Title

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ALL FOUR LIOUVILLE SIGN PATTERNS OCCUR IN THE BINARY DECOMPOSITIONS OF EVERY INTEGER $N \geq 11$

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ABSTRACT. Let λ be the Liouville function. We prove that for every integer $N \geq 2$ outside the set $\mathcal{E} = \{2, 3, 4, 5, 6, 9, 10\}$ and every $(s, t) \in \{\pm 1\}^2$ there are positive integers a, b with $a + b = N$, $\lambda(a) = s$ and $\lambda(b) = t$, and that each $N \in \mathcal{E}$ misses at least one of the four patterns. For this qualitative existence statement, we remove both the Generalised Riemann Hypothesis and the friability restriction from a theorem of Mangerel, and we settle the odd-total and positive-positive variants, raised by Mangerel, of a question of Shusterman whose even-total negative-negative case was settled recently in an anonymous repository. The main new ingredient is a rigidity statement at an odd prime p: if p has no representation $a + b$ with $\lambda(a) = \lambda(b) = \varepsilon$, then a single one-sided exclusion, together with the dilation laws $\lambda(2n) = \lambda(3n) = \lambda(5n) = -\lambda(n)$, already forces the odd completion of λ to be exactly multiplicative, which contradicts quadratic reciprocity. In addition, we give a short proof that every $N \geq 11$ has a pair of unequal signs, which recovers Mangerel's nonextremality theorem $|\sum_{a<N} \lambda(a)\lambda(N-a)| < N-1$. The same arguments show that, for a completely multiplicative function f with values ± 1 and $f(2) = f(3) = f(5) = -1$, a prime $p \geq 11$ can miss one of the four patterns only if f agrees with the Legendre symbol modulo p below $p/2$, so the Legendre symbol is the only obstruction. The main results are formalized in Lean 4 with mathlib, and the final theorem depends only on the standard logical axioms.

1. INTRODUCTION

Goldbach's conjecture asks whether every even integer greater than two is a sum of two primes, and it remains open in spite of the substantial progress made by sieve methods (e.g., Chen's theorem that every sufficiently large even integer is a prime plus a product of at most two primes [1]). A basic obstruction for these methods is the parity problem, i.e., the inability of classical sieves to distinguish integers with an odd number of prime factors from those with an even number (see, e.g., [4]). It is therefore natural to ask what can be said about the parity of the number of prime factors of the two summands, without asking the summands to be prime.

For a positive integer n let $\Omega(n)$ be the number of prime factors of n counted with multiplicity, and let $\lambda(n) = (-1)^{\Omega(n)}$ be the Liouville function. Every prime has $\lambda(p) = -1$, so Goldbach's conjecture implies that every even $N > 2$ can be written as $N = a + b$ with $\lambda(a) = \lambda(b) = -1$. This weaker statement was asked on MathOverflow in 2018 [13], and it is attributed to Shusterman in [10]. For $N \geq 2$ and $(s, t) \in \{\pm 1\}^2$ we say that N *exhibits the pattern* (s, t) if there are positive integers a, b with

$$a + b = N, \qquad \lambda(a) = s, \qquad \lambda(b) = t.$$

The two mixed patterns $(1, -1)$ and $(-1, 1)$ occur together, by interchanging a and b.

Previous work on this question proceeds in three steps. Mangerel [8] proved, by an elementary argument, that the reflected correlation

$$C(N) = \sum_{1 \leq a < N} \lambda(a)\lambda(N - a)$$

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satisfies $|C(N)| < N - 1$ for every $N \geq 11$, so that a mixed pair and an equal pair occur, although it does not specify which equal pattern occurs. In a subsequent preprint [10], Mangerel proved under the Generalised Riemann Hypothesis for Dirichlet L-functions that every sufficiently large even N exhibits $(-1,-1)$, and more generally that every N which is not p_0-friable (for an effectively computable p_0) exhibits all four patterns. Recently, an anonymous repository [2] gave an elementary and formally verified proof that every even $N > 2$ exhibits $(-1,-1)$. Its key step (restated as Theorem 2.2(b) below) is that for every prime $p > 3$ the integer $2p$ exhibits $(1,1)$; the argument uses two exclusions, one at p and one at $2p$, to force the odd completion of λ modulo p to be a quadratic character.

Our main result is the following.

Theorem A. *Let $\mathcal{E} = \{2,3,4,5,6,9,10\}$. Every integer $N \geq 2$ with $N \notin \mathcal{E}$ exhibits all four patterns $(s,t) \in \{\pm 1\}^2$. Each $N \in \mathcal{E}$ misses at least one pattern, namely*

N	2	3	4	5	6	9	10
missing	$(-,-)$, *mixed*	$(+,+)$, $(-,-)$	$(+,+)$	*mixed*	$(+,+)$	$(+,+)$	*mixed*

Theorem A is an unconditional qualitative strengthening of [10, Theorem 1.3]: it needs neither the Generalised Riemann Hypothesis nor a friability condition, and it holds for every N with an explicit and sharp list of exceptions. However, it does not give a lower bound for the number of representations comparable to [10, Proposition 1.4]. The two parts of Theorem A that are new are the equal patterns at odd totals and the pattern $(1,1)$ at even totals, and the first of these rests on the following statement about primes.

Theorem B. *Let $p \geq 5$ be a prime and $\varepsilon \in \{\pm 1\}$. Then there are positive integers a, b with $a + b = p$ and $\lambda(a) = \lambda(b) = \varepsilon$.*

The difficulty in Theorem B, compared with the even case treated in [2], is that a missing pair at an odd prime p gives only one exclusion. Indeed, at $2p$ a missing positive-positive pair also rules out negative-negative pairs at p (by halving), and the two exclusions together make the local identities of [2] close up. At an odd prime there is no second exclusion of this kind, and one of the identities used in [2] fails. However, we show that one exclusion is still enough once it is combined with the dilation law for the prime 5 (Theorem 3.7). The argument has two parts. A summation over two ranges shows that the doubling defect of the odd completion is supported on a very thin set, and the tripling defect is then constant along the orbits of an explicit piecewise-linear map Φ on the odd integers of $(p/3, p/2)$. Most orbits of Φ terminate at points where the defect vanishes. Some primes (e.g., 19, 67, 283, 1279) have periodic orbits, and on these the information coming from 2 and 3 alone is genuinely insufficient (Remark 3.8). Thus we use a short local argument based on the law $\lambda(5n) = -\lambda(n)$, which kills the defect at a suitable point of every orbit, and a well-founded measure replaces the distinction between terminating and periodic orbits.

