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A THEOREM ON INVARIANT AMPLITUDES*

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ABSTRACT

It is shown that any basis of covariant polynomials for a two particle scattering process yields invariant amplitudes free of kinematical singularities, provided (a) the total number of basis polynomials equals the number of spin space components of the scattering amplitude, and (b) the polynomials of each of the two parity signatures are separately linearly independent at all points where three of the particle four-momenta are linearly independent. This result allows one to directly identify good basis sets without going through the very tedious algebra involved in comparing them to the sets of Hepp and Williams. The latter are not useful for practical applications because the spinor indices belonging to different particles are coupled and these sets do not transform into themselves under the relevant discrete symmetry operations.

I. INTRODUCTION

The description of a scattering process in terms of invariant amplitudes is useful because they have simple analytic properties and can trivially satisfy the various symmetry requirements. For this purpose one must find a covariant basis that exhibits the appropriate symmetries and moreover ensures a decomposition into invariant amplitudes free of kinematical singularities. In this paper we prove a theorem that enables one to check by inspection whether a given basis is a satisfactory one.

Hepp[1] has rigorously proven that any set of holomorphic covariant functions $M(k)$ can be written in the form

$$M(k) = \sum_i A_i(s)\, Y^i(k) ,$$

where the "standard covariants" $Y^i(k)$ are matrices on the spinor indices and polynomials in the four-vectors $k = \{k_1,\cdots,k_n\}$. The invariant amplitudes $A_i(s)$, which are functions of the independent invariants $s = \{s_1,\cdots,s_m\}$ formed from the k_i's, are holomorphic except on the s-space image of k-space singularities of the functions $M(k)$. In general, a minimal set of standard covariants, i.e., a set whose number is just the number of values that the spinor indices take on, does not exist. But for the case of scattering amplitudes describing two incoming and two outgoing particles, and subject to the mass shell and four-momentum conservation constraints, Hepp proved that one can find a minimal set of $Y^i(k)$'s having positive and

negative signature under parity and whatever other discrete symmetry operations transform the particular process under consideration into itself.

To find the relevant standard covariants for any given $M(k)$ one must find a set of polynomials such that all covariant polynomials with the same spinor index types can be expressed in terms of them for all allowed values of the complex four-vectors k. Hepp's proof is, in effect, a proof that such a basis exists and that any such basis gives a holomorphic decomposition (one free of kinematical singularities) into invariant amplitudes of the given holomorphic functions $M(k)$. Any other form for physical scattering functions is equivalent to the $2j + 1$ spinor form, so Hepp has given a rigorous justification of the prescription originally developed by Hearn[2] for perturbation theory. Recently Scadron and Jones[3] have found some of the relations needed to apply this method to two-particle processes with arbitrary spins, and many other relations have been found by the author of this paper.[4]

Several examples of sets of covariants that give a holomorphic decomposition have been given by Hepp and independently by Williams,[5] but these do not transform into themselves under the discrete symmetry operations. To find a satisfactory basis it is sufficient to show how any one of these sets can be written as linear combinations of covariants having definite signature under P, C, T, and exchange symmetry, with coefficients that are polynomials in the invariants. For those processes in which two of the four particles are spinless this procedure is not too tedious and all the required covariants have been found.[4,6,7]

When more than two spinning particles are involved, the algebraic problems become rapidly unmanageable as the spins increase.[3,4,7,8] Thus, although the prescription of Hearn and Hepp solves the problem in principle, it does not solve it in practice.

The main aim of this work is to exhibit, in concrete terms, the origin of the difficulty in obtaining holomorphic invariant amplitudes and to develop a simple criterion that allows one to verify directly whether any given basis gives amplitudes that are free of kinematical singularities. The criterion is essentially this: any minimal set of covariant polynomials is a good set if all the covariants of positive parity signature are linearly independent and all the covariants of negative parity signature are linearly independent at all points where three of the particle momenta are linearly independent.

The core of the argument is a detailed description of the constraints that Lorentz covariance and analyticity impose in neighborhoods of points where only two of the particle four-momenta are linearly independent. Once the nature of the constraints is clearly understood, the essential requirements on the standard covariants will be easy to see and the proof of the theorem will be straightforward.

Aside from the Hearn-Hepp procedure, the only good criterion that has been previously given for justifying the absence of kinematical singularities in a set of invariant amplitudes is that of Williams.[5] However, Williams' arguments were for a particular set of covariants that has no simple relation to the discrete symmetries.

One reason that the standard covariants given by Hepp and Williams have not found practical application is that all spinor indices with the same properties under proper Lorentz transformations are coupled with Clebsch-Gordan coefficients.[9] These couplings join together parts having different discrete symmetry properties and also make the results impracticable for substitution into the unitarity relations. Any useful basis must avoid coupling spinor indices belonging to different particles, and this does not naturally emerge if one starts from the sets of Hepp or Williams. By our simple criterion, any proposed basis can be checked directly without comparing it to all the other polynomials having the same spinor index types or to the sets of Hepp or Williams.

To apply our criterion the linear independence of the basis sets of each parity signature must be checked. This problem, although not trivial, is simpler than the one involved in previous methods. A paper dealing with procedures for checking the linear independence of basis sets is being prepared.

The general background material needed for this work is scattered in various places and some of it is unpublished. Many of the essential results are obscured by the abstruse mathematical forms of the original papers, and there is no satisfactory account of the general situation anywhere in the literature. A secondary aim of this paper is therefore to provide a systematic simple description of the previous results that form the basis of the present work. These include a theorem of Stapp,[10,11] which states that scattering functions are covariant under proper

complex Lorentz transformations at all points where they are regular, and some basic properties of sets of complex four-vectors.[12] We also discuss and extend a theorem of Hepp[1] and Williams[5] that allows one to express a multisheeted invariant function of four-vectors as a function of invariants formed from them.

We confine our discussion to those cases in which all four particles are massive, since Zwanziger[13] has found all the standard covariants having definite parity signature when one or two are massless.

out of the physical region of the original process. This, in turn, leads to somewhat complicated crossing relations between the amplitudes for the various physical processes that are connected by analytic continuation. To simplify the crossing, analytic, and Lorentz transformation properties it is convenient to introduce the spinor amplitudes, or M functions.[14-17]

The M functions with lower undotted spinor indices assigned to outgoing particles and lower dotted indices to incoming particles are defined by

$$M\begin{pmatrix} K_b; & K_a \\ \alpha_b; & \dot{\alpha}_a \end{pmatrix} = D^{(j_b)}\left(\mathcal{B}(k_b)\right)_{\alpha_b}{}^{\alpha'_b} D^{(j_a)}\left(\mathcal{B}^*(k_a)\right)_{\dot{\alpha}_a}{}^{\dot{\alpha}'_a} S\begin{pmatrix} K_b; & K_a \\ \alpha'_b; & \dot{\alpha}'_a \end{pmatrix}, \quad (4)$$

where $\mathcal{B}(k_i)$, defined in (A.9), is the Hermitian matrix in SL(2,C) corresponding to the Hermitian "boost" $L(k_i)$ in (A.8) that carries $\bar{k}_i = (m_i,0)$ into k_i. Then, using $\Lambda = \Lambda(A,A^*)$, the following simple covariance property is easily verified from (3), (4), and (A.11):

$$M\begin{pmatrix} AK_b; & AK_a \\ \alpha_b; & \dot{\alpha}_a \end{pmatrix} = D^{(j_b)}(A)_{\alpha_b}{}^{\alpha'_b} D^{(j_a)}(A^*)_{\dot{\alpha}_a}{}^{\dot{\alpha}'_a} M\begin{pmatrix} K_b; & K_a \\ \alpha'_b; & \dot{\alpha}'_a \end{pmatrix} . \quad (5)$$

One can use instead the M functions with all lower undotted indices. It may be verified from (5), (A.4), and

$$M\begin{pmatrix} K_b; & K_a \\ \alpha_b; & \alpha_a \end{pmatrix} = D^{(j_a)}(\sigma\cdot k_a C^{-1}/m_a)_{\alpha_a}{}^{\dot{\alpha}'_a} M\begin{pmatrix} K_b; & K_a \\ \alpha_b; & \dot{\alpha}'_a \end{pmatrix} \quad (6)$$

that

$$M(\Lambda K_b;\ \Lambda K_a)_{\alpha_b;\ \alpha_a} = D^{(j_b)}(A)_{\alpha_b}{}^{\alpha'_b}\, D^{(j_a)}(A)_{\alpha_a}{}^{\alpha'_a}\, M(K_b;\ K_a)_{\alpha'_b;\ \alpha'_a}\ . \tag{7}$$

The choice of index type has no effect on the physics or on the analytic properties. The matrix that transforms a particular spinor index from one type to another is a holomorphic function of that particle's four-momentum.

For a given process, $M(K_b;\ K_a)$, where we suppress the spinor indices, is expressed as a sum of a "no-scattering" part $M^{ns}(K_b;\ K_a)$ and a "connected part"[18]

$$M^0(K_b;\ K_a) = i\,\delta^4(k_1 + k_2 - k_3 - k_4)\, M^c(K_b;\ K_a)\ . \tag{8}$$

Equation (8) defines M^c functions that are free of conservation delta functions. To avoid specifying spinor index types one may write their physical region covariance properties, of which (5) and (7) are particular examples, in the form

$$M^c(\Lambda(A,A^*)K) = \Lambda_s(A,A^*)\, M^c(K), \tag{9}$$

where $K = (K_b;\ K_a)$. The M^c functions are said to be $L_+^\uparrow$ covariant in the physical region of the process $t_1 + t_2 \to t_3 + t_4$. If all four particles are spinless, then $\Lambda_s(A,A^*) \equiv I$ on the right-hand side of (9) and the single M^c function is said to be $L_+^\uparrow$ invariant in the given physical region.

III. PROPERTIES IN COMPLEX FOUR-MOMENTA SPACE

A. Stapp's Theorem

Let $\mathcal{K}_3$ denote the set of points $k = \{k_1, k_2, k_3, k_4\}$ in complex four-vector space that satisfy the four mass-shell constraints $k_i \cdot k_i = (m_i)^2 > 0$ and the four conservation-law constraints $k_1 + k_2 = k_3 + k_4$. The M^c functions defined by (8) are assumed in S-matrix theory to be analytic functions over $\mathcal{K}_3$ except for dynamical singularities.[10,14,19] The set $\mathcal{K}_3$ is a subset of the space of 4 complex four-vectors. The notion of analyticity on such a subset is a standard mathematical concept.[20]

As usual we use the term holomorphic to designate the property of being analytic and single valued. A domain is a connected open set and, for our purposes, it is sufficient to regard the "domain of holomorphy" of a function as a union of sheets, a sheet being the maximum domain whose points map one-to-one onto points in $\mathcal{K}_3$.[21] The sheets may overlap; in fact, the boundaries of the sheets are somewhat arbitrary, and any point on the domain of holomorphy lies on the interior of some sheet. One does not include poles and branch points on any sheet, even though it is customary in physics to speak of a pole as lying on a particular sheet when it lies on the boundary of that sheet.

For a given scattering process there is an M^c function corresponding to each combination of values of the spinor indices. Following Stapp[10,11] we define the <u>domain of regularity</u>, $\mathcal{R}$, of this set of M^c functions to be the intersection of their domains of

holomorphy; i.e., $\mathcal{R}$ is the largest (multisheeted) domain to which all of these M^c functions can be simultaneously analytically continued. The M^c functions are known to be $L_+^\uparrow$ covariant [see Eq. (9)] at points lying in the physical region for which they were originally defined, but what can be said about covariance at other points on $\mathcal{R}$? Before stating Stapp's theorem, which gives a precise answer to our question, we require a few definitions.

A set of tensor-valued functions $F(k)$ is said to be $\mathcal{L}_+$ covariant on a set of points $\mathcal{S} \in \mathcal{K}_3$ if

$$F\big(\Lambda(A,B)k\big) = \Lambda_s(A,B)\, F(k) \tag{10}$$

whenever k and $\Lambda(A,B)k$ are in $\mathcal{S}$. As explained in Appendix A, the matrices A and B are in $SL(2,C)$ and $\Lambda(A,B) \in \mathcal{L}_+$, the group of proper complex Lorentz transformations. The notation on the right-hand side of (10) indicates that the matrix $D^{(j_i)}(A)$ acts from the left on the ith spinor index of $F(k)$ if it is a lower undotted one, whereas the matrix $D^{(j_i)}(B)$ acts on it from the left if it is a lower dotted one.

