mg}$, $basic(\varphi) \in \Lambda\mathsf{Mg}$. By Lemma 2.3.3, $\varphi \in \Lambda$. This finishes the completeness of Λ. □

Then we describe the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5\Pi})$.

Theorem 2.5.2. *The lattice* $\langle \mathsf{Next}\Pi(\mathsf{S5\Pi}), \subseteq \rangle$ *is isomorphic to the lattice of the open sets of* $\mathcal{S}$. *The isomorphism is* $\Lambda \mapsto S \setminus \text{Type}(\Lambda)$, *or in the other direction,* $X \mapsto \text{Log}(S \setminus X)$.

Proof. It is shown in the proof of Theorem 2.5.1 that for any $\Lambda \in \mathsf{Next}\Pi(\mathsf{S5\Pi})$, $\text{Type}(\Lambda)$ is the intersection of a filter of the basic opens of $\mathcal{S}$. By the basic theory of Stone spaces, this means that $\text{Type}(\Lambda)$ is always a closed set in $\mathcal{S}$. Also, for any $\Lambda_1, \Lambda_2 \in \mathsf{Next}\Pi(\mathsf{S5\Pi})$, obviously $\Lambda_1 \subseteq \Lambda_2$ iff $\text{Type}(\Lambda_1) \supseteq \text{Type}(\Lambda_2)$. This means that, if we can establish that for every closed set $X \subseteq S$, there is a $\Lambda \in \mathsf{Next}\Pi(\mathsf{S5\Pi})$ such that $X = \text{Type}(\Lambda)$, then the lattice structure of $\mathsf{Next}\Pi(\mathsf{S5\Pi})$ is precisely the inverse lattice of the closed sets of $\mathcal{S}$, or just the lattice of the open sets of $\mathcal{S}$, and the isomorphism will be given by $\Lambda \mapsto S \setminus \text{Type}(\Lambda)$.

Now take an arbitrary closed set X in $\mathcal{S}$. Then $\mathrm{Log}(X) \in \mathsf{Next\Pi}(\mathsf{S5\Pi})$ as it is the set of formulas in $\mathcal{L}\Pi$ valid on a class of complete simple S5 algebras. Then what remains to be shown is that $\mathrm{Type}(\mathrm{Log}(X)) = X$. Again, the direction $X \subseteq \mathrm{Type}(\mathrm{Log}(X))$ is trivial. Now take an arbitrary type $s \in S \setminus X$. Then we just need to show that $s \notin \mathrm{Type}(\mathrm{Log}(X))$. Since $\mathcal{S}$ is a Stone space, X is closed, and $s \notin X$, we know that s and X can be separated by a basic clopen. Then, we can find a $\varphi \in \mathcal{S}\mathrm{Basic}$ such that $X \subseteq \mathrm{Type}(\varphi)$ but $s \notin \mathrm{Type}(\varphi)$. But then, $\varphi \in \mathrm{Log}(X)$. Since $\mathrm{Type}(\mathrm{Log}(X)) = \bigcap\{\mathrm{Type}(\psi) \mid \psi \in \mathrm{Log}(X)\}$, we see that $s \notin \mathrm{Type}(\mathrm{Log}(X))$. This finishes the proof. $\square$

Since we have shown in the process of proving Theorem 2.4.9 that $\mathcal{S}$ is isomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$, we have the following corollary.

Corollary 2.5.3. *The lattice* $\langle \mathsf{Next\Pi}(\mathsf{S5\Pi}), \subseteq \rangle$ *is isomorphic to the lattice of open sets of the disjoint union of two copies of the one-point compactification of* $\mathbb{N}$*, which is further isomorphic to the lattice of filters of the direct product of two copies of the field of finite and cofinite sets in* $\mathbb{N}$*.*

Another corollary of this characterization of all logics in $\mathsf{Next\Pi}(\mathsf{S5\Pi})$ is that, in terms of computability, there are arbitrarily complex logics (coded as sets of natural numbers in some natural way). More precisely, for any $X \subseteq \mathbb{N}$, there is a $\Lambda \in \mathsf{Next\Pi}(\mathsf{S5\Pi})$ such that X and Λ are Turing-equivalent. But we prove a more general result regarding the Turing-degree of a logic in $\mathsf{Next\Pi}(\mathsf{S5\Pi})$.

Theorem 2.5.4. *We fix a natural encoding of* $\mathbb{N}^\infty$ *and* $\mathcal{L}\Pi$ *in* $\mathbb{N}$ *with the usual operations such as forming pairs or conjunctions computable. Then, For any* $\Lambda \in \mathsf{Next\Pi}(\mathsf{S5\Pi})$*,* Λ *is Turing-equivalent to* $\mathrm{Type}(\Lambda)$*.*

Proof. Let Λ be in $\mathsf{Next\Pi}(\mathsf{S5\Pi})$. To show that Λ and $\mathrm{Type}(\Lambda)$ are Turing-equivalent, we need to show that they can be reduced to each other. Crucially, we are not looking for a uniform reduction here, and the reduction algorithms depend on the following two questions:

- Is $\langle 0, \infty \rangle \in \mathrm{Type}(\Lambda)$ or not?
- Is $\langle 1, \infty \rangle \in \mathrm{Type}(\Lambda)$ or not?

This means we have four cases. Let us consider first the case where neither $\langle 0, \infty \rangle$ nor $\langle 1, \infty \rangle$ is in $\mathrm{Type}(\Lambda)$. Recall that $\mathrm{Type}(\Lambda)$ is a closed set in $\mathcal{S}$. Hence, $\mathrm{Type}(\Lambda)$ is now finite and computable, and we only need to reduce Λ to $\mathrm{Type}(\Lambda)$ as the other direction is trivial. To decide whether an input formula φ is in Λ or not, recall that by Lemma 2.4.7 and Theorem 2.5.1, $\Lambda = \mathrm{Log}(\mathrm{Type}(\Lambda))$. This means we only need to check if $\mathrm{Type}(\Lambda) \subseteq \mathrm{Type}(\varphi)$. Also, note that by computing *basic*(φ) and putting it into disjunctive normal form, we can compute a finite presentation of $\mathrm{Type}(\varphi)$ since it is a finite union of intervals in $\{0\} \times \mathbb{N}^\infty_+$ or $\{1\} \times \mathbb{N}^\infty$. Hence, to test if $\mathrm{Type}(\Lambda) \subseteq \mathrm{Type}(\varphi)$, we only need to check for every element in $\mathrm{Type}(\Lambda)$ (only finitely many such checks are needed) if it is in $\mathrm{Type}(\varphi)$.

For the other three cases, we first show how to reduce Type(Λ) to Λ. Note that in each case, we only need to answer if an element in $S_{fin} = (\{0\} \times \mathbb{N}_+) \cup (\{1\} \times \mathbb{N})$ is in Type(Λ), since whether $\langle 0, \infty \rangle$ or $\langle 1, \infty \rangle$ is in Type(Λ) is already answered by the discussion by cases and the answers can be hardwired into the final algorithm computing Type(Λ). Then it is enough to observe that for each $\langle i, n \rangle \in S_{fin}$, where $M_{-1}\top$ and $M_0\top$ are defined by $\top$:

- if $i = 0$, then $\langle i, n \rangle \in$ Type(Λ) iff the formula $\Box g \to (\neg M_{n-1}\top \vee M_{n+1}\top)$ is not in Λ since Type($\Box g \to (\neg M_{n-1}\top \vee M_{n+1}\top)) = S \setminus \langle 0, n \rangle$, and
- if $i = 1$, then $\langle i, n \rangle \in$ Type(Λ) iff the formula $\Diamond \neg g \to (\neg M_{n-1}\top \vee M_{n+1}\top)$ is not in Λ since Type($\Diamond \neg g \to (\neg M_{n-1}\top \vee M_{n+1}\top)) = S \setminus \langle 1, n \rangle$.

Now we show how to reduce Λ to Type(Λ). Note that our algorithm will depend on the above two questions. Let φ be an input formula. The general strategy is to test first whether Type$(\Lambda)_0 =$ Type$(\Lambda) \cap \{0\} \times \mathbb{N}_+^\infty$ is a subset of Type$(\varphi)_0 =$ Type$(\varphi) \cap \{0\} \times \mathbb{N}_+^\infty$, and then test whether Type$(\Lambda)_1 =$ Type$(\Lambda) \cap \{1\} \times \mathbb{N}^\infty$ is a subset of Type$(\varphi)_1 =$ Type$(\varphi) \cap \{1\} \times \mathbb{N}^\infty$. Clearly, $\varphi \in \Lambda$ iff both tests return an positive answer. To decide whether Type$(\Lambda)_0 \subseteq$ Type$(\varphi)_0$:

- If $\langle 0, \infty \rangle$ is in Type(Λ), the algorithm checks if the intervals in Type(φ) contains an interval with ∞ being the right endpoint. If not, then Type$(\Lambda)_0$ is not a subset of Type$(\varphi)_0$ since $\langle 0, \infty \rangle \notin$ Type$(\varphi)_0$. If Type$(\varphi)_0$ does contain an interval of the with ∞ being the right endpoint, let $\langle 0, n \rangle$ be its left endpoint. Then for every $m \in \mathbb{N}_+$ from 0 to $n - 1$, we test if either $\langle 0, m \rangle \notin$ Type$(\Lambda)_0$ or $\langle 0, m \rangle \in$ Type(φ). If the answer is positive for all such m, we know that Type$(\Lambda)_0 \subseteq$ Type$(\varphi)_0$, and otherwise Type$(\Lambda)_0 \not\subseteq$ Type$(\varphi)_0$, resulting in $\varphi \notin \Lambda$.
- If $\langle 0, \infty \rangle$ is not in Type(Λ), then using the topology of $\mathcal{S}$ and the fact that Type(Λ) is closed, our algorithm can check for every element in Type$(\Lambda)_0$ if it is in Type$(\varphi)_0$ (computable from φ) since Type$(\Lambda)_0$ in this case is a finite set.

The way to decide whether Type$(\Lambda)_1 \subseteq$ Type$(\varphi)_1$ is completely analogous. $\square$

Corollary 2.5.5. *For any set $X \subseteq \mathbb{N}$, there is a $\Lambda \in \mathsf{Next\Pi(S5\Pi)}$ that is Turing-equivalent to X.*

Proof. Note that $X^+ = \{\langle 1, n \rangle \mid n \in X\} \cup \{\langle 1, \infty \rangle\}$ is a closed set in $\mathcal{S}$. Letting $\Lambda =$ Log(X^+), as we have shown in the proof of Theorem 2.5.2, $X^+ =$ Type(Λ). Hence, by Theorem 2.5.4, Λ and X^+ are Turing-equivalent. But clearly, X is Turing-equivalent to X^+. So Λ is Turing equivalent to X. $\square$

Regarding the non-normal Π-logics extending $\mathsf{S5\Pi}$, we limit ourselves to merely point out that there are many such logics. Algebraically, non-normal modal logics come from *matrices* (see §1.5 of [108]), which are algebras of propositions with a set of designated truth values. To exhibit a non-normal Π-logic extending $\mathsf{S5\Pi}$, we can use just one particular structure. Let B

be the complete simple S5 algebra whose Boolean part is the direct product of the powerset algebra of $\mathbb{N}$ and the MacNeille completion of the free Boolean algebra with countably many generators. Note that $t(B) = \langle 1, \infty \rangle$. Now consider the following set:

$$\Lambda = \{\varphi \in \mathcal{L}\Pi \mid \forall V : \mathsf{Prop} \to B, \widehat{V}(\varphi) \geq g\}.$$

It is not hard to see that $\Lambda \supseteq \mathrm{Log}(B)$, as the latter collects formulas whose valuation stay at 1, hence necessarily above g. Also, Λ is a Π-logic. In particular, universalization is valid because if φ only evaluates to elements above (non-strictly) g, then $\forall\varphi$ evaluates to the meet of those elements above g, which must stay above g. Moreover, $\exists q(q \wedge \mathrm{atom}(q)) \in \Lambda$, as this formula evaluates precisely to g. However, $\Box\exists q(q \wedge \mathrm{atom}(q))$ is not in Λ, since $\Box g$ is $\bot$, because $g \neq 1$ in B: there is an atomless part in B. This means we obtained a non-normal Π-logic extending a normal Π-logic $\mathrm{Log}(B)$ which has no proper consistent normal extension: the only closed proper subset of $\{B\}$ in $\mathcal{S}$ is $\varnothing$. Obviously, for any complete simple S5 algebra B that has both a non-trivial atomless part and a non-trivial atomic part, we can obtain a non-normal Π-logic in the same fashion. We could also use the requirement that $\widehat{V}(\varphi) \geq \neg g$, which will result in non-normal Π-logics including $\neg g$ but not $\Box\neg g$.

2.6 Conclusion

In this chapter, we investigated Π-logics extending $\mathsf{S5\Pi}$. In particular, we see that complete simple S5 algebras are semantically adequate for all normal Π-logics extending $\mathsf{S5\Pi}$, that the lattice of these normal Π-logics is isomorphic to the lattice of the open sets of the type space $\mathcal{S}$ that is homeomorphic to the disjoint union of two copies of the one-point compactification of $\mathbb{N}$, that they can have arbitrarily high Turing-degree, and that they do not exhaust all the Π-logics extending $\mathsf{S5\Pi}$ as there are non-normal ones.

A major unresolved problem though, is the characterization of all Π-logics, instead of only the normal ones, extending $\mathsf{S5\Pi}$. We conjecture that a similar strategy can be used, though we need to be more careful about the choice of types. With an informative characterization, we may also be able to find a simple syntactical condition for a Π-logic extending $\mathsf{S5\Pi}$ to be normal and describe how the normal ones are distributed in the lattice of all Π-logics extending $\mathsf{S5\Pi}$.

Finally, we ask whether there is a way to prove all the results, especially the completeness of $\mathsf{S5\Pi}$ and stronger Π-logics, without using explicitly quantifier elimination. That this is important is because for many modal logics L, there is little hope that one can obtain a manageable quantifier elimination for $\mathsf{L\Pi}$. Hence, we need some technique that can be more easily generalized.

Chapter 3

Logics of Belief and Propositional Quantifiers

3.1 Introduction

In this chapter, we consider extending the modal logic KD45, commonly taken as the baseline system for belief, with propositional quantifiers that can be used to formalize natural language sentences such as "everything I believe is true" or "there is something that I neither believe nor disbelieve." Our main results are axiomatizations of the logics with propositional quantifiers of natural classes of complete Boolean algebras with an operator (BAOs) validating KD45. Among them is the class of complete, atomic, and completely multiplicative BAOs validating KD45. Hence, by duality, we also cover the usual method of adding propositional quantifiers to normal modal logics by considering their classes of Kripke frames. In addition, we obtain decidability for all the concrete logics we discuss.

The present work can be seen as sitting at the intersection of two strands of literature: the doxastic logic literature, since we are extending KD45, and the literature on modal logics with propositional quantifiers, since we are extending with propositional quantifiers. In both bodies of literature, algebraic approaches are not particularly popular. Moreover, KD45 was not discussed in the literature of modal logics with propositional quantifiers until very recently [12]. To explain our motivation and potential contribution to the two bodies of literature in more detail, we use two subsections below.

3.1.1 Dubious principles and possible-world semantics

Since Hintikka [88], modal logic has been indispensable for the study of intensional propositional operators like knowledge and belief. For the belief case, the system KD45 arose naturally as a baseline system. The reason may be that KD45 puts together the properties that we immediately recognize as what an ideal agent's belief (or an agent's ideal belief) should have: logical omniscience, consistency, and full introspection. Indeed, the modal rule and axioms in the standard axiomatization of KD45 can be matched precisely to these properties: the necessitation rule and K to logical omniscience, D to consistency, and 4 and 5 to introspection. The attitudes toward these idealizations vary (see, for example, more friendly views in [159] and Section 1.3 of [25] and much less friendly views in [148]), but the system KD45 remains central (for its most recent appearance, see [5] but also [6]).

Coming along with the syntactical formalism of modal logic is the possible-world semantics based on possible-worlds and accessibility relations (namely Kripke frames). The use of possible-world semantics is perhaps mainly fueled by the correspondence and completeness results for most philosophically interesting modal formulas. When deciding which axioms to use, if we accept that possible-world semantics in general is appropriate, we may first find out the axioms' corresponding frame conditions. To quote David Lewis in [122, p. 19], "instead of asking the baffling question whether whatever is actual is necessarily possible, we could try asking: is the relation R symmetric?" When we already have a strong intuition on which logic is the most appropriate (for whatever purpose), we may still want to use possible-world models to succinctly represent a consistent set of formulas describing a situation and then guide our syntactic reasoning in that situation. Completeness guarantees that this is always possible.

For the belief case, if we are not venturing below K, the standard possible-world semantics based on Kripke frames is always appropriate by Sahlqvist's completeness theorem [19, § 5.6], since the relevant axioms are D, $\mathsf{4}$, and $\mathsf{5}$, which are all Sahlqvist formulas. Moreover, all modal logics extending $\mathsf{KD45}$ are Kripke-complete in the sense that they are complete with respect to the classes of Kripke frames on which they are valid [144]. Even with the addition of dynamic operators as in [16], semantics based on possible-worlds is still largely appropriate, and many such extensions start with possible-world semantics. While it is well known that there are Kripke incomplete logics [96], meaning that no classes of Kripke frames can validate precisely the theorems in those logics, perhaps, when studying belief operators, Kripke frames are always enough for us, and there is nothing that can "banish" us from, to borrow from David Lewis again, "a doxastic logician's paradise"?

As another way of extending the language of Doxastic logic, consider propositional quantifiers. While we naturally quantify over propositions in both ordinary and philosophical discourses about belief, the addition of propositional quantifiers is not given much attention in the literature. Can we repeat the success story of the Kripke semantics here again, or are we in the situation that, with propositional quantifiers, we gain enough expressivity so that Kripke frames with their well-documented quirks in the literature on Kripke incompleteness lead to unwanted validities? Note that if there are formulas in the extended language such that, on the one hand, they are valid on Kripke frames validating a logic L, and on the other hand, we have strong reasons to at least treat them as optional and study and use extensions of L without them, Kripke frames must go.

Indeed, a number of new principles about belief that seem conceptually significant are formalizable in the extended language.

- "One believes that everything one believes is true" is formalized as $\mathtt{B}\forall p(\mathtt{B}p \to p)$.
- "If no matter what p stands for, one believes that φ, then one believes that no matter what p stands for, φ" is formalized as $\forall p \mathtt{B}\varphi \to \mathtt{B}\forall p\varphi$.
- "There is a proposition that the agent takes to be consistent and to settle everything" can be formalized as $\exists q(\widehat{\mathtt{B}}q \wedge \forall p(\mathtt{B}(q \to p) \vee \mathtt{B}(q \to \neg p)))$.

Conceptually, then, we can ask: if we would like to take all the idealizations encoded in $\mathsf{KD45}$ on board, should we also adopt or are we already committed to some of the principles above, once we add propositional quantifiers into our language?

Let us focus on the first principle, which we call Immod: "one believes that everything one believes is true." Even for idealized agents or idealized beliefs, as axiomatized by $\mathsf{KD45}$, it seems that Immod should not be included in a *logic* of belief. After all, the idealizations we are granting here are only about logic and introspection and do not warrant the truth of the uncertain beliefs that we choose to believe. Immod should be distinguished from "for every proposition p, one believes that if she believes that p then p" (with the "if ... then

..." here being the material implication). This principle, when formalized as $\forall p(\mathtt{B}(\mathtt{B}p \to p))$, is merely the universalization of a simple consequence of the negative introspection axiom. The crucial difference between this principle and Immod is that Immod says that one believes the totality of one's belief to be true, while $\forall p\mathtt{B}(\mathtt{B}p \to p)$ says only that for every proposition p, when considered individually, one believes that if p is believed, then p is true.

More concretely, we can take an agent who has credences about a real number x randomly generated (perhaps by an unending sequence of fair coin flips) from the interval $[0, 1]$. For all measurable $X \subseteq [0, 1]$, the agent's credence that $x \in X$ is just the measure of X. In addition, in this simple example, it seems not against our intuitive understanding of the concept of outright belief that the agent can simply believe precisely those propositions with credence 1.[1] Then, for all $a \in [0, 1]$, the agent believes that $x \in [0, 1] \setminus \{a\}$ since $[0, 1] \setminus \{a\}$ is measure 1. However, the agent does not believe that for all $a \in [0, 1]$, $x \in [0, 1] \setminus \{a\}$ since $\bigcap_{a\in[0,1]}([0, 1] \setminus \{a\}) = \varnothing$, which is not measure 1. Hence the agent in this situation does not believe that all her beliefs are true.

The above of course does not constitute a decisive argument that Immod is not valid for ideal agents or ideal beliefs axiomatized by $\mathsf{KD45}$. But we hope that at least we have demonstrated some interest that people might have in considering a logic without Immod. On the semantic side, though, as we will show in Section 3.2, if we adopt the standard possible-world semantics, Immod as formalized by $\mathtt{B}\forall p(\mathtt{B}p \to p)$ is valid on any Kripke frame that validates $\mathsf{KD45}$. Indeed, it is valid so long as the accessibility relation is shift-reflexive,[2] regardless of which domain of propositions (as represented by subsets of possible-worlds) we choose for the propositional quantifiers to range over and regardless of whether the domain varies from world to world. In other words, if we constrain ourselves with the standard possible-world semantics, the space of logics between $\mathsf{KD45}$ and $\mathsf{KD45}$ plus Immod is closed to us.

To allow for modesty above $\mathsf{KD45}$, we will turn to algebraic semantics. In algebraic semantics, propositions, instead of possible worlds, are first-class citizens that naturally form Boolean algebras when ordered by logical strength. Then, propositional quantifiers are interpreted in these algebras of propositions by the meet operation since, intuitively, for example, "everything I believe is true" is the conjunction of all instances of "if I believe that p then p." Specifically, we will use what were used in the first algebraic semantics for a $\mathsf{KD45}$ belief operator in [152]: proper filter algebras, except that we will consider only those whose underlying Boolean algebra is complete[3] in the sense that arbitrary, not just finite, meets and joins exist. We believe there can be an independent metaphysical argument for why the Boolean algebra of propositions should at least be complete, but we leave this for another occasion. For our purposes, the completeness condition is merely a condition with which

[1] Note that this does not rely on the agent's belief being reduced to credence in any way. However, see [32].

[2] A binary relation R is shift-reflexive if and only if for all x and y, xRy implies yRy. Shift-reflexivity follows from Euclidicity, the first-order correspondence of the axiom $\mathsf{5}$.

[3] Since the word "complete" is also used for saying that a logic is complete, we sometimes use "lattice complete" to express this idea.

we can show, in a way that does not use any special property of the belief operator $\mathtt{B}$, that all formulas, including those like $\forall p\varphi$, have well-defined semantic values. In other words, lattice completeness is a language-and-logic-blind condition guaranteeing that our algebraic semantics works.

While proper filter algebras allow modesty, they are not completely conceptually innocent beyond $\mathsf{KD45}$ though. A strengthened introspection axiom, which we call $\mathsf{4}^{\forall}$, is valid on these algebras. This new axiom $\mathsf{4}^{\forall}$ intuitively reads: if the agent believes every instance of φ, then the agent believes that she believes every instance of φ. In the formal language to be introduced in full later, $\mathsf{4}^{\forall}$ is $\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi$. However, unlike Immod, we find $\mathsf{4}^{\forall}$ well-motivated, especially when we are considering extending $\mathsf{KD45}$. Typically, and especially under idealization, we take our judgment about our internal state, like believing φ or not, as infallible. If so, it is not just that we are in a position to believe that we believe φ when we do believe φ. The aggregation of arbitrarily many such infallible judgments is still infallible (contrary to a large aggregation of credence 1 yet fallible propositions) and to be believed by us (or idealized versions of us). The formula $\mathsf{4}^{\forall}$ precisely formalizes this reasoning step.

Corresponding to this idea is the fact that a proper filter algebra works by keeping a proper filter of propositions in the underlying Boolean algebra as the filter of "believed propositions" and interprets $\mathtt{B}\varphi$ to either the top element or the bottom element depending on whether φ is interpreted as a "believed proposition" or not. If the proposition expressed by φ is "believed", then $\mathtt{B}\varphi$ is interpreted as the top element and otherwise the bottom element. More technically, proper filter algebras can be understood as Boolean algebras with an operator that validate $\mathsf{KD45}$ and also has the special property that the operator sends propositions to either the top element or the bottom element. Intuitively, then, from the agent's perspective, a formula $\mathtt{B}\varphi$ is as true as tautologies are once true and is as false as contradictions are once false.[4] Hence, it is not hard to check that $\mathsf{4}^{\forall}$ is valid, since we are essentially only considering the two-element Boolean algebra once we treat $\mathtt{B}\varphi$ as a whole.

But will this class of complete proper filter algebras validate any other formulas whose interpretation might be unwelcome? Our axiomatization suggests that the answer is no. We will show that the logic of complete proper filter algebras is axiomatized by $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, obtained by adding to $\mathsf{KD45}$ the usual Π-principles, namely those axioms about propositional quantifiers that are analogous to the axioms about first-order quantifiers, and then strengthening $\mathsf{4}$ to $\mathsf{4}^{\forall}$. Since the Π-principles encode only the quantificational axioms, like

[4]Of course a tautology like $p \to p$ and a formula like $\mathtt{B}p$ have different truth conditions, regardless of whether $\mathtt{B}p$ is true or not. So the proper filter algebras we consider are not representing propositions obtained by way of metaphysical (or a priori) equivalence where two sentences φ and ψ express the same proposition iff necessarily (or a priori) φ and ψ are either both true or both false. The proper filter algebras represent algebras of propositions obtained for a particular agent in a particular situation by stipulating that two sentences φ and ψ express the same proposition iff the agent is certain in that situation that either φ and ψ are both true or both false. For those who are unsatisfied with the restrictedness of proper filter algebras, we will show that we can, without changing the logic, consider all complete Boolean algebras with an operator validating $\mathsf{KD45}$. In this way, we can be more neutral on what count as "propositions". However, it is non-trivial to see that $\mathsf{4}^{\forall}$ is valid on all such algebras, and we devote the whole of Section 3.3 to this issue.

instantiation and universalization, the only conceptual leap in this axiomatization is from 4 to $4^{\forall}$.

3.1.2 Axiomatizability for modal logics with propositional quantifiers

Now we turn to a more technical side and connect our work to the literature on modal logics with propositional quantifiers. The systematic technical study of propositional quantifiers is arguably initiated in Fine's dissertation [54], though already in Kripke's [114], propositional quantifiers are discussed. Also around the same time as Fine's dissertation were Bull's [23] and Gabbay's [65]. Soon after his dissertation, Fine summarized and extended his results in [55]. From these early papers, we can already see a wide range of semantic choices, especially about the domain of propositions that $\forall p$ can quantify over (which is naturally encoded in general frames). Bull and Gabbay in the above-cited papers also identified two ways to refute Barcan's schema $\forall p\Box\varphi \to \Box\forall p\varphi$ through varying the domain of propositions for quantifiers across possible-worlds and through generalizing accessibility relations to neighborhood functions. In a completely non-technical paper [76], we also saw perhaps the earliest proposal of treating $\forall p$ as quantifying directly over objects in a lattice of propositions, a proposal perhaps inspired by the philosophical stance defended in that paper. Since then, there has been a steady stream of interest devoted to this topic, with general theoretical results focusing on expressive power under the standard possible-world semantics ([102, 116, 117, 29]), specific results mostly establishing non-axiomatizability ([3, 115, 111, 62, 64, 113, 110, 71, 160]) with the exception of [110] and [160], and more application-oriented works: [11, 9, 10, 12, 52, 8, 61].

A remarkable phenomenon when studying unimodal logic with propositional quantifiers on Kripke frames, where every set of possible-worlds counts as a proposition that $\forall p$ can quantify over, is the seeming existence of what we call an "axiomatizability boundary": there seems to be a line in the order structure of classes of Kripke frames of usual normal modal logics such that, below this line, the logics of those classes of frames with propositional quantifiers are extremely complex (often recursively equivalent to full second-order logic) and non-axiomatizable, while above this line, the logics with propositional quantifiers are suddenly decidable. Of course, we need to define what is "usual" for this "axiomatizability boundary" concept to make sense. A very preliminary step is to consider first the lattice of Kripke frame classes corresponding to the logics in the modal logic cube. We see that Fine's 1970 paper [55] sets the boundary between **S4** and **S5** and between **B** and **S5**. Kaminski's result [102] pushes the boundary further from **S4** to **S4.2**. However, where the boundary lies in the direction from **S5** to **KD45** and **KB5** remained open. In this chapter, we will show that the boundary can be pushed from the decidable side to **KD45**: the logic with propositional quantifiers of Kripke frames validating KD45 is decidable.

We may just focus on Kripke frames if we are only aiming at pushing the axiomatizability boundary. But also hard to ignore in this literature is a severe lack of an algebraic approach

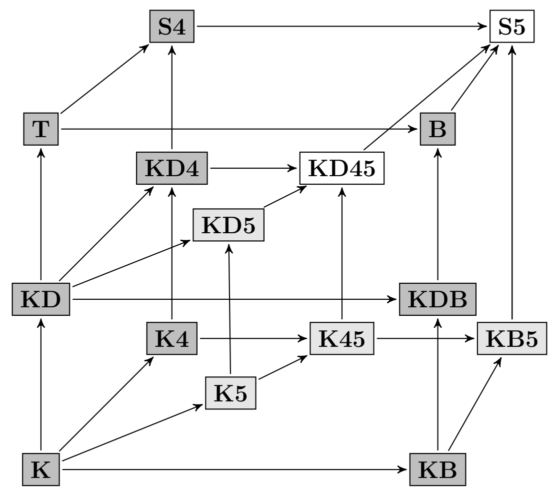


Figure 3.1: The frame class cube. Darker shade means the corresponding logic with propositional quantifiers is non-axiomatizable. No shade means decidability established, and light shade means decidability unknown.

(until very recently; [91], [97], and [40]). In particular, when propositional quantifiers are added to a modal logic L in the basic language, this is usually done by considering some class of Kripke frames on which L is valid and then generating the logic with propositional quantifiers of this class of Kripke frames. The main variability is in changing the domain of propositions for the propositional quantifiers to quantify over, and this is often achieved by considering general frames whose underlying Kripke frames are frames of L. Then in a general frame, the domain for interpreting propositional variables and for propositional quantifiers is naturally the set of admissible propositions. A problem with this approach, however, is that when we take $\forall p$ to mean "no matter what p stands for," which is the interpretation we are interested in here, the semantics must validate the full instantiation axiom $\forall p\varphi \to \varphi[\psi/p]$ where $\varphi[\psi/p]$ is the result of substituting p with ψ (with necessary renaming of bound propositional variables). In general frames, validating the full instantiation axiom often involves putting a so-called "closed under formula" condition, which seems to be dependent on the choice of L.

In the algebraic semantics for propositional quantifiers, the lattice completeness of an algebra of propositions ensures the well definedness of the semantic value of all formulas and the validity of the full instantiation axioms. While the lattice completeness condition is usually not necessary for this purpose, it is blind to the choice of language and logic. The semantics also directly models the intended interpretation of $\forall p$: the semantic value of $\forall p\varphi$ on an algebra is the meet of all the possible semantic values of φ as we reevaluate p to all elements in the algebra. Hence, the algebraic method, in contrast to the above possible-world-based method, of adding propositional quantifiers to L is to take the logic, in the language extended with propositional quantifiers, of the complete Boolean algebras with operators validating L. One can then investigate the result of imposing atomicity and/or complete multiplicativity. In particular, if both conditions are imposed, we recover the version of possible-world-based method of extension where all subsets count as propositions.

The algebraic approach poses also a series of natural open questions, and we will list some in the concluding section of this chapter. An example, relating to the above phenomenon of the "axiomatizability boundary", is this: how would a shift from Kripke frames to complete BAOs affect the boundary? Will the boundary move or even blur in the sense that we will see logics undecidable yet not as complex as theories like the second-order theory of arithmetic? In all the proofs of non-axiomatizability, atomicity is at least implicit in the set-up, if not directly used. It is not our ambition here to settle questions at this level of generality though. Our aim is merely to initiate this program by focusing on a very special case: the case of extending $\mathsf{KD45}$ with propositional quantifiers in an algebraic way. And we obtain the following results from a few more general theorems that we will establish along the way:

- If we consider all complete BAOs validating $\mathsf{KD45}$, the resulting logic is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$. Note that in principle we can consider the wider class of BAOs which happen to make the semantics well-defined and also validate $\mathsf{KD45}$. In particular, the Lindenbaum algebra of $\mathsf{KD45}\Pi$ is such an algebra. So if we drop the lattice completeness condition, we get $\mathsf{KD45}\Pi$. We will show using propositionally contingent frames to show that $\mathsf{KD45}\Pi \subsetneq \mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{Immod}$. Then it follows that lattice completeness is not inert: it strengthens $\mathsf{4}$ into $\mathsf{4}^{\forall}$.

- Imposing complete multiplicativity of $\mathtt{B}$ amounts to adding Immod (or Barcan's schema) to $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$.

- Imposing atomicity amounts to adding the axiom $\mathsf{At} = \exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q)))$ to $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$.

- Hence, if both conditions are imposed, the resulting logic is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{ImmodAt}$. By duality theory, then, this is the logic of serial, transitive, and Euclidean Kripke frames.

- Finally, all the logics above are decidable. Hence the "axiomatizability boundary" is pushed to $\mathsf{KD45}$ and does not change when we switch from Kripke frame to complete BAOs.

3.1.3 Organization

The rest of the chapter is organized as follows. In section 3.2, we formally define the language and algebraic semantics, and then introduce the necessary axioms and systems with some of their logical relations; we then show how the algebraic semantics can invalidate Immod while to a certain extent possible-world-based semantics cannot. However, we show that $\mathsf{4}^{\forall}$ can be invalidated using this kind of semantics once we allow propositional contingentism. In section

3.3, we show that $\mathsf{4}^{\forall}$, and hence also the logic $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, are valid on all complete algebras validating $\mathsf{KD45}$ (complete KD45 algebras). In section 3.4, we show that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ is complete with respect to the class of all complete proper filter algebras. Since complete proper filter algebras are also complete KD45 algebras, $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ axiomatizes the logic of both complete KD45 algebras and complete proper filter algebras and also any class of algebras in between. This is the longest section of the chapter, in which we need to prove two technical lemmas. The first lemma is an analog of the quantifier elimination used to show the completeness of $\mathsf{S5}\Pi$ by Fine. While we do not need a full quantifier elimination, we need to show that the quantifiers can be separated from unmodalized propositional variables and pulled out from the scopes of modal operators so that we can translate formulas into a first-order language about Boolean algebras with two named elements. The second lemma at its core says that the first-order logic of the quotients of complete Boolean algebras is just the first-order logic of Boolean algebras. While this seems to be a natural proposition of independent interest, to the best of our knowledge, it has not been shown previously. In section 3.5, we extract more results from the proofs in Section 3.4 and establish two general completeness theorems. From them, the logics resulting from imposing atomicity and complete multiplicativity to algebras naturally follow. We then show a general decidability theorem, from which the decidability of all the particular logics discussed follows. In the last section, Section 3.6, we conclude with directions of future research.

3.2 Syntax, semantics, logics, and the problem of Immod

The propositional language with a belief operator and propositional quantifiers is defined as follows.

Definition 3.2.1. Define the language $\mathcal{L}\Pi$ by the following grammar:

$$\varphi ::= p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \mathsf{B}\varphi \mid \forall p\varphi$$

where $p \in \mathsf{Prop}$, a set of propositional variables.[5] We adopt the usual abbreviations, and in particular we frequently write $\bot$ for $\neg\top$, $\widehat{\mathsf{B}}$ for $\neg\mathsf{B}\neg$, and $\exists p$ for $\neg\forall p\neg$. The free and bound occurrences of propositional variables are defined as in first-order logic. As is common in first-order logic, we write $\varphi(p)$ to note that $\varphi(\psi)$ is then the result of replacing the free occurrences of p in φ by ψ with necessary renaming of bound variables.

Now we turn to semantics. Algebraic semantics starts with a Boolean algebra of propositions, and every formula will be evaluated to one of the propositions in it. If we define Boolean algebras simply by the laws of conjunction and negation, then the semantics seems to lack motivation independent of the logic we want it to generate. However, it is also well

[5] In contrast, $\top$ can be viewed as a propositional constant.

known (see Chap. 4 of [34]) that they can be equivalently defined as partial orders with greatest lower bounds (meets), least upper bounds (joins), and complements, or more specifically, complemented distributive lattices. Thus, a Boolean algebra can be seen as representing propositions that form a complemented distributive lattice once ordered by their strength. Then $\top$, $\wedge$, $\vee$, and $\neg$ are interpreted uncontroversially as the top element, the meet (greatest lower bound) operation, the join (least upper bound) operation, and the complementation operation, respectively.

In the same fashion, $\forall p\varphi$ should express the proposition that is the meet of all propositions expressible as φ while the proposition expressed by p ranges over all propositions in the algebra. Since there are possibly infinitely many such propositions expressible by φ, we make a further assumption about the Boolean algebra of propositions we study here: they must be complete in the sense that every set of elements has a meet. As for the belief operator, the most general representation we can have is to use an arbitrary function on each algebra of propositions. But since our concern here is to study the logics of belief at least as strong as $\mathsf{KD45}$, we need to make corresponding assumptions on this function representing the belief operator. The following definition summarizes the assumptions we make.

Definition 3.2.2. A *KD45 algebra* is a pair $\mathcal{B} = \langle B, \Box \rangle$ where

- B is a non-trivial Boolean algebra with $\top$ being its top element, $\neg$ its complementation operation, and $\wedge$ its meet relation, and
- $\Box$ is a unary function on B such that for all $a, b \in B$,

$$\Box\top = \top,\ \Box(a \wedge b) = \Box a \wedge \Box b,\ \neg\Box\neg\top = \top,\ \Box a = \Box\Box a,\ \text{and}\ \neg\Box a = \Box\neg\Box a.$$

When we need to distinguish the operations from different algebras, we will subscript the operations by the algebra they are from. For example, we may write $\wedge_{\mathcal{B}}$ for the meet operation of the Boolean algebra part of $\mathcal{B}$ or write $\wedge_B$ for the meet operation of B. We also write $\leq$, possibly with subscripts, for Boolean lattice orderings. We will frequently write $x \in \mathcal{B}$ instead of $x \in B$, which is already an abbreviation of x being in the carrier set of B. The usual abbreviations for $\vee, \to, \leftrightarrow, \oplus, \setminus, \bot$, and $\Diamond$ apply too.

A *complete KD45 algebra* is a KD45 algebra whose Boolean algebra part is a complete Boolean algebra. We use $\bigwedge$ and $\bigvee$ for arbitrary meets and joins in complete Boolean algebras. Again, subscripts are added and dropped as needed.

Then the language $\mathcal{L}\Pi$ can be interpreted on any complete KD45 algebra. To express semantic substitution, for any function $f : X \to Y$ and any $x \in X$ and $y \in Y$, we write $f[y/x]$ for the function that is identical to f except that $f[y/x](x) = y$. This notation will be used by all the semantics we define in this chapter.

Definition 3.2.3. For any complete KD45 algebra $\mathcal{B}$, a *valuation* θ on $\mathcal{B}$ is a function from Prop to $\mathcal{B}$. Then a valuation θ on $\mathcal{B}$ can be uniquely extended to $\tilde{\theta} : \mathcal{L}\Pi \to \mathcal{B}$ recursively by the following clauses:

- $\tilde{\theta}(p) = \theta(p)$;
- $\tilde{\theta}(\neg\varphi) = \neg\tilde{\theta}(\varphi)$, $\tilde{\theta}(\varphi \wedge \psi) = \tilde{\theta}(\varphi) \wedge \tilde{\theta}(\psi)$, and $\tilde{\theta}(\top) = \top$;
- $\tilde{\theta}(\mathsf{B}\varphi) = \Box\tilde{\theta}(\varphi)$;
- $\tilde{\theta}(\forall p\varphi) = \bigwedge_{a\in\mathcal{B}} \widetilde{\theta[a/p]}(\varphi)$.

A formula $\varphi \in \mathcal{L}\Pi$ is *valid* in a complete KD45 algebra $\mathcal{B}$ if for all valuations θ on $\mathcal{B}$, $\tilde{\theta}(\varphi) = \top$; otherwise we call it *refutable* in $\mathcal{B}$. A formula φ is valid on a class of complete KD45 algebras if φ is valid on each member of that class, and a set of formulas is valid on a class whenever every formula in the set is valid on the class. As usual, validity is denoted by $\vDash$.

One problem with the Definition 3.2.3 is that it is very general, and little structure of these complete KD45 algebras is revealed in the definition. While we will study them in detail in Section 3.3, we now introduce a very concrete semantics whose structures in which we evaluate formulas can be seen as directly modeling doxastic scenarios of ideal agents.

Definition 3.2.4. A *proper filter algebra* $\mathcal{B}$ is a pair $\langle B, F\rangle$ where B is a Boolean algebra and F is a proper filter of that Boolean algebra. A *complete proper filter algebra* is a proper filter algebra whose Boolean algebra part is a complete Boolean algebra. We will write $F_{\mathcal{B}}$ if the context is not clear enough.

Definition 3.2.5. For any complete proper filter algebra $\mathcal{B} = \langle B, F\rangle$, a valuation θ is a function from Prop to $\mathcal{B}$. Any valuation θ on B extends to an $\mathcal{L}\Pi-$valuation $\tilde{\theta} : \mathcal{L}\Pi \to B$ given by:

- the same clauses for propositional variables $p \in \mathsf{Prop}$, connectives $\top, \neg, \wedge$, and $\forall p$ as in Definition 3.2.3, and
- $\tilde{\theta}(\mathsf{B}\varphi) = \top$ when $\tilde{\theta}(\varphi) \in F$ and otherwise $\tilde{\theta}(\mathsf{B}\varphi) = \bot$.

The concept of validity is defined as in Definition 3.2.3.

As we have discussed, a proper filter algebra can be seen as representing the propositions individuated by equivalence up to subjective certainty in a concrete doxastic scenario, with the proper filter representing the believed propositions in the scenario. That the believed propositions should form a proper filter comes from the assumption that the agent is logically competent and never believes in blatantly false propositions. That $\tilde{\theta}(\mathsf{B}\varphi)$ is always either $\top$ or $\bot$ depending on whether $\tilde{\theta}(\varphi)$ is in the filter of believed propositions or not comes from the assumption that the agent is sufficiently introspective. Proper filter algebras were first seen in [152] as models for beliefs.

The connection between proper filter algebras and KD45 algebras is this: proper filter algebras naturally correspond to those KD45 algebras whose $\Box$ operator's range is $\{\top, \bot\}$.

In [18], KD45 algebras are called pseudo-monadic algebras, and those with the said property are called well-connected ones, so here we call the above property well-connectedness too.

The correspondence can be easily specified. For any proper filter algebra $\langle B, F\rangle$, we can define a $\Box^F$ by $\Box^F a = \top$ if $a \in F$ and $\bot$ otherwise. Then $\langle B, \Box^F\rangle$ is the well-connected KD45 algebra corresponding to $\langle B, F\rangle$. Conversely, given a well-connected KD45 algebra $\langle B, \Box\rangle$, we can define $F^\Box = \{a \in B \mid \Box a = \top\}$. Then, $\langle B, F^\Box\rangle$ is the corresponding proper filter algebra. It is easy to verify that these two constructions are both bijections and are inverse of each other. Moreover, the semantic value of every formula is preserved for any valuation θ when we replace either a $\Box$ operator by the corresponding filter $F^\Box$ or vice versa. We can also show that the correspondence is a natural isomorphism between the category of proper filter algebras and the category of well-connected KD45 algebras. But for our purposes, this step is unnecessary.

With the semantics of interest defined, we now move on to define logics. The interpretation of $\forall p\varphi$ above in the algebraic semantics is informed by its intended reading: "for all proposition p, φ." Given this reading, even without formal semantics, the following axiom schemas and rules for propositional quantifiers, which we call the Π-principles, should be most certain:

- Dist : $\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$,
- Inst : $\forall p\varphi \to \varphi[\psi/p]$, where ψ is substitutable for p in φ and $\varphi[\psi/p]$ is the result of replacing all free occurrences of p in φ by ψ,
- Vacu : $\varphi \to \forall p\varphi$, if p is not free in φ,
- Univ : whenever φ is a theorem, $\forall p\varphi$ is also a theorem.

Just like a normal modal logic in full generality is defined as a set of formulas that contains all instances of propositional tautologies and the K axiom schema and is closed under the necessitation and modus ponens rules, we can similarly define normal Π-logics.

Definition 3.2.6. A *normal* Π-*logic* in a language $\mathcal{L} \supseteq \mathcal{L}\Pi$ is a set of formulas in $\mathcal{L}$ that contains all instances of propositional tautologies, K for B, and the Π-principles, and is closed under the necessitation rule Nec for B, the universalization rule Univ for $\forall p$ for all $p \in \mathsf{Prop}$, and modus ponens.

We will only consider normal Π-logics. When we put names of axiom schemas with K and Π together, we always mean the smallest normal Π-logic containing all instances of those axiom schemas. The ambient language should be clear from the context. For example, in this section we can write $\mathsf{K}\Pi$ for the smallest normal Π-logic and write $\mathsf{KD45}\Pi$ for the smallest normal Π-logic in $\mathcal{L}\Pi$ containing all instances of D, $\mathsf{4}$, and $\mathsf{5}$. In a later section where we prove the main completeness theorem, we will consider extended languages.

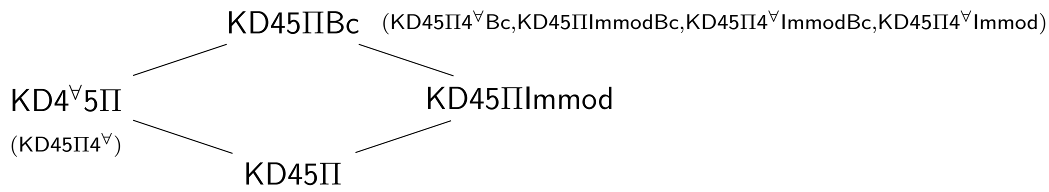


Figure 3.2: Normal Π-logics extending $\mathsf{KD45\Pi}$ generated by $\mathsf{4}^\forall$, Immod, and Bc.

Now that the syntax, semantics, and Π-logics are all formally defined, recall the three principles about belief we have seen in Section 3.1:

$$\mathsf{Immod} : \mathtt{B}\forall p(\mathtt{B}p \to p), \quad \mathsf{Bc} : \forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\varphi, \quad \mathsf{4}^\forall : \forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi.$$

Now we may have 8 normal Π-logics extending $\mathsf{KD45\Pi}$ by choosing which ones of the above three axiom schemas to add. But the following observation is immediate.

Proposition 3.2.7. $\mathsf{KD45\Pi4}^\forall = \mathsf{KD4}^\forall\mathsf{5\Pi}$ *and* $\mathsf{KD45\Pi Bc} = \mathsf{KD4}^\forall\mathsf{5\Pi Immod}$.

Proof. For the first equality, it is enough to show that we can prove all instances of $\mathsf{4}$ in $\mathsf{KD4}^\forall\mathsf{5\Pi}$. But for any φ, letting p be a propositional variable not free in φ, we have the following derivation:

1. $\mathtt{B}\varphi \to \forall p\mathtt{B}\varphi$ [Vacu]
2. $\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi$ [$\mathsf{4}^\forall$]
3. $\mathtt{B}\forall p\mathtt{B}\varphi \to \mathtt{BB}\varphi$ [Inst, K, *modus ponens*]
4. $\mathtt{B}\varphi \to \mathtt{BB}\varphi$ [*modus ponens*]

To show that $\mathsf{KD45\Pi Bc} = \mathsf{KD4}^\forall\mathsf{5\Pi Immod}$, it is enough to notice that $\mathsf{KD45\Pi}$ easily derives the following implications:

- $(\forall p\mathtt{BB}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) \to (\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi)$,
- $(\forall p\mathtt{B}(\mathtt{B}p \to p) \to \mathtt{B}\forall p(\mathtt{B}p \to p)) \to \mathtt{B}\forall p(\mathtt{B}p \to p)$,
- $((\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) \land \mathtt{B}\forall p(\mathtt{B}p \to p)) \to (\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\varphi)$. □

This proposition shows that there can be at most 4 different normal Π-logics extending $\mathsf{KD45\Pi}$: $\mathsf{KD45\Pi}$ itself, $\mathsf{KD4}^\forall\mathsf{5\Pi}$, $\mathsf{KD45\Pi Immod}$, and $\mathsf{KD45\Pi Bc}$. If they are all different, then we will have the simple 4-element Boolean algebra as shown in Figure 3.2. But are they all different?

With the algebraic semantics above, we can easily show that $\mathsf{KD4}^\forall\mathsf{5\Pi} \nvdash \mathsf{Immod}$, matching our intuition in the introduction that Immod is refutable even for ideally introspective agents.

Then we see that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ is strictly below $\mathsf{KD45}\Pi\mathsf{Bc}$, and consequently $\mathsf{KD45}\Pi$ must also be strictly below $\mathsf{KD45}\Pi\mathsf{Immod}$ by some simple Boolean reasoning. Thus with algebraic semantics, we can at least distinguish the lower left part from the upper right part. To show that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi \nvdash \mathsf{Immod}$, we only need a soundness theorem and countermodel. The soundness theorem is easy.

Theorem 3.2.8. *For any $\varphi \in \mathsf{KD4}^{\forall}\mathsf{5}\Pi$, φ is valid on all complete proper filter algebras.*

Proof. The only interesting axiom here is $\mathsf{4}^{\forall}$. Pick an arbitrary complete proper filter algebra $\mathcal{B}$ and a valuation θ on it. Now for any $a \in \mathcal{B}$, $\widetilde{\theta[a/p]}(\mathtt{B}\varphi)$ is either $\top$ or $\bot$. If there is an $a \in \mathcal{B}$ such that $\widetilde{\theta[a/p]}(\mathtt{B}\varphi) = \bot$, then $\tilde{\theta}(\forall p\mathtt{B}\varphi) = \bot$. Then trivially $\tilde{\theta}(\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) = \top$. On the other hand, if no such a exists, then $\tilde{\theta}(\forall p\mathtt{B}\varphi) = \top$, and hence $\tilde{\theta}(\mathtt{B}\forall p\mathtt{B}\varphi) = \top$. Then trivially $\tilde{\theta}(\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) = \top$. So $\tilde{\theta}(\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) = \top$ is valid either way. $\square$

To refute Immod in a countermodel, we first make the following useful and intuitive observation. It is intuitive because $\forall p(\mathtt{B}p \to p)$ says that "everything the agent believes is true," and the filter $F_{\mathcal{B}}$ of a proper filter algebra represents the set of propositions the agent believes.

Proposition 3.2.9. *For any complete proper filter algebra $\mathcal{B}$ and any valuation θ on it, $\tilde{\theta}(\forall p(\mathtt{B}p \to p)) = \bigwedge F_{\mathcal{B}}$.*

Proof. If $\tilde{\theta}(p) \in F_B$, then $\tilde{\theta}(\mathtt{B}p) = \top$, and hence $\tilde{\theta}(\mathtt{B}p \to p) = \tilde{\theta}(p) = \theta(p)$. If $\tilde{\theta}(p) \notin F_B$, then $\tilde{\theta}(\mathtt{B}p) = \bot$, and hence $\tilde{\theta}(\mathtt{B}p \to p) = \top$. Thus $\{\widetilde{\theta[a/p]}(\mathtt{B}p \to p) \mid a \in \mathcal{B}\}$ is precisely $F_{\mathcal{B}}$ (note that $\top$ must be in $F_{\mathcal{B}}$). Then $\tilde{\theta}(\forall p(\mathtt{B}p \to p))$ is the meet of this set, i.e., $\bigwedge F_{\mathcal{B}}$. $\square$

Given this observation, to refute $\mathsf{Immod} = \mathtt{B}\forall p(\mathtt{B}p \to p)$, we only need to find a complete proper filter algebra $\mathcal{B}$ such that the meet of $F_{\mathcal{B}}$ is not in $F_{\mathcal{B}}$: it is a non-principal filter.

Proposition 3.2.10. Immod *is not valid on all complete proper filter algebras.*

Proof. Let $\mathcal{B}$ be a complete proper filter algebra where its Boolean algebra is $\wp(\mathbb{N})$, and its filter $F_{\mathcal{B}}$ is the set of all cofinite sets. Fix an arbitrary valuation θ. Clearly $\bigwedge F_{\mathcal{B}} = \emptyset$, the bottom element. Using the previous proposition, $\tilde{\theta}(\forall p(\mathtt{B}p \to p)) = \bot$, but $\bot \notin F_{\mathcal{B}}$. Thus $\tilde{\theta}(\mathsf{Immod}) = \bot$. $\square$

In fact, in this algebra, $\tilde{\theta}(\exists p(\mathtt{B}p \wedge \neg p)) = \top \in F_B$, so $\tilde{\theta}(\mathtt{B}\exists p(\mathtt{B}p \wedge \neg p)) = \top$. In other words, this agent believes that there is a proposition she falsely believes.

Now we show the difficulty of invalidating Immod using possible-world-based semantics. The following semantics allow the full generality of propositional contingency in the sense that each possible-world has its own domain of propositions.

Definition 3.2.11. A *propositionally-contingent frame* (PC frame in short) is a tuple $F = \langle W, R, P\rangle$ where W is a non-empty set, R is a binary relation, and P is a function from W to $\wp(\wp(W))$. We may abuse notation and write "$w \in F$" for "$w \in W$".

A *PC model* based on F is a tuple $M = \langle F, w, V \rangle$ where $w \in W$ and V is a *valuation* function from Prop to $\wp(W)$. It is a *proper PC model* if V only takes value in $P(w)$.

For any formula $\varphi \in \mathcal{L}\Pi$ and any PC model $\langle F, w, V \rangle$, the truth of φ at $\langle F, w, V \rangle$ is defined inductively as follows:

$$
\begin{aligned}
\langle F, w, V \rangle \vDash p &\iff w \in V(p); \\
\langle F, w, V \rangle \vDash \neg\varphi &\iff \langle F, w, V \rangle \nvDash \varphi; \\
\langle F, w, V \rangle \vDash \varphi \wedge \psi &\iff \langle F, w, V \rangle \vDash \varphi \text{ and } \langle F, w, V \rangle \vDash \psi; \\
\langle F, w, V \rangle \vDash \mathsf{B}\varphi &\iff \text{for all } w' \in R(w), \langle F, w', V \rangle \vDash \varphi; \\
\langle F, w, V \rangle \vDash \forall p\varphi &\iff \text{for all } X \in P(w), \langle F, w, V[X/p] \rangle \vDash \varphi.
\end{aligned}
$$

Here $R(w) = \{w' \in W \mid wRw'\}$.

Finally, φ is *valid* on a PC frame F if φ is true at all *proper* PC models based on F: $\langle F, w, V \rangle \vDash \varphi$ for all $w \in F$ and $V : \mathsf{Prop} \to P(w)$.

Note that, to be fully general, we disregarded any notion of "coherence" one might want to impose on the P part of F (see [63] for some natural restrictions for F); we did not even require that $P(w)$ is a field-of-sets. Also note that validity is defined by proper PC models. Without restricting to proper PC models, the validity of even the most mundane instantiation axiom $\forall p\varphi \to \varphi$ when p is free in φ will be subject to great difficulty, and the validity of a formula with free variables may be different than the validity of its universal closure. However, our discussion below about Immod does not turn on the choice of proper PC models over all PC models since Immod has no free variables.

A lot of standard questions can be asked about this semantics. But for now, observe that for any PC frame $F = \langle W, R, P \rangle$, if R is shift-reflexive, meaning that every world in $R(w)$ is reflexive for all $w \in W$, then F validates Immod. To see this, first note that for every w in W such that wRw and any valuation V, $\langle F, w, V \rangle \vDash \mathsf{B}p \to p$ simply by the truth clause of B. Hence $\forall p(\mathsf{B}p \to p)$ is also always true on reflexive points by the truth clause of $\forall p$. Then, it is clear again from the truth clause of B that for any $w \in W$ such that every $w' \in R(w)$ is reflexive, $\langle F, w, V \rangle \vDash \mathsf{B}\forall p(\mathsf{B}p \to p)$. Thus, if R is shift-reflexive, Immod is validated. And in showing this, P is totally unused.

Does the above argument show that possible-world semantics is totally unusable if we want to model scenarios where Immod is false? If one is looking for intuitive and "clean" models, then the argument does suggest that possible-world semantics is not useful. The success of possible-world semantics is partly due to the easy first-order conditions corresponding to natural axioms. For the doxastic logic case, $\mathsf{D}, \mathsf{4}$, and $\mathsf{5}$ correspond to seriality, transitivity, and Euclidicity, respectively. And from Euclidicity alone, shift-reflexivity follows. The above argument shows that if we want to model failure of Immod while validating $\mathsf{KD45}$, we need to give up the appealing correspondence theory in the standard possible-world semantics. Nevertheless, this semantics can be of great use. First, we conjecture that with carefully chosen P and R, we can refute Immod while validating $\mathsf{KD45}\Pi$, though, it is likely that the R relation in that PC frame cannot be interpreted in a meaningful way. Second, we show here that this semantics can be used to invalidate $\mathsf{4}^\forall$ while validating $\mathsf{KD45\Pi}$ and Immod. Hence, $\mathsf{4}^\forall$ is not derivable in $\mathsf{KD45\Pi}$ (indeed, $\mathsf{KD45\Pi Immod}$), and Figure 3.2 is a complete description of the normal Π-logics extending $\mathsf{KD45\Pi}$ generated by $\mathsf{4}^\forall$, Immod, and Bc.

We will use the PC frame $F_{\neg\mathsf{4}^\forall} = \langle W, R, P\rangle$ where

$$\begin{aligned} W &= \{0,1,2\} \\ R &= \{\langle 0,1\rangle, \langle 0,2\rangle, \langle 1,1\rangle, \langle 1,2\rangle, \langle 2,1\rangle, \langle 2,2\rangle\} \\ P(0) &= \{\varnothing, \{0\}, \{1,2\}, \{0,1,2\}\} \\ P(1) &= P(2) = \wp(W). \end{aligned}$$

Here, world 0 sees the equivalence class formed by 1 and 2, and the R relation is serial, transitive, and Euclidean. This means that $\mathsf{KD45}$ and Immod as we commented above are automatically valid. To see that $\mathsf{4}^\forall$ is invalid, consider the formula $\varphi = \forall p\mathsf{B}(\mathsf{B}p \vee \mathsf{B}\neg p)$. Since $P(0)$ contains only the sets that cover either all or none of $\{1,2\} = R(0)$, it is easy to see that $\langle F_{\neg\mathsf{4}^\forall}, 0, V\rangle \vDash \varphi$ for any valuation V. However, $\langle F_{\neg\mathsf{4}^\forall}, 0, V\rangle \nvDash \mathsf{B}\varphi$, since $P(1)$ contains the set $\{1\}$ and as a result $\langle F_{\neg\mathsf{4}^\forall}, 1, V\rangle \nvDash \forall p\mathsf{B}(\mathsf{B}p \vee \mathsf{B}\neg p)$.

Now we are left with showing that $F_{\neg\mathsf{4}^\forall}$ validates the Π-principles, and obviously the only difficulty here is Inst. The validity of this axiom requires that there are enough propositions: essentially, it says that at any world, all propositions expressible by a formula exist, since its dual form is $\varphi[\psi/p] \to \exists p\varphi$. Here we give a sufficient condition for the validity of Inst. The condition is related to the coherence conditions in [63]. To formulate the condition, we first define automorphisms on PC frames.

Definition 3.2.12. Let $F = \langle W, R, P\rangle$ be a PC frame. Then an *automorphism* π of F is a function $\pi : W \to W$ such that:

- π is a permutation of W (injective and surjective);
- where $\pi[X] = \{\pi(x) \mid x \in X\}$ for any $X \subseteq W$, for any $w \in W$, $R(\pi(w)) = \pi[R(w)]$;
- where $\pi[A] = \{\pi[X] \mid X \in A\}$ for any $A \subseteq \wp(W)$, for any $w \in W$, $P(\pi(w)) = \pi[P(w)]$.

Let $Aut(F)$ be the group of all automorphisms of F.

For the last bullet point of the above definition, note that for any field-of-sets A on W and any permutation π on W, $\pi[A]$ is also a field-of-sets, though in full generality, $P(w)$ may not even be a field-of-sets. Now we show that automorphisms preserve the truth value of formulas.

Proposition 3.2.13. *Let $F = \langle W, R, P\rangle$ be a PC frame and $\pi \in Aut(F)$. Then, for any formula $\varphi \in \mathcal{L}\Pi$, any $w \in W$, and any valuation $V : \mathsf{Prop} \to \wp(W)$,*

$$\langle F, w, V\rangle \vDash \varphi \iff \langle F, \pi(w), \pi \circ V\rangle \vDash \varphi.$$

Here $\pi \circ V$ is defined by $(\pi \circ V)(p) = \pi[V(p)]$ for any $p \in \mathsf{Prop}$.

Proof. By induction on φ. The atomic case where φ is a propositional variable or constant is easy: $\langle F, w, V\rangle \vDash p$ iff $w \in V(p)$ iff $\pi(w) \in \pi[V(p)]$ iff $\pi(w) \in (\pi \circ V)(p)$.

For the modal case, the following chain of equivalence is clear:

$$\begin{aligned}
\langle F, w, V\rangle \vDash \mathtt{B}\varphi &\iff \text{for all } w' \in R(w), \langle F, w', V\rangle \vDash \varphi \\
&\iff \text{for all } w' \in R(w), \langle F, \pi(w'), \pi \circ V\rangle \vDash \varphi \\
&\iff \text{for all } w' \in \pi[R(w)], \langle F, w', \pi \circ V\rangle \vDash \varphi \\
&\iff \text{for all } w' \in R(\pi(w)], \langle F, w', \pi \circ V\rangle \vDash \varphi \\
&\iff \langle F, w', \pi \circ V\rangle \vDash \mathtt{B}\varphi.
\end{aligned}$$

For the propositional quantification case, the strategy is the same.

$$\begin{aligned}
\langle F, w, V\rangle \vDash \forall p\varphi &\iff \text{for all } X \in P(w), \langle F, w, V[p \mapsto X]\rangle \vDash \varphi \\
&\iff \text{for all } X \in P(w), \langle F, \pi(w), \pi \circ (V[p \mapsto X])\rangle \vDash \varphi \\
&\iff \text{for all } X \in P(w), \langle F, \pi(w), (\pi \circ V)[p \mapsto \pi[X]]\rangle \vDash \varphi \\
&\iff \text{for all } X \in \pi[P(w)], \langle F, \pi(w), (\pi \circ V)[p \mapsto X]\rangle \vDash \varphi \\
&\iff \text{for all } X \in P(\pi(w)), \langle F, \pi(w), (\pi \circ V)[p \mapsto X]\rangle \vDash \varphi \\
&\iff \langle F, \pi(w), (\pi \circ V)\rangle \vDash \forall p\varphi.
\end{aligned}$$

The Boolean cases are trivial. □

Now we formulate the sufficient condition with some helpful definitions.

Definition 3.2.14. Let $F = \langle W, F, P\rangle$ be a PC frame. For any subgroup G of $Aut(F)$, we define its *orbit-closed field-of-sets* $OF(G)$ by

$$OF(G) = \{X \subseteq W \mid \text{for any } \pi \in G, \pi[X] = X\}.$$

Using general group theory, it can be shown that $OF(G)$ is precisely the field-of-sets generated by arbitrary unions from the orbits of G, and hence $OF(G)$ is a complete and atomic Boolean algebra.

Also, for any w, we define $Aut(F)_w$ as the subgroup of all automorphisms in $Aut(F)$ that fix every set in $P(w)$:

$$Aut(F)_w = \{\pi \in Aut(F) \mid \pi[X] = X \text{ for all } X \in P(w)\}.$$

Finally, F is *propositionally rich* if for any w, $P(w) \supseteq OF(Aut(F)_w)$.

Proposition 3.2.15. *For any PC frame $F = \langle W, R, P\rangle$, if F is propositionally rich, then F validates* Inst. *Hence, propositionally rich PC frames validate the minimal normal* Π*-logic.*

Proof. Let φ and ψ be formulas such that ψ is substitutable for p in φ. Also, assume that $\langle F, w, V\rangle \vDash \forall p\varphi$ where $\langle F, w, V\rangle$ is a proper PC model, that is $V : \mathsf{Prop} \to P(w)$. Then we only need to show that $\langle F, w, V\rangle \vDash \varphi[\psi/p]$.