For the mixed patterns we give an independent argument that does not use [8].

Theorem C. *Every integer $N \geq 11$ can be written as $N = a + b$ with $a, b \geq 1$ and $\lambda(a) \neq \lambda(b)$.*

The proof of Theorem C observes that the property "all pairs have equal signs" passes to divisors, and that at a prime $p \geq 7$ it forces the *even* completion of λ to be exactly multiplicative by -1, 2 and 3, which again contradicts quadratic reciprocity. Combined with the equal pairs supplied by Theorem A, it recovers [8, Theorem 1.2].

Corollary 1.1 (Mangerel)**.** *For every $N \geq 11$ one has $|C(N)| < N - 1$.*

Our arguments use λ only through its complete multiplicativity, its values at 2, 3 and 5 and, in the very last step, its value at one prime $\ell < p/2$ that is a square modulo p. Stopping the same proofs before this last step gives a classification for arbitrary completely multiplicative functions: if $f\colon \mathbb{N} \to \{\pm 1\}$ is completely multiplicative with $f(2) = f(3) =$

$f(5) = -1$ and a prime $p \geq 11$ misses one of the four patterns for f, then $f(n) = \left(\frac{n}{p}\right)$ for $0 < n < p/2$ and $p \equiv 3$ or $1 \pmod 4$ according as an equal or a mixed pattern is missing (Theorem 7.2). This condition cannot be dropped, since the Legendre symbol itself misses two patterns at suitable primes (for example, at $p = 43$ and at $p = 29$), and in this sense Theorems B and C say that λ is not the Legendre symbol modulo p on $(0, p/2)$.

All the results above have been formalized in Lean 4 [7] with mathlib [12], building on the Lean development of [2]. The formal statement of Theorem A carries no hypothesis other than the stated range and exceptions, and its axiom audit reports only the standard logical axioms of Lean (Section 8). In addition, we checked Theorem A directly for every $N \leq 10^7$, and in this range every pattern already has a witness $a \leq 51$.

The paper is organized as follows. Section 2 collects the background results and the scaling lemma. Section 3 proves the one-exclusion rigidity theorem, which is the main technical part of the paper, and Section 4 deduces Theorem B. Section 5 proves Theorem C and Corollary 1.1, and Section 6 completes the proof of Theorem A. Section 7 extends the arguments to general completely multiplicative functions, Section 8 describes the formalization, and Section 9 discusses the role of the prime 5 and a three-colour analogue of Theorem A.

2. Preliminaries

The Liouville function is completely multiplicative and takes the value -1 at every prime, so $\lambda(qn) = -\lambda(n)$ for every prime q and every $n \geq 1$, and $\lambda(n) \in \{\pm 1\}$ for $n \geq 1$. The following scaling observation is used repeatedly.

Lemma 2.1 (Scaling)**.** *If N exhibits (s, t) and $c \geq 1$, then cN exhibits $(\lambda(c)s, \lambda(c)t)$.*

Proof. If $N = a + b$ with $\lambda(a) = s$ and $\lambda(b) = t$, then $cN = ca + cb$ with $\lambda(ca) = \lambda(c)s$ and $\lambda(cb) = \lambda(c)t$. □

For an odd integer $p \geq 3$ every nonzero residue $x \in \mathbb{Z}/p\mathbb{Z}$ has a unique *central representative* $r(x) \in (-p/2, p/2)$, and we write $|x| = |r(x)| \in [1, (p-1)/2]$. For $f\colon \mathbb{N} \to \mathbb{Z}$ we define the odd and even completions

$$F^-(x) = \operatorname{sgn}(r(x))\, f(|x|), \qquad F^+(x) = f(|x|), \qquad F^\pm(0) = 0.$$

Thus $F^-(-x) = -F^-(x)$ and $F^+(-x) = F^+(x)$, and both agree with f on the integers $1 \leq n < p/2$.

We use the following results of [2], all of which are formally verified there.

Theorem 2.2 ([2])**.** *(a) Every even $N > 2$ exhibits $(-1, -1)$.*

(b) For every prime $p > 3$, the integer $2p$ exhibits $(1, 1)$.

(c) (Extension lemma.) Let $p > 3$ be prime and let $F\colon \mathbb{F}_p \to \mathbb{Z}$ satisfy $F(0) = 0$, $F(1) = 1$, $F(x) \neq 0$ for $x \neq 0$, $F(cx) = F(c)F(x)$ for $c \in \{-1, 2, 3\}$ and all x, and $F(ab) = F(a)F(b)$ for all positive integers a, b with $2ab < p$. Then $F(xy) = F(x)F(y)$ for all $x, y \in \mathbb{F}_p$.

(d) Let $p > 3$ be a prime with $p \equiv 3 \pmod 4$. Then there is a prime ℓ with $2\ell < p$ which is a square modulo p.

Part (c) is proved in [2] through a descent on the least positive integer outside the group of exact multipliers, using only the multipliers 2 and 3 and an elementary short-representative lemma. We need the analogue of (d) for $p \equiv 1 \pmod 4$.

Lemma 2.3. *Let $p \geq 13$ be a prime with $p \equiv 1 \pmod 4$. Then there is a prime ℓ with $2\ell < p$ which is a square modulo p.*

Proof. If $p \equiv 1 \pmod 8$ then $\ell = 2$ works, by the second supplementary law. Otherwise $p \equiv 5 \pmod 8$, so $L = (p-1)/4$ is odd and $L > 1$. Let ℓ be a prime divisor of L; then ℓ

is odd, $2\ell \le 2L < p$, and $p \equiv 1 \pmod{\ell}$. Since $p \equiv 1 \pmod 4$, quadratic reciprocity gives $\left(\frac{\ell}{p}\right) = \left(\frac{p}{\ell}\right) = \left(\frac{1}{\ell}\right) = 1$. □

3. One exclusion at an odd modulus

In this section $p \ge 5$ is an odd integer (primality is not needed here) and $g\colon \mathbb{N} \to \mathbb{Z}$ satisfies

(G1) $g(n) \in \{\pm 1\}$ for $0 < n < p$;
(G2) $g(qn) = -g(n)$ for $q \in \{2, 3, 5\}$ whenever $n \ge 1$ and $qn < p$;
(G3) there are no positive integers a, b with $a + b = p$ and $g(a) = g(b) = 1$.

The intended example is $g = \varepsilon\lambda$, for which (G3) says that p does not exhibit $(\varepsilon, \varepsilon)$. The law (G2) for $q = 5$ is only used in Lemma 3.6.