The $\mathcal{L}_+$ orbit, $\mathcal{L}_+k$, of any point $k \in \mathcal{K}_3$ is the set of all points $\Lambda k = \{\Lambda k_1, \cdots, \Lambda k_4\}$ obtained by letting $\Lambda \in \mathcal{L}_+$ take on all possible values.

Stapp's Theorem:[10,11,22] Let the M^c functions for a given process be $L_+^\uparrow$ covariant and holomorphic on a real open connected set in $\mathcal{K}_3$ corresponding to physical points. Then the domain of regularity, $\mathcal{R}$, of these M^c functions has the following properties:

S1. $\mathcal{R}$ is a union of $\mathcal{L}_+$-invariant sheets $\mathcal{U}_a$; i.e., if the image in $\mathcal{K}_3$ of a sheet $\mathcal{U}_a$ contains a point k then this image contains all points on $\mathcal{L}_+ k$.

S2. The M^c functions are $\mathcal{L}_+$ covariant on each sheet $\mathcal{U}_a$. That is, if the image in $\mathcal{K}_3$ of a sheet $\mathcal{U}_a$ contains a point k, then

$$M^c\big(\Lambda(A,b)K\big) = \Lambda_s(A,B)\, M^c(K) \tag{11}$$

for each $\Lambda(A,B) \in \mathcal{L}_+$.

S3. Any bounded connected set of physical points on $\mathcal{R}$ is contained in some single $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a$.[23]

For a process involving four spinless particles one may replace $\mathcal{L}_+$ covariant by $\mathcal{L}_+$ invariant in statement S2. A function F(k) is $\mathcal{L}_+$ invariant on a set of points $\mathcal{S}$ in $\mathcal{K}_3$ if it satisfies (10) with $\Lambda_s(A,B) \equiv I$ whenever k and $\Lambda(A,B)k$ are in $\mathcal{S}$.

B. $\mathcal{L}_+$ Orbits in $\mathcal{K}_3$

We intend to investigate the constraints that $\mathcal{L}_+$ covariance, as specified by Stapp's theorem, imposes at certain points on the domain of regularity of the M^c functions. But first we need some properties of points k in $\mathcal{K}_3$ and their $\mathcal{L}_+$ orbits.

At any point $k = \{k_1, \cdots, k_4\}$ in $\mathcal{K}_3$ we may define the scalar invariants $k_i \cdot k_j$. For the case of an arbitrary number of particles, one should also consider the pseudoscalar invariants formed by contracting the completely antisymmetric tensor $\epsilon^{\mu\nu\lambda\rho}$ with the four momenta. The invariants taken together are then referred to as

$\mathcal{L}_+$ invariants, since they are invariant under any $\Lambda \in \mathcal{L}_+$, while the scalars alone are referred to as $\mathcal{L}$ invariants, since they are invariant under any $\Lambda \in \mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$, where $\mathcal{L}_-$ is the set of improper Lorentz transformations. In the case under consideration the pseudoscalars vanish identically, because four-momentum conservation allows at most three of the momenta to be linearly independent at any point. Consequently, it makes no difference whether we refer to two distinct points as having the same $\mathcal{L}_+$ invariants or the same $\mathcal{L}$ invariants, and we will use the former of these two terms.

All points on the same $\mathcal{L}_+$ orbit have the same $\mathcal{L}_+$ invariants. But one cannot always specify orbits by the values of their invariants, since two distinct points with the same $\mathcal{L}_+$ invariants do not necessarily lie on the same $\mathcal{L}_+$ orbit, as will be discussed below.

Consider any real or complex point $k = \{k_1, \cdots, k_n, k_{n+1}, \cdots, k_4\}$, where the vectors are ordered so that the first n are linearly independent at the point under consideration. Because of four-momentum conservation, $n \leq 3$, and, because the mass shell condition prevents the four-momenta from vanishing identically, $n \geq 1$. Let us define the Gram determinant

$$G(k_1,k_2,k_3) = \det \begin{pmatrix} (m_1)^2 & k_1 \cdot k_2 & k_1 \cdot k_3 \\ k_2 \cdot k_1 & (m_2)^2 & k_2 \cdot k_3 \\ k_3 \cdot k_1 & k_3 \cdot k_2 & (m_3)^2 \end{pmatrix} , \qquad (12)$$

and let r be the rank of this determinant at the point k. Hall and Wightman[12] gave the following relationship between the rank of the Gram determinant at any point and the number of linearly independent four-vectors at that point:

r	n
3	3
2	2 or 3
1	1 or 2

The possibility of having $n > r$ is, as will also be seen below, a consequence of the fact that one can have complex lightlike vectors in the space orthogonal to the first n vectors when $n < 3$.

From the considerations of Hall and Wightman regarding the properties of complex four-vectors we can make the following remarks about points and orbits in $\mathcal{K}_3$:

H1. (a) At any $r = n = 3$, 2, or 1 point, one can write

$$k_i = \sum_{j=1}^{r} a_{ij} k_j, \qquad \text{for } i=r+1,\cdots,3 , \tag{13}$$

where the a_{ij}'s are finite scalar coefficients. (Recall that the first $r = n$ vectors are linearly independent at the given point and that k_4 is globally determined by four-momentum conservation.)

(b) If k and k' are any two $r = n$ points with the same $\mathcal{L}_+$ invariants, they lie on the same $\mathcal{L}_+$ orbit.

H2. (a) There exist $r = 2$, $n = 3$ points with the same $\mathcal{L}_+$ invariants as any given $r = n = 2$ point. For example, consider the

point k determined by (13) with $r = 2$. In the spatial direction orthogonal to the two linearly independent vectors k_1 and k_2 one may define real unit space-like vectors $\hat{e}^1$ and $\hat{e}^2$ such that $\hat{e}^1 \cdot \hat{e}^2 = 0$. Then define

$$\omega_{\pm} = \hat{e}^1 \pm i\,\hat{e}^2 . \tag{14}$$

It follows that $\omega_{\pm} \cdot k_1 = \omega_{\pm} \cdot k_2 = \omega_{\pm} \cdot \omega_{\pm} = 0$. Now consider two points $k^{(+)}$ and $k^{(-)}$ such that $k^{(\pm)} = \{k_1, k_2, k_3^{(\pm)}, k_4^{(\pm)}\}$, where

$$k_3^{(\pm)} = k_3 + C\omega_{\pm} = \sum_{i=1}^{2} a_{3i} k_i + C\omega_{\pm} . \tag{15}$$

Here k_1, k_2, and k_3 are the same as for the $r = n = 2$ point in (13), while $C \neq 0$ is an arbitrary real or complex number. The points $k^{(+)}$ and $k^{(-)}$ are two distinct $n = 3$ points with the same $\mathcal{L}_+$ invariants as the $r = n = 2$ point obtained by putting $C = 0$ in (15). They are related by an improper Lorentz transformation that changes ω_+ into ω_-, while leaving k_1 and k_2 the same.

(b) The points $k^{(+)}$ and $k^{(-)}$ determined by (15) and the $r = n = 2$ point obtained by putting $C = 0$ in that equation all lie on different $\mathcal{L}_+$ orbits. Any other $r = 2$, $n = 3$ point with the same $\mathcal{L}_+$ invariants lies on either the $\mathcal{L}_+$ orbit of $k^{(+)}$ or the $\mathcal{L}_+$ orbit of $k^{(-)}$. Any point on one of these two orbits is related to any point on the other by means of an improper Lorentz transformation. As a consequence of Hl.(b), we may state that, for any set of values of the $\mathcal{L}_+$ invariants for which the rank of the Gram determinant is 2, there exist three different $\mathcal{L}_+$ orbits.

(c) Consider the limit $C \to 0$ in (15), which yields an $r = n = 2$ point with the same $\mathcal{L}_+$ invariants. This means that any neighborhood of an $r = n = 2$ point contains points of every $r = 2$, $n = 3$ orbit with the same $\mathcal{L}_+$ invariants.

H3. (a) Similar remarks enable one to construct an infinite number of $r = 1$, $n = 2$ orbits with limit points on a given $r = n = 1$ orbit. In this case, however, any two $r = 1$, $n = 2$ points related by an improper Lorentz transformation lie on the same $\mathcal{L}_+$ orbit.

(b) The occurrence of $r = 1$ points in $\mathcal{K}_3$ is possible only if the sum of some of the masses equals the sum of the others. This follows from the conservation of energy and the fact that every $r = n = 1$ orbit contains a point of the form $k_i = (\pm m_i, 0)$ for all $k_i \in k$.

H4. We define the little group $\mathcal{G}_+(k)$ of a point k to be the set of proper complex Lorentz transformations that leave k invariant; i.e., $\Lambda \in \mathcal{G}_+(k) \to \det \Lambda = 1$ and $\Lambda k = k$. At any point k with $n = 3$, the only matrix in $\mathcal{G}_+(k)$ is the unit matrix. However, if $r = n \leq 2$, $\mathcal{G}_+(k)$ is an infinite set.

C. The I_+-Saturated Kernel of the Domain of Regularity

We emphasize the fact that to a given point $k \in \mathcal{K}_3$ there can correspond many points on the domain of regularity, $\mathcal{R}$, of the M^c functions for a given process, although at most one point on any sheet $\mathcal{U}_a \subset \mathcal{R}$. In the remainder of this paper when we speak of a point k with certain values of r and n lying on $\mathcal{R}$ we actually mean a point on $\mathcal{R}$ whose image k in $\mathcal{K}_3$ has these values of r and n.

Thus Stapp's theorem in Part A of this section says essentially that (a) $\mathcal{R}$ is a union of $\mathcal{L}_+$ orbits; i.e., if a point k lies on $\mathcal{R}$ there is an $\mathcal{L}_+$ orbit, $\mathcal{L}_+k$, that lies on $\mathcal{R}$ and contains the given point, and (b) the M^c functions are $\mathcal{L}_+$ covariant on $\mathcal{R}$; i.e., the M^c functions at any two points k and $\Lambda(A,B)k$ on the same $\mathcal{L}_+$ orbit on $\mathcal{R}$ are related by (11). [If all four particles are spinless the M^c function is $\mathcal{L}_+$ invariant on $\mathcal{R}$; i.e., $\Lambda_s(A,B) = I$ in (11).]

If an $r = n < 3$ point k' lies on $\mathcal{R}$, then every $r \neq n$ orbit for which it is a limit point lies on $\mathcal{R}$. This is because there is some full neighborhood $\mathcal{N}$ in $\mathcal{R}$ of the point such that there is a one-to-one mapping between points in $\mathcal{N}$ and points in a neighborhood $N(k') \epsilon \mathcal{K}_3$--according to H.2 (c) and H.3 (a) $N(k')$ contains points of every $r \neq n$ orbit in $\mathcal{K}_3$ for which the $r = n$ point $k' \epsilon N(k')$ is a limit point. By Stapp's theorem $\mathcal{R}$ must contain the full $\mathcal{L}_+$ orbit of any point in $\mathcal{N}$.

If an $r \neq n$ point lies on $\mathcal{R}$, the $r = n$ limit points of its $\mathcal{L}_+$ orbit do not necessarily lie on $\mathcal{R}$.[1] The <u>I_+-saturated kernel</u>, $\mathcal{R}^{(+)}$, of $\mathcal{R}$ is the subset obtained by deleting from $\mathcal{R}$ all $r \neq n$ orbits whose $r = n$ limit points do not lie on $\mathcal{R}$.[1,5,11] All physical points on $\mathcal{R}$ lie on $\mathcal{R}^{(+)}$ because their image in $\mathcal{K}_3$ is real and the construction in (14) and (15) shows that $r \neq n$ points in $\mathcal{K}_3$ are always complex.