First, it is a standard exercise, similar to its counterpart in first-order logic, to show that $\langle F, w, V\rangle \vDash \varphi[\psi/p]$ iff $\langle F, w, V[p \mapsto [\![\psi]\!]^{F,V}]\rangle \vDash \varphi$ where $[\![\psi]\!]^{F,V} = \{w \in W \mid \langle F, w, V\rangle \vDash \psi\}$. But since $\langle F, w, V\rangle \vDash \forall p\varphi$, we have that for any $X \in P(w)$, $\langle F, w, V[p \mapsto X]\rangle \vDash \varphi$. Thus it is enough to show that $[\![\psi]\!]^{F,V} \in P(w)$. Since $P(w) \supseteq OF(Aut(F)_w)$ by propositional richness, it is enough to show that $[\![\psi]\!]^{F,V} \in OF(Aut(F)_w)$. But this is true given the invariance of the truth of ψ under automorphisms of F. For any $\pi \in Aut(F)_w$ and any $w \in W$, $w \in [\![\psi]\!]^{F,V}$ iff $\pi(w) \in [\![\psi]\!]^{F,\pi\circ V}$ by Proposition 3.2.13. Since V's range is in $P(w)$ and $\pi \in Aut(F)_w$, $\pi \circ V = V$. Thus, $w \in [\![\psi]\!]^{F,V}$ iff $\pi(w) \in [\![\psi]\!]^{F,V}$ for any $w \in W$. This means that $[\![\psi]\!]^{F,V} = \pi[[\![\psi]\!]^{F,V}]$. So $[\![\psi]\!]^{F,V} \in OF(Aut(F)_w)$ and is in $P(w)$. $\square$

Now we verify that $F_{\neg 4^\forall}$ is propositionally rich. First, $Aut(F_{\neg 4^\forall})$ has only two elements: the identity function id, and the permutation f that exchanges 1 and 2. Then note that $Aut(F_{\neg 4^\forall})_0 = Aut(F_{\neg 4^\forall})$, since $P(0) = \{\varnothing, \{0\}, \{1,2\}, \{0,1,2\}\}$, all of which are fixed by exchanging 1 and 2. Indeed, $P(0)$ contains precisely those sets that are fixed by exchanging 1 and 2. This means that $P(0) \supseteq OF(Aut(F_{\neg 4^\forall})_0)$. Also, $Aut(F_{\neg 4^\forall})_1 = Aut(F_{\neg 4^\forall})_2 = \{id\}$, but this is not important since $P(1)$ and $P(2)$ are as large as possible already, being both $\wp(\{0,1,2\})$. Trivially, $P(1) \supseteq OF(Aut(F_{\neg 4^\forall})_1)$ and $P(2) \supseteq OF(Aut(F_{\neg 4^\forall})_2)$. Thus, we verified that $F_{\neg 4^\forall}$ is propositionally rich. Hence, letting Γ be the set of formulas in $\mathcal{L}\Pi$ valid in $F_{\neg 4^\forall}$, $\Gamma \supseteq \mathsf{KD45\Pi Immod}$, and $4^\forall \notin \Gamma$. Thus:

Theorem 3.2.16. $4^\forall \notin \mathsf{KD45\Pi Immod}$*, and there are exactly 4 logics generated by adding some or none of* $4^\forall$, Immod*, and* Bc *to* $\mathsf{KD45\Pi}$*:* $\mathsf{KD45\Pi}$, $\mathsf{KD4^\forall 5\Pi}$, $\mathsf{KD45\Pi Immod}$, $\mathsf{KD45\Pi Bc}$.

3.3 Soundness of $4^\forall$ on complete KD45 algebras

In the last section, we have seen how complete proper filter algebras can be used to separate Immod from $\mathsf{KD4^\forall 5\Pi}$ and hence separate $\mathsf{KD45\Pi Immod}$ and $\mathsf{KD45\Pi Bc}$ from $\mathsf{KD45\Pi}$ and $\mathsf{KD4^\forall 5\Pi}$. We have also seen that complete proper filter algebras validate $4^\forall$ using the special property that the $\mathtt{B}$ operator always brings semantic values to either the top element or the bottom element.

While we separated $4^\forall$ from $\mathsf{KD45\Pi}$ using possible-world semantics with propositional contingency, it is also natural to ask whether we can do the same using algebraic semantics based on complete Boolean algebras at all. That is, whether we can refute $4^\forall$ if we do not assume the above special property about $\mathtt{B}$. In this section, we show that we cannot. In fact, $4^\forall$, and hence $\mathsf{KD4^\forall 5\Pi}$, are valid on all complete KD45 algebras. For this, we need to extract more structure from complete KD45 algebras and view the algebraic semantics from a different perspective.

Definition 3.3.1. For any KD45 algebra $\mathcal{B}$, let $\mathsf{fp}(\mathcal{B})$ be the set $\{a \in \mathcal{B} \mid a = \Box a\}$.

Lemma 3.3.2. *For any KD45 algebra $\mathcal{B}$, $\mathsf{fp}(\mathcal{B})$ has the following properties.*

- *First,* $\mathsf{fp}(\mathcal{B}) = \{\Box a \mid a \in \mathcal{B}\} = \{\Diamond a \mid a \in \mathcal{B}\}$.
- *Second,* $\mathsf{fp}(\mathcal{B})$ *is a subalgebra of* $\mathcal{B}$*. That is,* $\mathsf{fp}(\mathcal{B})$ *is closed under the complementation, the meet operation, and trivially the* $\Box$ *operator of* $\mathcal{B}$*.*
- *Third, while* $\langle \mathsf{fp}(\mathcal{B}), \top_{\mathcal{B}}, \neg_{\mathcal{B}}, \wedge_{\mathcal{B}} \rangle$ *is a complete Boolean algebra, it is not always a regular Boolean subalgebra of* $\mathcal{B}$*. In other words, when ordered as in* $\mathcal{B}$ *by* $\leq_{\mathcal{B}}$*,* $\mathsf{fp}(\mathcal{B})$ *form a complete Boolean lattice whose complementation operation and finite meet operation are the same as in* $\mathcal{B}$*. However, it is not always the case that* $\mathsf{fp}(\mathcal{B})$ *is closed under arbitrary meets in* $\mathcal{B}$*.*

Proof. The first two points follow directly from the definition of KD45 algebras. For the third point, one can easily verify that the join of $X \subseteq \mathsf{fp}(\mathcal{B})$ within $\mathsf{fp}(\mathcal{B})$ is $\Box \bigvee_{\mathcal{B}} X$. To see that for every $x \in X$, $x \leq \Box \bigvee_{\mathcal{B}} X$, note that since $X \subseteq \mathsf{fp}(\mathcal{B})$, $x = \Box x$. Note also that $\Box$ is monotone. Hence, given that $x \leq \bigvee_{\mathcal{B}} X$, $\Box x \leq \Box \bigvee_{\mathcal{B}} X$. Thus, $x \leq \Box \bigvee_{\mathcal{B}} X$. Now suppose $y \in \mathsf{fp}(\mathcal{B})$ and for all $x \in X$, $y \geq x$. Then $y \geq \bigvee_{\mathcal{B}} X$. And then $y = \Box y \geq \Box \bigvee_{\mathcal{B}} X$. Hence $\Box \bigvee_{\mathcal{B}} X$ is the least upper bound of X in $\mathsf{fp}(\mathcal{B})$.

An example where the join of a subset of $\mathsf{fp}(\mathcal{B})$ in $\mathcal{B}$ is not in $\mathsf{fp}(\mathcal{B})$ is given below. $\Box$

Definition 3.3.3. Let $\mathcal{N}$ be the set of non-principal ultrafilters in $\wp(\mathbb{N})$ and $W = \mathbb{N} \sqcup \mathcal{N}$. For any $A \subseteq \mathbb{N}$, define $\mathcal{N}(A) = \{f \in \mathcal{N} \mid A \in f\}$. Then for any subset $X \subseteq W$, define $\Box X$ by: $\Box X = (X \cap \mathbb{N}) \cup \mathcal{N}(X \cap \mathbb{N})$. Finally, let $\mathcal{B}_{\mathbb{N}} = \langle \langle \wp(W), W, W \setminus \cdot, \cap \rangle, \Box \rangle$.

Proposition 3.3.4. *$\mathcal{B}_{\mathbb{N}}$ is a complete KD45 algebra. Moreover, $\mathsf{fp}(\mathcal{B}_{\mathbb{N}})$ is not closed under arbitrary join in $\mathcal{B}_{\mathbb{N}}$.*

Proof. Clearly $\mathcal{B}_{\mathbb{N}}$ is complete since the Boolean algebra base is a powerset algebra. Now we show that $\Box$ satisfies all the relevant properties. Pick arbitrary $X, Y \in \mathcal{B}_{\mathbb{N}}$, and let $X_0 = X \cap \mathbb{N}, X_1 = X \cap \mathcal{N}, Y_0 = Y \cap \mathbb{N}, Y_1 = Y \cap \mathcal{N}$.

- $\Box X \cap \Box Y = (X_0 \cup \mathcal{N}(X_0)) \cap (Y_0 \cup \mathcal{N}(Y_0)) = (X_0 \cap Y_0) \cup (X_0 \cap \mathcal{N}(Y_0)) \cup (Y_0 \cap \mathcal{N}(X_0)) \cup (\mathcal{N}(X_0) \cap \mathcal{N}(Y_0)) = (X_0 \cap Y_0) \cup \mathcal{N}(X_0 \cap Y_0) = \Box(X_0 \cap Y_0)$. Here we used the fact that an ultrafilter contains X_0 and Y_0 iff it contains $X_0 \cap Y_0$.
- $\Box \varnothing = \varnothing \cup \mathcal{N}(\varnothing) = \varnothing$.
- $\Box\Box X = \Box(X_0 \cup \mathcal{N}(X_0)) = ((X_0 \cup \mathcal{N}(X_0)) \cap \mathbb{N}) \cup \mathcal{N}((X_0 \cup \mathcal{N}(X_0)) \cap \mathbb{N}) = X_0 \cup \mathcal{N}(X_0) = \Box X$.
- $\neg \Box X = \neg(X_0 \cup \mathcal{N}(X_0)) = (\mathbb{N} \setminus X_0) \cup (\mathcal{N} \setminus \mathcal{N}(X_0)) = (\mathbb{N} \setminus X_0) \cup \mathcal{N}(\mathbb{N} \setminus X_0) = (\neg X \cap \mathbb{N}) \cup \mathcal{N}(\neg X \cap \mathbb{N}) = \Box \neg X$. Hence by the previous part $\Box \neg \Box X = \Box\Box\neg X = \Box \neg X = \neg \Box X$.

Thus, $\mathcal{B}_{\mathbb{N}}$ is a complete KD45 algebra. Note also that for every $n \in \mathbb{N}$, $\{n\} \in \mathsf{fp}(\mathcal{B}_{\mathbb{N}})$. However, $\mathbb{N} = \bigcup_{n \in \mathbb{N}} \{n\}$ is not in $\mathsf{fp}(\mathcal{B}_{\mathbb{N}})$. $\Box$

Semantically, every formula obtained by combining propositional variables with propositional operators (Boolean or modal) and propositional quantifiers defines a function from valuations in $\mathcal{B}^{\mathsf{Prop}}$ to $\mathcal{B}$. Now we study these functions.

For simplicity and clarity, we fix an arbitrary complete KD45 algebra $\mathcal{B}$ and define $V = \mathcal{B}^{\mathsf{Prop}}$. Greek letters "$\theta$" and "$\gamma$" are used to denote valuations in V and "f" and "g" are used to denote functions from valuations in V to $\mathcal{B}$. We also use the notations from lambda calculus to define functions.

The following definition then defines the meaning of the operators in terms of how they generate functions from $\mathcal{B}^{\mathsf{Prop}}$ to $\mathcal{B}$.

Definition 3.3.5. First, for every $p \in \mathsf{Prop}$, define $[p] = \lambda\theta \in V, \theta(p)$. Then, for any $f, g : V \to \mathcal{B}$ and $p \in \mathsf{Prop}$, define:

$$\begin{aligned} f \wedge g &= \lambda\theta \in V, f(\theta) \wedge g(\theta); \\ \neg f &= \lambda\theta \in V, \neg f(\theta); \\ \Box f &= \lambda\theta \in V, \Box f(\theta); \\ \forall p f &= \lambda\theta \in V, \bigwedge\{f(\theta[a/p]) \mid a \in \mathcal{B}\}. \end{aligned}$$

Then $f \vee g$, $f \to g$, $f \leftrightarrow g$, $f \oplus g$, $\Diamond f$, and $\exists p f$ are defined in the obvious way. We call the set of functions generated by $\neg$, $\wedge$, $\Box$, and $\forall p$ for all $p \in \mathsf{Prop}$ from $\{[p] \mid p \in \mathsf{Prop}\}$ the set of *definable functions* on $\mathcal{B}$.

The above definition also gives us an alternative way to define the semantics for $\mathcal{L}\Pi$.

Definition 3.3.6. Recall that we have defined $[p] = \lambda\theta \in V, \theta(p)$. We can then extend this notation to all formulas in $\mathcal{L}\Pi$ inductively in the obvious way:

$$\begin{aligned} [\neg\varphi] &= \neg[\varphi]; \\ [\varphi \wedge \psi] &= [\varphi] \wedge [\psi]; \\ [\mathsf{B}\varphi] &= \Box[\varphi]; \\ [\forall p\varphi] &= \forall p[\varphi]. \end{aligned}$$

Proposition 3.3.7. *For any $\varphi \in \mathcal{L}\Pi$, $\tilde{\theta}(\varphi) = [\varphi](\theta)$ for all $\theta \in V$.*

Now we identify two properties of these functions from V to $\mathcal{B}$ that are important to us.

Definition 3.3.8. For any $f : V \to \mathcal{B}$, we say that f is *fixed* if its range is in $\mathsf{fp}(\mathcal{B})$ (that is, $\Box f = f$); and we say that f is *local* if for any $p \in \mathsf{Prop}$ and $\theta \in V$,

$$\text{if } b \le a_1 \leftrightarrow a_2, \text{ then } b \le f(\theta[a_1/p]) \leftrightarrow f(\theta[a_2/p]),$$

for all $b \in \mathsf{fp}(\mathcal{B})$ and $a_1, a_2 \in \mathcal{B}$.

The intuition behind locality is that for f to be local, what f is below a fixpoint $b \in \mathsf{fp}(\mathcal{B})$ (namely what $f \wedge b$ is) should only depend on what the arguments are below b (namely what $\theta(p) \wedge b$ is for all $p \in \mathsf{Prop}$). The above definition formalizes this intuition because in Boolean algebras, $x \leq y \leftrightarrow z$ iff $x \wedge y = x \wedge z$.

But why are these two properties important to us? Recall that what we are trying to show here is that $\forall p \mathtt{B}\varphi \to \mathtt{B}\forall p \mathtt{B}\varphi$ is valid on $\mathcal{B}$. If we can show that $[\forall p \mathtt{B}\varphi]$ is fixed, which means that $[\forall p \mathtt{B}\varphi] = \Box[\forall p \mathtt{B}\varphi]$, then we are done. This is because, with $[\forall p \mathtt{B}\varphi]$ being fixed, for any $\theta \in V$,

$$\tilde{\theta}(\forall p \mathtt{B}\varphi) = [\forall p \mathtt{B}\varphi](\theta) = (\Box[\forall p \mathtt{B}\varphi])(\theta) = [\mathtt{B}\forall p \mathtt{B}\varphi](\theta) = \tilde{\theta}(\mathtt{B}\forall p \mathtt{B}\varphi).$$

Obviously, then, $\tilde{\theta}(\forall p \mathtt{B}\varphi \to \mathtt{B}\forall p \mathtt{B}\varphi) = \top$. So our goal is now reduced to showing that $[\forall p \mathtt{B}\varphi]$ is fixed. Note that $[\forall p \mathtt{B}\varphi] = \forall p \Box[\varphi]$. It is trivial to see that $\Box[\varphi]$ is fixed. So one might hope that we can show that whenever f is fixed, $\forall p f$ is also fixed, and then claim victory. However, this is in general false, given the example we produced in Definition 3.3.3 above showing that the set of fixpoints $\mathsf{fp}(\mathcal{B})$ is in general not closed under arbitrary meets. One can construct an f whose range (when we vary the p coordinate of the input valuation) is precisely a set of fixpoints in $\mathsf{fp}(\mathcal{B})$ whose meet is not in $\mathsf{fp}(\mathcal{B})$. Then $\forall p f$ is not fixed. What is missing in this strategy of showing that $\forall p f$ is fixed whenever f is fixed is precisely locality. We will show first that $[\mathtt{B}\varphi]$ must be local in addition to being fixed. We will then show that if f is fixed and local, then $\forall p f$ is fixed and local.

To show that $[\mathtt{B}\varphi]$ is local, the following lemma, showing in fact that all definable functions are local, suffices.

Lemma 3.3.9. *We have the following closure properties for local functions.*

- *Projection functions of the form $[p]$ for some $p \in \mathsf{Prop}$ are local.*
- *Local functions are closed under Boolean combinations: if $f, g : V \to \mathcal{B}$ are local, then $f \wedge g$ and $\neg f$ are both local.*
- *Local functions are closed under $\Box$.*
- *Local functions are closed under $\forall p$ for all $p \in \mathsf{Prop}$.*

Hence, all definable functions are local.

Proof. The first two points are easy. For the third, consider $\Box f$ where f is local. Then for a $b \in \mathsf{fp}(\mathcal{B})$, we need to show that

$$\text{if } b \leq a_1 \leftrightarrow a_2, \text{ then } b \leq \Box f(\theta[a_1/p]) \leftrightarrow \Box f(\theta[a_2/p]).$$

Given that f is local, when $b \leq a_1 \leftrightarrow a_2$, $b \leq f(\theta[a_1/p]) \leftrightarrow f(\theta[a_2/p])$. Box both sides (using that $\Box$ is monotone), and we see that

$$b = \Box b \leq \Box(f(\theta[a_1/p]) \leftrightarrow f(\theta[a_2/p])) \leq \Box f(\theta[a_1/p]) \leftrightarrow \Box f(\theta[a_2/p]).$$

The first equality is due to that $b \in \mathsf{fp}(\mathcal{B})$, and the last inequality is by the normality of $\Box$: it commutes with finite meets, and $\Box(x \to y) \leq \Box x \to \Box y$.

Now, for the fourth point, consider $\forall p f$ where f is local. In this case, we need to work with an arbitrary $q \in \mathsf{Prop}$, an arbitrary $b \in \mathsf{fp}(\mathcal{B})$, and arbitrary $a_1, a_2 \in \mathcal{B}$ such that $b \leq a_1 \leftrightarrow a_2$, and show that

$$b \leq \bigwedge\{f(\theta[a_1/q][c/p]) \mid c \in \mathcal{B}\} \leftrightarrow \bigwedge\{f(\theta[a_2/q][c/p]) \mid c \in \mathcal{B}\}.$$

If $q = p$, then this is trivially true (the right-hand-side of the inequality is $\top$). So we now consider the case when $q \neq p$, in which case $\theta[a_1/q][c/p] = \theta[c/p][a_1/q]$ and $\theta[a_2/q][c/p] = \theta[c/p][a_2/q]$. For simplicity, let $\theta_c = \theta[c/p]$. Since we assumed that f is local, by definition, we have the following for all $c \in \mathcal{B}$:

$$b \leq f(\theta_c[a_1/q]) \leftrightarrow f(\theta_c[a_2/q]).$$

Thus, $b \leq \bigwedge\{f(\theta_c[a_1/q]) \leftrightarrow f(\theta_c[a_2/q]) \mid c \in \mathcal{B}\}$. Hence, all we need now is the following simple principle on complete Boolean algebras:

$$\bigwedge\{f(\theta_c[a_1/q]) \leftrightarrow f(\theta_c[a_2/q]) \mid c \in \mathcal{B}\} \leq \\ \bigwedge\{f(\theta_c[a_1/q]) \mid c \in \mathcal{B}\} \leftrightarrow \bigwedge\{f(\theta_c[a_2/q]) \mid c \in \mathcal{B}\}.$$

To see that in general, $\bigwedge_{i\in I}(x_i \leftrightarrow y_i) \leq \bigwedge_{i\in I} x_i \leftrightarrow \bigwedge_{i\in I} y_i$, note that it is enough to show that $\bigwedge_{i\in I}(x_i \leftrightarrow y_i) \wedge \bigwedge_{i\in I} x_i \leq \bigwedge_{i\in I} y_i$ and symmetrically that $\bigwedge_{i\in I} x_i \leftrightarrow y_i \wedge \bigwedge_{i\in I} y_i \leq \bigwedge_{i\in I} x_i$. Both of them are easy. $\Box$

That $[\mathsf{B}\varphi]$ is fixed is immediate from the following lemma.

Lemma 3.3.10. *We have the following closure properties for fixed functions.*

- *For any $f : V \to \mathcal{B}$, $\Box f$ is fixed.*
- *Fixed functions are closed under Boolean combinations.*
- *Not all fixed functions are closed under $\forall p$.*

Proof. Immediate from Lemma 3.3.2. $\Box$

The only missing piece then is the following lemma.

Lemma 3.3.11. *Fixed local functions are closed under $\forall p$. That is, if f is fixed and local, then $\forall p f$ is also fixed and local, for any $p \in \mathsf{Prop}$.*

Proof. Pick an arbitrary fixed and local $f : V \to \mathcal{B}$. Since fixed local functions are closed under Boolean combinations and local functions are closed under $\forall p$, without loss of generality, we only need to show that $\exists pf$ is also fixed.

The idea is the following. Pick an arbitrary $\theta \in V$. Then we show that $(\exists pf)(\theta)$ in fact has a witness: there exists $c \in \mathcal{B}$ such that $(\exists pf)(\theta) = f(\theta[c/p])$. Since f is fixed, $f(\theta[c/p]) \in \mathsf{fp}(\mathcal{B})$. So, $(\exists pf)(\theta)$, being just $f(\theta[c/p])$, is in $\mathsf{fp}(\mathcal{B})$. Since θ is arbitrarily chosen, this shows that $\exists pf$ is fixed. As a consequence, $\Box\exists pf = \exists pf$.

Hence, let us fix an arbitrary $\theta \in V$ and from now on write $f(a)$ for $f(\theta[a/p])$. Since f is fixed and local, we know that:

- For any $a \in \mathcal{B}$, $f(a) \in \mathsf{fp}(\mathcal{B})$.
- For any $b \in \mathsf{fp}(\mathcal{B})$ and any $a, a' \in \mathcal{B}$ such that $b \leq a \leftrightarrow a'$, $b \leq f(a) \leftrightarrow f(a')$.

Our goal, then, is to show that there is a $c \in \mathcal{B}$ such that $f(c) = \bigvee\{f(a) \mid a \in \mathcal{B}\}$.

To this end, let $F = \{f(a) \mid a \in \mathcal{B}\}$. We will show soon that F as a poset (with $\leq$ inherited from $\mathcal{B}$) has the following two properties:

- (Directed) For any $a, b \in F$, there is a $c \in F$ such that $a, b \leq c$.
- (Chain) For any ascending chain $\langle a_i \rangle_{i \leq \kappa}$ in F, there is $t \in F$ such that for all $i < \kappa$, $a_i \leq t$.

From these two conditions, it is easy to see that F has an x that is the greatest in F. By Zorn's lemma, F has a maximal element. By (Directed), the maximal element given by Zorn's lemma must also be the greatest element of F. Hence, the join of F is in F. Then, anything in $f^{-1}(\bigvee F)$ can serve as the witness for $(\exists pf)(\theta) = \bigvee F$.

Now we show the two properties. For (Directed), note that $F \subseteq \mathsf{fp}(\mathcal{B})$ since f is fixed. Thus, if we pick $b_1, b_2 \in F$ and $a_1, a_2 \in \mathcal{B}$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$, we can apply locality here. Indeed, let $a = (a_1 \wedge b_1) \vee (a_2 \wedge (b_2 \setminus b_1))$. Note that $b_2 \setminus b_1 \in \mathsf{fp}(\mathcal{B})$. It is also easy to see that $b_1 \leq a \leftrightarrow a_1$ and $b_2 \setminus b_1 \leq a \leftrightarrow a_2$. Then, $b_1 \leq f(a) \leftrightarrow f(a_1)$ and $b_2 \setminus b_1 \leq f(a) \leftrightarrow f(a_2)$. However, by the way we picked a_1 and a_2, $b_1 \leq f(a_1)$ and $b_2 \setminus b_1 \leq f(a_2)$. Thus, $b_1 \leq f(a)$, and $b_2 \setminus b_1 \leq f(a)$, and $b_1 \vee b_2 \leq f(a) \in F$.

For (Chain), we can use the same strategy. Pick an ascending chain $\langle b_i \rangle_{i \leq \kappa}$ in F for some cardinal κ with a corresponding sequence $\langle a_i \rangle_{i < \kappa}$ such that $f(a_i) = b_i$ for all $i < \kappa$. Then inductively define $\langle c_i \rangle_{i < \kappa}$ by

$$\begin{aligned}
c_0 &= a_0 \wedge b_0; \\
c_{i+1} &= c_i \vee (a_{i+1} \wedge (b_{i+1} \setminus b_i)); \\
c_\lambda &= \left(\bigvee_{i<\lambda} c_i \right) \vee \left(a_\lambda \wedge \left(b_\lambda \setminus f \left(\bigvee_{i<\lambda} c_i \right) \right) \right).
\end{aligned}$$

By an easy induction, we can see that for all $i < \kappa$, $c_i \leq b_i$, (note that $(a_\lambda \wedge (b_\lambda \setminus f(\bigvee_{i<\lambda} c_i))) \leq b_\lambda$) and that $\langle c_i \rangle_{i<k}$ is an ascending chain). Less easy is the following:

(1) For all $\lambda \leq \kappa$, $\bigvee_{i<\lambda} b_i \leq f(\bigvee_{i<\lambda} c_i)$.

We use strong induction here, and the base case is trivial. Now suppose as (IH) that for all $\delta < \lambda$, $\bigvee_{i<\delta} b_i \leq f(\bigvee_{i<\delta} c_i)$. Then our only goal is to show that $\bigvee_{i<\lambda} b_i \leq f(\bigvee_{i<\lambda} c_i)$.

- Say $\lambda = \alpha + 1$. Then $\bigvee_{i<\lambda} b_i = b_\alpha$ and $\bigvee_{i<\lambda} c_i = c_\alpha$ since $\langle b_i \rangle_{i<\kappa}$ and $\langle c_i \rangle_{i<\kappa}$ are both ascending chains. Hence we are just showing that $b_\alpha \leq f(c_\alpha)$. Now there are two cases.
  - Say $\alpha = \beta + 1$. Then $c_\alpha = c_\beta \vee (a_\alpha \wedge (b_\alpha \setminus b_\beta))$. By (IH) applied to α, $b_\beta = \bigvee_{i<\alpha} b_i \leq f(\bigvee_{i<\alpha} c_i) = f(c_\beta)$. Note also that $b_\beta \leq c_\beta \leftrightarrow c_\alpha$ and $b_\alpha \setminus b_\beta \leq a_\alpha \leftrightarrow c_\alpha$. By locality, then, $b_\beta \leq f(c_\beta) \leftrightarrow f(c_\alpha)$ and $b_\alpha \setminus b_\beta \leq f(a_\alpha) \leftrightarrow f(c_\alpha)$. However, by (IH) and the way we picked a_α, $b_\beta \leq f(c_\beta)$ and $b_\alpha \setminus b_\beta \leq b_\alpha \leq f(a_\alpha)$. Thus $b_\beta \leq f(c_\alpha)$ and $b_\alpha \setminus b_\beta \leq f(c_\alpha)$. Thus $b_\alpha \leq f(c_\alpha)$.
  - Say α is a limit ordinal. For convenience let $c_\beta = \bigvee_{i<\alpha} c_i$ and $b_\beta = f(c_\beta)$. Here $b_\beta \in \mathsf{fp}(\mathcal{B})$ since f is fixed. By definition, $c_\alpha = c_\beta \vee (a_\alpha \wedge (b_\alpha \setminus b_\beta))$. Now we can apply the same strategy again to show that $b_\alpha \leq f(c_\alpha)$.
- Then we consider the interesting case where λ is a limit ordinal. What we need to show here is that $\bigvee_{i<\lambda} b_i \leq f(\bigvee_{i<\lambda} c_i)$, which means for all $j < \lambda$, $b_j \leq f(\bigvee_{i<\lambda} c_i)$. To show this, pick an arbitrary $j < \lambda$. Then consider $b_j \wedge \bigvee_{i<\lambda} c_i$. Here we claim that this is just c_j. First, $b_j \wedge \bigvee_{i<\lambda} c_i = \bigvee_{i<\lambda}(b_j \wedge c_i)$. (This distributivity law holds on any complete Boolean algebra.) Also, for $i \leq j$, $b_j \wedge c_i = c_i$ since $c_i \leq b_i$ for all $i < \kappa$. Since $\langle c_i \rangle_{i<\kappa}$ is ascending, $\bigvee_{i\leq j}(b_j \wedge c_i) = c_j$. Thus we only need to show that for all i such that $j \leq i < \lambda$, $b_j \wedge c_i = c_j$. Obviously we need to do this by induction. The base case where $i = j$ is trivial (again, by $c_i \leq b_i$). For the inductive step:
  - $b_j \wedge c_{i+1} = (b_j \wedge c_i) \vee (b_j \wedge a_{i+1} \wedge (b_{i+1} \setminus b_i)) = (b_j \wedge c_i) \vee (a_{i+1} \wedge b_{i+1} \wedge \neg b_i \wedge b_j) = b_j \wedge c_i = c_j$. Here the first equality is by distributivity, the second by simple Boolean reasoning, the third by the fact that $j < i$ and hence $b_j \leq b_i$ and $b_j \wedge \neg b_i = \bot$, and the fourth by the induction hypothesis.
  - For a limit ordinal k between j and λ, $b_j \wedge c_k = (b_j \wedge \bigvee_{i<k} c_i) \vee (b_j \wedge a_k \wedge (b_k \setminus f(\bigvee_{i<k} c_i)))$. Now, by the induction hypothesis that for all i such that $j \leq i < k$, $b_j \wedge c_i = c_j$, we get that $b_j \wedge \bigvee_{i<k} c_i = c_j$. Recall that we are inside another induction with (IH) assumed. Applying (IH) to k, we get that $b_j \leq \bigvee_{i<k} b_i \leq f(\bigvee_{i<k} c_i)$. Hence, $b_j \wedge a_k \wedge (b_k \setminus f(\bigvee_{i<k} c_i)) = \bot$, and $b_j \wedge c_k = c_j$.

  So, we have shown that $b_j \wedge \bigvee_{i<\lambda} c_i = c_j$. Adding this to the fact that $c_j \leq b_j$, $b_j \leq \bigvee_{i<\lambda} c_i \leftrightarrow c_j$. By locality, $b_j \leq f(\bigvee_{i<\lambda} c_i) \leftrightarrow f(c_j)$. But $b_j \leq f(c_j)$ since we can apply (IH) to $j + 1$. Thus, $b_j \leq f(\bigvee_{i<\lambda} c_i)$, and we are done here. □

Now we put the three lemmas together to prove the main theorem of this section.

Theorem 3.3.12. *For any $\varphi \in \mathcal{L}\Pi$ and any complete KD45 algebra $\mathcal{B}$, $\mathsf{4}^\forall = \forall p \mathtt{B}\varphi \to \mathtt{B}\forall p \Box \varphi$ is valid on $\mathcal{B}$.*

Proof. Let $\mathcal{B}$ be an arbitrary complete KD45 algebra. Then all the definitions, propositions, and lemmas in this section apply to $\mathcal{B}$. A straightforward induction shows that $[\mathtt{B}\varphi]$ as defined in Definition 3.3.6 is a definable function from V to $\mathcal{B}$ according to Definition 3.3.5. By Lemma 3.3.9, it is local. Since $[\mathtt{B}\varphi] = \Box[\varphi]$, by Lemma 3.3.10, it is fixed. Thus, by Lemma 3.3.11, $[\forall p\mathtt{B}\varphi] = \forall p[\mathtt{B}\varphi]$ is also fixed as $[\mathtt{B}\varphi]$ is both fixed and local. Thus, $[\forall p\mathtt{B}\varphi] = \Box[\forall p\mathtt{B}\varphi] = [\mathtt{B}\forall p\mathtt{B}\varphi]$, and hence $[\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi] = [\forall p\mathtt{B}\varphi] \to [\mathtt{B}\forall p\mathtt{B}\varphi] = [\forall p\mathtt{B}\varphi] \to [\forall p\mathtt{B}\varphi]$, which is constantly $\top$ since in any Boolean algebra and for any element x in it, $x \to x$ is the top element. Then, according to Proposition 3.3.7, for any valuation θ on $\mathcal{B}$, $\tilde{\theta}(\mathsf{4}^\forall) = \tilde{\theta}(\forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi) = [\mathsf{4}^\forall](\theta) = \top$. Hence, $\mathsf{4}^\forall$ is valid on $\mathcal{B}$. ☐

Before we move on to the next section, let us briefly reflect on what this theorem tells us. First, we see that to separate $\mathsf{4}^\forall$ from $\mathsf{KD45\Pi}$ using algebraic semantics, we need to drop the completeness assumption. The difficulty here is that it is not known which meets and joins we need to make the semantics for all formulas with propositional quantifiers well defined. One trivial way to get a KD45 algebra on which the semantics is well defined is to use the Lindenbaum algebra of $\mathsf{KD45\Pi}$. However, showing that $\mathsf{4}^\forall$ in this algebra does not evaluate to $\top$ is plainly equivalent to showing that $\mathsf{KD45\Pi}$ does not prove $\mathsf{4}^\forall$, so this is hardly making any progress. What we need here is a less abstract way to build KD45 algebras on which the semantics is well defined, and this seems to require a less abstract sufficient condition for the well-definedness of the semantics that is strictly weaker than lattice completeness. Another way is to use the semantics with propositional contingency as we sketched at the end of Section 3.2. The difficulty there seems to be validating the Π-principles.

Second, we mentioned in Lemma 3.3.2 that there is a complete KD45 algebra $\mathcal{B}$ such that $\mathsf{fp}(\mathcal{B})$ is not closed under arbitrary meets in $\mathcal{B}$. If we examine the syntactical structure of $\mathsf{4}^\forall = \forall p\mathtt{B}\varphi \to \mathtt{B}\forall p\mathtt{B}\varphi$, it seems that $\mathsf{4}^\forall$ is a candidate formula whose validity on a KD45 algebra $\mathcal{B}$ would correspond to the condition that $\mathsf{fp}(\mathcal{B})$ is closed under arbitrary meets. After all, as we vary the valuation of p, $\mathtt{B}\varphi$ picks up a subset of $\mathsf{fp}(\mathcal{B})$. Then the validity of $\mathsf{4}^\forall$ says that this meet is below[6] the $\Box$ of this meet, which then implies, with a bit of manipulation like what we did in Lemma 3.3.2, that this meet is also a fixpoint. Of course, this is not to be: while the validity of $\mathsf{4}^\forall$ entails that some meets of fixpoints are still fixpoints, it does not correspond to the condition that the set of fixpoints is closed under *arbitrary* meets. What is at work here is that the expressivity constraint of the language is also a constraint on which sets of fixpoints $\mathtt{B}\varphi$ can pick up.

Can there be a way to syntactically capture the condition that the set of fixpoints is closed under arbitrary meets? One way is to add uninterpreted propositional operators. For example, let $\mathcal{L}\Pi O$ be the language extending $\mathcal{L}\Pi$ with a unary operator O. Then, a valuation θ assigns not only an element in $\mathcal{B}$ to each propositional variable, but also a unary function on $\mathcal{B}$ to O. The semantics of the formulas in $\mathcal{L}\Pi O$ under valuation θ can be defined in the obvious way. Then, the validity of $\forall p\mathtt{B}Op \to \mathtt{B}\forall p\mathtt{B}Op$ on $\mathcal{B}$ corresponds to $\mathsf{fp}(\mathcal{B})$ being closed under arbitrary meets. We leave further investigation of this formalism as future work.

[6] Through out this chapter, we use "below" in the weak sense when talking about elements in lattices.

3.4 Completeness of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ with respect to complete proper filter algebras

In this section, we show the following completeness theorem.

Theorem 3.4.1. *For any $\varphi \in \mathcal{L}\Pi$, if φ is valid on all complete proper filter algebras, then $\varphi \in \mathsf{KD4}^{\forall}\mathsf{5}\Pi$.*

Like most completeness proofs, we can show instead that any non-theorem φ of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ is refuted by a complete proper filter algebra. Then, one strategy particularly suitable for algebraic semantics, also used in [91], is the following:

- Construct the Lindenbaum-Tarski algebra $\mathcal{B}$ of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$. Automatically, $\mathcal{B}$ is a KD45 algebra, and φ is evaluated to a non-top element in $\mathcal{B}$ by a naturally defined valuation.
- Transform $\mathcal{B}$ into a complete proper filter algebra $\mathcal{C}$ while keeping φ evaluated to a non-top element. The transformation is typically by first turning $\mathcal{B}$ into a proper filter algebra and then using a construction like MacNeille completion.

The problem with this approach is that neither of the standard methods of constructing MacNeille completions of Boolean algebra with operators, namely the lower and upper MacNeille completions (see [70] or [82]), can be used here. Since proper filter algebras correspond to well-connected KD45 algebras, we can directly use the definition of lower or upper MacNeille completion. However, the upper MacNeille completion of a well-connected KD45 algebra is not necessarily a KD45 algebra (we leave this to the reader; the proper filter algebra based on the finite-cofinite algebra in the proof of Proposition 3.4.3 below can be used to show this as well). The lower MacNeille completion construction does preserve the property of being well-connected and KD45. When translated to proper filter algebras, we get the following definition.

Definition 3.4.2. The lower MacNeille completion $\overline{\mathcal{B}}$ of a proper filter algebra $\mathcal{B} = \langle B, F\rangle$ is $\langle \overline{B}, \uparrow F\rangle$ where $\overline{B}$ is the MacNeille completion of B (which can be assumed to have B as a subalgebra) and $\uparrow F = \{a \in \overline{B} \mid \exists b \in F, b \leq_{\overline{B}} a\}$.

Intuitively, the lower MacNeille completion of a proper filter algebra is obtained by first completing the Boolean algebra part and then extending the original filter minimally to be a filter in the completed Boolean algebra. The problem with the lower MacNeille completion construction is that it may change the semantic value of a sentence from a non-top element to the top element. This means that invalidity, or equivalently refutability, is not preserved.

Proposition 3.4.3. *There is a sentence $\varphi \in \mathcal{L}\Pi$ and a proper filter algebra $\mathcal{B}$ such that φ is invalid in $\mathcal{B}$ but is valid in the lower MacNeille completion $\overline{\mathcal{B}}$.*

Proof. Let B be the Boolean algebra of finite or cofinite subsets of $\mathbb{N}$ and F the set of cofinite sets of $\mathbb{N}$. Then $\mathcal{B} = \langle B, F \rangle$ is a proper filter algebra. The lower MacNeille completion $\overline{\mathcal{B}}$ is then $\langle \wp(\mathbb{N}), F \rangle$ where $\wp(\mathbb{N})$ is the powerset algebra of $\mathbb{N}$. Note that, importantly, the lower MacNeille completion does not change F.

Let φ be $\exists p(\neg \mathtt{B}p \wedge \neg \mathtt{B}\neg p)$. For any valuation θ on $\mathcal{B}$, $\tilde{\theta}(\varphi)$ is $\bot$, since $\tilde{\theta}(p) \in B$ is either finite or cofinite, and thus either $\tilde{\theta}(p)$ or $\tilde{\theta}(\neg p)$ is cofinite. Then either $\tilde{\theta}(\neg \mathtt{B}p)$ or $\tilde{\theta}(\neg \mathtt{B}\neg p)$ is $\bot$. This means that $\tilde{\theta}(\varphi) = \bot$. Now, let θ be a valuation on $\overline{\mathcal{B}}$ such that $\theta(p)$ is the set of even numbers. Then $\theta(p)$ is infinite and coinfinite. Hence $\tilde{\theta}(\neg \mathtt{B}p \wedge \neg \mathtt{B}\neg p)$ is $\top$, and so $\tilde{\theta}(\varphi) = \top$. Since φ is a sentence with no free variables, this means that φ is valid on $\overline{\mathcal{B}}$. $\square$

The lesson from the above example is that, to preserve the invalidity of $\exists p(\neg \mathtt{B}p \wedge \neg \mathtt{B}\neg p)$, we need to extend F, the filter of cofinite sets, into an ultrafilter. However, there seems to be no canonical way to do this: every ultrafilter seems as good as any other. It seems that to prove the completeness of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, we need a more detailed analysis of both the algebras and the system itself.

Our approach is based on the following observation: every complete proper filter algebra $\mathcal{B} = \langle B, F \rangle$ has a natural quotient Boolean algebra B/F, and a belief formula in the form of $\mathtt{B}\varphi$, where φ contains only Boolean connectives, is asserting that φ evaluates to the top element in the quotient B/F. Hence, $\mathtt{B}\varphi$ is talking about the quotient algebra B/F. With propositional quantifiers, $\mathcal{L}\Pi$ can in fact talk about B/F in a first-order way.

Given this observation, we have the following strategy, where we focus on just the sentences since for any formula, it is valid if and only if its universal closure, the result of bounding all free variables with universal quantifiers, is valid:

- For a sentence φ in $\mathcal{L}\Pi$ that is valid in all complete proper filter algebras, find a corresponding ψ in a first-order language for Boolean algebras. This ψ will be valid on all natural quotient Boolean algebras of all complete proper filter algebras.

- Show that if a first-order sentence is valid on all natural quotient Boolean algebras of complete proper filter algebras, then it is in fact valid on all non-trivial Boolean algebras. Consequently, ψ will have a proof in the usual first-order logic for all non-trivial Boolean algebras.

- Translate the first-order proof of ψ into a proof of φ in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$.

The main difficulties of implementing this strategy lie in the first two steps. First, in fact, not every sentence $\varphi \in \mathcal{L}\Pi$ can be translated into a first-order sentence to be evaluated on the natural quotient Boolean algebras. The reason is that a sentence in $\mathcal{L}\Pi$ can be evaluated to a proposition that is neither top nor bottom, yet a first-order sentence can only be true or false. To cope with this, several auxiliary languages will be introduced, so that we can separate the translatable part and the non-translatable part of a sentence in $\mathcal{L}\Pi$. It turns out that the non-translatable part is well behaved, so we can proceed with them.

In the second step, a theorem about the natural quotients of complete proper filter algebras is needed. Essentially, it has to be shown that the non-trivial quotients of complete

Boolean algebras are general enough to validate only those first-order formulas that are valid in all non-trivial Boolean algebras. In other words, the first-order logic of the non-trivial quotients of complete Boolean algebras is precisely the first-order logic of all non-trivial Boolean algebras. While the first-order logic remains the same, it should be mentioned here that there are interesting special properties of quotients of complete Boolean algebras, e.g., countable separation property (see [107], Lemma 5.27). Previous results about the quotients of complete Boolean algebras include [123], [45], and [153]. We will extend the result in [153] to fulfill our purpose.

The rest of this section is split into four subsections. In §3.4.1, we define a number of auxiliary languages, their semantics, and translations between some of them. In §3.4.2, we show how the completeness of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ follows from two lemmas resolving the two difficulties mentioned above. Then, the next two subsections, §3.4.3 and §3.4.4, are devoted to the two lemmas, respectively.

3.4.1 Auxiliary languages, semantics, and translations

Definition 3.4.4. Let $\mathcal{L}\Pi\mathsf{zg}$ denote the language extending $\mathcal{L}\Pi$ with two new propositional constants z and g. For more readability, we sometimes use overline instead $\neg$ for the negation of formulas and omit the $\wedge$ in a conjunction of literals. Now we define the following languages:

$$\begin{aligned} \mathcal{BL} &: t ::= p \mid \top \mid \neg t \mid (t \wedge t) && \text{where } p \in \mathsf{Prop}; \\ \mathcal{BL}\mathsf{zg} &: \varphi ::= \mathsf{z} \mid \mathsf{g} \mid p \mid \top \mid \neg\varphi \mid (\varphi \wedge \varphi) && \text{where } p \in \mathsf{Prop}; \\ \mathcal{L}\mathsf{B}\Pi\mathsf{zg} &: \varphi ::= \mathsf{B}t \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall p\varphi && \text{where } t \in \mathcal{BL}\mathsf{zg}, p \in \mathsf{Prop}. \end{aligned}$$

Very roughly speaking, z and g will be used to capture the non-translatable part, and $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ will be the translatable part. Recall that the main difficulty of translating $\mathcal{L}\Pi$ to a first-order logic is that there are sentences in $\mathcal{L}\Pi$ that evaluate to some intermediate proposition in some complete proper filter algebra. We will effectively show later that the Boolean combinations of z and g exhaust all possible semantic values that a sentence in $\mathcal{L}\Pi$ can take, and eventually every formula in $\mathcal{L}\Pi$ is provably equivalent to a Boolean combination of z, g, and formulas in $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$. It will be shown later in this section that $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ is translatable to a first-order language.

The next definition fixes the interpretation of the new constants z and g.

Definition 3.4.5. For any complete proper filter algebra $\mathcal{B} = \langle B, F\rangle$, define $z_{\mathcal{B}}, g_{\mathcal{B}}$ by

$$\begin{aligned} z_{\mathcal{B}} &= \bigwedge F, \\ g_{\mathcal{B}} &= \bigvee\{a \in \mathcal{B} \mid a \text{ is an atom and } a \leq z_{\mathcal{B}}\}. \end{aligned}$$

The subscripts of $z_{\mathcal{B}}$ and $g_{\mathcal{B}}$ may be dropped if the context of which algebra we are talking about is clear.

For any valuation θ on $\mathcal{B}$, we then extend it uniquely to a $\mathcal{L}\Pi\mathsf{zg}$-valuation $\tilde{\theta} : \mathcal{L}\Pi\mathsf{zg} \to B$, using the definition in Definition 3.2.5 plus two more clauses for z and g: $\tilde{\theta}(\mathsf{z}) = z$ and $\tilde{\theta}(\mathsf{g}) = g$.

Under this semantics, g and zg are semantically equivalent, but $\overline{\mathsf{g}}$ and $\mathsf{z}\overline{\mathsf{g}}$ are not. For symmetry, we will mostly use zg instead of g to contrast $\mathsf{z}\overline{\mathsf{g}}$.

It is important to see that for every complete proper filter algebra $\mathcal{B}$, $z_{\mathcal{B}}$ and $g_{\mathcal{B}}$ are expressible in $\mathcal{L}\Pi$. For this, we introduce a few more abbreviations.

Definition 3.4.6. Define the following abbreviations:

$$\begin{aligned} \langle \mathsf{z}\rangle\varphi &= \widehat{\mathsf{B}}(\mathsf{z} \wedge \varphi); \\ [\mathsf{z}]\varphi &= \mathsf{B}(\mathsf{z} \to \varphi); \\ at(\varphi) &= \langle \mathsf{z}\rangle\varphi \wedge \forall q([\mathsf{z}](\varphi \to q) \vee [\mathsf{z}](\varphi \to \neg q)). \end{aligned}$$

Here q is some variable not free in φ.

Proposition 3.4.7. *For any complete proper filter algebra $\mathcal{B}$ and valuation θ:*

1. $\tilde{\theta}(\langle \mathsf{z}\rangle\varphi) = \tilde{\theta}(\neg[\mathsf{z}]\neg\varphi)$.
2. $\tilde{\theta}(\langle \mathsf{z}\rangle\varphi)$ *and* $\tilde{\theta}([\mathsf{z}]\varphi)$ *are either* $\top$ *or* $\bot$.
3. $\tilde{\theta}([\mathsf{z}]\varphi) = \top$ *if and only if* $\tilde{\theta}(\varphi) \geq z$. *In other words,* $\tilde{\theta}([\mathsf{z}]\varphi) = \top$ *if and only if* $\tilde{\theta}(\varphi) \wedge z = z$.
4. $\tilde{\theta}(\langle \mathsf{z}\rangle\varphi) = \top$ *if and only if* $z \wedge \tilde{\theta}(\varphi)$ *is not* $\bot$.
5. $\tilde{\theta}(at(\varphi)) = \tilde{\theta}(\langle \mathsf{z}\rangle\varphi \wedge \neg\exists q(\langle \mathsf{z}\rangle(\varphi \wedge q) \wedge \langle \mathsf{z}\rangle(\varphi \wedge \neg q)))$ *where* q *is not free in* φ.
6. $\tilde{\theta}(at(\varphi))$ *is either* $\top$ *or* $\bot$. *It is the former if and only if* $z \wedge \tilde{\theta}(\varphi)$ *is an atom in (the Boolean algebra part of)* $\mathcal{B}$.
7. $z = \tilde{\theta}(\forall p(\mathsf{B}p \to p))$.
8. $g = \tilde{\theta}(\forall p(\mathsf{B}p \to p) \wedge \exists p(p \wedge at(p)))$.

Proof. The third item is not completely obvious, and we prove it here. Recall that $\tilde{\theta}([\mathsf{z}]\varphi) = \top$ iff $(z \to \tilde{\theta}(\varphi)) \in F$. For one direction, suppose that $z \leq \tilde{\theta}(\varphi)$. Then $(z \to \tilde{\theta}(\varphi)) = \top$. Since F is a filter, $(z \to \tilde{\theta}(\varphi)) \in F$. For the other direction, suppose that $(z \to \tilde{\theta}(\varphi)) \in F$. Note that z is the meet of F. So $z \leq (z \to \tilde{\theta}(\varphi))$. By the theory of Boolean algebra, this means that $z \wedge \neg(z \to \tilde{\theta}(\varphi)) = \bot$. Simplifying the left-hand-side, we have $z \wedge \neg\tilde{\theta}(\varphi) = \bot$. So $z \leq \tilde{\theta}(\varphi)$. The fourth item follows from the third item by simple duality reasoning.

The last item may also need some explanation. To start, unpack the semantics of $\tilde{\theta}(\forall p(\mathsf{B}p \to p) \wedge \exists p(p \wedge at(p)))$, and we see that it is

$$\begin{aligned}&\tilde{\theta}(\forall p(\mathsf{B}p \to p)) \wedge \bigvee\{\widetilde{\theta[a/p]}(p \wedge at(p)) \mid a \in \mathcal{B}\}\\ &= z \wedge \bigvee\{a \wedge \widetilde{\theta[a/p]}(at(p)) \mid a \in \mathcal{B}\}\\ &= \bigvee\{z \wedge a \wedge \widetilde{\theta[a/p]}(at(p)) \mid a \in \mathcal{B}\}.\end{aligned}$$

Now if a is an atom below z, then $z \wedge a = a$, which is still an atom. Then, given that $\widetilde{\theta[a/p]} = a$ and hence $z \wedge \widetilde{\theta[a/p]} = z \wedge a$, $\widetilde{\theta[a/p]}(at(p)) = \top$ according to item 6 above. Then $z \wedge a \wedge \widetilde{\theta[a/p]}(at(p)) = a$. This means that all atoms below z are included in the join. On the other hand, if a is not an atom below z, then $z \wedge a \wedge \widetilde{\theta[a/p]}(at(p)) = \bot$ since $\widetilde{\theta[a/p]}(at(p))$ is $\bot$, again by item 6 above. Thus the join is precisely the join of atoms below z. $\square$

It can also be shown that $\exists p(p \wedge at(p))$ expresses the join of those elements whose meet with z is an atom. However, taking this as a primitive seems to be less convenient for later work.

Now we focus on the fragment $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ and show in what sense it can be seen as talking about the natural quotients of complete proper filter algebras in a first-order way. To this end, we first define precisely what we mean by the natural quotient of a complete proper filter algebra $\mathcal{B}$.

Definition 3.4.8. For any complete proper filter algebra $\mathcal{B} = \langle B, F\rangle$, define its *natural quotient* $\mathcal{B}/F$ as the tuple $\langle B/F, \pi_{\mathcal{B}}(z), \pi_{\mathcal{B}}(g)\rangle$, where $\pi_{\mathcal{B}}$ is the quotient map from B to B/F. We may drop the subscript of $\pi_{\mathcal{B}}$ when its context is clear.

When viewed as a first-order structure, natural quotients of complete proper filter algebras are in the type of Boolean algebras augmented with two constants. Hence we define the following first-order language.

Definition 3.4.9. Let $\mathcal{FOL}$ be the first-order language defined as below.

$$\begin{aligned}\text{Terms} : t &::= p \mid \top \mid \neg t \mid (t \wedge t) && \text{where } p \in \mathsf{Prop},\\ \mathcal{FOL} : \varphi &::= (t = t') \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall p\varphi && \text{where } t, t' \in \text{Terms}.\end{aligned}$$

Let $\mathcal{FOL}\mathsf{zg}$ be the first-order language extending $\mathcal{FOL}$ by adding z and g as two constants.

Note that here we are intentionally reusing the symbols in $\mathcal{L}\Pi$, so that the translation between $\mathcal{FOL}\mathsf{zg}$ and $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ can be defined more directly. Note also that because we use the same symbols for meet in terms and conjunction in formulas, we need to bracket atomic formulas to avoid ambiguity. Now we define the standard first-order semantics for $\mathcal{FOL}\mathsf{zg}$.

Definition 3.4.10. For any triple $\langle B, z, g\rangle$ where B is a Boolean algebra and $z, g \in B$, a variable assignment $\theta : \mathsf{Prop} \to B$ can be extended uniquely to $\tilde{\theta}$ on the terms of $\mathcal{FOL}\mathsf{zg}$ by the following inductive clauses:

$$\tilde{\theta}(p) = \theta(p),\ \tilde{\theta}(\mathsf{g}) = g,\ \tilde{\theta}(\mathsf{z}) = z,\ \tilde{\theta}(\top) = \top,$$
$$\tilde{\theta}(\neg t) = \neg_B \tilde{\theta}(t),\ \tilde{\theta}((t \wedge s)) = \tilde{\theta}(t) \wedge_B \tilde{\theta}(s).$$

Then the semantics of $\mathcal{FOL}\mathsf{zg}$ is defined by

$$\begin{aligned}
\langle B, z, g\rangle, \theta \vDash (t = s) &\iff \tilde{\theta}(t) = \tilde{\theta}(s)\\
\langle B, z, g\rangle, \theta \vDash \neg\varphi &\iff \langle B, z, g\rangle, \theta \nvDash \varphi\\
\langle B, z, g\rangle, \theta \vDash (\varphi \wedge \psi) &\iff \langle B, z, g\rangle, \theta \vDash \varphi \text{ and } \langle B, z, g\rangle, \theta \vDash \psi\\
\langle B, z, g\rangle, \theta \vDash \forall p\varphi &\iff \langle B, z, g\rangle, \theta[a/p] \vDash \varphi \text{ for all } a \in B.
\end{aligned}$$

Syntactically, $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ and $\mathcal{FOL}\mathsf{zg}$ are almost identical. This can be seen from how simply the translations between them can be defined.

Definition 3.4.11. Let $\mathcal{T}$ be the function from $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ to $\mathcal{FOL}\mathsf{zg}$ such that $\mathcal{T}(\varphi)$ is the result of replacing all occurrences in φ of formulas of the form $\mathsf{B}\psi$ where $\psi \in \mathcal{BL}\mathsf{zg}$ by $(\psi = \top)$.

Let $\mathcal{T}'$ be the function from $\mathcal{FOL}\mathsf{zg}$ to $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ such that $\mathcal{T}'(\varphi)$ is the result of replacing all atomic formulas $(s = t)$ by $\mathsf{B}(s \leftrightarrow t)$.

For example, recall the sentence $\exists p(\neg\mathsf{B}p \wedge \neg\mathsf{B}\neg p)$ we used in Proposition 3.4.3 where we showed that lower MacNeille completion does not preserve its semantic value. It is not hard to see that $\mathcal{T}(\exists p(\neg\mathsf{B}p \wedge \neg\mathsf{B}\neg p)) = \exists p(\neg(p = \top) \wedge \neg(\neg p = \top))$. Then $\mathcal{T}(\exists p(\neg\mathsf{B}p \wedge \neg\mathsf{B}\neg p))$ is false on a $\langle B, z, g\rangle$ iff B is the 2-element Boolean algebra. This matches our observation there that $\exists p(\neg\mathsf{B}p \wedge \neg\mathsf{B}\neg p)$ is false (evaluates to $\bot$) on a complete proper filter algebra $\mathcal{B} = \langle B, F\rangle$ iff the filter F is an ultrafilter, or equivalently iff the quotient B/F is the 2-element Boolean algebra.

To make the intuition that $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ talks about natural quotients in the first-order way precise, we use the following lemma.

Lemma 3.4.12. *For any $\varphi \in \mathcal{L}\mathsf{B}\Pi\mathsf{zg}$, any complete proper filter algebra $\mathcal{B} = \langle B, F\rangle$, and any valuation θ on $\mathcal{B}$, $\tilde{\theta}(\varphi)$ is either $\top$ or $\bot$, and $\tilde{\theta}(\varphi) = \top$ if and only if $\langle B/F, \pi(z), \pi(g)\rangle, \pi \circ \theta \vDash \mathcal{T}(\varphi)$. As a corollary, when φ is a sentence, $\mathcal{B} \vDash \varphi$ iff $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\varphi)$.*

Proof. To avoid clutter, we omit the pair of parentheses immediately after π and also the circle for composing π with θ. Hence $\pi p = \pi(p)$ and $\pi\theta(p) = \pi(\theta(p)) = (\pi \circ \theta)(p)$. Now obviously we need an induction on $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$.

For any $\psi \in \mathcal{BL}\mathsf{zg}$, a simple induction shows that $\pi\tilde{\theta}(\psi) = \widetilde{\pi\theta}(\psi)$, since π is a quotient map and hence a homomorphism. By our algebraic semantics defined in Definition 3.2.5, $\tilde{\theta}(\mathsf{B}\psi)$ is either $\top_B$ or $\bot_B$ according to whether $\tilde{\theta}(\psi)$ is in the filter F or not. Also, since

$F = \pi^{-1}(\top_{B/F})$ by the definition of π in Definition 3.4.8, $\pi\tilde{\theta}(\psi) = \top_{B/F}$ if and only if $\tilde{\theta}(\psi) \in F$. Then

$$\begin{aligned}\tilde{\theta}(\mathtt{B}\psi) = \top_B &\iff \tilde{\theta}(\psi) \in F \iff \pi\tilde{\theta}(\psi) = \top_{B/F} \iff \widetilde{\pi\theta}(\psi) = \top_{B/F} \\ &\iff \langle B/F, \pi z, \pi g\rangle, \pi\theta \vDash \psi = \top \\ &\iff \langle B/F, \pi z, \pi g\rangle, \pi\theta \vDash \mathcal{T}(\mathtt{B}\psi).\end{aligned}$$

For formulas in $\mathcal{L}\mathtt{B}\Pi\mathtt{zg}$ of the form $\neg\varphi$ where $\varphi \in \mathcal{L}\mathtt{B}\Pi\mathtt{zg}$, note first that $\mathcal{T}(\neg\varphi) = \neg\mathcal{T}(\varphi)$. Also, $\tilde{\theta}(\neg\varphi)$ must be either $\top_B$ or $\bot_B$, given the induction hypothesis that $\tilde{\theta}(\varphi)$ is either $\top_B$ or $\bot_B$. Then, with the induction hypothesis that $\tilde{\theta}(\varphi) = \top_B$ iff $\langle B/F, \pi g, \pi z\rangle \vDash \mathcal{T}(\varphi)$, we have

$$\begin{aligned}\tilde{\theta}(\neg\varphi) = \top_B &\iff \tilde{\theta}(\varphi) = \bot_B \iff \tilde{\theta}(\varphi) \neq \top_B \\ &\iff \langle B/F, \pi g, \pi z\rangle \nvDash \mathcal{T}(\varphi) \\ &\iff \langle B/F, \pi g, \pi z\rangle \vDash \neg\mathcal{T}(\varphi) \\ &\iff \langle B/F, \pi g, \pi z\rangle \vDash \mathcal{T}(\neg\varphi).\end{aligned}$$

For formulas in $\mathcal{L}\mathtt{B}\Pi\mathtt{zg}$ of the form $\varphi_1 \wedge \varphi_2$, the situation is completely similar. We only need to do more replacements of equivalent claims coming from the induction hypothesis in this case.

Now consider $\forall q\varphi$. Note first that for all $a \in B$, by induction hypothesis we know that $\widetilde{\theta[a/q]}(\varphi)$ is either $\top_B$ or $\bot_B$, since the proof works for all valuations. Then, $\tilde{\theta}(\forall q\varphi) = \bigwedge\{\widetilde{\theta[a/q]}(\varphi)\}$ must be either $\top_B$ or $\bot_B$. Moreover, being a meet of elements that are either $\top_B$ or $\bot_B$, it is $\top_B$ iff for all $a \in B, \widetilde{\theta[a/q]}(\varphi) = \top_B$, which, using induction hypothesis, is equivalent to

$$\text{for all } a \in B, \langle B/F, \pi g, \pi z\rangle, \pi \circ (\theta[a/q]) \vDash \mathcal{T}(\varphi). \tag{3.1}$$

On the other hand, according to the semantics, $\langle B/F, \pi g, \pi z\rangle, \pi\theta \vDash \mathcal{T}(\forall q\varphi)$ iff

$$\text{for all } a \in B/F, \langle B/F, \pi g, \pi z\rangle, (\pi \circ \theta)[a/q] \vDash \mathcal{T}(\varphi). \tag{3.2}$$

Hence, all we need to show now is that (3.1) and (3.2) are equivalent. Too see that they are equivalent, note that for any $a \in B$, $\pi \circ (\theta[a/q])$ is the same function as $(\pi \circ \theta)[\pi(a)/q]$. Then, given that π is surjective, we are done. □

3.4.2 Logics in auxiliary languages and completeness proof

In the last subsection, we introduced a number of fragments of $\mathcal{L}\Pi\mathtt{zg}$, including $\mathcal{L}\mathtt{B}\Pi\mathtt{zg}$, which can be translated to the first-order language $\mathcal{FOL}\mathtt{zg}$ in a semantically meaningful way: a formula $\varphi \in \mathcal{L}\mathtt{B}\Pi\mathtt{zg}$ evaluates to $\top$ in a complete proper filter algebra $\mathcal{B}$ iff its translation

$\mathcal{T}(\varphi)$ is true on the natural quotient $\mathcal{B}/F_{\mathcal{B}}$. From this we see that semantically, $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$ is talking about the natural quotients of complete proper filter algebras in a first-order way.

In this subsection, we move to the logical side of this translation. We will first augment $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ with two definitional axioms for the two new constants z and g and obtain $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$. Then we provide a first-order logic FOLzg that is sound and complete with respect to a class of Boolean algebras with two extra named elements, which we call the class of zg-algebras. This class of zg-algebras is bigger than the class of natural quotients of complete proper filter algebras. However, we can show in a later section that the first-order logics of them are the same. In this section, the main task is to show that reasoning in FOLzg can be carried out in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ by the reverse translation $\mathcal{T}'$. Then, assuming that we can separate the translatable part $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$ and the non-translatable part and that FOLzg is not only the first-order logic of zg-algebras but also the first-order logic of the class of natural quotients of complete proper filter algebras, we show in this subsection that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is complete with respect to all complete proper filter algebras. Given that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is a definitional extension of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, the completeness of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ follows.

To start, we define the system $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$.

Definition 3.4.13. Define logic $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ by extending $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ with the following two axioms for z and g:

$$\begin{aligned}&\mathsf{z} : \mathsf{z} \leftrightarrow \forall p(\mathtt{B}p \to p),\\ &\mathsf{g} : \mathsf{g} \leftrightarrow (\forall p(\mathtt{B}p \to p) \wedge \exists p(p \wedge at(p))).\end{aligned}$$

The new axioms state the semantic definition of z and g, as shown in Proposition 3.4.7. Moreover, the right-hand side of the first axiom z is in $\mathcal{L}\Pi$, and the right-hand side of the second axiom g uses only z besides allowed constructions in $\mathcal{L}\Pi$. Hence $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is a definitional and conservative extension of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, and we only need to prove that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is the complete logic of complete proper filter algebras in $\mathcal{L}\Pi\mathsf{zg}$ according to the semantics defined in Definition 3.4.5.

Notation 3.4.14. In this and the next subsection, we will state many provability claims in the system $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$. We write $\vdash \varphi$ for φ being provable in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ and write $\varphi \dashv\vdash \psi$ for $\vdash \varphi \leftrightarrow \psi$. We treat $\dashv\vdash$ as a kind of equality between formulas so that in notation we chain them and use substitutions. We can do this because $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is a normal Π-logic, and thus $\dashv\vdash$ is a congruence relation with respect to all connectives and quantifiers.

Now we prove two lemmas that will be very useful. The first shows the importance of having $\mathsf{4}^{\forall}$, and the second shows the use of the constant z. To state the first lemma, we call a formula $\varphi \in \mathcal{L}\Pi\mathsf{zg}$ *fully modalized* when every propositional letter (those in $\mathsf{Prop} \cup \{\top, \mathsf{z}, \mathsf{g}\}$) is under the scope of some $\mathtt{B}$. It is not hard to see that the fully modalized formulas in $\mathcal{L}\Pi\mathsf{zg}$ can be characterized by the grammar $\varphi ::= \mathtt{B}\psi \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall p\varphi$, where $\psi \in \mathcal{L}\Pi\mathsf{zg}$.

Lemma 3.4.15. *For every fully modalized formula $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $\varphi \dashv\vdash \mathtt{B}\varphi \dashv\vdash \widehat{\mathtt{B}}\varphi$.*

Proof. First we show $\varphi \dashv\vdash \mathtt{B}\varphi$ by induction.

- The case where $\varphi = \mathtt{B}\psi$ is trivial by $\mathsf{KD45}$ since we are just showing that $\mathtt{B}\psi \dashv\vdash \mathtt{BB}\psi$.
- Suppose $\varphi = \psi_1 \wedge \psi_2$ where ψ_1 and ψ are fully modalized. Then $\psi_1 \dashv\vdash \mathtt{B}\psi_1$ and $\psi_2 \dashv\vdash \mathtt{B}\psi_2$. Then $\psi_1 \wedge \psi_2 \dashv\vdash \mathtt{B}\psi_1 \wedge \mathtt{B}\psi_2 \dashv\vdash \mathtt{B}(\psi_1 \wedge \psi_2)$.
- Suppose $\varphi = \neg\psi$ where ψ is fully modalized, and hence $\psi \dashv\vdash \mathtt{B}\psi$. Then again by $\mathsf{KD45}$, we have a chain of provable equivalences: $\neg\psi \dashv\vdash \neg\mathtt{B}\psi \dashv\vdash \mathtt{B}\neg\mathtt{B}\psi \dashv\vdash \mathtt{B}\neg\psi$. The last equivalence can be obtained by simply replacing $\mathtt{B}\psi$ by ψ. Since $\mathsf{KD4^\forall 5\Pi zg}$ is normal, we can certainly do such replacements.
- Suppose $\varphi = \forall p\psi$ where ψ is fully modalized and thus $\psi \dashv\vdash \mathtt{B}\psi$. Then $\forall p\psi \dashv\vdash \forall p\mathtt{B}\psi$. By $\mathsf{4^\forall}$, $\forall p\psi \dashv\vdash \mathtt{B}\forall p\mathtt{B}\psi$. Then we can replace $\mathtt{B}\psi$ by ψ again, and hence $\forall p\psi \dashv\vdash \mathtt{B}\forall p\varphi$.