We work with three integer-valued defects,

$$A(x) = g(x) - g(p - 2x), \qquad T_1(x) = g(x) - g(p - 3x), \qquad T_2(x) = g(x) + g(3x - p),$$

which we only evaluate for $p/4 < x < p/2$, $p/6 < x < p/3$ and $p/3 < x < p/2$ respectively. They take values in $\{-2, 0, 2\}$. If G denotes the odd completion of g, then $A(x) = G(2x) + G(x)$ on $(p/4, p/2)$, $T_1(x) = G(3x) + G(x)$ on $(p/6, p/3)$ and $T_2(x) = G(3x) + G(x)$ on $(p/3, p/2)$, while the laws (G2) say that $G(2x) + G(x) = 0$ for $0 < x < p/4$ and $G(3x) + G(x) = 0$ for $0 < x < p/6$. Therefore the vanishing of A, T_1 and T_2 on their ranges is equivalent to the exact laws $G(2x) = -G(x)$ and $G(3x) = -G(x)$ for all x (Section 4). Since p is odd, none of the endpoints $p/8$, $p/6$, $p/4$, $3p/8$, $5p/12$, $p/2$ is an integer, and all the ranges below are open.

Lemma 3.1 (Signs)**.** *The defects are nonnegative: $A(x) \ge 0$ for $p/4 < x < p/2$, $T_1(x) \ge 0$ for $p/6 < x < p/3$, and $T_2(x) \ge 0$ for $p/3 < x < p/2$.*

Proof. For $p/4 < x < p/2$ the pair $(2x, p - 2x)$ sums to p and $g(2x) = -g(x)$, so (G3) excludes $g(x) = -1$, $g(p - 2x) = 1$, which is the only configuration with $A(x) < 0$. The same argument with $(3x, p - 3x)$ gives $T_1(x) \ge 0$. For $p/3 < x < p/2$ put $u = p - 2x \in (0, p/3)$ and $v = 3x - p \in (0, p/2)$, so that $3u + 2v = p$. By (G2) we have $g(3u) = -g(u)$ and $g(2v) = -g(v)$, and (G3) applied to $(3u, 2v)$ excludes $g(u) = g(v) = -1$. In addition, $A(x) \ge 0$ gives $g(x) \ge g(u)$. Hence $T_2(x) = g(x) + g(v) \ge g(u) + g(v) \ge 0$. □

Lemma 3.2 (The commuting identity on four ranges)**.**
(R1) If $p/8 < x < p/6$, then $A(3x) = T_1(2x)$.
(R2) If $p/6 < x < p/4$, then $-A(p - 3x) = T_1(x) + T_2(2x)$.
(R3) If $p/4 < x < p/3$, then $A(x) = T_1(x) - T_2(p - 2x)$.
(R4) If $p/3 < x < 5p/12$, then $A(x) = T_2(x) - T_1(p - 2x)$; if $5p/12 < x < p/2$, then $A(3x - p) + A(x) = T_2(x)$.

Proof. Each identity follows by expanding the definitions and using (G2) once or twice. For example, in (R3) we have $3(p-2x)-p = 2(p-3x)$ with $2(p-3x) < p$, so $g(3(p-2x)-p) = -g(p-3x)$ and $T_1(x)-T_2(p-2x) = g(x)-g(p-2x)$. In (R2) we use $p-2(p-3x) = 3(2x)-p$ and $g(2x) = -g(x)$. In the first part of (R4) we use $p - 3(p - 2x) = 2(3x - p)$, and in the second part $p - 2(3x - p) = 3(p - 2x)$. Finally (R1) follows from $g(3x) = -g(x) = g(2x)$. (These are the specializations of the identity $A(3x) + A(x) = T(x) + T(2x)$ for the odd completion, which holds because multiplication by 2 and by 3 commute.) □

Corollary 3.3. *If $p/6 < x < p/4$ then $A(p - 3x) = T_1(x) = T_2(2x) = 0$. In particular T_1 vanishes on $(p/6, p/4)$ and T_2 vanishes on the even integers of $(p/3, p/2)$.*

Proof. By Lemma 3.1 the left-hand side of (R2) is at most 0 and the right-hand side is at least 0, and the three arguments lie in the ranges where the defects are nonnegative. □

Let Q, R and X be the sets of integers in $(p/4, p/3)$, $(p/3, p/2)$ and $(p/8, p/6)$, respectively, let $S \subset R$ be the set of multiples of 3 in $(3p/8, p/2)$, and let R_+ be the set of integers in $(5p/12, p/2)$.

Lemma 3.4 (Support of the doubling defect)**.** *We have $A(x) = 0$ for every $x \in Q$ and every $x \in R \setminus S$, and $A(3x - p) = 0$ for every $x \in R_+$.*

Proof. Sum (R3) over $x \in Q$ and (R4) over $x \in R$. The map $x \mapsto p - 2x$ is a bijection from Q onto the odd integers of R, and from $R \cap (p/3, 5p/12)$ onto the odd integers of $(p/6, p/3)$, on the part of which below $p/4$ the defect T_1 vanishes by Corollary 3.3. Using also that T_2 vanishes on the even integers of R, we obtain

$$\sum_{x \in Q} A(x) + \sum_{x \in R} A(x) + \sum_{x \in R_+} A(3x - p) = \sum_{z \in Q,\ z \text{ even}} T_1(z).$$

By (R1) and the bijections $x \mapsto 2x$ from X onto the even integers of Q and $x \mapsto 3x$ from X onto S, the right-hand side equals $\sum_{y \in S} A(y)$. Since $S \subset R$, cancelling this sum leaves

$$\sum_{x \in Q} A(x) + \sum_{x \in R \setminus S} A(x) + \sum_{x \in R_+} A(3x - p) = 0,$$

in which every term is nonnegative by Lemma 3.1 (note that $3x - p \in (p/4, p/2)$ for $x \in R_+$). Therefore every term vanishes. □

Let D be the set of odd integers in $(p/3, p/2)$. The next lemma shows that T_2 is transported along an explicit map on D.

Lemma 3.5 (The map Φ)**.** *Let $y \in D$.*

(i) If $y < 3p/8$, then $4y - p \in D$ and $T_2(y) = T_2(4y - p)$.
(ii) If $y > 3p/8$ and $3 \nmid y$, then $T_2(y) = 0$.
(iii) If $y > 3p/8$ and $y = 3k$, then $p - 4k \in D$ and $T_2(y) = T_2(p - 4k)$.