In the remainder of this paper the symbol $\mathcal{U}_a^{(+)}$ designates the set $\mathcal{U}_a \cap \mathcal{R}^{(+)}$, where $\mathcal{U}_a$ is some $\mathcal{L}_+$-invariant sheet on

$\mathcal{R}$; i.e., $\mathcal{U}_a^{(+)}$ is the set obtained by deleting from $\mathcal{U}_a$ all $r \neq n$ orbits whose $r = n$ limit points do not lie on $\mathcal{R}$. We refer to $\mathcal{U}_a^{(+)}$ as the I_+-saturated kernel of the $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a$.[25]

D. Kinematical Restrictions

The number of M^c functions for a given process is the same as the number of different combinations of values of the spinor indices. This is given by

$$N = \prod_{i=1}^{4} (2j_i + 1). \tag{16}$$

Equation (16) gives the number of independent scattering experiments at a fixed physical value of the four-momenta, at least on a dense subset of the physical points. Of course for the M^c functions under consideration discrete symmetries can lead to a relation between the results of various experiments, so that the number that are independently determined is less than the number in (16). Such restrictions will be ignored in this section and will be the concern of the next one.

We will show that at any point on $\mathcal{R}^{(+)}$ at which the rank of the Gram determinant is less than 3, $\mathcal{L}_+$ covariance leads to linear relationships between the M^c functions; i.e., there are kinematical restrictions at such a point.[26] For any $r = n = 2$ point k on $\mathcal{R}$ this statement follows by letting $\Lambda(\bar{A},\bar{B})$ be a matrix in the little group, $\mathcal{G}_+(k)$, defined in remark H4. Then the $\mathcal{L}_+$ covariance relation (11) becomes

$$M^c(K) = \Lambda_s(\bar{A},\bar{B})\, M^c(K). \tag{17}$$

We will also find kinematical restrictions at any $r = 2$, $n = 3$ point lying on $\mathcal{R}^{(+)}$. The trivial extension of our results to $r = 1$ points will not be needed in this paper.

We restrict our attention to a single point on any given orbit on $\mathcal{R}^{(+)}$, since the number of M^c functions whose values are independent is the same at all points on the orbit. Any $r = n = 2$ orbit on $\mathcal{R}$ contains a point whose image $k \in \mathcal{K}_3$ is such that each of the vectors $k_i \epsilon k$ has no components along the 1 and 2 axes. Then, from (A.1),

$$\sigma \cdot k_i = \begin{pmatrix} (k_i)^0 + (k_i)^3 & 0 \\ 0 & (k_i)^0 - (k_i)^3 \end{pmatrix} , \quad i=1,\cdots,4 . \tag{18}$$

Let us now make the following choice for the matrices $\bar{A}$ and $\bar{B} \in SL(2,C)$, where λ is any complex parameter:

$$\bar{A} = \begin{pmatrix} \exp(-\lambda/2) & 0 \\ 0 & \exp(\lambda/2) \end{pmatrix} , \tag{19a}$$

$$\bar{B} = \begin{pmatrix} \exp(\lambda/2) & 0 \\ 0 & \exp(-\lambda/2) \end{pmatrix} . \tag{19b}$$

From (A.2) it follows that $\Lambda(\bar{A},\bar{B})\, k_i = k_i$, for all $i = 1,\cdots,4$, when k is a point in $\mathcal{K}_3$ of the form in (18), so $\Lambda(\bar{A},\bar{B})$ belongs to the little group $\mathcal{G}_+(k)$.

Because of (A.3), we have,

$$D^{(j_i)}(\bar{A}) = \exp[-\lambda J_3^{(j_i)}] , \tag{20a}$$

$$D^{(j_i)}(\bar{B}) = \exp[\lambda J_3^{(j_i)}] . \tag{20b}$$

Suppose that the particles are ordered so that the first ℓ have lower undotted spinor indices and the remaining $4-\ell$ have lower dotted ones, the assignment of index types to individual particles being completely arbitrary. Then, because of (17) and (20), we have, at any point k of the form (18) lying on $\mathcal{R}$,

$$M^c_{(\alpha)(\dot{\beta})}(K) = \exp\left[-\lambda\left(\sum_{s=1}^{\ell}\alpha_s - \sum_{t=\ell+1}^{4}\beta_t\right)\right] M^c_{(\alpha)(\dot{\beta})}(K) , \tag{21}$$

where $(\alpha) = \alpha_1\cdots\alpha_\ell$ and $(\dot{\beta}) = \dot{\beta}_{\ell+1}\cdots\dot{\beta}_4$ are not to be confused with the outgoing and incoming spinor indices of the preceding section. Equation (21) requires that

$$M^c_{(\alpha)(\dot{\beta})}(K) = 0 \quad \text{if} \quad \sum_s \alpha_s - \sum_t \beta_t \neq 0 . \tag{22}$$

Therefore, the M^c functions have "kinematical zeroes" at any such point. At other $r = n = 2$ points the relationship among the values of the M^c functions, as given by (17), will be more complicated, but the number of such linear relationships will be the same as the number of "zeroes" in (22). It is convenient to continue to use the term "kinematical zeroes" to refer to the restrictions at these latter points.

Now consider any $r = 2$, $n = 3$ orbit on $\mathcal{R}^{(+)}$ for which the $r = n = 2$ point k on $\mathcal{R}$ that we have just considered is a limit point. There is a point on the orbit whose image $k^{(+)}$ or $k^{(-)}$ in $\mathcal{K}_3$ is such that each vector $k_i^{(\pm)} \in k^{(\pm)}$ has the form $k_i^{(\pm)} = k_i + C_i \omega_{\pm}$, where k_i is given by (18), C_i is a real or complex number, and $\omega_{\pm} = \hat{e}^1 \pm i\,\hat{e}^2$ is a complex light-like vector in the space orthogonal to the k_i. Choosing the real space-like vectors $\hat{e}^1$ and $\hat{e}^2$ to be parallel to the 1 and 2 axes respectively, we have from (18) and (A.1)

$$\sigma \cdot k_i^{(\pm)} = \begin{pmatrix} (k_i)^0 + (k_i)^3 & (1 \pm 1)C_i \\ (1 \mp 1)C_i & (k_i)^0 - (k_i)^3 \end{pmatrix}. \qquad (23)$$

Using $\Lambda(\bar{A},\bar{B})$ defined by (19) and (A.2), but restricting ourselves to real values of λ, we find that

$$\lim_{\lambda \to \pm\infty} \Lambda(\bar{A},\bar{B}) k_i^{(\pm)} = k_i; \qquad i=1,\cdots,4, \qquad (24)$$

with k_i given by (18). Since the M^c functions are continuous at the point k on $\mathcal{R}$, we have, using (11), (20), (24), and the obvious notation $K_i^{(\pm)} = (k_i^{(\pm)}, t_i)$,

$$\lim_{\lambda \to \pm\infty} M^c_{(\alpha)(\dot\beta)}\left(\Lambda(\bar A,\bar B)K^{(\pm)}\right) = M^c_{(\alpha)(\dot\beta)}(K)$$

$$= \lim_{\lambda \to \pm\infty} \exp\left[-\lambda\left(\sum_s \alpha_s - \sum_t \beta_t\right)\right] M^c_{(\alpha)(\dot\beta)}\left(K^{(\pm)}\right) \quad . \tag{25}$$

In particular, (25) implies that

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(\pm)}\right) = M^c_{(\alpha)(\dot\beta)}(K) \quad \text{if} \quad \sum_s \alpha_s = \sum_t \beta_t . \tag{26}$$

Furthermore, for the limit in (25) to be consistent with the result in (22) for the $r = n = 2$ point k on $\mathcal{R}$ we must have

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(+)}\right) = 0 \quad \text{if} \quad \sum_s \alpha_s < \sum_t \beta_t \ , \tag{27a}$$

$$M^c_{(\alpha)(\dot\beta)}\left(K^{(-)}\right) = 0 \quad \text{if} \quad \sum_s \alpha_s > \sum_t \beta_t \ . \tag{27b}$$

However, there is no kinematical restriction on the values of the M^c functions with $\sum_s \alpha_s > \sum_t \beta_t$ at a point on $\mathcal{R}^{(+)}$ whose image in $\mathcal{K}_3$ is $k^{(+)}$ and no such restriction on those with $\sum_s \alpha_s < \sum_t \beta_t$ at a point whose image is $k^{(-)}$. Either of Eqs. (27) is therefore sufficient to determine the number of kinematical zeroes at an $r = 2$, $n = 3$ point on $\mathcal{R}^{(+)}$, this number being exactly half the number given by (22) for an $r = n = 2$ point on $\mathcal{R}$.[27]

By adding up the number of M^c functions not restricted by (22) or (27) we get the number of such functions whose values are free of kinematical constraints at any $r = 2$ point on $\mathcal{R}^{(+)}$. The result is given in Table I, which is actually valid for M^c functions with any number of particles, but for which at most four have spin.[4]

Table I. Number of independent M^c functions at an $r = 2$ point.

These results are valid for the case when at most four particles have spins, although the total number of particles can be arbitrary, and are valid on the I_+-saturated kernel of the domain of regularity. Here r = rank of Gram determinant. We take $j_1 + j_2$ and $j_3 + j_4$ both to be integers, with $j_1 + j_2 \geq j_3 + j_4$ and $j_1 \geq j_2$, $j_3 \geq j_4$.

	Number of linearly independent 4-vectors, n	
	n = 2	n = 3
Case I $j_1 - j_2 \geq j_3 + j_4$	$(2j_2+1)(2j_3+1)(2j_4+1)$	$(j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)$
Case II $j_3 + j_4 \geq j_1 - j_2$ $j_1 - j_2 \geq j_3 - j_4$	$(2j_2+1)(2j_3+1)(2j_4+1)$ $-\frac{1}{3}(j_2+j_3+j_4-j_1)(j_2+j_3+j_4-j_1+1)$ $\cdot(j_2+j_3+j_4-j_1+2)$	$(j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)$ $\frac{1}{6}(j_2+j_3+j_4-j_1)(j_2+j_3+j_4-j_1+1)$ $\cdot(j_2+j_3+j_4-j_1+2)$
Case III $j_3 - j_4 \geq j_1 - j_2$	$(2j_4+1)[(2j_2+1)(2j_3+1)$ $-\frac{4}{3}j_4(j_4+1) - (j_2+j_3-j_4-j_1)$ $\cdot(j_2+j_3+j_4-j_1+1)]$	$(2j_4+1)[(j_1+1)(2j_2+1)(2j_3+1)$ $-\frac{2}{3}j_4(j_4+1) - \frac{1}{2}(j_2+j_3-j_4-j_1)$ $\cdot(j_2+j_3+j_4-j_1+1)]$

IV. DISCRETE SYMMETRIES

A. Identical Particles, PCT, P, T, and C

In this section we determine the number of independent M^c functions at various points on $\mathcal{R}^{(+)}$ if the functions have definite signature under the discrete symmetry operations. The main object is to establish the results at points where the number of linearly independent momenta, n, is 3, but the rank of the Gram determinant, r, is 2. The understanding derived from this discussion leads to the results of the following section.

Invariance under PCT and the connection between spin and statistics are both consequences of the basic principles of S-matrix theory[19] and of field theory.[24,28] In terms of the connected parts of the S matrix in canonical form, the PCT identity reads

$$\text{PCT}: \underset{\alpha_b;\,\alpha_a}{S^c(K_b;\,K_a)} = \eta_{\text{PCT}} \prod_{i=1}^{4} D^{(j_i)}(C)_{\alpha_i}{}^{\alpha_i'} \underset{\alpha_a';\,\alpha_b'}{S^c(\bar{K}_a;\,\bar{K}_b)}, \tag{28}$$

where η_{PCT} is a phase factor. Here $\bar{K}_i = (k_i, \bar{t}_i)$ with $\bar{t}_i = (m_i, j_i, -q_i)$ indicating an antiparticle. The spin-statistics connection states that exchanging the order of the momentum variables and spin components of any two identical initial or any two identical final particles of spin j changes the sign of the scattering function by $(-1)^{2j}$.

The assumption that transition probabilities are invariant under a change of direction of all spatial components of the four-momenta leads to the relation

$$P : S^c(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \eta_p \, S^c(\tilde{K}_b; \tilde{K}_a)_{\alpha_b; \dot{\alpha}_a} \tag{29}$$

in the physical region of the process $t_1 + t_2 \to t_3 + t_4$. Here $\tilde{K}_i = (\tilde{k}_i, t_i)$, with $\tilde{k}_i = (k_i^0, -\underset{\sim}{k}_i)$, and $\eta_p = \pm 1$, the "process intrinsic parity" is the product of the "particle intrinsic parities" of the particles occurring in the process.[29]

It is straightforward to show that if transition probabilities are invariant under time reversal, which involves exchanging initial and final states and changing the sign of all three vectors, one has for physical points

$$T : S^c(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \eta_T \, D^{(j_a)}(C)_{\alpha_a}{}^{\alpha'_a} \, D^{(j_b)}(C)_{\dot{\alpha}_b}{}^{\dot{\alpha}'_b} \, S^c(\tilde{K}_a; \tilde{K}_b)_{\alpha'_a; \dot{\alpha}'_b}, \tag{30}$$

where $\eta_T = +1$ is required for an elastic process.