Then for any fully modalized formula φ, noting that we just proved that $\varphi \dashv\vdash \mathtt{B}\varphi$, $\varphi \dashv\vdash \mathtt{B}\varphi \dashv\vdash \widehat{\mathtt{B}}\mathtt{B}\varphi \dashv\vdash \widehat{\mathtt{B}}\varphi$. ☐

Lemma 3.4.16. *For every $\varphi, \psi \in \mathcal{L}\Pi\mathsf{zg}$, the following are theorems of $\mathsf{KD4^\forall 5\Pi zg}$:*

$$\begin{aligned}
&[\mathsf{z}](\varphi \to \psi) \to ([\mathsf{z}]\varphi \to [\mathsf{z}]\psi),\\
&\mathsf{z} \to ([\mathsf{z}]\varphi \to \varphi),\\
&[\mathsf{z}]\varphi \to [\mathsf{z}][\mathsf{z}]\varphi,\\
&\langle\mathsf{z}\rangle\varphi \to [\mathsf{z}]\langle\mathsf{z}\rangle\varphi.
\end{aligned}$$

Moreover, if $\vdash \mathsf{z} \to \varphi$, then $\vdash \mathsf{z} \to [\mathsf{z}]\varphi$. This means that, assuming z, $[\mathsf{z}]$ is an $\mathsf{S5}$ modality.

Proof. The first, third, and last formulas are easy to derive in $\mathsf{KD4^\forall 5\Pi zg}$. For the second, recall that by the axiom z, $\vdash \mathsf{z} \leftrightarrow \forall p(\mathtt{B}p \to p)$. Hence, assuming z, we can deduce $\mathtt{B}(\mathsf{z} \to \varphi) \to (\mathsf{z} \to \varphi)$. But we can also derive $(\mathsf{z} \to \varphi) \to \varphi$ as we have z in hand. So we derive $\mathtt{B}(\mathsf{z} \to \varphi) \to \varphi$ and thus $[\mathsf{z}]\varphi \to \varphi$.

Finally, for the necessitation-like implication, suppose that $\vdash \mathsf{z} \to \varphi$. Then by necessitation, $\mathtt{B}(\mathsf{z} \to \varphi)$ is provable, but this is just $[\mathsf{z}]\varphi$. So certainly $\mathsf{z} \to [\mathsf{z}]\varphi$ is provable. ☐

Moving to the $\mathcal{FOL}\mathsf{zg}$ side, what we need is a first-order logic that is weak enough to be embedded using $\mathcal{T}'$ in $\mathsf{KD4^\forall 5\Pi zg}$, yet strong enough to include all validities of the natural quotients of complete proper filter algebras. It turns out that this logic is the logic of the following very general class of Boolean algebras with two named elements.

Definition 3.4.17. A *zg-algebra* A is a tuple $\langle A_0, z_A, g_A\rangle$ such that A_0 is a non-trivial Boolean algebra and $z_A, g_A \in A_0$, such that $z_A g_A$ is atomic (it is the join of atoms below it), $z_A\overline{g_A}$ is atomless (there are no atoms below it), and $g_A \leq z_A$.

Note that according to the definition, for any zg-algebra A, g_A is precisely the join of the atoms below z_A. Hence zg-algebras can alternatively be defined as Boolean algebras with an element z such that the join of the atoms below it exists and is denoted by g. It is not too hard to observe that all natural quotients of complete proper filter algebras are zg-algebras. On the other hand, there are certainly zg-algebras that are not isomorphic to the natural quotients of any complete proper filter algebra. An obvious way to construct such zg-algebras is to take zg-algebras whose restriction to z is not complete. By our definition, this is totally admissible for being a zg-algebra: the existence of just the join of atoms below z suffices. However, for any zg-algebra $A = \mathcal{B}/F_{\mathcal{B}}$ where $\mathcal{B}$ is some complete proper filter algebra, $A|_{z_A}$ must be a complete Boolean algebra since first $\mathcal{B}$ is complete and second $A|_{z_A}$ is isomorphic to $\mathcal{B}|_{z_{\mathcal{B}}}$. We will show that it is not a problem that zg-algebras forms a wider class than the class of natural quotients of complete proper filter algebras. The motivation of having a wider class is that this class of zg-algebras is first-order definable, and thus we get a complete first-order logic for free. The logic is presented below, and we omit the easy soundness and completeness proof since the class of zg-algebras is obviously defined by the non-logical axioms.

Proposition 3.4.18. *The validities of all zg-algebras in the language of $\mathcal{FOL}$zg under the semantics in Definition 3.4.10 is axiomatized by the logic* FOLzg *defined by the axiom schemas below and the usual modus ponens and universalization rule. In the group of logical axioms, φ, ψ are arbitrary formulas in $\mathcal{FOL}$zg, t is an arbitrary term, and p, q, r are arbitrary variables in* Prop*. In the second group of non-logical axioms, p, q, r are three specific and distinct variables in* Prop *while s, t still stand for arbitrary terms.*

Logical axioms

$$\text{All instances of propositional tautology schemas in } \mathcal{FOL}\mathsf{zg}$$
$$\forall p(\varphi \to \psi) \to (\forall p\varphi \to \forall p\psi)$$
$$\forall p\varphi \to \varphi[t/p] \text{ when } t \text{ is substitutable for } p \text{ in } \varphi$$
$$\varphi \to \forall p\varphi \text{ when } p \text{ is not free in } \varphi$$
$$(p = p) \wedge ((p = q) \to (q = p))$$
$$((p = q) \wedge (q = r)) \to (p = r)$$
$$(p = q) \to ((\neg p = \neg q))$$
$$(p = q) \to ((r \wedge p = r \wedge q) \wedge (p \wedge r = q \wedge r))$$

Non-logical axioms

$$\neg(\top \leqslant \bot) \tag{3.3}$$

$$(s = t) \text{ when } s \leftrightarrow t \text{ is a tautology} \tag{3.4}$$

$$(\mathsf{g} \leqslant \mathsf{z}) \tag{3.5}$$

$$\forall p(((p \leqslant \mathsf{zg}) \wedge (p \neq \bot)) \to \exists q((q \leqslant p) \wedge (q \neq \bot) \wedge \forall r((q \leqslant r) \vee (q \leqslant r)))) \tag{3.6}$$

$$\forall p(((p \leqslant \mathsf{z}\overline{\mathsf{g}}) \wedge (p \neq \bot)) \to \exists q((q \leqslant p) \wedge (pq \neq \bot) \wedge (p\overline{q} \neq \bot))) \tag{3.7}$$

Note that we are not using the usual Leibniz's law in this axiomatization. Instead, we have a group of axioms saying that the equality relation is a congruence relation. The usual Leibniz's law can be derived from them together with other axioms and rules. This is mainly for the ease of showing that the translations preserve theoremhood, since $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ *does not have Leibniz's law as one of its axioms.*

In the non-logical axioms above and also for the rest of the chapter, we use the following abbreviations in $\mathcal{FOL}\mathsf{zg}$*:* $(s \leqslant t) := ((s \to t) = \top)$ *and* $(s \neq t) := \neg(s = t)$*. Intuitively the abbreviation* $(s \leqslant t)$ *says that* s *is below* t *in the Boolean lattice order. Then the two axioms intuitively say that* zg *is atomic and* $\mathsf{z}\overline{\mathsf{g}}$ *is atomless respectively. Obviously then the non-logical axioms define zg-algebras.*

That FOLzg is weak enough to be embedded into $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is shown by the following three lemmas.

Lemma 3.4.19. *For any* $\varphi \in \mathcal{L}\mathsf{B}\Pi\mathsf{zg}$*,* $\varphi \leftrightarrow \mathcal{T}'(\mathcal{T}(\varphi))$ *is provable in* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$*. For any* $\varphi \in \mathcal{FOL}\mathsf{zg}$*,* $\varphi \leftrightarrow \mathcal{T}(\mathcal{T}'(\varphi))$ *is provable in* FOLzg

Proof. $\mathcal{T}'(\mathcal{T}(\varphi))$ turns every $\mathtt{B}\beta$ in φ first to $\beta = \top$ and then to $\mathtt{B}(\beta \leftrightarrow \top)$. But $\mathtt{B}\beta \leftrightarrow \mathtt{B}(\beta \leftrightarrow \top)$ is in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$. Similarly, $\mathcal{T}(\mathcal{T}'(\varphi))$ turns the $s = t$ in φ first to $\mathtt{B}(s \leftrightarrow t)$ and then to $((s \leftrightarrow t) = \top)$. But $(s = t) \leftrightarrow ((s \leftrightarrow t) = \top)$ is in FOLzg. □

Lemma 3.4.20. *For any axiom* φ *in Proposition 3.4.18 defining* $\mathcal{FOL}\mathsf{zg}$*,* $\mathcal{T}'(\varphi)$ *is provable in* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$*.*

Proof. The translations of the logical axioms are easily provable in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ since it is a normal Π-logic and, in particular it can do Boolean reasoning inside $\mathtt{B}$. For the rest, the only non-trivial axioms to be dealt with are (3.6) and (3.7). To derive the translation of (3.6) in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$, we now work in $\mathcal{L}\Pi\mathsf{zg}$. Let us first assume $p\mathsf{zg}$ in system. Then we have $p \wedge \forall p(\mathtt{B}p \to p) \wedge \exists p(p \wedge at(p))$. Instantiating $\exists p(p \wedge at(p))$ using x, we have $x \wedge at(x)$ that just abbreviates

$$x \wedge \langle \mathsf{z} \rangle x \wedge \forall r([\mathsf{z}](x \to r) \vee [\mathsf{z}](x \to \neg r)).$$

Instantiating $\forall r([\mathsf{z}](x \to r) \vee [\mathsf{z}](x \to \neg r))$ using p, we have $[\mathsf{z}](x \to p) \vee [\mathsf{z}](x \to \neg p)$. But the latter disjunct leads to contradiction since we have assumed z, which, according to Lemma 3.4.16, allows us to remove $[\mathsf{z}]$ and obtain $x \to \neg p$, contradicting the previously assumed $p\mathsf{zg}$ and $x \wedge at(x)$. Hence, we reject the second disjunct and derive $[\mathsf{z}](x \to p)$. Summing everything, we now have:

$$[\mathsf{z}](x \to p) \wedge \langle \mathsf{z} \rangle x \wedge \forall r([\mathsf{z}](x \to r) \vee [\mathsf{z}](x \to \neg r)).$$

Writing this without any abbreviation and using $\varphi_1 \to (\varphi_2 \to \varphi_3)$ being provably equivalent to $(\varphi_1 \wedge \varphi_2) \to \varphi_3$, we then have

$$\mathtt{B}((\mathsf{z} \wedge x) \to p) \wedge \widehat{\mathtt{B}}(\mathsf{z} \wedge x) \wedge \forall r(\mathtt{B}((\mathsf{z} \wedge x) \to r) \vee \mathtt{B}((\mathsf{z} \wedge x) \to \neg r)).$$

Then we can existentially quantify back to obtain

$$\varphi(p) := \exists q(\mathtt{B}(q \to p) \wedge \widehat{\mathtt{B}}q \wedge \forall r(\mathtt{B}(q \to r) \vee \mathtt{B}(q \to \neg r))),$$

as $\mathsf{z} \wedge x$ is a witness. The above process shows that $\vdash p\mathsf{zg} \to \varphi(p)$.

Now assume $\mathtt{B}(p \to \mathsf{zg}) \wedge \widehat{\mathtt{B}}p$ in system. Then clearly we can deduce $\widehat{\mathtt{B}}(p\mathsf{zg})$. Since we have just shown that $\vdash p\mathsf{zg} \to \varphi(p)$, we also obtain $\mathtt{B}(p\mathsf{zg} \to \varphi(p))$. Thus $\widehat{\mathtt{B}}(pzg) \to \widehat{\mathtt{B}}(\varphi(p))$ is provable. But $\varphi(p)$ is fully modalized. So by Lemma 3.4.15, $\widehat{\mathtt{B}}(\varphi(p)) \dashv\vdash \varphi(p)$. This means that $(\mathtt{B}(p \to \mathsf{zg}) \wedge \widehat{\mathtt{B}}p) \to \varphi(p)$ is provable, and hence, after universalization, $\vdash \forall p((\mathtt{B}(p \to \mathsf{zg}) \wedge \widehat{\mathtt{B}}p) \to \varphi(p))$.

Note that for any $s, t \in \mathcal{BL}\mathsf{zg}$, $\mathcal{T}(\mathtt{B}(s \to t)) = (s \leqslant t)$ and $\mathcal{T}(\widehat{\mathtt{B}}s) = (\neg s \neq \bot)$. The latter is easily seen to be provably equivalent to $(s \neq \top)$ in FOLzg. Thus, $\mathcal{T}(\forall p((\mathtt{B}(p \to \mathsf{zg}) \wedge \widehat{\mathtt{B}}p) \to \varphi(p)))$ is obviously provably equivalent to (3.6) in FOLzg. By Lemma 3.4.19, we are done in this case.

The translation of (3.7) can be derived in $\mathsf{KD4^{\forall}5\Pi zg}$ similarly. The key again is that once we assume z, $[\mathsf{z}]$ is an $\mathsf{S5}$ modality. $\square$

Lemma 3.4.21. *For any $\varphi \in \mathsf{FOLzg}$, $\mathcal{T}'(\varphi) \in \mathsf{KD4^{\forall}5\Pi zg}$.*

Proof. We show that for any derivation $\langle \varphi_1, \varphi_2, \cdots, \varphi_n \rangle$ of FOLzg, $\mathcal{T}'(\varphi_i)$ is a theorem of $\mathsf{KD4^{\forall}5\Pi zg}$ for all i from 1 to n by induction. For any i, if φ_i is an axiom in FOLzg, then by the previous lemma, $\mathcal{T}'(\varphi) \in \mathsf{KD4^{\forall}5\Pi zg}$. If φ_i is obtained by either modus ponens or universalization, notice that the same rule applies to the formulas after translation as the translation does not change the sentential form or the variables used. So $\mathcal{T}'(\varphi_i)$ can also be obtained from the rules. $\square$

Now we can show the completeness of $\mathsf{KD4^{\forall}5\Pi zg}$ from the following two lemmas.

Lemma 3.4.22. *Any sentence in $\mathcal{L}\Pi\mathsf{zg}$ is equivalent in $\mathsf{KD4^{\forall}5\Pi zg}$ to a sentence in the following form:*

$$(\overline{\mathsf{z}} \wedge \alpha) \vee (\mathsf{zg} \wedge \beta) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \gamma),$$

where α, β, γ are all sentences in $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$.

Lemma 3.4.23. *For any zg-algebra A, there is a complete proper filter algebra $\mathcal{B}$ such that $\mathcal{B}/F_{\mathcal{B}}$ is elementarily equivalent to A (satisfying the same formulas in $\mathcal{FOL}\mathsf{zg}$).*

Moreover, $\overline{z_{\mathcal{B}}}$ is not $\bot_{\mathcal{B}}$, $z_{\mathcal{B}}g_{\mathcal{B}}$ is $\bot_{\mathcal{B}}$ if and only if $z_A g_A$ is $\bot_A$, and $z_{\mathcal{B}}\overline{g_{\mathcal{B}}}$ is $\bot_{\mathcal{B}}$ if and only if $z_A\overline{g_A}$ is $\bot_A$.

Theorem 3.4.24. *Any sentence in $\mathcal{L}\Pi\mathsf{zg}$ that is valid on all complete proper filter algebras is also in $\mathsf{KD4^{\forall}5\Pi zg}$.*

Proof. By Lemma 3.4.22, we can assume that we are dealing with a sentence $\varphi = (\overline{\mathsf{z}} \wedge \alpha) \vee (\mathsf{zg} \wedge \beta) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \gamma)$ where α, β, γ are all sentences in $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$. To proceed, assume that φ is valid on all complete proper filter algebras.

By Lemma 3.4.12, for any complete proper filter algebra $\mathcal{B}$ and any valuation θ on $\mathcal{B}$, $\tilde{\theta}(\alpha), \tilde{\theta}(\beta), \tilde{\theta}(\gamma)$ are either $\top$ or $\bot$. Also, $\overline{z}, zg$, and $z\overline{g}$ disjointly decompose $\mathcal{B}$. This means that once $\tilde{\theta}(\overline{\mathsf{z}}) \neq \bot$, $\tilde{\theta}(\alpha)$ must be $\top$ since otherwise $\tilde{\theta}(\varphi)$ will lose a non-trivial $\tilde{\theta}(\overline{\mathsf{z}})$ and be strictly below $\top$. Similarly, $\tilde{\theta}(\mathsf{zg}) \neq \bot$ implies $\tilde{\theta}(\beta) = \top$, and $\tilde{\theta}(\mathsf{z}\overline{\mathsf{g}}) \neq \bot$ implies $\tilde{\theta}(\gamma) = \top$.

Focus on α first. Now we know that α is valid on all complete proper filter algebra $\mathcal{B}$ where $\overline{z_{\mathcal{B}}} \neq \bot$ since $\overline{\mathsf{z}}$ always evaluates to $\overline{z_{\mathcal{B}}}$. Then for any zg-algebra A, $A \vDash \mathcal{T}(\alpha)$ since by Lemma 3.4.23, for any zg-algebra A there is a complete proper filter algebra $\mathcal{B}$ with $\overline{z_{\mathcal{B}}} \neq \bot$ such that $\mathcal{B}/F_{\mathcal{B}}$ is elementarily equivalent to A, and by Lemma 3.4.12, $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\alpha)$ iff $\mathcal{B} \vDash \alpha$. Now that $\mathcal{T}(\alpha)$ is valid on all zg-algebras, by Proposition 3.4.18, $\mathcal{T}(\alpha) \in \mathcal{FOL}\mathsf{zg}$. By Lemma 3.4.21 and 3.4.19, then, $\vdash \alpha$, and thus $\vdash \mathsf{z} \to \alpha$.

The method applies to the cases with β and γ too. Take β as an example. Now β must be valid on all complete proper filter algebra $\mathcal{B}$ where $z_{\mathcal{B}}g_{\mathcal{B}}$ is non-trivial. If $\mathcal{T}(\beta)$ is refutable by some zg-algebra A with $z_A g_A$ non-trivial, then β will also be refutable by some complete proper filter algebra $\mathcal{B}$ with $z_{\mathcal{B}}g_{\mathcal{B}}$ non-trivial, using lemma 3.4.23 and 3.4.12. This means that $\mathcal{T}(\beta)$ is valid on any zg-algebra A with $z_A g_A$ non-trivial. In other words, $\neg(\mathsf{zg} = \bot) \to \mathcal{T}(\beta)$ is valid and hence provable in $\mathcal{FOL}\mathsf{zg}$. Translating back to $\mathcal{L}\Pi\mathsf{zg}$, $\vdash \widehat{\mathsf{B}}(\mathsf{zg}) \to \beta$. But $\vdash \mathsf{zg} \to \widehat{\mathsf{B}}(\mathsf{zg})$ since $\mathsf{z} \dashv\vdash \forall p(p \to \widehat{\mathsf{B}}p)$, and then we can instantiate this with zg. So $\vdash \mathsf{zg} \to \beta$. In the same fashion, it can be shown that $\vdash \mathsf{z}\overline{\mathsf{g}} \to \gamma$.

Taking stock, we have shown that $\vdash (\overline{\mathsf{z}} \to \alpha)$, $\vdash (\mathsf{zg} \to \beta)$, and $\vdash (\mathsf{z}\overline{\mathsf{g}} \to \gamma)$. But obviously $\vdash \overline{\mathsf{z}} \vee \mathsf{zg} \vee \mathsf{z}\overline{\mathsf{g}}$ since trivially $\vdash \mathsf{g} \to \mathsf{z}$. With some basic Boolean manipulation, $\vdash (\overline{\mathsf{z}} \wedge \alpha) \vee (\mathsf{zg} \wedge \beta) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \gamma)$. □

Theorem 3.4.1 follows since $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ is a conservative extension of $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, and a formula is valid if and only if its universal closure, which is a sentence, is valid.

3.4.3 Syntactical Reduction

In this section, we prove Lemma 3.4.22. The main idea is relativizing formulas by $\mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}$, and $\overline{\mathsf{z}}$. We also use ideas from the quantifier elimination for $\mathsf{S5}\Pi$. See the appendix of [91] and the original [54] for more about the quantifier elimination for $\mathsf{S5}\Pi$.

An important addition to the $\mathsf{S5}\Pi$ case is the following lemma, where the intuition is that if $\neg\widehat{\mathsf{B}}b$ is true, then b is unimportant and does not affect the semantic value of $\varphi(p)$ where $\varphi \in \mathcal{L}\mathsf{B}\Pi\mathsf{zg}$, when we replace p by either $p \vee b$ or $p \wedge \neg b$.

Lemma 3.4.25. *For any formula $\varphi(p) \in \mathcal{L}\mathsf{B}\Pi\mathsf{zg}$ where p is free and any propositional variable b not occurring in φ, the following are provable in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$:*

$$\mathsf{B}\neg b \to (\varphi(p) \leftrightarrow \varphi(p \vee b)), \quad \mathsf{B}\neg b \to (\varphi(p) \leftrightarrow \varphi(p \wedge \neg b)).$$

Proof. We only show the $p \vee b$ case here. The other case can be shown similarly. First, a simple induction shows that for any Boolean formula $\beta(p)$, $(\beta(p) \vee b) \dashv\vdash (\beta(p \vee b) \vee b)$.

The only non-trivial case is negation. If $(\beta(p) \vee b) \dashv\vdash (\beta(p \vee b) \vee b)$, then $\neg(\beta(p) \vee b) \dashv\vdash \neg(\beta(p \vee b) \vee b)$. Pushing $\neg$ inside, $(\neg\beta(p) \wedge \neg b) \dashv\vdash (\neg\beta(p \vee b) \wedge \neg b)$. Joining a b on both side and performing some Boolean manipulation, we see that $(\neg\beta(p) \vee b) \dashv\vdash (\neg\beta(p \vee b) \vee b)$.

Using the normality of $\mathtt{B}$, it is not hard to see that $\vdash \mathtt{B}\neg b \to (\mathtt{B}\varphi \leftrightarrow \mathtt{B}(\varphi \vee b))$: assuming $\mathtt{B}\neg b$, $\mathtt{B}(\varphi \vee b)$ implies $\mathtt{B}((\varphi \vee b) \wedge \neg b)$, which then implies $\mathtt{B}(\varphi \wedge \neg b)$ and hence also $\mathtt{B}\varphi$. The other direction is trivial. Applying this to the case of φ being $\beta(p)$, we see then that $\vdash \mathtt{B}\neg b \to (\mathtt{B}\beta(p) \leftrightarrow \mathtt{B}(\beta(p) \vee b))$. Using the claim we proved in the last paragraph, $\vdash \mathtt{B}\neg b \to (\mathtt{B}\beta(p) \leftrightarrow \mathtt{B}\beta(p \vee b))$. This forms the basis of a trivial induction on the formulas in $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$. □

Since our strategy is to relativize by zg, $\mathsf{z}\overline{\mathsf{g}}$, and $\overline{\mathsf{z}}$, we first introduce the necessary definitions and lemmas required for separating the zg and $\mathsf{z}\overline{\mathsf{g}}$ part. Then we move to the necessary preparation for separating the $\overline{\mathsf{z}}$ part. Then we show a simple lemma on when we can push $\exists p$ over conjunctions. After that, we combine everything together.

For the zg and $\mathsf{z}\overline{\mathsf{g}}$ part, we need the following abbreviations:

$$M_i\varphi := \exists p_1 \cdots p_i \left(\bigwedge_{1 \le i < j \le n} [\mathsf{z}](p_i \to \neg p_j) \wedge \bigwedge_{1 \le i \le n} at(p_i) \wedge \bigwedge_{1 \le i \le n} [\mathsf{z}](p_i \to \varphi) \right) \quad (\text{for } i \ge 1)\ ,$$
$$M_0\varphi := \top, \quad Q_i\varphi := M_i\varphi \wedge \neg M_{i+1}\varphi \ (\text{for } i \in \mathbb{N}), \quad N\varphi := \langle \mathsf{z} \rangle(\overline{g} \wedge \varphi).$$

As usual, the auxiliary variables are chosen to be distinct and unused in φ. Here M_i, Q_i, and N come from the quantifier elimination proof of [54], which requires the modality to be $\mathsf{S5}$. But by Lemma 3.4.16, the modality $[\mathsf{z}]$ used here is really $\mathsf{S5}$ if z is also present. Even if z is not assumed, $\mathsf{KD4^{\forall}5\Pi zg}$ still proves many intuitive principles. We summarize the results in the following lemma.

Lemma 3.4.26. $\mathsf{KD4^{\forall}5\Pi zg}$ *proves all instances of the following schemas. In the last group, q is required to be not free in φ and $\pm q$ can be either q or $\neg q$. Moreover, when $\pm q$ is taken to be q, m in the first four formulas is not* 0*, and when $\pm q$ is taken to be $\neg q$, n is not* 0*.*

$$\langle \mathsf{z} \rangle(\mathsf{g} \wedge \varphi) \leftrightarrow M_1\varphi \qquad M_i\varphi \leftrightarrow M_i(\mathsf{g} \wedge \varphi) \qquad Q_i\varphi \leftrightarrow Q_i(\mathsf{g} \wedge \varphi)$$

$$\begin{aligned}
&\exists q[Q_m(\varphi \wedge q) \wedge Q_n(\varphi \wedge \overline{q})] \leftrightarrow Q_{m+n}\varphi && \exists q[N(\varphi \wedge q) \wedge N(\varphi \wedge \overline{q})] \leftrightarrow N\varphi \\
&\exists q[Q_m(\varphi \wedge q) \wedge M_n(\varphi \wedge \overline{q})] \leftrightarrow M_{m+n}\varphi && \exists q[N(\varphi \wedge q) \wedge \neg N(\varphi \wedge \overline{q})] \leftrightarrow N\varphi \\
&\exists q[M_m(\varphi \wedge q) \wedge Q_n(\varphi \wedge \overline{q})] \leftrightarrow M_{m+n}\varphi && \exists q[\neg N(\varphi \wedge q) \wedge N(\varphi \wedge \overline{q})] \leftrightarrow N\varphi \\
&\exists q[M_m(\varphi \wedge q) \wedge M_n(\varphi \wedge \overline{q})] \leftrightarrow M_{m+n}\varphi && \exists q[\neg N(\varphi \wedge q) \wedge \neg N(\varphi \wedge \overline{q})] \leftrightarrow \neg N\varphi \\
&\bigvee_{i=0}^{n}(M_i(\varphi \wedge \psi) \wedge M_{n-i}(\varphi \wedge \overline{\psi})) \leftrightarrow M_n\varphi && ((N(\varphi \wedge \psi) \vee (N(\varphi \wedge \overline{\psi}))) \leftrightarrow N\varphi \\
&\bigvee_{i=0}^{n}(Q_i(\varphi \wedge \psi) \wedge Q_{n-i}(\varphi \wedge \overline{\psi})) \leftrightarrow Q_n\varphi && ((\neg N(\varphi \wedge \psi) \wedge \neg(N(\varphi \wedge \overline{\psi}))) \leftrightarrow \neg N\varphi
\end{aligned}$$

$$\exists q[\mathsf{zg} \wedge \varphi \wedge \pm q \wedge Q_m(\varphi \wedge q) \wedge Q_n(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{zg} \wedge \varphi \wedge Q_{m+n}\varphi)$$
$$\exists q[\mathsf{zg} \wedge \varphi \wedge \pm q \wedge M_m(\varphi \wedge q) \wedge Q_n(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{zg} \wedge \varphi \wedge M_{m+n}\varphi)$$
$$\exists q[\mathsf{zg} \wedge \varphi \wedge \pm q \wedge Q_m(\varphi \wedge q) \wedge M_n(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{zg} \wedge \varphi \wedge M_{m+n}\varphi)$$
$$\exists q[\mathsf{zg} \wedge \varphi \wedge \pm q \wedge M_m(\varphi \wedge q) \wedge M_n(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{zg} \wedge \varphi \wedge M_{m+n}\varphi)$$
$$\exists q[\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge \pm q \wedge N(\varphi \wedge q) \wedge N(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge N\varphi)$$
$$\exists q[\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge \pm q \wedge \neg N(\varphi \wedge q) \wedge N(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge N\varphi)$$
$$\exists q[\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge \pm q \wedge N(\varphi \wedge q) \wedge \neg N(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge N\varphi)$$
$$\exists q[\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge \pm q \wedge \neg N(\varphi \wedge q) \wedge \neg N(\varphi \wedge \overline{q})] \leftrightarrow (\mathsf{z}\overline{\mathsf{g}} \wedge \varphi \wedge \neg N\varphi)$$

Proof. Syntactical proofs of them are not interesting, and here we only briefly explain why they are valid, from which syntactical proofs can be extracted straightforwardly. Using Proposition 3.4.7, $M_i\varphi$ says that $\mathsf{z} \wedge \varphi$ contains at least i atoms. More precisely, for any complete proper filter algebra $\mathcal{B}$ and valuation θ on $\mathcal{B}$, $\tilde{\theta}(M_i\varphi) = \top$ if and only if $z_{\mathcal{B}} \wedge \tilde{\theta}(\varphi)$ contains at least i atoms, and otherwise $\tilde{\theta}(M_i\varphi) = \bot$. Similarly, $Q_i\varphi$ says that $\mathsf{z} \wedge \varphi$ contains exactly i atoms, and $N\varphi$ says that $\mathsf{z} \wedge \varphi$ contains an atomless part.

Note that $\tilde{\theta}(\mathsf{g}) = g_{\mathcal{B}}$ is the join of all atoms under $z_{\mathcal{B}}$ and hence atomic. So if $g_{\mathcal{B}} \wedge z_{\mathcal{B}} \wedge \tilde{\theta}(\varphi)$ is non-trivial, then $z_{\mathcal{B}} \wedge \tilde{\theta}(\varphi)$ must contain an atom, and the numbers of atoms below $z_{\mathcal{B}} \wedge \tilde{\theta}(\varphi)$ and $g_{\mathcal{B}} \wedge z_{\mathcal{B}} \wedge \tilde{\theta}(\varphi)$ respectively are the same. These two observations show the validity of the first group of three formulas.

The left six formulas in the second group are simply counting principles, and the right six formulas state obvious properties of atomless elements. Hence they are all valid. Note that they only consider the situation under z.

For the last group, note that by Boolean reasoning, $(\mathsf{z} \wedge \alpha) \leftrightarrow (\mathsf{z} \wedge \beta) \dashv\vdash \mathsf{z} \to (\alpha \leftrightarrow \beta)$. By Proposition 3.4.16, to prove the last group of formulas in $\mathsf{KD4}^{\forall}\mathsf{5\Pi zg}$, we only need to translate their proofs in $\mathsf{S5\Pi}$ to proofs in $\mathsf{KD4}^{\forall}\mathsf{5\Pi zg}$ by replacing the $\mathsf{S5}$ modality $\Box$ by $[\mathsf{z}]$. $\Box$

For the $\overline{\mathsf{z}}$ part, the only extra definition we need is the following.

Definition 3.4.27. Define the following abbreviations:

$$\langle\overline{\mathsf{z}}\rangle\varphi := \widehat{\mathsf{B}}(\overline{\mathsf{z}} \wedge \varphi), \quad [\overline{\mathsf{z}}]\varphi := \mathsf{B}(\overline{\mathsf{z}} \to \varphi).$$

Then define the following restricted version of $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$:

$$\mathcal{L}[\overline{\mathsf{z}}]\Pi : \varphi ::= [\overline{\mathsf{z}}]t \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \forall p\varphi$$

where $t \in \mathcal{BL}, p \in \mathsf{Prop}$.

Now we introduce the concept of a propositional variable being restricted. This helps us to distribute existential quantifiers over conjunctions in certain cases.

Definition 3.4.28. We say that p is *restricted* by a formula μ in φ just in case μ is substitutable for p in φ and $\vdash \forall p(\varphi(p) \leftrightarrow \varphi(p \wedge \mu))$.

Lemma 3.4.29. *$\exists p(\varphi \wedge \psi)$ is provably equivalent to $\exists p\varphi \wedge \exists p\psi$, if there are formulas μ, ν, such that*

- *p in φ is restricted by μ, p in ψ is restricted by ν, and*
- *$\neg(\mu \wedge \nu)$ is provable.*

Proof. One direction of the equivalence is trivial. For the other, the strategy is relativization. Suppose $\exists p\varphi(p) \wedge \exists p\psi(p)$ in system. Then we have $\varphi(p_1)$ and $\psi(p_2)$. By the assumption that p in φ is restricted by μ and that p in ψ is restricted by ν, we can derive $\varphi(p_1 \wedge \mu)$ and $\psi(p_2 \wedge \nu)$. Now we see that $\neg(\mu \wedge \nu)$ is provable. So, using Boolean reasoning and letting $\chi = (p_1 \wedge \mu) \vee (p_2 \wedge \nu)$, $\chi \wedge \mu \dashv\vdash p_1 \wedge \mu$ and $\chi \wedge \nu \dashv\vdash p_2 \wedge \nu$. Hence we now have a chain of provable equivalence: $\varphi(\chi) \dashv\vdash \varphi(\chi \wedge \mu) \dashv\vdash \varphi(p_1 \wedge \mu) \dashv\vdash \varphi(p_1)$. Similarly $\psi(\chi) \dashv\vdash \psi(p_2)$. Thus χ witness $\exists p(\varphi(p) \wedge \psi(p))$. $\square$

Now we start to combine everything together. A few extra notations are used. We fix an enumeration $\langle p_i \rangle_{i<|\mathsf{Prop}|}$ of Prop and write $\vec{p}$ or in general use vector notation for a finite subset of Prop. Then for $\vec{p} = \{p_{i_1}, p_{i_2}, \ldots, p_{i_n}\}$ with $i_1 < i_2 < \cdots < i_n$, $2^{\vec{p}}$ is the set of formulas in the form of $\pm p_{i_1} \wedge \pm p_{i_2} \wedge \cdots \wedge \pm p_{i_n}$. We call elements in $2^{\vec{p}}$ *cells*. And as usual, a conjunction of no formulas is $\top$, and a disjunction of no formulas is $\bot$.

Definition 3.4.30. A *state description* over $\vec{p}$ with *degree* $l \in \mathbb{N}$ is a conjunction of the following components:

- (choice of zg) one of $\overline{\mathsf{z}}, \mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}$,
- (propositional truth) one cell $c \in 2^{\vec{p}}$,
- ($\overline{\mathsf{z}}$ part) one $\mathcal{L}[\overline{\mathsf{z}}]\Pi$ formula δ whose free variables are all in $\vec{p}$,
- (zg part) for each cell $c \in 2^{\vec{p}}$, a formula $M_l c$ or $Q_i c$ for some $0 \leq i < l$,
- ($\mathsf{z}\overline{\mathsf{g}}$ part) for each cell $c \in 2^{\vec{p}}$, a formula Nc or $\neg Nc$.

We call the first two parts the *propositional part* and the rest the *modal part* of a state description. A *partial state description* over $\vec{p}$ of degree l is a formula missing one or more components above. If the only missing part is the propositional part, we also call it a *modal state description*.

Lemma 3.4.31. *Every free variable in a $\overline{\mathsf{z}}$ (respectively zg, $\mathsf{z}\overline{\mathsf{g}}$) part is restricted by $\overline{\mathsf{z}}$ (respectively zg, $\mathsf{z}\overline{\mathsf{g}}$).*

Proof. For the $\overline{\mathsf{z}}$ part, note that in any formula $\varphi \in \mathcal{L}[\overline{\mathsf{z}}]\Pi$, every free variable appears in a Boolean term which is then in a conjunction with $\overline{\mathsf{z}}$. We can distribute this $\overline{\mathsf{z}}$ into the Boolean term, assuming that Boolean term is already in negation normal form. Then every free variable appears either in the form of $\overline{\mathsf{z}} \wedge p$ or $\overline{\mathsf{z}} \wedge \neg p$. But $\overline{\mathsf{z}} \wedge p \dashv\vdash \overline{\mathsf{z}} \wedge (p \wedge \overline{\mathsf{z}})$ and $\overline{\mathsf{z}} \wedge \neg p \dashv\vdash \overline{\mathsf{z}} \wedge \neg(p \wedge \overline{\mathsf{z}})$.

For the zg part, take $M_i c$ for some $c \in 2^{\vec{p}}$ and $p \in \vec{p}$ for example. First note that by definition of M_i, c in $M_i c$ appears in a Boolean term directly following $[\mathsf{z}]$. So using a similar proof as in the previous case, p in $M_i c$ is restricted to z. Then note that in Lemma 3.4.26, $M_i c$ is provably equivalent to $M_i(\mathsf{g} \wedge c)$. So obviously p in $M_i c$ is restricted to g as well. Finally, it is not hard to see that in general if p in $\varphi(p)$ is restricted to both μ and ν, then it is also restricted to $\mu \wedge \nu$. Indeed, $\varphi(p)$ will first be equivalent to $\varphi(p \wedge \mu)$ and then to $\varphi((p \wedge \mu) \wedge \nu)$, but this is equivalent to $\varphi(p \wedge (\mu \wedge \nu))$. Thus p in $M_i c$ is restricted to zg.

The case for the $\mathsf{z\overline{g}}$ part is similar. □

Lemma 3.4.32. *For every partial state description φ over $\vec{p}$ with degree l, and for every finite set of variables $\vec{p}' \supseteq \vec{p}$ and every natural number $l' \geq l$, φ is provably equivalent in* $\mathsf{KD4^{\forall}5\Pi zg}$ *to a disjunction of state descriptions over $\vec{p}'$ with degree l'.*

Proof. Let φ, $\vec{p}$, l, $\vec{p}'$, and l' be arbitrarily given as above. Without loss of generality, we assume that $\vec{p}' = \vec{p} \cup \{p'\}$ since we can repeat the following process many times if necessary. Now, let ψ be the conjunction of the following $\psi_1, \psi_2, \psi_3, \psi_4$, and ψ_5:

- If φ has a choice of zg, let ψ_1 be this choice. Otherwise let ψ_1 be $\overline{\mathsf{z}} \vee \mathsf{zg} \vee \mathsf{z\overline{g}}$.
- If φ has a propositional truth $c \in 2^{\vec{p}}$, let ψ_2 be $(c \wedge p') \vee (c \wedge \overline{p'})$. Otherwise, let ψ_2 be $\bigvee 2^{\vec{p}'}$.
- If φ has a $\overline{\mathsf{z}}$ part, let ψ_3 be the $\overline{\mathsf{z}}$ part. Otherwise, let ψ_3 be $[\overline{\mathsf{z}}]\top$.
- If φ has no zg part, let ψ_4 be the disjunctive normal form of $\bigwedge_{c \in 2^{\vec{p}'}}(Q_0 c \vee Q_1 c \vee \cdots \vee Q_{l'-1} c \vee M_{l'} c)$ with $Q_i c$ and $M_{l'} c$ seen as atomic formulas.

  Otherwise, say φ has a zg part $\delta_{\mathsf{zg}} = \bigwedge_{c \in 2^{\vec{p}}} X_c c$ with X_c being either Q_i for some $i < l$ or M_l. If $\vec{p}'$ is just $\vec{p}$ (that is, $p' \in \vec{p}$), let ψ_4 be δ_{zg}. If $p' \notin \vec{p}$, construct ψ_4 by first replacing each $M_l c$ in δ_{zg} with $\bigvee_{i=0}^{l}(M_i(cp') \wedge M_{l-i}(c\overline{p'}))$ and each $Q_j c$ with $\bigvee_{i=0}^{j}(Q_i(cp') \wedge Q_{j-i}(c\overline{p'}))$. Then replace each $M_i c$ with c now being in $2^{\vec{p}'}$ and with $i < l'$ by $(Q_i c \vee Q_{i+1} c \vee \cdots \vee Q_{l'-1} c \vee M_{l'} c)$. Finally, take its disjunctive normal form with $Q_i c$ and $M_{l'} c$ for all $c \in 2^{\vec{p}'}$ seen as atomic formulas.
- If φ has no $\mathsf{z\overline{g}}$ part, let ψ_5 be the disjunctive normal form of $\bigwedge_{c \in 2^{\vec{p}'}}(Nc \vee \neg Nc)$ with Nc and $\neg Nc$ seen as atomic formulas.

  Otherwise, say φ has a $\mathsf{z\overline{g}}$ part $\delta_{\mathsf{z\overline{g}}} = \bigwedge_{c \in 2^{\vec{p}}} X_c c$ where X_c is either N or $\neg N$. If $\vec{p} = \vec{p}'$ (that is, $p' \in \vec{p}$), let ψ_5 be $\delta_{\mathsf{z\overline{g}}}$. If not, let ψ_5 be constructed by first replacing each Nc in $\delta_{\mathsf{z\overline{g}}}$ by $(N(cp') \wedge N(c\overline{p'})) \vee (N(cp') \wedge \neg N(c\overline{p'})) \vee (\neg N(cp') \wedge N(c\overline{p'}))$ and replacing each $\neg Nc$ in $\delta_{\mathsf{z\overline{g}}}$ by $\neg N(cp') \wedge \neg N(c\overline{p'})$. Then take its disjunctive normal form with Nc and $\neg Nc$ as atomic formulas.

Now it should not be too hard to see that ψ is provably equivalent to φ as each of ψ_i is provably equivalent to the respective part of φ if it exists, or to $\top$ otherwise. In particular, to see that $\psi_4 \dashv\vdash \delta_{\mathsf{zg}}$ and $\psi_5 \dashv\vdash \delta_{\mathsf{z\overline{g}}}$, use Lemma 3.4.26. Then, let φ' be the result of distributing the outermost conjunction over ψ_1 through ψ_5 over the disjunctions in them. Clearly φ' is now a disjunction of state descriptions over $\vec{p}'$ with degree l', and φ' is provably equivalent to φ. $\square$

Lemma 3.4.33. *Any formula $\varphi \in \mathcal{L}\Pi\mathsf{zg}$ with $\vec{p}$ being its set of free variables is provably equivalent in $\mathsf{KD4}^{\forall}5\Pi\mathsf{zg}$ to a disjunction of state descriptions over $\vec{p}$.*

Proof. By induction. Since we are only after provable equivalence, we can pretend that our language has $\vee$, $\widehat{\mathsf{B}}$, and $\exists$ as primitives. For the base cases, note that:

- every propositional variable p in Prop is a partial state description in $\{p\}$ with degree 0;
- $\top$ is a partial state description in $\{\}$ with degree 0;
- since $\mathsf{z} \dashv\vdash (\mathsf{zg} \vee \mathsf{z}\overline{\mathsf{g}})$, z is equivalent to a disjunction of two partial state descriptions over $\{\}$ with degree 0;
- since $\vdash \mathsf{g} \to \mathsf{z}$ and $\mathsf{g} \dashv\vdash \mathsf{zg}$, g is also provably equivalent to a disjunction of partial state descriptions over $\{\}$ with degree 0.

Hence we can apply the previous lemma to clear the base cases.

Now suppose $\varphi = \varphi_1 \vee \varphi_2$, and let $\vec{p}_1$ and $\vec{p}_2$ be the set of free variables in φ_1 and φ_2, respectively. Then $\vec{p} = \vec{p}_1 \cup \vec{p}_2$ is the set of free variables of φ. By the induction hypothesis, there is a disjunction ψ_1 of state descriptions over $\vec{p}_1$ with some degree l_1 provably equivalent to φ_1 and a disjunction ψ_2 of state descriptions over $\vec{p}_2$ with some degree l_2 provably equivalent to φ_2. Without loss of generality, we can assume that $l_1 \geq l_2$. Now we first use the previous Lemma 3.4.32 to turn ψ_1 and ψ_2 into disjunctions of state descriptions over $\vec{p}$ with degree l_1 and obtain α and β. Then $\alpha \vee \beta$ is the formula we need in this case.

For the negation case, suppose $\varphi = \neg\psi$ with $\vec{p}$ being the set of free variables in ψ. Then $\vec{p}$ is also the set of free variables in φ. Using the induction hypothesis, let α be a disjunction of state descriptions over $\vec{p}$ with some degree l that is provably equivalent to ψ. Then using Lemma 3.4.32, let β be a disjunction of state descriptions over $\vec{p}$ with degree l that is provably equivalent to $\top$. Then let γ be the disjunction of the state descriptions over $\vec{p}$ with degree l that are in β but not in α. Then γ is a disjunction of state descriptions over $\vec{p}$ with degree l that is provably equivalent to φ.

Now suppose $\varphi = \widehat{\mathsf{B}}\psi$ with $\vec{p}$ being the set of free variables in ψ and hence also φ. By the induction hypothesis, ψ is provably equivalent to a disjunction of state descriptions $\bigvee_{i \in I} \psi_i$. Then $\widehat{\mathsf{B}}\psi \dashv\vdash \widehat{\mathsf{B}} \bigvee_{i \in I} \psi_i \dashv\vdash \bigvee_{i \in I} \widehat{\mathsf{B}}\psi_i$. Hence, we only need to show that for each state description ψ_i, $\widehat{\mathsf{B}}\psi_i$ is equivalent to a partial state description, which can then be turned to a disjunction of state descriptions. Let $\psi_i = b \wedge c \wedge d$ so that $b \in \{\overline{\mathsf{z}}, \mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}\}, c \in 2^{\vec{p}}$, and d is the modal part of ψ_i. By Lemma 3.4.15, $\psi_i \dashv\vdash (b \wedge c \wedge \mathsf{B}d)$ since d is fully modalized. Then it is a standard exercise of modal logic to show that $\widehat{\mathsf{B}}\psi_i \dashv\vdash \widehat{\mathsf{B}}(b \wedge c) \wedge \mathsf{B}d$, which is then provably equivalent to $\widehat{\mathsf{B}}(b \wedge c) \wedge d$. Now there are three cases:

- if $b = \overline{\mathsf{z}}$, then $\widehat{\mathsf{B}}(b \wedge c)$ is just $\langle\overline{\mathsf{z}}\rangle c$. But then $\langle\overline{\mathsf{z}}\rangle c \wedge d$ is a partial state description, as $\langle\overline{\mathsf{z}}\rangle c$ is a $\mathcal{L}[\overline{\mathsf{z}}]\Pi$ formula and can be included in the $\overline{\mathsf{z}}$ part.
- if $b = \mathsf{zg}$, then $\widehat{\mathsf{B}}(b \wedge c) \dashv\vdash \langle\mathsf{z}\rangle(\mathsf{g} \wedge c)$. But $\langle\mathsf{z}\rangle(g \wedge c) \dashv\vdash M_1 c$, and $M_1 c \wedge d$ can be turned into a partial state description, as there is a $Q_i c/M_l c$ formula in d, and $M_1 c$ can be merged in to that formula, resulting in $\perp$ or the original d.
- if $b = \mathsf{z}\overline{\mathsf{g}}$, then $\widehat{\mathsf{B}}(b \wedge c) \dashv\vdash \langle\mathsf{z}\rangle(\overline{\mathsf{g}} \wedge c)$. But $\langle\mathsf{z}\rangle(\overline{g} \wedge c) \dashv\vdash Nc$ and hence can be merged into d.

In sum, $\widehat{\mathsf{B}}\psi_i$ is provably equivalent to a partial state description missing a choice of a zg part and a propositional truth part.

For $\exists$, like $\widehat{\mathsf{B}}$, we only need to show that $\exists q\psi_i$ is provably equivalent to a partial state description over $\vec{p}\setminus\{q\}$ where ψ_i is a state description over $\vec{p}$. The case where $q \notin \vec{p}$ is trivial so we assume here that $q \in \vec{p}$. Let ψ_i be $b \wedge c \wedge \delta_{\overline{\mathsf{z}}} \wedge \delta_{\mathsf{zg}} \wedge \delta_{\mathsf{z}\overline{\mathsf{g}}}$, where $b \in \{\overline{\mathsf{z}}, \mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}\}$, $c \in 2^{\vec{p}}$, $\delta_{\overline{\mathsf{z}}}$ is the $\overline{\mathsf{z}}$ part of ψ_i (a $\mathcal{L}[\overline{\mathsf{z}}]\Pi$ formula), δ_{zg} is the zg part, and $\delta_{\mathsf{z}\overline{\mathsf{g}}}$ is the $\mathsf{z}\overline{\mathsf{g}}$ part. As noted in Lemma 3.4.31, $\delta_{\overline{\mathsf{z}}}, \delta_{\mathsf{zg}}$, and $\delta_{\mathsf{z}\overline{\mathsf{g}}}$ are restricted by $\overline{\mathsf{z}}$, zg, and $\mathsf{z}\overline{\mathsf{g}}$ respectively. Then by repeated use of Lemma 3.4.29, we have the following cases:

- If $b = \overline{\mathsf{z}}$, then $\exists q\psi_i \dashv\vdash \exists q(\overline{\mathsf{z}} \wedge c \wedge \delta_{\overline{\mathsf{z}}}) \wedge \exists q\delta_{\mathsf{zg}} \wedge \exists q\delta_{\mathsf{z}\overline{\mathsf{g}}}$.
- If $b = \mathsf{zg}$, then $\exists q\psi_i \dashv\vdash \exists q\delta_{\overline{\mathsf{z}}} \wedge \exists q(\mathsf{zg} \wedge c \wedge \delta_{\mathsf{zg}}) \wedge \exists q\delta_{\mathsf{z}\overline{\mathsf{g}}}$.
- If $b = \mathsf{z}\overline{\mathsf{g}}$, then $\exists q\psi_i \dashv\vdash \exists q\delta_{\overline{\mathsf{z}}} \wedge \exists q\delta_{\mathsf{zg}} \wedge \exists q(\mathsf{z}\overline{\mathsf{g}} \wedge c \wedge \delta_{\mathsf{z}\overline{\mathsf{g}}})$.

Hence what remains to be shown is that in each of these three cases, the three conjuncts on the right-hand side of the $\dashv\vdash$ claim are provably equivalent to a $\overline{\mathsf{z}}$ part, a zg part, and a $\mathsf{z}\overline{\mathsf{g}}$ part, possibly with a corresponding choice of zg and a propositional truth c, respectively.

First consider the two possibilities $\exists q\delta_{\overline{\mathsf{z}}}$ and $\exists q(\overline{\mathsf{z}} \wedge c \wedge \delta_{\overline{\mathsf{z}}})$. We need to show that they are provably equivalent to some $\overline{\mathsf{z}}$ part. Now $\exists q\delta_{\overline{\mathsf{z}}}$ is already a $\mathcal{L}[\overline{\mathsf{z}}]\Pi$ formula and thus a $\overline{\mathsf{z}}$ part, so there is nothing further to show. For $\exists q(\overline{\mathsf{z}} \wedge c \wedge \delta_{\overline{\mathsf{z}}})$, depending on whether q appears in c positively or negatively, we have two cases (the f below is the result of excluding the literal of q in c).

- If q appears positively, we have $\exists q(\overline{\mathsf{z}} \wedge f \wedge q \wedge \delta_{\overline{\mathsf{z}}})$. This is provably equivalent to $\overline{\mathsf{z}} \wedge f \wedge \exists q\delta_{\overline{\mathsf{z}}}$. The direction from left to right is trivial. For the other direction, if q is not free in $\delta_{\overline{\mathsf{z}}}$, it is also trivial. So assume now that q is free in $\delta_{\overline{\mathsf{z}}}(q)$. First instantiate $\exists q\delta_{\overline{\mathsf{z}}}(q)$ with a fresh a and obtain $\delta_{\overline{\mathsf{z}}}(a)$. Now recall that $\overline{\mathsf{z}} \dashv\vdash \exists r(r \wedge \mathsf{B}\neg r)$. Since we already have $\overline{\mathsf{z}}$, we can now instantiate with a fresh propositional variable b and get $b \wedge \mathsf{B}\neg b$. By Lemma 3.4.25, we derive $\delta_{\overline{\mathsf{z}}}(a) \leftrightarrow \delta_{\overline{\mathsf{z}}}(a \vee b)$ (recall that $\delta_{\overline{\mathsf{z}}}(a)$ is in $\mathcal{L}\mathsf{B}\Pi\mathsf{zg}$), and hence also $\delta_{\overline{\mathsf{z}}}(a \vee b)$. But $\vdash b \to (a \vee b)$, so we also obtain $a \vee b$. Then $a \vee b$ witnesses $\exists q(q \wedge \delta_{\overline{\mathsf{z}}}(q))$. Summing up the process, we have shown that $\vdash (\overline{\mathsf{z}} \wedge \exists q\delta_{\overline{\mathsf{z}}}) \to \exists q(q \wedge \delta_{\overline{\mathsf{z}}})$. Then it is easy to see that $\vdash (\overline{\mathsf{z}} \wedge f \wedge \exists q\delta_{\overline{\mathsf{z}}}) \to \exists q(\overline{\mathsf{z}} \wedge f \wedge q \wedge \delta_{\overline{\mathsf{z}}})$ since q does not appear in $\overline{\mathsf{z}}$ and f.

- $\exists q(\overline{\mathsf{z}} \wedge f \wedge \neg q \wedge \delta_{\overline{\mathsf{z}}})$. This is very similar to the previous case. We only need to prove the direction from $\overline{\mathsf{z}} \wedge f \wedge \exists \delta_{\overline{\mathsf{z}}}$ to $\exists q(\overline{\mathsf{z}} \wedge f \wedge \neg q \wedge \delta_{\overline{\mathsf{z}}})$ and in fact only that $\vdash (\overline{\mathsf{z}} \wedge \exists q \delta_{\overline{\mathsf{z}}}) \to \exists q(\neg q \wedge \delta_{\overline{\mathsf{z}}})$. We can also assume that q is free in $\delta_{\overline{\mathsf{z}}}(q)$. Instantiating $\exists q \delta_{\overline{\mathsf{z}}}$ and $\exists r(r \wedge B\neg r)$ (equivalent to $\overline{\mathsf{z}}$) with fresh a and b, we get $\delta_{\overline{\mathsf{z}}}(a)$ and $b \wedge \mathtt{B}\neg b$. By Lemma 3.4.25, we get $\delta_{\overline{\mathsf{z}}}(a \wedge \neg b)$. Also, $\vdash b \to \neg(a \wedge \neg b)$. Hence $a \wedge \neg b$ witnesses $\exists q(\neg q \wedge \delta_{\overline{\mathsf{z}}}(q))$, and we are done in this case.

So, the two formulas involving $\delta_{\overline{\mathsf{z}}}$ are indeed provably equivalent to formulas that can serve as the $\overline{\mathsf{z}}$ part of some state description over $\vec{p} \setminus \{q\}$.

For the cases involving δ_{zg} and $\delta_{\mathsf{z}\overline{\mathsf{g}}}$, note that both of them are conjunctions of formulas that are restricted to one of $c \in 2^{\vec{p}}$. Considering this, we can push $\exists q$ further down, with results in the following cases where again f is the result of restricting c to literals using only things in $\vec{p} \setminus \{q\}$:

- $\exists q(Q_i/M_l(fq) \wedge Q_j/M_l(f\overline{q}))$,
- $\exists q(\mathsf{zg} \wedge f \wedge \pm q \wedge Q_i/M_l(fq) \wedge Q_j/M_l(f\overline{q}))$,
- $\exists q(\pm N(fq) \wedge \pm N(f\overline{q}))$,
- $\exists q(\mathsf{z}\overline{\mathsf{g}} \wedge f \wedge \pm q \wedge \pm N(fq) \wedge \pm N(f\overline{q}))$.

They are all addressed in Lemma 3.4.26. $\square$

Now Lemma 3.4.22 follows from the previous lemma. Too see this, observe first that the modal parts of any state description are $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$ formulas. Further, when there are no free variables, the propositional truth part will be $\top$ in any state description. So a state description over $\{\}$ can be seen as simply a conjunction of one of $\overline{\mathsf{z}}, \mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}$, and a $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$ formula. Then for any sentence φ in $\mathcal{L}\Pi$, since it has no free variables, it is provably equivalent in $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}$ to a disjunction $\bigvee_{i \in I}(a_i \wedge b_i)$ such that for all $i \in I$, $a_i \in \{\overline{\mathsf{z}}, \mathsf{zg}, \mathsf{z}\overline{\mathsf{g}}\}$ and $b_i \in \mathcal{L}\mathtt{B}\Pi\mathsf{zg}$. But then we only need to extract the a_i's according to what they are. Formally, letting $I_{\overline{\mathsf{z}}} = \{i \in I \mid a_i = \overline{\mathsf{z}}\}$, $I_{\mathsf{zg}} = \{i \in I \mid a_i = \mathsf{zg}\}$, and $I_{\mathsf{z}\overline{\mathsf{g}}} = \{i \in I \mid a_i = \mathsf{z}\overline{\mathsf{g}}\}$, $\varphi \dashv\vdash (\overline{\mathsf{z}} \wedge \bigvee_{i \in I_{\overline{\mathsf{z}}}} b_i) \vee (\mathsf{zg} \wedge \bigvee_{i \in I_{\mathsf{zg}}} b_i) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \bigvee_{i \in I_{\mathsf{z}\overline{\mathsf{g}}}} b_i)$. This formula is in the required form in Lemma 3.4.22.

3.4.4 Quotients of complete Boolean algebras

In this subsection we prove Lemma 3.4.23. The main idea is to show that every Boolean algebra is elementarily equivalent to a κ-field-of-sets for a large enough cardinal κ to be specified later and then invoke a theorem saying that every κ-field-of-sets can be realized as a quotient of a complete Boolean algebra for large enough κ. To show that every Boolean algebra is elementarily equivalent to a κ-field-of-sets, we show that every Tarski invariant, which fully describes the first-order properties of Boolean algebras, is realized by a κ-field-of-sets.

First we define κ-regular subalgebras and κ-field-of-sets.

Definition 3.4.34. A Boolean algebra B is a *κ-regular subalgebra* of C if B is a subalgebra of C and for any $X \subseteq B$ with $|X| < \kappa$, whenever $\bigwedge_C X$ exists, it is also in B. We write $B \preccurlyeq_\kappa C$ for B being a κ-regular subalgebra of C.

we say an embedding $f : B \hookrightarrow C$ is *κ-regular* if for every $X \subseteq B$ such that $|X| < \kappa$, whenever $\bigwedge_C f[X]$ exists, $\bigwedge_B X$ also exists and $f(\bigwedge_B X) = \bigwedge_C f[X]$. Or equivalently, f is κ-regular if the image of f is a κ-regular subalgebra of C. We write $f : B \hookrightarrow_\kappa C$ when f is a κ-regular embedding from B to C and write $B \hookrightarrow_\kappa C$ when there is such a κ-regular embedding.

Definition 3.4.35. A Boolean algebra B is a *κ-field-of-sets* if there is a set D such that $B \hookrightarrow_\kappa \wp(D)$. Here $\wp(D)$ is the powerset algebra of D.

Proposition 3.4.36. *For any cardinal κ, the property of being a κ-field-of-sets is closed taking κ-regular subalgebras and is closed under taking arbitrary direct product.*

Proof. First, clearly, if $A \hookrightarrow_\kappa B$ and $B \hookrightarrow_\kappa C$, then $A \hookrightarrow_\kappa C$. Hence if $A \preccurlyeq_\kappa B$ and $B \hookrightarrow_\kappa \wp(D)$, then $A \hookrightarrow_\kappa \wp(D)$. Thus κ-field-of-sets is closed under taking κ-regular subalgebras.

Now consider an indexed set $\{B_i\}_{i \in I}$ of κ-field-of-sets with $f_i : B_i \hookrightarrow_\kappa \wp(D_i)$ for each $i \in I$. Then it is not hard to see that $\Pi_{i \in I} B_i \hookrightarrow_\kappa \Pi_{i \in I} \wp(D_i)$. This is because, letting π_i be the natural projection map from $\Pi_i B_i$ to B_i, for every $X \subseteq \Pi_i B_i$, $\bigwedge X = \langle \bigwedge \pi_i[X] \rangle_{i \in I}$, if any side of this equation exists. In other words, meets can be computed componentwisely. But $\Pi_{i \in I} \wp(D_i)$ is isomorphic to $\wp(\bigcup_{i \in I}(\{i\} \times D_i))$. Hence $\Pi_{i \in I} B_i$ is also a κ-field-of-sets. $\square$

Due to the fact that we need to deal with zg-algebras instead of just Boolean algebras, sometimes we need to make sure that the cokernels of the quotient maps we use have a trivial meet. We now introduce notations for this and prove two lemmas about it.

Definition 3.4.37. We say a surjective homomorphism $f : A \twoheadrightarrow B$ is *meet-trivial* if its cokernel $f^{-1}(\top_B)$ has a trivial meet: $\bigwedge f^{-1}(\top_B)$ is $\bot_A$. We write $f : A \circ\!\!\twoheadrightarrow B$ when $f : A \twoheadrightarrow B$ and f is meet-trivial, and we write $A \circ\!\!\twoheadrightarrow B$ when there is a meet-trivial surjective homomorphism from A to B. In the later case, we also say that B is a meet-trivial homomorphic image of A.

Proposition 3.4.38. *For any Boolean algebras A, B, and C, if $f : A \twoheadrightarrow B$ and $g : B \circ\!\!\twoheadrightarrow C$, then $(g \circ f) : A \circ\!\!\twoheadrightarrow C$.*

Proof. Let $f : A \twoheadrightarrow B$ and $g : B \circ\!\!\twoheadrightarrow C$ be given. To show that $g \circ f : A \circ\!\!\twoheadrightarrow C$, by definition, we only need to show that $\bigwedge F = \bot_A$ where $F = (g \circ f)^{-1}(\top_C)$. Suppose not and let a be a non-trivial lower bound of F in A. Then first we can show that $f(a) \neq \bot_B$ since if otherwise $f(a) = \bot_B$, then $f(\neg a) = \top_B$, meaning that $g(f(\neg a)) = \top_C$ and that $\neg a \in F$, which obviously contradicts the assumption that a is below everything, in particular $\neg a$, in F and that $a \neq \bot_A$. Since f is a homomorphism, $f(a)$ is a lower bound of $f[F]$. Then we only need to note that $g^{-1}(\top_C) = f[F]$, and hence $f(a)$ is a non-trivial lower bound of $g^{-1}(\top_C)$, contradicting $g : B \circ\!\!\twoheadrightarrow C$. $\square$

Proposition 3.4.39. *For any κ-field-of-sets B where κ is a regular cardinal, meaning that the cofinality $cf(\kappa) = \kappa$, there is a κ-field-of-sets C such that $C \multimap\!\!\!\!\rightarrow B$.*

Proof. Let κ and B be given as above. Then consider the following subset of $B^\kappa = \Pi_{i<\kappa} B$:

$$C = \{f \in B^\kappa \mid \exists \alpha \in \kappa, \forall \beta \in \kappa, \text{ if } \beta \geq \alpha \text{ then } f(\beta) = f(\alpha)\}.$$

The set C collects what we may call the eventually constant elements in B^κ. For every $f \in C$, let $\lim f$ be the limit of f defined in the obvious way. Now we show that C with operations inherited from B^κ is a κ-regular subalgebra of B^κ.

1. C is closed under negation. This is trivial.

2. C is closed under taking meets of sets of cardinality smaller than κ. Take any $\{f_i\}_{i \in I} \subseteq C$ with $|I| < \kappa$. Let α_i for each $i \in I$ be the smallest ordinal in κ such that for any β such that $\kappa > \beta \geq \alpha_i$, $f_i(\beta) = f_i(\alpha_i)$. Then let $f = \bigwedge f_i$ in B^κ. Now because $\mathrm{cf}(\kappa) = \kappa$, $\alpha = \sup_{i \in I} \alpha_i < \kappa$. Thus for any β such that $\kappa > \beta \geq \alpha$ and any $i \in I$, $f_i(\beta) = f_i(\alpha_i) = f_i(\alpha)$. Hence for any $\kappa > \beta \geq \alpha$,

$$f(\beta) = \bigwedge_{i \in I} f_i(\beta) = \bigwedge_{i \in I} f_i(\alpha) = f(\alpha).$$

   Then we know that $f \in C$.

This also shows that for any set $\{f_i\}_{i \in I} \subseteq C$ with $|I| < \kappa$, $\lim \bigwedge_{i \in I} f_i = \bigwedge_{i \in I} \lim f_i$.

Using Proposition 3.4.36, C is now also a κ-field-of-sets since $C \preccurlyeq_\kappa B^\kappa$ and B is a κ-field-of-sets. Now consider the set $F = \{f \in C \mid \lim f = \top_B\}$. Observe that F is a filter in C. Also, $\bigwedge F = \bot_C$. To see this, consider the sequence $\langle f_i \rangle_{i \in \kappa}$ defined by

$$f_i(\alpha) = \begin{cases} \bot_B & \alpha < i \\ \top_B & \alpha \geq i. \end{cases}$$

Each f_i is in F, yet the only $f \in C$ that is below all the f_i's is the constantly 0_B function, which is $\bot_C$.

Now note that lim as a function from C to B is a surjective homomorphism and by definition $\lim^{-1}(\top_B) = F$. Hence $\lim : C \multimap\!\!\!\!\rightarrow B$. □

Now we start to show that for every non-trivial Boolean algebra A, there is a $(2^{\omega_0})^+$-field-of-sets B which is elementarily equivalent to A. To this end, we first recall the Tarski invariants.

Definition 3.4.40. For any Boolean algebra B, call an element $b \in B$ *atomic* if b is the join of the atoms below it, and *atomless* if there are no atoms below it. If an element is the join of an atomic element and an atomless element, we call it *separable.* Denote the set of separable element in B by $S(B)$. It is easy to see that $S(B)$ is an ideal, which is generated by the atomic and atomless elements.

