

# Lawrence Berkeley National Laboratory

## Recent Work

**Title**

Aspects of Large N Gauge Theory Dynamics as seen by String Theory

**Permalink**

<https://escholarship.org/uc/item/792311vj>

**Journal**

Physical Review D, 58(10)

**Author**

Ooguri, H.

**Publication Date**

1998-07-02



# ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY

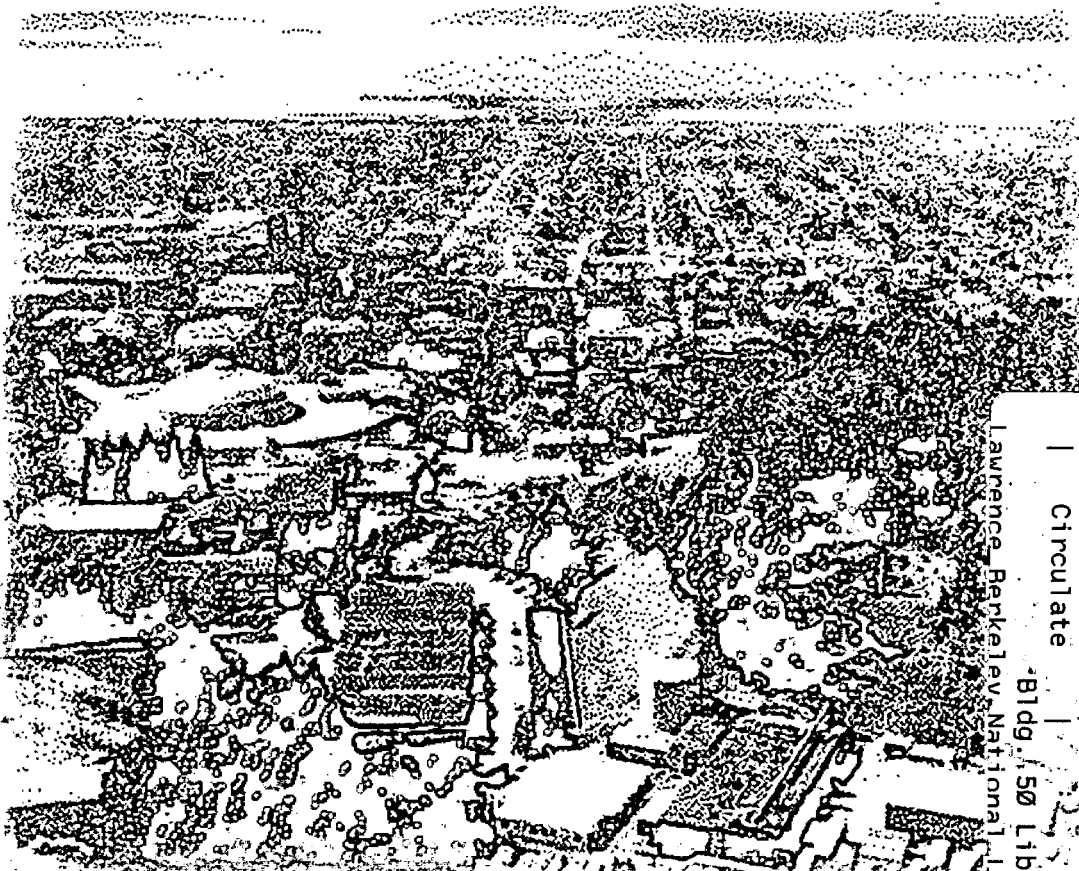
## Aspects of Large N Gauge Theory Dynamics as Seen by String Theory

David J. Gross and Hiroshi Ooguri

Physics Division

July 1998

Submitted to  
*Physical Review D*



Lawrence Berkeley National Laboratory

REFERENCE COPY	_____
Does Not Circulate	_____
Bldg. 50 Library - Ref.	_____
Copy	1

LBLN-41850

## **DISCLAIMER**

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

# Aspects of Large N Gauge Theory Dynamics as Seen by String Theory

David J. Gross<sup>1</sup> and Hiroshi Ooguri<sup>1,2,3</sup>

<sup>1</sup>Institute for Theoretical Physics  
University of California  
Santa Barbara, CA 93106

<sup>2</sup>Department of Physics  
University of California  
Berkeley, CA 94720

<sup>3</sup>Theory Group  
Ernest Orlando Lawrence Berkeley National Laboratory  
Berkeley, CA 94720

July 1998

ITP/NSF-98-064, UCB/PTH-98/27, LBNL-41850  
hep-th/9805129

## Aspects of Large $N$ Gauge Theory Dynamics as Seen by String Theory

David J. Gross<sup>1</sup> and Hiroshi Ooguri<sup>1,2,3</sup>

<sup>1</sup> Institute for Theoretical Physics, University of California,  
Santa Barbara, CA 93106

<sup>2</sup> Department of Physics, University of California,  
Berkeley, CA 94720

<sup>3</sup> Theory Group, Lawrence Berkeley National Laboratory,  
Berkeley, CA 94720

### Abstract

In this paper we explore some of the features of large  $N$  supersymmetric and nonsupersymmetric gauge theories using Maldacena's duality conjectures. We shall show that the resulting strong coupling behavior of the gauge theories is consistent with our qualitative expectations of these theories. Some of these consistency checks are highly nontrivial and give additional evidence for the validity of the proposed dualities.

# 1 Introduction

The newest, and perhaps most interesting, of the dualities of string theory is that conjectured by Maldacena, which relates the large  $N$  expansion of conformal field theory in  $d$  dimensions to string theory in a  $AdS_{d+1} \times M$  spacetime background (where  $AdS_{d+1}$  is  $(d + 1)$ -dimensional Anti de-Sitter space and  $M$  is a compact space) [1]. The dictionary that relates these dual descriptions identifies the  $1/N$  expansion of the field theory to the perturbative expansion of the string theory, and the strong coupling expansion of the field theory to the  $\alpha'$  expansion of the string theory. This conjecture offers the exciting possibility of using perturbative string theory to explore the large  $N$  limit of field theory.

The simplest case of Maldacena's conjecture is the duality between large  $N$  supersymmetric, conformally invariant,  $SU(N)$  gauge theory in four dimensions (with coupling  $g_{YM}^2$ ) and type IIB string theory expanded about an  $AdS_5 \times S^5$  background. Here the string coupling,  $g_{st}$ , is proportional to  $g_{YM}^2$ ;  $N$  equals, in the string theory, the magnitude of the five-form flux on the five-sphere; and  $(g_{YM}^2 N)^{1/4}$  is proportional to the radius of curvature of the background  $AdS_5$  space. One can therefore hope to calculate gauge theory correlation functions, for large  $N$  and large  $(\lambda = g_{YM}^2 N)$ , in terms of weak coupling string theory in the semiclassical approximation — *i.e.* supergravity.

The precise relation between the gauge theory correlation functions and the supergravity effective action has been given by [2, 3], following the earlier works [4]. In particular this prescription determines the dimensions of operators in the conformal field theory in terms of the masses of particle in the string theory. This correspondence has been checked for the duality between  $SU(N)$  gauge theory in four dimensions and type IIB string theory expanded about an  $AdS_5 \times S^5$  background; where it was shown that there is a precise correspondence between the chiral fields of the conformal gauge theory and the finite mass string states in the above limit, including the complete infinite tower of massive Kaluza-Klein states of ten-dimensional supergravity on the 5-sphere [5, 6]

In [7, 8], it was shown that the strong coupling limit of the large Wilson loop for large  $N$  can be evaluated using semiclassical string theory, thereby obtaining the interaction energy between infinitely massive quarks and anti-quarks (external sources in the fundamental representation of  $SU(N)$ ), sep-

arated by distance  $R$  as

$$E_{q\bar{q}} = -\frac{4\pi^2}{\Gamma(1/4)^4} \frac{\sqrt{2\lambda}}{R}, \quad (1)$$

a result that is completely consistent with our limited understanding of the gauge theory, wherein the  $1/R$  behavior is dictated by conformal invariance. The proportionality to  $\sqrt{\lambda}$  suggests that the Coulomb forces is somewhat reduced from the weakly coupled value of  $\lambda$ . Similar calculations have been performed for the monopole-monopole and monopole-quark potential, yielding, as expected,  $S$ -dual expressions [9].

One can regard Maldacena's duality as realizing the long sought goal of finding the master field representation of large  $N$  gauge theory correlation functions. What is most surprising from this point of view is that the master field lives in a compactified ten-dimensional space-time, and corresponds to supersymmetric type IIB string theory. That there should exist a string representation of the  $\mathcal{N} = 4$  conformally invariant large  $N$  gauge theory is somewhat surprising, since the traditional arguments for such a representation have been for confining theories, whereas here we have a string theory for the Coulomb phase of the gauge theory. Thus, even though the Wilson loop is given by the minimal area classical string configuration spanning the loop, the fact that the loop can meander into the extra dimensions and the nature of the geometry of  $AdS$  space lead to a  $1/R$  potential in this case.