It may happen that PT is a symmetry, even if P and T are not. Then the PCT identity (28) requires charge-conjugation invariance,

$$C : S^c(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \eta_C \, S^c(\overline{K}_b; \overline{K}_a)_{\alpha_b; \dot{\alpha}_a}. \tag{31}$$

The relevant symmetry relations for M^c functions with spinor indices of the types introduced in (4) and (6) follow easily from (28)-(31) and are given in Table II. If a given symmetry is valid for a certain physical process, it is valid for the analytically continued functions and, consequently, for the processes related by crossing.[30]

Table II. Effect of invariance under various discrete symmetry operations on the M^c functions

N_a = number of fermions in state a

Symmetry	$M^c(K_b; K_a)$ $\alpha_b; \alpha_a$	$M^c(K_b; K_a)$ $\alpha_b; \dot{\alpha}_a$
PCT	$= (-1)^{N_a} \eta_{PCT} M^c(\bar{K}_a; \bar{K}_b)$ $\alpha_a; \alpha_b$	$= (-1)^{N_a} \eta_{PCT} M^c(\bar{K}_a; \bar{K}_b)$ $\alpha_a; \alpha_b$ $= \eta_{PCT} D^{(j_a)}\left(C \frac{\tilde{\sigma}\cdot k_a}{m_a}\right)_{\dot{\alpha}_a \alpha'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b} C\right)_{\alpha_b \dot{\alpha}'_b} M^c(\bar{K}_a; \bar{K}_b)$ $\alpha'_a; \dot{\alpha}'_b$
P	$= \eta_P M^c(\tilde{K}_b; \tilde{K}_a)$ $\dot{\alpha}_b; \dot{\alpha}_a$ $= \eta_P D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b \dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a \dot{\alpha}'_a} M^c(\tilde{K}_b; \tilde{K}_a)$ $\alpha'_b; \alpha'_a$	$= (-1)^{N_a} \eta_P M^c(\tilde{K}_b; \tilde{K}_a)$ $\dot{\alpha}_b; \alpha_a$ $= \eta_P D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b \dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma^T\cdot k_a}{m_a}\right)_{\dot{\alpha}_a \alpha'_a} M^c(\tilde{K}_b; \tilde{K}_a)$ $\alpha'_b; \dot{\alpha}'_a$
T	$= (-1)^{N_a} \eta_T M^c(\tilde{K}_a; \tilde{K}_b)$ $\dot{\alpha}_a; \dot{\alpha}_b$ $= (-1)^{N_a} \eta_T D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a \dot{\alpha}'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b \dot{\alpha}'_b} M^c(\tilde{K}_a; \tilde{K}_b)$ $\alpha'_a; \alpha'_b$	$= \eta_T D^{(j_a)}(C)_{\alpha_a \alpha'_a} D^{(j_b)}(C)_{\dot{\alpha}_b \dot{\alpha}'_b} M^c(\tilde{K}_a; \tilde{K}_b)$ $\alpha'_a; \dot{\alpha}'_b$
PT	$= (-1)^{N_a} \eta_{PT} M^c(K_a; K_b)$ $\alpha_a; \alpha_b$	$= (-1)^{N_a} \eta_{PT} M^c(K_a; K_b)$ $\dot{\alpha}_a; \alpha_b$ $= \eta_{PT} D^{(j_a)}\left(C \frac{\tilde{\sigma}\cdot k_a}{m_a}\right)_{\dot{\alpha}_a \alpha'_a} D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b} C\right)_{\alpha_b \dot{\alpha}'_b} M^c(K_a; K_b)$ $\alpha'_a; \dot{\alpha}'_b$
C	$= \eta_c M^c(\bar{K}_b; \bar{K}_a)$ $\alpha_b; \alpha_a$	$= \eta_c M^c(\bar{K}_b; \bar{K}_a)$ $\alpha_b; \dot{\alpha}_a$

B. Functions with Definite Parity Signature

Even when spatial inversion is not a symmetry of the process under consideration, one can find it useful to decompose the M^c functions for two incoming and two outgoing particles into parts having positive and negative parity signature. In the physical region of the process $t_1 + t_2 \to t_3 + t_4$ let us define the functions

$$M_\epsilon^{\,c}(K_b;\, K_a)_{\alpha_b;\,\alpha_a} = \frac{1}{2}\left[M^c(K_b;\, K_a)_{\alpha_b;\alpha_a} + \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}'_a} M^c(\tilde{K}_b;\, \tilde{K}_a)_{\alpha'_b;\,\alpha'_a}\right] \quad (32)$$

for $\epsilon = \pm 1$. With the aid of (A.12) we find that

$$M_\epsilon^{\,c}(K_b;\, K_a)_{\alpha_b;\,\alpha_a} = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}'_a} M_\epsilon^{\,c}(\tilde{K}_b;\, \tilde{K}_a)_{\alpha'_b;\,\alpha'_a}, \quad (33)$$

which indicates that the functions defined by (32) have definite parity signature (see Table II).

One may analytically continue the functions $M_+^{\,c}$ and $M_-^{\,c}$ defined by (32) over all of $\mathcal{R}^{(+)}$ and the following decomposition is valid at all points k on that domain:

$$M^c(K_b;\, K_a)_{\alpha_b;\,\alpha_a} = M_+^{\,c}(K_b;\, K_a)_{\alpha_b;\,\alpha_a} + M_-^{\,c}(K_b;\, K_a)_{\alpha_b;\,\alpha_a} . \quad (34)$$

By comparing (33) and (34) with Table II, we see that if spatial inversion symmetry is valid with $\eta_p = \pm 1$ then $M_{\mp}^{\,c} \equiv 0$.

According to the remarks in Sec. III, Part B, the points k and $\tilde{k}$ in the analytically continued relation (33) lie on the same $\mathcal{L}_+$ orbit on $\mathcal{R}^{(+)}$ unless they are $r = 2$, $n = 3$ points. If they are $r = 2$, $n = 3$ points then remark H2.(b) shows that they lie on two different $\mathcal{L}_+$ orbits on $\mathcal{R}^{(+)}$ having the same $r = n = 2$ limit points. It is well known that $\mathcal{L}_+$ covariance leads to linear relations between the functions having the same parity signature at all $r = n$ points. We will review this result in order to extend it to $r = 2$, $n = 3$ points.

Because of the $D^{(j_i)}(\sigma \cdot k_i/m_i)$ matrices that act on the M_ϵ^c functions under the parity operation defined by (33), it is much simpler to use the connected parts of the S-matrix elements to count the number of independent functions. The S^c functions have kinematical singularities arising from the "boost" matrices that relate them to the M^c functions, as in (4). These singularities, unlike those of the M^c functions, are not Lorentz invariant, so we can always find points on any orbit at which the S^c functions are analytic if the M^c functions are holomorphic.[31]

By application of the proper boost matrices to (32) and (33), we obtain

$$S_{\pm}^{c}(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \frac{1}{2}\left[S^c(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} \pm S^c(\tilde{K}_b; \tilde{K}_a)_{\alpha_b; \dot{\alpha}_a}\right], \tag{35}$$

where

$$S_{\epsilon}^{c}(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \epsilon\, S_{\epsilon}^{c}(\tilde{K}_b; \tilde{K}_a)_{\alpha_b; \dot{\alpha}_a}. \tag{36}$$

To see the restrictions at $r = n = 3$ or 2 points on $\mathcal{R}$ it is best to choose a point $k = \{k_b; k_a\}$ on a given orbit such that all spatial components are normal to the 2 axis; then a rotation of π about this axis carries $\tilde{k} \in \mathcal{K}_3$ into k. Since the matrix $A(k_i)$ in (A.11) corresponding to the Wigner rotation is equal to A when A is unitary, and since the required rotation matrix for spin j_i is now simply the matrix $D^{(j_i)}(C)$ in (A.5), (3) gives us

$$S_\epsilon{}^c(K_b; K_a)_{\alpha_b; \dot{\alpha}_a} = \epsilon(-1)^{\sum_{i=1}^{4}(j_i-\alpha_i)} S_\epsilon{}^c(K_b; K_a)_{-\alpha_b; -\dot{\alpha}_a}. \tag{37}$$

Equation (37) allows us to conclude that at any $r = 3$ point on the domain of regularity of the M^c functions, disregarding possible restrictions due to other symmetry operations, the number of $M_\epsilon{}^c$ functions whose values are independent is

$$N_\epsilon = \frac{1}{2}\prod_{i=1}^{4}(2j_i + 1), \tag{38}$$

if there are fermions involved in the process. On the other hand, the number is

$$N_\epsilon = \frac{1}{2}\left[\prod_{i=1}^{4}(2j_i + 1) + \epsilon(-1)^{\sum_{i=1}^{4} j_i}\right] \tag{39}$$

if all the particles are bosons.

In order to obtain the restrictions at $r = n = 2$ points on $\mathcal{R}$ we note that, in terms of the $S_\epsilon{}^c$ functions, (22) becomes

$$S_\epsilon{}^c(K_b;\ K_a)_{\alpha_b;\ \alpha_a} = 0 \qquad \text{if } \alpha_1 + \alpha_2 \neq \alpha_3 + \alpha_4. \tag{40}$$

Then (37) restricts the number of $M_\epsilon{}^c$ functions whose values are independent to exactly half the number allowed by (40) alone, if some of the particles are fermions, and to that number plus $\frac{1}{2}\epsilon(-1)^{\sum_{i=1}^{4} j_i}$, if all the particles are bosons. The number allowed by (40) alone has already been given in Table I.

From remark H2. (b) in Sec. III, we know that an $r = 2$, $n = 3$ point is related to its spatial inverse only by an improper Lorentz transformation. However, on $\mathcal{R}^{(+)}$ the limit (25) must be valid for the $M_\epsilon{}^c$ functions also and we do get restrictions at $r = 2$, $n = 3$ points. In particular, in place of (26) we have

$$M_\epsilon{}^c(K_b;\ K_a)_{\alpha_b;\ \alpha_a} = M_\epsilon{}^c\left(K_b^{(\pm)};\ K_a^{(\pm)}\right)_{\alpha_b;\ \alpha_a}, \quad \text{for} \quad \sum_{i=1}^{4} \alpha_i = 0, \tag{41}$$

if $k^{(\pm)}$, as defined by (23), is an $r = 2$, $n = 3$ point on $\mathcal{R}^{(+)}$ and k is an $r = n = 2$ limit point of $\mathcal{L}_+ k^{(\pm)}$ of the form (18).

Similarly, in place of (27) we get

$$M_\epsilon{}^c\left(K_b^{(+)};\ K_a^{(+)}\right)_{\alpha_b;\ \alpha_a} = 0 \qquad \text{if} \quad \sum_{i=1}^{4} \alpha_i < 0, \tag{42a}$$

$$M_\epsilon{}^c\left(K_b^{(-)};\ K_a^{(-)}\right)_{\alpha_b;\ \alpha_a} = 0 \qquad \text{if} \quad \sum_{i=1}^{4} \alpha_i > 0, \tag{42b}$$

but there are no restrictions on the components not accounted for by (41) or (42).

By inspection of (41), we see that at the $r = 2$, $n = 3$ point under consideration, the M_ϵ^c functions satisfying $\sum_i \alpha_i = 0$ are subject to the same restrictions as they are subject to at the $r = n = 2$ limit point of the orbit. Therefore, the number of independent functions with $\sum_i \alpha_i = 0$ is the same as the number mentioned in the sentence following (40). Adding to this the number of functions not restricted by either (41) or (42), we find that the number of M_ϵ^c functions whose values are not subject to any linear restriction among themselves is exactly the same at any $r = 2$, $n = 3$ point on $\mathcal{R}^{(+)}$ as the number at any $r = 3$ point on $\mathcal{R}$ --this number is given by either (38) or (39).