Then for any non-trivial Boolean algebra B, we can define a sequence of Boolean algebras:

$$B^{(0)} = B, B^{(i+1)} = B^{(i)}/S(B^{(i)}).$$

With the above sequence, define $\mathrm{Inv}(B)$ for every non-trivial Boolean algebra B as follows:

$$m(B) = \begin{cases} k & \text{if } B^{(k)} \text{ is non-trivial and } B^{(k+1)} \text{ is trivial} \\ \infty & \text{if for all } k \in \omega, B^{(k)} \text{ is non-trivial.} \end{cases}$$

$$n_0(B) = \begin{cases} \infty & \text{if } m(B) \in \mathbb{N} \text{ and } B^{(m(B))} \text{ has infinitely many atoms} \\ l & \text{if } m(B) \in \mathbb{N} \text{ and } B^{(m(B))} \text{ has } l \in \mathbb{N} \text{ many atoms.} \end{cases}$$

$$n(B) = \begin{cases} 0 & \text{if } m(B) \notin \mathbb{N} \\ n_0(B) & \text{if } m(B) \in \mathbb{N} \text{ and } B^{(m(B))} \text{ is atomic} \\ -n_0(B) & \text{if } m(B) \in \mathbb{N} \text{ and } B^{(m(B))} \text{ is not atomic.} \end{cases}$$

$$\mathrm{Inv}(B) = \langle m(B), n(B) \rangle.$$

We also define $\mathrm{Inv}(B) = \langle -1, 0 \rangle$ when B is trivial. Finally let Inv be the set of all possible invariant, i.e., $\mathsf{Inv} = \{\mathrm{Inv}(B) \mid B \text{ a Boolean algebra}\}$.

Proposition 3.4.41. *For any two Boolean algebras A and B, they are elementarily equivalent if and only if $Inv(A) = Inv(B)$. In fact, for any two Boolean algebras with extra distinguished elements, $\langle A, a_1, a_2, \cdots a_n \rangle$, $\langle B, b_1, b_2, \cdots b_n \rangle$, they are elementarily equivalent in the first-order language of Boolean algebras extended with n constants to be interpreted by the corresponding distinguished elements if and only if for each $f \in 2^n$, $Inv(A|_{f[\vec{a}]}) = Inv(B|_{f[\vec{b}]})$. Here for any $f \in 2^n$, $f[\vec{a}]$ is defined as the element $\bigwedge_{i \in f^{-1}(1)} a_i \wedge \bigwedge_{i \in f^{-1}(0)} \neg a_i$ and $f[\vec{b}]$ is defined similarly.*

Proof. See Chap.5.5 of [31]. □

Hence our goal now is to construct a $(2^{\omega_0})^+$-field-of-sets B for each invariant $c \in \mathsf{Inv}$ such that $\mathrm{Inv}(B) = c$. To start, we need at least an atomic and an atomless κ-field-of-sets. An atomic κ-field of sets can be easily found, such as the Boolean algebra of two elements. Now we construct an atomless κ-field-of-sets.

Proposition 3.4.42. *For each regular infinite cardinal κ, there exists an atomless κ-field-of-sets L.*

Proof. We construct a κ-field-of-sets in the powerset algebra of $\{0,1\}^\kappa$. For any $f, g \in \{0,1\}^\kappa$ and $i \in \kappa$, write $f =_i g$ when $f(j) = g(j)$ for all $j < i$. Also let $[f]_i$ be the equivalence class that f is in under $=_i$, i.e. $\{g \in \{0,1\}^\kappa \mid g =_i f\}$.

Then it is not hard to see that the set of all subsets of $\{0,1\}^\kappa$ that are closed under $=_i$ for some $i \in \kappa$ forms an atomless κ-field-of-sets under the inherited complementation and intersection.

- Both empty set and $\{0,1\}^\kappa$ are closed under $=_1$.
- Clearly if X is closed under $=_i$, then $\{0,1\}^\kappa \setminus X$ is also closed under $=_i$.
- For any family of $\kappa_0 < \kappa$ many sets $\{X_i\}_{i\in\kappa_0}$ such that each X_i is closed under $=_{\lambda_i}$ where $\lambda_i \in \kappa$, consider their intersection. Let $\mu = \sup_{i\in\kappa_0} \lambda_i$. By the regularity of κ, $\mu \in \kappa$, and obviously each X_i is also closed under $=_\mu$ since $=_\alpha$ refines $=_\beta$ if $\alpha \geq \beta$. Then the intersection $\bigcap\{X_i \mid i \in \kappa_0\}$ is also closed under $=_\mu$.
- For any non empty $X \subseteq \{0,1\}^\kappa$ that is closed under $=_\lambda$, pick $f \in X$ and then we can easily split $[f]_\lambda \subseteq X$ into two non-empty parts: $\{g \in [f]_\lambda \mid g(\lambda+1) = 0\}$ and $\{g \in [f]_\lambda \mid g(\lambda+1) = 1\}$. Both parts are non-empty subsets of X and are closed under $=_{\lambda+1}$. So X is not an atom in the Boolean algebra we construct. $\square$

Now fix κ as an infinite regular cardinal, $\mathbf{2}$ a two-element Boolean algebra, and L the atomless κ-field-of-sets constructed above. The next step is to show that for every κ-field-of-sets B, there is a $U(B)$ which is also a κ-field-of-sets, and moreover $U(B)/S(U(B)) \cong B$. Since we are constructing a κ-field-of-sets that has B as a homomorphic image with some requirement on the kernel of the homomorphism, the construction here is very similar to the one we did in Proposition 3.4.39.

For a κ-field-of-sets B, we construct $U(B)$ as follows. First, since B is a κ-field-of-sets, without loss of generality, we can assume that $B \preccurlyeq_\kappa \wp(\rho)$ with ρ a large enough cardinal. Then we have $e : B \hookrightarrow_\kappa (\mathbf{2}\times L)^\rho$ where e is such that for all $b \in B$ and $\lambda < \rho$, $e(b)(\lambda) = \top_{\mathbf{2}\times L}$ if and only if $\lambda \in b$, and otherwise $e(b)(\lambda) = \bot_{\mathbf{2}\times L}$. In other words, $e(b)$ is the characteristic function of b using $\{\bot_{\mathbf{2}\times L}, \top_{\mathbf{2}\times L}\}$ instead of $\{0,1\}$ as the codomain. By Proposition 3.4.36, $(\mathbf{2} \times L)^\rho$ is a κ-field-of-sets. Now, as in the proof of Proposition 3.4.39, we can now define the set of eventually constant functions $C = \{f \in ((\mathbf{2} \times L)^\rho)^\kappa \mid \exists\alpha < \kappa\forall\beta < \kappa, \beta > \alpha \Rightarrow f(\beta) = f(\alpha)\}$. Then we know that $C \preccurlyeq_\kappa ((\mathbf{2} \times L)^\rho)^\kappa$ and hence is a κ-field-of-sets, and in addition $\lim : C \twoheadrightarrow (\mathbf{2} \times L)^\rho$. However, since we need a κ-field-of-sets with B, not $(\mathbf{2} \times L)^\rho$, as its homomorphic image, we need to take a κ-regular subalgebra of C. Indeed, we only need to take $U(B) = \lim^{-1}(e[B])$. Essentially, $U(B)$ is the pullback of e and lim. This is illustrated by the following commutative diagram:

$$\begin{tikzcd}
B \arrow[r, hook, "e"] & (\mathbf{2} \times L)^\rho \\
U(B) \arrow[u, two heads] \arrow[r, hook, "id"] & C. \arrow[u, two heads, "\lim"']
\end{tikzcd}$$

Lemma 3.4.43. *For any κ-field-of-sets B, $U(B)$ as defined above is also a κ-field-of-sets, and $U(B)/S(U(B)) \cong B$.*

Proof. To show that $U(B)$ is a κ-field-of-sets, it is enough to show that $U(B) \preccurlyeq_\kappa ((\mathbf{2}\times L)^\rho)^\kappa$. That is, we only need to show that for all $X \subseteq U(B)$ such that $|X| < \kappa$, $\bigwedge_{((\mathbf{2}\times L)^\rho)^\kappa} X$ is also in $U(B)$ (it always exists as $((\mathbf{2}\times L)^\rho)^\kappa$ is a κ-field-of-sets). To show that $\bigwedge_{((\mathbf{2}\times L)^\rho)^\kappa} X \in U(B)$, we only need to show that $\lim \bigwedge_{((\mathbf{2}\times L)^\rho)^\kappa} X \in e[B]$. Since κ is regular, $\lim \bigwedge_{((\mathbf{2}\times L)^\rho)^\kappa} X = \bigwedge_{(\mathbf{2}\times L)^\rho} \lim[X]$. Since $X \subseteq U(B)$, $\lim[X] \subseteq e[B]$. Since $|X| < \kappa$, $|\lim[X]| < \kappa$. Then indeed $\bigwedge_{(\mathbf{2}\times L)^\rho} \lim[X] \in e[B]$ since e is a κ-regular embedding, and hence $e[B] \preccurlyeq_\kappa (\mathbf{2}\times L)^\rho$.

To show that $U(B)/S(U(B)) \cong B$, it is enough to show that $\lim^{-1}(\perp_{(\mathbf{2}\times L)^\rho})$, the kernel of lim, is precisely $S(U(B))$, the set of separable elements in $U(B)$. To this end, note first that in both $(\mathbf{2}\times L)^\rho$ and $((\mathbf{2}\times L)^\rho)^\kappa$, the join of atoms exists and can be easily described. Let $\mathbf{a}$ be the constantly $\langle \top_\mathbf{2}, \perp_L\rangle$ function in $(\mathbf{2}\times L)^\rho$ and $f_\mathbf{a}$ be the constantly $\mathbf{a}$ function in $((\mathbf{2}\times L)^\rho)^\kappa$. Then clearly $\mathbf{a}$ is the join of atoms in $(\mathbf{2}\times L)^\rho$, and $f_\mathbf{a}$ is the join of atoms in $((\mathbf{2}\times L)^\rho)^\kappa$. Similarly, let $\mathbf{l}$ be the constantly $\langle \perp_\mathbf{2}, \top_L\rangle$ function in $(\mathbf{2}\times L)^\rho$ and $f_\mathbf{l}$ be the constantly $\mathbf{l}$ function in $((\mathbf{2}\times L)^\rho)^\kappa$. Then $\mathbf{l}$ is the join of atomless elements in $(\mathbf{2}\times L)^\rho$, and $f_\mathbf{l}$ is the join of atomless elements in $((\mathbf{2}\times L)^\rho)^\kappa$.

Now, to show that $S(U(B)) \subseteq \lim^{-1}(\perp_{(\mathbf{2}\times L)^\rho})$, it is enough to show that every atomic and every atomless elements in $U(B)$ are in $\lim^{-1}(\perp_{(\mathbf{2}\times L)^\rho})$ since the $S(U(B))$ is the ideal generated by those elements and a kernel is always an ideal. Let f be an atomic element in $U(B)$. First, we claim that $f \leq f_\mathbf{a}$. Suppose not. Then there would be $i < \kappa, j < \rho$ such that $f(i)(j) \wedge \top_L$ is non-trivial. Let g be the function in $((\mathbf{2}\times L)^\rho)^\kappa$ such that $g(i')(j') = \perp_{\mathbf{2}\times L}$ unless $i' = i$ and $j' = j$, in which case $g(i')(j') = f(i)(j) \wedge \top_L$. Then $g \leq f$, $\lim g = \perp_{(\mathbf{2}\times L)^\rho}$, and hence $g \in U(B)$. But obviously g is atomless in $U(B)$ since we can simply keep decreasing $g(i)(j)$ using the fact that L is atomless, and the resulting function's limit is still $\perp_{(\mathbf{2}\times L)^\rho}$, meaning that the function itself is still in $U(B)$. This contradicts that f is atomic. So $f \leq f_\mathbf{a}$, and hence $\lim f \leq \mathbf{a}$. But recall that f is from $U(B)$ and hence $\lim f \in e[B]$, which means that for each $i < \rho$, $(\lim f)(i) \in \{\top_{\mathbf{2}\times L}, \perp_{\mathbf{2}\times L}\}$. Now for each $i < \rho$, $\mathbf{a}(i) < \top_{\mathbf{2}\times L}$. Obviously then, the only element in $e[B]$ that is below $\mathbf{a}$ is $\perp_{(\mathbf{2}\times L)^\rho}$, and hence $\lim f = \perp_{(\mathbf{2}\times L)^\rho}$. So we are done showing that every atomic element in $U(B)$ is in the kernel of lim. To show that every atomless element in $U(B)$ is in the kernel of lim the strategy is exactly the same. If $f \in U(B)$ is atomless, then we can show similarly that $f \leq f_\mathbf{l}$. Then $\lim f$, being both below $\mathbf{l}$ and also inside $e[B]$, must be $\perp_{(\mathbf{2}\times L)^\rho}$. So this f is also in the kernel of lim.

To show that $\lim^{-1}(\perp_{(\mathbf{2}\times L)^\rho}) \subseteq S(U(B))$, pick an arbitrary $f \in U(B)$ such that $\lim f = \perp_{(\mathbf{2}\times L)^\rho}$. Then $f \wedge f_\mathbf{a}$ is also in $U(B)$ as $\lim(f \wedge f_\mathbf{a})$ must also be $\perp_{(\mathbf{2}\times L)^\rho}$. For similar reasons, $f \wedge f_\mathbf{l} \in U(B)$ too. Now clearly $f \wedge f_\mathbf{a}$ is atomic in $U(B)$ since it is the join of $\{g_{i,j} \mid (f \wedge f_\mathbf{a})(i)(j) = \langle \top_\mathbf{2}, \perp_L\rangle\}$ where $g_{i,j}$ is the function that always returns $\perp_{\mathbf{2}\times L}$ expect that $g_{i,j}(i)(j) = \langle \top_\mathbf{2}, \perp_L\rangle$. Each $g_{i,j}$ is obviously in $U(B)$ and is atomic. Hence $f \wedge f_\mathbf{a}$ is a join of atoms in $U(B)$ and hence atomic. Similarly $f \wedge f_\mathbf{l}$ is atomless in $U(B)$ as it is the join of the elements of the form $h_{i,j}$ below it where $h_{i,j}$ always returns $\perp_{\mathbf{2}\times L}$ except that $h_{i,j}(i)(j) = \langle \perp_\mathbf{2}, \top_L\rangle$. Each $h_{i,j}$ is in $U(B)$ and is atomless. Hence $f \wedge f_\mathbf{l}$ is atomless. But then, f is separable by definition since $f = (f \wedge f_\mathbf{a}) \vee (f \wedge f_\mathbf{l})$. $\square$

Now we can sum the above up and obtain the following proposition.

Proposition 3.4.44. *For every Boolean algebra A, there is a $(2^{\omega_0})^+$-field-of-sets B which is elementarily equivalent to A.*

Proof. Let κ be $(2^{\omega_0})^+$. It is a successor cardinal, so it is regular. By Proposition 3.4.41, it is enough to show that for every $c \in \mathsf{Inv}$, there is a $(2^{\omega_0})^+$-field-of-sets B such that $\mathrm{Inv}(B) = c$. Now Inv can be partitioned into three parts: $\{\langle -1, 0\rangle\}$, $\{\langle m, n\rangle \mid m \in \mathbb{N}, n \in \mathbb{Z}^\infty\}$, and $\{\langle \infty, 0\rangle\}$. For $\langle -1, 0\rangle$, we use $\wp(\varnothing)$. For the second part, we use a simple induction on the first coordinate.

1. For non-zero $n \in \mathbb{N}$, $\mathrm{Inv}(\wp(n)) = \langle 0, n\rangle$, and $\mathrm{Inv}(\wp(\mathbb{N})) = \langle 0, \infty\rangle$. For $\langle 0, 0\rangle$, use the L from Proposition 3.4.42, which is an atomless κ-field-of sets. For invariants $\langle 0, -n\rangle$ $(n > 0)$ and $\langle 0, -\infty\rangle$, use $\wp(n) \times L$ and $\wp(\mathbb{N}) \times L$, respectively.

2. Suppose for any $n \in \mathbb{Z}^\infty$, there is a κ-field-of-sets B_n such that $\mathrm{Inv}(B_n) = \langle m, n\rangle$. Then for $\langle m + 1, n\rangle$ for any $n \in \mathbb{Z}^\infty$, use $U(B_n)$, since by Lemma 3.4.43, $U(B_n)$ is a κ-field-of-sets, $U(B_n)/S(U(B_n)) \cong B_n$, and thus $\mathrm{Inv}(U(B_n)) = \mathrm{Inv}(B_n) + \langle 1, 0\rangle = \langle m + 1, n\rangle$.

For the invariant $\langle \infty, 0\rangle$, take the product $B = \Pi_{i\in\mathbb{N}} U^i(\mathbf{2}) = \mathbf{2} \times U(\mathbf{2}) \times U(U(\mathbf{2})) \times U(U(U(\mathbf{2}))) \cdots$. That B is a κ-field-of-sets follows from Proposition 3.4.36. Also, since $B/S(B) = \Pi_{i\in\mathbb{N}} U^i(\mathbf{2})/S(U^i(\mathbf{2})) = \mathbf{1} \times \Pi_{i\in\mathbb{N},i>0} U^{i-1}(\mathbf{2})$, $B/S(B)$ is isomorphic to B. ($\mathbf{1}$ is the trivial algebra, and it appears here as the result of $\mathbf{2}/S(\mathbf{2})$.) This means that for any $n \in \mathbb{N}$, $B^{(n)}$ is isomorphic to B, which means that $\mathrm{Inv}(B) = \langle \infty, 0\rangle$. □

The only missing link now is the following proposition, shown in [153].

Proposition 3.4.45 (Vermeer 1996)**.** *Every $(2^{\omega_0})^+$-field-of-sets is a quotient of a complete Boolean algebra.*

With this, we can prove the following lemma, which leads to a proof of Lemma 3.4.23 that also takes care of the requirements for the distinguished elements z and g.

Lemma 3.4.46. *For every Boolean algebra A there is a non-trivial complete Boolean algebra C with a filter $H \subseteq C$ such that $\bigwedge H = \bot$, and that A is elementarily equivalent to C/H.*

Proof. If A is trivial, let D be the two-element Boolean algebra and H the improper filter in D. Now pick an arbitrary non-trivial Boolean algebra A. By Proposition 3.4.44, there is a $(2^{\omega_0})^+$-field-of-sets B that is elementarily equivalent to A. Then we only need to find a complete Boolean algebra C such that $C \multimapdotboth\!\!\twoheadrightarrow B$.

Notice that $(2^{\omega_0})^+$ is a successor cardinal, so it is regular. Then, by Proposition 3.4.39, there is a $(2^{\omega_0})^+$-field-of-sets B' such that $B' \multimapdotboth\!\!\twoheadrightarrow B$. By Proposition 3.4.45, then, there is a complete Boolean algebra C such that $C \twoheadrightarrow B'$. But then, by Proposition 3.4.38, $C \multimapdotboth\!\!\twoheadrightarrow B$. Since A is non-trivial, B and hence C must also be non-trivial. □

for Lemma 3.4.23. Pick an arbitrary zg-algebra $\langle A, z, g\rangle$. We decompose A as $A|_{\overline{z}} \times A|_{zg} \times A|_{z\overline{g}}$ since $g \leq z$. Since we are only after elementary equivalence, by Proposition 3.4.41, it is enough to find $\langle A', z', g'\rangle$ such that $\mathrm{Inv}(A'|_{\overline{z'}}) = \mathrm{Inv}(A|_{\overline{z}})$, $\mathrm{Inv}(A'|_{z'g'}) = \mathrm{Inv}(A|_{zg})$, and $\mathrm{Inv}(A'|_{z'\overline{g'}}) = \mathrm{Inv}(A|_{z\overline{g}})$, and that $\langle A', z', g'\rangle$ is the natural quotient of a complete proper filter algebra.

By the definition of zg-algebra, $A|_{zg}$ is atomic and $A|_{z\overline{g}}$ is atomless. Let A'_2 and A'_3 be the MacNeille completion of $A|_{zg}$ and $A|_{z\overline{g}}$, respectively. Note that MacNeille completion does not change the number of atoms. Thus $\mathrm{Inv}(A'_2) = \mathrm{Inv}(A|_{zg})$, and $\mathrm{Inv}(A'_3) = \mathrm{Inv}(A|_{z\overline{g}})$.

To figure out $A'|_{\overline{z}}$, we invoke Lemma 3.4.46. By that lemma, there is a non-trivial complete Boolean algebra C with a filter H, such that C/H is elementarily equivalent to $A|_{\overline{z}}$, and that $\bigwedge H = \bot_C$. Let $A'_1 = C/H$.

Now let $\langle A', z', g'\rangle = \langle A'_1 \times A'_2 \times A'_3, \langle \bot, \top, \top\rangle, \langle \bot, \bot, \top\rangle\rangle$. Then by construction, $\langle A', z', g'\rangle$ is also a zg-algebra and is elementarily equivalent to $\langle A, z, g\rangle$.

Then let $\mathcal{B} = \langle B, F\rangle$ where $B = C \times A'_2 \times A'_3$ and $F = H \times \{\top\} \times \{\top\}$. For this $\mathcal{B}$, we need to establish two points.

- First, $\mathcal{B}$ is a complete proper filter algebra. To see this, note first that $C \times A'_2 \times A'_3$ is a complete Boolean algebra as each of the three components are. Then note that F is a proper filter. It is obviously a filter. It is proper because if not, A'_1, A'_2, and A'_3 are all trivial, and hence A' and A are trivial since A' and A are elementarily equivalent. But A is a zg-algebra and zg-algebras are non-trivial. Note though that any two of A'_1, A'_2, and A'_3 can be trivial together.

- Also, the natural quotient of $\mathcal{B}$, $\mathcal{B}/F$, is precisely $\langle A', z', g'\rangle$. That $A' = B/F$ is a simple Boolean algebra exercise. The next thing to note is that $z_{\mathcal{B}} = \bigwedge F = \langle \bot, \top, \top\rangle \in B$ since $\bigwedge H = \bot_C$. Hence $\pi_F(z_{\mathcal{B}}) = \langle \bot, \top, \top\rangle \in A'$, which is precisely z'. Also, the join of atoms below $z_{\mathcal{B}}$ in B is precisely $\langle \bot, \top, \bot\rangle$ as A'_2 by construction is atomic and A'_3 is atomless.

The extra constraints in Lemma 3.4.23 are also satisfied. $\mathcal{B}|_{z_{\mathcal{B}}} \cong C$ is always non-trivial by construction. $\mathcal{B}|_{z_{\mathcal{B}} g_{\mathcal{B}}} \cong A'_2$ is trivial if and only if $A|_{zg}$ is trivial since the construction method is MacNeille completion. By the same reason, $\mathcal{B}|_{z_{\mathcal{B}} \overline{g_{\mathcal{B}}}} \cong A'_3$ is trivial if and only if $A|_{z\overline{g}}$ is trivial. □

3.5 Stronger logics and decidability

In the previous section, our only goal was the completeness theorems Theorem 3.4.1 and Theorem 3.4.24. However, the method we used to show them in fact supports a full analysis of the expressivity of $\mathcal{L}\Pi$ on complete proper filter algebras and the normal Π-extension of $\mathsf{KD4^{\forall}5\Pi}$, similar to the one in [40]. In light of the space such a general analysis would take, in this section we focus only on several natural concrete cases in which we only add one formula, or equivalently finitely many formulas, to $\mathsf{KD4^{\forall}5\Pi}$. Since $\mathsf{KD4^{\forall}5\Pi zg}$ is a definitional extension of $\mathsf{KD4^{\forall}5\Pi}$, in this section we move between $\mathsf{KD4^{\forall}5\Pi}$ and $\mathsf{KD4^{\forall}5\Pi zg}$ freely in stating the results, noting that to obtain the results in $\mathcal{L}\Pi$, one only needs to replace z and g by their definitions in the axioms z and g in Definition 3.4.13.

Before we start, let us introduce a bit of notation. For any $X \subseteq \mathcal{L}\Pi\mathsf{zg}$, let $\mathsf{CPFA}(\Gamma)$ be the class of complete proper filter algebras validating every formula in Γ. As usual we write

$\mathsf{CPFA}(\varphi)$ for $\mathsf{CPFA}(\{\varphi\})$ and write CPFA for $\mathsf{CPFA}(\varnothing)$, the class of all complete proper filter algebras. Then for any class K of complete proper filter algebras, we write $\mathrm{Log}(\mathsf{K})$ for the set of formulas in $\mathcal{L}\Pi\mathsf{zg}$ that are valid in all complete proper filter algebras in K. Finally, as usual, for any $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, we write $\mathsf{KD4^{\forall}5\Pi zg}\varphi$ for the smallest normal Π-logic extending $\mathsf{KD4^{\forall}5\Pi zg}$ with φ. Then, we first define the following semantics-preserving mapping between complete proper filter algebras.

Definition 3.5.1. For any complete proper filter algebras $\mathcal{B}$ and $\mathcal{B}'$ and any function f from $\mathcal{B}$ to $\mathcal{B}'$, we say f is a *complete homomorphism* if

- f is a complete Boolean homomorphism: $f(\neg a) = \neg f(a)$ and $f(\bigwedge X) = \bigwedge f[X]$;
- for any $a \in \mathcal{B}$, $a \in F_{\mathcal{B}}$ iff $f(a) \in F_{\mathcal{B}'}$.

Proposition 3.5.2. *If $f : \mathcal{B} \to \mathcal{B}'$ is a complete homomorphism, then for any valuation θ on $\mathcal{B}$ and $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $f(\tilde{\theta}(\varphi)) = \widetilde{f \circ \theta}(\varphi)$.*

Proof. If $\varphi \in \mathcal{L}\Pi$, a simple induction suffices. For $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, note that we have the definitional axioms z and g that are sound. $\square$

Now we prove the following general completeness theorem.

Theorem 3.5.3. *For any $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $\mathsf{KD4^{\forall}5\Pi zg}\varphi = \mathrm{Log}(\mathsf{CPFA}(\varphi))$.*

Proof. That $\mathsf{KD4^{\forall}5\Pi zg}\varphi \subseteq \mathrm{Log}(\mathsf{CPFA}(\varphi))$ is trivial by soundness. Now pick an arbitrary $\psi \in \mathrm{Log}(\mathsf{CPFA}(\varphi))$. Without loss of generality we assume that both φ and ψ are sentences. Since we will only be dealing with sentences whose semantic values do not depend on particular valuations, we use the notation $\mathcal{B}(\chi)$ for the semantic value of any sentence χ in $\mathcal{B}$. By necessitation and modus ponens in $\mathsf{KD4^{\forall}5\Pi zg}$, it is enough to show that $\vdash (\varphi \wedge \mathsf{B}\varphi) \to \psi$. By the completeness theorem, then, it is enough to show that for every $\mathcal{B} \in \mathsf{CPFA}$, $\mathcal{B} \vDash (\varphi \wedge \mathsf{B}\varphi) \to \psi$.

Pick an arbitrary complete proper filter algebra $\mathcal{B}$. If $\mathcal{B}(\varphi) \notin F_{\mathcal{B}}$, then we are done since $\mathcal{B}(B\varphi)$ and hence $\mathcal{B}(\varphi \wedge B\varphi)$ in this case is $\bot$. So now we focus on the case where $\mathcal{B}(\varphi) \in F_{\mathcal{B}}$ and let $v = \mathcal{B}(\varphi)$. Consider $\mathcal{B}'$ defined by restricting $\mathcal{B}$ to v: $\mathcal{B}' = \langle \mathcal{B}|v, \{a \wedge v \mid a \in F_{\mathcal{B}}\}\rangle$. It is not hard to see that $h : \mathcal{B} \to \mathcal{B}'$ defined by $h(a) = a \wedge v$ is a complete homomorphism. By Proposition 3.5.2, we see that (1) $\mathcal{B}'(\varphi) = h(\mathcal{B}(\varphi)) = v = \top_{\mathcal{B}'}$ and also that (2) $\mathcal{B}'(\psi) = h(\mathcal{B}(\psi))$. From (1), it follows that $\mathcal{B}' \vDash \varphi$. By assumption, $\mathcal{B}' \vDash \psi$. Hence, by (2), $h(\mathcal{B}(\psi)) = \mathcal{B}(\psi) \wedge v = v$ and thus $\mathcal{B}(\psi) \geq v$. This means that $\mathcal{B}(\psi) \geq \mathcal{B}(\varphi \wedge B\varphi)$ and that $\mathcal{B} \vDash (\varphi \wedge B\varphi) \to \psi$.

$\square$

While of course there is a limit to the expressivity of $\mathcal{L}\Pi\mathsf{zg}$, many natural classes of complete proper filter algebras corresponds to the validity of sentences in $\mathcal{L}\Pi\mathsf{zg}$. We give some examples below.

Corollary 3.5.4.
- $\mathsf{CPFA}(\mathtt{z})$ *is the class of complete proper filter algebras with trivial filters. Hence its logic is* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zgz}$. *In* $\mathcal{L}\Pi$, *the logic is* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\forall p(\mathtt{B}p \to p)$.
- $\mathsf{CPFA}(\mathtt{Bz})$ *is the class of complete proper filter algebras with principal filters. Hence its logic is* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zgBz}$. *In* $\mathcal{L}\Pi$, *the logic is* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{Immod}$.
- *The class of complete proper filter algebras with ultrafilters is defined by* $\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$. *Hence its logic in* $\mathcal{L}\Pi$ *is* $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$.

Now we consider an undefinable property: atomicity. To see that it is not definable by the validity of any formula in $\mathcal{L}\Pi\mathsf{zg}$, we first establish a general proposition.

Proposition 3.5.5. *For any complete proper filter algebras* $\mathcal{B}$ *and* $\mathcal{B}'$, *if* $\overline{z_{\mathcal{B}}}$ *and* $\overline{z_{\mathcal{B}'}}$ *are non-trivial and* $\mathcal{B}/F_{\mathcal{B}}$ *and* $\mathcal{B}'/F_{\mathcal{B}'}$ *are elementarily equivalent, then* $\mathrm{Log}(\mathcal{B}) = \mathrm{Log}(\mathcal{B}')$.

Proof. Let $\mathcal{B}$ and $\mathcal{B}'$ with the suppositions above be given. Observe first that when passing from $\mathcal{B}$ to $\mathcal{B}/F_{\mathcal{B}}$, the Boolean structure of $\mathcal{B}$ below $z_{\mathcal{B}}$ and the Boolean structure of $\mathcal{B}/F_{\mathcal{B}}$ below $z_{\mathcal{B}/F_{\mathcal{B}}}$ are the same. In fact, if B is a complete Boolean algebra, F a filter in B, and z the meet of F, then B/F is isomorphic to $(B|_{\overline{z}}/F|_{\overline{z}}) \times B|_z$ where $F|_{\overline{z}} = \{a \wedge \neg z \mid a \in F\}$. Thus, the zg in $\mathcal{B}$ is trivial iff the zg in $\mathcal{B}/F_{\mathcal{B}}$ is trivial, and the same goes for $z\overline{g}$ and for $\mathcal{B}'$. Since $\mathcal{B}/F_{\mathcal{B}}$ and $\mathcal{B}'/F_{\mathcal{B}'}$ are elementarily equivalent, zg (resp. $z\overline{g}$) in $\mathcal{B}$ is non-trivial iff zg (resp. $z\overline{g}$) in $\mathcal{B}'$ is non-trivial. Since we also assumed that the $\overline{z}$ in both $\mathcal{B}$ and $\mathcal{B}'$ are non-trivial, in sum, the triviality of $\overline{z}$, zg, and $z\overline{g}$ in $\mathcal{B}$ and $\mathcal{B}'$ are the same, respectively.

Now, recall that by Lemma 3.4.22, for any sentence $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, we can assume that $\varphi = (\overline{\mathsf{z}} \wedge \varphi_{\overline{\mathsf{z}}}) \vee (\mathsf{zg} \wedge \varphi_{\mathsf{zg}}) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \varphi_{\mathsf{z}\overline{\mathsf{g}}})$ where $\varphi_{\overline{\mathsf{z}}}$, φ_{zg}, and $\varphi_{\mathsf{z}\overline{\mathsf{g}}}$ are all in $\mathcal{L}\mathtt{B}\Pi\mathsf{zg}$. This means, given Lemma 3.4.12 and that the natural quotients of $\mathcal{B}$ and $\mathcal{B}'$ are elementarily equivalent, $\mathcal{B} \vDash \chi$ iff $\mathcal{B}' \vDash \chi$ for all $\chi \in \{\varphi_{\overline{\mathsf{z}}}, \varphi_{\mathsf{zg}}, \varphi_{\mathsf{z}\overline{\mathsf{g}}}\}$. By the simple reasoning we have used in the beginning of the proof of Theorem 3.4.24, $\mathcal{B} \vDash \varphi$ iff $\mathcal{B}' \vDash \varphi$. To show the same for formulas with free variables, take the universal closure of them. □

Proposition 3.5.6. *There are no* $\Gamma \subseteq \mathcal{L}\Pi\mathsf{zg}$ *such that* $\mathsf{CPFA}(\Gamma)$ *is precisely the class of atomic complete proper filter algebras.*

Proof. Let $\mathcal{B}_0 = \langle \wp(\mathbb{N}), F_0 \rangle$ where F_0 is a non-principal ultrafilter of $\wp(\mathbb{N})$. Then let $\mathcal{B}_1 = \langle L, F_1 \rangle$ where L is a complete atomless Boolean algebra and F_1 is an ultrafilter in L. Note that for both $i \in \{0, 1\}$, $z_{\mathcal{B}_i}$ is $\perp$ and the natural quotient $\mathcal{B}_i/F_i$ is isomorphic to $\langle \mathbf{2}, \perp, \perp \rangle$ where $\mathbf{2}$ is a two-element Boolean algebra. By Proposition 3.5.5, for any $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $\mathcal{B}_0 \vDash \varphi$ if and only if $\mathcal{B}_1 \vDash \varphi$. But $\mathcal{B}_0$ is atomic yet $\mathcal{B}_1$ is not. □

However, the undefinability of atomicity in complete proper filter algebras does not preclude axiomatization. An obvious validity on atomic complete proper filter algebras is $\mathsf{z} \to \mathsf{g}$ since z must be below the join of atoms below z. It turns out that we can just append this to $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}\varphi$ to obtain the logic of the atomic algebras in $\mathsf{CPFA}(\varphi)$. To show this, first note that we can strengthen Lemma 3.4.46 so that the non-trivial complete Boolean algebra C is also atomic. This can be done simply by using the canonical extension C^{δ}, the powerset algebra of the set of ultrafilters of C, rather than C as the final result of that lemma, since by Sikorski's extension lemma and C being complete, $C^\delta \twoheadrightarrow C$ (for a proof, see [73], Theorem 5, Chapter 13). We can then chain the surjective morphisms and see that $C^\delta \looparrowright\!\!\twoheadrightarrow B$ where B is elementarily equivalent to an arbitrarily given Boolean algebra. But then, the Lemma 3.4.23 is also strengthened so that besides all other requirements, $\overline{z_\mathcal{B}}$ can be atomic. In sum, the completeness theorem is now strengthened into the following: if $\varphi \in \mathcal{L}\Pi\mathsf{zg}$ is valid on all complete proper filter algebras $\mathcal{B}$ such that $\overline{z_\mathcal{B}}$ is atomic, then φ is already in $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}$. To formulate results below, let us use $\mathsf{CPFA}^{\overline{z}\mathrm{at}}(\Gamma)$ to denote the class of complete proper filter algebras such that $\overline{z}$ is atomic and every formula in Γ is validated and use $\mathsf{CPFA}^{\mathrm{at}}(\Gamma)$ for the class of complete proper filter algebras that are atomic and validates everything in Γ.

Theorem 3.5.7. *For every formula $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}\varphi = \mathrm{Log}(\mathsf{CPFA}^{\overline{z}\mathrm{at}}(\varphi))$. For CPFA^{at}, we have that $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}((\mathsf{z} \to \mathsf{g}) \wedge \varphi) = \mathrm{Log}(\mathsf{CPFA}^{\mathrm{at}}(\varphi))$.*

Proof. To show that $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}\varphi = \mathrm{Log}(\mathsf{CPFA}^{\overline{z}\mathrm{at}}(\varphi))$, using Theorem 3.5.3, we only need to show that $\mathrm{Log}(\mathsf{CPFA}^{\overline{z}\mathrm{at}}(\varphi)) \subseteq \mathrm{Log}(\mathsf{CPFA}(\varphi))$. This clearly follows from the fact that for every $\mathcal{B} \in \mathsf{CPFA}$ there is a $\mathcal{B}' \in \mathsf{CPFA}^{\overline{z}\mathrm{at}}$ such that $\mathrm{Log}(\mathcal{B}) = \mathrm{Log}(\mathcal{B}')$. If $\mathcal{B}$ is such that $\overline{z}_\mathcal{B}$ is trivial, then $\mathcal{B}$ itself is in $\mathsf{CPFA}^{\overline{z}\mathrm{at}}$ and we are done. If $\overline{z}_\mathcal{B}$ is not trivial, then apply the strengthened Lemma 3.4.23 to $\mathcal{B}/F_\mathcal{B}$ and obtain $\mathcal{B}'$. By the strengthening, $\mathcal{B}' \in \mathsf{CPFA}^{\overline{z}\mathrm{at}}$. Moreover, Lemma 3.4.23 states that the zg (resp. $z\overline{g}$) in $\mathcal{B}'$ is non-trivial iff the zg (resp. $z\overline{g}$) in $\mathcal{B}/F_\mathcal{B}$ is non-trivial. This means that Proposition 3.5.5 can be applied to $\mathcal{B}$ and $\mathcal{B}'$, and from it we have that $\mathrm{Log}(\mathcal{B}) = \mathrm{Log}(\mathcal{B}')$.

To show that $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}((\mathsf{z} \to \mathsf{g}) \wedge \varphi) = \mathrm{Log}(\mathsf{CPFA}^{\mathrm{at}}(\varphi))$, note that $\mathsf{CPFA}^{\overline{z}\mathrm{at}}(\mathsf{z} \to \mathsf{g}) = \mathsf{CPFA}^{\mathrm{at}}$ and thus $\mathsf{CPFA}^{\mathrm{at}}(\varphi) = \mathsf{CPFA}^{\overline{z}\mathrm{at}}((\mathsf{z} \to \mathsf{g}) \wedge \varphi)$, since for any $\mathcal{B} \in \mathsf{CPFA}$, $z_\mathcal{B} g_\mathcal{B}$ is already atomic, so if $z_\mathcal{B} \to g_\mathcal{B}$ is $\top$, $z_\mathcal{B}$ is atomic in $\mathcal{B}$. $\square$

While $\mathsf{z} \to \mathsf{g}$ is simple in $\mathcal{L}\Pi\mathsf{zg}$, its translation back to $\mathcal{L}\Pi$ with z and g replaced by their axiomatic definitions is rather long (recall Definition 3.4.6 and Definition 3.4.13). Note that in complete simple S5 algebras, atomicity can defined easily with the following sentence (with $\Box$ replaced by $\mathtt{B}$):

$$\mathsf{At} : \exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q))).$$

Moreover, this formula has a natural interpretation: there is a world proposition, a proposition that is true and entails all true propositions. We now show that At can replace $\mathsf{z} \to \mathsf{g}$ in axiomatization since they are logically equivalent, though with $\mathtt{B}$ interpreted as belief, At falls short to say exactly that there is a world proposition.

Proposition 3.5.8. *In $\mathsf{KD4}^\forall 5\Pi\mathsf{zg}$, $\mathsf{At} \leftrightarrow (\mathsf{z} \to \mathsf{g})$ is derivable.*

Proof. We mimic Fitch-style natural deduction below and do the left-to-right direction first.

1. Suppose $\mathsf{At} = \exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q)))$ and z.
2. By the definitional axiom z, we have $\forall p(\mathtt{B}p \to p)$.

3. Setting up an existential elimination, consider the following derivation:

   a) Suppose $p \wedge \forall q(q \to \mathtt{B}(p \to q))$. Then we have p and $\forall q(q \to \mathtt{B}(p \to q))$. We also have $\mathsf{z} \wedge p$ now.

   b) Using the dual of $\forall p(\mathtt{B}p \to p)$, we have $\widehat{\mathtt{B}}(\mathsf{z} \wedge p) = \langle \mathsf{z} \rangle p$.

   c) Introduce the variable q and consider the following derivation:

      i. We have the tautology $q \vee \neg q$.

      ii. Suppose q, then instantiating $\forall q(q \to \mathtt{B}(p \to q))$ by q itself, we have $q \to \mathtt{B}(p \to q)$ and hence $\mathtt{B}(p \to q)$. By monotonicity for $\mathtt{B}$, we have $[\mathsf{z}](p \to q)$.

      iii. Suppose $\neg q$, then instantiating $\forall q(q \to \mathtt{B}(p \to q))$ by $\neg q$, we have $\neg q \to \mathtt{B}(p \to \neg q)$ and hence $\mathtt{B}(p \to \neg q)$. By monotonicity for $\mathtt{B}$, we have $[\mathsf{z}](p \to \neg q)$.

      iv. Thus, by disjunction elimination, we have $[\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q)$.

      By universalization, we have $\forall q([\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q))$.

   d) Thus we have $p \wedge \langle \mathsf{z} \rangle p \wedge \forall q([\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q)) = p \wedge at(p)$

   Thus, we have $\exists p(p \wedge at(p))$. Together with z and using the definitional axiom g, we have g.

Now for the right-to-left direction, we need to show that At follows from both g and $\neg\mathsf{z}$. We show that At follows from g first.

1. Suppose g. Then we have $\forall p(\mathtt{B}p \to p) \wedge \exists p(p \wedge at(p))$ by axiom g. So we also have $\forall p(\mathtt{B}p \to p)$ and $\exists p(p \wedge at(p))$. By axiom z, we also have z.

2. Setting up an existential elimination, consider the following derivation:

   a) Suppose $(p \wedge at(p)) = (p \wedge \langle z \rangle p \wedge \forall q([\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q)))$.

   b) Now we have p, $\langle z \rangle p$, and $\forall q([\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q))$.

   c) Introduce the variable q and consider the following derivation:

      i. Suppose q. Then we have $\mathsf{z} \wedge p \wedge q$. Using the dual form of $\forall p(\mathtt{B}p \to p)$, we have $\widehat{\mathtt{B}}(\mathsf{z} \wedge p \wedge q) = \langle \mathsf{z} \rangle (p \wedge q)$. By the normality of $\mathtt{B}$, we have $\neg[\mathsf{z}](p \to \neg q)$.

      ii. Instantiating $\forall q([\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q)))$ by q, we have $[\mathsf{z}](p \to q) \vee [\mathsf{z}](p \to \neg q))$. By disjunctive syllogism, we have $[\mathsf{z}](p \to q) = \mathtt{B}(\mathsf{z} \to (p \to q))$. By the normality of $\mathtt{B}$, we have $\mathtt{B}((\mathsf{z} \wedge p) \to q)$.

      By universalization, we have $\forall q(q \to \mathtt{B}((\mathsf{z} \wedge p) \to q))$.

   d) Since we had z and p, we have $(\mathsf{z} \wedge p)$. By existential introduction with $(\mathsf{z} \wedge p)$ being a witness, we have $\exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q))$.

   With existential elimination, we have $\exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q))$, which is At.

Then we show that At follows from $\neg\mathsf{z}$.

1. Suppose $\neg\mathsf{z}$. By axiom z, we have $\neg\forall p(\mathtt{B}p \to p)$ and hence $\exists p(p \wedge \mathtt{B}\neg p)$.
2. Setting up an existential elimination, consider the following derivation:
   a) Suppose $p \wedge \mathtt{B}\neg p$. Then we have p and $\mathtt{B}\neg p$.
   b) Introduce the variable q and consider the following derivation:
      i. Suppose q.
      ii. Now note that we had $\mathtt{B}\neg p$. By the normativity of $\mathtt{B}$, we have $\mathtt{B}(p \to q)$.
      iii. So we have $q \to \mathtt{B}(p \to q)$.

      Universalizing over q, we have $\forall q(q \to \mathtt{B}(p \to q))$.
   c) Putting p and $\forall q(q \to \mathtt{B}(p \to q))$ together, we have $p \wedge \forall q(q \to \mathtt{B}(p \to q))$. So with p being a witness, we have $\exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q)))$.

   By existential elimination, we derive $\exists p(p \wedge \forall q(q \to \mathtt{B}(p \to q)))$, which is At. □

In light of the above derivation, it is also true that for any $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{zg}(\mathsf{At}\wedge\varphi) = \mathrm{Log}(\mathsf{CPFA}^{\mathrm{at}}(\varphi))$, and for any $\varphi \in \mathcal{L}\Pi$, $\mathsf{KD4}^{\forall}\mathsf{5}\Pi(\mathsf{At} \wedge \varphi) = \mathrm{Log}(\mathsf{CPFA}^{\mathrm{at}}(\varphi))$. We collect some special cases in the following corollary:

Corollary 3.5.9.
- *The logic of the class of atomic complete proper filter algebras with trivial filters in language $\mathcal{L}\Pi$ is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \to p)$.*
- *The logic of the class of atomic complete proper filter algebras with principal filters in language $\mathcal{L}\Pi$ is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{AtImmod}$.*
- *The logic of the class of atomic complete proper filter algebras with ultrafilters in language $\mathcal{L}\Pi$ is $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$.*

Before we move on to decidability, note that since $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ is sound on the class of all complete KD45 algebras, the above general completeness theorems, when phrased in $\mathcal{L}\Pi$ (since we did not define the semantics of z and g on those algebras), hold for complete KD45 algebras too.

For decidability, the situation is simple: all the logics mentioned above are decidable. To see this, we first state a general theorem linking the decidability of logics in the form of $\mathrm{Log}(\mathsf{K})$ to the decidability of the first-order theories of some classes of zg-algebras coming from K.

Theorem 3.5.10. *For any $\mathsf{K} \subseteq \mathsf{CPFA}$, Let $Q\mathsf{K}_0 = \{\mathcal{B}/F_{\mathcal{B}} \mid \mathcal{B} \in \mathsf{K} \text{ and } \overline{z_{\mathcal{B}}} = \bot\}$ and $Q\mathsf{K}_1 = \{\mathcal{B}/F_{\mathcal{B}} \mid \mathcal{B} \in \mathsf{K} \text{ and } \overline{z_{\mathcal{B}}} \neq \bot\}$. Then, if $Q\mathsf{K}_0$ and $Q\mathsf{K}_1$ have a decidable first-order theory in $\mathcal{FOL}\mathsf{zg}$, then $\mathrm{Log}(\mathsf{K})$ is decidable.*

Proof. For any $\varphi \in \mathcal{L}\Pi\mathsf{zg}$, to decide whether $\varphi \in \mathrm{Log}(\mathsf{K})$, we can first take its universal closure and then turn it into a sentence of the form

$$(\overline{\mathsf{z}} \wedge \alpha) \vee (\mathsf{zg} \wedge \beta) \vee (\mathsf{z}\overline{\mathsf{g}} \wedge \gamma)$$

with $\alpha, \beta, \gamma \in \mathcal{L}\mathtt{B}\Pi\mathsf{zg}$. Obviously this process is decidable. Then, following similar reasoning done in the proof of Theorem 3.4.24, $\varphi \in \mathrm{Log}(\mathsf{K})$ if and only if for all $\mathcal{B} \in \mathsf{K}$, the following hold.

- Either $z_{\mathcal{B}}g_{\mathcal{B}} = \bot$ or $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\beta)$.
- Either $z_{\mathcal{B}}\overline{g_{\mathcal{B}}} = \bot$ or $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\gamma)$.
- Either $\overline{z_{\mathcal{B}}} = \bot$ or $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\alpha)$.

It is not hard to see that $z_{\mathcal{B}}g_{\mathcal{B}} = \bot$ if and only if $\mathcal{B}/F_{\mathcal{B}} \vDash ((\mathsf{z} \wedge \mathsf{g}) = \bot)$, and similarly $z_{\mathcal{B}}\overline{g_{\mathcal{B}}} = \bot$ if and only if $\mathcal{B}/F_{\mathcal{B}} \vDash ((\mathsf{z} \wedge \neg\mathsf{g}) = \bot)$. This is because $z_{\mathcal{B}} = \bigwedge F_{\mathcal{B}}$, so all distinctions below $z_{\mathcal{B}}$ are preserved under quotienting through $F_{\mathcal{B}}$. However, there is no analog for $\overline{z_{\mathcal{B}}}$. It may well be that $\mathcal{B}/F_{\mathcal{B}} \vDash \neg\mathsf{z} = \bot$ while $\overline{z_{\mathcal{B}}} > \bot$. This happens whenever $F_{\mathcal{B}}$ is principal, and this is why we need to take care of two classes of natural quotients. Using the observations we collected, now $\varphi \in \mathrm{Log}(\mathsf{K})$ if and only if the following hold.

- For all $\mathcal{B} \in \mathsf{K}$ such that $\overline{z_{\mathcal{B}}} = \bot$, $\mathcal{B}/F_{\mathcal{B}} \vDash ((\mathsf{zg} = \bot) \vee \mathcal{T}(\beta)) \wedge ((\mathsf{z}\overline{\mathsf{g}} = \bot) \vee \mathcal{T}(\gamma))$.
- For all $\mathcal{B} \in \mathsf{K}$ such that $\overline{z_{\mathcal{B}}} \neq \bot$, $\mathcal{B}/F_{\mathcal{B}} \vDash \mathcal{T}(\alpha) \wedge ((\mathsf{zg} = \bot) \vee \mathcal{T}(\beta)) \wedge ((\mathsf{z}\overline{\mathsf{g}} = \bot) \vee \mathcal{T}(\gamma))$.

Thus we are now deciding if two formulas, obtained effectively from φ, are in the first-order theories of $Q\mathsf{K}_0$ and $Q\mathsf{K}_1$ respectively. By assumption the two theories are decidable. Hence whether $\varphi \in \mathrm{Log}(\mathsf{K})$ is decidable. □

Theorem 3.5.11. *The following logics are decidable:*

- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\forall p(\mathtt{B}p \to p)$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathtt{B}\forall p(\mathtt{B}p \to p)$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \to p)$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\mathtt{B}\forall p(\mathtt{B}p \to p)$,
- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$.

Proof. As is argued above, each of them comes from a well-behaved class of complete proper filter algebras. Take $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ for example. It is the logic of CPFA. Using the notation above in Theorem 3.5.10, we only need to argue that $Q\mathsf{CPFA}_0$ and $Q\mathsf{CPFA}_1$ have decidable theories in $\mathcal{FOL}\mathsf{zg}$.

- $Q\mathsf{CPFA}_0$ is just the class of complete zg-algebras with z being the top element. This is because that if $\mathcal{B} = \langle B, F \rangle$ is such that $\overline{z_{\mathcal{B}}} = \bot$, then $z_{\mathcal{B}} = \top$, and hence F is the trivial filter. Thus $\mathcal{B}/F = \langle B, \top, g \rangle$ where g is the join of atoms. It is well known that the first-order theory of non-trivial complete Boolean algebras is decidable. To decide whether $\varphi \in \mathcal{FOL}\mathsf{zg}$ is valid in $Q\mathsf{CPFA}_0$, we only need to test whether the formula $(x = \top \wedge at(y)) \to \varphi[x/\mathsf{z}, y/\mathsf{g}]$ in $\mathcal{FOL}$ is valid in all non-trivial complete Boolean algebras, where x and y are two fresh variables and $at(y)$ states that y is the join of all atoms (which is expressible in $\mathcal{FOL}$).

- By Lemma 3.4.23, we see that the theory of $Q\mathsf{CPFA}_1$ is precisely the theory of all zg-algebras: FOLzg. This theory is decidable since the theory of all non-trivial Boolean algebras is well known to be decidable, and to test whether $\varphi \in \mathcal{FOL}\mathsf{zg}$ is in FOLzg, we only need to test whether the formula $at(x, y) \to \varphi[x/\mathsf{z}, y/\mathsf{g}]$ is valid in all non-trivial Boolean algebras, where x and y are fresh variables and $at(x, y)$ states that y is the join of the atoms below x.

The argument above clearly generalizes to all other cases, noting also that the first-order theory of atomic Boolean algebras, the first-order theory of complete and atomic Boolean algebras, and the first-order theory of two-element Boolean algebras are all decidable. We briefly sketch the $\mathcal{FOL}\mathsf{zg}$ theories we need for the other logics.

- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\forall p(\mathtt{B}p \to p)$ is the logic of complete proper filter algebras with the trivial filter. Calling this class K, the theory of $Q\mathsf{K}_0$ is the theory of complete zg-algebras with z being $\top$, and the theory of $Q\mathsf{K}_1$ is the inconsistent theory since $Q\mathsf{K}_1$ is empty.

- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathtt{B}\forall p(\mathtt{B}p \to p)$ is the logic of complete proper filter algebras with a principal filter. Calling this class K, the theory of $Q\mathsf{K}_0$ is the theory of complete zg-algebras with z being $\top$, and the theory of $Q\mathsf{K}_1$ is also the theory of complete zg-algebras with z being $\top$.

- $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathtt{B}\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$ is the logic of complete proper filter algebras with an ultrafilter. Calling this class K, the theory of $Q\mathsf{K}_0$ is the theory of two-element zg-algebras with z being $\top$, and the theory of $Q\mathsf{K}_1$ is the theory of two-element zg-algebras.

- For $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}$, the relevant $\mathcal{FOL}\mathsf{zg}$-theories are the theory of atomic and complete zg-algebras with z being $\top$ and the theory of zg-algebras with g being equal to z.

- For $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \to p)$, the relevant $\mathcal{FOL}\mathsf{zg}$-theories are the theory of atomic and complete zg-algebras and the inconsistent theory.

- For $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\mathtt{B}\forall p(\mathtt{B}p \to p)$, the relevant $\mathcal{FOL}\mathsf{zg}$-theory is the theory of atomic and complete zg-algebras with z being $\top$.

- For $\mathsf{KD4}^{\forall}\mathsf{5}\Pi\mathsf{At}\forall p(\mathtt{B}p \vee \mathtt{B}\neg p)$, the relevant $\mathcal{FOL}\mathsf{zg}$-theories are the theory of two-element zg-algebras with z being $\top$ and the theory of two-element zg-algebras. $\square$

3.6 Conclusion

In the previous sections, we have studied complete KD45 algebras, complete proper filter algebras, and logics in $\mathcal{L}\Pi$ extending $\mathsf{KD45}$ based on these algebras. It turns out that $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ is the weakest logic we can have if we use algebraic semantics based on complete Boolean algebras of propositions to extend $\mathsf{KD45}$ with propositional quantifiers. Beyond $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$, the semantics based on complete proper filter algebras is adequate for many logics, and we can even show some general completeness theorems. Moreover, the semantics is arguably intuitive for the language $\mathcal{L}\Pi$ as many properties of the algebras can be easily defined by the language, with atomicity being an exception, and we can determine decidability easily in many cases if the logic is coming from a class of complete proper filter algebras.

To conclude, we mention some directions of future research. First, noting that the set of measure 1 set in any probability space is always a proper filter in the algebra of events and that probability spaces are commonly used to model subjective credences, we may consider interpreting $\mathcal{L}\Pi$ on probability spaces and obtain a logic of "credence 1". The first difficulty for this is that in a probability space $\langle X, B, \mu\rangle$ with B the algebra of events, in most realistic cases, B is not lattice complete. To overcome this, it would be good to pin down exactly what is required for the well-definedness of the semantics of $\mathcal{L}\Pi$ and see how widely applicable the requirement is. Once this is done, to obtain the logic, our strategy above suggests that we need to study the natural quotient of $\langle B, F\rangle$ by F, the filter of measure 1 sets. It is well known that if B is a σ-algebra and μ is countably additive, then B/F is lattice complete. Roughly speaking, then, the first-order theory of the natural quotients of countably additive probability spaces by their filter of measure 1 sets is at least the first-order theory of complete Boolean algebras. On the other hand, if we do not assume countable additivity, then there seems to be little constraint on what the quotient could be. These two observations suggest that $\mathcal{L}\Pi$ is able to distinguish countably additive probability spaces from merely finitely additive probability spaces.

Second, we can include more modal operators in the language, each of which is interpreted by a proper filter. This is of course not the most general way to extend our language with multiple modalities. But if not careful, we may suddenly find ourselves on the other side of the axiomatizability boundary. Also, some special cases of this semantics may be of conceptual significance. For example, there can be two modal operators, one for "necessarily", which is interpreted using the trivial filter containing only the top element, and the other for "actually", interpreted by a complete ultrafilter, which is necessarily generated by an atom, or just an ultrafilter, if one would like to drop the assumption that there is an "actual world". Without the modality for necessity, the logic would be extending $\mathsf{KD4}^{\forall}\mathsf{5}\Pi$ with both Immod and $\forall p(\mathsf{B}p \lor \neg\mathsf{B}p)$ or just $\forall p(\mathsf{B}p \lor \neg\mathsf{B}p)$, depending on whether the ultrafilter is principal or not, as we have shown above. Another example is when the modal operators are belief operators of different agents, where the beliefs of all agents are publicly known to all agents, so that one agent believes that p if and only if any other agent believes that the former agent believes that p. We conjecture that the general idea of relativization to $\overline{z}$, zg, and $z\overline{g}$ on both the logic and the algebra sides can be generalized to deal with multiple filters too.

Third, in Section 3.2, we introduced semantics for $\mathcal{L}\Pi$ based on frames with propositional contingency and showed that while they cannot be used to refute Immod while maintaining $\mathsf{KD45}\Pi$ easily, they can be used to refute $\mathsf{4}^\forall$ while maintaining $\mathsf{KD45}\Pi$, something that cannot be done with complete BAOs. Many questions can be asked about these propositionally contingent frames. Relating to the logics above $\mathsf{KD45}$, we ask, first, whether we can refute Immod while validating $\mathsf{KD45}\Pi$ and what the accessibility relation will look like in a countermodel for this, and second, if we require that the accessibility relations have the properties corresponding to $\mathsf{KD45}$ (serial, transitive, and Euclidean), what will be the resulting logic and is it $\mathsf{KD45}\Pi\mathsf{Immod}$ if we also assume propositional richness?

Let us now consider the general method of extending normal modal logics with propositional quantifiers through complete algebras and raise some natural questions here. Let $\mathcal{L}$ be the quantifier free fragment $\mathcal{L}\Pi$, and let $\mathsf{CAlg}(\Gamma)$ be the class of lattice complete BAOs validating all formulas in Γ, with $\mathsf{CAlg}(\varphi)$ and CAlg abbreviating $\mathsf{CAlg}(\{\varphi\})$ and $\mathsf{CAlg}(\varnothing)$, respectively. Then, let $\mathrm{Log}(\mathsf{K})$ be the set of sentences in $\mathcal{L}\Pi$ validated by every member of K. Once these two operators are defined, a series of standard questions can be asked. Most notably is the question of characterizing the fixed points of this Galois connection, namely the classes of algebras of the form $\mathsf{CAlg}(\Gamma)$ and the sets of sentences of the form $\mathrm{Log}(\mathsf{K})$. But from the perspective of extending normal modal logics with propositional quantifiers, the natural object of study is $\mathrm{Log} \circ \mathsf{CAlg}$, an operator from $\wp(\mathcal{L})$ to $\wp(\mathcal{L}\Pi)$. A theorem we have shown in this chapter is that $\mathrm{Log}(\mathsf{CAlg}(\mathsf{KD45})) = \mathsf{KD4}^\forall\mathsf{5}\Pi$, where we see that axiom $\mathsf{4}$ is strengthened into $\mathsf{4}^\forall$. Note, however, that $\mathrm{Log}(\mathsf{CAlg}(\mathsf{S5})) = \mathsf{S5}\Pi$, in which case there is no strengthening of the axioms in $\mathsf{S5}$. In other words, for $\mathsf{S5}$, the syntactic way of extending it with propositional quantifiers by adding Π-principles and the semantic way of extending it by going through complete BAOs result in the same logic, while for $\mathsf{KD45}$ this is not so. In general, let us call a normal modal logic L in $\mathcal{L}$ $\mathcal{C}\Pi$-complete if $\mathrm{Log}(\mathsf{CAlg}(\mathsf{L})) = \mathsf{L}\Pi$; that is, the syntactic way and the semantic way of extending L to a Π-logic in $\mathcal{L}\Pi$ are the same judging from the final result. Then, we can ask what accounts for the distinction that $\mathsf{KD45}$ is $\mathcal{C}\Pi$-incomplete yet $\mathsf{S5}$ is $\mathcal{C}\Pi$-complete, and more generally we can ask whether there is a more logical or intrinsic way to characterize $\mathcal{C}\Pi$-(in)completeness.

The name "$\mathcal{C}\Pi$-completeness" we chose for the property is inspired by the well-studied property of $\mathcal{C}$-completeness of normal modal logics in $\mathcal{L}$. Recall that by definition, using our notation, a normal modal logic $\mathsf{L} \subseteq \mathcal{L}$ is $\mathcal{C}$-complete if and only if $\mathrm{Log}(\mathsf{CAlg}(\mathsf{L})) \cap \mathcal{L} = \mathsf{L}$. Given that the definitions of these two properties are similar in form, one might hope that there are some logical relations between them. However, if $\mathsf{4}^\forall$ is not in $\mathsf{KD45}\Pi$, as we believe, then $\mathcal{C}$-completeness does not imply $\mathcal{C}\Pi$-completeness since $\mathsf{KD45}$ is well known to be $\mathcal{C}$-complete (in fact Kripke complete or more algebraically $\mathcal{CAV}$-complete). The other direction is also not obvious. Suppose L is $\mathcal{C}$-incomplete. Then there is a $\varphi \in (\mathrm{Log}(\mathsf{CAlg}(\mathsf{L})) \cap \mathcal{L}) \setminus \mathsf{L}$. If we can show that $\varphi \notin \mathsf{L}\Pi$ then we will be done. However, this is not obvious as while $\varphi \in \mathcal{L}$, it may well be that $\mathsf{L}\Pi$ is not conservative over L and $\mathsf{L}\Pi \cap \mathcal{L} \supsetneq \mathsf{L}$ with φ witnessing the inequality. In general, we can call a normal modal logic L in $\mathcal{L}$ Π-conservative if $\mathsf{L}\Pi \cap \mathcal{L} = \mathsf{L}$. Then, it is easy to observe that $\mathcal{C}$-incompleteness plus Π-conservativity imply $\mathcal{C}\Pi$-incompleteness. However, it seems unlikely that $\mathcal{C}$-incompleteness and Π-conservativity can coexist, since the

Π-principles intuitively should help derive validities in complete BAOs that normal modal logics cannot. At any rate, the logical relations among the above three properties about normal modal logics regarding how they can be extended (with or without propositional quantifiers) using complete BAOs seem intricate and may be worthy of future research.

Finally, we would like to point out that our proof of the completeness theorem relies heavily on a syntactic reduction that can hardly be generalized below $\mathsf{KD45}$ since once we introduced z and g, by the end of the process, we see that all quantifiers are outside the scope of the modal operators, and moreover there is only one layer of modal operators. Once we let go of the $\mathsf{4}$ and $\mathsf{5}$ axioms, we can hardly achieve this result. Our strategy may still work when we study $\mathrm{Log}(\mathsf{CAlg}(\mathsf{K45}))$, but a method more generalizable is clearly needed if we want to venture further.

Chapter 4

The Logic of Comparative Cardinality

4.1 Introduction

Reasoning about the relative size of infinite sets has been a source of puzzles since at least Galileo [66]. Any consistent extension of the notion of relative size from finite to infinite sets must give us very different principles in the infinite compared to the finite. For two key principles that hold in the finite—that proper subsets are smaller than their supersets, and that sets in one-to-one correspondence have the same size—are inconsistent in the infinite. Cantor's theory of infinite cardinalities [27] maintains the latter principle at the expense of the former, while the more recent theory of infinite numerosities [126] does the reverse.

For logicians, a precise definition of relative size of sets raises an obvious question: can we completely axiomatize reasoning about the relative size of finite sets, of infinite sets, and of arbitrary sets in a formal set-theoretic language? Just as the laws for reasoning about intersection, union, and complementation of sets are captured by the laws of Boolean algebra, what are the laws one must add to Boolean algebra to capture reasoning about the relative size of sets according to the given definition?

In this chapter, we answer this question for a particular language and definition of relative size. Our language (see Definition 4.2.1) allows us to build terms using the standard set-theoretic operations of intersection, union, and complementation, and to express that a set s is at least as big as a set t: $|s| \geq |t|$. Thus, we work with a comparative notion of size, prior to the reification of sizes as cardinal numbers. The semantics is given by the Cantorian definition: $|s| \geq |t|$ is true iff there is an injection from t into s.

This language has an alternative interpretation in terms of the relative *likelihood* of events, instead of the relative size of sets. We will exploit this connection to prove some of our main results. In essence, Cantorian reasoning about the relative size of *finite* size is the same as *probabilistic* reasoning about the relative likelihood of events, while Cantorian reasoning about the relative size of *infinite* sets is the same as what is called *possibilistic* reasoning [46] about the relative likelihood of events. Each type of likelihood reasoning has been axiomatized by itself [145, 67, 24, 30]. If we reinterpret these results in terms of cardinality, then reasoning about the comparative cardinality of finite sets and reasoning about the comparative cardinality of infinite sets have each been axiomatized by themselves.

The goal of this chapter is to bridge the divide between finite and infinite and axiomatize reasoning about the cardinality of arbitrary sets. In § 4.2, we define our formal language and its interpretation in fields of sets. We then present our first axiomatization, which uses two extra predicates Fin and Inf to express that a set is finite or infinite. The axiomatization without these predicates is more complicated and saved for later. Both axiomatizations use the so-called finite cancellation axiom schema, which encodes an infinite sequence of axioms of exponentially growing length. In section § 4.3, we show how this schema can be replaced with the combination of a simple axiom and a simple rule. In § 4.4, we define models based on Boolean algebras to be used later and adapt to this context the representation theorem in the classic paper [109] by Kraft, Pratt, and Seidenberg. We also show the effective finite model property and as a corollary the decidability of our two logics (with or without the Fin and Inf predicates). In § 4.5, we construct canonical models from maximally consistent sets, as is common in proofs of completeness, leading in § 4.6 to the completeness of the system with extra predicates. In § 4.7, we first show in what sense finiteness and infiniteness of a set can be defined in the language with only cardinality comparisons between set terms. Then we finally define the axiomatic system mentioned in § 4.2 without the two extra predicates and prove its soundness and completeness. Lastly, we end with open problems in § 4.8.