Proof. (i) Here $y < 3p/8 < 5p/12$ and $y \in R \setminus S$, so the first part of (R4) and Lemma 3.4 give $T_2(y) = T_1(z)$ with $z = p - 2y \in Q$. Then (R3) at z and $A(z) = 0$ give $T_1(z) = T_2(p - 2z) = T_2(4y - p)$. For (ii) and (iii) note that $p - 2y < p/4$, so $T_1(p - 2y) = 0$ when $y < 5p/12$ (Corollary 3.3) and $A(3y - p) = 0$ when $y > 5p/12$ (Lemma 3.4); in both cases (R4) gives $T_2(y) = A(y)$. If $3 \nmid y$ then $y \in R \setminus S$ and $A(y) = 0$. If $y = 3k$ then $k \in X$, (R1) gives $A(3k) = T_1(2k)$ with $2k \in Q$, and (R3) at $2k$ together with $A(2k) = 0$ gives $T_1(2k) = T_2(p - 4k)$. The images in (i) and (iii) are odd and lie in $(p/3, p/2)$ by direct computation. □

The map $\Phi\colon y \mapsto 4y - p$ (for $y < 3p/8$) and $y \mapsto p - 4y/3$ (for $y > 3p/8$, $3 \mid y$) can have periodic orbits, for instance $7 \mapsto 9 \mapsto 7$ for $p = 19$ and $451 \mapsto 525 \mapsto 579 \mapsto 507 \mapsto 603 \mapsto 475 \mapsto 621 \mapsto 451$ for $p = 1279$. The next lemma, which is the only place where the law for $q = 5$ is used, shows that the defect vanishes at the points where a periodic orbit re-enters the lower branch.

Lemma 3.6 (A local contradiction)**.** *Let $y \in D$ with $y < 3p/8$ and $y \equiv p \pmod 4$. Then $T_2(y) = 0$.*

Proof. Suppose that $T_2(y) \neq 0$, so that $T_2(y) = 2$ by Lemma 3.1. Put

$$w = \frac{3y - p}{2}, \qquad s = \frac{p - y}{2}, \qquad z = \frac{5y - p}{4}, \qquad k = \frac{p - 5w}{2}.$$

Since $y \equiv p \pmod 4$ and y is odd, s is even, z is an integer, w is odd and k is an integer. From $p/3 < y < 3p/8$ one checks that $w \in (0, p/16)$, $s \in (5p/16, p/3)$, $z \in (p/6, 7p/32)$ and $k \in (11p/32, p/2)$, and that $2s$, $5s/2$, $p - 5w$ and $3z$ all lie in $(p/2, p)$. In addition, $2s + y = p$, $5s/2 + z = p$, $(p - 5w) + 5w = p$ and $3z + k = p$.

Since $3y \equiv 2w \pmod p$ with $2w < p/2$, we have $T_2(y) = g(y) - g(w)$, so $g(y) = 1$ and $g(w) = -1$. On the one hand, (G3) for the pair $(2s, y)$ gives $g(2s) = -1$, hence $g(s) = 1$, hence $g(s/2) = -1$, hence $g(5s/2) = 1$, and (G3) for $(5s/2, z)$ gives $g(z) = -1$. On the other hand, $g(5w) = -g(w) = 1$, so (G3) for $(p - 5w, 5w)$ gives $g(2k) = -1$, hence $g(k) = 1$, and (G3) for $(3z, k)$ gives $g(3z) = -1$, hence $g(z) = 1$. This contradiction proves the lemma. □

Theorem 3.7 (One-exclusion rigidity)**.** *Assume (G1)–(G3), and assume in addition that $3 \nmid p$ and $7 \nmid p$. Then*

$$g(p - 2n) = g(n) \quad \text{for } p/4 < n < p/2, \qquad g(3n - p) = -g(n) \quad \text{for } p/3 < n < p/2.$$

Proof. We first show that $T_2(y) = 0$ for every $y \in D$. For $y \in D$ define

$$\mu(y) = \begin{cases} 2p - y, & y < 3p/8, \\ p - |7y - 3p|, & y > 3p/8, \end{cases}$$

so that $\mu(y) > p$ on the lower branch and $0 < \mu(y) \le p$ on the upper branch. We argue by strong induction on $\mu(y)$. If $y < 3p/8$ and $y \equiv p \pmod 4$, Lemma 3.6 applies. If $y < 3p/8$ and $y \not\equiv p \pmod 4$, then $T_2(y) = T_2(4y - p)$ by Lemma 3.5(i), and $\mu(4y - p) < \mu(y)$, since either $4y - p$ is again on the lower branch and larger than y, or it is on the upper branch. If $y > 3p/8$ and $3 \nmid y$, Lemma 3.5(ii) applies. Finally let $y = 3k > 3p/8$ and $y' = p - 4k$, so that $T_2(y) = T_2(y')$ by Lemma 3.5(iii). If $y' < 3p/8$, then $y' \equiv p \pmod 4$ and Lemma 3.6 gives $T_2(y') = 0$. If $y' > 3p/8$, then

$$|7y' - 3p| = 4|p - 7k| > 3|7k - p| = |7y - 3p|,$$

where the strict inequality uses $7 \nmid p$, so that $\mu(y') < \mu(y)$ and the induction hypothesis applies.

Together with Corollary 3.3, this gives $T_2 = 0$ on all of $(p/3, p/2)$, which is the second identity. For the first, $A(n) = 0$ for $n \in Q$ and $n \in R \setminus S$ by Lemma 3.4, and for $n \in S$ the proof of Lemma 3.5(ii),(iii) shows $A(n) = T_2(n) = 0$. Since $3 \nmid p$, the sets Q and R cover all integers in $(p/4, p/2)$. □

Remark 3.8. The law for $q = 5$ cannot be dropped from this argument. For the weaker hypotheses in which (G2) is imposed only for $q \in \{2, 3\}$ (and (G3) only for the pairs $(m, p - m)$ with $2 \mid m$ or $3 \mid m$), a 2-SAT computation finds admissible functions g with nonzero defects at the primes $p = 1051, 1279, 9397, 16411$ (and at $p = 7$), all of which carry a periodic orbit of Φ. Once the law for 5 is added, every such defect scenario becomes unsatisfiable, in agreement with Lemma 3.6. We do not claim that every proof of Theorem B must use the prime 5. There are 21 primes below 2×10^6 at which Φ has a periodic orbit, and each such prime divides a number of the form $4^n + 3^j$ (e.g., $13 \cdot 1279 = 4^7 + 3^5$), because modulo p the two branches of Φ multiply by 4 and by $-4/3$, respectively.