Although the duality between  $SU(N)$  gauge theory in four dimensions and type IIB string theory expanded about an  $AdS_5 \times S^5$  background is of great academic interest, the most exciting extension of Maldacena's conjecture is to non-supersymmetric gauge theories, especially to the physically relevant case of four-dimensional, non-supersymmetric gauge theory—namely QCD<sub>4</sub>. As Witten has shown [10], it is reasonable to extend the conjecture to cases where supersymmetry is broken, thereby obtaining properties of non-supersymmetric gauge theories in the large  $N$  limit. For example, one can easily extend the duality to discuss the finite temperature behavior of the  $\mathcal{N} = 4$  gauge theory, by compactifying the (Euclidean) time direction of the background space time of  $AdS$  on a circle of radius  $\propto 1/\text{temperature}$ , in which case supersymmetry is broken by the boundary conditions on the circle. One can argue that, since supersymmetry is broken, the fermions and the scalars acquire a mass and, at least for large temperature decouple, thus

yielding a duality to high temperature QCD. Witten showed that in this case one derives many of the expected features of high temperature gauge theory; including a non-zero expectation value of temporal (Polyakov) loops, an area law for spatial Wilson loops and a mass gap (*i.e.* a magnetic mass).

Finally, Witten has proposed a strategy to study ordinary four dimensional QCD at zero temperature using string theory [10]. This can be done by using Maldacena's conjecture to relate the large  $N$  limit of the  $SU(N)$ -type  $(2,0)$  theory in  $\mathbf{R}^6$  to  $M$  Theory on  $AdS_7 \times S_4$  and dimensionally reducing these to four dimensions. (Throughout this paper, we consider theories on Euclidean signature spaces, and  $AdS_7$  here means its Euclideanized version.) To do this and to break the supersymmetry one sets the  $(2,0)$  theory on  $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{R}^4$  with supersymmetry breaking boundary condition on the fermions around one of the  $\mathbf{S}^1$ 's.

An obvious candidate for its  $M$  Theory dual would be obtained by periodically identifying points on  $AdS_7$  corresponding to the periodicity's of the  $\mathbf{S}^1 \times \mathbf{S}^1$  and by imposing the supersymmetry breaking boundary condition on the fermions by hand. There is, however, another candidate which obeys the same boundary condition. It is the Anti-de Sitter Schwarzschild solution constructed by Hawking and Page (for  $AdS_4$  case) [15]. The supersymmetry breaking boundary condition is automatically imposed by the Schwarzschild geometry. It turned out that, the classical action for the  $AdS$  Schwarzschild solution is smaller than that of the vacuum  $AdS_7$ , and therefore is dominant in the large  $N$  limit [10].

To make contact with four dimensional QCD we must shrink the radii of the two circles to zero in a certain limit. In this construction, the six-dimensional  $(2,0)$  theory is regarded as a regularization of the four-dimensional QCD. The ultraviolet cut-off scale is therefore set by the size of the compact space  $\mathbf{S}^1 \times \mathbf{S}^1$ .

Denote the radius of the supersymmetry preserving circle by  $R_1$  and that of the supersymmetry breaking one by  $R_2$ . The gauge coupling constant  $g_{YM}$  of  $QCD_4$  is given by the ratio of the radii  $g_{YM}^2 = R_1/R_2$ . In the 't Hooft limit, where one keeps  $g_{YM}^2 N$  to be finite, the circle  $\mathbf{S}_{R_1}^1$  shrinks to zero as one takes  $N \rightarrow \infty$ . This corresponds to the IIA limit of  $M$  Theory as  $\mathbf{S}_{R_1}^1$  is the supersymmetry preserving circle. Therefore one could have started with the theory on  $N$  D4 branes in the IIA theory, wrapped around a circle with nonsupersymmetric boundary conditions, rather than the six-dimensional theory. We will take this approach throughout the paper. QCD is then



regarded as the dimensional reduction of the five dimensional theory at high temperature, with coupling  $g_{YM}^2 = g_5^2 T$ , where  $g_5$  is the five dimensional coupling and  $T$  the temperature (inverse radius) of the circle.

Witten has argued that Wilson loops exhibit a confining area law behavior in this geometry for large  $N$  and large  $g_{YM}^2 N$ . However, as he points out, this does not establish that QCD is a confining theory. The gauge theory so constructed has an ultraviolet cutoff ( $\propto T$ ) and the coupling  $g_{YM}$  should be thought of as the bare coupling at distances corresponding to  $1/T$ . The string tension will turn out, for large  $\lambda = g_{YM}^2 N$  (as we shall show below), to be proportional to  $\lambda T^2$ . To construct four dimensional QCD we must take

$$T \rightarrow \infty \quad \text{and} \quad \lambda \rightarrow \frac{b}{\ln\left(\frac{T}{\Lambda_{\text{QCD}}}\right)}, \quad (2)$$

where  $\Lambda_{\text{QCD}}$  is the QCD mass scale. Presumably we would find, were we able to calculate the small  $\lambda$  behavior of the tension, that the tension behaves as  $\exp[-\frac{2b}{\lambda}]T^2 \sim \Lambda_{\text{QCD}}^2$ . This calculation is beyond our control at the moment. For small  $\lambda$  the background geometry develops singular behavior and the supergravity approximation surely breaks down. To deal with this continuum limit one would have to be able to calculate the properties of string theory with background Ramond-Ramond charge in a rather singular background.

Thus, for the time being, the Maldacena-Witten conjecture only informs us about the behavior of large  $N$  QCD, with a fixed ultraviolet cutoff in the strong coupling (large  $\lambda$ ) regime. The resulting physics should be compared best with strong coupling lattice gauge theory, where the lattice spacing  $a$  is analogous to  $1/T$ , the radius of the fifth dimension. What is remarkable here is that the short distance cutoff, unlike in the case of lattice, does not destroy the rotational or Lorentz symmetry of the theory. Indeed, at short distances we see a higher dimensional theory with more symmetry, indeed enough symmetry to render the theory finite. We are using the six dimensional, ultraviolet finite, (2,0) theory to define the theory in the ultraviolet, yet its infrared behavior should be qualitatively the same as QCD.

In this paper we shall explore some of the features of large  $N$  supersymmetric and nonsupersymmetric gauge theories using the above duality conjectures. We shall show that the resulting strong coupling behavior of the gauge theories is, in all cases, consistent with our qualitative expectations of these theories. Some of these consistency checks are highly nontrivial and give additional evidence for the validity of the proposed dualities.

First we shall explore, in the next section, the connected correlation function of Wilson loops. This kind of calculation can be used for many purposes among which are the evaluation of the electric mass (or screening length) of high temperature QCD, the glue ball spectrum of confining gauge theories and the demonstration that in the confining phase of QCD monopoles are condensed. In particular we outline how the glueball spectrum of this version of strong coupling QCD could be calculated.

In Section III, we generalize the discussion of QCD to the case where the  $\theta$  parameter is non-zero and argue that we can demonstrate oblique confinement.

In Section IV we generalize the evaluation of Wilson loops in the fundamental representation to higher representations. Here we find that the string theory naturally produces the behavior of higher representations that we would expect in a confining theory—a result that depends critically on the master field being described by fermionic strings.

In Section V we argue that one can also use the duality to discuss heavy quark baryonic states and determine the effective energy of  $N$  fundamental representation quarks in a singlet state for large  $N$ . The construction of the baryon is possible because of the Chern-Simons term in the action for supergravity on *AdS*. The same arguments allow us to show that the interaction energy between any finite number of quarks is zero for the conformally invariant supersymmetric four dimensional gauge theory and infinite for the confining theory.

Finally, we conclude with a discussion of the possibility of a large  $N$  phase transition. If such a phase transition exists the power of the conjectured duality would be significantly weaker.

While this paper was being typed, we learned of the work [11] where a similar construction of baryons is given.

## 2 Confinement, Monopole Condensation and Glueball

In this section, we first review the works [10, 12, 13, 14] where it was shown how confinement in strong coupling QCD<sub>p</sub> can be seen in the dual description based on *AdS* supergravity. In particular, they demonstrated the area law

behavior of the Wilson loop expectation value. We then discuss implications of this result and clarify an issue that was raised in [12, 13, 14] on the apparent divergence of the electric and magnetic masses. It turns out that this is related to the computation of the mass gap suggested in [10]. We discuss how one can compute glueball masses in this description.

According to Maldacena's conjecture [1, 16], the  $(p+1)$ -dimensional maximally supersymmetric gauge theory realized as the low energy dynamics of  $N$   $Dp$  branes ( $p \leq 5$ ) is dual to type II string theory on the near horizon geometry of the  $Dp$  brane, as given by

$$l_s^{-2} ds^2 = \sqrt{\frac{gN}{u^{7-p}}} du^2 + \sqrt{\frac{u^{7-p}}{gN}} \sum_{i=0}^p dx_i^2 + \sqrt{gNu^{p-3}} d\Omega_{8-p}^2, \quad (3)$$

where  $l_s$  is the string length,  $d\Omega_{8-p}$  is the line element of  $S^{8-p}$ , and  $g$  is related to the Yang-Mills coupling constant. We have neglected numerical factors that are not relevant to the following discussion. For  $p \neq 3$ , the dilaton  $\phi$  depends on  $u$  and is given by

$$e^\phi = g \left( \frac{gN}{u^{7-p}} \right)^{\frac{3-p}{4}}. \quad (4)$$

In particular, for  $p = 3$ , the near horizon geometry (3) is  $AdS$  and the dilaton (4) is constant, corresponding to the fact that the theory on  $D3$  brane is conformal.