C. Restrictions in Special Cases

Invariance of a scattering process under any discrete symmetry other than spatial inversion can restrict the number of independent M^c functions only in special cases in which the point in momentum space resulting from the symmetry operation lies on the same $\mathcal{L}_+$ orbit as the original point. We will consider the restrictions for $r = 3$ points by working in the physical region of a particular center-of-mass system with the 1 and 3 axes orientated as in Fig. 1 and with the 2 axis pointing out of the paper. The restrictions thus obtained are easily extended to all $r = 3$ points on $\mathcal{R}$ and also to the points on $\mathcal{R}^{(+)}$ with $r < 3$.

For example, for a scattering process of the form $t_1 + t_1 \to t_3 + t_3$ the exchange symmetry for identical particles leads to a restriction, when one simultaneously exchanges the orders of the initial particles among themselves and of the final particles among themselves. In the reference system of Fig. 1, a rotation of π about the 2 axis carries the transformed momenta back into the original orientation and we have

$$E : \underset{\alpha_3,\alpha_4;\,\mathring{\alpha}_1,\mathring{\alpha}_2}{S^c(K_3,K_4;\,K_1,K_2)} = (-1)^{\sum_{i=1}^{4}(j_i+\alpha_i)} \underset{-\alpha_4,-\alpha_3;-\mathring{\alpha}_2,-\mathring{\alpha}_1}{S^c(K_3,K_4;\,K_1,K_2)}, \quad \text{for } t_1 = t_2 \text{ and } t_3 = t_4 . \tag{43}$$

The functions for the crossed process $t_1 + \bar{t}_3 \to \bar{t}_1 + t_3$ have the same number of independent components as those allowed by (43). This restriction could also have been obtained by applying the PCT relation (28) in the new channel. For a process of the form $t + t \to t + t$ with $t = \bar{t}$, the PCT relation and the symmetry under the exchange of identical particles simultaneously lead to restrictions in the same channel. In the reference frame of Fig. 1 a rotation of π about the 3 axis carries the point on the right-hand side of (28) into that on the left and we obtain

$$PCT : \underset{\alpha_3,\alpha_4;\,\mathring{\alpha}_1,\mathring{\alpha}_2}{S^c(K_3,K_4;\,K_1,K_2)} = \underset{-\alpha_1,-\alpha_2;-\mathring{\alpha}_3,-\mathring{\alpha}_4}{S^c(K_3,K_4;\,K_1,K_2)},$$

$$\text{for } t_1 = t_2 = t_3 = t_4 = t = \bar{t} . \tag{44}$$

Time-reversal invariance restricts the number of independent components only in an elastic two-particle process. In the reference frame of Fig. 1, the required exchange of the four-momenta is brought about by a rotation of π about the 1 axis and (30) becomes, with $\eta_T = 1$,

$$T : S^c(K_3,K_4;\ K_1,K_2)_{\alpha_3,\alpha_4;\ \dot{\alpha}_1,\dot{\alpha}_2}$$

$$= (-1)^{\alpha_1+\alpha_2-\alpha_3-\alpha_4}\, S^c(K_3,K_4;\ K_1,K_2)_{\alpha_1,\alpha_2;\ \dot{\alpha}_3,\dot{\alpha}_4}, \qquad \text{for } t_1 = t_3 \text{ and } t_2 = t_4. \tag{45}$$

Equations (37) and (43)-(45) are sufficient to find all possible restrictions at $r = 3$ points on $\mathcal{R}$.

In Tables III, IV, and V we have listed the number of independent M_ϵ^c functions having definite signature under the relevant symmetry operations, for those cases in which spatial inversion is not the only possible symmetry that can give a restriction. In Table IV we have allowed only those terms satisfying $(-1)^{2(j_1+j_3)} = 1$ when taking (43) into account. This is because the additive quantum numbers independent of the Lorentz group must be the same for

t_1 and t_3 in a process of the form $t_1 + t_1 \to t_3 + t_3$, and in nature all strongly interacting particles with half odd-integer spin apparently have odd baryon number, while those with integer spin have even baryon number.

Of course, when spatial inversion is a symmetry for all processes in nature only terms with $\epsilon = P = +1$ are nonvanishing in any of the tables,[29] and when time reversal is a symmetry for the elastic processes in Tables III and V only terms with $T = +1$ are nonvanishing. If neither P nor T is a symmetry, but PT is, then only terms with $PT = +1$ are nonvanishing in Tables III and V.[32]

Any process not accounted for in Tables III, IV, and V for which a discrete symmetry other than spatial inversion can restrict the number of independent M^c functions is related through crossing to one in the table. For example, in the process $t_1 + \bar{t}_1 \to t_2 + \bar{t}_2$ charge-conjugation invariance can yield a restriction, but this is exactly the same restriction given by PT symmetry for the process $t_1 + t_2 \to t_1 + t_2$.

Although Tables III, IV, and V were derived for $r = 3$ points, they also tell us how many independent M_ϵ^c functions having the given symmetry properties there are at $r = 2$, $n = 3$ points on $\mathcal{R}^{(+)}$. We have seen at the end of Part B of this section that $\mathcal{L}_+$ covariance allows the number of independent M_ϵ^c functions to be the same at an $r = 2$, $n = 3$ point on $\mathcal{R}^{(+)}$ as at the $r = 3$ points, the latter of which form a dense subdomain of $\mathcal{R}^{(+)}$.

The tables derived from the considerations of this section are important for the application of the criteria to be presented in the next one. In checking any set of covariant polynomials to see whether they give invariant amplitudes free of kinematical singularities it is useful to know in advance how many of any discrete symmetry signature one should have.

Table III: Number of independent M_ϵ^c functions having definite signature under T and PT for the process $t_1 + t_2 \to t_1 + t_2$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta. Exceptional cases in which exchange symmetry or PTC (or both) must give a restriction are: (1) $t_1 = t_2$; (2) $t_1 = \bar{t}_2$; (3) both $t_1 = \bar{t}_1$ and $t_2 = \bar{t}_2$. These exceptions are either given directly by the processes in Tables IV and V or related to them through crossing

(a) If at least one of the incoming particles is a fermion:			
$\epsilon = P$	T	PT	Number of terms
+	+	+	$\frac{1}{4}(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2+3)$
+	-	-	$\frac{1}{4}(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2-1)$
-	+	-	$\frac{1}{4}(2j_1+1)^2(2j_2+1)^2$
-	-	+	$\frac{1}{4}(2j_1+1)^2(2j_2+1)^2$
(b) If both particles are bosons:			
$\epsilon = P$	T	PT	Number of terms
+	+	+	$\frac{1}{4}[(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2+3)+1]$
+	-	-	$\frac{1}{4}[(2j_1+1)(2j_2+1)(4j_1j_2+2j_1+2j_2-1)+1]$
-	+	-	$\frac{1}{4}[(2j_1+1)^2(2j_2+1)^2-1]$
-	-	+	$\frac{1}{4}[(2j_1+1)^2(2j_2+1)^2-1]$

Table IV: Number of independent M_ϵ^c functions for the process $t_1 + t_1 \to t_3 + t_3$ with $t_1 \neq t_3$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta. Since the additive quantum numbers that are independent of the Lorentz group must be the same for t_1 and t_3, such a process apparently occurs in nature only when t_1 and t_3 are both fermions or both bosons. The table is valid even when $t_1 = \bar{t}_1$.

(a) When particles t_1 and t_3 are fermions:	
$\epsilon = P$	Number of terms
+	$\frac{1}{4}(2j_1+1)(2j_3+1)(4j_1j_3+2j_1+2j_3+3)$
-	$\frac{1}{4}(2j_1+1)^2(2j_3+1)^2$
(b) When t_1 and t_3 are bosons:	
$\epsilon = P$	Number of terms
+	$(2j_1j_3+j_1+j_3+1)^2$
-	$(2j_1j_3+j_1+j_3)(2j_1j_3+j_1+j_3+1)$

Table V. Number of independent M_ϵ^c-functions having definite signature under T and PT for the process $t + t \to t + t$.

These results hold at any point on the I_+-saturated kernel of the domain of regularity where there are three linearly independent four-momenta.

(a) When particle t is a fermion:			
ϵ = P	T	PT	Number of terms
+	+	+	$\frac{1}{8}(2j + 1)(8j^3 + 12j^2 + 18j + 7)$
+	-	-	$\frac{1}{8}(2j + 1)(8j^3 + 12j^2 + 2j - 1)$
-	+	-	$\frac{1}{8}(2j + 1)^4$
-	-	+	$\frac{1}{8}(2j + 1)^4$
(b) When particle t is a boson. If, in addition $t = \bar{t}$, then only terms with PT = + are allowed by the PCT theorem.			
ϵ = P	T	PT	Number of terms
+	+	+	$2j^4 + 4j^3 + 6j^2 + 4j + 1$
+	-	-	$2j^2(j + 1)^2$
-	+	-	$j(j + 1)(2j^2 + 2j + 1)$
-	-	+	$j(j + 1)(2j^2 + 2j + 1)$

V. INVARIANT AMPLITUDES

A. Extension of the Theorem of Hepp and Williams

We have been considering the M^c functions for two incoming and two outgoing particles with spin as functions of their four-momenta on the mass shell. The above discussion of kinematical constraints on functions having definite parity signature, together with a result of Hepp and Williams, will enable us to describe the analytic properties in terms of $\mathcal{L}_+$ invariants. Because of the mass shell conditions and four-momenta conservation, one can form only two independent $\mathcal{L}_+$ invariants. These can be taken to be two of the three linearly related Mandelstam invariants: $s = (k_1 + k_2)^2$, $t = (k_1 - k_3)^2$, and $u = (k_1 - k_4)^2$.

Let us first consider a process for which all four particles are spinless; that is, one for which the M^c function is $\mathcal{L}_+$ invariant,

$$M^c(K) = M^c(\Lambda K), \tag{46}$$

for any point k on the domain of regularity $\mathcal{R}$.

According to a theorem of Hepp[1] and Williams[5] any such function can be expressed as a holomorphic function of the independent $\mathcal{L}_+$ invariants on the image of the I_+-saturated kernel $\mathcal{U}_a^{(+)}$ of each sheet $\mathcal{U}_a \subset \mathcal{R}$:

$$M^c(K) = A\big(s(k),t(k)\big) \equiv A(s,t). \tag{47}$$

Thus (47) defines an analytic function A over the invariants s and t. The domain of regularity of this function is the image of $\mathcal{R}^{(+)}$.[33]

Actually the restriction to the image of the I_+-saturated kernel is not necessary--the domain of regularity of the function A defined by (47) is the image of the full domain of regularity of M^c. This extension of the result of Hepp and Williams arises from the fact that $\mathcal{R}$ is automatically I_+ saturated; i.e., $\mathcal{R}^{(+)} = \mathcal{R}$:

Lemma 1. Let the domain of regularity of the function F be a domain $\mathcal{R}(F)$ lying over $\mathcal{K}_3$. Suppose the function F is $\mathcal{L}_+$ invariant on $\mathcal{R}(F)$. Then $\mathcal{R}(F)$ is I_+ saturated; i.e., the $r = n$ limit points of every $r \neq n$ orbit on $\mathcal{R}(F)$ also lie on $\mathcal{R}(F)$.

The proof of the above lemma, due to H. P. Stapp and this author, is given in Appendix B.

B. Standard Covariants

We would like to generalize the preceding considerations of this section to the cases in which one or more of the four particles has spin. That is, we would like to be able to express the M_ϵ^c functions for any process with two incoming and two outgoing particles in terms of invariant functions that are holomorphic in the $\mathcal{L}_+$ invariants, s and t, everywhere on the image of the subset $\mathcal{U}_a^{(+)}$ of each $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a \subset \mathcal{R}$. The spin dependence and, consequently, all the kinematical properties of the M_ϵ^c functions will be accounted for by polynomials in the four-momenta referred to as "standard covariants." First we will precisely define the latter.

Definition. Consider the M^c functions for a scattering process involving two incoming particles of spins j_1 and j_2 and two outgoing particles of spins j_3 and j_4. A set of spinor functions $Y_{+1}^{(g)}(k)$, for $g = 1,\cdots,\bar{N}_+$, and $Y_{-1}^{(g)}(k)$, for $g = 1,\cdots,\bar{N}_-$, where

$\bar{N}_+ + \bar{N}_- = \prod_{i=1}^{4} (2j_i + 1)$, is said to be a set of <u>standard covariants</u> for this process (and the processes related by crossing) if they satisfy the following five properties:

SC1. They are polynomials in the four momenta for the process, subject to the mass shell and four-momentum conservation constraints. (They are therefore holomorphic everywhere.)