4. Equal-sign pairs at primes

Proof of Theorem B. For $p = 5$ we have $5 = 1 + 4 = 2 + 3$, and for $p = 7$ we have $7 = 1 + 6 = 2 + 5$. Let $p \ge 11$ be prime and suppose that p does not exhibit $(\varepsilon, \varepsilon)$. Then $g = \varepsilon\lambda$ satisfies (G1)–(G3), and $3 \nmid p$, $7 \nmid p$. By Theorem 3.7 (after dividing by ε),

$$\lambda(p - 2n) = \lambda(n) \ \ (p/4 < n < p/2), \qquad \lambda(3n - p) = -\lambda(n) \ \ (p/3 < n < p/2). \tag{4.1}$$

Let F be the odd completion of λ modulo p. We claim that $F(2x) = -F(x)$ and $F(3x) = -F(x)$ for all $x \in \mathbb{F}_p$. By oddness it is enough to take $x = n$ with $0 < n < p/2$. If $4n < p$ then $F(2n) = \lambda(2n) = -\lambda(n)$; if $4n > p$ then $2n \equiv -(p - 2n)$ and $F(2n) = -\lambda(p - 2n) = -\lambda(n)$ by (4.1). If $6n < p$ then $F(3n) = \lambda(3n) = -\lambda(n)$; if $3n > p$ then $3n \equiv 3n - p$ and $F(3n) = \lambda(3n - p) = -\lambda(n)$. In the remaining range $p/6 < n < p/3$ we apply the case

already proved to $2n$ or to $p-2n$ (both in $(p/3, p/2)$ according as $n < p/4$ or $n > p/4$), and use that multiplication by 2 commutes with multiplication by 3.

Thus F satisfies the hypotheses of Theorem 2.2(c): $F(-1) = -1$, $F(2) = F(3) = -1$ are exact multipliers, and $F(ab) = \lambda(ab) = \lambda(a)\lambda(b) = F(a)F(b)$ whenever $2ab < p$. Hence F is multiplicative on $\mathbb{F}_p$, so $F(y^2) = 1$ for every $y \neq 0$. Since $F(-1) = -1$, the residue -1 is not a square and $p \equiv 3 \pmod 4$. Let ℓ be the prime given by Theorem 2.2(d). Then $1 = F(\ell) = \lambda(\ell) = -1$, a contradiction. □

5. Unequal pairs

We say that N is *all-equal* if $\lambda(N-a) = \lambda(a)$ for every $0 < a < N$. Theorem C states that no $N \geq 11$ is all-equal. The integers 2, 5 and 10 are all-equal.

Lemma 5.1 (Divisors)**.** *If dc is all-equal with $c \geq 1$ and $d \geq 2$, then d is all-equal.*

Proof. For $0 < a < d$ apply the hypothesis to ca; since $dc - ca = c(d-a)$, we obtain $\lambda(c)\lambda(d-a) = \lambda(c)\lambda(a)$, and $\lambda(c) = \pm 1$. □

Proposition 5.2. *No prime $p \geq 7$ is all-equal.*

Proof. Suppose that p is all-equal and let G be the even completion of λ modulo p. We first show that $G(2x) = -G(x)$ and $G(3x) = -G(x)$ for every $x \in \mathbb{F}_p$; by evenness it is enough to take $x = n$ with $0 < n < p/2$. If $4n < p$ or $6n < p$ the claims follow from $\lambda(2n) = \lambda(3n) = -\lambda(n)$. If $4n > p$, then $G(2n) = \lambda(p-2n) = \lambda(2n) = -\lambda(n)$ by the all-equal property. If $p/6 < n < p/3$, then similarly $G(3n) = \lambda(p-3n) = \lambda(3n) = -\lambda(n)$. If $p/3 < n < p/2$, then $3n \equiv 3n - p$ with $0 < 3n - p < p/2$, and using the all-equal property at $6n - 2p$ and at $2n$ we find

$$\lambda(3n-p) = -\lambda(6n-2p) = -\lambda(3p-6n) = \lambda(p-2n) = \lambda(2n) = -\lambda(n).$$

Therefore G satisfies the hypotheses of Theorem 2.2(c) with $G(-1) = 1$ and $G(2) = G(3) = -1$, and G is multiplicative on $\mathbb{F}_p$. By Theorem 2.2(d) when $p \equiv 3 \pmod 4$, and by Lemma 2.3 when $p \equiv 1 \pmod 4$ (note that $p \neq 5, 9$), there is a prime ℓ with $2\ell < p$ which is a square modulo p. Hence $1 = G(\ell) = \lambda(\ell) = -1$, a contradiction. □

Proof of Theorem C. We first record that 3, 4, 6, 7, 8, 9, 20 and 25 are not all-equal, as shown by the pairs $1+2$, $1+3$, $1+5$, $3+4$, $1+7$, $1+8$, $1+19$ and $3+22$, respectively. We prove by strong induction that no $N \geq 11$ is all-equal. If $N = 2m$ is even and all-equal, then m is all-equal by Lemma 5.1; if $m \geq 11$ this contradicts the induction hypothesis, if $6 \leq m \leq 9$ it contradicts the list above, and $m = 10$ gives $N = 20$, which is on the list. If N is an odd prime, Proposition 5.2 applies. Otherwise let d be the least prime factor of an odd composite N, so that $d \geq 3$, $d^2 \leq N$, and d is all-equal by Lemma 5.1. If $d \geq 7$ this contradicts Proposition 5.2, and $d = 3$ contradicts the list. If $d = 5$, then $e = N/5 \geq 5$ is odd and all-equal by Lemma 5.1, which contradicts the induction hypothesis when $e \geq 11$ and the list when $e \in \{7, 9\}$; if $e = 5$, then $N = 25$, which is on the list. □

Proof of Corollary 1.1. Each of the $N-1$ terms of $C(N)$ equals ± 1. By Theorem C at least one term equals -1, and by Theorem A (whose equal-sign part does not use Corollary 1.1) at least one term equals $+1$. Hence $|C(N)| \leq N - 3$. In fact the terms for a and $N-a$ coincide, so the -1 term given by Theorem C occurs at least twice, and $C(N) \leq N - 5$. Similarly, if N is odd, the equal pair given by Theorem A yields two terms equal to $+1$; if N is even, the term with $a = N/2$ equals $+1$, and at least one of the $(1,1)$ and $(-1,-1)$ decompositions given by Theorem A has $a \neq N/2$ and yields two further terms equal to $+1$. In both cases $C(N) \geq -(N-5)$, so that $|C(N)| \leq N - 5$. □

6. Proof of Theorem A

Let $N \geq 2$ with $N \notin \mathcal{E}$.