Witten [10] proposed to study non-supersymmetric  $QCD_p$  by compactifying the supersymmetric theory in  $(p+1)$  dimensions on a circle and break the supersymmetry by imposing anti-periodic boundary conditions on the fermions. In the dual type II theory this corresponds to considering the  $AdS$  Schwarzschild geometry

$$l_s^{-2} ds^2 = \sqrt{\frac{gN}{u^{7-p}}} \frac{du^2}{1 - u_0^{7-p}/u^{7-p}} + \sqrt{\frac{u^{7-p}}{gN}} \left( 1 - u_0^{7-p}/u^{7-p} \right) d\tau^2 + \sqrt{\frac{u^{7-p}}{gN}} \sum_{i=1}^p dx_i^2 + \sqrt{gNu^{p-3}} d\Omega_{8-p}^2, \quad (5)$$

with the dilaton  $\phi$  given by (4). We can regard  $(\tau, x_1, \dots, x_p)$  as coordinates for the  $(p+1)$ -dimensional gauge theory. To make the horizon at  $u = u_0$

regular, the coordinate  $\tau$  has to be periodically identified as  $\tau \rightarrow \tau + 1/T$  with  $T$  being related to  $u_0$  by

$$u_0 = (gNT^2)^{\frac{1}{5-p}}. \quad (6)$$

Since the circle in the  $\tau$ -direction is contractible at  $u = u_0$ , the boundary condition on the fermions around the circle is automatically anti-periodic, breaking the supersymmetry. For large  $T$ , the  $(p+1)$ -dimensional theory becomes effectively  $p$ -dimensional, the fermions and scalars decouple, and the theory should resemble QCD $_p$  in the infra-red.

If QCD $_p$  is confining, the vacuum expectation value of the Wilson loop operator  $W(C)$  should exhibit area law behavior. In [10, 12, 13] this was shown to be the case, for large  $gN$ , by evaluating the classical action of string worldsheet bounded by a loop on  $\mathbf{R}^p$  located at  $u = \infty$ . Because of the  $u$ -dependent factor  $\sqrt{u^{7-p}/gN}$  in front of  $\sum_i dx_i^2$  in the metric (5), it is energetically favorable for the worldsheet to drop near the horizon  $u = u_0$  before spreading out in the  $\mathbf{R}^p$  direction. At the horizon, the  $u$ -dependent factor becomes

$$\sqrt{\frac{u_0^{7-p}}{gN}} = (gN)^{\frac{1}{5-p}} T^{\frac{7-p}{5-p}}, \quad (7)$$

where we used (6). Therefore the area dependent part of the Wilson loop expectation value becomes

$$\langle W(C) \rangle = \exp(-(gN)^{\frac{1}{5-p}} T^{\frac{7-p}{5-p}} A(C)), \quad (8)$$

where  $A(C)$  is the area bounded by the loop  $C$ . Since the QCD $_p$  coupling constant  $g_{YM}$  is related to  $g$  by  $g_{YM}^2 = gT$ , the string tension derived from the above formula is

$$(\text{tension})_p = (g_{YM}^2 N)^{\frac{1}{5-p}} T^{\frac{6-p}{5-p}}. \quad (9)$$

For  $p = 3, 4$ , this agrees with the formulae derived in [12, 13].

In four dimensions, it is expected that confinement is associated with magnetic monopole condensation. It is interesting to see that this in fact happens in this construction<sup>1</sup>. To discuss QCD $_4$ , we start with the five-dimensional theory on D4 branes. The magnetic monopole in five dimensions

---

<sup>1</sup>While this work was in progress, we received [14] where a related issue was discussed.

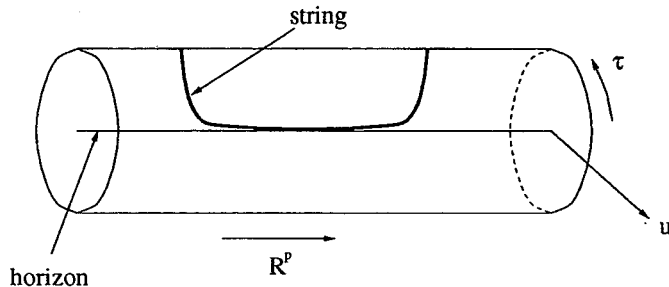


Figure 1: The string drops to the horizon first before spreading in the  $\mathbf{R}^p$  direction.

is a string which is realized as a D2 brane ending on a D4 brane [17, 18]. The monopole in four dimensions is obtained by wrapping the string around the compactifying  $\mathbf{S}^1$ . It is now straightforward to compute the potential between a monopole ( $m$ ) and an anti-monopole ( $\bar{m}$ ). Consider a pair of  $m$  and  $\bar{m}$  traveling along the  $x_1$ -axis in  $\mathbf{R}^4$  and separated in  $x_2$  direction by distance  $L$ . In the large  $gN$  limit, the force between them is mediated by a D2 brane bounded by  $\mathbf{S}^1$  times the trajectories of  $m$  and  $\bar{m}$ , which are located at  $u = \infty$ . Away from the boundaries, the D2 brane can spread in the  $u$ -direction. In its classical configuration,  $u$  would be a function of  $x_2$  only because of the symmetry of the problem. If we use  $(\tau, x_1, x_2)$  as the coordinates on the D2 brane, the induced metric on the brane is then

$$\begin{aligned}
 G_{\tau,\tau} &= \sqrt{\frac{u^3}{gN}} \left( 1 - \frac{u_0^3}{u^3} \right) \\
 G_{11} &= \sqrt{\frac{u^3}{gN}}, \quad G_{22} = \sqrt{\frac{gN}{u^3}} \frac{(du/dx_2)^2}{1 - u_0^3/u^3} + \sqrt{\frac{u^3}{gN}}.
 \end{aligned} \tag{10}$$

By taking into account the dilaton configuration (4), which in this case is

$$e^\phi = g \left( \frac{u^3}{gN} \right)^{1/4}, \tag{11}$$

the D2 brane action per unit length in the  $x_1$  direction becomes

$$E_{m\bar{m}} = \int_0^L d\tau dx_2 e^{-\phi} \sqrt{G_{\tau\tau} G_{11} G_{22}}$$

$$= \frac{1}{gT} \int_0^L dx_2 \sqrt{\left(\frac{du}{dx_2}\right)^2 + \frac{1}{gN}(u^3 - u_0^3)}, \quad (12)$$

and it gives the potential energy for the  $m$ - $\bar{m}$  pair.

The next task would be to minimize this action. In fact, essentially the same problem has already appeared in [12, 13] where the correlation function of temporal Wilson loops in five dimensions was studied. There one considers a string, rather than the D2 brane, wrapping in the  $\tau$ -direction and spreading in the  $x_2$ -direction. Because of (10) and (11), we have

$$e^{-\phi} \sqrt{G_{\tau\tau} G_{11} G_{22}} = \frac{1}{g} \sqrt{G_{\tau\tau} G_{22}}. \quad (13)$$

Therefore the classical action of the string is equal to  $g$  times that of the D2 brane discussed in the above paragraph. Therefore, we can borrow the result of [12, 13] to discuss the  $m$ - $\bar{m}$  correlation.

The new feature of this problem is a classical instability of the D2-brane worldvolume. When the distance  $L$  between  $m$  and  $\bar{m}$  is less than a certain critical distance  $L_{\text{crit}}$ , which is equal to  $1/T$  times some numerical factor, there is a D2 brane configuration minimizing the action (12) and connecting  $m$  and  $\bar{m}$ . If  $L$  exceeds this critical distance, there is no connected D2 brane configuration minimizing the action (Fig. 2). This happens because the  $G_{\tau\tau}$  component of the induced metric (10) can be made arbitrary small by going near the horizon  $u = u_0$  reflecting the fact that the compactification circle along  $\tau$  is contractible in the  $AdS$  Schwarzschild geometry. Since the circle is contractible, the D2 brane can split into two pieces each of which has a topology of a disk and is bounded by the trajectory of  $m$  or  $\bar{m}$ . Therefore, for  $L > L_{\text{crit}}$ , the potential between  $m$  and  $\bar{m}$  becomes constant and the force between them vanishes. This suggests that the magnetic monopole is completely screened. In this construction, therefore, confinement is in fact accompanied by monopole condensation. If we view this system as finite temperature  $\text{QCD}_{p+1}$ , such a complete screening indicates that the magnetic mass is infinite. This is somewhat puzzling and we will address this issue later in this section.

A similar classical instability also shows up when one studies correlation functions of Wilson loops. If one considers two Wilson loops in  $\mathbf{R}^4$  and repeats the above analysis to compute their correlation function, one finds that, beyond a certain critical distance determined by the size of the loops,

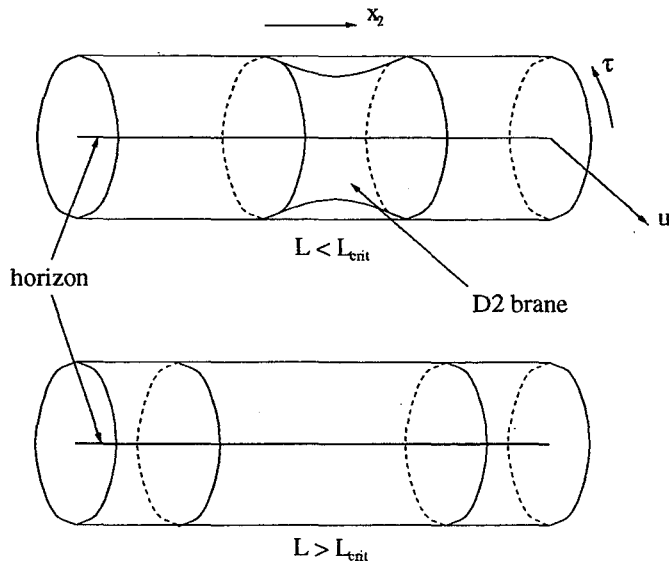


Figure 2: For  $L > L_{\text{crit}}$ , there is no volume-minimizing D2 brane configuration connection the  $m$ - $\bar{m}$  pair

the correlation function vanishes identically. Once again, this is because the loops are contractible and the string stretched between the loops becomes classically unstable beyond the critical distance (Fig. 3). This result is again somewhat puzzling since one would expect that the Wilson loops correlation for large distance would be characterized by glueball exchange. This result seems to indicate that the glueball mass in  $\text{QCD}_p$  is infinite. To address this issue, it is useful to look into the nature of the classical instability and discuss what happens at the critical distance and beyond.