Comparison to related work Two strands of work related to ours are worth mentioning. The first is the study of computable fragments of set theory, as in [26, 53]. For example, consider the quantifier-free language with intersection and set difference as binary functions and membership, inclusion, and equality as binary predicates. When this language is interpreted on the universe of all sets in the obvious way, the satisfiability problem is decidable; in fact, more functions and predicates can be added without loss of decidability [26]. In particular, a cardinality comparison predicate can be added, resulting in a language very similar to ours [53]. However, the language is still different from ours, due to its lack of set-theoretic complementation. Moreover, the cited works do not provide any axiomatization.

Another strand is the work on extending syllogistic logic with cardinality comparison initiated by Lawrence Moss (see [129] for an introduction). In this setting, the language consists of sentences of the form "all x are y", "some x are y", "there are at least as many x as y", and "there are more x than y" with variables interpreted as subsets of an arbitrary set. In [128, 130], axiomatizations of the valid sentences (on finite or infinite domains) are provided. However, in this setting there are no sentential Boolean connectives, nor Boolean set operators except complementation. Thus, the expressivity of the syllogistic language with cardinality comparisons is much weaker than ours, though with the consequent advantage of having a tractable satisfiability problem.

4.2 Formal setup and statement of main result

Definition 4.2.1. Given a countably infinite set Φ of *set labels*, the *set terms* t and *formulas* φ of the language $\mathcal{L}$ are generated by the following grammar:

$$\begin{aligned} t &::= a \mid t^c \mid (t \cap t) \\ \varphi &::= |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi), \end{aligned}$$

where $a \in \Phi$. The other sentential connectives $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual, and we use $\varphi \oplus \psi$ as an abbreviation for $(\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$. Standard set-theoretic notation may be defined as follows:

- $\varnothing := t \cap t^c$;
- $t \subseteq s := |\varnothing| \geq |t \cap s^c|$;
- $t = s := (t \subseteq s \wedge s \subseteq t)$ and $t \neq s := \neg(t = s)$;

- $t \not\subseteq s := \neg(t \subseteq s)$ and $t \subsetneq s := (t \subseteq s \wedge s \not\subseteq t)$.

We also use $|s| \leq |t|$ for $|t| \geq |s|$, $|s| > |t|$ for $\neg|t| \geq |s|$, and $|s| = |t|$ for $|s| \geq |t| \wedge |t| \geq |s|$. For any $\Delta \subseteq \Phi$, let $\mathcal{L}(\Delta)$ be the fragment of $\mathcal{L}$ using only set labels in Δ, and let $T(\Delta)$ be the set of set terms generated by Δ.

Our models consist essentially of a collection of sets, some of which are assigned set labels from Φ.

Definition 4.2.2. A *field of sets* is a pair $\langle X, \mathcal{F} \rangle$ where X is a nonempty set and $\mathcal{F}$ is a collection of subsets of X closed under intersection and set-theoretic complementation. A *field of sets model* is a triple $\mathcal{M} = \langle X, \mathcal{F}, V \rangle$ where $\langle X, \mathcal{F} \rangle$ is a field of sets and $V : \Phi \to \mathcal{F}$.

The satisfaction relation is defined in the obvious way.

Definition 4.2.3. Given a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V \rangle$, we define a function $\widehat{V}$, which assigns to each set term a set in $\mathcal{F}$, by:

- $\widehat{V}(a) = V(a)$ for $a \in \Phi$;
- $\widehat{V}(t^c) = X \setminus \widehat{V}(t)$;
- $\widehat{V}(t \cap s) = \widehat{V}(t) \cap \widehat{V}(s)$.

We then define a satisfaction relation $\vDash$ as follows:

- $\mathcal{M} \vDash |t| \geq |s|$ iff there is an injection from $\widehat{V}(s)$ into $\widehat{V}(t)$;
- $\mathcal{M} \vDash \neg\varphi$ iff $\mathcal{M} \nvDash \varphi$;
- $\mathcal{M} \vDash \varphi \wedge \psi$ iff $\mathcal{M} \vDash \varphi$ and $\mathcal{M} \vDash \psi$.

Given a class K of field of sets models, φ *is valid over* K iff $\mathcal{M} \vDash \varphi$ for all $\mathcal{M} \in \mathsf{K}$.

In Definition 4.7.12, we will define the cardinality comparison logic $\mathsf{CardCompLogic}$. Our main result is that this logic is sound and complete.

Theorem 4.2.4. *The cardinality comparison logic* $\mathsf{CardCompLogic}$ *is sound and complete with respect to field of sets models.*

The logic is somewhat complicated, so we will leave its definition for later. For now, it will be helpful to consider the expanded language $\mathcal{L}_{\text{Fin,Inf}}$ that adds predicates Fin and Inf that pick out the finite and infinite sets, respectively. Then the logic $\mathsf{CardCompLogic}$ can be obtained by eliminating Fin and Inf.

Definition 4.2.5. Let $\mathcal{L}_{\mathrm{Fin,Inf}}$ be the language extending $\mathcal{L}$ with two new unary predicates Fin and Inf using the following grammar:

$$\begin{array}{rcl} t & ::= & a \mid t^c \mid (t \cap t) \\ \varphi & ::= & \mathrm{Fin}(t) \mid \mathrm{Inf}(t) \mid |t| \geq |t| \mid \neg\varphi \mid (\varphi \wedge \varphi), \end{array}$$

where $a \in \Phi$.

Satisfaction can then be extended from $\mathcal{L}$ to $\mathcal{L}_{\mathrm{Fin,Inf}}$ as follows.

Definition 4.2.6. For any field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$, define the satisfaction relation $\vDash$ for $\mathcal{L}_{\mathrm{Fin,Inf}}$ with the following two new clauses:

- $\mathcal{M} \vDash \mathrm{Fin}(t)$ iff $\widehat{V}(t)$ is finite;
- $\mathcal{M} \vDash \mathrm{Inf}(t)$ iff $\widehat{V}(t)$ is infinite.

It will be convenient for later use to divide the logic of cardinality comparison with Fin and Inf into two parts, the first of which gives basic comparison principles such as transitivity and the second of which involves additional principles such as that infinite sets are larger than finite sets.

Definition 4.2.7. The basic comparison logic BasicCompLogic is the logic for $\mathcal{L}$ (or $\mathcal{L}_{\mathrm{Fin,Inf}}$) with the following axiom schemas and rules:

(BC1) all substitution instances of classical propositional tautologies;

(BC2) $\neg\, |\varnothing| \geq |\varnothing^c|$;

(BC3) $|s| \geq |t| \vee |t| \geq |s|$;

(BC4) $(|s| \geq |t| \wedge |t| \geq |u|) \to |s| \geq |u|$;

(BC5) $|\varnothing| \geq |s \cap t^c| \to |t| \geq |s|$;

(BC6) $(|\varnothing| \geq |s| \wedge |\varnothing| \geq |t|) \to |\varnothing| \geq |s \cup t|$;

(BC7) if φ and $\varphi \to \psi$ are theorems, so is ψ;

(BC8) if $t = 0$ is provable in the equational theory of Boolean algebras, then $|\varnothing| \geq |t|$ is a theorem.

Definition 4.2.8. The logic $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, the cardinality comparison logic with predicates Fin and Inf, consists of the axioms and rules of the basic comparison logic BasicCompLogic together with the following axiom schemas:

(A1) $\mathrm{Fin}(s) \oplus \mathrm{Inf}(s)$;

(A2) $\mathrm{Fin}(\varnothing) \wedge ((\mathrm{Fin}(s) \wedge \mathrm{Fin}(t)) \to \mathrm{Fin}(s \cup t))$;

(A3) $(\mathrm{Fin}(t) \wedge s \subseteq t) \to \mathrm{Fin}(s)$;

(A4) $(\mathrm{Fin}(s) \wedge \mathrm{Inf}(t)) \to |t| > |s|$;

(A5) $\bigwedge_{i=1}^{n}(\mathrm{Fin}(s_i) \wedge \mathrm{Fin}(t_i)) \to \mathsf{FC}_n(s_1, \cdots, s_n, t_1, \cdots, t_n)$ (for all $n \geq 1$);

(A6) $(\mathrm{Inf}(s) \wedge |s| \geq |t| \wedge |s| \geq |u|) \to |s| \geq |t \cup u|$;

Here $\mathsf{FC}_n(s_1, \cdots, s_n, t_1, \ldots, t_n)$ is what we call the *finite cancellation axiom.* To define this formula, first for each m such that $1 \leq m \leq n$, define the term S_m as the union of the terms of the form $s_1^{c_1} \cap s_2^{c_2} \cap \cdots \cap s_n^{c_n}$ where exactly m many c_i's are c and the rest are empty. Similarly define T_m with s replaced by t. Intuitively, S_n denotes the set of elements which are in exactly m many sets among the sets denoted by $s_1, s_2, \ldots, s_n$.

Then $\mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots, t_n)$ is defined by

$$\left(\bigwedge_{i=1}^{n} S_i = T_i\right) \to \left(\left(\bigwedge_{i=1}^{n-1} |s_i| \geq |t_i|\right) \to |t_n| \geq |s_n|\right).$$

The first four axioms set up the relations between finite sets and infinite sets—for example, that finite sets are smaller than infinite sets. Axioms (A5) and (A6) describe the distinct behavior of finite and infinite cardinal arithmetic. To understand (A5), suppose the condition expressed by $\bigwedge_{i=1}^{n} S_i = T_i$ is true, assuming that the sets denoted by the s_i's and t_i's are all finite. Note that to compute the sum K of the cardinalities of the sets denoted by the s_i's, instead of the most straightforward way of adding their cardinalities, we can consider how much each element contributes to K: if an element e is in k_e many sets denoted by the s_i's, then the contribution of this e is k_e, and then K is the sum of the k_e's. Thus,

$$\sum_{i=1}^{n} |V(s_i)| = \sum_{i=1}^{n} i \times |V(S_i)|$$

as S_i is precisely the set of elements that lie in exactly i many sets denoted by s_i's. The same holds for the t_i's. Then given that $\bigwedge_{i=1}^{n} S_i = T_i$ is true,

$$\sum_{i=1}^{n} |V(s_i)| = \sum_{i=1}^{n} i \times |V(S_i)| = \sum_{i=1}^{n} i \times |V(T_i)| = \sum_{i=1}^{n} |V(t_i)|.$$

Hence it is not hard to see that the consequent

$$\left(\bigwedge_{i=1}^{n-1} |s_i| \geq |t_i|\right) \to |t_n| \geq |s_n|$$

must be true in the same model.

Example 4.2.9. Let $A, B, C, D, E \subseteq X$ be disjoint and finite. Then it follows from

- $|A| \geq |B \cup C|$,
- $|B \cup E| \geq |A \cup C|$, and
- $|C \cup D| \geq |A \cup B|$,

that $|D \cup E| \geq |A \cup B \cup C|$. To see this, we only need to add the cardinalities of the inequalities on both sides, which leads to $|A| + |B| + |C| + |D| + |E| \geq |A| + |A| + |B| + |B| + |C| + |C|$. Hence by canceling $|A| + |B| + |C|$ since they are finite, we get $|D| + |E| \geq |A| + |B| + |C|$. Thus $|D \cup E| \geq |A \cup B \cup C|$. In our system, this reasoning is captured by

$$\mathsf{FC}_4(a, b \cup e, c \cup d, a \cup b \cup c, b \cup c, a \cup c, a \cup b, d \cup e)$$

with $a, b, c, d, e \in \Phi$, as the antecedent of FC_4 follows from the assumption that these five sets are disjoint, which can be expressed by formulas like $|a \cap b| = \varnothing$.

Finally, (A6) captures the distinct absorption property (or non-additivity) of infinite sets. In terms of the analogy with relative likelihood, it is (A5) that matches probabilistic reasoning, as is shown in [109] and [142], while (A6) matches what is called possibilistic reasoning [46].

Theorem 4.2.10. $\mathsf{CardCompLogic}_{\mathrm{Fin},\mathrm{Inf}}$*, the cardinality comparison logic with predicates* Fin *and* Inf*, is sound and complete with respect to field of sets models.*

Remark 4.2.11. Admittedly, (A5) is an infinite sequence of axioms that are long and somewhat complicated. We remark here that (A5) can be replaced by the combination of the following axiom and rule:

(A7) $(\mathrm{Fin}(s) \wedge \mathrm{Fin}(t)) \to (|s| \geq |t| \leftrightarrow |s \cap t^c| \geq |t \cap s^c|)$;

(A8) where $a|t$ abbreviates $|t \cap a| = |t \cap a^c|$ for $a \in \Phi$, if $a|t \to \varphi$ is derivable, then φ is derivable, assuming that a does not occur in t or in φ.

Axiom (A7) is sometimes called the quasi-additivity axiom. Intuitively, it means that taking unions with a disjoint set does not change the ordering of sets by cardinality. Rule (A8) is slightly non-standard. Intuitively, $a|t$ says that set a splits t into two parts of the same size. In both [109] and [24], this is expressed by "a polarizes t". Hence (A8) is called the "polarizability rule" in [24] ((A8) also appeared, but not as a rule in a formal system, in [109]). Proof theoretically, (A8) allows us to assume without loss of generality that any set can be polarized with a fresh set when proving some formula φ. Semantically, the idea behind (A8) is the invariance of truth under duplication: for any sets a, b, $|a| \succeq |b|$ iff $|a \times \{0, 1\}| \succeq |b \times \{0, 1\}|$, and a is finite iff $a \times \{0, 1\}$ is finite. This warrants the use of (A8), as to show that φ is true on all field-of-sets models, it is enough to focus on those models that come from duplication, in which every set in the field-of-sets can be polarized. The

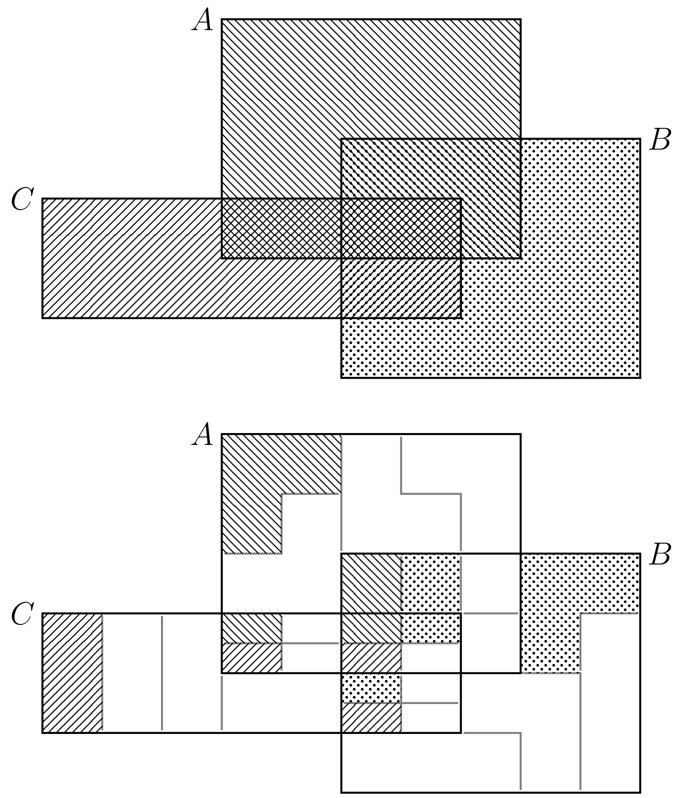


Figure 4.1: Polarization and set addition

power of (A8) lies in the fact that with it, we can simulate the addition of overlapping sets so that we can count overlaps correctly. Hence all instances of (A5) are provable from the system with (A5) replaced by (A7) and (A8). This point is already made implicitly in [109] and explicitly in [24], but we give a direct syntactic proof in section § 4.3. To get a flavor of the strategy, see Figure 4.1 where the shaded area in the second picture has a cardinality equal to one fourth of the sum of the cardinalities of the three larger sets (say A, B, and C). To see this, first polarize all minimal regions like $A \cap B^c \cap C^c$ and $A \cap B \cap C$ into four parts, and then for regions that are contained in m of the sets A, B, and C, select m parts of those regions. For example, as shown in the diagram, three of the four parts of region $A \cap B \cap C$ are selected, while only one of the four in $A \cap B^c \cap C^c$ is selected. So we have replaced the non-disjoint union of the shapes A, B, and C by a disjoint union of the shaded areas, while keeping the same area up to a factor of 1/4.

For the logic in the language $\mathcal{L}$, without the predicates Fin and Inf, it takes more work to capture the difference between finite and infinite cardinal arithmetic. The cardinality comparison logic CardCompLogic will also extend the same basic comparison logic BasicCompLogic. The key idea for the extra axioms is that of a set being "witnessed to be finite" or "witnessed to be infinite." For example, if

$$\mathcal{M} \vDash |s| \geq |s \cup t| \wedge |s \cap t| = |\varnothing|,$$

then s must denote an infinite set in $\mathcal{M}$. Since s denotes an infinite set, anything true of infinite sets must be true of s. One can think of the cardinality comparison logic $\mathsf{CardCompLogic}$ as being the same as the cardinality comparison logic $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ except with $\mathrm{Fin}(t)$ and $\mathrm{Inf}(t)$ being replaced by formulas that witness t to be finite or infinite, respectively.

Of course the definition of a set being finite or infinite would be superfluous if we were to restrict our models so that every set in the field of sets is finite (or infinite except for the empty set). In fact, letting $\mathsf{FinCardCompLogic}$ be the result of adding to $\mathsf{BasicCompLogic}$ all instances of $\mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots, t_n)$, $\mathsf{FinCardCompLogic}$ is sound and complete with respect to all field of sets models where the underlying set is finite. Similarly, the system $\mathsf{InfCardCompLogic}$ defined by adding $(|s| \geq |t| \wedge |s| \geq |u|) \to |s| \geq |t \cup u|$ is sound and complete with respect to all field of sets models where all nonempty sets in the field are infinite. We will not formally prove these two completeness results since the strategy we use to prove the completeness of $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ can be readily adapted.

4.3 Polarizability rule and finite cancellation axiom schema

Let $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ be the system obtained by adding axiom schemas and rules (A1)–(A4), (A6), (A7), and (A8) to $\mathsf{BasicCompLogic}$. In this section, we discuss how (A5) can be derived in $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. First, we verify that the rule (A8) is sound in the sense that if the premise $a|t \to \varphi$ is valid, then the conclusion φ is also valid.

Proposition 4.3.1. *The polarizability rule* (A8) *is sound on field of sets models.*

Proof. Suppose φ is not valid, so $\neg\varphi$ is satisfiable. Our goal is to show that $a|t \wedge \neg\varphi$ is satisfiable. Let Δ be the set of set labels in φ or t. Then by the constraint of (A7), $a \notin \Delta$. The strategy is simple: take a model $\mathcal{M}$ of $\neg\varphi$; construct the disjoint union $\mathcal{N}$ of two copies of $\mathcal{M}$ with a valued to exactly one copy of $\mathcal{M}$; then we have that both $a|t$ and $\neg\varphi$ are true in $\mathcal{N}$.

Formally, take a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V\rangle$ that makes $\neg\varphi$ true. Let $X' = X \times \{0,1\}$. Define a duplication function $d : \wp(X) \to \wp(X')$ by $d(S) = S \times \{0,1\}$. Then $d \circ V$ is a valuation on $\langle X', \wp(X')\rangle$. Define V' such that if $a \in \Delta$, then $V'(a) = d(V(a))$, and otherwise $V'(a) = X \times \{0\}$. Let $\mathcal{N} = \langle X', \wp(X'), V'\rangle$. A simple induction shows that for any term $t \in T(\Delta)$, $\widehat{V'}(t) = d(\widehat{V}(t)) = \widehat{V}(t) \times \{0,1\}$. This implies that for any two terms $s, t \in T(\Delta)$, $\mathcal{M} \vDash |s| \geq |t|$ iff $\mathcal{N} \vDash |s| \geq |t|$. In addition, a set S is finite iff $S \times \{0,1\}$ is finite. Another simple induction then shows that $\mathcal{M}$ and $\mathcal{N}$ satisfy the same formulas in $\mathcal{L}_{\mathrm{Fin,Inf}}$ using only labels in Δ. In particular, $\mathcal{N} \vDash \neg\varphi$. Since $a \notin \Delta$, we have $\widehat{V'}(a) = X \times \{0\}$ while $\widehat{V'}(t) = \widehat{V}(t) \times \{0,1\}$. Thus, $\mathcal{N} \vDash a|t$. $\square$

While it is not hard to understand the content of the polarizability rule itself, it is harder to see what it can prove and how it can be used.

Kraft, Pratt, and Seidenberg famously observed in [109] that without the polarizability rule (A8), the remaining system does not capture all valid reasoning patterns for finite sets, contrary to a conjecture of de Finetti [58].

For compact notation, we use the standard set theoretical definition of $n = \{0, 1, \dots, n-1\}$ and do not distinguish a sequence of length n and a function with domain n. We let ${}^n2$ denote the set of such functions/sequences with codomain 2. Then FC_n can be defined under this notation by the following.

Definition 4.3.2. For each $n, m \in \mathbb{N}$, sequence $\vec{s} = \langle s_0, \cdots, s_{n-1}\rangle$ of n terms, and $f \in {}^n2$, define the term

$$\vec{s}[f] = \bigcap\{s_i \mid f(i) = 1\} \cap \bigcap\{s_i^c \mid f(i) = 0\}$$

and the term

$$\mathsf{N}_m(\vec{s}) = \bigcup\{\vec{s}[f] \mid f : n \to 2 \text{ and } |f^{-1}(1)| = m\}.$$

For each $f \in {}^n2$, $\vec{s}[f]$ is intuitively a "definable" atom (in the Boolean algebra of terms constructible from $\vec{s}$), and $\mathsf{N}_m(\vec{s})$ is then the union of atoms that appear in exactly m terms in $\vec{s}$. Given two sequences $\vec{s}$ and $\vec{t}$ of n terms, we can then define the formula

$$\vec{s} \mathrel{\mathsf{E}} \vec{t} = \bigwedge_{0 \le i \le n} (\mathsf{N}_i(\vec{s}) = \mathsf{N}_i(\vec{t})).$$

Recall that equality between terms is defined in Definition 4.2.1. Then

$$\mathsf{FC}_n(\vec{s}, \vec{t}) = \vec{s} \mathrel{\mathsf{E}} \vec{t} \to ((\bigwedge_{i<n-1} |s_i| \ge |t_i|) \to |t_{n-1}| \ge |s_{n-1}|).$$

Consequently (A5) is now

$$(\mathrm{Fin}(\vec{s}) \wedge \mathrm{Fin}(\vec{t})) \to \mathsf{FC}_n(\vec{s}, \vec{t}), \tag{A5}$$

where Fin is extended to sequences of terms in the obvious way.

Now we show that we can derive (A5) in $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. The main strategy is to repeatedly use (A8) so that (A7) can be applied. In fact, Kraft, Pratt, and Seidenberg already sketched a proof of this in [109] for their Theorem 5. More specifically, the idea is the following, assuming that we are dealing with only finite set terms (for convenience we often speak loosely of terms as sets, say that one set is a subset of another when the relevant formula involving terms is provable, etc.):

1. Given two sequences $\vec{s}$ and $\vec{t}$ of length n, use the polarizability rule (A8) to keep polarizing atomic terms (minimal regions in the Venn diagram) definable from the terms in $\vec{s}$ and $\vec{t}$ until each atomic term is split into $2^n \ge 2n$ pieces of equal size.

2. Now each s_i and t_i are unions of definable atomic terms. For each s_i, define s'_i to be the union of the ith piece of the definable atomic terms that are subsets of s_i (so for example, if there are just s_1 and s_2, then s'_1 is the union of the first piece of $s_1 \cap s_2$ and the first piece of $s_1 \cap s_2^c$). Similarly define t'_i by using the $(i+n)$th pieces.

3. Then intuitively s'_i and t'_i are disjoint representatives of s_i and t_i: for any $i \neq j$, s'_i, s'_j, t'_i, and t'_j are all disjoint, and for each i, $|s'_i| = \frac{1}{2^n}|s_i|$ and $|t'_i| = \frac{1}{2^n}|t_i|$.

4. Recall that intuitively, when $\vec{s} \; \mathsf{E} \; \vec{t}$, we have $\sum_{i<n} |s_i| = \sum_{i<n} |t_i|$. This means $\sum_{i<n} |s'_i| = \sum_{i<n} |t'_i|$ as we just need to scale both sides by $\frac{1}{2^n}$. Also, since now the primed versions of s_i and t_i are disjoint, the sum of the sizes is just the size of the union. So intuitively we should get $\left|\bigcup_{i<n} s'_i\right| = \left|\bigcup_{i<n} t'_i\right|$. Indeed, this is derivable from $\vec{s} \; \mathsf{E} \; \vec{t}$.

5. Using (A7), which deals with disjoint unions, we can then derive $(\bigwedge_{i<n-1} |s'_i| \geq |t'_i|) \rightarrow |t'_{n-1}| \geq |s'_{n-1}|$. But recall that intuitively $|s'_i|$ and $|t'_i|$ are just $\frac{1}{2^n}$ of $|s_i|$ and $|t_i|$. Formally, this means that $|s_i| \geq |t_i| \leftrightarrow |s'_i| \geq |t'_i|$ is derivable for any $i < n$. So the real consequent of $\mathsf{FC}_n(\vec{s}, \vec{t})$ is derivable.

The rest of this section implements the sketch above formally in system $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$. We start with a lemma showing that for disjoint finite sets, cardinality comparison works as intended. Note that we have proved that theorems are closed under substitution in Lemma 4.5.1. Hence we will use substitution freely without explicit reference.

Lemma 4.3.3. *For any sequence $\vec{s}$ of n terms, define the disjointness of terms in $\vec{s}$ by*

$$\mathsf{D}(\vec{s}) := \bigwedge_{0 \leq i < j < n} (s_i \cap s_j) = \varnothing.$$

Then $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ derives the following with $\vec{s}$ a sequence of $2n$ terms:

$$\big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})\big) \rightarrow \Big(\big(\bigwedge_{i<n} |s_i| \geq |s_{i+n}|\big) \rightarrow |\bigcup_{i<n} s_i| \geq |\bigcup_{i<n} s_{i+n}|\Big); \tag{4.1}$$

$$\big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})\big) \rightarrow \Big(\big(\bigwedge_{i<n} |s_i| = |s_{i+n}|\big) \rightarrow |\bigcup_{i<n} s_i| = |\bigcup_{i<n} s_{i+n}|\Big); \tag{4.2}$$

$$\Big(\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s}) \wedge |\bigcup_{i<n} s_i| = |\bigcup_{i<n} s_{i+n}|\Big) \rightarrow \Big(\big(\bigwedge_{i<n-1} |s_i| \geq |s_{i+n}|\big) \rightarrow |s_{n-1}| \leq |s_{2n-1}|\Big). \tag{4.3}$$

Proof. Note that (4.2) follows directly from (4.1). Also, we need only prove the case for $n = 2$, as the general formula can then be derived inductively.

Suppose now that $\mathsf{D}(\vec{s}) \wedge \mathrm{Fin}(\vec{s})$ holds with $n = 2$. Then consider the following three terms: $s_{01} = s_0 \cup s_1, s_{12} = s_1 \cup s_2$, and $s_{23} = s_2 \cup s_3$. Using $\mathsf{BasicCompLogic}$, we have

$$s_{01} \cap s_{12}^c = s_0, \qquad s_{12} \cap s_{01}^c = s_2,$$
$$s_{12} \cap s_{23}^c = s_1, \qquad s_{23} \cap s_{12}^c = s_3.$$

So by (A7), we have

$$|s_{01}| \geq |s_{12}| \leftrightarrow |s_0| \geq |s_2|, \quad |s_{12}| \geq |s_{23}| \leftrightarrow |s_1| \geq |s_3|.$$

Hence, we get $(|s_0| \geq |s_2| \wedge |s_1| \geq |s_3|) \to |s_{01}| \geq |s_{12}|$. This shows (4.1). Also, when $|s_0| \geq |s_2|$, suppose further that $\neg|s_3| \geq |s_1|$, that is, $|s_1| > |s_3|$. Then $|s_{01}| \geq |s_{12}| > |s_{23}|$. Hence $|s_{01}| > |s_{23}|$, contradicting $|s_{01}| = |s_{23}|$.

For induction, we just need to consider the union of the first $n-1$ sets, the nth set, the next $n-1$ sets, and the last set as a four-set sequence. □

Proposition 4.3.4. $\mathsf{CardCompLogic}'_{\mathrm{Fin,Inf}}$ *derives* (A5).

Proof. Take an arbitrary sequence $\vec{s}$ of $2n$ terms. Let $\vec{s}_<$ be the sequence of the first n terms in $\vec{s}$ and $\vec{s}_>$ that of the last n terms. Similarly, for any function $f \in {}^{2n}2$, define $f_<$ to be the restriction of f on n and $f_>$ the restriction of f on $\{n, n+1, \cdots, 2n-1\}$. Our final goal is to derive

$$(\mathrm{Fin}(\vec{s}) \wedge \vec{s}_< \mathsf{E}\, \vec{s}_>) \to \Big(\big(\bigwedge_{i<n-1} |s_i| \geq |s_{i+n}|\big) \to |s_{n-1}| \leq |s_{2n-1}|\Big). \tag{4.4}$$

As we mentioned above, our strategy will be to "disjointify" $\vec{s}$ so that we can use (4.3) in Lemma 4.3.3. This is done by constructing in each s_i a subset s'_i so that $\langle s'_i \rangle_{i<2n}$ is a sequence of pairwise disjoint sets while each s'_i is $\frac{1}{2^n}$ of s_i. Then Lemma 4.3.3 can be applied.

More formally, our plan is to use the polarizability rule (A8) to construct a term s'_i for each $i < 2n$ so that the following three formulas are derivable:

$$(\bigwedge_{i<2n} s'_i \subseteq s_i) \wedge \mathsf{D}(\langle s'_i \rangle_{i<2n}); \tag{4.5}$$

$$\mathrm{Fin}(\vec{s}) \to (\bigwedge_{i<n} (|s'_i| \geq |s'_{i+n}| \leftrightarrow |s_i| \geq |s_{i+n}|)); \tag{4.6}$$

$$\vec{s}_< \mathsf{E}\, \vec{s}_> \to |\bigcup_{i<n} s'_i| = |\bigcup_{i<n} s'_{i+n}|. \tag{4.7}$$

Once the three formulas are derived, it is then quite obvious that the system can derive (4.4) with the help of (4.3). □

Hence the rest of this proof is devoted to the construction of s'_i and s'_{i+n} and the derivation of (4.5)–(4.7) above. Indented passages marked with a vertical line give details that may be skipped on a first reading.

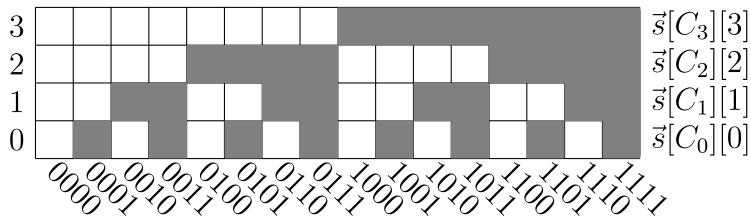


Figure 4.2: Polarization and construction when $n = 2$. Squares in the same row can be of different sizes. But squares in the same column must be of the same size.

Polarization and construction. By repeated use of (A8), for any $f \in {}^{2n}2$, we can also assume that $\vec{s}[f]$ is polarized into 2^n many pieces. Let us enumerate the partitions of $\vec{s}[f]$ by $\vec{s}[f][i]$ with $i < 2^n$. Let us also generalize the notation of $\vec{s}[f][i]$ to $\vec{s}[F][I]$ where $F \subseteq {}^{2n}2, I \subseteq 2^n$, defined by

$$\bigcup\{\vec{s}[f][i] \mid f \in F, i \in I\}.$$

Then we abbreviate $\vec{s}[\{f\}][I]$ as $\vec{s}[f][I]$ and $\vec{s}[F][\{i\}]$ as $\vec{s}[F][i]$.

Now define $C_i = \{f \in {}^{2n}2 \mid f(i) = 1\}$ for $i < 2n$. The equation $\vec{s}[C_i] = s_i$ is in the equational theory of Boolean algebras and hence is derivable in our system. Then for any $i < 2n$, our s_i' used in the outline above is defined by $\vec{s}[C_i][i]$ (note that for any $n \geq 1$, $2^n \geq 2n$). In Figure 4.2, we use a grid to illustrate the partition resulting from polarization. Each column is an $\vec{s}[f]$ for some $f \in {}^{2n}2$. Each cell is then an $\vec{s}[f][i]$. We shade $\vec{s}[C_i][i]$ for $i = 0, 1, 2, 3$, each in its own row; note that they are disjoint, and each is $1/4$ the size of the corresponding $\vec{s_i}$. This is essentially Figure 4.1 but since there are 4 sets, we choose not to draw a Venn diagram in the usual way.

The indented passage provides more details on the construction of $\vec{s}[f][i]$:

> We can prepare for each $\vec{s}[f]$ and each natural number $l < n$ a set of 2^l many fresh set labels. For convenience, we can just use functions in ${}^{l}2$. Now, we can first assume that the empty function ε polarizes $\vec{s}[f]$: $\varepsilon|\vec{s}[f]$. This gives us two sets: $\vec{s}[f] \cap \varepsilon$ and $\vec{s}[f] \cap \varepsilon^c$. Then we can inductively polarize the generated sets.
>
> In fact, for each function $g \in {}^{l}2$ with $l < n$, we can define
>
> $$\vec{s}[f][g] = \vec{s}[f] \cap \bigcap_{k<l} {g_{<k}}^{c \cdot g(j)},$$
>
> where $g_{<k}$ is g restricted to k and $c \cdot x$ is c if x is 1 and empty otherwise. Then (A8) allows us to assume that for all $0, 1$ sequences g of length at most $n-1$, $g|\vec{s}[f][g]$, or equivalently by our definition, $|\vec{s}[f][\langle g, 0\rangle]| = |\vec{s}[f][\langle g, 1\rangle]|$. Of course, due to the restriction of (A8), we need to arrange those formulas so that those with shorter variables come first. Fix

an enumeration $\langle g_i \rangle$ of $\bigcup_{l<n} {}^{l}2$ so that if g_j extends g_i then $j \geq i$. Then formally we are using (A8), so that it suffices to prove

$$g_{2^n-1}|\vec{s}[f][g_{2^n-1}] \to (\cdots (g_i|\vec{s}[f][g_i] \to (\cdots (g_0|\vec{s}[f][g_0] \to \varphi) \cdots)) \cdots)$$

when we want to prove φ. Hence, from now on, we have that each $\vec{s}[f]$ is polarized into 2^n many pieces, enumerated by $\vec{s}[f][g]$ with $g \in {}^{n}2$.

Deriving formula (4.5). Since for all $f \in C_i$, $\vec{s}[f][i] \subseteq s_i$ is obviously derivable, we have $\vec{s}[C_i][i] \subseteq s_i$. Hence the first part of (4.5) can be derived.

Disjointness is slightly less trivial. Recall that by our definition of $\vec{s}[f][i]$, for any $f \in {}^{2n}2$, $\vec{s}[f][i] \cap \vec{s}[f][j] = \varnothing$ is derivable when $i \neq j$. Thus when relativized to each $\vec{s}[f]$, $\vec{s}[C_i][i]$ and $\vec{s}[C_j][j]$ are disjoint for $i \neq j$. Some simple Boolean equational theory will then show that $\vec{s}[C_i][i]$ and $\vec{s}[C_j][j]$ themselves are disjoint.

Deriving formula (4.6). Assume $\mathrm{Fin}(\vec{s})$. Note that for any $f \in {}^{2n}2$ that is not constantly 0, there is an $i < 2n$ such that $\vec{s}[f] \subseteq s_i$ is derivable: just pick i with $f(i) = 1$. Hence, using (A2) and (A3), for any $I \subseteq 2^n$ and $F \subseteq {}^{2n}2$ with the constantly 0 function not in F, the system derives $\mathrm{Fin}(\vec{s}[F][I])$. Then, by repeated use of Lemma 4.3.3, the system derives that for any $i, j < 2^n$ and $f \in {}^{2n}2$ with f not constantly 0, $|\vec{s}[f][i]| = |\vec{s}[f][j]|$.

Recall how we defined $\vec{s}[f][i]$ by polarization. We can in fact use a simple induction on $0 < l < n$ to show that for each l and $g_0, g_1 \in {}^{l}2$, $|\vec{s}[f][g_0]| = |\vec{s}[f][g_1]|$ is derivable. The base case is when $l = 1$ and $g_0 = \langle 0 \rangle, g_1 = \langle 1 \rangle$. Here what we need to show is already assumed when we apply (A8): $\varepsilon|\vec{s}[f]$, as this is defined precisely as $|\vec{s}[f][\langle 0 \rangle]| = |\vec{s}[f][\langle 1 \rangle]|$. To go from l to $l+1$, note that any function in ${}^{l+1}2$ is obtained by appending a 0 or 1 to functions in ${}^{l}2$. So it is enough to show that for any $g_0, g_1 \in {}^{l}2$, the four sets in the sequence

$$\vec{t} = \langle \vec{s}[f][\langle g_0, 0 \rangle], \vec{s}[f][\langle g_0, 1 \rangle], \vec{s}[f][\langle g_1, 0 \rangle], \vec{s}[f][\langle g_1, 1 \rangle] \rangle$$

are of equal size. In the previous (unindented) paragraph, we have derived $\mathrm{Fin}(\vec{s}[F][I])$ for any F and I, and hence we have derived $\mathrm{Fin}(\vec{t})$. It is also obvious that the system can derive $\mathsf{D}(\vec{t})$ using the equational theory of Boolean algebras. By the induction hypothesis, we also have that the union of the first two and the last two are of equal size. Hence we can apply (4.3) to $\vec{t}$ and obtain

$$\vec{s}[f][\langle g_0, 0 \rangle] \geq \vec{s}[f][\langle g_1, 0 \rangle] \to \vec{s}[f][\langle g_1, 1 \rangle] \geq \vec{s}[f][\langle g_0, 1 \rangle].$$

By switching the first two and the second two sets in $\vec{t}$ and applying (4.3) again, we get

$$\vec{s}[f][\langle g_0, 0 \rangle] \leq \vec{s}[f][\langle g_1, 0 \rangle] \to \vec{s}[f][\langle g_1, 1 \rangle] \leq \vec{s}[f][\langle g_0, 1 \rangle].$$

Now $|\vec{s}[f][\langle g_0, 0 \rangle]| = |\vec{s}[f][\langle g_0, 1 \rangle]|$ and $|\vec{s}[f][\langle g_1, 0 \rangle]| = |\vec{s}[f][\langle g_1, 1 \rangle]|$ are derivable since we have assumed when using the polarizability rule (A8) that $g_0|\vec{s}[f][g_0]$ and $g_1|\vec{s}[f][g_1]$. With the transitivity of $\geq$ encoded by axiom (BC3), we can derive that the four sets involved are all equal in size. This shows that the 2^n subsets of $\vec{s}[f]$ obtained by polarization are of equal size whenever f is not constantly 0.

Since C_i does not contain the constantly 0 function and $\vec{s}[C_i][j]$ is the disjoint union of all $\vec{s}[f][j]$ with $f \in C_i$, using (4.2), we have $|\vec{s}[C_i][j]| = |\vec{s}[C_i][k]|$ for any $i < n$ and $j, k < 2^n$.

Now we can start to derive the consequent of (4.6). Fix an $i < n$. The idea is simple: $|\vec{s}[C_i][i]| > |\vec{s}[C_{i+n}][i+n]|$ iff for any j, $|\vec{s}[C_i][j]| > |\vec{s}[C_{i+n}][j]|$. Summing over j, this is equivalent to $|\vec{s}[C_i]| > |\vec{s}[C_{i+n}]|$. Of course, the equivalences must be derived by Lemma 4.3.3 and in particular (4.3).

First, since both C_i and C_{i+n} do not include the constantly 0 function, we can derive $\mathrm{Fin}(\vec{s}[C_i][j])$ and $\mathrm{Fin}(\vec{s}[C_{i+n}][j])$. With (A7), we have for all $j < 2^n$, $|\vec{s}[C_i][j]| \geq |\vec{s}[C_{i+n}][j]| \leftrightarrow |\vec{s}[C_i \setminus C_{i+n}][j]| \geq |\vec{s}[C_{i+n} \setminus C_i][j]|$. Let $\vec{t}$ be the sequence of 2×2^n terms with the first 2^n terms being $\langle \vec{s}[C_i \setminus C_{i+n}][j] \rangle_{j<2^n}$ and the rest being $\langle \vec{s}[C_{i+n} \setminus C_i] \rangle_{j<2^n}$. Also let $\vec{t'}$ be the same as $\vec{t}$ except that the first 2^n terms and the last 2^n terms are switched.

Then $\mathsf{D}(\vec{t})$ and $\mathsf{D}(\vec{t'})$ are derivable. This is because for any two terms, if they do not share the same second coordinate, then they are certainly disjoint. But if they do share the same second coordinate, then they are of the form $\vec{s}[C_i \setminus C_{i+n}][j]$ and $\vec{s}[C_{i+n} \setminus C_i][j]$, which are disjoint. Obviously we also have $\mathrm{Fin}(\vec{t})$ and $\mathrm{Fin}(\vec{t'})$.

Now, from left to right, suppose $|\vec{s}[C_i][i]| \geq |\vec{s}[C_{i+n}][i+n]|$. Then, for any $j < 2^n$, we have $|\vec{s}[C_i][j]| \geq |\vec{s}[C_{i+n}][j]|$. By (A7), this implies $|\vec{s}[C_i \setminus C_{i+n}][j]| \geq |\vec{s}[C_{i+n} \setminus C_i][j]|$. Then we can apply (4.2) to $\vec{t}$ and obtain $|\vec{s}[C_i \setminus C_{i+n}]| = |\vec{s}[C_{i+n} \setminus C_i]|$. But by (A7) again, this gives us $|\vec{s}[C_i]| \geq |\vec{s}[C_{i+n}]|$.

From right to left, assume $|\vec{s}[C_i]| \geq |\vec{s}[C_{i+n}]|$ and suppose for contradiction that $|\vec{s}[C_{i+n}][i+n]| > |\vec{s}[C_i][i]|$. Then for any $j < 2^n$, we have $|\vec{s}[C_{i+n}][j]| > |\vec{s}[C_i][j]|$. By (A7), this implies $|\vec{s}[C_{i+n} \setminus C_i][j]| > |\vec{s}[C_i \setminus C_{i+n}][j]|$. Thus, in sequence $\vec{t'}$ the first 2^n terms are strictly larger than the last 2^n terms, respectively. By (BC3), $>$ implies $\geq$. Hence, by (4.1), $|\vec{s}[C_{i+n}]| \geq |\vec{s}[C_i]|$, as they are the unions of the first and last 2^n terms, respectively. Together with the assumption, we have $|\vec{s}[C_{i+n}]| = |\vec{s}[C_i]|$. At this point, we can apply (4.3) and obtain $|\vec{s}[C_{i+n} \setminus C_i][2^n-1]| \leq |\vec{s}[C_i \setminus C_{i+n}][2^n-1]|$. With (A7), this contradicts $|\vec{s}[C_{i+n}][2^n-1]| > |\vec{s}[C_i][2^n-1]|$, which is derived from $|\vec{s}[C_{i+n}][i+n]| > |\vec{s}[C_i][i]|$.

Deriving formula (4.7). First, note that

$$\begin{aligned}\bigcup_{i<n} \vec{s}[C_i][i] &= \bigcup_{i<n} \bigcup_{f \in C_i} \vec{s}[f][i] = \bigcup_{f \in {}^{2n}2} \vec{s}[f][f_<^{-1}(1)],\\ \bigcup_{i<n} \vec{s}[C_{i+n}][i+n] &= \bigcup_{i=n}^{2n-1} \bigcup_{f \in C_i} \vec{s}[f][i] = \bigcup_{f \in {}^{2n}2} \vec{s}[f][f_>^{-1}(1)].\end{aligned} \tag{4.8}$$

Now assume $\vec{s}_< \mathsf{E}\, \vec{s}_>$. Recall that our goal is to derive $|\bigcup_{i<n} \vec{s}[C_i][i]| = |\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]|$.

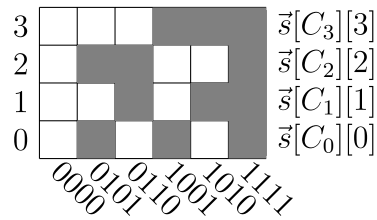


Figure 4.3: The grid with $n = 2$ when $\vec{s}_< \mathsf{E}\, \vec{s}_>$. Recall that squares in the same column are of the same size. It is not hard to see then that $|\vec{s}[C_0][0] \cup \vec{s}[C_1][1]| = |\vec{s}[C_2][2] \cup \vec{s}[C_3][3]|$ by comparing them in each column.

Our strategy is the following. When we assume $\vec{s}_< \mathsf{E}\, \vec{s}_>$, we can show that for any $f \in {}^{2n}2$, treated as a sequence of 0's and 1's, if the number of 1's in the first n places of f and the number of 1's in the last n places of n are not equal, then $\vec{s}[f] = \varnothing$ can be derived. We can call f "balanced" when this condition is satisfied; when f is not balanced, $\vec{s}[f] = \varnothing$ can be derived. However, for those balanced f, when restricted to $\vec{s}[f]$, $\bigcup_{i<n} \vec{s}[C_i][i]$ and $\vec{s}[C_i][i]| = |\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]|$ are of the same size. For a simple illustration, see Figure 4.3. Then summing over all balanced f, we obtain the required formula.

Pick an arbitrary $f \in {}^{2n}2$ and let $k_< = |f_<^{-1}(1)|, k_> = |f_>^{-1}(1)|$. Then it is easy to see that the system can derive the following through the equational theory of Boolean algebras:

$$\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<) \wedge \vec{s}[f] \subseteq \mathsf{N}_{k_>}(\vec{s}_>).$$

Also by the definition of N in Definition 4.3.2 and by using the equational theory of Boolean algebras, $\mathsf{N}_i(\vec{s}_>) \cap \mathsf{N}_j(\vec{s}_>) = \varnothing$ and $\mathsf{N}_i(\vec{s}_>) \subseteq (\mathsf{N}_j(\vec{s}_>))^c$ are derivable when $i \neq j$. Hence, if $k_< \neq k_>$, then $\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<)$ and also $\vec{s}[f] \subseteq (\mathsf{N}_{k_<}(\vec{s}_>))^c$. Since we have assumed $\vec{s}_< \mathsf{E}\, \vec{s}_>$, we have $\mathsf{N}_{k_<}(\vec{s}_<) = \mathsf{N}_{k_<}(\vec{s}_>)$. This means that we can derive $\vec{s}[f] \subseteq \mathsf{N}_{k_<}(\vec{s}_<) \wedge \vec{s}[f] \subseteq (\mathsf{N}_{k_<}(\vec{s}_<))^c$ and then $\vec{s}[f] = \varnothing$.

Now we derive that $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$. When $k_< \neq k_>$, we derive $\vec{s}[f] = \varnothing$. Then trivially $|\vec{s}[f][f_<^{-1}(1)]| = |\varnothing| = |\vec{s}[f][f_>^{-1}(1)]|$.

If $k_< = k_>$, then let $k = k_< = k_>$ and consider the sequence $\vec{t}$ where the terms are $\langle \vec{s}[f][i] \rangle_{f \in C_i}$:

- $\mathsf{D}(\vec{t})$ is derivable using the equational theory of Boolean algebras.
- $\vec{t}$ has $2k$ terms; the union of the first k terms is $\vec{s}[f][f_<^{-1}(1)]$ and the union of the last k terms is $\vec{s}[f][f_>^{-1}(1)]$;
- we showed when we derived (4.6) that for any i, j, $|\vec{s}[f][i]| = |\vec{s}[f][j]|$; hence for any $i < k$, $|t_i| = |t_{i+k}|$.

Given these three points, we can apply (4.2) to $\vec{t}$ and derive the equation $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$.

In sum, we have derived for any $f \in {}^{2n}2$ the equation $|\vec{s}[f][f_<^{-1}(1)]| = |\vec{s}[f][f_>^{-1}(1)]|$. Then we can apply (4.2) to the sequence where the first 2^{2n} terms are $\langle \vec{s}[f][f_<^{-1}(1)] \rangle_{f \in {}^{2n}2}$ and the last 2^{2n} are $\langle \vec{s}[f][f_>^{-1}(1)] \rangle_{f \in {}^{2n}2}$. Hence it is derivable that the unions of each, which by (4.8) are just $\bigcup_{i<n} \vec{s}[C_i][i]$ and $\bigcup_{i<n} \vec{s}[C_{i+n}][i+n]$, are of equal size.

This completes the whole proof.

4.4 Other types of models

While our interest is in field of sets models, it is convenient to think in terms of an abstract Boolean algebra rather than a concrete field of sets. For then we do not have to worry about which particular elements a set contains, but instead we only have to consider the cardinality of the set. This will help us focus on the structures related to the truth of formulas and to show the effective finite model property. In the following, we use $\wedge, \vee$, and $'$ for meet, join, and complementation in arbitrarily picked Boolean algebras. For specifically constructed Boolean algebras, the symbols for the operations may change and we will usually specify only the complementation and meet operation. The lattice ordering of a Boolean algebra will be denoted by $\leq$ (below) and $\geq$ (above[1]), possibly with a subscript to show which Boolean algebra we are talking about. Since the models defined below are all Boolean algebras with extra structure, we call them *algebra-based models* and call them finite when the underlying Boolean algebra is finite. As a convenient notation, for any models $\mathcal{B}, \mathcal{C}$ and set L of formulas, we write $\mathcal{B} \equiv_L \mathcal{C}$ when for any $\varphi \in L$, $\mathcal{B} \vDash \varphi$ iff $\mathcal{C} \vDash \varphi$.

4.4.1 Measure algebra models

The first step is to forget the elements in sets and only keep their Boolean structure and their cardinality. This gives us the following definition.

Definition 4.4.1. A *measure algebra* is a pair $\langle B, \mu \rangle$ where B is a Boolean algebra and μ is a function assigning a cardinal to each element of B such that

- if $a \wedge b = \bot$, then $\mu(a \vee b) = \mu(a) + \mu(b)$, and
- $\mu(b) = 0$ iff $b = \bot$.

We call such a cardinal-valued function μ with which $\langle B, \mu \rangle$ is a measure algebra a *cardinal measure* on B.

A *measure algebra model* is a triple $\mathcal{B} = \langle B, \mu, V \rangle$ where $\langle B, \mu \rangle$ is a measure algebra and V is a function from Φ to B. This V can be extended to a function $\widehat{V}$ from $T(\Phi)$ to B

[1] We use "below" and "above" in the weak sense.

as in Definition 4.2.3 but using the Boolean complement and meet in place of set-theoretic complement and intersection.

Note that μ is only finitely additive, which is good enough because the language is finitary and unable to express countable additivity.

Definition 4.4.2. Given a measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$, we define the satisfaction relation $\vDash$ as follows, where $\varphi, \psi \in \mathcal{L}$ and $s, t \in T(\Phi)$:

- $\mathcal{B} \vDash |t| \geq |s|$ iff $\mu(V(t)) \geq \mu(V(s))$;
- $\mathcal{B} \vDash \neg\varphi$ iff $\mathcal{B} \nvDash \varphi$;
- $\mathcal{B} \vDash \varphi \wedge \psi$ iff $\mathcal{B} \vDash \varphi$ and $\mathcal{B} \vDash \psi$.

We also have the following two clauses for $\mathcal{L}_{\text{Fin,Inf}}$ sentences:

- $\mathcal{B} \vDash \text{Inf}(t)$ iff $\mu(\widehat{V}(t))$ is infinite;
- $\mathcal{B} \vDash \text{Fin}(t)$ iff $\mu(\widehat{V}(t))$ is finite.

We can turn any field of sets $\langle X, \mathcal{F} \rangle$ into a measure algebra $\langle B, \mu \rangle$ by setting $B = \langle \mathcal{F}, X \setminus \cdot, \cap \rangle$ and $\mu(a) = |a|$ for $a \in B$. It is easy to check that this is a measure algebra. If we have a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V \rangle$ then we get a measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$ using the same valuation V; moreover, $\mathcal{M} \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}$ by a simple induction.

On the other hand, given a *finite* measure algebra model $\mathcal{B}$, we can turn it into a field of sets model $\mathcal{M}$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}$. Since the cardinal measure functions in measure algebras are only finitely additive, the construction will fail for infinite measure algebra models.

Proposition 4.4.3. *For any finite measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$, there is a field of sets model $\mathcal{M} = \langle X, \mathcal{F}, V' \rangle$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}$.*

Proof. Since B is finite, let $a_1, \ldots, a_n$ be the atoms of B. Let $S_1, \ldots, S_n$ be disjoint sets with $|S_i| = \mu(a_i)$. Let $X = \bigcup_{i=1}^n S_i$ and let $\mathcal{F}$ be the field of sets generated by $S_1, \ldots, S_n$ under complementation (in X) and intersection. The map $f(a_i) = S_i$ extends to an isomorphism between B and $\mathcal{F}$, which maps an element $a_{i_1} \vee \cdots \vee a_{i_\ell}$ to $S_{i_1} \cup \cdots \cup S_{i_\ell}$. We have

$$\begin{aligned} \mu(a_{i_1} \vee \cdots \vee a_{i_\ell}) &= \mu(a_{i_1}) + \cdots + \mu(a_{i_\ell}) \\ &= |S_{i_1}| + \cdots + |S_{i_\ell}| \\ &= |S_{i_1} \cup \cdots \cup S_{i_\ell}|. \end{aligned}$$

Thus, $|f(a)| = \mu(a)$. Let $V' = f \circ V$. Then it is easy to see that for any set term s, $|V(s)| = \mu(V'(s))$, since f is an isomorphism preserving cardinalities. A simple induction then shows that $\langle X, \mathcal{F}, V' \rangle \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}$. $\square$

Thus, there is no loss of generality when focusing on measure algebra models if we consider only finite models, and as we will see later in this section, there is also no loss of generality in restricting to finite models.

4.4.2 Comparison algebra models

As one often does to prove the completeness of some logic, we will use a canonical model construction. In building the canonical model, we start with a maximally consistent set of sentences in our language $\mathcal{L}$ or $\mathcal{L}_{\text{Fin,Inf}}$, which encodes only comparisons between set terms and their being finite or infinite in the case of $\mathcal{L}_{\text{Fin,Inf}}$. Hence it will be convenient to forget about the cardinals we assign to each element of the Boolean algebra and to remember only the comparisons between elements. When we need to work with $\mathcal{L}_{\text{Fin,Inf}}$, we also need the model to contain a set of distinguished elements.

Definition 4.4.4. A *comparison algebra* is a pair $\langle B, \succeq \rangle$ where B is a Boolean algebra and $\succeq$ is a total preorder on B such that

- for all $a, b \in B$, $a \geq_B b$ implies $a \succeq b$, and
- $\bot_B \not\succeq b$ for all $b \in B \setminus \{\bot_B\}$.

A *labeled comparison algebra* is a triple $\langle B, \succeq, F \rangle$ where $\langle B, \succeq \rangle$ is a comparison algebra and $F \subseteq B$. A *comparison algebra model* is a triple $\mathcal{B} = \langle B, \succeq, V \rangle$ where $\langle B, \succeq \rangle$ is a comparison algebra and V a function from Φ to B. Similarly, a *labeled comparison algebra model* is a labeled comparison algebra model together with a valuation. The valuation V can be extended in the usual way to a valuation $\widehat{V}$ from $T(\Phi)$ to B.

Here the relation $\succeq$ is intended to interpret "at least as great in cardinality as" and F is intended to interpret "being a finite set". Hence we have the required constraints for $\succeq$ in the above definition: it is a total preorder, extends the Boolean lattice order (set inclusion relation), and makes the bottom element (the empty set) the strictly smallest set. Formally, the interpretation is given by the satisfaction relation.

Definition 4.4.5. Given a comparison algebra model $\mathcal{B} = \langle B, \succeq, V \rangle$, we define the satisfaction relation $\vDash$ for $\mathcal{L}$ as follows, where $s, t \in T(\Phi)$:

- $\mathcal{B} \vDash |s| \geq |t|$ iff $\widehat{V}(s) \succeq \widehat{V}(t)$;
- usual clauses for $\neg$ and $\wedge$.

Given a labeled comparison algebra model $\mathcal{B} = \langle B, \succeq, F, V \rangle$ the satisfaction relation $\vDash$ can be extended to $\mathcal{L}_{\text{Fin,Inf}}$ with the extra clauses:

- $\mathcal{B} \vDash \text{Fin}(s)$ iff $\widehat{V}(s) \in F$;
- $\mathcal{B} \vDash \text{Inf}(s)$ iff $\widehat{V}(s) \notin F$.

While $\succeq$ is intended to compare cardinality and F is intended to include exactly finite elements, the requirements given above are not enough to let us know that it is cardinality that $\succeq$ is comparing and that elements in F are precisely those that are finite. Given a measure algebra $\langle B, \mu \rangle$, we can easily build a comparison algebra $\langle B, \succeq \rangle$ by taking $a \succeq b$ if and only if $\mu(a) \geq \mu(b)$ and further a labeled measure algebra $\langle B, \succeq, F \rangle$ by taking $F = \{b \in B \mid \mu(b) \text{ is finite}\}$. But for the other direction, to fix that $\succeq$ is comparing cardinality and F captures finiteness, we need some extra conditions. We state this in terms of a representation theorem.

Definition 4.4.6. A comparison algebra $\langle B, \succeq \rangle$ (labeled comparison algebra $\langle B, \succeq, F \rangle$) is *represented by* a cardinal measure μ on B if for all $a, b \in B$, we have $a \succeq b$ iff $\mu(a) \geq \mu(b)$ (and $F = \{b \in B \mid \mu(b) \text{ is finite}\}$). We also say $\langle B, \succeq \rangle$ or $\langle B, \succeq, F \rangle$ is represented by a measure algebra $\mathcal{B}'$ when it is represented by μ and $\mathcal{B}' = \langle B, \mu \rangle$. A (labeled) comparison algebra model is representable if its (labeled) comparison algebra part is representable.

Clearly if a finite (labeled) comparison algebra model $\langle \mathcal{B}, V \rangle$ is represented by a measure algebra $\langle \mathcal{B}', V \rangle$, then $\mathcal{B} \equiv_{\mathcal{L}_{\mathrm{Fin,Inf}}} \mathcal{B}'$. Hence if φ is satisfiable on a finite representable (labeled) comparison algebra model, then, in light of Proposition 4.4.3, φ is satisfiable on a field of sets model.

Before proving the full representation, we recall the following classic theorem on when an ordering is representable by a probability measure.

Theorem 4.4.7 (Kraft, Pratt, Seidenberg [109], Theorem 2)**.** *For any finite Boolean algebra B with $\top$ as the top element and $\bot$ the bottom element and any binary relation $\succeq$ on B, there is a probability measure μ on B such that for all $a, b \in B$, $a \succeq b$ iff $\mu(a) \geq \mu(b)$, if and only if the following conditions are satisfied:*

- *not $\bot \succeq \top$;*
- *for all $b \in B$, $b \succeq \bot$;*
- *$\succeq$ is transitive, and for any $a, b \in B$, $a \succeq b$ or $b \succeq a$;*
- *for any two sequences of elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ from B of equal length, if every atom of B is below (in the order of the Boolean algebra) exactly as many a's as b's, and if $a_i \succeq b_i$ for all $i \in \{1, \ldots, n-1\}$, then $b_n \succeq a_n$.*

The fourth condition, known also as *finite cancellation*, is precisely the truth condition of FC_n. Put more algebraically, if we represent elements in B as their characteristic functions over the atoms of B and further identify those functions with vectors of 0's and 1's, then the vector sum of a's being the same as the vector sum of b's implies that the sum of the probabilities of a's is also equal to the sum of the probabilities of b's. This is of course because vector sums count overlaps properly, unlike unions. Dana Scott used this representation in [142] and provided a lucid proof of the above theorem. It can also be observed from the proof in [109] (see Corollary 2) or [142] that the probability measure μ can be turned into an additive function to non-negative rational numbers and then to natural numbers by scaling, since μ is obtained by solving a finite system of (possibly strict) linear inequalities with rational coefficients.

With this component dealing with finite elements, we can prove the representation theorem for both finite and infinite elements.

Theorem 4.4.8. *A finite labeled comparison algebra $\mathcal{B} = \langle B, \succeq, F\rangle$ is represented by some cardinal measure μ on B if and only if the following conditions hold:*

(1) F is an ideal;

(2) elements in F satisfy the finite cancellation condition in Theorem 4.4.7: for any two sequences of elements $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ from F of equal length, if every atom of B is below (in the order of the Boolean algebra) exactly as many a's as b's, and if $a_i \succeq b_i$ for all $i \in \{1, \ldots, n-1\}$, then $b_n \succeq a_n$;

(3) for any $a, b, c \in B$ such that $a \notin F$, if $a \succeq b$ and $a \succeq c$, then $a \succeq b \vee_B c$;

(4) for any $a, b \in B$, if $a \in F$ and $b \notin F$, then $b \succeq a$ and not $a \succeq b$.

It is then easy to see that a finite comparison algebra $\mathcal{B} = \langle B, \succeq\rangle$ is represented by μ if and only if there exists an $F \subseteq B$ such that $\langle B, \succeq, F\rangle$ is represented by μ.

Proof. For the proof, we use the following definitions. For any Boolean algebra B and $b \in B$, let $At(B)$ be the set of all atoms in B and $At(b)$ the set of atoms below b. Given a preorder $\succeq$ on B and $b \in B$, let $[b]$ be $\{b' \in B \mid b' \succeq b \text{ and } b \succeq b'\}$. Then define $Rank(b)$ for $b \in B$ to be the number of atoms strictly smaller than b in the order $\succeq$, modulo equivalence, i.e., the cardinality of the set $\{[a] \mid a \in At(B), b \succ a\}$. Finally, let $\max At(b)$ be any one of the $\succeq$-maximal elements in $At(b)$, if there is such an element.

Since B is finite, F is a finite ideal and hence principal. Then the quotient $B|_F$ becomes a finite Boolean algebra with a binary relation that satisfies all the conditions required in Theorem 4.4.7 because we required that the finite cancellation condition holds for all elements in F. Hence there is an additive measure function μ_0 from $B|_F$ to $\mathbb{N}$ such that for any $b_1, b_2 \in B|_F$, we have $b_1 \succeq b_2$ iff $\mu_0(b_1) \geq \mu_0(b_2)$.

If $B = B|_F$ then we are done, so we now consider the case where $B \neq B|_F$. Consider an arbitrary element b outside $B|_F$. Because B is finite, it is atomic and hence $At(b)$ must contain an atom that is not in $B|_F$. Then by (4) this atom is strictly greater in $\succeq$ than all atoms in $B|_F$. Hence $\max At(b)$ is outside $B|_F$ (it exists because B is finite).

By definition, $\max At(b)$ is an atom under b. Thus $b \succeq \max At(b)$. But because $b \notin F$, we can also show that $\max At(b) \succeq b$ using condition (3). To see this, list $At(b)$ as $b_1, b_2, \ldots, b_n$. Then we have the following inductive argument:

- $\max At(b) \succeq b_1$;

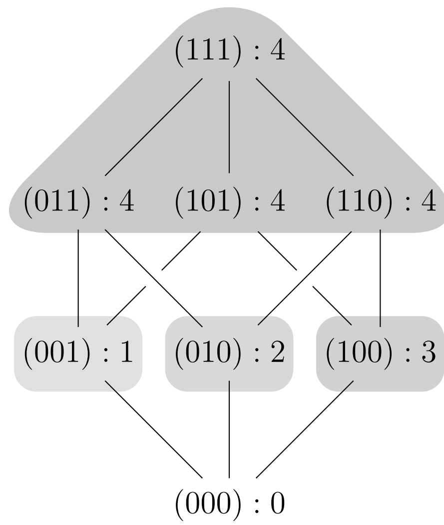


Figure 4.4: A non-representable comparison algebra

- supposing $\max At(b) \succeq \bigvee_{i=1}^{k} b_i$, then together with $\max At(b) \succeq b_{k+1}$ and condition (3), $\max At(b) \succeq \bigvee_{i=1}^{k} b_i \vee b_{k+1} = \bigvee_{i=1}^{k+1} b_i$.

Hence at the end of the induction we have $\max At(b) \succeq \bigvee_{i=1}^{n} b_i = b$.

Now define a measure μ on B as follows:

$$\mu(b) = \begin{cases} \mu_0(b) & b \in B|_F \\ \aleph_{Rank(\max At(b))} & b \notin B|_F. \end{cases}$$

It is not hard to see that this is indeed a measure function on B. Moreover, we now show that for any $b_1, b_2 \in B$, we have $b_1 \succeq b_2$ iff $\mu(b_1) \geq \mu(b_2)$:

- if both $b_1, b_2 \in B|_F$, then we can use μ_0;
- if both $b_1, b_2 \notin B|_F$, then $b_1 \succeq b_2$ iff $\max At(b_1) \succeq \max At(b_2)$ iff $Rank(\max At(b_1)) \geq Rank(\max At(b_2))$ iff $\mu(b_1) \geq \mu(b_2)$;
- if $b_1 \in B|_F$ and $b_2 \notin B|_F$, then $b_2 \succeq b_1$ by condition (4), but it is also trivially true that $\mu(b_2) \geq \mu(b_1)$. □

The above theorem again only works for finite models. Representation of infinite models requires different conditions and techniques to prove. We will review this as an open problem in § 4.8.

Example 4.4.9. Figure 4.4.2 presents a comparison algebra that cannot be represented by any cardinal measure. The number after the colon in each node is the rank of that node in $\succeq$; this determines the preorder $\succeq$, as $x \succeq y$ iff the rank of x is at least that of y. We also group the nodes of the same rank into shaded areas. We can then see that if $\succeq$ were representable by a cardinal measure, all nodes would be finite. To see this, note that (110) is the join of (010) and (100), but (110) is also strictly greater than both of them, which implies that (110) is finite. Now (110) is of rank 4, which means it is as large as any other set, so all the sets would be finite. However, we can also see that, letting $a_1 = (101), a_2 = (010), b_1 = (110), b_2 = (001)$, the finite cancellation condition fails. First, every atom is below exactly one of a_1 and a_2 and also exactly one of b_1 or b_2. Thus, the antecedent of finite cancellation is true. But the consequent is false, as $a_1 \succeq a_2$ but not $b_2 \succeq b_1$. The non-representability of this comparison algebra also implies, by the previous theorem, that the required ideal does not exist.