The pattern $(-1,-1)$ *at even* N is Theorem 2.2(a), since $N \neq 2$.

The pattern $(1,1)$ *at even* $N = 2m$. If $m = 1$ we use $1+1$. If m is a prime, then $m > 3$ since $N \neq 4, 6$, and Theorem 2.2(b) applies. If $m = rk$ is composite with r prime and $k \geq 2$, then $2k > 2$ exhibits $(-1,-1)$ by Theorem 2.2(a), and Lemma 2.1 with $c = r$ gives $(1,1)$ at $2rk = N$.

The equal patterns at odd N, where $N \neq 3, 9$. If N has a prime factor $r \geq 5$, write $N = rc$. Given ε, Theorem B gives the pattern $(\lambda(c)\varepsilon, \lambda(c)\varepsilon)$ at r, and Lemma 2.1 gives $(\varepsilon, \varepsilon)$ at N. Otherwise $N = 3^j$ with $j \geq 3$; since $27 = 1 + 26 = 7 + 20$ exhibits both equal patterns, Lemma 2.1 with $c = 3^{j-3}$ covers N.

The mixed patterns. For $N \geq 11$ this is Theorem C, and interchanging the summands gives the other mixed pattern. The remaining values are $N = 7 = 3+4$ and $N = 8 = 1+7$.

Sharpness. For $N \in \mathcal{E}$ one lists all decompositions: $2 = 1+1$; $3 = 1+2$; $4 = 1+3 = 2+2$; $5 = 1+4 = 2+3$; $6 = 1+5 = 2+4 = 3+3$; $9 = 1+8 = 2+7 = 3+6 = 4+5$; $10 = 1+9 = 2+8 = 3+7 = 4+6 = 5+5$. Using $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = 1$ and $\lambda(2) = \lambda(3) = \lambda(5) = \lambda(7) = \lambda(8) = -1$ one reads off the table in Theorem A. □

7. Completely multiplicative functions

The proofs of Sections 3–5 use the Liouville function only through its complete multiplicativity, its values $\lambda(2) = \lambda(3) = \lambda(5) = -1$, and, in the last line of each proof, the value $\lambda(\ell) = -1$ at a prime quadratic residue $\ell < p/2$. In this section we record what the same arguments give when this last step is omitted. We need the following elementary classification.

Lemma 7.1. *Let p be an odd prime and let $F\colon \mathbb{F}_p \to \mathbb{Z}$ be multiplicative with $F(x) \in \{\pm 1\}$ for $x \neq 0$. If $F(c) = -1$ for some $c \in \mathbb{F}_p$, then $F(x) = \left(\frac{x}{p}\right)$ for every $x \neq 0$.*

Proof. For $y \neq 0$ we have $F(y^2) = F(y)^2 = 1$, so F equals 1 on the nonzero squares, and in particular c is not a square. If $x \neq 0$ is not a square, then cx is a square, and $1 = F(cx) = F(c)F(x) = -F(x)$. □

Theorem 7.2. *Let $f\colon \mathbb{N} \to \mathbb{Z}$ be completely multiplicative with $f(n) \in \{\pm 1\}$ for $n \geq 1$ and $f(2) = f(3) = -1$, and let p be a prime.*

(a) If $f(5) = -1$, $p \geq 11$, $\varepsilon \in \{\pm 1\}$, and there are no positive integers a, b with $a+b = p$ and $f(a) = f(b) = \varepsilon$, then $p \equiv 3 \pmod 4$ and $f(n) = \left(\frac{n}{p}\right)$ for $0 < n < p/2$.

(b) If $p \geq 7$ and $f(a) = f(p-a)$ for every $0 < a < p$, then $p \equiv 1 \pmod 4$ and $f(n) = \left(\frac{n}{p}\right)$ for $0 < n < p/2$.

Proof. (a) The function $g = \varepsilon f$ satisfies (G1)–(G3), since $f(qn) = f(q)f(n) = -f(n)$ for $q \in \{2, 3, 5\}$, and $3 \nmid p$, $7 \nmid p$. Theorem 3.7 then gives (4.1) with λ replaced by f, and the first paragraph of the proof of Theorem B, which uses only $f(2) = f(3) = -1$, shows that the odd completion F of f satisfies $F(2x) = -F(x)$ and $F(3x) = -F(x)$ for all $x \in \mathbb{F}_p$. Since $F(ab) = f(a)f(b) = F(a)F(b)$ whenever $2ab < p$, Theorem 2.2(c) shows that F is multiplicative on $\mathbb{F}_p$. Since $F(-1) = -1$, Lemma 7.1 with $c = -1$ gives $F(x) = \left(\frac{x}{p}\right)$ for $x \neq 0$; hence $f(n) = F(n) = \left(\frac{n}{p}\right)$ for $0 < n < p/2$, and $\left(\frac{-1}{p}\right) = -1$, i.e., $p \equiv 3 \pmod 4$.

(b) The proof of Proposition 5.2 uses only $f(2) = f(3) = -1$ and complete multiplicativity up to the point where the even completion G of f is shown to be multiplicative on $\mathbb{F}_p$. Since $G(2) = -1$, Lemma 7.1 with $c = 2$ gives $G(x) = \left(\frac{x}{p}\right)$ for $x \neq 0$, so $f(n) = \left(\frac{n}{p}\right)$ for $0 < n < p/2$, and $\left(\frac{-1}{p}\right) = G(-1) = G(1) = 1$ gives $p \equiv 1 \pmod 4$. □

Mangerel [9] proves a related rigidity statement of a different, approximate nature: a completely multiplicative ± 1-valued function whose exponential sums satisfy a Gauss-sum-like dilation symmetry must essentially coincide with a real character. Theorem 7.2 may be viewed as an exact additive analogue at a single prime.