The instability of minimal surfaces has been known for a long time. It was L. Euler who showed that a minimal surface bounded by a two concentric circle in  $\mathbf{R}^3$  is given by a catenoid. Let us put the two circles of radius  $R_0$  at  $z = \pm L/2$ . Euler's catenoid is given by

$$\sqrt{x^2 + y^2} = R_{\text{min}} \cosh\left(\frac{z}{R_{\text{min}}}\right), \quad (14)$$

where  $R_{\text{min}}$  is the minimum radius of the catenoid, which is a function of the

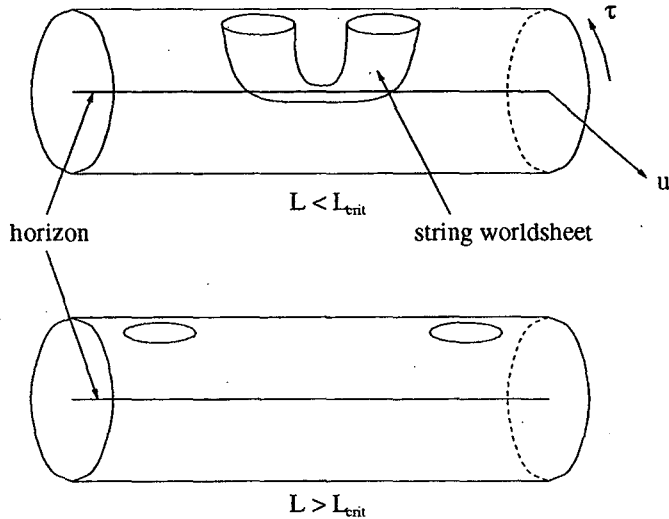


Figure 3: For  $L > L_{\text{crit}}$ , there is no area-minimizing string worldsheet connecting the two Wilson loops. The critical distance  $L_{\text{crit}}$  is determined by the size of the loop.

distance  $L$  between the circles and the radius  $R_0$  determined by the relation

$$R_0 = R_{\text{min}} \cosh\left(\frac{L}{2R_{\text{min}}}\right). \quad (15)$$

When the two loops coincide ( $L = 0$ ), obviously this formula gives  $R_{\text{min}} = R_0$ . As one increases  $L$ , the minimum radius  $R_{\text{min}}$  decreases. As shown in Fig. 4, however, there is a critical value of  $L_{\text{crit}} = 1.325R_0$ . For  $L > L_{\text{crit}}$ , there is no solution to (15). There the only minimal surface is a pair of disks bounded by the two circles, called the Goldschmit discontinuous solution. At  $L = L_{\text{crit}}$ , the catenoid becomes unstable. A small perturbation would make the surface to pinch and split into the two disks.

At  $L < \tilde{L}_{\text{crit}} = 1.056R_0$ , the area of the catenoid is smaller than that of the Goldschmit solution and therefore the catenoid is absolutely stable. At  $L = \tilde{L}_{\text{crit}}$ , the areas of the two solutions coincide and, for  $\tilde{L}_{\text{crit}} < L < L_{\text{crit}}$ , the catenoid becomes more voluminous than the Goldschmit solution. Therefore the transition from the catenoid to the Goldschmit solution at  $L = L_{\text{crit}}$  is of



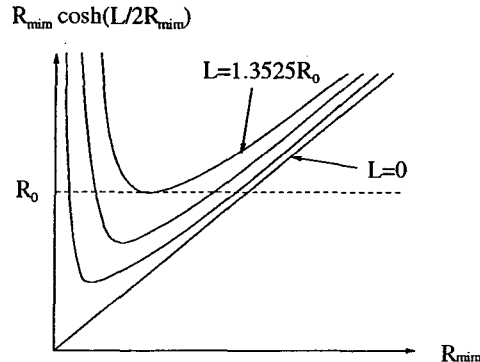


Figure 4: For  $0 < L < 1.3525R_0$ , the solid curve  $R = R_{\min}\cosh(L/2R_{\min})$  intersects twice with the dotted line  $R = R_0$ , determining the minimum radius  $R_{\min}$  of the catenoid. For  $L > L_{\text{crit}}$ , there is no intersection, indicating that a catenoid solution does not exist.

the first order.

What does this mean for the Wilson loop correlation function? When the distance between the loops  $C_1$  and  $C_2$  is less than the critical distance  $L < L_{\text{crit}}$ , the main contribution to the connected part of the correlation function  $\langle W(C_1)W(C_2) \rangle$  comes from the classical string connecting  $C_1$  and  $C_2$ . At  $L = L_{\text{crit}}$ , the string worldsheet becomes unstable and starts to collapse. Before the surface becomes disjoint, however, the supergravity approximation breaks down when the radius of the cylinder becomes of the order of the string length  $l_s$ . After that, quantum fluctuations of the surface start to support the worldsheet against the total collapse, and the two disks would be connected by a thin tube of a string scale  $l_s$ . For large  $L$ , the thin tube is represented by the supergraviton exchange between the two disks. Therefore the correlation between the Wilson loops does not completely vanish, but are mediated by the supergraviton exchange between the disks (Fig. 5). This indicates that the supergravitons in the  $AdS_{p+2}$  Schwarzschild blackhole geometry should be identified with the glueballs of  $QCD_p$ .

Another way to obtain glueball masses would be to compute correlation functions of local operators in  $QCD_p$  and look for particle poles. According to [2, 3], a two-point correlation function of local operators in the

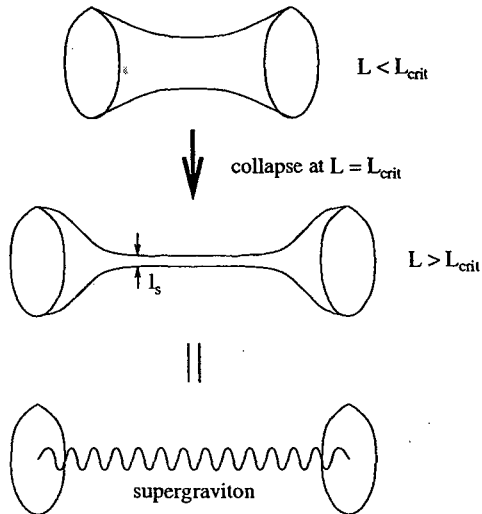


Figure 5: The string worldsheet connecting the Wilson loops collapses at  $L = L_{\text{crit}}$  and is replaced by the supergraviton exchange.

$(p + 1)$ -dimensional supersymmetric gauge theory is obtained by computing the Green's function of the corresponding supergraviton (or its Kaluza-Klein cousin) on  $AdS_{p+2}$ . Similarly a correlation function in  $QCD_p$  should be related to a Green's function on the  $AdS_{p+2}$  Schwarzschild geometry. The glueball masses computed in this way would be the same as the one that appeared in the Wilson loop correlators in the above paragraph as they are both extracted from the supergraviton propagator.

Since the bulk geometry is invariant under translation in the  $S^1 \times \mathbf{R}^p$  direction, we can expand the supergraviton wave  $\phi(u, \tau, x)$  in the Fourier modes as

$$\phi(u, \tau, x) = \sum_n \int \frac{d^p k}{(2\pi)^p} \tilde{\phi}_{n,k}(u) e^{in\tau + ikx}. \quad (16)$$

Each Fourier mode corresponds to a particle pole of the correlation function on  $\mathbf{R}^p$  with mass  $k^2$ . Those with  $n \neq 0$  are Kaluza-Klein modes on  $S^1$  and are not of interest for  $QCD_p$ . For a given  $k^2$ , the Fourier mode  $\tilde{\phi}_{n,k}(u)$  obeys the second order ordinary differential equation for  $u$ . Witten showed in [10] that the equation has a regular solution only for discrete values of  $k^2$ , suggesting

the mass gap in  $\text{QCD}_p$ .

In order to actually compute the glueball masses, one has to solve the differential equation. In the  $AdS_{p+2}$  geometry, the differential equation for the supergraviton has three regular singularities and therefore can be solved analytically using the hypergeometric function [4, 2]. For the Schwarzschild geometry, the differential equation has four regular singularities (for  $\text{QCD}_3$ ), with the additional singularity coming from the horizon, and requires numerical work. Various aspects of the glueball spectrum are currently under study using this technique [19].