SC2. They are $\mathcal{L}_+$ covariant; i.e.,

$$Y_\epsilon^{(g)}{}_{\alpha_b;\,\alpha_a}\big(\Lambda(A,B)k\big) = D^{(j_b)}(A)_{\alpha_b}{}^{\alpha'_b}\, D^{(j_a)}(A)_{\alpha_a}{}^{\alpha'_a}\, Y_\epsilon^{(g)}{}_{\alpha'_b;\,\alpha'_a}(k) \quad . \tag{48}$$

SC3. They have definite signature under the spatial inversion operation:

$$Y_\epsilon^{(g)}{}_{\alpha_b;\dot{\alpha}_a}(k) = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}'_a} Y^{(g)}{}_{\dot{\alpha}'_b;\alpha'_a}(\tilde{k}) \, . \tag{49}$$

SC4. The functions of each of the two parity signatures are separately linearly independent at all $n = 3$ points. That is, if we form the functions

$$\Gamma_\epsilon(k)_{\alpha_b;\,\alpha_a} = \sum_{g=1}^{\bar{N}_\epsilon} \gamma_\epsilon^{(g)}\, Y_\epsilon^{(g)}{}_{\alpha_b;\,\alpha_a}(k) \quad , \tag{50}$$

where the $\gamma_\epsilon^{(g)}$'s are real or complex numbers, then at any $n = 3$ point $k = \{k_b;\, k_a\}$, the only solution to the equations $\Gamma_\epsilon(k)_{\alpha_b;\alpha_a} = 0$ for all values of $\{\alpha_b;\, \alpha_a\}$ is $\gamma_\epsilon^{(g)} = 0$, for all $g = 1,\cdots,\bar{N}_\epsilon$.

SC5. If the number of M^c functions whose values are independent is restricted by any discrete symmetry other than parity then each of the spinor functions $Y_\epsilon^{(g)}(k)$ has definite signature, +1 or -1, under this operation (the form of the symmetry operation is the same as that for the M^c functions in Table II).

The above properties of the standard covariants lead to the following lemma, the proof of which is given in Appendix C:

Lemma 2. Consider a set of standard covariants for the M^c functions describing a process with two incoming and two outgoing particles. Then,

L1. At any $r = 3$ point, the standard covariants of parity signature +1 are linearly independent of those of parity signature -1.

L2. For each value of ϵ, $\overline{N}_\epsilon = N_\epsilon$ as given by (38) if there are some fermions involved in the process, or $\overline{N}_\epsilon = N_\epsilon$ as given by (39) if all the particles are bosons.

L3. In those cases in which property SC5 holds, the number of standard covariants having a given signature under any of the applicable symmetry operations is in agreement with the number in Tables III, IV, and V.

The choice of a set of standard covariants for any process is by no means unique. If we have found a set $Y_\epsilon^{(g)}(k)$, for $g = 1,\cdots,N_\epsilon$, satisfying the required properties, and if we can write

$$Y_\epsilon^{(g)}(k) = \sum_{g'=1}^{N_\epsilon} f_\epsilon^{gg'}(s,t)\, \overline{Y}_\epsilon^{(g')}(k), \tag{51}$$

such that the coefficients $f_\epsilon^{gg'}(s,t)$ are globally holomorphic functions of the Mandelstam invariants with $\det\left(f_\epsilon^{gg'}(s,t)\right)$ nowhere zero, then the $Y_\epsilon^{(g)}(k)$'s also form a set of standard covariants.

C. Invariant Amplitudes for Scattering Functions With Spin

We are now ready to consider the possibility of expanding the M_ϵ^c functions for a given process in terms of standard covariants having the same discrete symmetry signatures, using the properties of the standard covariants in the definition and lemma of Part B of this section. Our results are expressed by the following theorem:

Theorem. Consider the M^c functions describing a process with two incoming particles and two outgoing particles. On the I_+-saturated kernel $\mathcal{R}^{(+)}$ of the domain of regularity $\mathcal{R}$ of the M^c functions one may write the following global decompositions:

$$M_\epsilon^c = \sum_{g=1}^{N_\epsilon} A_\epsilon^{(g)} Y_\epsilon^{(g)}, \tag{52}$$

where the $Y_\epsilon^{(g)}$'s are any set of standard covariants for the process. The "invariant amplitudes" $A_\epsilon^{(g)}$, for $g = 1,\cdots,N_\epsilon$, are holomorphic functions of the Mandelstam invariants s and t on the image of the I_+-saturated kernel $\mathcal{U}_a^{(+)}$ of each $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a \subset \mathcal{R}$.

If the M_ϵ^c functions for a particular value of ϵ are identically zero, the above result is trivial with vanishing $A_\epsilon^{(g)}$'s. In our proof of the theorm for nontrivial cases we will first completely ignore those cases in which the number of independent M_ϵ^c-function components is less than the number in (38) or (39) at $n = 3$ points on $\mathcal{R}^{(+)}$. The necessary modifications for the exceptional cases will be easy to make.

For our proof we will first try writing on the subset $\mathcal{U}_a^{(+)}$ of a particular sheet $\mathcal{U}_a \subset \mathcal{R}$

$$M_\epsilon{}^c(K)_{\alpha_b;\,\alpha_a} = \sum_{g=1}^{N_\epsilon} a_\epsilon{}^{(g)}(k)\; Y_\epsilon{}^{(g)}(k)_{\alpha_b;\,\alpha_a} \quad , \tag{53}$$

which corresponds to (52), except that we regard the invariant amplitudes $a_\epsilon{}^{(g)}$ as functions of the four-momenta for the present. We will show that (53) is invertible; that is, we will solve for the amplitudes $a_\epsilon{}^{(g)}$ in terms of the $M_\epsilon{}^c$ functions and show that this does not introduce any singularities not present in the $M_\epsilon{}^c$ functions themselves. Finally, the theorem of Hepp and Williams will allow us to express the analytic properties of the invariant amplitudes in terms of $\mathcal{L}_+$ invariants. Recall that, according to the last paragraph, we are ignoring the exceptional cases for the present.

Consider the scalars

$$y_{\epsilon',\epsilon}^{(g',g)}(k) = Y_{\epsilon'}{}^{(g')}(k)_{\alpha_b;\alpha_a}\; Y_\epsilon{}^{(g)}(k)^{\alpha_b;\alpha_a} \quad . \tag{54}$$

Each of the above invariant functions is holomorphic everywhere on $\mathcal{K}_3$ and the determinant formed from them, $\det\left(y_{\epsilon',\epsilon}^{(g',g)}\right)$, cannot vanish at any $r = 3$ point. The only way the determinant at such a point could vanish would be for some of the standard covariants to be linearly dependent, contradicting statements SC4 and L1. Since the standard covariants are not all linearly independent at an $r = 2$ point, $\det\left(y_{\epsilon',\epsilon}^{(g',g)}\right)$ must vanish at such a point.

The determinant just introduced consists of four blocks, the upper left-hand one having components of the form $y_{+1,+1}^{(g',g)}$, the

lower right-hand one $\mathcal{Y}^{(g',g)}_{-1,-1}$, the upper right-hand one $\mathcal{Y}^{(g',g)}_{+1,-1}$, and the lower left-hand one $\mathcal{Y}^{(g',g)}_{-1,+1}$. However, it follows from (49) that $\mathcal{Y}^{(g',g)}_{\epsilon,-\epsilon} \equiv 0$, since one has an invariant function of three independent four-vectors that has negative signature under spatial inversion, and such a function vanishes identically. Consequently, the determinant is factorizable:

$$\det\left(\mathcal{Y}^{(g',g)}_{\epsilon',\epsilon}\right) = \left[\det\left(\mathcal{Y}^{(g',g)}_{+1,+1}\right)\right]\left[\det\left(\mathcal{Y}^{(g',g)}_{-1,-1}\right)\right], \tag{55}$$

and neither $\det\left(\mathcal{Y}^{(g',g)}_{+1,+1}\right)$ nor $\det\left(\mathcal{Y}^{(g',g)}_{-1,-1}\right)$ can vanish at any $r = 3$ point. Since (55) must vanish at $r = 2$ points we must have $\det\left(\mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}\right) \propto G^N(k_1,k_2,k_3)$, where k_1,k_2, and k_3 are any three of the momenta and N is some integer.

The $\mathcal{L}_+$-invariant functions

$$\mathcal{M}^{(g)}_{\epsilon',\epsilon}(k) = M_{\epsilon'}{}^{c}_{\alpha_b;\alpha_a}(k)\, Y_{\epsilon}{}^{(g)}{}^{\alpha_b;\alpha_a}_{(k)}; \quad g = 1,\cdots,N_\epsilon, \tag{56}$$

are holomorphic everywhere on the particular domain $\mathcal{U}_a{}^{(+)}$ under consideration with $\mathcal{M}^{(g)}_{\epsilon,-\epsilon} \equiv 0$. From (53), (54), and (56) we obtain the set of equations

$$\sum_{g'} \mathcal{a}_{\epsilon}{}^{(g')}(k)\, \mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}(k) = \mathcal{M}^{(g)}_{\epsilon,\epsilon}(k); \quad g = 1,\cdots,N_\epsilon. \tag{57}$$

At least at $r = 3$ points, where $\det\left(\mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}\right)$ cannot vanish, (57) is soluble for the invariant functions $\mathcal{a}_{\epsilon}{}^{(g)}$ in terms of the $\mathcal{M}^{(g)}_{\epsilon,\epsilon}$'s. The solution of (57) has the form

$$\mathcal{a}_{\epsilon}{}^{(g)}(k) = \frac{\mathcal{J}_{\epsilon}{}^{(g)}(k)}{G^N(k_1,k_2,k_3)}; \quad g = 1,\cdots,N_\epsilon, \tag{58}$$

where each $\mathcal{J}_{\epsilon}^{(g)}$ is some combination of the $\mathcal{Y}_{\epsilon,\epsilon}^{(g',g)}$'s and the $\mathcal{M}_{\epsilon,\epsilon}^{(g)}$'s.

The numerator $\mathcal{J}_{\epsilon}^{(g)}$ on the right-hand side of (58) is holomorphic on the given domain $\mathcal{U}_a^{(+)}$. The only possible singularities of the invariant amplitudes on the left-hand side of (58) on this domain $\mathcal{U}_a^{(+)}$ are poles at points where the rank of the Gram determinant is less than three.

Let us look at (53) once more. We have already seen that the invariant amplitudes on the right-hand side of the equation are holomorphic on the image in $\mathcal{K}_3$ of the $r = 3$ points on the domain $\mathcal{U}_a^{(+)}$. Since these $r = 3$ points form a dense subdomain of $\mathcal{U}_a^{(+)}$, as we approach any $r < 3$ point on $\mathcal{U}_a^{(+)}$ the limit of the right-hand side of (53) must exist and be equal to the value of the left-hand side at the given point. If the limit point is an $r = 2$, $n = 3$ point, the standard covariants are all linearly independent at the point and there can be no cancelling singularities in the invariant amplitudes; i.e., the limit at such a point must exist for each invariant amplitude in (58) separately and not just for the right-hand side of (53) as a whole.

The above considerations show that there exists some neighborhood of each $r = 2$, $n = 3$ point on the domain $\mathcal{U}_a^{(+)}$ under consideration such that $\mathcal{J}_{\epsilon}^{(g)}$ in (58) has the form $\mathcal{J}_{\epsilon}^{(g)}(k) = \mathcal{Y}_{\epsilon}^{(g)}(k)G^N$, where $\mathcal{Y}_{\epsilon}^{(g)}$ is holomorphic in the given neighborhood. From (58), $\mathcal{A}_{\epsilon}^{(g)}(k) = \mathcal{Y}_{\epsilon}^{(g)}(k)$ is holomorphic in that neighborhood.