4.4.3 Effective finite model property

In the previous two subsections, we saw that our representation theorems (Proposition 4.4.3 and Theorem 4.4.8) only work on finite models. However, to make a formula true, we only need finite models. In fact, we can effectively bound the size of satisfying models for any formula that is satisfiable by some (possibly infinite) field of sets model. Since the construction of a finite satisfying model will be used later, we provide a systematic treatment, starting with the following definition.

Definition 4.4.10. Let $\mathcal{B}$ be an algebra-based model and $\Delta \subseteq \Phi$. $\mathcal{B}$ is *adapted to* Δ if $\widehat{V}$ is surjective from $T(\Delta)$ to the underlying Boolean algebra B in $\mathcal{B}$.

The importance of this definition is that for any algebra-based model $\mathcal{B}$ that is adapted to Δ, every element $b \in \mathcal{B}$ is named in the sense that there exists $t \in T(\Delta)$ such that $\widehat{V}(t) = b$. It is easy to see that an algebra-based model adapted to a finite set Δ is finite. To be more precise, when $\Delta \subseteq \Phi$ is finite, let $T_0(\Delta)$ be the set of all distinct terms in Δ in disjunctive normal form with no repetition of conjuncts or disjuncts. $T_0(\Delta)$ is finite, and using Boolean identities, for every term $t \in T(\Delta)$, there is a term $t' \in T_0(\Delta)$ such that $\widehat{V}(t)$ is always the same as $\widehat{V}(t')$. Thus, in any model, $\widehat{V}(T(\Delta)) = \widehat{V}(T_0(\Delta))$ is finite.

Proposition 4.4.11. *Fix a finite set $\Delta \subseteq \Phi$. For any measure (resp. comparison, labeled comparison) algebra model $\mathcal{B}$, there is a measure (resp. comparison, labeled comparison) algebra model $\mathcal{B}_\Delta$ that is adapted to Δ and satisfies $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$.*

Proof. For any measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$, define $\mathcal{B}_\Delta = \langle B_\Delta, \mu_\Delta, V_\Delta\rangle$ where the B_Δ is the subalgebra of B with $\widehat{V}(T(\Delta))$ as the carrier set, μ_Δ is the restriction of μ to $\widehat{V}$, and V_Δ is defined as:

$$V_\Delta(a) = \begin{cases} V(a) & a \in \Delta \\ \bot_B & a \notin \Delta. \end{cases}$$

Similarly, for any comparison algebra model $\mathcal{B} = \langle B, \succeq, V \rangle$, we can define $\mathcal{B}_\Delta = \langle B_\Delta, \succeq_\Delta, V_\Delta \rangle$ where now $\succeq_\Delta$ is the restriction of $\succeq$ to $\widehat{V}(T(\Delta))$. It is not hard to see that $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$. For labeled comparison algebras, we just need to further define $F_\Delta = F \cap \widehat{V}(T(\Delta))$. $\square$

Now we prove the effective finite model property.

Theorem 4.4.12. *For any formula $\varphi \in \mathcal{L}_{\text{Fin,Inf}}$, φ is satisfied by some field of sets model if and only if it is satisfied by a labeled comparison algebra model $\langle B, \succeq, F, V \rangle$ such that:*

1. *B is finite with at most $2^{|\Delta|}$ many atoms where Δ is the set of set labels appearing in φ, and*

2. *$\langle B, \succeq, F \rangle$ is representable (i.e., satisfies the conditions listed in Theorem 4.4.8).*

The right-to-left direction does not require any bound on the size of B, so long as it is finite. In addition, it is decidable whether a finite labeled comparison algebra model is representable. The complexity of deciding whether a finite labeled comparison algebra model is representable is NP in the size of the underlying Boolean algebra of the labeled comparison algebra model.

Proof. The right-to-left direction is immediate by Theorem 4.4.8 and Proposition 4.4.3. For the left-to-right direction, suppose $\mathcal{M} = \langle X, \mathcal{F}, V \rangle \vDash \varphi$, and let Δ be the set of set labels appearing in φ. As we described above, the field of sets model can be naturally turned into a measure algebra model $\mathcal{B} = \langle \mathcal{F}, \mu, V \rangle$ and then into a labeled comparison algebra model $\mathcal{C} = \langle \mathcal{F}, \succeq, F, V \rangle$ such that $\mathcal{M} \equiv_{\mathcal{L}_{\text{Fin,Inf}}} \mathcal{B}, \mathcal{C}$. By adapting $\mathcal{C}$ to Δ using Proposition 4.4.11, we obtain $\mathcal{C}_\Delta$ that satisfies the same formulas in $\mathcal{L}_{\text{Fin,Inf}}(\Delta)$, which includes φ. Note that $\mathcal{C}_\Delta$ is represented by $\mathcal{B}_\Delta$. Also, since $\mathcal{B}_\Delta$ is adapted to Δ, the size of the Boolean algebra base of $\mathcal{B}_\Delta$ is at most $2^{2^{|\Delta|}}$, and there are at most $2^{|\Delta|}$ many atoms.

To decide whether a finite labeled comparison algebra $\langle B, \succeq, F, V \rangle$ is representable, the only non-trivial part is to verify whether $B|_F$ satisfies the finite cancellation condition. However, rather than using this characterization, it is easier to naively check the definition of representability; this is an integer linear programming problem with at most $2^{|\Delta|}$ (the number of atoms in B) many variables and $(2^{2^{|\Delta|}})^2$ many inequalities, the coefficients of which are all in $\{0, 1, -1\}$. According to a standard result on integer linear programming (see [134]), the complexity is NP in the size of B. $\square$

As a simple corollary, the sets of sentences in $\mathcal{L}$ or $\mathcal{L}_{\text{Fin,Inf}}$ valid on all field of sets models are decidable, as to decide whether φ is satisfiable (or equivalently whether $\neg\varphi$ is valid) we can enumerate all labeled comparison algebra models of size up to $2^{2^{|\Delta|}}$ and for each one first check if it is representable and then check if φ is satisfied. If φ is never satisfied in this procedure, then φ is in fact unsatisfiable.

4.5 Canonical comparison algebra models

This section is devoted to the construction of canonical comparison algebra models. In Definition 4.2.7, we formulated a logic that captures our basic intuitions about the comparison of "sizes" of sets. For example, (BC2) says that the complement of the empty set is strictly larger than the empty set, which we should assume as otherwise all set terms would simply be empty. (BC3) and (BC4) together provide the order structure of the "sizes" of sets: it is a total preorder. (BC5) and (BC6) gives two basic interactions between set construction and size comparison: (BC5) says that if a set is a subset of another, then the size of the subset should be no greater than that of the superset, and (BC6) says that the union of two empty sets (those with sizes no greater than the empty set) is still empty. (This is the union of the empty set with itself, though two different set terms denote it.)

Those axioms may capture some notion of "size" comparison, but they are not enough to completely capture the notion of "cardinality" comparison. Loosely speaking, we may treat "cardinality" as a special kind of "size", less general but perhaps more interesting, due to the distinct behaviors of finite and infinite cardinalities. The following theorem says that the basic comparison logic BasicCompLogic captures precisely the notion of "size" comparison in comparison algebra models defined in Definition 4.4.4. This will be useful, as to show that CardCompLogic captures the notion of "cardinality" comparison, we only need to show that the difference between the two logics captures the difference between the models: the extra properties identified in Theorem 4.4.8 that make a comparison algebra model *representable* by a cardinal measure.

Lemma 4.5.1. BasicCompLogic *derives the following for terms s, t, u, s', t':*

1. *$s = t$ if it is provable in the equational theory of Boolean algebras.*

2. *$\subseteq$ is a preorder: $s \subseteq s$, $(s \subseteq t \wedge t \subseteq u) \to s \subseteq u$.*

3. *$=$ is an equivalence relation: $s = s$, $s = t \to t = s$, $(s = t \wedge t = u) \to s = u$.*

4. *$\subseteq$ works as the subset relation: $s \subseteq t \to t^c \subseteq s^c$, $s \subseteq t \to (s \cap u) \subseteq t$, $(s \subseteq t \wedge s \subseteq u) \to s \subseteq (t \cap u)$, and $(s \subseteq u \wedge t \subseteq u) \to (s \cup t) \subseteq u$.*

5. *$=$ is a congruence relation: $s = t \to s^c = t^c$ and $(s = s' \wedge t = t') \to (s \cap t) = (s' \cap t')$. With axioms in* $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$*, $s = t \to (\mathrm{Fin}(s) \leftrightarrow \mathrm{Fin}(t))$ and $s = t \to (\mathrm{Inf}(s) \leftrightarrow \mathrm{Inf}(t))$ are also derivable.*

6. *Substitution of equal terms: $s = t \to (\varphi \leftrightarrow \psi)$ for all formulas φ and ψ in $\mathcal{L}$ where ψ is obtained from φ by replacing one or more occurrences of s by t. When using* $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$*, this substitution schema is valid for all $\varphi \in \mathcal{L}_{\mathrm{Fin,Inf}}$.*

Proof. If $s = t$ is provable in the equational theory of Boolean algebras, then so are $0 = s \cap t^c$ and $0 = t \cap s^c$. Then by axiom (BC8), both $|\varnothing| \geq |s \cap t^c|$ and $|\varnothing| \geq |t \cap s^c|$ are provable.

But they are abbreviated by $s \subseteq t$ and $t \subseteq s$. Putting them together, we have that $s = t$ is provable.

Note that $s = s$ is obviously provable. So we have $s \subseteq s$ as well. Now assume $s \subseteq t$ and $t \subseteq u$. They abbreviate $|\varnothing| \geq |s \cap t^c|$ and $|\varnothing| \geq |t \cap u^c|$. By (BC6), we have $|\varnothing| \geq |(s \cap t^c) \cup (t \cap u^c)|$. Note that the following is in the equational theory of Boolean algebras by distinguishing cases t and t^c:

$$0 = (s \cap u^c) \cap ((s \cap t^c) \cup (t \cap u^c))^c.$$

Hence by (BC7), we have $(s \cap u^c) \subseteq ((s \cap t^c) \cup (t \cap u^c))$. Then by (BC5), we have $|s \cap u^c| \leq |(s \cap t^c) \cup (t \cap u^c)|$. Combining this with what we derived by (BC6) above, using (BC4) we have $|\varnothing| \geq |s \cap u^c|$, which is just $s \subseteq u$.

Part 3 follows directly from part 2 with just a few Boolean manipulations. Part 4 is also not hard using the same technique we used in part 2. The congruence over complementation and union in part 5 follows from part 4 by Boolean manipulations. The congruence over Fin is an easy consequence of (A3). Note also that using (A1), (A4), and (BC5), we have $(\mathrm{Inf}(s) \wedge s \subseteq t) \to \mathrm{Inf}(t)$. So we can also easily derive the congruence over Inf.

Finally, to show substitution, we need to use two inductions. First, an induction on terms using part 5 will show that for any four terms s, t, u_0, u_1 with u_1 being the result of replacing some occurrences of s in u_0 by t, we can derive $s = t \to u_0 = u_1$. Obviously for any terms s, t, we can derive $s = t \to |s| = |t|$. So we proved substitution for atomic sentences in $\mathcal{L}$. When we are in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, part 5 also provides substitution for the rest of the atomic sentences in $\mathcal{L}_{\mathrm{Fin,Inf}}$. Then a simple induction on formulas will do, since $\leftrightarrow$ is again congruential over $\neg$ and $\wedge$. □

Theorem 4.5.2. *For any set X of $\mathcal{L}$-sentences that is maximally consistent relative to* $\mathsf{BasicCompLogic}$*, there exists a comparison algebra model $\mathcal{C}^X$ such that $\mathcal{C}^X \vDash X$.*

Proof. For terms s and t, define $s \subseteq_X t$ iff $s \subseteq t \in X$, $s \succeq_X t$ iff $|s| \geq |t| \in X$, $s =_X t$ iff $s = t \in X$, and $s \simeq_X t$ iff $s \succeq_X t$ and $t \succeq_X s$, for all $s, t \in T(\Phi)$. We also write $s \equiv t$ when $s = t$ is provable in the equational theory of Boolean algebras.

Note that by the maximality of X and Lemma 4.5.1, $=_X$ is a congruence relation on $T(\Phi)$. Let $\mathcal{B}^X = \langle T(\Phi)/_{=_X}, \cdot^c, \sqcap\rangle$ be the homomorphic image of the term algebra $T(\Phi)$, with the homomorphism $[\cdot]_{=_X}$ that sends a term to its equivalence class under $=_X$, and $[s]^c_{=_X} = [s^c]_{=_X}, [s]_{=_X} \sqcap [t]_{=_X} = [s \cap t]_{=_X}$. Since $=_X$ extends $\equiv$ (by Lemma 4.5.1 again), $\mathcal{B}^X$ is a Boolean algebra. The bottom element $\bot_{\mathcal{B}^X}$ is obviously $[\varnothing]_{=_X}$ as it is the meet of $[t]_{=_X}$ and $[t^c]_{=_X}$, but the latter is just $[t]^c_{=_X}$. Regarding the Boolean lattice ordering in $\mathcal{B}^X$, note that:

$$\begin{aligned}
[s]_{=_X} \leq_{\mathcal{B}^X} [t]_{=_X} &\Leftrightarrow [s]_{=_X} \sqcap [t]^c_{=_X} = \bot_{\mathcal{B}^X} \\
&\Leftrightarrow [s \cap t^c]_{=_X} = [\varnothing]_{=_X} \\
&\Leftrightarrow s \cap t^c =_X \varnothing.
\end{aligned}$$

It is also not hard to see that $s \cap t^c =_X \varnothing$ iff $s \subseteq_X t$ since $\mathsf{BasicCompLogic}$ derives $s \cap t^c = \varnothing \leftrightarrow s \subseteq t$. Hence $\leq_{\mathcal{B}^X}$ is just $\subseteq_X$.

Now we add a comparison structure to $\mathcal{B}^X$. Note that by (BC5), $\simeq_X$ extends $=_X$. So we can take the quotient $\succeq_X/_{=_X}$, so that $[s]_{=_X} \succeq_X/_{=_X} [t]_{=_X}$ iff $s \succeq_X t$. Let $\mathcal{C}^X = \langle \mathcal{B}^X, \succeq_X/_{=_X}, [\cdot]_{=_X} \rangle$. Now we show that $\mathcal{C}^X$ is a comparison algebra model:

- We have just shown that $\leq_{\mathcal{B}^X}$ is identical to $\subseteq_X$. By (BC5), $\succeq_X$ extends $\supseteq_X$. So $\succeq_X$ extends $\geq_{\mathcal{B}^X}$.
- Suppose $\bot_{\mathcal{B}^X} \succeq_X/_{=_X} [s]_{=_X}$. Then by definition, $\varnothing \succeq_X s$. What we need is $[s]_{=_X} = \bot_{\mathcal{B}^X}$. To show this, we just need $\mathsf{BasicCompLogic}$ to derive $|\varnothing| \geq |s| \to s = \varnothing$ as the antecedent is given by $\varnothing \succeq_X s$. First, $\varnothing \subseteq s$ is derivable trivially. To derive $s \subseteq \varnothing$, we need $|\varnothing| \geq |s \cap \varnothing^c|$ by definition. Obviously $s \cap \varnothing \equiv s$. So we can substitute and then use the assumption that $|\varnothing| \geq |s|$.

Finally, we verify that $\varphi \in X$ iff $\mathcal{C}^X \vDash \varphi$. For the atomic case, consider a formula $|s| \geq |t|$ for arbitrary $s, t \in T(\Phi)$. Then $|s| \geq |t| \in X$ iff $s \succeq_X t$ iff $[s]_{=_X} \succeq_X/_{=_X} [t]_{=_X}$ iff $\mathcal{C}^X \vDash |s| \geq |t|$. The induction is trivial. $\square$

4.6 Completeness with predicates for infinite and finite sets

In this short section, we will prove Theorem 4.2.10, which says that the cardinality comparison logic with Fin and Inf, $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, is sound and complete with respect to field of sets models. It is not hard to check that it is sound with respect to field of sets models as well as measure algebra models and labeled comparison algebra models. For completeness, we follow the standard strategy by starting with a consistent formula φ, building a canonical labeled comparison model satisfying φ, adapting it to the set labels appearing in φ so that we obtain a finite model, and finally using the fact that the canonical model must also satisfy all the axioms to show that it is representable. By Theorem 4.4.12, this means φ is satisfied by a field of sets model.

Theorem 4.2.10. $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ *is sound and complete with respect to the class of all measure algebra models and also the class of all field of sets models.*

Proof. Soundness is almost trivial. For completeness, we show that every formula φ that is consistent in $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ is also satisfied by a measure algebra model. Since φ is consistent, let X be a maximally consistent set that contains φ.

Since $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$ includes $\mathsf{BasicCompLogic} \subseteq \mathcal{L}$, $X|_{\mathcal{L}} = X \cap \mathcal{L}$ is a maximally consistent set for the logic $\mathsf{BasicCompLogic}$ in $\mathcal{L}$. By the canonical model theorem (Theorem 4.5.2),

$$\mathcal{C} = \langle \langle T(\Phi)/_{=_{X|_{\mathcal{L}}}}, \cdot^c, \sqcap \rangle, \succeq_{X|_{\mathcal{L}}}/_{=_{X|_{\mathcal{L}}}}, [\cdot]_{=_{X|_{\mathcal{L}}}} \rangle$$

is a comparison algebra model and $\mathcal{C} \vDash X|_{\mathcal{L}}$. Now we need to build an $F \subseteq T(\Phi)/_{X|_{\mathcal{L}}}$ to interpret Fin and Inf. Define $[s]_{=_{X|_{\mathcal{L}}}} \in F$ iff $\text{Fin}(s) \in X$. For this F to be well defined, we need to show that if $s =_{X|_{\mathcal{L}}} t$, then $\text{Fin}(s) \in X$ iff $\text{Fin}(t) \in X$. As shown in the beginning of the proof of Theorem 4.5.2, $=_{X|_{\mathcal{L}}}$ is extended by $\simeq_{X|_{\mathcal{L}}}$. Thus, once $s =_{X|_{\mathcal{L}}} t$, both $|s| \geq |t|$ and $|t| \geq |s|$ are in $X|_{\mathcal{L}}$. By axiom (A3) in $\mathsf{CardCompLogic}_{\text{Fin,Inf}}$ and the maximality of X, this implies $\text{Fin}(s) \in X$ iff $\text{Fin}(t) \in X$.

So we can define $F = \{[s]_{=_{X|_{\mathcal{L}}}} \mid \text{Fin}(s) \in X\}$. Then

$$\mathcal{C}^F = \langle \langle T(\Phi)/_{=_{X|_{\mathcal{L}}}}, \cdot^c, \sqcap \rangle, \succeq_{X|_{\mathcal{L}}}/_{=_{X|_{\mathcal{L}}}}, F, [\cdot]_{=_{X|_{\mathcal{L}}}} \rangle$$

is a labeled comparison algebra model. For any $s, t \in T(\Phi)$, $\mathcal{C}^F \vDash |s| \geq |t|$ iff $\mathcal{C} \vDash |s| \geq |t|$ iff $|s| \geq |t| \in X$; and $\mathcal{C}^F \vDash \text{Fin}(s)$ iff $\text{Fin}(s) \in X$. Because of axiom (A1) in $\mathsf{CardCompLogic}_{\text{Fin,Inf}}$, $\text{Inf}(s) \in X$ iff $\text{Fin}(s) \notin X$. So it follows that $\mathcal{C}^F \vDash \text{Inf}(s)$ iff $s \notin F$ iff $\text{Fin}(s) \notin X$ iff $\text{Inf}(s) \in X$. Then a simple inductive argument on $\mathcal{L}_{\text{Fin,Inf}}$ shows that $\mathcal{C}^F \vDash X$. Hence, in particular, $\mathcal{C}^F \vDash \varphi$. In other words, φ is satisfied by the labeled comparison algebra model $\mathcal{C}^F$ that also satisfies the axioms of $\mathsf{CardCompLogic}_{\text{Fin,Inf}}$.

Now we adapt $\mathcal{C}^F$ to the finite set Δ of the set labels appearing in φ. By Proposition 4.4.11, the resulting model $\mathcal{C}^F_\Delta$ is finite and satisfies the same formulas in $\mathcal{L}_{\text{Fin,Inf}}$ as $\mathcal{C}^F$ does. This implies that:

- $\mathcal{C}^F_\Delta \vDash \varphi$.
- Given that $\mathcal{C}^F_\Delta$ is adapted to Δ, every element $b \in \mathcal{C}^F_\Delta$ is equal to $\widehat{V}(t)$ for some $t \in T(\Delta)$, where V is the valuation in $\mathcal{C}^F_\Delta$.
- $\mathcal{C}^F_\Delta$ is representable, using Theorem 4.4.8. As we noted before Proposition 4.4.11, it is finite. Now we need to check the four conditions listed in Theorem 4.4.8. Condition (1) is guaranteed since all instances of (A2) and (A3) are theorems and hence true of $\mathcal{C}^F_\Delta$, which means we can apply (A2) and (A3) to every element as they are all named by terms. Hence it is clear that condition (1) is true. Similarly conditions (2), (3), and (4) are guaranteed by axioms (A5), (A6), and (A4), respectively.

Then by Theorem 4.4.12, φ is satisfied by a field of sets model. This completes the proof of completeness. □

4.7 Completeness without predicates for infinite and finite sets

In this section, we define the logic $\mathsf{CardCompLogic}$ and prove Theorem 4.2.4.

Theorem 4.2.4. *The cardinality comparison logic* $\mathsf{CardCompLogic}$ *is sound and complete with respect to field of sets models.*

By earlier results, it suffices to consider only measure algebra models. The key idea is to find two formulas Fin and Inf in $\mathcal{L}$ to replace the two primitive predicates, Fin and Inf, added in $\mathcal{L}_{\text{Fin,Inf}}$. The general strategy is as follows:

1. Show that there is a way to define formulas $\mathsf{Fin}(u)$ and $\mathsf{Inf}(u)$ in $\mathcal{L}$, instead of adding the two extra predicates, to capture the finiteness or infiniteness of u on all adapted measure algebra models except a very special class of models, which we call *flexible* models in § 4.7.1. We use the name "flexible" because in those models we can change the cardinality of an element to be anything finite or infinite without changing the comparative structure.
2. Then, in § 4.7.2, we give the axioms for $\mathsf{CardCompLogic}$ using the formulas Fin and Inf defined in § 4.7.1, and we show that any (adapted) comparison algebra model satisfying these axioms can be turned into a measure algebra model. This uses Theorem 4.4.8 and splits into two cases depending on whether the model is flexible or not.
3. Finally, in § 4.7.3, given a formula φ consistent with $\mathsf{CardCompLogic}$, we use the canonical model construction of Theorem 4.5.2 to build a comparison algebra model satisfying φ. Then, using previous results in this section, we can assume that the model is adapted and hence turn it into a measure algebra model.

4.7.1 Flexible models

In this subsection, we define flexible models and show how they appear when we try to define Fin and Inf in $\mathcal{L}$. Essentially, flexible models are models where our definition, or in fact any definition to capture Fin and Inf in language $\mathcal{L}$, fails. This is because we can make the cardinality of an element in a flexible model anything we like, be it finite or infinite, without changing the formulas in $\mathcal{L}$ satisfied by that model.

Definition 4.7.2. A finite measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$ is *flexible* if there is an atom a in B whose measure is strictly smaller than the measure of all other atoms in B, and a is the only atom in B with finite measure, if there is any atom with finite measure.

The following two propositions show why we call such models flexible.

Proposition 4.7.3. *If $\mathcal{B} = \langle B, \mu, V\rangle$ is a flexible finite measure algebra model, then for any non-bottom element $b \in B$, we have*

$$\mu(b) = \max\{\mu(a) \mid a \in At(B), a \leq b\}.$$

Proof. Write b as a finite join of the atoms below it: $b = a_1 \vee \cdots \vee a_n$. Then $\mu(b) = \mu(a_1) + \cdots + \mu(a_n)$. If b is an atom, then $\mu(b) = \mu(a_1)$; otherwise, $n \geq 2$ and at least one of $\mu(a_1), \ldots, \mu(a_n)$ is infinite, so

$$\mu(b) = \max\{\mu(a_1), \ldots, \mu(a_n)\} = \max\{\mu(a) \mid a \in At(B), a \leq b\}. \qquad \square$$

Proposition 4.7.4. *For any flexible finite measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$ and cardinal κ, if a_0 is the atom of B with the smallest measure, then there is a flexible finite measure algebra model $\mathcal{C} = \langle B, \nu, V\rangle$ such that:*

1. $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$;
2. $\nu(a_0) = \kappa$.

In other words, for any flexible finite measure algebra model, the measure of the smallest atom does not matter, if we are only concerned with the truth of formulas in $\mathcal{L}(\Delta)$.

Proof. First, $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$ when for any $b_1, b_2 \in B$, we have $\mu(b_1) \geq \mu(b_2)$ iff $\nu(b_1) \geq \nu(b_2)$.

But notice that by Proposition 4.7.3, both $\mu(b)$ and $\nu(b)$ are calculated by taking maximums of the measures of the atoms below b, given that the $\langle B, \nu, V\rangle$ is flexible. This means the condition for equivalence can be weakened to: for any two atoms $a_1, a_2 \in B$, we have $\mu(a_1) \geq \mu(a_2)$ iff $\nu(a_1) \geq \nu(a_2)$. So we only need to have an order-preserving map for the measures of all atoms in B while keeping the flexibility.

Thus, we can define ν on the atoms of B as follows:

$$\nu(a) = \begin{cases} \kappa & \text{if } a = a_0, \text{ the smallest atom} \\ \aleph_{\kappa+i} & \text{if } |\{\mu(a') \mid a' \in At(B), \mu(a') < \mu(a)\}| = i \end{cases}$$

where to compute $\aleph_{\kappa+i}$ we view κ as an ordinal, i.e., as the least order type of a well-order of size κ. It is not hard to verify that $\mathcal{C} = \langle B, \nu, V\rangle$ is still a flexible finite measure algebra model and that for any $a_1, a_2 \in At(b)$, we have $\mu(a_1) \leq \mu(a_2)$ iff $\nu(a_1) \leq \nu(a_2)$. $\square$

Example 4.7.5. Figure 4.7.1 displays a particular flexible measure algebra (flexible model without the valuation). The comparison structure (illustrated by shaded areas in the same way as in Example 4.4.9) is the same regardless of what the cardinal κ is so long as $1 \leq \kappa \leq \aleph_1$.

Now we capture Fin and Inf in the language $\mathcal{L}$ by the following.

Definition 4.7.6. When $\Delta \subseteq \Phi$ is finite, define the $\mathsf{Fin}_\Delta(u)$ for any set term $u \in T(\Delta)$ as:

$$\mathsf{Fin}_\Delta(u) := |\varnothing| \geq |u| \vee \bigvee_{s,t \in T_0(\Delta)} |s \cup t| \geq |u| \wedge |s \cup t| > |s| \geq |t|$$

and then define $\mathsf{Inf}_\Delta(u)$ for any set term $u \in T(\Delta)$ as

$$\mathsf{Inf}_\Delta(u) := \bigvee_{s,t \in T_0(\Delta)} (t \not\subseteq s \ \wedge \ |u| \geq |s| \geq |s \cup t|)\,.$$

Here r_i ranges over elements in R, and s_i, t_i range over elements in sequences S and T, respectively. When no confusion arises, we may drop the subscript Δ.

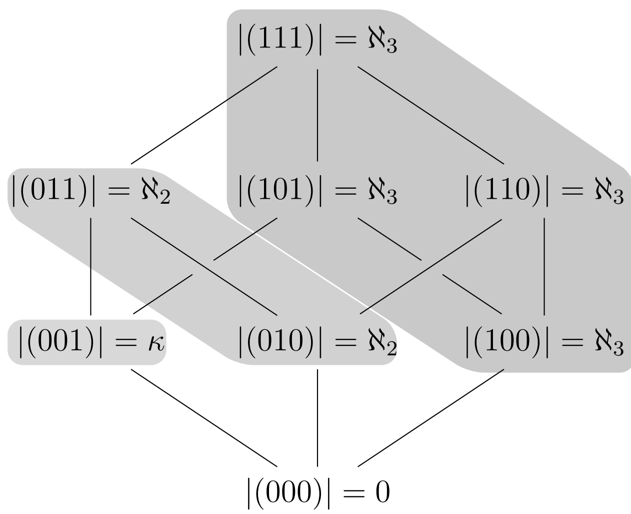


Figure 4.5: A flexible measure algebra

To understand this definition, recall that by basic cardinal arithmetic, the distinct feature of infinite sets is so-called absorption: if a set X is infinite, then $|X| \geq |X \cup Y|$ whenever $|X| \geq |Y|$, even if Y is not a subset of X. On the other hand, when Y is not a subset of X and yet $|X| \geq |X \cup Y|$, then X must be infinite. Hence, we can witness X's finiteness by a set Y such that $|X \cup Y| > |X| \geq |Y|$. Note that this also shows that Y, and thus $X \cup Y$, is finite, and moreover any set smaller than $X \cup Y$ is also finite. Similarly, we can witness X's infiniteness by a set Y that is not a subset of X yet for which $|X| \geq |X \cup Y|$. Then, any set as large as X must also be infinite. Our definitions of Fin_Δ and Inf_Δ are based on these simple observations.

The following two propositions tell us precisely to what extent these formulas capture Fin and Inf. In sum, their truth forces the respective properties (finiteness and infiniteness) on adapted models, but not vice versa. However, (in adapted models) the other direction fails only on the smallest atom of flexible models. This is the best we can do in $\mathcal{L}$, due to Proposition 4.7.4 and the existence of flexible models.

Proposition 4.7.7. *Fix a finite $\Delta \subseteq \Phi$. For any adapted measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$:*

1. *if $\mathcal{B} \vDash \mathsf{Fin}_\Delta(u)$, then $\mu(\widehat{V}(u))$ is finite;*
2. *if $\mathcal{B} \vDash \neg\mathsf{Fin}_\Delta(u)$ and yet $\mu(\widehat{V}(u))$ is finite, then $\mathcal{B}$ is flexible and $\widehat{V}(u)$ is the smallest atom in B.*

Proof. As we reasoned above, the first claim is easy. For the second claim, assume for contradiction that $\widehat{V}(u)$ is not an atom in B. $\widehat{V}(u)$ cannot be the bottom element, since then $\mathsf{Fin}_\Delta(u)$ is trivially true. So $\widehat{V}(u)$ is neither the bottom nor an atom. This means there are two non-bottom elements $a, b \in B$ that are below $\widehat{V}(u)$, whose join is $\widehat{V}(u)$, whose meet is bottom, and $\mu(a) \geq \mu(b)$. Since $\mu(\widehat{V}(u))$ is finite, $\mu(a)$ and $\mu(b)$ must also be finite. Then $\mu(a) \geq \mu(b)$ but $\mu(a) < \mu(a \vee b) = \mu(a) + \mu(b)$, since b is not bottom and $\mu(b) > 0$. From this, we have:

- $\mu(a \vee b) > \mu(a) \geq \mu(b)$;
- $\widehat{V}(u) = a \vee b$.

So $\mathcal{B} \vDash \mathsf{Fin}_\Delta(u)$, a contradiction. Here, we are using the fact that $\widehat{V}(T(\Delta)) = \widehat{V}(T_0(\Delta)) = B$ to get terms s and t with $\widehat{V}(s) = a$ and $\widehat{V}(t) = b$.

Now assume again for contradiction that $\widehat{V}(u)$ is not the only atom finite in measure and in particular that $a \in At(B)$, $a \neq \widehat{V}(u)$, and $\mu(a)$ is finite. Then $a \wedge \widehat{V}(u) = 0$ and $\mu(a \vee \widehat{V}(u)) = \mu(a) + \mu(\widehat{V}(u)) > \mu(\widehat{V}(u))$, since a is not bottom and $\mu(a) > 0$. Then we again have a witness for $\mathsf{Fin}_\Delta(u)$, depending on which of $\widehat{V}(u)$ and a is larger. For example, if a is larger, then $\widehat{V}(u)$ is smaller than a, a does not absorb a smaller element $\widehat{V}(u)$, and $\widehat{V}(u)$ is smaller than the join of $\widehat{V}(u)$ and a. Thus, we contradict $\mathcal{B} \vDash \neg\mathsf{Fin}_\Delta(u)$.

In summary, we have that $\widehat{V}(u)$ is the only finite atom in B, which immediately shows that $\widehat{V}(u)$ is the smallest atom in B and $\mathcal{B}$ is flexible. $\square$

Proposition 4.7.8. *Fix a finite $\Delta \subseteq \Phi$. For any adapted measure algebra model $\mathcal{B} = \langle B, \mu, V\rangle$:*

1. *if $\mathcal{B} \vDash \mathsf{Inf}_\Delta(u)$, then $\mu(\widehat{V}(u))$ is infinite;*
2. *if $\mathcal{B} \vDash \neg\mathsf{Inf}_\Delta(u)$ and yet $\mu(\widehat{V}(u))$ is infinite, then $\mathcal{B}$ is flexible and $\widehat{V}(u)$ is the smallest atom in B.*

Proof. After expanding the semantics, it is easy to see that $\mathsf{Inf}_\Delta(u)$ expresses that there is an element $b \in B$ such that b absorbs an element c not contained in b, and $\widehat{V}(u)$ is no smaller than b in measure. The elements b and c are obtained by first picking out the true disjunct $(t \not\subseteq s \wedge |u| \geq |s| \geq |s \cup t|)$ of $\mathsf{Inf}_\Delta(u)$ and then taking b and c to be just $\widehat{V}(s)$ and $\widehat{V}(t)$, respectively. Then $t \not\subseteq s$ being true means that c is not contained in b, and $|s| \geq |s \cup t|$ means c absorbs b. Hence $\mu(b)$ must be infinite, and since $|u| \geq |s|$, $\mu(\widehat{V}(u))$ is infinite too.

For the second claim, first suppose that there is an atom $a \in At(B)$ with $\mu(a)$ finite. Then for any atom $b \in At(B)$ that is infinite in measure, $a \not\subseteq b$ and $\mu(b) \geq \mu(b \vee a)$. Now $\mu(\widehat{V}(u))$ is infinite, so $\widehat{V}(u)$ must have an infinite atom b below it. Then $\mathsf{Inf}_\Delta(u)$ is witnessed by b and a, a contradiction. Thus, there is no finite atom in B.

Now suppose there is no strictly smallest atom—for any atom b, there is another atom a such that $\mu(b) \geq \mu(a)$. Then $a \not\subseteq b$, and because they are both infinite in measure, $\mu(b) \geq \max\{\mu(b), \mu(a)\} = \mu(b \vee a)$. So by the same reasoning as the previous case, $\mathsf{Inf}_\Delta(u)$ is witnessed, and we have a contradiction. Thus, there is also a strictly smallest infinite atom a_0 in B.

Finally, suppose $\widehat{V}(u) \neq a_0$. Then $\widehat{V}(u)$ must be above another atom b, as it is infinite in measure and cannot be bottom. But then $a_0 \not\subseteq b$ and $\mu(b) \geq \max\{\mu(a_0), \mu(b)\} = \mu(b \vee a_0)$. So $\mathsf{Inf}_\Delta(u)$ is still witnessed. In sum, $\widehat{V}(u)$ is the strictly smallest infinite atom in B, so $\mathcal{B}$ is flexible. □

4.7.2 Representation using axioms of the language

It is now time to give the axioms for cardinality comparison in $\mathcal{L}$ that are not already in BasicCompLogic.

Definition 4.7.9. Where $\Delta \subseteq \Phi$ is finite, define $\mathsf{Axiom}(\Delta)$ as the set containing all of the following formulas for all $u, s, t \in T_0(\Delta)$:

(C1) $\neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$;

(C2) $(\neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)) \to \bigwedge_{t \in T_0(\Delta)}(|u| \geq |t| \to (t = \varnothing \vee t = u))$;

(C3) $\bigwedge_{i=1}^{n}(\mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)) \to \mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots t_n)$;

(C4) $\mathsf{Inf}_\Delta(u) \to ((|u| \geq |s| \wedge |u| \geq |t|) \to |u| \geq |s \cup t|)$;

(C5) $(\mathsf{Inf}_\Delta(s) \wedge \mathsf{Fin}_\Delta(t)) \to |s| > |t|$,

where $n \geq 1$, and $s_1, \ldots, s_n, t_1, \ldots, t_n \in T_0(\Delta)$ are also all arbitrary.

Given that Fin and Inf do not fully capture Fin and Inf, we cannot use $\mathsf{Fin}_\Delta(u) \oplus \mathsf{Inf}_\Delta(u)$ like axiom (A1) for Fin and Inf, since it is outright invalid among all adapted measure algebra models. Instead, we have (C1) and (C2) here. Put together, they ensure that the only case when $\mathsf{Fin}_\Delta(u)$ and $\mathsf{Inf}_\Delta(u)$ fail to capture Fin and Inf is when we are in a flexible model and u is the smallest atom.

As we have done for $\mathsf{CardCompLogic}_{\mathrm{Fin,Inf}}$, we prove a representation theorem using $\mathsf{Axiom}(\Delta)$. In other words, we show that $\mathsf{Axiom}(\Delta)$ is enough to force the comparison relation $\succeq$ in an adapted comparison algebra model to be a comparison of cardinalities. To start, we need the following straightforward lemma.

Lemma 4.7.10. *Fix a finite $\Delta \subseteq \Phi$. For any comparison algebra model $\mathcal{B}$ and $s, t \in T(\Delta)$:*

1. *$\mathcal{B} \vDash |\varnothing| \geq |t|$ iff $V(t) = \bot_B$;*
2. *$\mathcal{B} \vDash t \subseteq s$ iff $V(t) \leq_B V(s)$;*
3. *$\mathcal{B} \vDash t = s$ iff $V(t) = V(s)$;*
4. *if $\mathcal{B} \vDash \mathsf{Fin}_\Delta(t) \wedge s \subseteq t$, then $\mathcal{B} \vDash \mathsf{Fin}_\Delta(s)$;*

5. *if* $\mathcal{B} \vDash \mathsf{Fin}_\Delta(s) \wedge \mathsf{Fin}_\Delta(t)$*, then* $\mathcal{B} \vDash \mathsf{Fin}_\Delta(s \cup t)$*;*

6. *if* $\mathcal{B} \vDash \mathsf{Inf}_\Delta(t) \wedge |s| \geq |t|$*, then* $\mathcal{B} \vDash \mathsf{Inf}_\Delta(s)$*.*

Proof. The first item follows from the requirement that $\perp_B \not\succeq b$ for all $b \in B \setminus \{\perp_B\}$. The second and third follow easily.

For the fourth item, suppose $\mathcal{B} \vDash \mathsf{Fin}_\Delta(t) \wedge s \subseteq t$. Then by definition, we have terms r_i as the witnesses of the finiteness of s. It is easy to see that $r_i \cap s$'s are witnesses of the finiteness of s. Similarly for the fifth item, the witnesses of $s \cup t$ are just the union of witnesses for s and witnesses for t. The sixth item is even easier, as the same witness works. $\square$

Theorem 4.7.11. *Fix a finite* $\Delta \subseteq \Phi$*. Let* $\mathcal{B}$ *be an adapted comparison algebra model such that* $\mathcal{B} \vDash \mathsf{Axiom}(\Delta)$*. Then there is a* μ *such that* $m(\mathcal{B}) = \langle B, \mu, V\rangle$ *is a finite measure algebra model representing* $\mathcal{B}$ *and hence* $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} m(\mathcal{B})$*.*

Proof. Since $\mathcal{B}$ is adapted to Δ, every element in B is named by some term in $T(\Delta)$. Thus, given $b \in B$ we may write $\varphi(b)$ for the sentence $\varphi(t_b)$ where $t_b \in T(\Delta)$ and $V(t_b) = b$. By axiom (C1) in $\mathsf{Axiom}(\Delta)$, there are two cases:

Case 1 there is a $b \in B$ such that $\mathcal{B} \vDash \neg(\mathsf{Fin}_\Delta(b) \vee \mathsf{Inf}_\Delta(b))$;

Case 2 for any $b \in B$, $\mathcal{B} \vDash \mathsf{Fin}_\Delta(b) \oplus \mathsf{Inf}_\Delta(b)$.

In both cases, we want to obtain an $F \subseteq B$, so that using F as the labeling set, $\langle B, \succeq, F, V\rangle$ is a labeled comparison algebra model satisfying all the conditions in Theorem 4.4.8.

Case 1: First, we show that there is a unique atom $a_0 \in At(B)$ such that $\mathcal{B} \vDash \neg(\mathsf{Fin}_\Delta(a_0) \vee \mathsf{Inf}_\Delta(a_0))$, and that for all other atoms b, we have $b \succ a_0$. Suppose for $a_0, a_1 \in B$ that we have both $\mathcal{B} \vDash \neg\mathsf{Fin}_\Delta(a_i) \wedge \neg\mathsf{Inf}_\Delta(a_i)$ for $i \in \{0, 1\}$. Then by axiom (C2) and Lemma 4.7.10, for any $b \in B$, if $a_0 \succeq b$, then $b = a_0$ or $b = \perp_B$. But for any $b \in B$ that is below a_0 in the Boolean algebra, i.e., $b \leq a_0$, it is also true that $b \preceq a_0$, by the definition of a comparison algebra. So whenever $b \leq a_0$, we have $b \preceq a_0$ and hence b is a_0 or $\perp_B$. This means that a_0 is an atom in B, as it must not be $\perp_B$, since $\mathcal{B} \vDash \mathsf{Fin}_\Delta(\perp_B)$. In exactly the same fashion we can show that a_1 is also an atom. Now $\succeq$ is a total preorder. So either $a_0 \succeq a_1$ or $a_1 \succeq a_0$. But in either case, they must be equal, as they cannot be $\perp_B$, but axiom (C2) says they are either equal or are bottom. Thus, there is a unique atom a_0 such that $\mathcal{B} \vDash \neg\mathsf{Fin}_\Delta(a_0) \wedge \neg\mathsf{Inf}_\Delta(a_0)$.

Building on the previous conclusion, for any $a \in At(B) \setminus \{a_0\}$, we have $\mathcal{B} \vDash \mathsf{Fin}_\Delta(a) \oplus \mathsf{Inf}_\Delta(a)$. The second step is to show that in fact $\mathcal{B} \vDash \mathsf{Inf}_\Delta(a)$. Consider $a \cup a_0$. If $\mathcal{B} \vDash (a \cup a_0) \succ a$, then we have $\mathcal{B} \vDash (a \cup a_0) \succ a \succeq a_0 \wedge (a \cup a_0) \succeq a_0$. It is not hard to see that then $\mathcal{B} \vDash \mathsf{Fin}_\Delta(a_0)$, a contradiction. Hence $\mathcal{B} \vDash a \succeq (a \cup a_0)$ instead. However, $\mathcal{B} \vDash a_0 \not\subseteq a$ since a_0 and a are distinct atoms. So $\mathcal{B} \vDash \mathsf{Inf}_\Delta(a)$.

Thus we see that there is no element in B satisfying Fin except the bottom element. So define $F = \{\perp_B\}$. We can show that Theorem 4.4.8 can be applied to $\langle B, \succeq, F, V\rangle$. In fact, the only nontrivial condition is the third condition: for any $a, b, c \in B$ with $a \notin F$, if $a \succeq b$ and $a \succeq c$, then $a \succeq b \vee c$. There are two cases:

- $a = a_0$: then a is the smallest atom, and b, c are either $\bot_B$ or a_0 and so is $b \vee c$; hence $a \succeq b \vee c$;
- $a \neq a_0$: as we have shown, now $\mathcal{B} \vDash \mathsf{Inf}_\Delta(a)$; thus by axiom (C4) in $\mathsf{Axioms}(\Delta)$, $a \succeq b \vee c$.

Hence we can invoke Theorem 4.4.8 to build μ.

Case 2: Define $F = \{b \in B \mid \mathcal{B} \vDash \mathsf{Fin}_\Delta(b)\}$. By Lemma 4.7.10, F is an ideal. Axioms (C3), (C4), and (C5) in $\mathsf{Axioms}(\Delta)$ ensure conditions (2), (3), and (4) in Theorem 4.4.8. Thus, Theorem 4.4.8 applies again. □

4.7.3 Completeness

Finally we are in a position to present the logic of cardinal comparison $\mathsf{CardCompLogic}$.

Definition 4.7.12. Let $\mathsf{CardCompLogic}$ be the logic for $\mathcal{L}$ with the following axioms and rules:

1. all axioms and rules in $\mathsf{BasicCompLogic}$;
2. for any finite $\Delta \subseteq \Phi$, all formulas in $\mathsf{Axioms}(\Delta)$;

Now we show that $\mathsf{CardCompLogic}$ is sound and complete with respect to all measure algebra models and also field of sets models, completing our proof of Theorem 4.2.4.

Theorem 4.7.13 (Soundness). $\mathsf{CardCompLogic}$ *is sound with respect to the class of all measure algebra models. Since every field of sets model can be equivalently turned into a measure algebra model in an obvious way,* $\mathsf{CardCompLogic}$ *is also sound on the class of all field of sets models.*

Proof. The only non-trivial axioms are in $\mathsf{Axioms}(\Delta)$ for an arbitrary finite Δ. Given an arbitrary measure algebra model $\mathcal{B} = \langle B, \mu, V \rangle$, by Proposition 4.4.11 there is a measure algebra model $\mathcal{B}_\Delta$ which is adapted to Δ and which satisfies $\mathcal{B}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{B}$. So to show that $\mathcal{B} \vDash \mathsf{Axioms}(\Delta)$, it is enough to show $\mathcal{B}_\Delta \vDash \mathsf{Axioms}(\Delta)$.

Hence we only need to show that all sentences in $\mathsf{Axioms}(\Delta)$ are true on any adapted measure algebra model $\mathcal{B}$. Fix an arbitrary such $\mathcal{B} = \langle B, \mu, V \rangle$; we check that all the sentences in $\mathsf{Axioms}(\Delta)$ are true in $\mathcal{B}$:

- $\neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$: By Propositions 4.7.7 and 4.7.8, $\mathcal{B} \vDash \mathsf{Fin}_\Delta(u)$ implies that $\mu(\widehat{V}(u))$ is finite, and $\mathcal{B} \vDash \mathsf{Inf}_\Delta(u)$ implies that $\mu(\widehat{V}(u))$ is infinite. But $\mu(\widehat{V}(u))$ cannot be both finite and infinite. Therefore, $\mathcal{B} \vDash \neg(\mathsf{Fin}_\Delta(u) \wedge \mathsf{Inf}_\Delta(u))$.
- $(\neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)) \to \bigwedge_{t \in T_0(\Delta)}(|u| \geq |t| \to (t = \varnothing \vee t = u))$: Suppose $\mathcal{B} \vDash \neg\mathsf{Fin}_\Delta(u) \wedge \neg\mathsf{Inf}_\Delta(u)$. Then by the second part of Proposition 4.7.7, if $\mu(\widehat{V}(u))$ is infinite, then $\widehat{V}(u)$ is the strictly smallest atom in B. Then it follows that
$$\mathcal{B} \vDash \bigwedge_{t \in T_0(\Delta)} (|u| \geq |t| \to (t = \varnothing \vee t = u)).$$

Similarly, by Proposition 4.7.8, if $\mu(\widehat{V}(u))$ is finite, the above statement holds as well. So indeed the formula is valid.

- $\bigwedge_{i=1}^{n}(\mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)) \to \mathsf{FC}_n(s_1, \ldots, s_n, t_1, \ldots t_n)$: By Proposition 4.7.7, when $\mathcal{B} \vDash \mathsf{Fin}_\Delta(s_i) \wedge \mathsf{Fin}_\Delta(t_i)$, $\widehat{V}(s_i)$ and $\widehat{V}(t_i)$ are indeed finite. But as we have explained both in § 4.2 and immediately after Theorem 4.4.7, the consequent (finite cancellation axiom) is clearly valid for elements of finite cardinality.

- $\mathsf{Inf}_\Delta(u) \to ((|u| \geq |s_1| \wedge |u| \geq |s_2|) \to |u| \geq |s_1 \cup s_2|)$: By Proposition 4.7.8, $\widehat{V}(u)$ is infinite, and the consequent expresses a simple property of elements of infinite cardinality.

- $(\mathsf{Inf}_\Delta(s_1) \wedge \mathsf{Fin}_\Delta(s_2)) \to |s_1| > |s_2|$: Using Propositions 4.7.7 and 4.7.8 again, this says that when $\mu(\widehat{V}(s_1))$ is infinite and $\mu(\widehat{V}(s_2))$ is finite, then $\mu(\widehat{V}(s_1))$ is greater than $\mu(\widehat{V}(s_2))$, which is trivial. □

Theorem 4.7.14 (Completeness)**.** CardCompLogic *is complete with respect to the class of all field of sets models: every valid formulas is derivable in* CardCompLogic*.*

Proof. We show that any formula that is consistent is satisfied by a measure algebra model. By Proposition 4.4.3, it is also satisfied by a field of sets model.

Suppose φ is consistent. Then let X be a maximally consistent set of CardCompLogic containing φ. Using the canonical model theorem (Theorem 4.5.2), we obtain a comparison algebra model $\mathcal{C} \vDash X$.

Now let Δ be the set of all set labels appearing in φ. Then Δ is finite. By Proposition 4.4.11, there is a comparison algebra model $\mathcal{C}_\Delta$ that is adapted to Δ and that satisfies $\mathcal{C}_\Delta \equiv_{\mathcal{L}(\Delta)} \mathcal{C}$. Since $\mathsf{Axioms}(\Delta) \subseteq X$, $\mathcal{C} \vDash \mathsf{Axioms}(\Delta)$, and since $\mathsf{Axioms}(\Delta) \cup \{\varphi\} \subseteq \mathcal{L}(\Delta)$, $\mathcal{C}_\Delta \vDash \mathsf{Axioms}(\Delta) \cup \{\varphi\}$. Now the representation theorem (Theorem 4.7.11) can be applied to $\mathcal{C}_\Delta$, and we obtain a measure algebra model $\mathcal{B}$ such that $\mathcal{B} \equiv_{\mathcal{L}(\Delta)} \mathcal{C}_\Delta$. Thus $\mathcal{B} \vDash \varphi$. So φ is satisfied on a measure algebra model. □

Thus, the question with which we began—what are the laws one must add to Boolean algebra to capture reasoning about the relative size of sets according to Cantor's definition?—is answered by the laws of CardCompLogic.

4.8 Open problems

In our proofs, we quickly passed to finite models, that is, models with only finitely many sets (some of which may of course be infinite). For example, our representation theorem (Theorem 4.4.8) applies only to finite models, and in Theorems 4.2.10 and 4.2.4, we proved completeness rather than strong completeness.

Problem 4.8.1. Find a logic that is sound and strongly complete with respect to field of sets models.

Problem 4.8.2. Prove a representation theorem for infinite comparison algebras.

We will give some examples that show the difficulties that arise here. First, such a logic cannot be compact. Indeed (in the language $\mathcal{L}_{\mathrm{Fin,Inf}}$), with distinct set terms $\langle s_n \rangle_{n\in\omega}$ and t, the following set of formulas is finitely satisfiable in field of sets models, but not satisfiable:

$$\{|s_n| < |s_{n+1}| \mid n \in \omega\} \cup \{|s_n| \leq |t| \mid n \in \omega\} \cup \{\mathrm{Fin}(t)\}.$$

One can give similar examples in the language $\mathcal{L}$. Then to obtain a strongly complete logic, one might add an infinitary rule stating that if the sentences in

$$\{|s_n| < |s_{n+1}| \mid n \in \omega\} \cup \{|s_n| \leq |t| \mid n \in \omega\}$$

are derivable, then so is $\mathrm{Fin}(t)$.

Another interesting example comes from the fact that the relation of cardinality comparison must be well-founded (assuming the axiom of choice). Thus, if $\langle s_n \rangle_{n\in\omega}$ is a sequence of distinct set terms, then the set of sentences

$$\{|s_{n+1}| < |s_n| \mid n \in \omega\}$$

is not satisfiable in field of sets models, but it is again finitely satisfiable.

Note also that finite cardinalities are just natural numbers, whose ratios are all rational. However, with infinitely many formulas, we can express that the sizes of two sets are of irrational ratio. To do this, define the following set of formulas:

$$\begin{aligned} A \;=\; & \{|a_i| = |a_j|, |b_i| = |b_j| \mid i,j \in \omega\} \cup \\ & \{|a_i \cap a_j| = \varnothing \wedge |b_i \cap b_j| = \varnothing \mid i,j \in \omega, i \neq j\}. \end{aligned}$$

Intuitively, this says that the a_i's are disjoint and of the same size, and the same holds for the b_i's. Then we can approximate any ratio by using a_i's and b_i's. For example, consider sequences $\langle l_i \rangle_{i\in\omega}, \langle r_i \rangle_{i\in\omega}, \langle n_i \rangle_{i\in\omega}$ of natural numbers such that l_i/n_i approaches $\sqrt{2}$ from below and r_i/n_i from above. Then let

$$B = \{|\bigcup_{k<l_i} a_k| < |\bigcup_{k<n_i} b_k| < |\bigcup_{k<r_i} a_k| \mid i \in \omega\}.$$

The set $A \cup B$ is then finitely satisfiable, but not satisfiable as it forces the ratio of $|a_0|$ and $|b_0|$ to be $\sqrt{2}$.

As the last example of non-compactness, suppose that we allow more than countably many set labels in the language. Let $|\Phi| = \aleph_1$. Then the set of sentences

$$\{|a| \neq |b| \mid a, b \in \Phi, a \neq b\} \cup \{\mathrm{Fin}(a) \mid a \in \Phi\}$$

is not satisfiable. However, any countable subset is satisfiable.

A natural extension of our language is to add the powerset operation. In this case, one must replace the complement operation with the relative complement operation $s \setminus t$. Then a field of sets model (with powerset) is a collection $\mathcal{F}$ of sets closed under intersection, union, relative complement, and powerset, together with a valuation of the set labels.

Problem 4.8.3. Axiomatize the logic of cardinality comparison with the powerset operation.

In this language, one can consider principles such as

$$|s| < |t| \to |\mathcal{P}(s)| < |\mathcal{P}(t)|,$$

which is true under GCH but is independent of ZFC [100]. It would be interesting to have a logic for comparing such principles.

Chapter 5

Logics of Imprecise Comparative Probability

5.1 Introduction

While the standard probability calculus remains the dominant formal framework for representing uncertainty across numerous disciplines, a small but significant tradition in philosophy, economics, computer science, and statistics has contended that the precision inherent in assigning "sharp" probabilities to uncertain events is often inappropriate. The reasons are several. One obvious concern is the psychological reality of arbitrarily precise real-valued judgments ([20, 103, 106, 75, 149]). As [149] expresses the concern, "Almost everyone who has thought about the problems of measuring beliefs in the tradition of subjective probability or Bayesian statistical procedures concedes some uneasiness with the problem of always asking for the next decimal of accuracy in the prior estimation of a probability" (p. 160). Another quite distinct concern is that even for a certain kind of idealized agent free of computational or representational limitations, in many important cases the available evidence somehow underdetermines the "right" probability function to have, and it would be epistemically unfitting to opt for any one of them ([28, 121, 101, 105]).

A number of alternative formal frameworks have been advanced (see, e.g., [80]). Our focus here is on two especially prominent alternatives. Some authors favor a sort of generalization of the probability calculus, allowing uncertainty to be measured by *sets* of probability functions ([75, 121, 154, 146]; see [21] for a philosophical overview). This *imprecise probability* framework retains many of the benefits of standard Bayesian representation and reasoning—indeed allowing the standard picture to emerge as a special case—while also affording a wider range of epistemic attitudes. Philosophical questions about imprecise probability have generated a great deal of discussion in recent years (see, e.g., [101, 141, 137, 22, 131]). A second line of work renounces the demand for explicit numerical judgments altogether, arguing that qualitative, especially *comparative*, judgments should be the primitive building blocks for the theory of uncertainty ([103, 106, 57, 87]; see [104] for a philosophical overview). Aside from being intuitively simpler and arguably closer to "ordinary" expressions of uncertainty, some authors have argued that this setting of *comparative probability* is perhaps uniquely suited to solving notable epistemic puzzles ([56, 38, 50]). Others have sought more ameliorative reconciliations between the quantitative and qualitative approaches so as to capitalize on the advantages of each (see, e.g., [150] and [49]).

Our aim in this chapter is neither to weigh in on the debate between precise and imprecise versions of probabilism, nor to adjudicate between the quantitative and the qualitative alternatives, but rather to shed light on the connections between them. Only quite recently have even the most basic questions about such connections been clarified ([138, 2, 1, 83]). This is of interest from all perspectives. If one takes sets of probability measures as primitive, it would nevertheless be desirable to understand some of the core qualitative commitments implicit in this representation, including how such commitments relate to those of precise probability and other frameworks. Most conspicuously, the generalization to sets of measures brings with it a rejection of the infamous comparability principle (also sometimes called *opinionation* or *totality*), according to which every two events ought to be compared in probability. Indeed, rejection of this principle has served as one of the primary arguments against precise probabilism. As Keynes [103] expressed it a century ago:

> Is our expectation of rain, when we start out for a walk, always more likely than not, or less likely than not, or as likely as not? I am prepared to argue that on some occasions none of these alternatives hold, and that it will be an arbitrary matter to decide for or against the umbrella. If the barometer is high, but the clouds are black, it is not always rational that one should prevail over the other in our minds, or even that we should balance them. (p. 30)

Aside from the rejection of comparability, are there other differences between the precise and imprecise probabilistic frameworks that surface in this qualitative setting? Likewise, we can ask about various additional qualitative notions aside from the usual "weak" comparison 'at least as likely as'. For example, whereas the strict version of this judgment, 'more likely than', is easily definable in the precise setting in terms of weak comparison, this is no longer the case in the imprecise setting (see Section 5.2 below), raising new questions about the qualitative principles characterizing this distinctive kind of *unanimity* operator.

If, on the other hand, one takes qualitative judgments as primitive, this has the potential advantage of discarding principles forced upon us by (even imprecise) probabilistic representations. This may be desirable, e.g., if one is solely concerned with certain epistemic virtues such as maximizing accuracy ([60]). At the same time, there are also arguments that purport to show why an agent who maintains only comparative judgments would not want to violate qualitative probabilistic principles ([59, 60, 99]). For example, suppose that we operationalize a judgment of the form 'A is more likely than B' in terms of a disposition to opt for a prospect that pays some positive dividend conditional on A over one that pays the same amount conditional on B. Moreover, suppose that satisfying this preference is worth some cost, while judgments of the form 'A and B are equally likely' engender no such disposition. Then one can show that an agent will be forced into choosing strictly dominated actions (worse than some other available option no matter how the world turns out) if and only if the agent's judgments fail to comport with any set of probability measures ([99]). Arguments like these highlight the importance of gaining a better understanding of what compatibility of comparative judgments with imprecise probability means.

In this chapter, we take a logical approach, studying a sequence of increasingly expressive qualitative formal systems, all interpreted over sets of probability measures. To illustrate the type of reasoning we would like to systemize, consider the following examples.

Example 5.1.1. A patient learns from her doctor of the existence of a gland in the human body and of a disease previously unknown to her.[1] The doctor informs her that if her gland is swollen, then it is more likely than not that she has the disease. Subsequently the patient's gland is examined, and she learns that it is swollen. As a result, she comes to think it is more likely than not that she has the disease.

[1]This example is inspired by van Benthem's [13, p. 164, p. 166] example of the hypochondriac.

How should we model the patient's evolving uncertainty? A natural approach is to represent her relevant uncertainty using the following set of four possible states:

$$\{\langle \text{swollen, disease} \rangle, \langle \text{swollen, no disease} \rangle, \langle \text{not swollen, disease} \rangle, \langle \text{not swollen, no disease} \rangle\}.$$

Initially, the patient knows nothing about the gland or the disease. We represent this ignorance using the set $\mathcal{P}$ of all probability measures on the state space above. Next, when her doctor informs her that if her gland is swollen, then it is more likely than not that she has the disease, we eliminate from her set of measures all measures except those for which the probability of disease conditional on a swollen gland is greater than the probability of no disease conditional on a swollen gland. This gives us a new set $\mathcal{P}'$ of measures. Finally, when she has the gland examined and learns that it is swollen, we condition each measure in $\mathcal{P}'$ on the information that the gland is swollen, giving us a final set $\mathcal{P}''$ of measures. All measures in $\mathcal{P}''$ give a higher probability to disease than no disease.

How should one model the example using the standard representation of an agent's uncertainty with a single probability measure? First, the standard representation forces the agent to have sharp probabilities that her gland is swollen and that she has the disease, even when she just learns of their existence and knows nothing else about them. It also forces her to have a sharp conditional probability for having the disease conditional on her gland being swollen, before the doctor tells her anything about the connection between the two. Suppose she thinks that disease and no disease are equally likely conditional on her gland being swollen. What do we then do with her probability measure when the doctor informs her that if her gland is swollen, then it is more likely than not that she has the disease? One idea would be to replace her probability measure with the "closest" measure for which the conditional probability of disease given a swollen gland is greater than that of no disease given a swollen gland; but the existence of a unique closest such measure is clearly problematic. Another idea is that we must give up the simple state space above. Instead, we must use a complicated state space involving possibilities for what her doctor might say to her. On this approach, the patient must start out with a sharp conditional probability for having the disease conditional on her doctor uttering at time t the words "if your gland is swollen, then it is more likely than not that you have the disease." Assuming this conditional probability is greater than .5, it follows that conditional on the doctor *not* uttering those words at time t, the probability she assigns to having the disease will be less than .5. In order to allow that time t may pass in silence without the patient changing her probability for disease, we must introduce still further distinctions in the state space, beyond the distinction that the doctor may or may not utter the indicated words at t.

Though we will not argue that the modeling approach with a single probability measure is unworkable, in this chapter we wish to explore the multi-measure approach sketched above. We will fully formalize the swollen gland example in Section 5.6.2. There we will even model the patient's becoming aware of the distinction between having a swollen gland and not having a swollen gland and of the distinction between having the disease and not having the disease, creating the state space and set $\mathcal{P}$ of measures above.

The next example is one in which it is essential to consider the possibilities for what an informant may say. It was made famous by vos Savant [140] in the Monty Hall version of the puzzle posed by Selvin [147]. We will present the earlier but mathematically equivalent Three Prisoners version of the puzzle from Gardner [68, 69].

Example 5.1.2. The following is Diaconis and Zabell's [37, p. 30] description of the Three Prisoners puzzle (also see [36] and [80]):

> Of three prisoners a, b, and c, two are to be executed, but a does not know which. He therefore says to the jailer, "Since either b or c is certainly going to be executed, you will give me no information about my own chances if you give me the name of one man, either b or c, who is going to be executed." Accepting this argument, the jailer truthfully replies, "b will be executed." Thereupon a feels happier because before the jailer replied, his own chance of execution was two-thirds, but afterward there are only two people, himself and c, who could be the one not executed, and so his chance of execution is one-half.

Under what conditions could a's reasoning possibly be sound? Imagine there are four relevant ways the world could be: w_{ab}, w_{ac}, w_{bc}, and w_{cb}, where in w_{ij} prisoner i is the one who lives and prisoner j is the one who the jailer says will be executed. Assuming that each prisoner is equally likely to be spared, we can assume w_{bc} and w_{cb} both have probability one-third, and the disjunction "w_{ab} *or* w_{ac}" has probability one-third. Concerning the relative probability of w_{ab} and w_{ac}, we could apply a principle of indifference and proclaim that the jailer is equally likely to announce b or announce c, in case a is the one to be spared. It is then easy to compute that the conditional probability of being spared after learning that b will be executed (and thus w_{ac} and w_{bc} can be eliminated as possibilities) is still one-third. In this case a learns nothing from the jailer's announcement.

By contrast, if for whatever reason a thinks the jailer is certain to tell him it is b who will be executed when a is the one to be spared, then learning b will be executed does rationally lead a to conclude that he now has a one-half chance of survival.

There is an intuition in this scenario that the right way to respond to the evidence is to leave the relative likelihood of w_{ab} and w_{ac} open: to represent a's uncertainty in terms of the set of all probability measures that assign one-third to each of w_{bc}, w_{cb}, and the disjunction "w_{ab} *or* w_{ac}." In this case the probabilities of w_{ab} and w_{ac} each range from zero to one-third, under the constraint that their sum is one-third. Updating each such measure by eliminating w_{ac} and w_{bc} results in a range of posterior probability values for a surviving, from zero to one-half. Thus, the probability that a is spared (the disjunction "w_{ab} *or* w_{ac}") has *dilated* ([154]) from precisely one-third to the entire interval $[0, 1/2]$.

Examples 5.1.1 and 5.1.2 illustrate some important aspects of imprecise probabilistic reasoning, which surface already in a purely qualitative setting. By the end of this chapter, we will be able to formalize Examples 5.1.1 and 5.1.2 in a dynamic logic of updating imprecise comparative probability (Examples 5.6.5 and 5.6.21).

The outline of the chapter is as follows. In Section 5.2, we consider the pure order-theoretic setting of comparative probability and prove a representation theorem extending previous results in the literature. The theorem concerns both a weak and a strict comparative relation together represented by a set of probability measures (Theorem 5.2.7). In Section 5.3, we turn to the logical setting and review some completeness theorems for logics of precise and imprecise probability with a single weak comparative relation (Theorems 5.3.7, 5.3.9), as well as alternative interpretations of these systems (Theorems 5.3.10, 5.3.11). In Section 5.4, we consider a logical language that includes both weak and strict comparative relations and, using the representation in Theorem 5.2.7, prove a corresponding completeness theorem (Theorem 5.4.4). Section 5.5 explores the addition of a primitive "possibility" operator asserting the existence of a probability measure with a given property, culminating again in a complete axiomatization (Theorem 5.5.5), plus an analysis of complexity (Theorem 5.5.12).