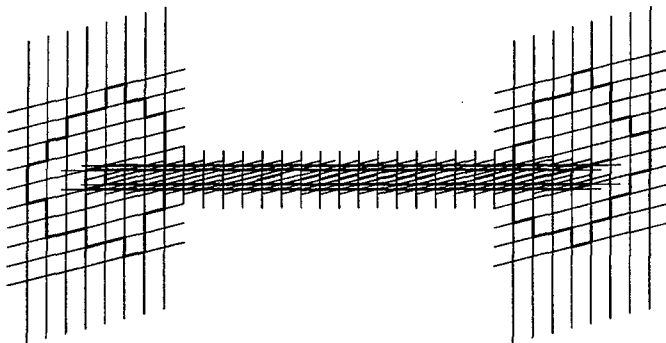


Figure 6: The glueball in the strong coupling lattice QCD.

The glueball masses computed in this way are quantized in the unit of the temperature  $T$ . One might have worried that in the effective large  $N$  string theory of glueballs the mass gap (the lowest mass of glueballs) would vanish — corresponding to the massless mode of the string. In strong coupling lattice QCD, the leading contribution to the Wilson loop correlator comes from a thin rectangular tube of the size of the minimum lattice spacing  $a$ , as shown in Fig. 6. Therefore the glueball masses are of the order of  $1/a$  in the strong coupling. Indeed, with the standard Wilson lattice action, the glueball mass, for strong coupling is given by  $m_{\text{glue}} = 4/a \log(g^2 N)[1 + O(1/g^2 N)]$ . To make contact with the real world, one would have to sum the strong coupling expansion to obtain  $m_{\text{glue}} = 4/af(g^2 N)$ , and then let  $g^2 N \rightarrow b/\log(1/\Lambda_{\text{QCD}}a)$  as  $a \rightarrow 0$ , obtaining (hopefully) a finite result proportional to the QCD scale  $\Lambda_{\text{QCD}}$ . In the  $AdS$  picture  $1/T$  plays the role of  $a$ , and

as the string shrinks to distances of order  $1/T$  the fluctuations in the extra dimensions produce a finite mass gap proportional to  $T$ . Thus  $m_{\text{glue}} \propto Tf(\lambda)$ . As in the case of the string tension discussed above, the computation of the  $4d$  glueball spectrum would require control of the string theory in a singular background with R-R charge.

### 3 $\theta$ Parameter and Oblique Confinement

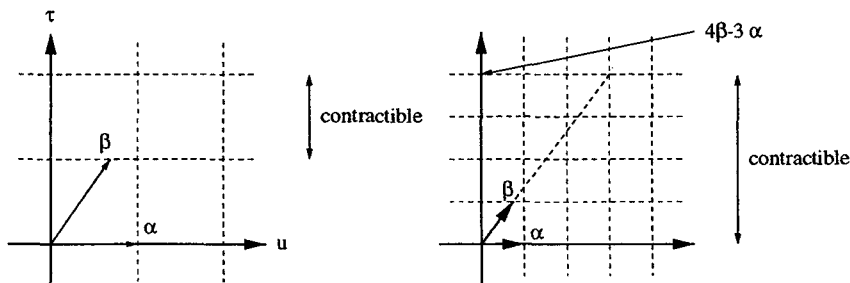


Figure 7: When  $\theta/2\pi$  is a rational number, one can find a contractible cycle on the torus.

It is interesting to generalize the discussion of the previous section to the case of  $\text{QCD}_4$  with non-zero  $\theta$  parameter. Its  $M$  Theory dual can be constructed as follows. The  $\theta$  parameter couples to  $\int \text{tr} F \wedge F$  in four dimensions, which is the D0 brane charge on the D4 brane. Therefore, if we view  $\text{QCD}_4$  as the high temperature theory of the theory in five dimensions,  $\theta$  can be interpreted as the chemical potential for D0 branes. In  $M$  Theory, this is geometrically realized as a rotation of the supersymmetric circle by  $\theta$  as one goes around the supersymmetry breaking circle once. As shown in [10], the  $M$  Theory dual of  $\text{QCD}_4$  with  $\theta = 0$  is the  $AdS_7$  Schwarzschild solution times  $S^4$ , given by

$$N^{-2/3} ds^2 = (u^2 - 1/u^4) d\tau^2 + g_{YM}^2 u^2 d\rho^2 + \frac{du^2}{u^2 - 1/u^4} + u^2 \sum_i dx_i^2 + d\Omega_4^2, \quad (17)$$

with periodicities

$$\begin{aligned}\alpha - \text{cycle} &: (\tau, \rho) \rightarrow (\tau, \rho + 2\pi) \\ \beta - \text{cycle} &: (\tau, \rho) \rightarrow (\tau + 2\pi, \rho).\end{aligned}\tag{18}$$

We can turn on  $\theta$  in this geometry by introducing a twist as

$$\begin{aligned}\alpha - \text{cycle} &: (\tau, \rho) \rightarrow (\tau, \rho + 2\pi) \\ \beta - \text{cycle} &: (\tau, \rho) \rightarrow (\tau + 2\pi, \rho + \theta).\end{aligned}\tag{19}$$

For non-zero  $\theta$ , the  $\beta$ -cycle is not contractible. Correspondingly the monopole condensation does not take place. In fact the  $m\text{-}\bar{m}$  potential would obey the area law in this case. If  $\theta = 2\pi p/q$  for some co-prime integers  $(p, q)$ , however, the cycle  $(q\beta - p\alpha)$  becomes contractible in the Schwarzschild geometry (Fig. 7). A membrane ending on this cycle gives a dyon of an electric charge  $-p$  and a magnetic charge  $q$ , and this dyon is screened since the membrane worldvolume can collapse. In this case, confinement is associated with the condensation of this  $(-p, q)$  dyon, corresponding to oblique confinement.

## 4 Higher Representations

Another interesting generalization is to consider Wilson loops for higher representations of  $SU(N)$ . Using the Frobenius formula,

$$\chi_R(U) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) \prod_{i=1}^{K_\sigma} \text{tr } U^{k_i},\tag{20}$$

one can relate the character  $\chi_R(U)$  of a representation  $R$  to a product of traces in the fundamental representation of  $SU(N)$  as expressed in the right-hand side of the equation. Here  $n$  is the number of boxes of the Young tableau for  $R$ ,  $\chi_R(\sigma)$  is the character of the permutation  $\sigma$  in the representation of the symmetric group  $S_n$  associated to the same Young tableau. A permutation  $\sigma$  can be expressed as a product of cycles, and  $k_1, \dots, k_{K_\sigma}$  are lengths of the cycles. Therefore, the Wilson loops expectation values for higher representations are related to multiply wound loops in the fundamental representation. The latter are computable by studying string worldsheet ending on such loops.

Let us summarize what we expect for the Wilson loops for higher representations from the field theory analysis. In QCD<sub>2</sub>, they are all computed exactly [20] as

$$\langle \chi_R(U) \rangle = (\dim R) \exp(-g_{YM}^2 C_2(R) A(C)), \quad (21)$$

where  $C_2(R)$  is the quadratic Casimir of  $R$  given by

$$C_2(R) = nN + \sum_i r_i^2 - \sum_i c_i^2 - \frac{n^2}{N}, \quad (22)$$

with  $r_i$  and  $c_i$  being the lengths of the rows and the columns of the Young tableau for  $R$ .

It would be instructive to look at some examples. For  $n = 2$ , there are symmetric ( $S$ ) and anti-symmetric ( $A$ ) representations. Their quadratic Casimirs are

$$\begin{aligned} C_2(S) &= 2N + 2 - \frac{4}{N} \\ C_2(A) &= 2N - 2 - \frac{4}{N}. \end{aligned} \quad (23)$$

The Frobenius formula (20) gives

$$\begin{aligned} \chi_S(U) &= \frac{1}{2}[(\text{tr}U)^2 + \text{tr}U^2] \\ \chi_A(U) &= \frac{1}{2}[(\text{tr}U)^2 - \text{tr}U^2], \end{aligned} \quad (24)$$

which can be inverted as

$$\begin{aligned} \langle \text{tr}U^2 \rangle &= \chi_S(U) - \chi_A(U) \\ &= N^2 e^{-2g_{YM}^2 N A(C)} \times \left( -4g_{YM}^2 A + \frac{1}{N} + \dots \right). \end{aligned} \quad (25)$$

In general, (21) combined with the Frobenius formula gives

$$\frac{1}{N^n} \langle \text{tr}U^n \rangle = (-1)^{n-1} g_{YM}^{2n-2} \left( A^{n-1} + O\left(\frac{1}{g_{YM}^2 N}\right) \right) \exp(-ng_{YM}^2 N A(C)). \quad (26)$$

The factor  $(-1)^{n-1} A^{n-1}$  is closely related to the presence of the quadratic Casimir in the string tension and it is reasonable to expect that this factor would appear in other confining theories as well.