Remark 7.3. Both parts of Theorem 7.2 are sharp. Let p be a prime and let f be the completely multiplicative function with $f(q) = \left(\frac{q}{p}\right)$ for the primes $q \neq p$ and $f(p) = -1$, so that $f(n) = \left(\frac{n}{p}\right)$ whenever $p \nmid n$. If $p \equiv 3 \pmod 8$ and $\left(\frac{3}{p}\right) = \left(\frac{5}{p}\right) = -1$ (for example, $p = 43$, 67 and 163), then $f(2) = f(3) = f(5) = -1$ and $f(p-a) = \left(\frac{-1}{p}\right) f(a) = -f(a)$ for every $0 < a < p$, so p misses both equal patterns for f. Similarly, if $p \equiv 5 \pmod 8$ and $\left(\frac{3}{p}\right) = -1$ (for example, $p = 29$ and 101), then $f(2) = f(3) = -1$ and $f(p-a) = f(a)$ for every $0 < a < p$, so p misses both mixed patterns. Consequently, Theorems B and C hold for λ precisely because λ differs from $\left(\frac{\cdot}{p}\right)$ at some prime below $p/2$, as guaranteed by Theorem 2.2(d) and Lemma 2.3. We note that this is also consistent with our numerical checks (Section 9), in which the smallest pattern count relative to $(N-1)/4$ occurs at $N = 163$, a prime at which $\lambda(n) = \left(\frac{n}{163}\right)$ for every $n < 41$.

8. Formal verification

Theorems A–C, Corollary 1.1 (in the form $|C(N)| < N - 1$), Theorem 3.7 and Theorem 7.2 have been formalized in Lean 4 (version `v4.34.0-rc2`) with mathlib (commit `de2ef682`), on top of the Lean development of [2] (commit `da3c528c`), whose build and axiom audit we reproduced first. The formalization consists of two libraries.

- `SignPatterns` (179 lines) proves the pattern $(1, 1)$ at even totals and the sharpness statements at 2, 4, 6 and 10.
- `OddCase` (1714 lines, seven files) proves Section 3 for an abstract function g (`OneExcl.exact_bands`), Theorem B (`equal_pair_at_prime`), Theorem C (`mixedPair_of_ge`), Corollary 1.1 (`mangerel_nonextremality`) and Theorem A (`all_four_patterns_unconditional`), together with the sharpness at 3, 5 and 9. The file `General.lean` proves Theorem 7.2 for an arbitrary completely multiplicative f (`legendre_of_no_equal_pair` and `legendre_of_no_mixed_pair`), with Lemma 7.1 in place of the final contradiction.

The final statement reads

```
theorem all_four_patterns_unconditional {N : ℕ} (hN : 2 ≤ N)
    (h2 : N ≠ 2) (h3 : N ≠ 3) (h4 : N ≠ 4) (h5 : N ≠ 5) (h6 : N ≠ 6)
    (h9 : N ≠ 9) (h10 : N ≠ 10) {s t : ℤ} (hs : s = 1 ∨ s = -1)
    (ht : t = 1 ∨ t = -1) : SignPair N s t
```

where `SignPair N s t` means $\exists\, a\, b > 0$ with $a + b = N$, $\lambda(a) = s$ and $\lambda(b) = t$, and λ is mathlib's `ArithmeticFunction.liouville`. The command `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound` for this theorem and for every result listed above; no `sorry`, `native_decide` or additional axiom occurs. The formalization uses Theorem 2.2 through its Lean proofs in [2], so no external result enters as a hypothesis. The formalization also made two small points explicit: Lemma 3.4 needs only that p is odd (not $3 \nmid p$), and the law for $q = 5$ is used only in Lemma 3.6. In the formal proof, the induction in Theorem 3.7 is carried out on the measure μ exactly as above. The Lean source, the pinned dependency manifest and the axiom audit of a fresh build are available at `https://github.com/lsmeng/liouville-sign-patterns` (commit `25210d9`); the background development of [2] is fetched as a pinned dependency rather than copied. The sharpening $|C(N)| \leq N - 5$ in the proof of Corollary 1.1 and the remarks of Section 9 are not formalized.

9. Remarks and questions

Numerical checks. We verified Theorem A directly for every $N \le 10^7$, and an independent reimplementation confirmed it for $N \le 10^6$. In this range the least witnesses are small (at most 51 for $(1,1)$, 50 for $(-1,-1)$ and 24 for the mixed patterns), which suggests that a bound such as $a \ll \log N$ for the least witness may hold, although our method, which argues by contradiction, gives no information on the size of a.

Quantitative bounds. Theorem A gives one decomposition of each pattern, but no lower bound for their number, in contrast to [10, Proposition 1.4], which assumes the Generalised Riemann Hypothesis. Krishnamoorthy [6] has recently obtained cancellation in Goldbach-type sums for λ, with applications to sign patterns, and Cantarini, Gambini and Zaccagnini [5] study weighted averages of related convolutions. We note that an unconditional lower bound of positive proportion is likely to be difficult, for the following reason. Let $p \equiv 3 \pmod 4$ be prime, let $E(p)$ be the number of $0 < a < p$ with $\lambda(a) = \lambda(p-a)$, and let $D = \{0 < n < p : \ \lambda(n) \neq \left(\frac{n}{p}\right)\}$. Since $\left(\frac{p-n}{p}\right) = -\left(\frac{n}{p}\right)$, the signs $\lambda(a)$ and $\lambda(p-a)$ agree exactly when one of a and $p-a$ lies in D, so that

$$E(p) = 2\big(|D| - |D \cap (p - D)|\big) \le 2|D|,$$

and for $p \equiv 1 \pmod 4$ the same identity holds for the number of unequal-sign decompositions. Consequently, $E(p)$ is small whenever λ agrees with the Legendre symbol on most of $(0,p)$. This happens at $p = 163$, where $\lambda(n) = \left(\frac{n}{163}\right)$ for every $n < 41$, $|D| = 21$ and $E(163) = 38$, the smallest value of $E(p)/p$ ($\sim$0.23) that we found. More generally, $|D| \le p \sum_{q<p,\ (q/p)=1} 1/q$, since every $n \in D$ has a prime factor q with $\left(\frac{q}{p}\right) = 1$. An exceptional real zero of $L(s, \left(\frac{\cdot}{p}\right))$ is expected to force $\left(\frac{q}{p}\right) = -1$ for all but few primes $q \in [p^{\varepsilon}, p]$ (see, e.g., [11] and the references therein); if this bias were strong enough that $\sum_{p^{\varepsilon} \le q < p,\ (q/p)=1} 1/q = o(1)$, and if in addition no prime below p^{ε} were a quadratic residue modulo p, then $|D| = o(p)$ and $E(p) = o(p)$. Hence a bound $E(p) \ge \eta p$ for all large primes p would exclude this scenario, and this appears to be beyond current methods; Mangerel makes a related observation about exceptional moduli in [10, Remark 6]. The construction of Remark 7.3 suggests the natural order of magnitude: if f agrees with $\left(\frac{\cdot}{p}\right)$ at every prime except $f(\ell) = -1$ for one prime quadratic residue ℓ, then D is the set of n with $v_\ell(n)$ odd and $D \cap (p - D) = \emptyset$, so that the number of equal-sign decompositions for f is $2\#\{0 < n < p : \ v_\ell(n) \text{ odd}\} = 2p/(\ell+1) + O(\log p/\log \ell)$. This suggests that a bound of the shape $E(p) \gg p/\ell(p)$, with $\ell(p)$ the least prime quadratic residue modulo p, may be the natural unconditional target, although our arguments, which proceed by contradiction, do not give it.