The $\mathcal{L}_+$-invariant sheets whose union is $\mathcal{R}$ can be chosen to overlap, and the particular sheet $\mathcal{U}_a$ for which the decomposition (53) was carried out was arbitrary. Thus, the above procedure defines a unique set of functions $a_\epsilon^{(g)}$, for $g = 1,\cdots,N_\epsilon$. The domain of regularity of each function $a_\epsilon^{(g)}$ is a domain $\mathcal{R}(a_\epsilon^{(g)})$ over $\mathcal{K}_3$, and $a_\epsilon^{(g)}$ is $\mathcal{L}_+$ invariant on $\mathcal{R}(a_\epsilon^{(g)})$. We have seen that each domain $\mathcal{R}(a_\epsilon^{(g)})$ contains all $n = 3$ points on $\mathcal{R}^{(+)}$. Because of Lemma 1 each domain $\mathcal{R}(a_\epsilon^{(g)})$ also contains all $r = n = 2$ points on $\mathcal{R}$.

Because of the theorem of Hepp and Williams we may express each invariant amplitude as a function of $\mathcal{L}_+$ invariants on the image of the I_+-saturated kernel $\mathcal{U}_a^{(+)}$ of each sheet $\mathcal{U}_a \subset \mathcal{R}$:

$$a_\epsilon^{(g)}(k) = A_\epsilon^{(g)}\big(s(k),t(k)\big) \equiv A_\epsilon^{(g)}(s,t); \quad g = 1,\cdots,N_\epsilon, \tag{59}$$

with the possible exception of $r = 1$ points on the domain $\mathcal{U}_a^{(+)}$. But, at $r = 1$ points $k_i \cdot k_j = \pm m_i m_j$ for all values of i and j. Consequently, such points are isolated in the space of the $\mathcal{L}_+$ invariants. It is well known that an analytic function of several complex variables cannot have isolated singularities.[34]

Equation (59) therefore defines functions $A_\epsilon^{(g)}$, each of whose domain of regularity is a domain over the space of the Mandelstam invariants s and t. This domain is the image of all points on $\mathcal{R}^{(+)}$, so the proof of the theorem is completed--for those cases in which the number of independent M_ϵ^c functions is given by (38) or (39).[35]

In the exceptional cases in which properties SC5 and L3 are applicable the scalars $\mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}$ in (54) formed from standard covariants that have opposite signatures under any of the applicable discrete symmetries vanish identically. Then the determinant in (55) splits up further; that is, $\det\left(\mathcal{Y}^{(g',g)}_{\epsilon,\epsilon}\right)$, for each value of ϵ, can itself be written as a product of smaller determinants. Furthermore, when the $M_\epsilon{}^c$ functions have definite signature under the symmetries in question, the scalars in (56) involving standard covariants with different symmetry properties also vanish identically. The solution of (53) proceeds very much as before, except that now only the standard covariants having the correct symmetry properties need be used in the expansion, and the set of equations to be solved is of smaller order. Since the standard covariants continue to be linearly independent at $r = 2$, $n = 3$ points, which was the crucial factor in our previous proof, we have no singularities in the invariant amplitudes at $r < 3$ points on $\mathcal{R}^{(+)}$.

We have seen in Sec. IV, Part B, that $\mathcal{L}_+$ covariance requires the various $M_\epsilon{}^c$ functions to satisfy certain linear relations at all points on $\mathcal{R}^{(+)}$. Because of the properties of the standard covariants the decomposition on the right-hand side of (52) automatically satisfies these relations. Therefore, there is no point at which some linear combination of the invariant amplitudes must vanish in order for this decomposition to satisfy the required kinematical constraints; i.e., the invariant amplitudes are free of "kinematical zeroes."

VI. SUMMARY AND DISCUSSION

The main results of this paper are as follows:

(a) If the domain of regularity $\mathcal{R}$ of the scattering functions M^c for a $2 \to 2$ process contains a point at which only two of the external four-momenta are linearly independent then, as is well known, $\mathcal{R}$ contains also points having the same scalar invariants but with three linearly independent momenta. At any of these points $\mathcal{L}_+$ covariance requires the number of linearly independent components of M^c to be less than the dimensionality of the spin space.

(b) Let $\mathcal{P}$ be the parity operation for the M^c functions and let $M_\epsilon{}^c = \frac{1}{2}[M^c + \epsilon \mathcal{P} M^c]$. The number of linearly independent components of the functions $M_\epsilon{}^c$ is the same at all points on their domain of regularity at which there are three linearly independent momenta. This result continues to hold if one imposes additional discrete symmetry requirements.

(c) If one expresses the individual functions $M_+{}^c$ and $M_-{}^c$ as sums of covariant polynomials times invariant functions, then these invariant functions will be holomorphic in the Mandelstam invariants s and t except at the image of the singularities in four-momenta space of the corresponding $M_\epsilon{}^c$ functions, provided (a) the total number of basis polynomials equals the dimensionality of the spin space and (b) the basis polynomials for each of the two parity signatures are separately linearly independent at all points at which the number of linearly independent momenta is three.

Our result allows the awkward comparison to the basis sets of Hepp and Williams to be avoided. It reduces the problem to the essential one of the linear independence properties of the proposed basis set.

In another paper we intend to discuss the problem of checking the linear independence of the polynomials of each parity signature. Several theorems that greatly simplify the practical procedure will be given, together with many practical applications.

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APPENDIX A: SPINOR CALCULUS AND LORENTZ TRANSFORMATIONS

This appendix serves to clarify the notation of the main part of the paper and other sources should be consulted for more complete details.[4,5,14-17,24,36] We use the superscript T to indicate the transpose of any matrix and $\dagger$ to indicate the Hermitian conjugate. The Pauli matrices are

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.1}$$

and the convention for any real or complex four-vector z is that $z^\mu = (z^0, \underset{\sim}{z})$.

Any two matrices A and $B \in SL(2,C)$, the group of two dimensional unimodular matrices, define a $\Lambda(A,B) \in \mathcal{L}_+$, the group of proper complex Lorentz transformations, through the relation

$$\sigma \cdot \Lambda(A,B) z = A \, \sigma \cdot z B^T. \tag{A.2}$$

In particular, if $B = A^*$, $\Lambda(A,A^*) \in L_+^\uparrow$ is a real proper orthochronous Lorentz transformation.

The full complex Lorentz group is $\mathcal{L} = \mathcal{L}_+ \cup \mathcal{L}_-$, where any $\Lambda_I \in \mathcal{L}_-$ is an improper Lorentz transformation with $\det \Lambda_I = -1$. In contrast to the real Lorentz group, which has four components because the unit matrix, $I \in L_+^\uparrow$, and the simultaneous reflection of all four

coordinate axes, $-I \in L_+^{\downarrow}$, are not related by any continuous transformation, the complex Lorentz group has only two components, $\mathcal{L}_+$ and $\mathcal{L}_-$, because I and $-I$ are connected by a continuous path in $\mathcal{L}_+$.

By the usual methods, one obtains a $2j+1$ by $2j+1$ matrix $D^{(j)}(A)$ corresponding to any $A \in SL(2,C)$. There always exist real parameters $\underset{\sim}{\phi}$ and $\underset{\sim}{\lambda}$ such that one can write $D^{(j)}(A) = D^{(j)}(H)\, D^{(j)}(V)$ with

$$D^{(j)}(V) = \exp\left(-i\, \underset{\sim}{\phi} \cdot \underset{\sim}{J}^{(j)}\right), \tag{A.3a}$$

$$D^{(j)}(H) = \exp\left(\underset{\sim}{\lambda} \cdot \underset{\sim}{J}^{(j)}\right). \tag{A.3b}$$

The $J^{(j)}$'s are the familiar generators of rotations.

The generalization of (A.2) to arbitrary integer or half odd integer j is then

$$D^{(j)}\left(\sigma \cdot \Lambda(A,B)z\right) = D^{(j)}(A)\, D^{(j)}(\sigma \cdot z)\, D^{(j)}(B^{\dagger}), \tag{A.4a}$$

or, more specifically,

$$D^{(j)}\left(\sigma \cdot \Lambda(A,B)z\right)_{\alpha\dot{\beta}} = D^{(j)}(A)_{\alpha}^{\ \alpha'}\, D^{(j)}(B)_{\dot{\beta}}^{\ \dot{\beta}'}\, D^{(j)}(\sigma \cdot z)_{\alpha'\dot{\beta}'}, \tag{A.4b}$$

which serves to clarify the meaning of lower dotted and lower undotted spinor indices. In particular, for real Lorentz transformations a dotted spinor index transforms like the complex conjugate of an undotted one.

The matrix $D^{(j)}(C)$ is defined by

$$D^{(j)}(C^{-1})^{\alpha\beta} = D^{(j)}(C^{-1})^{\dot\alpha\dot\beta} = (-1)^{j-\alpha}\,\delta^{\alpha,-\beta}, \tag{A.5a}$$

$$D^{(j)}(C)_{\alpha\beta} = D^{(j)}(C)_{\dot\alpha\dot\beta} = (-1)^{2j}\,D^{(j)}(C^{-1})^{\alpha\beta}. \tag{A.5b}$$

An important property is

$$D^{(j)}(C)\,D^{(j)}(A)\,D^{(j)}(C^{-1}) = D^{(j)}(A^{T^{-1}}). \tag{A.6}$$

$D^{(j)}(C^{-1})$, acting from the left on any lower spinor index, turns it into an upper spinor index. Upper undotted spinor indices are therefore acted on from the left by $D^{(j)}(A^{T^{-1}})$ and upper dotted ones are acted on from the left by $D^{(j)}(B^{T^{-1}})$. Contraction of an upper undotted index with a lower undotted one, or of the corresponding dotted indices with each other, yields a scalar.

In addition to (A.6) one has

$$D^{(j)}(C)\,D^{(j)}(\sigma^{T}\cdot z)\,D^{(j)}(C^{-1}) = D^{(j)}(\tilde\sigma\cdot z), \tag{A.7}$$

where $\tilde\sigma_\mu = \sigma^\mu = (\sigma_0, -\underset{\sim}{\sigma})$.

Let k be any real four-vector on the mass shell; that is, $k^2 = m^2$, where $m \neq 0$ is the mass of the particle under consideration. Its rest-frame value is $\bar k = (m,0)$ and the "boost" $L(k)$ is the Hermitian matrix in $L_+^\uparrow$ defined by

$$k = L(k)\,\bar k. \tag{A.8}$$

One may write $L(k) = \Lambda\big(\mathcal{B}(k), \mathcal{B}^*(k)\big)$, where the Hermitian matrix $\mathcal{B}(k) \in SL(2,C)$ is given by

$$\mathcal{B}(k) = (\sigma \cdot k/m)^{\frac{1}{2}} = [2m(m + k^0)]^{-\frac{1}{2}} [m + k^0 + \underset{\sim}{\sigma} \cdot \underset{\sim}{k}]. \qquad (A.9)$$

Corresponding to any $\Lambda \in L_+^{\uparrow}$ and any real four-momentum k on the mass shell, one may define the "Wigner rotation"

$$R(k,\Lambda) = L^{-1}(\Lambda k)\, \Lambda L(k), \qquad (A.10)$$

which is well known to those familiar with the unitary representations of the inhomogeneous Lorentz group. One may write $R(k,\Lambda) = \Lambda(A(k),A(k)^*)$, where $A(k) \in SU(2)$ is given by

$$A(k) = \mathcal{B}^{-1}(\Lambda k)\, A\, \mathcal{B}(k). \qquad (A.11)$$

The following relation is valid for any complex value of k on the mass shell:

$$D^{(j)}(\sigma \cdot k/m)\, D^{(j)}(\tilde{\sigma} \cdot k/m) = I. \qquad (A.12)$$

APPENDIX B: PROOF OF LEMMA 1

The details given here are due to H. P. Stapp and this author. Some definitions we make use of are:

$\mathcal{K}_\ell$: The points $k = \{k_1,\cdots,k_{\ell+1}\}$ in complex four-vector space subject to the conditions $k_i \cdot k_i = (m_i)^2 > 0$; $i = 1,\cdots,\ell+1$, and $\sum_{i=1}^{\ell+1} \epsilon_i k_i = 0$, where $\epsilon_i = \pm 1$.

I_+: The mapping that takes sets in k space to their images $S(k)$ in the space of the $\mathcal{L}_+$ invariants; i.e., $S(k) = I_+(k) = \big(I(k),P(k)\big)$, where $I(k)$ is the set of all inner products formed from the k_i's and $P(k)$ is the set of all pseudoscalars formed from them.