The prefactor in (26) has the form of the  $\alpha'$  expansion of the string theory. One might hope therefore to be able to derive the leading term  $(-1)^{n-1} A^{n-1}$  without detailed knowledge of string theory on  $AdS_{p+2}$ . Let us see how far we can go. To evaluate  $\langle \text{tr} U^n \rangle$  using Maldacena's duality, one sets the Wilson loop  $C$  at  $u = \infty$  and considers a string worldsheet whose boundary winds around  $C$   $n$ -times. With this boundary condition, the minimal surface in the  $AdS_{p+1}$  Schwarzschild geometry (5) should look as follows. As in the case of  $n = 1$  in section 2, because of the  $u$ -dependent factor in front of  $\sum_i dx_i^2$ , the worldsheet quickly drops to the horizon. After that, the worldsheet is allowed to spread in the  $\mathbf{R}^p$  direction. For fixed  $u$ , the metric in the  $\mathbf{R}^p$  direction is flat, so that we can use flat space intuition. It is possible to construct a smooth surface whose boundary winds around a circle  $n$ -times, but one can always reduce an area of such a surface by shrinking a part of the surface and creating branch points. The minimal surface constructed in this way should then have a form of  $n$ -disks on the top of each other connected by  $(n - 1)$   $Z_2$ -branch points <sup>2</sup>. The classical action for the minimal surface is then

$$S_{n\text{-winding}} = n(gN)^{\frac{1}{5-p}} T^{\frac{7-p}{5-p}} A(C). \quad (27)$$

To understand the prefactor  $(-1)^{n-1} A^{n-1}$ , we have to go beyond the supergravity approximation. In general, stringy corrections are difficult to control since we know little about the formulation of string theory in a background with Ramond-Ramond (R-R) charges. Fortunately, we only need to study the 0-th order in the  $\alpha'$ -expansion to understand the prefactor, consequently we can neglect the R-R charges as well as the spacetime curvature. Therefore, we can treat the string worldsheet as described by the free fields of the NS/R string. All we have to do then is to compute the disk amplitude of free string theory with  $(n - 1)$  branch points. To the 0-th order in the  $\alpha'$ -expansion, this simply amounts to computing determinants of the Laplace operators on the disk with  $(n - 1)$  branch points and summing over all possible locations of the branch points. The factor  $A^{n-1}$  is then easy to explain as it corresponds to the entropy factor associated to the positions of the branch points.

The sign factor  $(-1)^{n-1}$  is more interesting. It cannot come from the determinant factor since the worldsheet does not have obvious unstable modes.

---

<sup>2</sup>In  $\mathbf{R}^3$ , it is known that a minimal surface bounded by any regular curve is smooth without branch point [21]. The boundary contour in our problem does not satisfy an assumption of this theorem as it winds around the identical circle  $C$   $n$ -times.

We claim that it is a consequence of the GSO projection of the superstring theory. To the 0-th order in  $\alpha'$ , the R-R background does not affect the worldsheet theory and the distinction between NS and R states is well defined.

When  $n = 1$  (no branch point), the fermions around the contour  $C$  obey NS-NS boundary conditions. Let us remind ourselves why this is the case. As a function of a coordinate  $z$  on the disk ( $|z| \leq 1$ ), the fermion  $\psi(z)$  in this case is single valued since there is no singularity in the interior of the disk. To study the boundary condition around the contour  $C$  at  $|z| = 1$ , it is appropriate to use another coordinate  $\theta$  defined by  $z = e^{i\theta}$ . Since the fermion is a spin-1/2 field, we have to multiply the transition function  $\sqrt{dz} = e^{\frac{i}{2}\theta} \sqrt{d\theta}$ . This means that a fermion obeys anti-periodic boundary conditions around  $C$ , i.e. it is in the NS-NS sector.

For  $n > 1$ , we must take into account the presence of the branch points. Near  $|z| = 1$ , we can use the covering coordinate  $w$  which is related to  $z$  by  $z = w^n$ . To change coordinates from  $z$  to  $w$ , we have to multiply the transition function  $\sqrt{dz} = w^{\frac{n-1}{2}} \sqrt{dw}$ . To study the boundary condition around  $C$ , we use the coordinate  $\theta$  defined by  $w = e^{i\theta}$ , and multiply the transition function  $w^{\frac{n-1}{2}} \sqrt{dw} = e^{i\frac{n}{2}\theta} \sqrt{d\theta}$  to the fermions. It is then clear that a fermion will obey anti-periodic or periodic boundary conditions around  $C$  depending on whether  $n$  is odd or even. Therefore, the closed string emitted from the Wilson loop  $C$  is in the NS-NS or in the R-R sector depending on the parity of  $n$ .

It is known that the GSO projection requires that amplitudes in the R-R sector in this case should be multiplied by the sign factor  $(-1)$  relative to that in the NS-NS sector. This was observed, for example, in [22] and was found to be responsible for the cancellation between the exchange forces of the NS-NS and the R-R fields. The string worldsheet in the  $AdS$  Schwarzschild geometry has a quasi-cylindrical region where the worldsheet quickly drops from  $u = \infty$  to the horizon  $u = u_0$  to save the energy. If we look at this region in the open string channel, to reproduce the fermion boundary condition for even  $n$ , one has to insert the fermion number parity operator  $(-1)^F$  as one goes around the loop  $C$ . It is then clear that the GSO projection require the sign factor  $(-1)$  for the corresponding amplitude. It is interesting to note that this result depends critically on the fact that the master field is described by fermionic string. This confirms a long-standing conjecture as to the fermionic nature of the large  $N$  string theory that describes a confining



gauge theory [23, 20].

So far we have discussed the confining case, but the same argument should be applicable to the conformal case as well, leading to the sign factor  $(-1)^{n-1}$  in  $\langle \text{tr} U^n \rangle$ . In order to see whether this is what one naturally expects, it is useful to first point out some puzzling feature of Maldacena's computation of the Wilson loop in the conformal case [7]. For a rectangular Wilson loop of sides  $L$  and  $R$  ( $L \gg R$ ) in the  $\mathcal{N} = 4$  theory in four dimensions, he finds

$$\frac{1}{N} \langle \text{tr} U \rangle = \exp \left( + \frac{4\pi^2}{\Gamma(1/4)^4} \frac{\sqrt{2g_{YM}^2 NL}}{R} \right) \quad (28)$$

for the fundamental representation. Since  $U$  is a unitary matrix, there is an upper bound on its expectation value  $\frac{1}{N} \langle \text{tr} U \rangle \leq 1$  which contradicts with the sign in the exponent in the right-hand side of (28). One possible resolution of this puzzle would be that the unitarity bound is violated due to a renormalization of the operator  $U$ . The renormalization of  $U$  corresponds to the mass renormalization of the quark going around the loop, and we expect it to be zero in the  $\mathcal{N} = 4$  theory. However the nonrenormalization theorem assumes a supersymmetric regularization, which would typically violate the inequality of this type. With an explicit ultraviolet cutoff,  $\Lambda$ , which may break the supersymmetry but preserves the inequality, the Wilson loop expectation value would be

$$\frac{1}{N} \langle \text{tr} U \rangle = \exp \left( -cL\Lambda + \frac{4\pi^2}{\Gamma(1/4)^4} \frac{\sqrt{2g_{YM}^2 NL}}{R} \right) \quad (29)$$

for some positive constant  $c$ . If the mass renormalization  $c\Lambda$  for a representation  $R$  is proportional to its quadratic Casimir, which is reasonable, one would find

$$\langle \text{tr} U^n \rangle \sim (-1)^{n-1} (L^{n-1} + \dots) \exp \left( -nNc_0L\Lambda + \frac{4\pi^2 n}{\Gamma(1/4)^4} \frac{\sqrt{2g_{YM}^2 NL}}{R} \right), \quad (30)$$

obtaining the sign factor  $(-1)^{n-1}$  again. The factor  $L^{n-1}$ , as opposed to  $A^{n-1}$  in the confining case, indicates that the twist operators are constrained to stay along the minimum  $u$  point in the string worldsheet. It would be interesting to understand this phenomenon better from the point of view of the string theory in the  $AdS$  background.

## 5 Baryons and Stringy Exclusion Principle

The  $M$  Theory dual of  $\text{QCD}_4$  can also be used to construct hadrons with heavy quarks and study their properties. To obtain mesons, one simply starts with a pair of quark and anti-quark represented by Wilson lines of opposite orientations on a D4 brane separated from  $N$  D4 branes. The quark anti-quark pair is then connected by a string extended through the  $AdS$  geometry (the Schwarzschild solution or  $AdS$  depending on whether one study the pure QCD or the superconformal theory). The string generates a potential between the quarks, which is either linear (confining case) or is inversely proportional to the distance between the quarks (conformal case). One can then study the non-relativistic quantum mechanics of the quarks in this potential to compute the meson spectrum. This is essentially the same analysis as the Bag model.

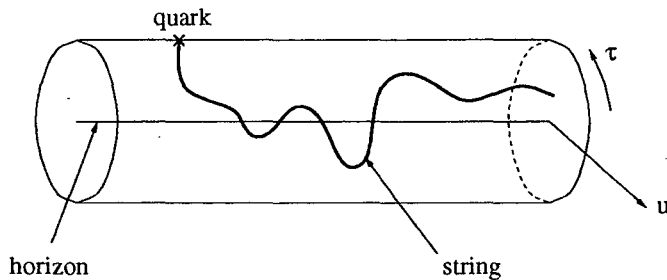


Figure 8: In the Schwarzschild geometry, there is no point where the open string attached to the quark can end.

There is no free quark of finite energy in the confining case; whereas in the conformal case finite mass colored states exist. How do we see this in the supergravity picture? Both are a consequence of the fact that the string attached to the quark must end somewhere. In the conformal case, the string can end at  $u = 0$ . The energy of such a string is simply equal to the BPS mass of the quark [7]. In the confining case, the bulk geometry has the topology of a solid cylinder with the horizon  $u = u_0$  as its axis as shown in Fig. 8. In this case there is no point where the open string can end and the string has to stretch to  $x = \infty$  in the  $\mathbf{R}^p$  direction, costing an infinite amount of

energy.