In Section 5.6, we turn to modeling the dynamics of learning. In Section 5.6.1, we add to our language an update operator whose semantics is given by a process of discarding from one's set of measures any measure assigning zero probability to the learned proposition and then conditioning the remaining measures on the learned proposition. With this we can model updating on pure comparative probability formulas (through the discarding part), as well as non-probabilistic (ontic) formulas (through the conditioning part) and mixed probabilistic-ontic formulas. The language also allows the formalization of basic comparative conditional probabilities. Yet we prove that the extended language is in fact no more expressive than the previous system from Section 5.5: the extended language can be completely axiomatized by a set of "reduction axioms" (Theorem 5.6.8). Finally, in Section 5.6.2, we add a second dynamic operator for *becoming aware* of a new proposition (recall how the patient becomes aware of the existence of the gland and disease in Example 5.1.1). When an agent becomes aware of a new proposition, we form a new state space by splitting each state in her old state space in two, one where the new proposition is true and the other where it is false, and we form a new set of probability measures by taking all measures on the new set of propositions that when restricted to just the old propositions coincide with some old measure. We show that this language is more expressive than our previous languages, allowing us to express any linear inequality with integer coefficients about the probability of formulas.

What emerges is a landscape of increasingly expressive logical systems, consistent with both precise and imprecise probabilistic representations, simple but sufficiently powerful to model sophisticated reasoning about uncertainty. Perhaps surprisingly, the computational complexity of reasoning (e.g., determining validity or consistency) in each of the "static" systems is no worse than for the classical propositional calculus. The complexity of reasoning in the dynamic logic of updating sets of probability measures is an open problem, as is the complexity and axiomatization of the dynamic logic of becoming aware.

5.2 Representation

Before introducing any explicit logical calculus, in this section we consider the pure order-theoretic setting of comparative probability. A comparative notion of probability is most naturally formalized as a binary relation on an algebra of events. However, not all binary relations can be intuitively interpreted as comparing how likely events are, just as not all functions from events to $[0, 1]$ can be interpreted as assigning quantitative probabilities. Taking the usual axiomatization of quantitative probability for granted, a natural question—posed early on by [58]—is what would be a set of axioms that are intuitive and in harmony with those quantitative axioms.

This question was first solved for finite event algebras by [109]. Given a binary relation $\succsim$ on $\wp(W)$, where W is a finite set, and a probability measure μ on $\wp(W)$, we say that $\succsim$ is *precisely represented by* μ if for all $X, Y \subseteq W$, $X \succsim Y$ iff $\mu(X) \geq \mu(Y)$.

Theorem 5.2.1 ([109]). *Let W be a nonempty finite set and $\succsim$ a binary relation on $\wp(W)$. Then $\succsim$ is precisely represented by some probability measure on $\wp(W)$ if and only if:*

- *$\varnothing \not\succsim W$, $\{w\} \succsim \varnothing$ for all $w \in W$, and for all $A, B \in \wp(W)$, $A \succsim B$ or $B \succsim A$, and*
- *$\succsim$ satisfies the finite cancellation condition (FC): letting $\mathbf{1}_X$ denote the characteristic function of X, for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ such that $\sum_{i=1}^n \mathbf{1}_{A_i} = \sum_{i=1}^n \mathbf{1}_{B_i}$ (additions are done in the vector space $\mathbb{R}^W$), if for all $i < n$, $A_i \succsim B_i$, then $B_n \succsim A_n$.*

Following the same paradigm, we can consider a comparative notion of imprecise probability and ask the following question: which binary relations on a finite algebra of events can be naturally interpreted as an imprecise version of the at-least-as-likely-as relation? More precisely, given a binary relation $\succsim$ on $\wp(W)$, where W is a finite set, and a set $\mathcal{P}$ of probability measures on $\wp(W)$, we say that $\succsim$ is *imprecisely represented as the weak relation by* $\mathcal{P}$ if for all $X, Y \subseteq W$, $X \succsim Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$. The following analogue of Theorem 5.2.1 was proved by Ríos Insua [138] (also see [2]).

Theorem 5.2.2 ([138]). *Let W be a nonempty finite set and $\succsim$ a binary relation on $\wp(W)$. Then $\succsim$ is imprecisely represented as the weak relation by some set $\mathcal{P}$ of probability measures on $\wp(W)$ if and only if:*

- *$\varnothing \not\succsim W$, $\{w\} \succsim \varnothing$ for all $w \in W$, and*
- *$\succsim$ satisfies the generalized finite cancellation condition (GFC): for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{n-1} \mathbf{1}_{A_i} + k\mathbf{1}_{A_n} = \sum_{i=1}^{n-1} \mathbf{1}_{B_i} + k\mathbf{1}_{B_n}$, if for all $i < n$, $A_i \succsim B_i$, then $B_n \succsim A_n$.*[2]

[2]Note that n can be 1, in which case the condition simply expresses the reflexivity of $\succsim$.

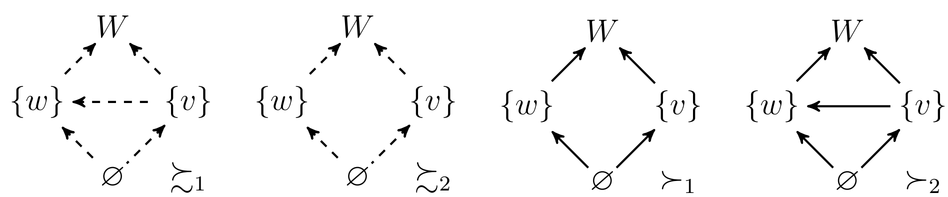
Remark 5.2.3. Harrison-Trainor et al. [83] prove that there are relations $\succsim$ satisfying the conditions of Theorem 5.2.1 *except for the comparability principle* (that for all $A, B \in \wp(W)$, $A \succsim B$ or $B \succsim A$) and which fail to satisfy the GFC condition in Theorem 5.2.2. Thus, it is necessary to strengthen FC to GFC when dropping comparability to obtain Theorem 5.2.2.

A subtlety not covered by Theorem 5.2.2 is that given a set $\mathcal{P}$ of probability measures, there are two natural ways to generate a strict relation, corresponding to the strict and the weak dominance relation in game theory:

- X *strictly dominates* Y in $\mathcal{P}$ iff for all $\mu \in \mathcal{P}$, $\mu(X) > \mu(Y)$;
- X *weakly dominates* Y in $\mathcal{P}$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$, and there is a $\mu \in \mathcal{P}$ such that $\mu(X) > \mu(Y)$.

When $\succsim$ is represented as the weak relation by $\mathcal{P}$, it is easy to see that X weakly dominates Y iff $X \succsim Y$ but $Y \not\succsim X$. However, we cannot pin down the strict dominance relation simply from the weak relation $\succsim$ or vice versa, as shown by the following example.

Example 5.2.4. Let $W = \{w, v\}$ and consider the four binary relations $\succsim_1, \succsim_2, \succ_1, \succ_2$ pictured below from left to right (for dashed arrows, reflexive and transitive arrows are omitted; for solid arrows, transitive arrows are omitted).



If all we know about a set $\mathcal{P}$ of probability measures on $\wp(W)$ is that its weak relation is $\succsim_1$, then both $\succ_1$ and $\succ_2$ may be $\mathcal{P}$'s strict dominance relation. For example, we can define a probability measure $\mu_{w<v}$ on $\wp(W)$ that favors v so that $\mu_{w<v}(\{w\}) = 1/3$. Then let $\mu_{w=v}$ be the uniform distribution on $\wp(W)$: $\mu_{w=v}(\{w\}) = \mu_{w=v}(\{v\}) = 1/2$. Then for both $\{\mu_{w<v}, \mu_{w=v}\}$ and $\{\mu_{w<v}\}$, their weak relation is $\succsim_1$. Yet the strict dominance relation of the former is $\succ_1$ while the strict dominance relation of the latter is $\succ_2$.

Similarly, if all we know about $\mathcal{P}$ is that its strict dominance relation is $\succ_1$, then both $\succsim_1$ and $\succsim_2$ may be its weak relation. For this, define a probability measure $\mu_{w>v}$ that favors w so that $\mu_{w>v}(\{w\}) = 2/3$. Then we see that the strict dominance relation of both $\{\mu_{w<v}, \mu_{w=v}\}$ and $\{\mu_{w<v}, \mu_{w>v}\}$ is $\succ_1$ while the weak relation of the former is $\succsim_1$ and the weak relation of the latter is $\succsim_2$.

In light of these considerations, we introduce the following definition that accounts for both relations; cf. Konek [104, p. 275, footnote 4], who suggests that the study of comparative probability ought to start with pairs $\langle \succsim, \succ \rangle$ because an agent who judges that X is at least as likely as Y but withholds judgment about whether Y is at least as likely as X does not necessarily judge that X is strictly more likely than Y.

Definition 5.2.5. Given a pair $\langle \succsim, \succ \rangle$ of binary relations on $\wp(W)$ and a set $\mathcal{P}$ of probability measures on $\wp(W)$, we say that $\langle \succsim, \succ \rangle$ is *represented by* $\mathcal{P}$ iff for all $X, Y \subseteq W$,

- $X \succsim Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) \geq \mu(Y)$, and
- $X \succ Y$ iff for all $\mu \in \mathcal{P}$, $\mu(X) > \mu(Y)$.

Remark 5.2.6. Define $X \succcurlyeq Y$ as *not* $Y \succ X$, i.e., there is some $\mu \in \mathcal{P}$ such that $\mu(X) \geq \mu(Y)$ (cf. the notion of *justifiable preference* in [119]). Then the pair $\langle \succsim, \succcurlyeq \rangle$ of weak relations is what Giarlotta and Greco [72] call a *necessary and possible preference*.

The following theorem characterizes the representable relation pairs.

Theorem 5.2.7. *Let W be a nonempty finite set and $\succsim, \succ$ two binary relations on $\wp(W)$. Then $\langle \succsim, \succ \rangle$ is represented by a set $\mathcal{P}$ of probability measures on $\wp(W)$ if and only if:*

- *$\succ$ is irreflexive and $\succ \subseteq \succsim$;*
- *$W \succ \varnothing$, and $\{w\} \succsim \varnothing$ for all $w \in W$;*
- *$\succsim$ satisfies (GFC) and $\succ$ satisfies the strict generalized finite cancellation condition (SGFC): for any two finite sequences $\langle A_i \rangle_{i=1}^n, \langle B_i \rangle_{i=1}^n$ of events in $\wp(W)$ and $k \in \mathbb{N} \setminus \{0\}$ such that $\sum_{i=1}^{n-1} \mathbf{1}_{A_i} + k\mathbf{1}_{A_n} = \sum_{i=1}^{n-1} \mathbf{1}_{B_i} + k\mathbf{1}_{B_n}$, if for all $i < n$, $A_i \succsim B_i$ and there is $i < n$ with $A_i \succ B_i$, then $B_n \succ A_n$.*

The rest of this section is devoted to the proof of Theorem 5.2.7. The proof is adapted from the proof of Theorem 5.2.2 above in [2], which also generalizes the proof in [142] for Theorem 5.2.1 (also see [127, § 3.3] for a representation theorem concerning $\langle \succsim, \succ \rangle$ in the setting of precise probability). For this, pick a nonempty finite set W and a pair $\langle \succsim, \succ \rangle$ satisfying the conditions (the necessity of the conditions is easy). The main strategy is to reframe the representability of $\langle \succsim, \succ \rangle$ in terms of the existence of solutions to some systems of homogeneous linear inequalities in the vector space $\mathbb{R}^W$. Hence we use vectors in $\Delta(W) = \{\mu \in \mathbb{R}^W \mid \mu \cdot \mathbf{1}_W = 1 \text{ and for all } w \in W, \mu(w) \geq 0\}$ as probability measures.

Define $D_{\succsim} = \{\mathbf{1}_A - \mathbf{1}_B \mid A, B \subseteq W, A \succsim B\}$ and $D_{\succ} = \{\mathbf{1}_A - \mathbf{1}_B \mid A, B \subseteq W, A \succ B\}$. Intuitively, $D_{\succsim}$ contains vectors that always receive non-negative measures and $D_{\succ}$ contains vectors that always receive positive measures. Given the conditions satisfied by $\succsim$ and $\succ$, we can prove the following lemmas.

Lemma 5.2.8. *If $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim}$, then $f \in D_{\succsim}$.*

Proof. Suppose $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim}$. Since all the vectors are in $\{-1, 0, 1\}^W$, we can assume that all coefficients are rational since a system of linear inequalities with rational coefficients has a solution if and only if it has a rational solution. Then we can clear the denominators and obtain a $k \in \mathbb{N} \setminus \{0\}$ such that kf is simply a sum of vectors in $D_{\succsim}$ possibly with repetitions: $\sum_{i=1}^n g_i$. Since f and the g_i's are in $D_{\succsim}$, we can find subsets A_i, B_i for $i = 1 \ldots n+1$ of W such that

- $g_i = \mathbf{1}_{A_i} - \mathbf{1}_{B_i}$ for $i = 1 \dots n$ and $f = \mathbf{1}_{B_{n+1}} - \mathbf{1}_{A_{n+1}}$ (take $B_{n+1} = f^{-1}(1)$ and $A_{n+1} = f^{-1}(-1)$), and
- $A_i \succsim B_i$ for $i = 1 \dots n$.

Then given that $kf = \sum_{i=1}^{n} g_i$, we have $\sum_{n=1}^{n} \mathbf{1}_{A_i} + k\mathbf{1}_{A_{n+1}} = \sum_{i=1}^{n} \mathbf{1}_{B_i} + k\mathbf{1}_{B_{n+1}}$. Hence we can apply (GFC) to $\langle A_i \rangle_{i=1}^{n+1}$ and $\langle B_i \rangle_{i=1}^{n+1}$ and see that $B_{n+1} \succsim A_{n+1}$. Therefore, $f = \mathbf{1}_{B_{n+1}} - \mathbf{1}_{A_{n+1}} \in D_{\succsim}$. □

Lemma 5.2.9. *If $f \in \{-1, 0, 1\}^W$ is a non-negative linear combination of vectors in $D_{\succsim} \cup D_{\succ}$ with a coefficient for a vector in $D_{\succ}$ being positive, then $f \in D_{\succ}$.*

Proof. Similar to the proof of the previous lemma. The only change in this case is that when we find k and express kf as a sum of vectors in $D_{\succsim} \cup D_{\succ}$, at least one vector in $D_{\succ}$ must figure in the sum since initially the non-negative linear combination resulting in f has a positive coefficient for a vector in $D_{\succ}$. Then we can find sets A_i's and B_i's similarly and apply (SGFC) to see that f must be in $D_{\succ}$ already. □

Now define

$$\mathcal{P} = \{\mu \in \Delta(W) \mid \forall f \in D_{\succsim},\ \mu \cdot f \geq 0 \text{ and } \forall f \in D_{\succ},\ \mu \cdot f > 0\}.$$

Our goal is to show that $\langle \succsim, \succ \rangle$ is represented by this $\mathcal{P}$. Note that one direction is done already: for any $A, B \subseteq W$,

- if $A \succsim B$, then by the definition of $\mathcal{P}$, for all $\mu \in \mathcal{P}$, $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) \geq 0$, which means that $\mu \cdot \mathbf{1}_A \geq \mu \cdot \mathbf{1}_B$;
- similarly, if $A \succ B$, then for all $\mu \in \mathcal{P}$, $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) > 0$, which means that $\mu \cdot \mathbf{1}_A > \mu \cdot \mathbf{1}_B$.

Hence all that are left to prove are the following two claims:

(a) If $A \not\succsim B$, then there is a $\mu \in \mathcal{P}$ such that $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) < 0$;

(b) If $A \not\succ B$, then there is a $\mu \in \mathcal{P}$ such that $\mu \cdot (\mathbf{1}_A - \mathbf{1}_B) \leq 0$.

For (a), it is enough to prove that for all $f \in \{-1, 0, 1\}^W$, if $f \notin D_{\succsim}$, then there is $\mu \in \mathcal{P}$ such that $\mu \cdot -f > 0$, since for any $A, B \subseteq W$, we have $\mathbf{1}_A - \mathbf{1}_B \in \{-1, 0, 1\}^W$. Hence take such an $f \in \{-1, 0, 1\}^W \setminus D_{\succsim}$. We need to find a μ such that (i) $\mu \in \mathcal{P}$ and (ii) $\mu \cdot -f > 0$. Given the definition of $\mathcal{P}$, this amounts to the existence of a solution to the following system of homogeneous linear inequalities (where we write $[D]$ for the matrix containing as columns the vectors in a set D of vectors):

$$[D_{\succsim}]^\top \vec{x} \geq \vec{0}, \quad [D_{\succ} \cup \{-f\}]^\top \vec{x} > \vec{0}. \tag{5.1}$$

The existence of a μ satisfying (i) and (ii) is equivalent to the existence of a solution to the above system of inequalities because by assumption, $W \succ \varnothing$ and $\{w\} \succsim \varnothing$ for all $w \in W$, which means that $\mathbf{1}_W \in D_\succ$ and $\mathbf{1}_{\{w\}} \in D_\succsim$ for all $w \in W$, so any solution can be scaled to be an element in $\mathcal{P}$. The condition of the existence of a solution is given by a special case of Motzkin's Transposition Theorem (see [133]).

Theorem 5.2.10 (Motzkin's Transposition Theorem)**.** *The linear inequality system $M_1\vec{x} \geq 0, M_2\vec{x} > \vec{0}$ has a solution if and only if there is no solution to the system $M_1^\top \vec{y_1} + M_2^\top \vec{y_2} = \vec{0}, \vec{y_1} \geq \vec{0}, \vec{y_2} \geq \vec{0}, \vec{y_2} \neq \vec{0}$.*

Suppose toward a contradiction that there is no solution to (5.1). Then by Motzkin's Transposition Theorem, there are non-negative $\vec{y_1}, \vec{y_2}$ with $\vec{y_2}$ non-trivial such that $[D_\succsim]^\top \vec{y_1} + [D_\succ \cup \{-f\}]^\top \vec{y_2} = \vec{0}$. In other words, $\vec{0}$ is a non-negative linear combination of vectors in $D_\succsim \cup D_\succ \cup \{-f\}$ with one of the vectors in $D_\succ \cup \{-f\}$ having a positive coefficient. Now there are two possibilities: either $-f$ has a positive coefficient or not. If not, then $\vec{0}$ is a non-negative linear combination of vectors in $D_\succsim \cup D_\succ$ with a vector in $D_\succ$ having a positive coefficient. Then, by Lemma 5.2.9, $\vec{0} \in D_\succ$. This contradicts the assumption that $\succ$ is irreflexive. If $-f$ has a positive coefficient, then f is a linear combination of vectors in $D_\succsim \cup D_\succ = D_\succsim$ since $\succ \subseteq \succsim$. By Lemma 5.2.8, $f \in D_\succsim$, but we picked f specifically from outside $D_\succsim$. Hence, either way, we have a contradiction. This completes the proof of (a).

The proof of (b) is almost identical. It is enough to show that for any $f \in \{-1, 0, 1\}^W \setminus D_\succ$, the following has a solution:

$$[D_\succsim \cup \{-f\}]^\top \vec{x} \geq \vec{0}, \quad [D_\succ]^\top \vec{x} > \vec{0}.$$

If there is no solution, then by Motzkin's Transposition Theorem, $\vec{0}$ is a non-negative linear combination of vectors in $D_\succsim \cup \{-f\} \cup D_\succ$ with at least one vector in $D_\succ$ having a positive coefficient. Again, we consider whether $-f$ has a positive coefficient or not. If not, then $\vec{0}$ should again be in $D_\succ$, which is not the case. If indeed $-f$ has a positive coefficient, then f is a linear combination of vectors in $D_\succsim \cup D_\succ$ with at least one vector in $D_\succ$ having a positive coefficient. By Lemma 5.2.9, $f \in D_\succ$, contradicting the way we picked f. Hence (b) is also proved, which completes the proof of Theorem 5.2.7.

Remark 5.2.11. The sets $D_\succsim$ and $D_\succ$ used in the proof above are reminiscent of an alternative but also prominent way of modeling uncertainty in the imprecise probability literature: sets of desirable gambles (see [155] and chapters in [4] for introductions). For any event $A \subseteq W$, we may interpret it as a gamble that returns a unit of utility for the states in A and returns nothing for states outside A. In other words, we can understand comparing the likelihoods of two events A and B as comparing the two corresponding gambles $\mathbf{1}_A$ and $\mathbf{1}_B$, which in turn reduces to the question of whether the gamble $\mathbf{1}_A - \mathbf{1}_B$ is acceptable/desirable. However, there are two important differences between our setting and the desirable gambles approach commonly presented in the literature.

First, since we are only comparing propositions, we do not need to appeal to the desirability of gambles not in $\{-1, 0, 1\}^W$. In the literature on desirable gambles, all gambles in $\mathbb{R}^W$ are considered, and that is partly the reason for succinct axioms for coherent sets of desirable gambles such as closure under positive scaling and pairwise addition. The same cannot be done when restricting to $\{-1,0,1\}^W$, since for example $\mathbf{1}_W + \mathbf{1}_W$ is no longer in $\{-1,0,1\}^W$. Also, it is not hard to see that different coherent sets of desirable gambles have the same intersection with $\{-1,0,1\}^W$. This means that using sets of desirable gambles in $\mathbb{R}^W$, we may encode more information than needed for comparing propositions.

Second, we model an agent's uncertainty with a pair of binary relations, and hence when translated to sets of desirable gambles, we use a pair of sets of desirable gambles instead of a single one. This can be easily seen from the proof above: we constructed a pair $\langle D_{\succsim}, D_{\succ}\rangle$ from $\langle \succsim, \succ \rangle$. If we disregard the previous difference and consider all gambles in $\mathbb{R}^W$, our approach can be understood as generalizing representation by a single set of *almost desirable* gambles (using the terminology in [33]) by pairing it with another set of gambles that can be interpreted as *strictly desirable* gambles. However, the axiomatic requirement for this set is weaker than the requirement for "sets of strictly desirable gambles" in [33]. More importantly, our axiomatic requirement concerns two sets jointly, as can be seen from Lemma 5.2.9. In this way, we achieve greater generality (expressivity) than merely using a set of almost desirable gambles. We leave further comparison between these two approaches to imprecise probability for future work.

5.3 The Logic $\mathsf{IP}(\succsim)$

In this section and the following sections, we turn to the formalization of imprecise comparative probabilistic reasoning in logical systems. The representation theorems of Section 5.2 lead to completeness theorems for these logical systems.

The logics we consider form a hierarchy of increasing expressive power of their languages. The least expressive language we will consider is the following.

Definition 5.3.1. The language $\mathcal{L}(\succsim)$, generated from a nonempty set Prop of propositional variables, is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi)$$

where $p \in \mathsf{Prop}$. A *propositional* (or *Boolean*) *formula* is a formula generated from Prop using only $\neg$ and $\wedge$. We define the other propositional connectives $\vee$, $\to$, $\leftrightarrow$, $\top$, and $\bot$ as usual. Finally, we define $\varphi \succnsim \psi$ as $(\varphi \succsim \psi) \wedge \neg(\psi \succsim \varphi)$ and $\varphi \approx \psi$ as $(\varphi \succsim \psi) \wedge (\psi \succsim \varphi)$.

We will consider several semantics for this language, each of which builds on the standard possible world models for propositional logics.

Definition 5.3.2. A *propositional model* is a pair $\mathcal{M} = \langle W, V\rangle$ where W is a nonempty set and $V : \mathsf{Prop} \to \wp(W)$. We may abuse notation and write '$w \in \mathcal{M}$' to mean $w \in W$.

The first semantics for $\mathcal{L}(\succsim)$ that we will consider, which may be considered its "intended semantics," equips a propositional model with one or more probability measures, as follows.

Definition 5.3.3. An *imprecise probabilistic model* (IP model) is a pair $\langle \mathcal{M}, \mathcal{P} \rangle$ where $\mathcal{M} = \langle W, V \rangle$ is a propositional model and $\mathcal{P}$ is a set of finitely additive probability measures on a field $\mathcal{F}$ of subsets of W such that $V(p) \in \mathcal{F}$ for each $p \in \mathsf{Prop}$. A *precise probabilistic model* is an imprecise probabilistic model $\langle \mathcal{M}, \mathcal{P} \rangle$ such that $|\mathcal{P}| = 1$.

The key part of the truth definition of formulas of $\mathcal{L}(\succsim)$ in IP models matches the notion of imprecise representation from Section 5.2: $\varphi \succsim \psi$ is true just in case *according to all the probability measures in* $\mathcal{P}$, the probability of the set of worlds where φ is true is at least as great as the probability of the set of worlds where ψ is true.

Definition 5.3.4. Given an IP model $\langle \mathcal{M}, \mathcal{P} \rangle$, $w \in \mathcal{M}$, and $\varphi \in \mathcal{L}(\succsim)$, we define $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ and $[\![\varphi]\!]^{\mathcal{M},\mathcal{P}} = \{w \in W \mid \mathcal{M}, \mathcal{P}, w \vDash \varphi\}$ as follows:

1. $\mathcal{M}, \mathcal{P}, w \vDash p$ iff $w \in V(p)$;
2. $\mathcal{M}, \mathcal{P}, w \vDash \neg\varphi$ iff $\mathcal{M}, \mathcal{P}, w \nvDash \varphi$;
3. $\mathcal{M}, \mathcal{P}, w \vDash (\varphi \wedge \psi)$ iff $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ and $\mathcal{M}, \mathcal{P}, w \vDash \psi$;
4. $\mathcal{M}, \mathcal{P}, w \vDash \varphi \succsim \psi$ iff for all $\mu \in \mathcal{P}$, $\mu([\![\varphi]\!]^{\mathcal{M},\mathcal{P}}) \geq \mu([\![\psi]\!]^{\mathcal{M},\mathcal{P}})$.

If α is a propositional formula, we may write '$V(\alpha)$' for $[\![\alpha]\!]^{\mathcal{M},\mathcal{P}}$ to emphasize that the set of worlds where α is true does not depend on the set $\mathcal{P}$ of probability measures.

Finally, given a class K of IP models, φ is *valid with respect to* K iff for all $\langle \mathcal{M}, \mathcal{P} \rangle \in \mathsf{K}$ and $w \in \mathcal{M}$, we have $\mathcal{M}, \mathcal{P}, w \vDash \varphi$.

An easy induction shows that for any formula φ, the set of worlds where φ is true belongs to the algebra $\mathcal{F}$ of measurable sets.

Lemma 5.3.5. *For every IP model* $\langle \mathcal{M}, \mathcal{P} \rangle$ *and* $\varphi \in \mathcal{L}(\succsim)$, *we have* $[\![\varphi]\!]^{\mathcal{M},\mathcal{P}} \in \mathcal{F}$.

Below we define logics that are sound and complete with respect to the classes of imprecise probabilistic models and precise probabilistic models, respectively. To do so, we first need to define a syntactic abbreviation that allows us to express the finite cancellation condition of Theorem 5.2.1 using formulas of our language. Given formulas $\varphi_1, \dots, \varphi_n, \psi_1, \dots, \psi_n \in \mathcal{L}(\succsim)$, and $1 \leq k \leq n$, define C_k to be the disjunction of all conjunctions

$$f_1\varphi_1 \wedge \cdots \wedge f_n\varphi_n \wedge g_1\psi_1 \wedge \cdots \wedge g_n\psi_n$$

where exactly k of the f's and k of the g's are the empty string, and the rest are $\neg$. Thus, C_k is true at a state $w \in W$ iff exactly k of the φ's and k of the ψ's are true at w. Then let

$$(\varphi_1, \dots, \varphi_n) \equiv (\psi_1, \dots, \psi_n) := C_1 \vee \cdots \vee C_n,$$

which is true at a state $w \in W$ iff the number of φ's true at w is exactly the same as the number of ψ's true at w. Using these abbreviations, we can express the finite cancellation condition with the axiom schema (**A4**) below.

Definition 5.3.6. The set of theorems of $\mathsf{SP}(\succsim)$ (the *logic of sharp probability*) is the smallest subset of $\mathcal{L}(\succsim)$ that contains all tautologies of propositional logic, is closed under *modus ponens* (if $\varphi \in \mathsf{SP}(\succsim)$ and $\varphi \to \psi \in \mathsf{SP}(\succsim)$, then $\psi \in \mathsf{SP}(\succsim)$) and *necessitation* (if $\varphi \in \mathsf{SP}(\succsim)$, then $\varphi \succsim \top \in \mathsf{SP}(\succsim)$), and contains all instances of the following axiom schemas for all $n \in \mathbb{N}$:[3]

(**A0**) $(\varphi \succsim \psi) \vee (\psi \succsim \psi)$;

(**A1**) $\varphi \succsim \bot$;

(**A2**) $\varphi \succsim \varphi$;[4]

(**A3**) $\neg(\bot \succsim \top)$;

(**A4**) $\big((\varphi_1 \succsim \psi_1) \wedge \cdots \wedge (\varphi_n \succsim \psi_n) \wedge (\varphi_1, \ldots, \varphi_n, \varphi') \equiv (\psi_1, \ldots, \psi_n, \psi') \succsim \top\big) \to (\psi' \succsim \varphi')$;

(**A5**) $(\varphi \succsim \psi) \to \big((\varphi \succsim \psi) \succsim \top\big)$;

(**A6**) $\neg(\varphi \succsim \psi) \to \big(\neg(\varphi \succsim \psi) \succsim \top\big)$.

The representation result in Theorem 5.2.1 may be used to prove the following completeness theorem for $\mathsf{SP}(\succsim)$.

Theorem 5.3.7 ([145, 67])**.** *For all $\varphi \in \mathcal{L}(\succsim)$: φ is a theorem of $\mathsf{SP}(\succsim)$ if and only if φ is valid with respect to the class of all precise probabilistic models.*

To obtain a complete logic for *imprecise* probabilistic models, we express the generalized finite cancellation conditions of Theorem 5.2.2 using formulas of our language as follows.

Definition 5.3.8. The logic $\mathsf{IP}(\succsim)$ (the *logic of imprecise probability*) is defined in the same way as $\mathsf{SP}(\succsim)$ except without axiom (**A0**) and with (**A4**) replaced by:

(**A4′**) $$\big((\varphi_1 \succsim \psi_1) \wedge \cdots \wedge (\varphi_n \succsim \psi_n) \wedge (\varphi_1, \ldots, \varphi_n, \underbrace{\varphi', \ldots, \varphi'}_{k\,\text{times}}) \equiv (\psi_1, \ldots, \psi_n, \underbrace{\psi', \ldots, \psi'}_{k\,\text{times}}) \succsim \top\big) \to (\psi' \succsim \varphi').$$

The representation result in Theorem 5.2.2 may be used to prove the following completeness theorem for $\mathsf{IP}(\succsim)$.

Theorem 5.3.9 ([1])**.** *For all $\varphi \in \mathcal{L}(\succsim)$: φ is a theorem of $\mathsf{IP}(\succsim)$ if and only if φ is valid with respect to the class of all imprecise probabilistic models.*

[3] The labeling of axioms here follows [1].

[4] Axiom (**A2**) is redundant given (**A0**), but below we consider a logic that drops (**A0**). In fact, (**A2**) is also derivable from the $n = 0$ case of (**A4**) and (**A4′**), but we include (**A2**) to match [1].

In Section 5.4 we will give a completeness proof that shows how the proof of Theorem 5.3.9 goes as well.

Finally, we mention two non-probabilistic semantics for the logics $\mathsf{SP}(\succsim)$ and $\mathsf{IP}(\succsim)$, respectively. First, van der Hoek [89] proved that $\mathsf{SP}(\succsim)$ is also the logic of cardinality comparisons between finite sets (for the logic of cardinality comparisons between sets of arbitrary cardinality, see [41]).

Theorem 5.3.10 ([89])**.** *The logic $\mathsf{SP}(\succsim)$ is also sound and complete with respect to finite propositional models $\mathcal{M} = \langle W, V\rangle$ with the semantics*

$$M, w \vDash \varphi \succsim \psi \text{ iff there is an injection } f : [\![\psi]\!]^{\mathcal{M}} \to [\![\varphi]\!]^{\mathcal{M}}.$$

To obtain a similar semantics for $\mathsf{IP}(\succsim)$, Holliday and Icard [95] use *preferential models* $M = (W, \succeq, V)$ where $\succeq$ is a preorder (a reflexive and transitive binary relation) on W. Intuitively, $\succeq$ is a likelihood relation on worlds, which is then lifted to a likelihood relation on events as in the following theorem (also see [84]).

Theorem 5.3.11 ([85])**.** *The logic $\mathsf{IP}(\succsim)$ is also sound and complete with respect to finite preferential models $M = (W, \succeq, V)$ with the semantics*

$$M, w \vDash \varphi \succsim \psi \text{ iff there is a } \succeq\text{-inflationary injection } f : [\![\psi]\!]^{M} \to [\![\varphi]\!]^{M},$$

where f is $\succeq$-inflationary if $f(w) \succeq w$ for each $w \in \mathrm{dom}(f)$.

For a discussion of conceptual issues in the choice of probabilistic vs. non-probabilistic semantics for $\mathsf{IP}(\succsim)$ in the context of natural language semantics, see [95] and Section 3.4 of [94].

5.4 The Logic $\mathsf{IP}(\succsim, \succ)$

Our first step beyond the existing literature on logics of imprecise comparative probability is to add to our formal language the primitive strict operator $\succ$ from Section 5.2.

Definition 5.4.1. The language $\mathcal{L}(\succsim, \succ)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi)$$

where $p \in \mathsf{Prop}$. As before, we define $\varphi \gnsim \psi$ as $(\varphi \succsim \psi) \wedge \neg(\psi \succsim \varphi)$. Let $\mathcal{L}(\succ)$ be the fragment of $\mathcal{L}(\succsim, \succ)$ in which $\succsim$ does not occur.

Definition 5.4.2. We extend the semantics of Definition 5.3.4 to $\mathcal{L}(\succsim, \succ)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \varphi \succ \psi$ iff for all $\mu \in \mathcal{P}$, $\mu([\![\varphi]\!]^{\mathcal{M},\mathcal{P}}) > \mu([\![\psi]\!]^{\mathcal{M},\mathcal{P}})$.

It follows from Example 5.2.4 that the formula $p \succ q$ is not equivalent to any formula of $\mathcal{L}(\succsim)$, including $p \gnsim q$, while the formula $p \succsim q$ is not equivalent to any formula of $\mathcal{L}(\succ)$.

In the following, we first present a sound and complete logic for $\mathcal{L}(\succsim, \succ)$ whose axioms match the conditions of the representation result in Theorem 5.2.7. Then we discuss the expressivity of this language, including how it is more expressive than $\mathcal{L}(\succsim)$.

5.4.1 Logic

Definition 5.4.3. The logic $\mathsf{IP}(\succsim,\succ)$ is the smallest subset of $\mathcal{L}(\succsim,\succ)$ that contains all tautologies of propositional logic, is closed under *modus ponens* (if $\varphi \in \mathsf{IP}(\succsim,\succ)$ and $\varphi \to \psi \in \mathsf{IP}(\succsim,\succ)$, then $\psi \in \mathsf{IP}(\succsim,\succ)$) and *necessitation* (if $\varphi \in \mathsf{IP}(\succsim,\succ)$, then $\varphi \succsim \top \in \mathsf{IP}(\succsim,\succ)$), and contains all instances of the following axiom schemas for all $n \in \mathbb{N}$:

(**B1**) $\varphi \succsim \bot$;

(**B2**) $\top \succ \bot$;

(**B3**) $(\varphi \succ \psi) \to (\varphi \succsim \psi)$;

(**B4**) $\neg(\varphi \succ \varphi)$;

(**B5**) $$\big(\varphi_1, \dots, \varphi_n, \underbrace{\varphi', \dots, \varphi'}_{k\,\text{times}}\big) \equiv (\psi_1, \dots, \psi_n, \underbrace{\psi', \dots, \psi'}_{k\,\text{times}}) \succsim \top\big) \to \big((\textstyle\bigwedge_{i=1}^n (\varphi_i \succsim \psi_i)) \to (\psi' \succsim \varphi')\big);$$

(**B6**) $$\big((\varphi_1, \dots, \varphi_n, \underbrace{\varphi', \dots, \varphi'}_{k\,\text{times}}) \equiv (\psi_1, \dots, \psi_n, \underbrace{\psi', \dots, \psi'}_{k\,\text{times}}) \succsim \top\big) \to \big((\textstyle\bigwedge_{i=1}^n (\varphi_i \succsim \psi_i) \wedge \bigvee_{i=1}^n (\varphi_i \succ \psi_i)) \to (\psi' \succ \varphi')\big);$$

(**B7**) $(\varphi \succsim \psi) \to ((\varphi \succsim \psi) \succsim \top)$;

(**B8**) $\neg(\varphi \succsim \psi) \to (\neg(\varphi \succsim \psi) \succsim \top)$;

(**B9**) $(\varphi \succ \psi) \to ((\varphi \succ \psi) \succsim \top)$;

(**B10**) $\neg(\varphi \succ \psi) \to (\neg(\varphi \succ \psi) \succsim \top)$.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.4.4 (Soundness and Completeness)**.** *For all $\varphi \in \mathcal{L}(\succsim,\succ)$: φ is a theorem of $\mathsf{IP}(\succsim,\succ)$ if and only if φ is valid with respect to the class of all imprecise probabilistic models.*

The soundness direction is easy to check. For completeness, as usual, pick an arbitrary formula γ consistent in $\mathsf{IP}(\succsim,\succ)$ and let p be the set of propositional variables appearing in γ and $\mathcal{L}_0$ the restriction of $\mathcal{L}(\succsim,\succ)$ to p. Then extend $\{\gamma\}$ to a set Γ that is maximally consistent in $\mathsf{IP}(\succsim,\succ)$ with respect to $\mathcal{L}_0$. Now our goal is to build an IP model of γ by extracting information from Γ. To this end, we view $\mathcal{L}_0$ as a term algebra of the type of Boolean algebras expanded with two binary operations. Then define $\blacksquare\varphi$ by $\varphi \wedge (\varphi \succsim \top)$, $F = \{\varphi \in \mathcal{L}_0 \mid \blacksquare\varphi \in \Gamma\}$, and define a binary relation $\sim$ on $\mathcal{L}_0$ by $\varphi \sim \psi$ iff $(\varphi \leftrightarrow \psi) \in F$.

Lemma 5.4.5. *F contains $\top$ and is closed under deduction in $\mathcal{L}_0$: whenever $\varphi \to \psi \in \mathcal{L}_0$ is a theorem of $\mathsf{IP}(\succsim, \succ)$ and $\varphi \in F$, then $\psi \in F$ too. Also, $\sim$ is an equivalence relation extending the provable equivalence relation on $\mathcal{L}_0$ and is congruential over $\neg, \wedge, \succsim$, and $\succ$: for all $\varphi, \psi, \chi \in \mathcal{L}_0$, if $\varphi \sim \psi$, then $\neg\varphi \sim \neg\psi$, $(\varphi \wedge \chi) \sim (\psi \wedge \chi)$, $(\chi \wedge \varphi) \sim (\chi \wedge \psi)$, $(\varphi \succsim \chi) \sim (\psi \succsim \chi)$, $(\chi \succsim \varphi) \sim (\chi \succsim \psi)$, $(\varphi \succ \chi) \sim (\psi \succ \chi)$, and $(\chi \succ \varphi) \sim (\chi \succ \psi)$.*

Proof. When $n = 0$, (**B5**) together with necessitation shows that for every φ, $\varphi \succsim \varphi$ is a theorem. Then clearly $\top \in F$. To show that F is closed under deduction in $\mathcal{L}_0$, noting that Γ is clearly closed under deduction in $\mathcal{L}_0$ due to its being a maximally consistent set, it is enough to show that whenever $\varphi \to \psi \in \mathsf{IP}(\succsim, \succ)$, we have $(\varphi \succsim \top) \to (\psi \succsim \top) \in \mathsf{IP}(\succsim, \succ)$ too. For this, apply (**B5**) to $\langle \varphi, \psi \wedge \neg\varphi, \top \rangle$ and $\langle \top, \bot, \psi \rangle$.

Since F is closed under deduction in $\mathcal{L}_0$ and contains $\top$, F also contains all theorems of $\mathsf{IP}(\succsim, \succ)$ in $\mathcal{L}_0$. Hence it is easy to show that $\sim$ is an equivalence relation extending the provable equivalence relation on $\mathcal{L}_0$ that is congruential over $\neg$ and $\wedge$. To show that $\sim$ is congruential over $\succsim$ and $\succ$, using again that Γ is closed under deduction in $\mathcal{L}_0$, we only need to show that the following are derivable:

- $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi))$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to (((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi)) \succsim \top)$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi))$;
- $((\varphi \leftrightarrow \psi) \succsim \top) \to (((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi)) \succsim \top)$.

In fact, the second and the fourth follow from the first and the third using (**B7**) to (**B10**), the closure of $(\cdot \succsim \top)$ under deduction, and Boolean reasoning. The first and the third are again simple exercises using (**B5**) and (**B6**), respectively. $\square$

Lemma 5.4.6. *$\mathcal{B} = \mathcal{L}_0/{\sim}$ is a Boolean algebra expanded with two binary operations which we denote again by $\succsim$ and $\succ$. Moreover, by axioms (**B7**) to (**B10**), for any $a, b \in \mathcal{B}$, $a \succsim b$ is either the top element or the bottom element, and so is $a \succ b$. In addition, $\mathcal{B}$ is finite.*

Proof. Since $\sim$ is a congruence extending the provable equivalence relation and $\mathsf{IP}(\succsim, \succ)$ has all Boolean reasoning principles, $\mathcal{B}$ is a Boolean algebra. To see that $a \succsim b$ is either the top element or the bottom element, pick any $\varphi, \psi \in \mathcal{L}_0$ such that $[\varphi]_\sim = a$ and $[\psi]_\sim = b$. Then note that either $\varphi \succsim \psi \in \Gamma$ or $\neg(\varphi \succsim \psi) \in \Gamma$. In the former case, given (**B7**), we have $(\varphi \succsim \psi) \in F$ and hence $a \succsim b = [\varphi \succsim \psi]_\sim$ is the top element. In the latter case, using (**B8**), $\neg(a \succsim b)$ is the top element, which means that $a \succsim b$ is the bottom element. The same reasoning goes for $a \succ b$, using (**B9**) and (**B10**). Finally, to see that $\mathcal{B}$ is finite, note first that it has a finite set of generators: $[\mathsf{p}]_\sim = \{[p]_\sim \mid p \in \mathsf{p}\}$. Since we have just shown that $\succsim$ and $\succ$ only bring elements to either the top element or the bottom element, in generating $\mathcal{B}$ from $[\mathsf{p}]_\sim$ we can use only the Boolean operations. Hence the Boolean reduct of $\mathcal{B}$ is a finitely generated Boolean algebra, which must be finite. $\square$

Since (the Boolean reduct of) $\mathcal{B}$ is a finite Boolean algebra, it is isomorphic to the powerset algebra of its set of atoms. However, to facilitate the proof of the completeness theorem of the next section, we take the set that includes all possible truth-assignments of propositional variables in p.

Definition 5.4.7. Let $W_\mathsf{p} = \{0,1\}^\mathsf{p}$ and $V_\mathsf{p} : \mathsf{Prop} \to \wp(W)$ be the natural valuation function defined by $V_\mathsf{p}(p) = \{f \in W_\mathsf{p} \mid f(p) = 1\}$ when $p \in \mathsf{p}$ and $V_\mathsf{p}(p) = \varnothing$ when $p \notin \mathsf{p}$. Finally, let $\mathcal{M}_\mathsf{p} = \langle W_\mathsf{p}, V_\mathsf{p} \rangle$.

In this way, $\wp(W_\mathsf{p})$ is essentially the free Boolean algebra generated by the images of p under V_p. The difference between $\wp(W_\mathsf{p})$ and the Boolean reduct of $\mathcal{B}$ is that $\mathcal{B}$ might be missing some of the atoms in the sense that some truth-assignments to p may be inconsistent in $\mathcal{B}$. However, from the probabilistic point of view, it is enough to make them impossible probabilistically by assigning them 0 probability. This gives us the advantage of always using the same $\mathcal{M}_\mathsf{p}$ when satisfying any consistent subset of $\mathcal{L}_0$.

To connect $\mathcal{M}_\mathsf{p}$ to $\mathcal{B}$, first let π be the natural Boolean quotient map π from $\wp(W_\mathsf{p})$ to $\mathcal{B}_0$ such that $\pi(V_\mathsf{p}(p)) = [p]_\sim$. This map is uniquely given since $\wp(W_\mathsf{p})$ is the free Boolean algebra generated by $\{V(p) \mid p \in \mathsf{p}\}$ and $\mathcal{B}$ is generated by $\{[p]_\sim \mid p \in \mathsf{p}\}$ using Boolean operations. Then, on $\wp(W_\mathsf{p})$, we define two binary relations:

- $X \succsim_\Gamma Y$ iff $\pi(X) \succsim \pi(Y)$ is the top element of $\mathcal{B}$;
- $X \succ_\Gamma Y$ iff $\pi(X) \succ \pi(Y)$ is the top element of $\mathcal{B}$.

Then it is not hard to show the following using the axioms (**B1**) to (**B6**).

Lemma 5.4.8. *$\langle \succsim_\Gamma, \succ_\Gamma \rangle$ satisfies all the conditions required in Theorem 5.2.7.*

Proof. Note that for every $a \in \mathcal{B}$, $a = [\varphi]_\sim$ for some $\varphi \in \mathcal{L}_0$. Hence any quantification over $\mathcal{B}$, and by the quotient map π, any quantification over $\wp(W_\mathsf{p})$ as well, can be simulated by quantification over $\mathcal{L}_0$. Since the axioms are schematic, (**B1**) to (**B4**) directly translate the first two bullet points of Theorem 5.2.7.

For (GFC) and (SGFC), it is enough to note that for any two finite sequences $\langle A_i \rangle_{i=1}^n$ and $\langle B_i \rangle_{i=1}^n$ of sets in $\wp(W_\mathsf{p})$ such that $\sum_{i=1}^n \mathbf{1}_{A_i} = \sum_{i=1}^n \mathbf{1}_{B_i}$, we can find two sequences $\langle \varphi_i \rangle_{i=1}^n$ and $\langle \psi_i \rangle_{i=1}^n$ of formulas in $\mathcal{L}_0$ such that:

- for all $i = 1 \dots n$, we have $[\varphi_i]_\sim = \pi(A_i)$ and $[\psi_i]_\sim = \pi(B_i)$, which implies that $A_i \succsim_\Gamma B_i$ iff $\varphi_i \succsim \psi_i \in \Gamma$ and that $A_i \succ_\Gamma B_i$ iff $\varphi_i \succ \psi_i \in \Gamma$;
- $[(\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n)]_\sim = [\top]_\sim$ and hence $(\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n) \in F$, which in turn implies that $((\varphi_1, \dots, \varphi_n) \equiv (\psi_i, \dots, \psi_n) \succsim \top) \in \Gamma$.

The existence of these formulas means that we can use (**B5**) and (**B6**) to show (GFC) and (SGFC), respectively. $\square$

Hence, by Theorem 5.2.7, we obtain a set $\mathcal{P}_\Gamma$ of probability measures on $\wp(W_\mathsf{p})$ such that

- $X \succsim_\Gamma Y$ iff for all $\mu \in \mathcal{P}_\Gamma$, $\mu(X) \geq \mu(Y)$, and
- $X \succ_\Gamma Y$ iff for all $\mu \in \mathcal{P}_\Gamma$, $\mu(X) > \mu(Y)$.

From this, we can show the following truth lemma.

Lemma 5.4.9. *For all $\varphi \in \mathcal{L}_0$, $\pi(\llbracket\varphi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) = [\varphi]_\sim$.*

Proof. By a simple induction on $\mathcal{L}_0$. The only cases of interest are the inductive steps for $\succsim$ and $\succ$. Note that $\llbracket\varphi \succsim \psi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$ is either W_{p} or $\varnothing$. Similarly, we have shown that $[\varphi \succsim \psi]_\sim$ is either $[\top]_\sim$ or $[\bot]_\sim$. Then the only missing connection is the following:

$$\begin{aligned}
\llbracket\varphi \succsim \psi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} = W_{\mathsf{p}} &\iff \forall \mu \in \mathcal{P}_\Gamma, \mu(\llbracket\varphi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \geq \mu(\llbracket\psi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \\
&\iff \llbracket\varphi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} \succsim_\Gamma \llbracket\psi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle} \\
&\iff (\pi(\llbracket\varphi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}) \succsim \pi(\llbracket\psi\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle})) = [\top]_\sim \\
&\iff ([\varphi]_\sim \succsim [\psi]_\sim) = [\top]_\sim \\
&\iff [\varphi \succsim \psi]_\sim = [\top]_\sim.
\end{aligned}$$

The proof for the case with $\varphi \succ \psi$ is almost identical. □

Now note that $[\gamma]_\sim$ is not the bottom element in $\mathcal{B}$, since otherwise $[\neg\gamma]_\sim$ would be the top element, and then $\blacksquare\neg\gamma \in \Gamma$, which means $\neg\gamma \in \Gamma$ too, rendering Γ inconsistent. Hence $\llbracket\gamma\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$ is nonempty because $\pi(\varnothing)$ must be $[\bot]_\sim$, which is not $[\gamma]_\sim$. Take a $w \in \llbracket\gamma\rrbracket^{\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle}$. Then $\langle \mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma\rangle, w \vDash \gamma$, and we are done.

To sum up, we now have the following strengthening of the completeness theorem, noting that there are only finitely many logically inequivalent formulas all using only a finite set p of propositional variables (see Lemma 5.6.11).

Proposition 5.4.10. *For any finite subset p of Prop with $\mathcal{L}_0$ being the set of formulas in $\mathcal{L}(\succsim, \succ)$ using only the propositional variables in p, and for any $\Gamma \subseteq \mathcal{L}_0$ that is consistent relative to $\mathsf{IP}(\succsim, \succ)$, there is a set $\mathcal{P}_\Gamma$ of probability measures on $\wp(W_{\mathsf{p}})$ and a $w \in W_{\mathsf{p}}$ such that $\mathcal{M}_{\mathsf{p}}, \mathcal{P}_\Gamma, w \vDash \gamma$ for all $\gamma \in \Gamma$.*

Before we discuss the expressivity of $\mathcal{L}(\succsim, \succ)$, we comment on the logic of *precise* probabilistic models. While $\succ$ is not definable in $\mathcal{L}(\succsim, \succ)$ with respect to all IP models, with respect to precise probabilistic models, $\varphi \succ \psi$ can be defined simply by $\neg(\psi \succsim \psi)$. Hence we can define the logic $\mathsf{SP}(\succsim, \succ)$ as follows.

Definition 5.4.11. The logic $\mathsf{SP}(\succsim, \succ)$ is the smallest subset of $\mathcal{L}(\succsim, \succ)$ that is closed under *modus ponens* (if $\varphi \in \mathsf{SP}(\succsim, \succ)$ and $\varphi \to \psi \in \mathsf{SP}(\succsim, \succ)$, then $\psi \in \mathsf{SP}(\succsim, \succ)$) and *necessitation* (if $\varphi \in \mathsf{SP}(\succsim, \succ)$ then $\varphi \succsim \top \in \mathsf{SP}(\succsim, \succ)$), contains all instances of tautologies of propositional logic, all instances of the axiom schemas (**A1**) to (**A6**) for $\mathsf{SP}(\succsim)$, and all instances of the axiom schema (**A7**) $(\varphi \succ \psi) \leftrightarrow \neg(\psi \succsim \varphi)$.

Then the following completeness theorem for $\mathsf{SP}(\succsim, \succ)$ can be shown in the same way that we just showed the completeness of $\mathsf{IP}(\succsim, \succ)$ using instead the representation result in Theorem 5.2.1. It will be used in the completeness proof for $\mathsf{IP}(\succsim, \succ, \Diamond)$ in the next section.

Proposition 5.4.12. *For any finite subset* p *of* Prop *with* $\mathcal{L}_0$ *being the set of formulas in* $\mathcal{L}(\succsim, \succ)$ *using only the propositional variables in* p*, and for any* $\Gamma \subseteq \mathcal{L}_0$ *that is consistent relative to* $\mathsf{SP}(\succsim, \succ)$*, there is a probability measure* μ_Γ *on* $\wp(W_{\mathsf{p}})$ *and a* $w \in W_{\mathsf{p}}$ *such that* $\mathcal{M}_{\mathsf{p}}, \{\mu_\Gamma\}, w \vDash \gamma$ *for all* $\gamma \in \Gamma$*.*

5.4.2 Expressivity

In this subsection we discuss the expressivity of $\mathcal{L}(\succsim)$ and $\mathcal{L}(\succsim, \succ)$ in distinguishing IP models. Given Example 5.2.4, it should not be surprising that $\mathcal{L}(\succsim, \succ)$ is more expressive than $\mathcal{L}(\succ)$. But here we precisely characterize the expressivity of these languages.

Definition 5.4.13. For any probability measure μ defined on a field F of sets, let $\succsim_\mu$ and $\succ_\mu$ be binary relations on F such that for any $X, Y \in F$, $X \succsim_\mu Y$ iff $\mu(X) \geq \mu(Y)$, and $X \succ_\mu Y$ iff $\mu(X) > \mu(Y)$. In addition, for any set $\mathcal{P}$ of probability measures defined on F, let $\succsim_{\mathcal{P}} = \bigcap\{\succsim_\mu \mid \mu \in \mathcal{P}\}$ and $\succ_{\mathcal{P}} = \bigcap\{\succ_\mu \mid \mu \in \mathcal{P}\}$.

For IP models $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$, we say that they are $\succsim$*-order-similar in* p, a subset of Prop, if for any Boolean formulas α, β using only letters in p,

$$[\![\alpha]\!]^{\langle W,V\rangle} \succsim_{\mathcal{P}} [\![\beta]\!]^{\langle W,V\rangle} \text{ iff } [\![\alpha]\!]^{\langle W',V'\rangle} \succsim_{\mathcal{P}'} [\![\beta]\!]^{\langle W',V'\rangle}.$$

We say that they are *order-similar in* $\mathsf{p} \subseteq \mathsf{Prop}$ if in addition to the above biconditional for $\succsim$, it is also true that for any Boolean formulas α, β using only letters in p,

$$[\![\alpha]\!]^{\langle W,V\rangle} \succ_{\mathcal{P}} [\![\beta]\!]^{\langle W,V\rangle} \text{ iff } [\![\alpha]\!]^{\langle W',V'\rangle} \succ_{\mathcal{P}'} [\![\beta]\!]^{\langle W',V'\rangle}.$$

A special case for ($\succsim$-)order-similarity is worth mentioning.

Proposition 5.4.14. *Let* $\langle W, V, \mathcal{P}\rangle$ *and* $\langle W, V, \mathcal{P}'\rangle$ *be IP models and* p *a subset of* Prop*. Let* F *be the field of sets generated by the image of* p *under* V*. Then* $\langle W, V, \mathcal{P}\rangle$ *and* $\langle W, V, \mathcal{P}'\rangle$ *are* $\succsim$*-order-similar (resp. order-similar) in* p *iff* $\succsim_{\mathcal{P}}|_F = \succsim_{\mathcal{P}'}|_F$ *(resp.* $\succsim_{\mathcal{P}}|_F = \succsim_{\mathcal{P}'}|_F$ *and* $\succ_{\mathcal{P}}|_F = \succ_{\mathcal{P}'}|_F$*).*

Proposition 5.4.15. *Let* $\langle W, V, \mathcal{P}\rangle$ *and* $\langle W', V', \mathcal{P}'\rangle$ *be IP models and* w, w' *worlds in* W *and* W'*, respectively. Then* w *and* w' *satisfy the same formulas in* $\mathcal{L}(\succsim, \succ)$ *(resp.* $\mathcal{L}(\succsim)$*) using only propositional variables in* $\mathsf{p} \subseteq \mathsf{Prop}$ *iff*

1. *w and w' satisfy the same propositional variables in p, and*

2. *$\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$ are order-similar (resp. $\succsim$-order-similar) in p.*

Proof. The left-to-right direction is trivial since failure of either 1 or 2 directly translates to a formula in the appropriate language with respect to which w and w' disagrees. For the right-to-left direction, note first that any comparative formula χ of the form $\varphi \succsim \psi$ or $\varphi \succ \psi$ is true at one world iff it is true at all worlds. This means that a formula φ with χ occurring is equivalent to $(\chi \wedge \varphi[\chi/\top]) \vee (\neg\chi \wedge \varphi[\chi/\bot])$ where $\varphi[\chi/\top]$ is the result of replacing χ by $\top$ in φ, and similarly for $\varphi[\chi/\bot]$. By repeated use of this method, it is not hard to see that every formula in $\mathcal{L}(\succsim, \succ)$ using only letters in p is semantically equivalent to a Boolean combination of formulas of one of the following types:

- a propositional variables in p,
- $\alpha \succsim \beta$ where α, β are Boolean formulas using only letters in p, and
- $\alpha \succ \beta$ where α, β are Boolean formulas using only letters in p.

The case with $\mathcal{L}(\succsim)$ is similar (without the last kind of formula in the above list). The proposition then follows easily. $\square$

Now we can translate Example 5.2.4 to a pair of pointed IP models that $\mathcal{L}(\succsim, \succ)$ can distinguish but $\mathcal{L}(\succsim)$ cannot. Let $W = \{w, v\}$ and V be the valuation such that $V(p) = \{w\}$ and $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Let $\mu_{w<v}$ and $\mu_{w=v}$ be defined as in Example 5.2.4. Then by Propositions 5.4.15 and 5.4.14, $\mathcal{L}(\succsim)$ cannot distinguish $\langle W, V, \{\mu_{w<v}, \mu_{w=v}\}\rangle, w$ from $\langle W, V, \{\mu_{w<v}\}\rangle, w$, since $\succsim_{\{\mu_{w<v}, \mu_{w=v}\}}$ and $\succsim_{\{\mu_{w<v}\}}$ are the same on $\wp(W)$. However, $\neg p \succ p$ distinguishes the pointed models.

5.5 The Logic $\mathsf{IP}(\succsim, \succ, \Diamond)$

In this section, we further extend our language with a *possibility modal* $\Diamond$. In the context of natural language semantics, one proposal for the meaning of "possibly φ" in *precise* probabilistic models is that φ has non-zero probability [118, §4.4]. In imprecise probabilistic models, we could require either (a) that all measures in $\mathcal{P}$ give φ non-zero probability or (b) that at least some measure in $\mathcal{P}$ gives φ non-zero probability. We adopt the weaker interpretation (b) of "possibly φ" (not as a proposal in natural language semantics, but because it suits our technical purposes in the next section). In addition to making claims about the possibility of factual states of affairs, e.g., "It is possible that it is raining," we would like to be able to make claims about the possibility of likelihood relations, e.g., "It is possible that hail is more likely than lightning tonight." According to the formal semantics given below, the latter will be true when there exists a probability measure in $\mathcal{P}$ such that *according to that measure* hail is more likely than lightning.

Definition 5.5.1. The language $\mathcal{L}(\succsim, \succ, \Diamond)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi$$

where $p \in \mathsf{Prop}$. We define $\Box\varphi := \neg\Diamond\neg\varphi$.

Definition 5.5.2. We extend the semantics of Definition 5.4.2 to $\mathcal{L}(\succsim, \succ, \Diamond)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \Diamond\varphi$ iff there is a $\mu \in \mathcal{P}$ such that $\mu([\![\varphi]\!]^{\mathcal{M},\{\mu\}}) \neq 0$.

Note that with $\Diamond$ added, we no longer need $\succ$ as a primitive in the language, since $\varphi \succ \psi$ is definable as $\neg\Diamond(\psi \succsim \varphi)$, but we keep $\succ$ as a primitive for convenience.

In the following, we first present a sound and complete logic for the valid formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$. Then we briefly comment on the logic's complexity. Finally, we show how $\mathcal{L}(\succsim, \succ, \Diamond)$ is more expressive than $\mathcal{L}(\succsim, \succ)$ and characterize the expressivity of $\mathcal{L}(\succsim, \succ, \Diamond)$.

5.5.1 Logic

An important logical fact about the set of valid formulas of $\mathcal{L}(\succsim, \succ, \Diamond)$ is that it is not closed under *uniform substitution* of arbitrary formulas for propositional variables.

Example 5.5.3. The formula $(p \succ \bot) \to \Diamond(p \succ \bot)$ is valid but

$$(\neg((p \succsim q) \vee (q \succsim p)) \succ \bot) \to \Diamond(\neg((p \succsim q) \vee (q \succsim p)) \succ \bot)$$

is not valid. The reason is that there is no single probability measure that can make true the non-comparability formula $\neg((p \succsim q) \vee (q \succsim p))$.

While the failure of uniform substitution can complicate efforts to axiomatize a set of validities (cf. [92, 93]), we will completely axiomatize the validities of $\mathcal{L}(\succsim, \succ, \Diamond)$ with the logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ defined below.

Definition 5.5.4. The logic $\mathsf{SP}(\succsim, \succ, \Diamond)$ is the smallest subset of $\mathcal{L}(\succsim, \succ, \Diamond)$ that is (i) closed under modus ponens, uniform substitution, and the rule of replacement of provable equivalents, and (ii) contains all theorems of $\mathsf{SP}(\succsim, \succ)$ and $\Diamond p \leftrightarrow (p \succ \bot)$.

The logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ is the smallest subset of $\mathcal{L}(\succsim, \succ, \Diamond)$ that is (i) closed under modus ponens, the rule of replacement of provable equivalents, and the rule that if $\varphi \in \mathsf{SP}(\succsim, \succ, \Diamond)$, then $\Box\varphi \in \mathsf{IP}(\succsim, \succ, \Diamond)$, and (ii) contains all substitution instances in $\mathcal{L}(\succsim, \succ, \Diamond)$ of the theorems in $\mathsf{IP}(\succsim, \succ)$ and also all instances of the following axiom schemas, where α and β are propositional:

(C1) $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$;

(C2) $\Diamond\top$;

(C3) $\Box\varphi \to (\Box\varphi \succsim \top)$;

(C4) $\Diamond\varphi \to (\Diamond\varphi \succsim \top)$;

(C5) $\Box\varphi \leftrightarrow \Box(\varphi \succsim \top)$;

(C6) $(\alpha \succsim \beta) \leftrightarrow \Box(\alpha \succsim \beta)$;

(**C7**) $(\alpha \succ \beta) \leftrightarrow \Box(\alpha \succ \beta)$.

As an example derivation using the above axioms and rules, especially the universalization rule over $\mathsf{SP}(\succsim, \succ, \Diamond)$ by $\Box$, we derive the $\mathsf{4}$ axiom as follows:

- By the axiom $\Diamond p \leftrightarrow (p \succ \bot)$ in $\mathsf{SP}(\succsim, \succ, \Diamond)$, we have $\Diamond\neg\varphi \to (\neg\varphi \succ \bot)$. Recall that in $\mathsf{SP}(\succsim, \succ)$, $\succ$ is definable by $\succsim$. So we have $\Diamond\neg\varphi \to \neg(\bot \succsim \neg\varphi)$. With the cancelation axiom (**A4**), we have $\Diamond\neg\varphi \to \neg(\varphi \succsim \top)$. Contraposing this, we have $(\varphi \succsim \top) \to \Box\varphi$.
- Since we can universalize theorems in $\mathsf{SP}(\succsim, \succ, \Diamond)$, we now have $\Box((\varphi \succsim \top) \to \Box\varphi)$. By axiom (**C5**), $\Box\varphi \to \Box(\varphi \succsim \top)$. By axiom (**C1**), we can connect the two implications and obtain $\Box\varphi \to \Box\Box\varphi$, which is the $\mathsf{4}$ axiom.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.5.5 (Soundness and Completeness)**.** *For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$: φ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond)$ if and only if φ is valid with respect to the class of all imprecise probabilistic models.*

To prove Theorem 5.5.5, we first show that (1) there is no need for a $\Diamond$ to scope over a $\Diamond$ and (2) there is no need for a $\succsim$ or $\succ$ to scope over a $\Diamond$. In other words, we will find a significantly simpler fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$, which we call $\mathcal{L}_{\text{Simp}}$, such that every formula in $\mathcal{L}(\succsim, \succ, \Diamond)$ is provably equivalent to a formula in $\mathcal{L}_{\text{Simp}}$ in $\mathsf{IP}(\succsim, \succ, \Diamond)$.

Definition 5.5.6. Define $T_{-\Diamond} : \mathcal{L}(\succsim, \succ, \Diamond) \to \mathcal{L}(\succsim, \succ)$ by:

- $T_{-\Diamond}(p) = p$ for all $p \in \mathsf{Prop}$;
- $T_{-\Diamond}(\neg\varphi) = \neg T_{-\Diamond}(\varphi)$;
- $T_{-\Diamond}(\varphi \wedge \psi) = T_{-\Diamond}(\varphi) \wedge T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\varphi \succsim \psi) = T_{-\Diamond}(\varphi) \succsim T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\varphi \succ \psi) = T_{-\Diamond}(\varphi) \succ T_{-\Diamond}(\psi)$;
- $T_{-\Diamond}(\Diamond\varphi) = \neg(\bot \succsim T_{-\Diamond}(\varphi))$.

Lemma 5.5.7. *For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $\varphi \leftrightarrow T_{-\Diamond}(\varphi)$ is in $\mathsf{SP}(\succsim, \succ, \Diamond)$. Moreover, $T_{-\Diamond}(\varphi)$ uses the same propositional variables as φ does.*

Proof. A simple induction with repeated use of replacement of equivalents suffices. $\Box$

Lemma 5.5.8. *In $\mathsf{IP}(\succsim, \succ, \Diamond)$, formulas of the form $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ are theorems. In addition, $\Box$ is a normal operator: for any $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $(\Box\varphi \wedge \Box(\varphi \to \psi)) \to \Box\psi$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$, and whenever φ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$, so is $\Box\varphi$.*

Proof. To derive $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, it is enough to derive $\Diamond\varphi \leftrightarrow \Diamond\neg\neg\varphi$. But this is clearly derivable by replacement of equivalents since $\Diamond\varphi \leftrightarrow \Diamond\varphi$ and $\varphi \leftrightarrow \neg\neg\varphi$ are theorems. $\Box$

Definition 5.5.9. Let $\mathcal{L}_{\mathrm{Simp}}$ be the fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$ generated from Prop and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}(\succsim, \succ)\}$ by $\neg$ and $\wedge$.