$\mathcal{M}_{\ell+}$: The space of $\mathcal{L}_+$ invariants corresponding to the points of $\mathcal{K}_\ell$; i.e., $\mathcal{M}_{\ell+} \equiv I_+(\mathcal{K}_\ell)$.

To prove Lemma 1 we need

Williams' Lemma[37] (Open mappings from $\mathcal{K}_3$ to $\mathcal{M}_{3+}$). The I_+ image of a neighborhood of a point $k \in \mathcal{K}_3$ is a neighborhood of $S(k) = I_+(k)$ in $\mathcal{M}_{3+}$; i.e., the map I_+: $\mathcal{K}_3 \to \mathcal{M}_{3+}$ is open.

Proof of Lemma 1. According to remark H1.(b) in Part B of Sec. III there is a one-to-one mapping between orbits in $\mathcal{K}_3$ and points $S \in \mathcal{M}_{3+} \equiv I_+(\mathcal{K}_3)$. Thus $\mathcal{F}(S)$ defined by $\mathcal{F}\big(S(k)\big) = F(k)$ is uniquely defined for all $r = 3$ points k lying on $\mathcal{R}(F)$. The set of points $r \leq 2$ is a set of codimension 1 in invariant space, since it is defined by $G(k) = G(S(k)) \equiv G(S) = 0$. If any $r = n$

Because of (49) and (50);

$$\Gamma_{\epsilon\,\alpha_b;\alpha_a}(k) = \epsilon\, D^{(j_b)}\left(\frac{\sigma\cdot k_b}{m_b}\right)_{\alpha_b\dot{\alpha}'_b} D^{(j_a)}\left(\frac{\sigma\cdot k_a}{m_a}\right)_{\alpha_a\dot{\alpha}'_a}\,\Gamma(\tilde{k})_{\alpha'_b;\alpha'_a} \quad . \qquad (C.4)$$

If we multiply each spinor index in (C.3) from the left by $D^{(j_i)}\left(\frac{\sigma\cdot k_i}{m_i}\right)$ and then make use of (C.4), we get:

$$\Gamma_{+\,\alpha_b;\alpha_a}(k) - \Gamma_{-\,\alpha_b;\alpha_a}(k) = 0 \, . \qquad (C.5)$$

The consistency of (C.1) and (C.5) requires that

$$\Gamma_{\epsilon\,\alpha_b;\alpha_a}(k) = 0 \, , \qquad (C.6)$$

for both $\epsilon = +1$ and $\epsilon = -1$, for all choices of $\{\alpha_b;\ \alpha_a\}$. But, as mentioned after (50) in property SC4, the linear independence of the standard covariants of the same parity signature at any $n = 3$ point means that (C.6) cannot be true there for nonzero $\gamma_\epsilon^{(g)}$'s. Consequently, (C.1) cannot be true and statement L1 of the lemma is valid.[39]

We now consider statement L2 of the lemma. First note that each component $Y_\epsilon^{(g)}{}_{\alpha_b;\alpha_a}(k)$, for a fixed value of g, but different values of $\{\alpha_b;\ \alpha_a\}$, is actually a different function. However, as was the case for the $M_\epsilon^{\,c}$ functions in Sec. IV, (49) means that at most N_ϵ of their values, where N_ϵ is given by (38) or (39), whichever is appropriate, can actually be chosen independently at any $n = 3$ point. Thus, the number of standard covariants $Y_\epsilon^{(g)}(k)$ that are linearly independent

for a given ϵ cannot be greater than N_ϵ; i.e., $\bar{N}_\epsilon \leq N_\epsilon$. Since $\bar{N}_+ + \bar{N}_- = \prod_{i=1}^{4} (2j_i + 1) = N_+ + N_-$, one must have $\bar{N}_\epsilon = N_\epsilon$.

Finally, consider statement L3 of the lemma. If, for example, $m_1 = m_3$ and $m_2 = m_4$, we require that standard covariants have definite signature under the simultaneous exchanges $(k_1,\alpha_1) \leftrightarrow (k_3,\alpha_3)$ and $(k_2,\alpha_2) \leftrightarrow (k_4,\alpha_4)$, which is the same as the PT operation for the M^c functions in Table II. Then the same considerations that led to Table III tell us how many linearly independent covariants at most can have a particular signature under this operation, and considerations such as those in the proof of L2 show that this equals the actual number of such standard covariants. Note that, by choosing our covariants to have definite PT signature, we automatically assure that they have definite signature under T. The above considerations are easily extended to the case $m_1 = m_2$ and $m_3 = m_4$, when the covariants are chosen to have definite signature under the simultaneous exchanges $(k_1,\alpha_1) \leftrightarrow (k_2,\alpha_2)$ and $(k_3,\alpha_3) \leftrightarrow (k_4,\alpha_4)$, and to the case of all equal masses, when definite signatures under both types of exchanges mentioned in this paragraph are chosen.

FOOTNOTES AND REFERENCES

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6. D. N. Williams (Department of Physics, University of Michigan, Ann Arbor, Michigan, private communication, has found these standard covariants in the same form as given in Ref. 4. The derivation given by the author of this paper in Ref. 4 is not the original one and does not start from the results of Hepp. Most of the results of Williams and this author have also been found by K. Hardenberg, K. H. Mütter, and W. R. Theis, Nuovo Cimento 62A, 385 (1969).

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21. Rigorous definitions are to be found in Refs. 5 and 11. See also A. S. Wightman in Dispersion Relations and Elementary Particles, edited by C. de Witt and R. Omnes (John Wiley and Sons, Inc., New York, 1960), p. 227.

22. The theorem, as given here, is valid for M^c functions describing the interaction of $\ell + 1$ particles when $\mathcal{K}_3$ is replaced by $\mathcal{K}_\ell$ ($\mathcal{K}_\ell$ is defined in Appendix B). The M^c functions are "connected parts" (see Ref. 18) that are free of four-momentum conservation delta functions.

23. In its most general form, Stapp's theorem (Refs. 10 and 11) guarantees $\mathcal{L}_+$ covariance on the entire multisheeted domain of regularity of a set of tensor-valued functions originally defined to be holomorphic and either $L_+^\uparrow$ covariant or $\mathcal{L}_+$ covariant on some arbitrarily small real or complex domain that does not have to be on the mass shell. It is a generalization of earlier work showing that functions originally defined to be holomorphic and $L_+^\uparrow$ covariant on the so-called "future tube," such as the Wightman

functions in axiomatic field theory, have a single-sheeted $\mathcal{L}_+$-covariant extension to the "extended tube," which is the union of the $\mathcal{L}_+$ orbits of all points on the "future tube." See A. S. Wightman, J. Indian Math. Soc. 24, 625 (1960), and R. Jost in Lectures on Field Theory and the Many-Body Problem, edited by E. R. Caianiello (Academic Press, New York, 1961), p. 127. See also Ref. 24.

24. R. F. Streater and A. S. Wightman, PCT, Spin, and Statistics, and All That (W. A. Benjamin Inc., New York, 1964).

25. The usual definition of the I_+-saturated kernel $\mathcal{U}_a^{(+)}$ of an $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a$ on the domain of regularity, $\mathcal{R}$, of the M^c functions for a given process is: $\mathcal{U}_a^{(+)}$ is the largest subset of $\mathcal{U}_a$ such that if $\mathcal{U}_a^{(+)}$ contains a point then $\mathcal{U}_a^{(+)}$ contains every point whose image in $\mathcal{K}_3$ has the same $\mathcal{L}_+$ invariants (see Refs. 1, 5, and 11). However, for a function defined over $\mathcal{K}_3$ our definition of the I_+-saturated kernel of an $\mathcal{L}_+$-invariant sheet $\mathcal{U}_a$ is equivalent to the usual one. This is a consequence of Lemma 4 of Ref. 5.

26. A general discussion of linear relationships for physical values of the four-momenta using helicity amplitudes is given by J. Daboul, Linear Symmetries of Scattering Amplitudes, Temple University preprint, Nov. 1969.

27. We will not discuss the restrictions that Lorentz covariance places on the derivatives of the M^c functions at $r = 2$ points. The considerations here will prove to be sufficient for specifying the

properties of the "standard covariants" introduced in Sec. 5 when the decomposition of the M^c functions into invariant amplitudes is considered.

28. R. Jost, The General Theory of Quantized Fields (American Mathematical Society, Providence, Rhode Island, 1965).

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30. For a discussion of the crossing properties see Refs. 17 and 19.

31. Note that, because of the kinematical branch points in the "boost" matrices, the mapping from the M^c functions to the S^c functions at any point on the domain of regularity of the former is not single-valued. This fact is not of any importance for the application we have in mind here.

32. We call attention to the fact that in Table III the total number of terms is $(2j_1 + 1)^2(2j_2 + 1)^2$, in agreement with (16), but the total number is less in Tables IV and V, because S-matrix theory and field theory forbid the occurrence of terms with the wrong signature under exchange symmetry and under PCT.

33. In its more general form, the theorem of Hepp and Williams says that one can express any multisheeted invariant function of any number of four-vectors (regardless of whether or not mass-shell

constraints are present) as a holomorphic function of the independent $\mathcal{L}_+$ invariants on the image of the I_+-saturated kernel of its domain of regularity. However, with more than four functionally independent vectors (e.g., a scattering amplitude with more than five particles), it is not possible to use the same set of $\mathcal{L}_+$ invariants globally; in other words one must use "local" $\mathcal{L}_+$-invariant coordinates. The result of Hepp and Williams is a generalization to arbitrary domains of a theorem of Hall and Wightman (Ref. 12) that enables one to express an $\mathcal{L}$-invariant single-sheeted function regular everywhere on the "extended tube" as a function of "local" $\mathcal{L}$-invariant coordinates. (See also Ref. 23.)

34. R. C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables (Prentice-Hall, Inc., Edgewood Cliffs, New Jersey, 1965), p. 21.

35. Our theorem generalizes the approach of Williams (Ref. 5). Williams' arguments were for his particular basis and depended on a certain physically irrelevant decomposition of the scattering functions into "tensor" and "pseudotensor" parts.

36. A. S. Wightman in Dispersion Relations and Elementary Particles, edited by C. de Witt and R. Omnes (John Wiley and Sons, Inc., New York, 1960), p. 159.

37. Lemma 5 and Appendix IV of Ref. 5.

38. Reference 34, p. 19.

39. It is worth noting that if k is an $r = 2$, $n = 3$ point, then $\tilde{k}$ lies on a different $\mathcal{K}_+$ orbit, according to H2.(b), and (C.2) is not valid for any choice of $A_k \in SL(2,C)$. In this case it is possible to satisfy (C.1) with $\Gamma_{+\,\alpha_b;\alpha_a}(k) \neq 0$ and $\Gamma_{-\,\alpha_b;\alpha_a}(k) \neq 0$, in contrast to the result (C.6) for an $r = 3$ point. The standard covariants of signature +1 are not linearly independent of those of signature -1 at any $r = 2$, $n = 3$ point; at such a point the total number of linearly independent standard covariants in the two sets is the same as the number given in Table I.

FIGURE CAPTIONS

Fig. 1. Special reference frame used for obtaining discrete symmetry restrictions. The 2 axis points out of the paper.

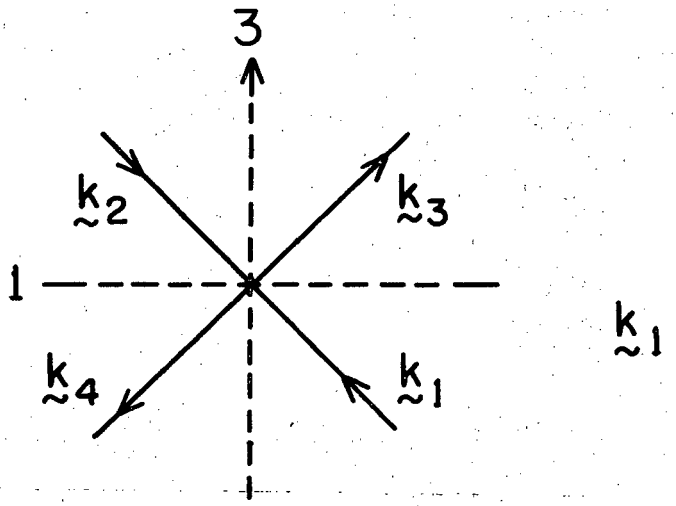

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Fig. 1.

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