This raises the interesting question of how one could construct a baryon, a  $SU(N)$  singlet bound state of  $N$  quarks, in this picture. We have to find a way to tie together  $N$  strings emerging from the quarks. In the flat space this is not possible because of the conservation of the NS-NS two-form charge. A quark and an anti-quark can be connected by a string since the NS-NS charge is canceled at the two end points. It is not obvious how this can be done with  $N$  end points with the same charges. It is an amazing consequence of supergravity that it is possible to do so in the  $AdS$  space. In the conformal case, the dual supergravity on  $AdS_5$  has the Chern-Simons term for the  $SL(2, Z)$  doublet two-forms  $B_{\mu\nu}^a$  ( $a = NSNS, RR$ )

$$S_{CS_5} = \frac{N}{2} \epsilon_{ab} \int_{AdS_5} B^a \wedge dB^b. \quad (31)$$

This comes from the fact that the equation of motion for  $B^a$  in ten dimensions contains a coupling to the 5-form [24] as

$$D^\mu \partial_{[\mu, B_{\nu\rho]}^a} = -g \epsilon_{ab} F_{\mu\nu}{}^{\rho\sigma\kappa} \partial_\rho B_{\sigma\kappa}^b. \quad (32)$$

A Lagrangian density which gives such an equation of motion would be<sup>3</sup>

$$\mathcal{L}_B = \frac{1}{2g} |dB^a|^2 + \frac{\epsilon_{ab}}{2} F \wedge B^a \wedge dB^b. \quad (33)$$

On  $AdS_5 \times \mathbf{S}^5$ , the 5-form  $F$  carries  $N$  units of flux on  $\mathbf{S}^5$ . Thus, for  $B^a$ 's which are constant on  $\mathbf{S}^5$ , the ten-dimensional Lagrangian (33) implies the Chern-Simons term (31) in five dimensions. Similarly the 3-form  $C_{\mu\nu\rho}$  in the eleven-dimensional supergravity has the Chern-Simons term  $\int C \wedge dC \wedge dC$ , which upon compactification on  $\mathbf{S}^4$ , gives the seven-dimensional Chern-Simons term on  $AdS_7$

$$S_{CS_7} = \frac{N}{2} \int_{AdS_7} C \wedge dC. \quad (34)$$

As we now explain, these Chern-Simons terms make it possible for the  $N$  strings to combine together and end on a point in the bulk  $AdS$  geometry.

---

<sup>3</sup> Although the complete Lagrangian for the IIB supergravity in ten dimensions is not known, for the purpose of the discussion here, we only need a Lagrangian for the two-form. Since we are only considering the classical supergravity, any Lagrangian which gives (32) should be good enough.

Before explaining why the  $N$  strings can join together in  $AdS$ , it would be instructive to review a similar but more familiar phenomenon in the three-dimensional gauge theory with the Chern-Simons term,

$$S_{CS_3} = \frac{N}{2} \int AdA \quad (35)$$

for an abelian gauge field  $A$ . Suppose the electric charge of the theory is quantized so that we allow a gauge transformation  $A \rightarrow A + d\theta$  with  $\theta$  defined modulo integer. Let us perform a gauge transformation by  $\theta$  which has a discontinuity  $\delta\theta = 1$  across a two-dimensional surface  $D_2$  with the topology of disk. The variation of the Chern-Simons term under this gauge transformation is

$$\begin{aligned} \delta S_{CS_3} &= N \int_{AdS_3} d(\theta dA) \\ &= N \int_{D_2} dA = N \oint_C A, \end{aligned} \quad (36)$$

where  $C$  is the boundary of the disk  $D_2$ . Thus, if there is a Wilson loop  $C$  carrying  $N$  units of charge, we can absorb it into the Chern-Simons term by the gauge transformation. For the same reason, if we put  $N$  particles of unit charge on the top of each other, the composite particle decouples from the gauge field  $A$ .

The above story can be immediately generalized to the case of string with the Chern-Simons term (31). The supergravity action is invariant under the gauge transformation  $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_{[\mu, \lambda_{\nu]}$ . Since the  $B$ -field charges are quantized with the unit charges carried by the fundamental string and the D string, the gauge transformation parameter  $\lambda$  does not have to be a single-valued vector field on  $AdS_5$ , but its integral  $\oint_C \lambda$  around a closed contour  $C$  can jump by an integer amount. Suppose such a discontinuity of  $\lambda$  occurs across a four-dimensional subspace  $M_4$  of  $AdS_5$ . Since the discontinuity  $\delta\lambda$  has to be such that  $\oint_C \delta\lambda$  is an integer and therefore is invariant under smooth deformation of the contour  $C$ ,  $\delta\lambda$  is closed  $d\delta\lambda = 0$  on  $M_4$  and it can locally be written as  $\delta\lambda = d\theta$  with  $\theta$  is defined modulo integers. Under such a gauge transformation, the Chern-Simons term (31) changes as

$$\begin{aligned} \delta S_{CS_5} &= N \epsilon_{ab} \int_{AdS_5} d(\lambda^a dB^b) \\ &= N \epsilon_{ab} \int_{M_4} \delta\lambda^a dB^b = N \epsilon_{ab} \int_{M_4} d(\theta^a dB^b). \end{aligned} \quad (37)$$

Suppose further that  $M_4$  has the topology of  $\mathbf{S}^1 \times M_3$  with  $M_3$  being a three-dimensional space bounded by a two-dimensional surface  $\Sigma$  and that  $\theta^{(RR)}$  jumps by 1 across a point on  $\mathbf{S}^1$  (times  $M_3$ ) while  $\theta^{(NSNS)}$  is continuous. The gauge variation of  $S_{CS_5}$  can then be written as an integral of  $B$  over  $\Sigma$ ,

$$\delta S_{CS_5} = N \int_{\Sigma} B^{(NSNS)}. \quad (38)$$

Therefore  $N$  string worldsheets wrapping on a contractible surface  $\Sigma$  in  $AdS_5$  can be absorbed into the Chern-Simons term (31) by the gauge transformation. This also means that, if we put  $N$  strings on the top of each other, it decouples from  $B^{(NSNS)}$ .

The string can be viewed as a soliton of the supergravity with the  $B^{(NSNS)}$ -charge [25]. The fact that  $N$  strings decouple from  $B^{(NSNS)}$  suggests that one can construct a supergravity solution in which  $N$  strings join together at a point in  $AdS_5$ <sup>4</sup>. This defines a baryon in the conformal case. The baryon in  $QCD_3$  is obtained by simply compactifying this picture on  $\mathbf{S}^1$ . Because of the Chern-Simons term (31),  $N$  strings can end on a point in the Schwarzschild geometry. The resulting string configuration is very similar to the one suggested much earlier in [26].

It is obvious that this phenomenon also holds for closed strings —  $N$  closed strings can join together and disappear in  $AdS_5$ . This may be viewed as a higher dimensional generalization of the stringy exclusion principle pointed out by Maldacena and Strominger [27]. They showed that there is an upper bound on the number of BPS particles in  $AdS_3$ . Although they derived this result by studying the spectrum of chiral primary fields in the dual conformal field theory in two dimensions, it is also possible to show this using the Chern-Simons term for the  $SU(2) \times SU(2)$  gauge field in  $AdS_3$  [28]. The mechanism we described in the above is a natural generalization of this to  $AdS_5$  with the gauge field being replaced by  $B^{(NSNS)}$ .

Let us turn to the case of  $QCD_4$ . To construct baryons in this case, we have to start with a membrane of the supergravity in  $AdS_7$ . Wrapping the membrane on the supersymmetry preserving circle gives a string on the  $AdS_7$  Schwarzschild geometry. The membrane carries a charge with respect to the 3-form  $C$ , and the Chern-Simons term (34) can create a membrane with  $N$

---

<sup>4</sup>While this paper was being typed, we learned of the work [11] where it is shown that  $N$  strings can end on the 5-brane wrapping on  $\mathbf{S}^5$  and localized on  $AdS_5$ . This verifies our claim that such a supergravity solution should exist in  $AdS_5$ .

units of charges. Therefore, once again,  $N$  open strings can join together in the bulk.

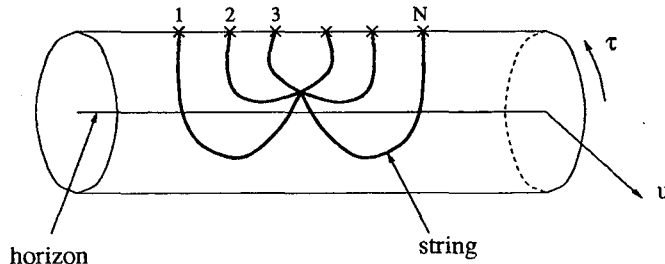


Figure 9: Baryon is constructed as  $N$  strings joining together at a point in the bulk.