In the following, for any $\mathsf{p} \subseteq \mathsf{Prop}$, we append $[\mathsf{p}]$ to the name of a language to denote the set of formulas in that language using only variables in p.

Lemma 5.5.10. *For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$, there is a $T(\varphi) \in \mathcal{L}_{Simp}$ such that $\varphi \leftrightarrow T(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. Moreover, $T(\varphi)$ and φ use the same propositional variables.*

Proof. By induction on $\mathcal{L}(\succsim, \succ, \Diamond)$. The base case is trivial: we can simply define $T(p) = p$. The Boolean cases are also trivial: we can define $T(\neg\varphi) = \neg T(\varphi)$ and $T(\varphi\wedge\psi) = T(\varphi)\wedge T(\psi)$. For the $\Diamond$ case, define $T(\Diamond\varphi) = \Diamond T_{-\Diamond}(\varphi)$. To see that $\Diamond\varphi$ is provably equivalent to $\Diamond T_{-\Diamond}(\varphi)$, first note that by Lemma 5.5.7, $\varphi \leftrightarrow T_{-\Diamond}(\varphi) \in \mathsf{SP}(\succsim, \succ, \Diamond)$. But then $\Box(\varphi \leftrightarrow T_{-\Diamond}(\varphi)) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. By the normality of $\Box$, we have $\Diamond\varphi \leftrightarrow \Diamond T_{-\Diamond}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$.

To find the appropriate $T(\varphi \succsim \psi)$, given that the required $T(\varphi)$ and $T(\psi)$ in $\mathcal{L}_{\mathrm{Simp}}$ have been found, we need to extract all $\Diamond$'ed formulas in $T(\varphi) \succsim T(\psi)$ so that they are no longer in the scope of the main connective $\succsim$ in $T(\varphi) \succsim T(\psi)$. Clearly this can be done by iteratively using the following claim:

(*) for any $\chi \in \mathcal{L}(\succsim, \succ)$ and $\varphi, \psi \in \mathcal{L}_{\mathrm{Simp}}$,

$$(\varphi \succsim \psi) \leftrightarrow (\Diamond\chi \wedge (\varphi[\Diamond\chi/\top] \succsim \psi[\Diamond\chi/\top])) \vee (\neg\Diamond\chi \wedge (\varphi[\Diamond\chi/\bot] \succsim \psi[\Diamond\chi/\bot])))$$

is in $\mathsf{IP}(\succsim, \succ, \Diamond)$.

The claim is provable using (**C3**) and (**C4**) and instances (**B5**). Note that since φ, ψ are in $\mathcal{L}_{\mathrm{Simp}}$, they are Boolean combinations of propositional variables and formulas of the form $\Diamond\chi$ where $\chi \in \mathcal{L}(\succsim, \succ)$. List all the $\Diamond$'ed formulas appearing in φ or ψ as $\delta_1, \delta_2, \ldots, \delta_n$. Then for any $f \in \{0,1\}^n$, let δ_f be $\bigwedge_{i=1}^{n} \neg^{f(i)}\delta_i$ where $\neg^0\delta_i$ is $\neg\delta_i$ and $\neg^1\delta_i$ is simply δ_i. Moreover, let $\varphi[f]$ be $\varphi[\delta_1/\top^{f(1)}, \cdots, \delta_n/\top^{f(n)}]$ and similarly for $\psi[f]$, where $\top^{f(i)} = \top$ if $f(i) = 1$ and $\top^{f(i)} = \bot$ if $f(i) = 0$. With this notation, it is not hard to see that by repeatedly applying (*), $\varphi \succsim \psi$ is provably equivalent to $\bigvee_{f\in\{0,1\}^n}(\delta_f \wedge (\varphi[f] \succsim \psi[f]))$ and then also to $\bigvee_{f\in\{0,1\}^n}(\delta_f \wedge \Box(\varphi[f] \succsim \psi[f]))$ since for any f, $\varphi[f]$ and $\psi[f]$ are propositional since we have replaced all the $\Diamond$'ed formulas by either $\top$ or $\bot$ and by axiom (**C6**) we can add a $\Box$ there. The formula $\bigvee_{f\in\{0,1\}^n}(\delta_f \wedge \Box(\varphi[f] \succsim \psi[f]))$ is the desired $T(\varphi \succsim \psi)$ since it is clearly in $\mathcal{L}_{\mathrm{Simp}}$ now. The definition of $T(\varphi \succ \psi)$ is almost identical: we can simply replace $\varphi[f] \succsim \psi[f]$ by $\varphi[f] \succ \psi[f]$. In this case, we use (**C7**) instead. $\Box$

Now we are ready to prove the soundness and completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$. Soundness is clear as usual. For completeness, pick an arbitrary γ that is consistent relative to $\mathsf{IP}(\succsim, \succ, \Diamond)$, and let p be the set of propositional variables used in γ. Then take an arbitrary Γ that is maximally consistent containing γ. Following the standard strategy, let $\Sigma = \{(\varphi \gtrsim \top) \mid \Box\varphi \in \Gamma, \varphi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]\}$. Note that $\Sigma \subseteq \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]$. Also, Σ must be consistent relative to $\mathsf{SP}(\gtrsim, \succ)$ since otherwise there are formulas $(\varphi_1 \gtrsim \top), (\varphi_2 \gtrsim \top), \dots, (\varphi_n \gtrsim \top)$ in Σ such that $((\varphi_1 \gtrsim \top) \wedge \cdots \wedge (\varphi_n \gtrsim \top)) \to \bot$ is in $\mathsf{SP}(\gtrsim, \succ)$. But then by the rules of $\mathsf{IP}(\gtrsim, \succ, \Diamond)$ and the normality of $\Box$, we have that $(\Box(\varphi_1 \gtrsim \top) \wedge \cdots \wedge \Box(\varphi_n \gtrsim \top)) \to \Box\bot$ is in $\mathsf{IP}(\gtrsim, \succ, \Diamond)$. Since $\Box\varphi$ is provably equivalent to $\Box(\varphi \gtrsim \top)$ by (**C5**), we have that $\Box\bot$ is in Γ according to the maximality of Γ, rendering Γ inconsistent since we have (**C2**).

Now let $\mathcal{D} = \{\Sigma \cup \{\neg(\varphi \gtrsim \top)\} \mid \neg\Box\varphi \in \Gamma, \varphi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]\}$. Note that for each $\Delta = \Sigma \cup \{\neg(\varphi \gtrsim \top)\} \in \mathcal{D}$, Δ is also a set of formulas in $\mathcal{L}(\gtrsim, \succ)[\mathsf{p}]$. Moreover, Δ must be consistent relative to $\mathsf{SP}(\gtrsim, \succ)$ as well. If not, then since Σ is consistent, we must have formulas $(\varphi_1 \gtrsim \top), \dots, (\varphi_n \gtrsim \top)$ in Σ such that $((\varphi_1 \gtrsim \top) \wedge \cdots \wedge (\varphi_n \gtrsim \top)) \to (\varphi \gtrsim \top) \in \mathsf{SP}(\gtrsim, \succ)$. Then by reasoning similar to that above, $\Box(\neg\varphi \gtrsim \top)$ and hence $\Box\neg\varphi$ are in Γ using (**C5**), rendering Γ inconsistent.

Thus, for each $\Delta \in \mathcal{D}$, according to Proposition 5.4.12, there is a probability measure μ_Δ on $\wp(W_\mathsf{p})$ and a $w \in W_\mathsf{p}$ such that $\mathcal{M}_\mathsf{p}, \{\mu_\Delta\}, w \vDash \Delta$. Note that since all formulas in Δ are comparison formulas of the form $\varphi \gtrsim \top$ or its negation, it does not matter what w is. Hence we have that $\mathcal{M}_\mathsf{p}, \{\mu_\Delta\} \vDash \Delta$. Take $\mathcal{P}$ to be the set $\{\mu_\Delta \mid \Delta \in \mathcal{D}\}$. Then we are left only to show that there is a $w \in W_\mathsf{p}$ such that $\mathcal{M}_\mathsf{p}, \mathcal{P}, w \vDash \varphi$ for all $\varphi \in \Gamma \cap \mathcal{L}(\gtrsim, \succ, \Diamond)[\mathsf{p}]$.

Let w_0 be the element in $W_\mathsf{p} = \{0,1\}^\mathsf{p}$ defined by $w_0(p) = 1$ iff $p \in \Gamma$ for all $p \in \mathsf{p}$. Then we are ready to show the following truth lemma.

Lemma 5.5.11. *For all $\varphi \in \mathcal{L}(\gtrsim, \succ, \Diamond)[\mathsf{p}]$, $\mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash \varphi$ iff $\varphi \in \Gamma$.*

Proof. It is enough to show that for all $\varphi \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$, $\mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash \varphi$ iff $\varphi \in \Gamma$. This is because for any $\varphi \in \mathcal{L}(\gtrsim, \succ, \Diamond)[\mathsf{p}]$, according to Lemma 5.5.10, $\varphi \in \Gamma$ iff $T(\varphi) \in \Gamma$ with $T(\varphi) \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$. But then

$$T(\varphi) \in \Gamma \iff \mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash T(\varphi) \iff \mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash \varphi.$$

The first equivalence holds by the fact that $T(\varphi) \in \mathcal{L}_{\text{Simp}}[\mathsf{p}]$ and the truth lemma we will show below in this fragment. The second is by soundness.

We now focus on the fragment $\mathcal{L}_{\text{Simp}}[\mathsf{p}]$. Since the generating operations of this fragment are Boolean, the inductive cases are trivial. The atomic case for propositional variables in p is also trivial by the definition of w_0. Hence we are left to show that for any $\varphi \in \{\Diamond\psi \mid \psi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]\}$, we have $\varphi \in \Gamma$ iff $\mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash \varphi$. In other words, we only need to show that for all $\varphi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]$, we have $\Diamond\varphi \in \Gamma$ iff $\mathcal{M}_\mathsf{p}, \mathcal{P}, w_0 \vDash \Diamond\varphi$.

- Suppose $\Diamond\varphi \notin \Gamma$, so $\Box\neg\varphi \in \Gamma$. Then $(\neg\varphi \gtrsim \top) \in \Sigma$ since $\neg\varphi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]$, which means $(\neg\varphi \gtrsim \top) \in \Delta$ for all $\Delta \in \mathcal{D}$. Then, for any $\mu_\Delta \in \mathcal{P}$, $\mathcal{M}_\mathsf{p}, \{\mu_\Delta\} \vDash \neg\varphi \gtrsim \top$ since $(\neg\varphi \gtrsim \top) \in \Delta$, which in turn means that $\mu_\Delta([\![\varphi]\!]^{\mathcal{M}_\mathsf{p}, \{\mu_\Delta\}}) = 0$. This is precisely the condition for $\Diamond\varphi$ to be false at $\mathcal{M}_\mathsf{p}, \mathcal{P}, w_0$.

- Suppose $\Diamond\varphi \in \Gamma$, so $\neg\Box\neg\varphi \in \Gamma$. Then there is a Δ such that $\neg(\neg\varphi \gtrsim \top) \in \Delta$ again because $\neg\varphi \in \mathcal{L}(\gtrsim, \succ)[\mathsf{p}]$. For this μ_Δ then, $\mathcal{M}_\mathsf{p}, \{\mu_\Delta\} \nvDash \neg\varphi \gtrsim \top$. In other words, $\mu_\Delta([\![\varphi]\!]^{\mathcal{M}_{\mathsf{p}},\{\mu_\Delta\}}) \neq 0$. The existence of this $\mu_\Delta \in \mathcal{P}$ shows that $\Diamond\varphi$ is true at $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0$. $\square$

Given the above truth lemma, $\mathcal{M}_{\mathsf{p}}, \mathcal{P}, w_0 \vDash \gamma$ since $\gamma \in \Gamma$ and $\gamma \in \mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$. Hence we have successfully found a model for the arbitrarily chosen consistent γ, completing the proof of the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$.

5.5.2 Complexity

In this section, we briefly comment on the complexity of the consistency problem of the logic $\mathsf{IP}(\succsim, \succ, \Diamond)$ or equivalently the satisfiability problem of $\mathcal{L}(\succsim, \succ, \Diamond)$. First, adapting the proof of Theorem 9 in [85], it is not hard to see that the satisfiability problem for a conjunction of literals where we take formulas in both Prop and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}(\succsim, \succ)\}$ as atomic formulas is in NP (note that Theorem 2.6 in [51], used in the proof of [85], allows strict inequalities). Hence the satisfiability problem for $\mathcal{L}_{\mathrm{Simp}}$ is also in NP. Then to see that the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond)$ is in NP, it is enough to show that every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$ is equivalent to a disjunction of formulas in $\mathcal{L}_{\mathrm{Simp}}$ where each disjunct's length is bounded by $O(|\varphi|)$. In our proof of Lemma 5.5.10 above, this is done by extracting $\Diamond$ from the scope of $\succsim$ and $\succ$ and eliminating $\Diamond$ in the scope of $\Diamond$. Note that the elimination of $\Diamond$ in the scope of $\Diamond$ can be done before the extraction: given an input formula φ, replace each subformula $\Diamond\chi$ not in the scope of any $\Diamond$ by $\Diamond T_{-\Diamond}(\chi)$. The resulting formula, which we call φ', is clearly at most four times longer than φ. Then we only need to run the process of (1) extracting $\Diamond$'ed formulas in the scope of $\succsim$ or $\succ$ and (2) adding a $\Box$ to a $\succsim$ formula or a $\succ$ formula when both arguments to the $\succsim$ or $\succ$ no longer contain modal operators. This process, while introducing disjunctions exponentially, only grows the length of the disjuncts by at most a constant for each extracting operation. The number of total extracting operations is clearly at most the length of the input formula φ'. Thus, we obtain the following.

Theorem 5.5.12. *The complexity of the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond)$ is NP-complete.*

5.5.3 Expressivity

Reflecting the failure of uniform substitution, for any purely propositional formula α, $\Diamond\alpha$ is already expressible in $\mathcal{L}(\succsim)$.

Lemma 5.5.13. *Let α, β be propositional formulas. Then:*

1. *$\Diamond\alpha$ is equivalent to $\neg(\bot \succsim \alpha)$;*
2. *$\Diamond(\alpha \succsim \beta)$ and $\Diamond\neg(\beta \succ \alpha)$ are both equivalent to $\neg(\beta \succ \alpha)$;*
3. *$\Diamond(\beta \succ \alpha)$ and $\Diamond\neg(\alpha \succsim \beta)$ are both equivalent to $\neg(\alpha \succsim \beta)$.*

However, $\Diamond\varphi$ is not in general expressible without $\Diamond$.

Example 5.5.14. The formula $\Diamond(p \approx \neg p)$ is not equivalent to any formula of $\mathcal{L}(\succsim, \succ)$. Consider again the propositional model $\mathcal{M} = \langle W, V\rangle$ where $W = \{w, v\}$ and $V(p) = \{w\}$ while $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Then let $\mathcal{P}$ be the set of all probability measures on $\wp(W)$ and $\mathcal{P}'$ the set of all probability measure μ on $\wp(W)$ except the ones that give equal probability to $\{w\}$ and $\{v\}$. Then $\succsim_{\mathcal{P}}$ and $\succsim_{\mathcal{P}'}$ (resp. $\succ_{\mathcal{P}}$ and $\succ_{\mathcal{P}'}$) are the same on $\wp(W)$ and are pictured below:

$$\begin{tikzcd}
 & W & \\
\{w\} \arrow[ur, dashed] & & \{v\} \arrow[ul, dashed] \\
 & \varnothing \arrow[ul, dashed] \arrow[ur, dashed] &
\end{tikzcd}$$

Thus, using Propositions 5.4.15 and 5.4.14, for any $\varphi \in \mathcal{L}(\succsim, \succ)$, $\mathcal{M}, \mathcal{P}, w \vDash \varphi$ iff $\mathcal{M}, \mathcal{P}', w \vDash \varphi$. Yet $\mathcal{M}, \mathcal{P}, w \vDash \Diamond(p \approx \neg p)$ while $\mathcal{M}, \mathcal{P}', w \nvDash \Diamond(p \approx \neg p)$.

Now we characterize the expressivity of $\mathcal{L}(\succsim, \succ, \Diamond)$ precisely.

Proposition 5.5.15. *Let $\langle W, V, \mathcal{P}\rangle$ and $\langle W', V', \mathcal{P}'\rangle$ be IP models and w, w' worlds in W and W', respectively. Let p be a subset of Prop. Then w and w' satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$ using only propositional variables in p if*

1. *w and w' satisfy the same propositional variables in p,*
2. *for any $\mu \in \mathcal{P}$, there is $\mu' \in \mathcal{P}'$ such that $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are order-similar in p, and*
3. *for any $\mu' \in \mathcal{P}'$, there is $\mu \in \mathcal{P}$ such that $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are order-similar in p.*

The converse also holds if in addition p is finite.

Proof. The left-to-right direction is again easy. For the only non-obvious case, suppose for example that the second clause fails: there is a $\mu \in \mathcal{P}$ such that for any $\mu' \in \mathcal{P}'$, $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are not order-similar in p. Then let $\{\alpha_i\}_{1\leq i\leq n}$ be a finite set of Boolean formulas such that every Boolean formula using only letters in p is logically equivalent to some α_i (such a set can be found using disjunctive normal forms). We can now describe μ in full relative to p by the conjunction $\chi = \bigwedge_{1\leq i,j\leq n} s_i(\alpha_i \succsim \alpha_j)$ where s_i is empty if $\mu([\![\alpha_i]\!]^{\langle W,V\rangle}) \succsim \mu([\![\alpha_j]\!]^{\langle W,V\rangle})$ and is $\neg$ otherwise. Indeed, by the definition of order-similarity, whenever $\langle W, V, \{\mu\}\rangle$ and $\langle W', V', \{\mu'\}\rangle$ are not order-similar in p, at any world in W', χ is false. This means that w' would falsify $\Diamond\mu$, but w satisfies $\Diamond\mu$, showing that the two worlds disagree on a formula in $\mathcal{L}(\succsim, \succ, \Diamond)$.

The right-to-left direction follows from the normal form lemma, Lemma 5.5.10. If the last two clauses hold, then for any formula of the form $\Diamond\varphi$ where $\varphi \in \mathcal{L}(\succsim, \succ)[\mathsf{p}]$, $\Diamond\varphi$ is true at $\mathcal{M}, \mathcal{P}, w$ iff it is true at $\mathcal{M}, \mathcal{P}', w'$. By the first clause, the two pointed IP models also satisfy the same propositional variables in p. Then by a simple induction, they satisfy the same formulas in $\mathcal{L}_{\mathrm{Simp}}[\mathsf{p}]$. But by Lemma 5.5.10, this is enough for them to satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$. □

The special case where the two IP models share the same propositional model is again worth spelling out.

Proposition 5.5.16. *Let $\mathcal{M} = \langle W, V \rangle$ be a propositional model, w and w' two worlds in W, and $\mathcal{P}$ and $\mathcal{P}'$ nonempty sets of probability measures defined on fields of sets extending $V[\mathsf{Prop}]$. Let p be a subset of Prop and F the field of sets on W generated by $V[\mathsf{p}]$. Then $\mathcal{M}, \mathcal{P}, w$ and $\mathcal{M}, \mathcal{P}', w'$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond)[\mathsf{p}]$ if*

- *w and w' satisfy the same propositional variables in p,*
- *for any $\mu \in \mathcal{P}$, there is $\mu' \in \mathcal{P}'$ such that $\succsim_\mu|_F = \succsim_{\mu'}|_F$, and*
- *for any $\mu' \in \mathcal{P}'$, there is $\mu \in \mathcal{P}$ such that $\succsim_\mu|_F = \succsim_{\mu'}|_F$.*

The converse also holds if in addition p is finite.

5.6 Dynamics

In this section, we consider two kinds of information dynamics in the context of imprecise probability. The first is a standard notion of *updating* a set of probability measure on new evidence (see, e.g., [80, p. 81]) where we can eliminate both possible worlds (keeping only the worlds compatible with the evidence) and probability measures (keeping only the probability measures that give the evidence a positive probability measure). Usually, especially in a Bayesian framework, such updates are all we need for information dynamics, since we can always model agents with a universal and all-inclusive state space, anticipating all distinctions that could be made among states. However, there are numerous examples where an agent is not initially aware of a distinction. In Example 5.1.1, the agent is not initially aware of the gland and hence the distinction between a swollen and normal gland. When the doctor tells the agent about the gland, we can model the agent as first learning the mere existence of a new proposition—the swollen gland proposition—and then learning how this proposition relates probabilistically to her having the disease. Without imprecise probability, we face the perennial question of how to assign a probability for such a new proposition. Given imprecise probability, however, we can simply choose the set of all probability measures that are compatible with one of the old probability measures. This models how an agent can "initialize" her uncertainty toward a newly introduced proposition.

In the next two subsections, we discuss the two dynamic operators in more detail. For the update operator, we show how it does not add expressivity to the language $\mathcal{L}(\succsim, \succ, \Diamond)$, and we present a sound and complete logic following the standard "reduction axiom" strategy in dynamic epistemic logic. For the operators modeling the introduction of new propositions, however, we show that they significantly increase expressivity, and we leave the axiomatization of the valid formulas as an open question.

5.6.1 Updating Probabilities and the Logic $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$

In this subsection, we introduce the update operator $\langle\,\rangle$ that models learning the truth of a proposition. Given an initial set $\mathcal{P}$ of probability measures, after learning some proposition $U \subseteq W$ with certainty, we update the set $\mathcal{P}$ to the set

$$\mathcal{P}_U = \{\mu(\cdot \mid U) : \mu \in \mathcal{P}, \mu(U) > 0\},$$

where $\mu(\cdot \mid U)$ is defined by conditionalization as usual: for any $V \subseteq W$, $\mu(V \mid U) = \frac{\mu(V \cap U)}{\mu(U)}$.

Since we have a formal language with comparative probability operators, we can model updating on sentences containing not only factual formulas but also comparative probability formulas (cf. [156, 158, 132]), as in "it is raining, and it is more likely that there will be hail than it is that there will be lightning" $(r \wedge (h \succ \ell))$. Intuitively, if Ann tells Bob that "hail is more likely than lightning," she is not telling Bob something about *his own* epistemic state (which he already knows, in the models of this chapter) but is rather recommending that he update his epistemic state to one according to which hail is more likely than lightning—which he can do by discarding from his set of measures any measure according to which hail is not more likely than lightning.[5] Our semantics below, developed in the style of *dynamic epistemic logic* (see, e.g., [43, 13]), will allow such updates in response to comparative probability claims.

Definition 5.6.1. The language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi \mid \langle\varphi\rangle\varphi$$

where $p \in \mathsf{Prop}$. We read $\langle\alpha\rangle\varphi$ as "(update with α is possible and) after update with α, φ is the case." As usual, $[\alpha]\varphi$ abbreviates $\neg\langle\alpha\rangle\neg\varphi$.

Definition 5.6.2. We extend the semantics of Definition 5.5.2 to $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ as follows:

- $\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle\psi$ iff there is a $\mu \in \mathcal{P}$ such that $\mu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\mu\}}) \neq 0$ and $\mathcal{M}, \mathcal{P}_\varphi, w \vDash \psi$,

where

$$\mathcal{P}_\varphi = \{\nu(\cdot \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) : \nu \in \mathcal{P} \text{ and } \nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \neq 0\}.$$

Lemma 5.6.3. *The semantics for $[\varphi]\psi$ is as follows:*

- $\mathcal{M}, \mathcal{P}, w \vDash [\varphi]\psi$ *iff if there is a $\mu \in \mathcal{P}$ such that $\mu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\mu\}}) \neq 0$, then $\mathcal{M}, \mathcal{P}_\varphi, w \vDash \psi$.*

[5] Another possible interpretation is that there is some objectively correct probability measure, and Ann is telling Bob a fact about that measure, which he wants his probabilities to ultimately match.

The following lemma states how updating with a formula $\varphi \succsim \psi$, if possible, results in restricting one's set of measures to just those that individually satisfy $\varphi \succsim \psi$.

Lemma 5.6.4. *For any IP model $\langle \mathcal{M}, \mathcal{P} \rangle$ and $\varphi, \psi \in \mathcal{L}(\succsim, \succ, \Diamond)$, $\mathcal{P}_{\varphi \succsim \psi} = \varnothing$ or*

$$\mathcal{P}_{\varphi \succsim \psi} = \{\nu \in \mathcal{P} : \mathcal{M}, \{\nu\} \vDash \varphi \succsim \psi\}.$$

Let us see how this framework can be used to formalize the three prisoners scenario from Example 5.1.2.

Example 5.6.5. Let e_i and s_i stand for 'prisoner i will be *executed*' and 'the jailer *says* that prisoner i will be executed', respectively. Define a propositional model $\mathcal{M} = \langle W, V \rangle$ with

$$W = \{w_{ab}, w_{ac}, w_{bc}, w_{cb}\}$$

where at w_{ij}, prisoner i is the only prisoner who lives and prisoner j is the prisoner who the jailer says will be executed, so

$$V(e_a) = \{w_{bc}, w_{cb}\}, V(e_b) = \{w_{ab}, w_{ac}, w_{cb}\}, V(e_c) = \{w_{ab}, w_{ac}, w_{bc}\},$$

$$V(s_b) = \{w_{ab}, w_{cb}\}, V(s_c) = \{w_{ac}, w_{bc}\}.$$

Since prisoner a knows that each prisoner is equally likely to be executed but has no idea about how the jailer is likely to answer his question about which of b or c will be executed (except that the jailer is certain to give a true answer), prisoner a's epistemic state may be modeled by the following set of probability measures:

$$\mathcal{P} = \{\mu : \mu(\{w_{ab}, w_{ac}\}) = \mu(\{w_{bc}\}) = \mu(\{w_{cb}\}) = 1/3\}.$$

Then the following formulas together capture what is distinctive about the puzzle, all coming out true in this model. First, we can state that each prisoner is equally likely to be spared—indeed that each has one-third chance:

$$\alpha := \big(\bot \succsim (e_a \wedge e_b \wedge e_c)\big) \wedge \Big(\big((e_a \wedge e_b) \vee (e_a \wedge e_c) \vee (e_b \wedge e_c)\big) \succsim \top\Big) \wedge (e_a \approx e_b) \wedge (e_b \approx e_c).$$

Second, we can state that the jailer only announces truthfully one of s_b and s_c:

$$\beta := ((s_b \to e_b) \succsim \top) \wedge ((s_c \to e_c) \succsim \top) \wedge (\bot \succsim (s_b \wedge s_c)).$$

Given the dynamic operator, we can also express a fact about how a's uncertainty is affected upon learning that b is to be executed. After this announcement, a's credences *dilate* from a sharp two-thirds probability to including the possibilities that he is sure to be executed and that he has merely one-half probability of being executed:

$$\langle s_b \rangle \big(\Diamond(e_a \succsim \top) \wedge \Diamond(e_a \approx \neg e_a)\big).$$

If, however, a first updates with the information that the jailer is following a protocol of reporting b or reporting c with equal probability in the case that a is to be spared, then dilation no longer occurs. In fact, the probability of e_a remains at two-thirds, and for instance the following formula is true:

$$\langle(\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c)\rangle\langle s_b\rangle\big((e_a \succ e_c) \wedge (e_a \succ \neg e_a) \wedge (\top \succ e_a)\big).$$

Finally, were a to update with the information that the jailer would certainly announce e_b in case e_a were false, then the probabilities of e_a, e_b, and e_c would all remain equally likely:

$$\langle\bot \gtrsim (\neg e_a \wedge s_c)\rangle\alpha.$$

But after learning that b will be executed, the probability of e_a decreases to one-half:

$$\langle\bot \gtrsim (\neg e_a \wedge s_c)\rangle\langle s_b\rangle(e_a \approx \neg e_a).$$

It is important to note that we do not have to resort to the particular model above to model the prisoner case. Indeed, the following formulas are true at any pointed IP model and hence also provable in the complete logic to be presented:

$$(\alpha \wedge \beta) \to [(\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c)]\langle s_b\rangle\big((e_a \succ e_c) \wedge (e_a \succ \neg e_a) \wedge (\top \succ e_a)\big) \tag{5.2}$$

$$(\alpha \wedge \beta) \to [\bot \gtrsim (\neg e_a \wedge s_c)](\alpha \wedge \langle s_b\rangle(e_a \approx \neg e_a)) \tag{5.3}$$

$$\begin{aligned}&(\alpha \wedge \beta) \to \\ &\quad ((\Diamond(\bot \gtrsim (\neg e_a \wedge s_b)) \wedge \Diamond(\bot \gtrsim (\neg e_a \wedge s_c))) \to \langle s_b\rangle(\Diamond(e_a \gtrsim \top) \wedge \Diamond(e_a \approx \neg e_a)))\,.\end{aligned} \tag{5.4}$$

In (5.2) and (5.3), we have to use $[\,]$ instead of $\langle\,\rangle$ since there are models that satisfy $\alpha \wedge \beta$ but do not contain probability measures satisfying either $(\neg e_a \wedge s_b) \approx (\neg e_a \wedge s_c)$ or $\bot \gtrsim (\neg e_a \wedge s_c)$, unlike the particular model above using the all-inclusive $\mathcal{P}$. To cope with this, we need to use the box version of the update operator. In formula (5.4), the extra premise $\Diamond(\bot \gtrsim (\neg e_a \wedge s_b)) \wedge \Diamond(\bot \gtrsim (\neg e_a \wedge s_c))$ is again required since dilation crucially relies on $\mathcal{P}$ containing both a measure assigning 0 to $\neg e_a \wedge s_b$ and a measure assigning 0 to $\neg e_a \wedge s_c$. In our current language, using the $\Diamond$ operator is the most straightforward way to express this. An equivalent way is to use $\neg((\neg e_a \wedge s_b) \succ \bot) \wedge \neg((\neg e_a \wedge s_b) \succ \bot)$. However, the $\Diamond$ in $\Diamond(e_a \approx \neg e_a)$ is necessary: there is no formula in $\mathcal{L}(\gtrsim, \succ)$ that is equivalent to $\Diamond(e_a \approx \neg e_a)$.

To obtain a complete logic for reasoning about updating sets of probability measures, we follow the standard "reduction axiom" strategy used in dynamic epistemic logic: identify a set of valid biconditionals that allow us to reduce any formula containing the dynamic operators $\langle\varphi\rangle$ to an equivalent formula of $\mathcal{L}(\gtrsim, \succ, \Diamond)$ without dynamic operators, which can then be handled by the complete logic for $\mathcal{L}(\gtrsim, \succ, \Diamond)$.

Definition 5.6.6. The logic $\mathsf{IP}(\gtrsim, \succ, \Diamond, \langle\,\rangle)$ is the smallest set of $\mathcal{L}(\gtrsim, \succ, \Diamond, \langle\,\rangle)$ formulas that is (i) closed under modus ponens and the rule of replacement of equivalents, and (ii) contains all theorems of $\mathsf{IP}(\gtrsim, \succ, \Diamond)$ as well as all instances of the following axiom schemas where $p \in \mathsf{Prop}$ and α and β are propositional:

(**R0**) $\langle\varphi\rangle p \leftrightarrow (\Diamond\varphi \wedge p)$;

(**R1**) $\langle\varphi\rangle\Diamond\psi \leftrightarrow \Diamond\langle\varphi\rangle\psi$;

(**R2**) $\langle\varphi\rangle\neg\psi \leftrightarrow (\Diamond\varphi \wedge \neg\langle\varphi\rangle\psi)$;

(**R3**) $\langle\varphi\rangle(\psi \wedge \chi) \leftrightarrow (\langle\varphi\rangle\psi \wedge \langle\varphi\rangle\chi)$;

(**R4**) $\langle\varphi\rangle(\alpha \succsim \beta) \leftrightarrow (\Diamond\varphi \wedge \Box((\varphi \wedge \alpha) \succsim (\varphi \wedge \beta)))$;

(**R5**) $\langle\varphi\rangle(\alpha \succ \beta) \leftrightarrow (\Diamond\varphi \wedge \Box((\varphi \succ \bot) \rightarrow ((\varphi \wedge \alpha) \succ (\varphi \wedge \beta))))$.

Example 5.6.7. In a given model, we may ask if *after the agent updates with the information that it is raining and that hail is more likely than lightning tonight* the agent judges that it is at least as likely that a window will break as it is that the power will go out:

$$\langle r \wedge (h \succ l)\rangle(w \succsim p).$$

This is equivalent, in light of the reduction axiom (**R4**), to

$$\Diamond(r \wedge (h \succ l)) \wedge \Box\big(\big((r \wedge (h \succ l)) \wedge w\big) \succsim \big((r \wedge (h \succ l)) \wedge p\big)\big),$$

which is in turn equivalent to

$$\Diamond(r \wedge (h \succ l)) \wedge \Box\big((h \succ l) \rightarrow \big((r \wedge w) \succsim (r \wedge p)\big)\big),$$

i.e., there is some measure that gives r non-zero probability and gives h greater probability than l, and every measure that gives h greater probability than l also makes the probability of w conditional on r at least as great as the probability of p conditional on r.

The rest of this section is devoted to the proof of the following theorem.

Theorem 5.6.8 (Soundness and Completeness)**.** *For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: φ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ if and only if φ is valid with respect to the class of all imprecise probabilistic models.*

The soundness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is less trivial than the soundness of the previous systems. More importantly, we will use its soundness to prove its completeness, similar to the proof of completeness of other dynamic epistemic logics axiomatized by reduction axioms.

Proposition 5.6.9. *For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: if φ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, then φ is valid with respect to the class of all imprecise probabilistic models.*

Proof. Clearly it is enough to check the validity of (**R0**) to (**R5**).

- For (**R0**), note that the valuation of p is invariant under the updating.

- For (**R1**), the key is to treat $\langle\varphi\rangle\Diamond$ as a whole, whence the semantics of $\langle\varphi\rangle\Diamond\psi$ at $\mathcal{M}, \mathcal{P}, w$ is that there is a $\mu \in \mathcal{P}_\varphi$ such that $\mu(\llbracket\psi\rrbracket^{\mathcal{M},\{\mu\}}) > 0$. But given the construction of $\mathcal{P}_\varphi$, this is precisely saying that there is a $\nu \in \mathcal{P}$ such that $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0$ and that, letting $\mu = \nu(\cdot \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}})$, we have $\mu(\llbracket\psi\rrbracket^{\mathcal{M},\{\mu\}}) > 0$. Now note that for any $\nu \in \mathcal{P}$ such that $\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}} > 0$, letting $\mu = \nu(\cdot \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}})$, we have $\llbracket\langle\varphi\rangle\psi\rrbracket^{\mathcal{M},\{\nu\}} = \llbracket\psi\rrbracket^{\mathcal{M},\{\mu\}}$ since $\{\nu\}_\varphi = \{\mu\}$. Hence the truth condition of $\langle\varphi\rangle\Diamond\psi$ is transformed into the existence of $\nu \in \mathcal{P}$ such that $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0$ and that $\llbracket\langle\varphi\rangle\psi\rrbracket^{\mathcal{M},\{\nu\}} > 0$. But this is precisely the truth condition of $\Diamond\langle\varphi\rangle\psi$.

- For (**R2**), the key insight is that at $\mathcal{M}, \mathcal{P}, w$, assuming that there is a $\nu \in \mathcal{P}$ such that $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0$, we have:

$$\begin{aligned}
\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle\neg\psi &\iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \neg\psi \\
&\iff \mathcal{M}, \mathcal{P}_\varphi, w \nvDash \psi \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \neg\langle\varphi\rangle\psi.
\end{aligned}$$

- For (**R3**), the idea is similar to the above.

- For (**R4**), it is enough to observe the following chain of equivalences assuming that there is a $\nu \in \mathcal{P}$ such that $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0$:

$$\begin{aligned}
\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle(\alpha \succsim \beta) &\iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \alpha \succsim \beta \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu(\llbracket\alpha\rrbracket^{\mathcal{M},\mathcal{P}_\varphi}) \geq \mu(\llbracket\beta\rrbracket^{\mathcal{M},\mathcal{P}_\varphi}) \\
&\iff \forall\mu \in \mathcal{P}_\varphi, \mu(V(\alpha)) \geq \mu(V(\beta)) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \mid \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P} \text{ such that } \nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P}, \nu(V(\alpha) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \\
&\iff \forall\nu \in \mathcal{P}, \mathcal{M}, \{\nu\} \vDash (\varphi \wedge \alpha) \succsim (\varphi \wedge \beta) \\
&\iff \forall\nu \in \mathcal{P}, \nu(\llbracket(\varphi \wedge \alpha) \succsim (\varphi \wedge \beta)\rrbracket^{\mathcal{M},\{\nu\}}) = 1 \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \Box((\varphi \wedge \alpha) \succsim (\varphi \wedge \beta)).
\end{aligned}$$

  Note that the last three equivalences extensively use the fact that a Boolean combination of comparison formulas is true at a world if and only if it is true at all worlds. The sixth equivalence is true because when $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) = 0$, it trivially holds that $\nu(V(\alpha) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) \geq \nu(V(\beta) \cap \llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}})$.

- For (**R5**), the strategy is the same—it is enough to observe the following chain of equivalences assuming that there is a $\nu \in \mathcal{P}$ such that $\nu(\llbracket\varphi\rrbracket^{\mathcal{M},\{\nu\}}) > 0$:

$$\mathcal{M}, \mathcal{P}, w \vDash \langle\varphi\rangle(\alpha \succ \beta) \iff \mathcal{M}, \mathcal{P}_\varphi, w \vDash \alpha \succ \beta$$

$$
\begin{aligned}
&\iff \forall \mu \in \mathcal{P}_\varphi, \mu(\llbracket \alpha \rrbracket^{\mathcal{M},\mathcal{P}_\varphi}) > \mu(\llbracket \beta \rrbracket^{\mathcal{M},\mathcal{P}_\varphi}) \\
&\iff \forall \mu \in \mathcal{P}_\varphi, \mu(V(\alpha)) > \mu(V(\beta)) \\
&\iff \forall \nu \in \mathcal{P} \text{ such that } \nu(\llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \mid \llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) > \nu(V(\beta) \mid \llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) \\
&\iff \forall \nu \in \mathcal{P} \text{ such that } \nu(\llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) > 0, \\
&\qquad \nu(V(\alpha) \cap \llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) > \nu(V(\beta) \cap \llbracket \varphi \rrbracket^{\mathcal{M},\{\nu\}}) \\
&\iff \forall \nu \in \mathcal{P}, \text{ if } \mathcal{M}, \{\nu\} \vDash \varphi \succ \bot \text{ then} \\
&\qquad \mathcal{M}, \{\nu\} \vDash (\varphi \wedge \alpha) \succ (\varphi \wedge \beta) \\
&\iff \forall \nu \in \mathcal{P}, \mathcal{M}, \{\nu\} \vDash (\varphi \succ \bot) \to ((\varphi \wedge \alpha) \succ (\varphi \wedge \beta)) \\
&\iff \forall \nu \in \mathcal{P}, \nu(\llbracket (\varphi \succ \bot) \to ((\varphi \wedge \alpha) \succ (\varphi \wedge \beta)) \rrbracket^{\mathcal{M},\{\nu\}}) = 1 \\
&\iff \mathcal{M}, \mathcal{P}, w \vDash \Box((\varphi \succ \bot) \to ((\varphi \wedge \alpha) \succ (\varphi \wedge \beta))).
\end{aligned}
$$

Again, the last four equivalences extensively use the fact that a Boolean combination of comparison formulas is true at a world if and only if it is true at all worlds. □

For completeness, we first show that the axioms allow us to provably-equivalently reduce any formula in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ to a fragment $\mathcal{L}_{\text{Simpd1}}$ that is even simpler than the fragment $\mathcal{L}_{\text{Simp}}$: the comparison formulas in the scope of any $\Diamond$ must not have nested comparison.

Definition 5.6.10. Let $\mathcal{L}_{\text{Bool}}$ be the set of propositional formulas. In other words, this is the fragment generated from Prop by $\neg$ and $\wedge$.

Let $\mathcal{L}_{\text{Compd1}}$ be the fragment of $\mathcal{L}(\succsim, \succ)$ with no nesting of $\succsim$ and $\succ$. In other words, this is the fragment generated from Prop and $\{(\alpha \succsim \beta), (\alpha \succ \beta) \mid \alpha, \beta \in \mathcal{L}_{\text{Bool}}\}$ by $\neg$ and $\wedge$.

Finally, let $\mathcal{L}_{\text{Simpd1}}$ be the fragment of $\mathcal{L}(\succsim, \succ, \Diamond)$ generated from Prop and $\{\Diamond\varphi \mid \varphi \in \mathcal{L}_{\text{Compd1}}\}$ by $\neg$ and $\wedge$.

Lemma 5.6.11. *For every $\varphi \in \mathcal{L}(\succsim, \succ)$, there is a $T_{Compd1}(\varphi) \in \mathcal{L}_{Compd1}$ such that $\varphi \leftrightarrow T_{Compd1}(\varphi) \in \mathsf{IP}(\succsim, \succ)$. Moreover, φ and $T_{Compd1}(\varphi)$ use the same propositional variables.*

Proof. We use a standard argument for extracting comparisons embedded in comparisons. Formally, an induction over $\mathcal{L}(\succsim, \succ)$ is needed. The base case and the inductive cases for $\neg$ and $\wedge$ are trivial as we can simply define $T_{\text{Compd1}}(p) = p$, $T_{\text{Compd1}}(\neg\varphi) = \neg T_{\text{Compd1}}(\varphi)$, and $T_{\text{Compd1}}(\varphi \wedge \psi) = T_{\text{Compd1}}(\varphi) \wedge T_{\text{Compd1}}(\psi)$.

For the non-trivial cases for $\succsim$ and $\succ$, we only need the following theorems in $\mathsf{IP}(\succsim, \succ)$

for any $\alpha, \beta \in \mathcal{L}_{\text{Bool}}$ and $\varphi, \psi \in \mathcal{L}_{\text{Compd1}}$:

$$
\begin{aligned}
&(\varphi \succsim \psi) \leftrightarrow \\
&\quad (((\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succsim \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succsim \psi[\alpha \succsim \beta/\bot])));\\
&(\varphi \succsim \psi) \leftrightarrow \\
&\quad (((\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succsim \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succsim \psi[\alpha \succsim \beta/\bot])));\\
&(\varphi \succ \psi) \leftrightarrow \\
&\quad (((\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succ \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succsim \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succ \psi[\alpha \succsim \beta/\bot])));\\
&(\varphi \succ \psi) \leftrightarrow \\
&\quad (((\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\top] \succ \psi[\alpha \succsim \beta/\top])) \vee (\neg(\alpha \succ \beta) \wedge (\varphi[\alpha \succsim \beta/\bot] \succ \psi[\alpha \succsim \beta/\bot]))).
\end{aligned}
$$

They are proven mainly by (**B7**) to (**B10**). The key idea is to first derive the following:

$$
\begin{aligned}
(\alpha \succsim \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succsim \beta/\top]) \succsim \top);\\
\neg(\alpha \succsim \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succsim \beta/\bot]) \succsim \top);\\
(\alpha \succ \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succ \beta/\top]) \succsim \top);\\
\neg(\alpha \succ \beta) &\to ((\varphi \leftrightarrow \varphi[\alpha \succ \beta/\bot]) \succsim \top).
\end{aligned}
$$

Together with $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succsim \chi) \leftrightarrow (\psi \succsim \chi))$ and $((\varphi \leftrightarrow \psi) \succsim \top) \to ((\varphi \succ \chi) \leftrightarrow (\psi \succ \chi))$, the required equivalences can easily be derived. □

Proposition 5.6.12. *For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond)$ there is a $T_{Simpd1}(\varphi) \in \mathcal{L}_{Simpd1}$ such that $\varphi \leftrightarrow T_{Simpd1}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond)$.*

Proof. The result of replacing all $\Diamond\chi$ in $T_{\text{Simp}}(\varphi)$ by $\Diamond T_{\text{Compd1}}(\chi)$ is the desired $T_{\text{Simpd1}}(\varphi)$. □

Proposition 5.6.13. *For every $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ there is a $T_{Simpd1}(\varphi) \in \mathcal{L}_{Simpd1}$ such that $\varphi \leftrightarrow T_{Simpd1}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$.*

Proof. By induction. Given Proposition 5.6.12 and the rule of replacement of equivalents, the only non-trivial case is to show that there is a $T_{\text{Simpd1}}(\langle\varphi\rangle\psi)$ that is provably equivalent to $\langle\varphi\rangle\psi$ in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ where φ, ψ are in $\mathcal{L}_{\text{Simpd1}}$. By repeated use of (**R1**) to (**R3**) and the rule of replacement of equivalents, obviously we can push the $\langle\varphi\rangle$ into ψ over Boolean connectives and $\Diamond$ and obtain a Boolean combination of formulas of the form $\langle\varphi\rangle p$ or of the form $\langle\varphi\rangle(\alpha \succsim \beta)$ or $\langle\varphi\rangle(\alpha \succ \beta)$ since in $\mathcal{L}_{\text{Simpd1}}$, $\succsim$ and $\succ$ only scope over propositional formulas. All three kinds of formulas can be replaced by formulas in $\mathcal{L}(\succsim, \succ, \Diamond)$ provably equivalently. Then we apply T_{Simpd1} again to finish off (to eliminate any $\Diamond$'s appearing inside $\Diamond$'s). □

With the above reduction method, the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ follows.

Proposition 5.6.14. *For all $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$: if φ is valid with respect to the class of all imprecise probabilistic models, then φ is a theorem of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$.*

Proof. Let φ be any valid formula in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$. Then by the soundness of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$ and the fact that $\varphi \leftrightarrow T_{\mathrm{Simpd1}}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, $T_{\mathrm{Simpd1}}(\varphi)$ is also valid. But $T_{\mathrm{Simpd1}}(\varphi) \in \mathcal{L}_{\mathrm{Simpd1}} \subseteq \mathcal{L}(\succ, \succ, \Diamond)$. By the completeness of $\mathsf{IP}(\succsim, \succ, \Diamond)$, $T_{\mathrm{Simpd1}}(\varphi) \in \mathsf{IP}(\succsim, \succ, \Diamond)$. By the definition of $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$, it contains all theorems of $\mathsf{IP}(\succsim, \succ, \Diamond)$. Hence $T_{\mathrm{Simpd1}}(\varphi)$ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$. Then by Boolean reasoning, φ is in $\mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle)$. $\square$

Although the reduction axioms for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ allow us to reduce the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ to that for $\mathcal{L}(\succsim, \succ, \Diamond)$, which is in NP (Theorem 5.5.12), it does not immediately follow that the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ is in NP, due to the blowup in the length of formulas during the reduction process. A similar obstacle occurs in the case of the simplest dynamic epistemic logic (public announcement logic), in which case a solution is to use a satisfiability-preserving reduction with only polynomial blowup instead of the standard validity-preserving reduction with exponential blowup ([125]). Whether this or other techniques apply to $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ we leave as an open problem.

Problem 5.6.15. Determine the complexity of the satisfiability problem for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

5.6.2 Introducing a New Proposition

In the previous subsection, we considered the dynamic update operator that concerns learning the truth of a proposition. In this subsection, we consider the complementary dynamics of learning the mere existence of a proposition and then being maximally uncertain about it in the way of imprecise probability (cf. [101]). Our goal is to show that this kind of information dynamics is expressively helpful, especially in formalizing examples in a natural way, and we leave the complete axiomatization of its logic as an open question.

Definition 5.6.16. The language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \succsim \varphi) \mid (\varphi \succ \varphi) \mid \Diamond\varphi \mid \langle\varphi\rangle\varphi \mid I_p^+\varphi \mid I_p^-\varphi$$

where $p \in \mathsf{Prop}$. We read $I_p^+\varphi$ as "letting p be a true proposition that is newly introduced to the agent, φ"; similarly, $I_p^-\varphi$ reads "letting p be a false proposition that is newly introduced to the agent, φ". We also take $I_p\varphi$ as an abbreviation of $(I_p^+\varphi \wedge I_p^-\varphi)$.

We treat both I_p^+ and I_p^- as a kind of propositional quantifier, since they change the meaning (denotation) of p, and we define free and bound propositional variables in the obvious way. For any $\varphi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$, let $\mathsf{Prop}(\varphi)$ be the set of freely occurring propositional variables in φ.

Now we specify the semantics for I^+ and I^-. First, we define how a model changes when we introduce a new proposition.

Definition 5.6.17. Given a non-empty set W, a field of sets $\mathcal{F}$ on W, a valuation V such that $V(p) \in \mathcal{F}$ for all $p \in \mathsf{Prop}$, and a set of finitely additive probability measure $\mathcal{P}$ on $\mathcal{F}$, we interpret $\mathcal{F}$ as the collection of the "old" propositions. Our goal is to define the result of adding a "new" proposition P. Intuitively, we first split each $w \in W$ into $\langle w, 1\rangle$ and $\langle w, 0\rangle$ corresponding to P being true and false, respectively, while keeping the truth value of the old propositions. For the probability measures, we take all probability measures defined on both the old and new propositions that, when restricted to just the old propositions, coincide with some old probability measure. The following gives the formal details.

- Let $\mathcal{F} \times 2 = \{X \times \{0,1\} \mid X \in \mathcal{F}\}$, which is a field of sets on $W \times \{0,1\}$.
- Let $\mathsf{Split}(\mathcal{F})$ be the smallest field of sets on $W \times \{0,1\}$ extending $\mathcal{F} \cup \{W \times \{0\}\}$.
- Let $V \times 2$ be defined such that $V \times 2(p) = V(p) \times \{0,1\}$ for all $p \in \mathsf{Prop}$; note that $V \times 2(p) \in \mathcal{F} \times 2$ for all $p \in \mathsf{Prop}$.
- For any $p \in \mathsf{Prop}$, let V^{+p} be defined such that
$$V^{+p}(q) = \begin{cases} V(q) \times \{0,1\} & \text{if } q \neq p \\ W \times \{1\} & \text{if } q = p\,; \end{cases}$$
note that $V^{+p}(q) \in \mathsf{Split}(\mathcal{F})$ for all $q \in \mathsf{Prop}$.
- For any finitely additive measure μ on $\mathcal{F}$, define $\mu \times 2$, a finitely additive measure on $\mathcal{F} \times 2$, by $\mu \times 2(X \times \{0,1\}) = \mu(X)$ for all $X \in \mathcal{F}$.
- Let $\mathcal{P} \times 2 = \{\mu \times 2 \mid \mu \in \mathcal{P}\}$.
- Let $\mathsf{Split}(\mathcal{P})$ be the set of all finitely additive measures μ on $\mathsf{Split}(\mathcal{F})$ such that $\mu|_{\mathcal{F}\times 2} \in \mathcal{P} \times 2$.

Using the above definition, given a propositional variable $p \in \mathsf{Prop}$ and a propositional model $\mathcal{M} = \langle W, V\rangle$, let $\mathcal{M} \times 2 = \langle W \times \{0,1\}, V \times 2\rangle$ and $\mathcal{M}^{+p} = \langle W \times \{0,1\}, V^{+p}\rangle$. Then if $\langle \mathcal{M}, \mathcal{P}\rangle$ is an IP model, so is $\langle \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P})\rangle$, and $\langle \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P})\rangle$ represents the result of adding a new proposition, now denoted by p, to $\langle \mathcal{M}, \mathcal{P}\rangle$.

Remark 5.6.18. In the algebraic theory of Boolean algebras, there is a standard operation of freely adjoining a new element to a Boolean algebra: for any Boolean algebra $\mathcal{B}$ and any $a \notin \mathcal{B}$, there is a unique up to isomorphism Boolean algebra $\mathcal{B}^{+a}$ such that

- $\mathcal{B}$ is a subalgebra of $\mathcal{B}^{+a}$, and every element in $\mathcal{B}^{+a}$ is generated from $\mathcal{B} \cup \{a\}$;
- for any $b \in \mathcal{B}$, $b \wedge a$ and $b \wedge \neg a$ are not the bottom element in $\mathcal{B}^{+a}$.

The operation of $\mathsf{Split}(\mathcal{F})$ is precisely the dual of this algebraic operation.

Hence, if we use an algebraic model $\langle \mathcal{B}, V, \mathcal{P}\rangle$ where $\mathcal{B}$ is a Boolean algebra (of propositions of which the agent is currently aware), V a valuation function from Prop to $\mathcal{B}$, and $\mathcal{P}$ a set of finitely additive functions from $\mathcal{B}$ to $[0,1]$, we can easily define the result of adding a new proposition $a \notin \mathcal{B}$ to be denoted by p as $\langle \mathcal{B}^{+a}, V', \mathcal{P}'\rangle$ where V' coincides with V on Prop except that $V'(p) = a$ and $\mathcal{P}' = \{\mu : \mathcal{B}^{+a} \to [0,1] \mid \mu \text{ is finitely additive and } \mu|_{\mathcal{B}} \in \mathcal{P}\}$.

Remark 5.6.19. The model construction from $\langle \mathcal{M}, \mathcal{P} \rangle$ to $\langle \mathcal{M}^p, \mathsf{Split}(\mathcal{P}) \rangle$ can also be viewed as an event-model update from (probabilistic) dynamic epistemic logic ([14]). The event model contains two events $\{1, 0\}$ corresponding to whether the new proposition is true or not with no preconditions, and the agent is maximally ignorant about these two events: at any of the old worlds, she cannot distinguish between these two events, is completely ignorant about the relative likelihood of these two events, and does not observe which event happens. Using the terminology from [14], the agent is maximally and imprecisely ignorant about the occurrence probability of these two events and makes no observation about these two events.

Definition 5.6.20. The semantics of I_p^+ and I_p^- are given by

$$\mathcal{M}, \mathcal{P}, w \vDash I_p^- \varphi \text{ iff } \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P}), \langle w, 0 \rangle \vDash \varphi\,,$$
$$\mathcal{M}, \mathcal{P}, w \vDash I_p^+ \varphi \text{ iff } \mathcal{M}^{+p}, \mathsf{Split}(\mathcal{P}), \langle w, 1 \rangle \vDash \varphi\,.$$

Now let us put the new operators to work. We first use them to formalize the medical example (Example 5.1.1).

Example 5.6.21. The following sentence is valid and represents the medical example if we take p to mean that the agent has the disease (that is, the proposition introduced by I_p is that the agent has the disease) and q to mean that the gland is swollen (that is, the proposition introduced by I_q is that the gland is swollen).

$$I_p \langle \neg p \succ p \rangle I_q \langle (q \wedge p) \succ (q \wedge \neg p) \rangle \langle q \rangle (p \succ \neg p). \tag{5.5}$$

We interpret the first update by $\neg p \succ p$ as the result of the agent observing that she is not feeling uncomfortable and hence believing that her not having the disease is more likely than her having it. The second update represents what the agent learns from the doctor, and the third update represents a medical examination revealing that her gland is swollen.

The above simple sentence does not capture more nuanced probabilistic relationships between p and q such as that conditioning on q, p is twice as likely as $\neg p$ or that the medical examination does not reveal q but only a signal that is probabilistically related to q. But with the new operator I, we can easily say these things. For example, to express that p is twice as likely as $\neg p$ conditioning on q, we may introduce two new propositions (like two coin flips) by I_r and I_s at the beginning of the formula (note that our syntax forbids embedding I in updates) and later add after I_q the update $\langle ((q \wedge r \wedge s) \approx (q \wedge r \wedge \neg s)) \wedge ((q \wedge \neg r \wedge s) \approx (q \wedge \neg r \wedge \neg s)) \wedge ((q \wedge r \wedge s) \approx (q \wedge \neg r \wedge s)) \wedge (\bot \succsim (q \wedge \neg r \wedge \neg s)) \rangle$, which says that conditioning on q the two coin flips are fair and independent but the two tail situation is impossible (perhaps because the two coins will be retossed if they are both tails up). Then, using $\langle (q \wedge p) \approx (q \wedge s) \rangle$, we essentially say that p's probability conditioning on q is $2/3$ and thus twice as likely as $\neg p$. To express that the medical examination only provides an informative signal related to q, we may again introduce a new proposition t representing that signal and then let the agent learn the probabilistic relationship between t and q.

Example 5.6.22. For the prisoner example, recall that α is the formula

$$\big(\bot \succsim (e_a \wedge e_b \wedge e_c)\big) \wedge \big(((e_a \wedge e_b) \vee (e_a \wedge e_c) \vee (e_b \wedge e_c)\big) \succsim \top\big) \wedge (e_a \approx e_b) \wedge (e_b \approx e_c),$$

saying that two of the prisoners will be executed and the probabilities for the three situations are equal. Recall also that β is the following formula

$$((s_b \to e_b) \succsim \top) \wedge ((s_c \to e_c) \succsim \top) \wedge (\bot \succsim (s_b \wedge s_c)),$$

saying that the jailer will announce one and only one prisoner to be executed truthfully. Then the following formula is valid and represents the dilation when a hears that the jailer announces that b will be executed:

$$I_{e_a} I_{e_b} I_{e_c} \langle \alpha \rangle I_{s_b} I_{s_c} \langle \beta \rangle \langle s_b \rangle (\Diamond(e_a \succsim \top) \wedge \Diamond(e_a \approx \neg e_a)).$$

As we have seen in Example 5.6.21, $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is capable of expressing numerical relationships. Leveraging this capability, it is easy to observe that $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is more expressive than $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$.

Example 5.6.23. Consider a propositional model $\mathcal{M} = \langle W, V \rangle$ where $W = \{w, u\}$ has two worlds, $V(p) = \{w\}$, and $V(q) = \varnothing$ for all $q \in \mathsf{Prop} \setminus \{p\}$. Then let μ_1 be a probability measure on $\wp(W)$ such that $\mu_1(\{w\}) = 0.6$, and let μ_2 also be a probability measure on $\wp(W)$ such that $\mu_2(\{w\}) = 0.9$. Then it is easy to see that $\mathcal{M}, \{\mu_1\}, w$ and $\mathcal{M}, \{\mu_2\}, w$ satisfy the same formulas in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$. However, the following formula

$$I_q I_r \langle ((q \wedge r) \approx (q \wedge \neg r)) \wedge ((q \wedge r) \approx (\neg q \wedge r)) \wedge ((q \wedge r) \approx (\neg q \wedge \neg r)) \rangle (p \succ \neg (q \wedge r)),$$

which intuitively says that p is more likely than not getting two heads up from two randomly and independently flipped fair coins, is true at $\mathcal{M}, \{\mu_2\}, w$, but false at $\mathcal{M}, \{\mu_1\}, w$.

Indeed, we will show that $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ can express any linear inequality with integer coefficients about the probability of formulas. For this, we first introduce some notation.

Definition 5.6.24. Let Γ be a finite set of formulas, $C(\Gamma)$ the set of all clauses (conjunctions of the form $\bigwedge_{\varphi \in \Gamma} \pm\varphi$ where $\pm$ is either the empty string or $\neg$), and p a propositional variable. Then define $(p|\Gamma)$ to be the formula

$$\bigwedge_{\psi \in C(\Gamma)} ((\psi \wedge p) \approx (\psi \wedge \neg p)).$$

Intuitively, $(p|\Gamma)$ says that p represents a fair coin flip independent of all events expressible using formulas in Γ.

Proposition 5.6.25. *For any sequences $\langle \varphi_i \rangle_{i=1\ldots n}$ and $\langle \psi_i \rangle_{i=1\ldots m}$ of formulas in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ and any sequences $\langle a_i \rangle_{i=1\ldots n}$ and $\langle b_i \rangle_{i=1\ldots m}$ of natural numbers, there is a formula $\chi \in \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ such that for any IP model $\mathcal{M}, \mathcal{P}, w$,*

$$\mathcal{M}, \mathcal{P}, w \vDash \chi \text{ iff } \forall \mu \in \mathcal{P}, \sum_{i=1}^{n} a_i \mu([\![\varphi_i]\!]^{\mathcal{M},\mathcal{P}}) \geq \sum_{i=1}^{m} b_i \mu([\![\psi_i]\!]^{\mathcal{M},\mathcal{P}}).$$

Proof. The central idea is already in [109] and is also described in Section 2 of [41]: we use I operators to introduce new propositions that evenly partition the logical space spanned by φ_i's so that we can take the union of multiple copies of the partitioned φ_i's to simulate addition.

Let l be the smallest natural number such that 2^l is larger than the sum of all a_i's and b_i's and pick propositional variables $\langle p_i\rangle_{i=1\dots l}$ not occurring in any of the φ_i's and ψ_i's. Then let C list all logically inequivalent clauses made from p_i's. Since $|C| = 2^l$ and 2^l is larger than the sum of all coefficients, let f be a function from $\{1,\dots n\} \times \{0\} \cup \{1,\dots m\} \times \{1\}$ to $\wp(C)$ such that $f(x) \cap f(y) = \varnothing$ whenever $x \neq y$ and $|f(i,0)| = a_i$ and $|f(i,1)| = b_i$. Let Γ be set of all φ_i's and ψ_i's. Then consider the following formula:

$$I^+_{p_1} I^+_{p_2} \cdots I^+_{p_l} \langle (p_1|\Gamma) \wedge (p_2|\Gamma \cup \{p_1\}) \cdots (p_l|\Gamma \cup \{p_1, p_2, \cdots, p_{l-1}\})\rangle \\ (\bigvee_{i=1}^{n} \bigvee_{c\in f(i,0)} (\varphi_i \wedge c)) \succsim (\bigvee_{i=1}^{n} \bigvee_{c\in f(i,1)} (\psi_i \wedge c)). \quad (5.6)$$

This is the required formula since after the introduction of new propositions and the announcement, the probability of $\bigvee_{c\in f(i,0)}(\varphi_i \wedge c)$ (resp. $\bigvee_{c\in f(i,1)}(\psi_i \wedge c)$) is precisely $a_i/2^l$ (resp. $b_i/2^l$) times the probability of φ_i (resp. ψ_i). Canceling out the common denominator 2^l, we see that the inequality expressed by formula (5.6) is the required one. □

Therefore, we see that with the new operators I^+_p and I^-_p, $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ is capable of expressing quantitative (and in particular arbitrary additive) information. This also means that we cannot use the same reduction strategy we used for $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ to axiomatize the logic in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$. However, we conjecture that there is a computable translation from $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ to $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle)$ that preserves satisfiability. Such a translation can then be coded as rules instead of axioms that completely axiomatize the logic.

Problem 5.6.26. Find an axiomatization of the set of valid formulas in $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$.

Problem 5.6.27. Determine the complexity of the satisfiability problem for the language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$.

5.7 Conclusion

In this chapter, we have investigated a hierarchy of languages

$$\mathcal{L}(\succsim) \subseteq \mathcal{L}(\succsim, \succ) \subseteq \mathcal{L}(\succsim, \succ, \Diamond) \subseteq \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle) \subseteq \mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$$

and matching complete logics for imprecise comparative probabilistic reasoning in the first four languages:

$$\mathsf{IP}(\succsim) \subseteq \mathsf{IP}(\succsim, \succ) \subseteq \mathsf{IP}(\succsim, \succ, \Diamond) \subseteq \mathsf{IP}(\succsim, \succ, \Diamond, \langle\,\rangle).$$

The first four languages have straightforward extensions to the multi-agent setting, in which each agent i has their own comparative probability relations $\succsim_i$ and $\succ_i$, allowing us to formalize statements such as "Ann judges it more likely than not that Bob thinks hail is more likely than lightning": $(h \succ_b l) \succ_a \neg(h \succ_b l)$. A multi-agent version of the language $\mathcal{L}(\succsim)$ was already studied in [1]. Generalizing the other languages in this chapter to the multi-agent setting presents no major challenges, although the complexity of the resulting multi-agent logics goes beyond that of the single-agent versions, just as the complexity of the basic epistemic logic $\mathsf{S5}$ jumps from NP to PSPACE when moving from the single-agent to multi-agent setting (see [81]). When generalizing the language $\mathcal{L}(\succsim, \succ, \Diamond, \langle\,\rangle, I)$ to the multi-agent setting, there is a distinction between introducing a new proposition to every agent publicly and introducing a new proposition for only one agent so that she becomes privately aware of it. Our semantics naturally generalizes to model all agents publicly becoming aware of a new proposition, but the modeling of some agent's privately becoming aware of a new proposition requires a different treatment.

Further extensions to the language are natural to consider, such as adding comparative conditional probability formulas $(\varphi \mid \psi) \succsim (\alpha \mid \beta)$ (resp. $(\varphi \mid \psi) \succ (\alpha \mid \beta)$) expressing that the conditional probability of φ given ψ is at least as great as (resp. greater than) the conditional probability of α given β for every measure in one's set of measures, which is not expressible in the languages of this chapter (see [124]). For precise probabilistic models, such a quarternary operator is investigated in, e.g., [44, § 2.6] and [151] (and recently in [87] using so-called Popper functions), but the interpretation in imprecise probabilistic models seems yet to be explored. Allowing inequalities of probabilistic products $(\varphi \times \psi) \succsim (\alpha \times \beta)$ would allow even greater expressivity (such an extension in the precise case is also considered in [44, §2.4]).

More generally, the systems in this chapter are part of a much broader hierarchy of probabilistic languages, ranging from the very simple $\mathcal{L}(\succsim)$ all the way to highly expressive probabilistic languages encompassing full quantified real number arithmetic [79]. In addition to their inherent theoretical interest, probabilistic logics have emerged as a foundational tool for many central computational tasks, from core knowledge representation [139], to reasoning about strategic interaction [35, 15], to causal inference (witness *do-calculus*, which is built on top of a probability calculus; see, e.g., [135, 7, 98]). Furthermore, applications in these contexts have motivated some of the very systems presented here (e.g., [1]). Understanding the capacities and limitations of such systems may well be an important step toward further integration of explicit probabilistic tools in these and other domains.

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