Numerically the counts are much larger. For all primes $10^3 < p \le 10^7$ the normalized deviation $\sqrt{p}\,(E(p)/p - 1/2)$ has mean $\sim$0 and standard deviation $\sim$0.707, close to $1/\sqrt{2}$ (the value for $(p-1)/2$ independent signs), and it stays in $[-3.4, 3.5]$. Above $p = 10^3$ we found no correlation between $E(p)/p$ and $\ell(p)$ or the class number $h(-p)$. This behaviour is consistent with the conjecture of Corrádi and Kátai that $C(N) = o(N)$ (see [8]), and it suggests that $C(p)$ is typically of size $\sqrt{p}$.

A three-colour analogue. Theorem A controls $\Omega(a)$ and $\Omega(b)$ modulo 2. A natural family of analogues asks, for a fixed $k \ge 3$, whether every sufficiently large N can be written as $a + b$ with $\Omega(a) \equiv \Omega(b) \equiv 1 \pmod k$. For a single N and $k > \log_2 N$ this condition is Goldbach's condition $\Omega(a) = \Omega(b) = 1$. However, the fixed-$k$ statements do not approach Goldbach's conjecture without a uniform control of their thresholds in k, and they are nested only along divisibility of the moduli (e.g., $\Omega = 4$ satisfies the condition for $k = 3$ but not for $k = 2$), so we regard them as a separate family of problems rather

than as steps towards Goldbach. Our computations up to 2×10^6 are consistent with a finite set of exceptions for each fixed k:

k	3	4	5	6
number of exceptions	4	10	21	61
largest exception	17	57	145	413

For $k = 3$ the problem reduces to primes. By Lemma 2.1, a decomposition with $\Omega(a) \equiv \Omega(b) \equiv r \pmod 3$ at n gives one with residue $r + \Omega(c)$ at cn, and a finite computation over the 36,864 exponent vectors of the smooth integers that are not already covered by a complete divisor shows the following: if every prime $p \geq 103$ admits all three diagonal residues $r = 0, 1, 2$, then the exceptions for $k = 3$ are exactly $\{2, 3, 11, 17\}$. We verified this prime hypothesis for every prime $103 \leq p \leq 2 \times 10^6$. In addition, the version for almost all N (for every fixed k and every pair of residues) follows from standard bounds for exponential sums of aperiodic multiplicative functions (see, e.g., [3]).

However, the argument of Section 3 does not appear to extend to $k = 3$. With $\omega = e^{2\pi i/3}$, the orthogonality relations give

$$9\, r_{1,1}(N) = N - 1 + 4\,\mathrm{Re}\big(\omega^2 T(N)\big) + 2\,\mathrm{Re}\big(\omega C_{1,1}(N)\big) + 2\, C_{1,2}(N),$$

where $r_{1,1}(N)$ counts the decompositions with $\Omega(a) \equiv \Omega(b) \equiv 1 \pmod 3$, $T(N) = \sum_{a<N} \omega^{\Omega(a)} = o(N)$, and $C_{1,1}$, $C_{1,2}$ are the reflected correlations of ω^{Ω} with ω^{Ω} and with its conjugate, respectively. Thus a missing pattern only forces a correlation of size $\gg N$, whereas for $k = 2$ it forces the correlation $C(N)$ to be within $o(N)$ of its extreme value $-(N-1)$, and every step of Section 3 (nonnegative defects whose sum vanishes, exact dilation laws, a character) is a consequence of this extremality. In agreement with this, in a model in which only the laws for a few small primes are imposed and the values at the remaining primes are left free, a single exclusion at p admits many solutions that are not related to cubic characters, whereas for $k = 2$ the laws for 2, 3 and 5 suffice for every p. In addition, a cubic character χ always satisfies $\chi(-1) = 1$, so the mechanism of Section 4, in which the odd completion is forced to be an odd quadratic character, has no direct cubic analogue; in any case a cubic character cannot agree with $\omega^{\Omega(n)}$ for all $n < p/2$ when p is large, since $\sum_{n<p/2} \omega^{\Omega(n)} \neq 0$ by the Selberg–Delange method.

Question. Does every prime $p \geq 103$ admit, for each $r \in \{0, 1, 2\}$, a decomposition $p = a+b$ with $\Omega(a) \equiv \Omega(b) \equiv r \pmod 3$? (By the reduction above, a positive answer would imply that 2, 3, 11 and 17 are the only integers $N \geq 2$ with no decomposition $N = a+b$ satisfying $\Omega(a) \equiv \Omega(b) \equiv 1 \pmod 3$.)

Parity. Our arguments use the complete multiplicativity of λ on all integers below p, through the dilation laws and the extension lemma of [2], and this is likely the reason why they are not affected by the parity problem of sieve theory. However, the same feature suggests that they do not extend to problems in which one summand is restricted to primes or to integers with a bounded number of prime factors, since there the dilation laws are no longer available on a dense set of integers.

Statement on the use of AI tools

The author conceived and directed this project. In particular, the author chose to extend the even-total result of [2] to odd totals and to all four sign patterns, proposed to remove the dependence on Mangerel's nonextremality theorem so that the final statement carries no external hypothesis, required a complete formal verification, and made the key decisions on which approaches to pursue and which to abandon (e.g., the three-colour analogue and the sieved and quantitative variants discussed in Section 9). The work was carried out with substantial assistance from AI systems, mainly Anthropic's Claude models used through Claude Code, which were used to develop and check detailed arguments, to write and

run the numerical computations, to write and debug the Lean formalization, and to draft parts of the text. Additional independent checks were obtained from other AI systems (Kimi and Gemini). The author has checked the mathematical arguments and takes full responsibility for the content of this paper. In addition, the correctness of the main results does not rest on these checks alone: they are verified by the Lean formalization described in Section 8, whose axiom audit reports only the standard axioms. The Lean development of [2], on which this work builds, was reportedly also obtained with AI assistance.

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