Thus, in both the conformal and confining cases, a baryon is constructed as  $N$  heavy quarks joined together by  $N$  open strings ending at a point in the  $AdS$  geometry. The total energy of the baryon is the sum of the geodesic length of the strings. In the confining case, it is proportional to a sum of four-dimensional distances between the quarks and the location of the string junction (projected to  $\mathbf{R}^4$ ). In the conformal case, it is a sum of (distance) $^{-1}$ . Although the location of the string junction is dynamical, in the large  $N$  limit, we can use the Born-Oppenheimer approximation and regard it as a fixed point in the  $AdS$  geometry. The  $N$  quarks then move independently under the potential given by the string stretched between them and the junction. The mass spectrum of the baryon can then be obtained by solving a one-body problem of the quark in the potential.

It would perhaps be worthwhile to point out that the existence of the  $N$  string junction does not contradict with the vanishing of the  $q$ - $q$  potential in the conformal case. If you have a  $q$ - $q$  pair, in addition to the obvious string configuration where a string stretched from each quark ends on  $u = 0$ , one may consider joining them at a point from which  $(N - 2)$  strings come out. This may seem to give a non-trivial potential between the quarks. However the  $(N - 2)$  strings should end somewhere, and the only place they can end is at  $u = 0$ . For  $N > 4$ , it is clear that it costs less energy if we move the  $N$  string junction toward  $u = 0$ . We then end up recovering the obvious

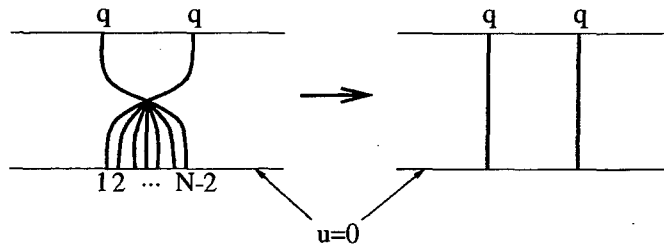


Figure 10: It is energetically favorable to move the  $N$  string junction to  $u = 0$ , leaving the two separate strings.

configuration where the two strings separately end at  $u = 0$ , without a non-trivial potential between  $q$ - $q$  (Fig. 10).

A similar configuration can be considered in the confining case. In this case, however, there is no place where the  $(N - 2)$  strings can end, except at  $x = \infty$  in  $\mathbf{R}^p$  costing infinite energy to create (Fig. 11), as expected from the quark confinement. The same discussion holds for any non-singlet combination of quarks.

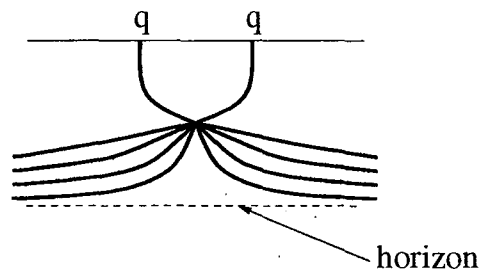


Figure 11: A similar configuration in the confining case costs infinite energy, as expected from the quark confinement.

## 6 Discussion

In the previous sections we have discussed many aspects of the dynamics of large  $N$  gauge theory for strong coupling, as calculated using the dual supergravity, string or  $M$  theory. In all cases the results are remarkably consistent with our intuition and expectations. This strengthens our faith in the validity of the duality between these pictures. However, there is no strong evidence to date that these pictures actually overlap — they could be descriptions of two quite different phases of the same theory.

Consider the duality between the conformally invariant  $\mathcal{N} = 4$  gauge theory in four dimensions and string theory in the  $AdS_5 \times S^5$  background. The strongest form of this duality is to claim that these two are equivalent for all values of  $N$  and  $\lambda$ . However, perturbative string theory is defined as an asymptotic expansion in  $g_s = g_{YM}^2 = \lambda/N$ . So perhaps a safer conjecture is the equivalence of these two formulations for  $N = \infty$ , the classical limit of string theory and the planar limit of the gauge theory, or in the asymptotic  $1/N$  expansion.

Here one is on firmer ground since the weak coupling expansion of the  $N = \infty$  conformally invariant gauge theory might very well converge [29, 30] and there is no reason to expect the  $1/\lambda \sim$  curvature expansion of string theory to diverge. If so, one could then imagine, in principle, using the large  $\lambda$  expansion (or even better the exact solution) of classical string theory to define the gauge theory for all coupling. If one could do this for the compactified non-supersymmetric theory as well, then one could construct continuum QCD, by taking  $\lambda$  to zero à la asymptotic freedom, as one lets the compactification radii vanish. So far there is no strong evidence for even this form of the conjecture; since the calculations of quantities in string theory can only be done for large  $\lambda$  and in the gauge theory only for small  $\lambda$ .

There is, however, a possibility that the conjecture has to be weakened even further. Namely, it is possible that the gauge theory picture is valid for weak coupling, the string theory for strong coupling, and there is no region where they are both valid — *i.e.* there is a phase transition at  $\lambda = \lambda_{cr}$ . This is a ubiquitous phenomenon in large  $N$  theories [31], including the one-plaquette model [32], QCD<sub>2</sub> on the sphere [33, 34] and is believed to be the case (even for finite  $N$ ) for lattice QCD. In the case of lattice QCD such a transition leads one to suspect that the effective string theory that can be deduced from the strong coupling expansion cannot be extended to



the continuum theory. If this were the case here as well, it would mean, unfortunately, that these dualities are not as powerful as one might hope, and in particular one might not be able to use them to construct the master field, string theory, of QCD. On the other hand, the existence of such a phase transition would make the conjectured duality seem more reasonable, by eliminating some of the paradoxical aspects of the duality [35].

### Acknowledgments

We would like to thank Tom Banks, Gary Horowitz, Emil Martinec, Yaron Oz, John Schwarz and many other participants of ITP Program, *Duality in String Theory*, for useful discussions. The work of DG is supported in part by NSF grants NSF-PHY-94-07194 and NSF-PHY-97-22022. The work of HO is supported in part by NSF grant NSF-PHY-95-14797 and by DOE grant DE-AC03-76SF00098, and also by NSF grant NSF-PHY-94-07194.

## References

- [1] J. Maldacena, hep-th/9711200.
- [2] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, hep-th/9802109.
- [3] E. Witten, hep-th/9802150.
- [4] I.R. Klebanov, hep-th/9702076; S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, hep-th/9703040; M.R. Douglas, J. Polchinski and A. Strominger, hep-th/9703031.
- [5] G. Horowitz and H. Ooguri, hep-th/9802116.
- [6] S. Ferrara, C. Fronsdal and A. Zaffaroni, hep-th/9803039; L. Andrianopoli and S. Ferrara, hep-th/9803171.
- [7] J. Maldancena, hep-th/9803002.
- [8] S.-J. Rey and J. Yee, hep-th/9803001
- [9] J. A. Minahan, hep-th/9803111.
- [10] E. Witten, hep-th/9803131.
- [11] E. Witten, hep-th/9805112.
- [12] A. Brandhuber, N. Itzhaki, J. Sonnenschein and S. Yankielowicz, hep-th/9803138; hep-th/9803263.
- [13] S.-J. Rey, S. Theisen and J. Yee, hep-th/9803135.
- [14] M. Li, hep-th/9803252; hep-th/9804175.
- [15] S. Hawking and D. Page, Commun. Math. Phys. **87** (1983) 577.
- [16] N. Itzhaki, J. Maldacena, J. Sonnenschein and S. Yankielowicz, hep-th/9802042.
- [17] D.-E. Diaconescu, hep-th/9608163, Nucl. Phys. **B503** (1997) 220.
- [18] A. Hanany and E. Witten, hep-th/9611230, Nucl. Phys. **B492** (1997) 152.

- [19] C. Csaki, H. Ooguri, Y. Oz, H. Robins, J. Tannenhauser, J. Terning, work in progress.
- [20] D.J. Gross and W. Taylor IV, hep-th/9303046, Nucl. Phys. **B403** (1993) 395.
- [21] R. Osserman, Ann. Math. **91** (1970) 550.
- [22] J. Polchinski, hep-th/9510017, Phys. Rev. Lett, **75** (1995) 4724.
- [23] A. A. Migdal, Phys. Rep. 102 (1983) 199.
- [24] J. H. Schwarz, Nucl. Phys. B226 (1983) 269.
- [25] A. Dabholkar, G. Gibbons, J. A. Harvey and F. Ruiz Ruiz, Nucl. Phys. B340 (1990) 33.
- [26] E. Witten, Nucl. Phys. B160 (1979) 57.
- [27] J. Maldacena and A. Strominger, hep-th/9804085.
- [28] E. Martinec, hep-th/9804111; Talk given at ITP Program, *Duality in String Theory*, <http://www.itp.ucsb.edu/online/dual/martinec1>.
- [29] G. 't Hooft, "Planar Diagram Field Theories," in *Progress in Gauge Field Theory*, NATO Adv. Study Inst. Series, eds. G. 't Hooft et al, (Plenum, 1984).
- [30] T. Banks and M. Green, hep-th/9804170.
- [31] E. Brezin and D. J. Gross, Phys. Lett. B97 (1980) 120.
- [32] D. J. Gross and E. Witten, Phys. Rev. D21 (1980) 446.
- [33] M. R. Douglas and V. A. Kazakov, hep-th/9305047, Phys. Lett. B319 (1993) 219.
- [34] D. J. Gross and A. Matytsin, Nucl. Phys. B419 (1994) 50, hep-th/9404004.
- [35] C. Vafa, hep-th/9804172.

**ERNEST ORLANDO LAWRENCE BERKELEY NATIONAL LABORATORY**  
**ONE CYCLOTRON ROAD | BERKELEY, CALIFORNIA 94720**