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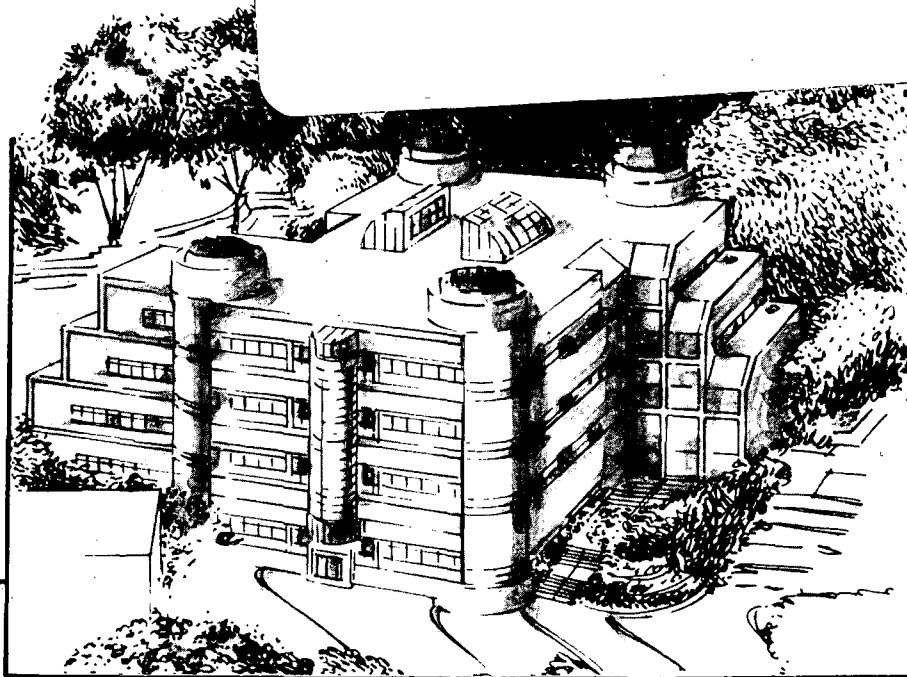
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## On the Numerical Integration of Viscoelastic Flow Equations

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by

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# ON THE NUMERICAL INTEGRATION OF VISCOELASTIC FLOW EQUATIONS

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## Abstract

Various numerical techniques are compared in the integration of the upper-convected Maxwell (UCM) constitutive model on the basis of known kinematics. They include Galerkin's method, streamline upwinding schemes and the method of characteristics. The results show the superiority of the latter in flows endowed with stress singularities. Solutions of the full set of momentum, mass and constitutive equations are computed with the method of characteristics and a Picard iterative scheme. Convergence difficulties are addressed.

## 1. Introduction

In the past, Galerkin finite element methods (GFEM's) have been the techniques of choice in the numerical discretization of viscoelastic flow equations (see [1] and [2] for detailed reviews). GFEM's, however, are known to produce oscillatory results in flow problems possessing stress singularities or boundary layers, with the consequence that solutions can only be obtained for small values of the Weissenberg number. (The Weissenberg number  $We$  is a dimensionless group determining the elastic character of the flow [1,2]). One cause for the numerical instabilities is related to the hyperbolic nature of differential viscoelastic constitutive models [1].

In the first part of the present communication, various methods are compared in the prediction of viscoelastic stresses on the basis of known kinematics for the planar stick-slip problem. The methods include Galerkin's technique, streamline upwinding (SU), streamline-upwind Petrov-Galerkin (SUPG), and streamline integration (SI) applied to the integration of the UCM constitutive model. In the second part, a Picard iterative method is used together with SI to produce solutions of the full set of governing equations for an Oldroyd-B fluid. Convergence behavior of the iterations is discussed. For a complete written report, see [3].

## 2. Governing Equations

Isothermal creeping flow of an Oldroyd-B fluid is governed by the following equations :

$$\lambda \overset{\nabla}{T}_v + T_v = 2 \mu_v D, \quad (1)$$

$$\nabla \cdot (2 \mu_N D + T_v - p I) + f = 0, \quad (2)$$

$$\nabla \cdot u = 0 \quad (3)$$

Here  $\lambda$  is a relaxation time,  $\mu_v$  and  $\mu_N$  are viscoelastic and Newtonian viscosity coefficients respectively,  $D$  is the rate of deformation tensor,  $u$  is the velocity vector,  $p$  is pressure,  $f$  is a body force,  $T_v$  is the viscoelastic extra-stress tensor, and the superscript  $\nabla$  refers to the upper-convected derivative [1]. By setting appropriate parameters to zero, these equations also

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describe Newtonian and Maxwell (UCM) flows. For fixed steady-state kinematics, the UCM equation (1) constitutes a set of first-order hyperbolic equations for  $T_v$ .

### 3. Numerical Techniques

The governing equations (1-3) are usually solved in terms of the primary variables  $T_v$ ,  $u$ , and  $p$  by means of a GFEM [1]. The unknown fields are approximated by the finite sums :

$$T_v = \sum_i T_{vi} \phi_i, \quad u = \sum_j u_j \psi_j, \quad p = \sum_k p_k \pi_k, \quad (4)$$

where  $T_{vi}$ ,  $u_j$ , and  $p_k$  are unknown nodal values, and  $\phi_i$ ,  $\psi_j$ , and  $\pi_k$  are finite element basis functions. For planar flows, the standard Galerkin weak form of (1-3) is

$$\langle \phi_i; \lambda \nabla T_v + T_v - 2 \mu_v D \rangle = 0, \quad (5)$$

$$\langle \nabla \psi_j^T; 2 \mu_N D + T_v - p I \rangle = \langle \psi_j; t \rangle + \langle \psi_j; f \rangle, \quad (6)$$

$$\langle \pi_k; \nabla \cdot u \rangle = 0, \quad (7)$$

where the brackets  $\langle ; \rangle$  and  $\langle \langle ; \rangle \rangle$  denote the  $L^2$  inner products over the flow domain and the domain boundary respectively, and  $t$  is the surface traction. In this work, the elements used are isoparametric quadrilaterals, extra-stress and pressure approximations are bilinear, and velocity interpolations are biquadratic. All approximations are of class  $C^0$ . Following [4], the extra-stress is calculated on a mesh where each quadrilateral is subdivided uniformly into  $N \times N$  sub-elements, with  $N$  between 1 and 4.

#### 3.1. Integration of the Constitutive Equation

GFEM's are known to be unstable in the solution of hyperbolic problems [5]. Various upwinding techniques have been introduced recently to stabilize finite element solutions of hyperbolic systems, most notably streamline upwinding (SU) and streamline-upwind Petrov-Galerkin (SUPG) methods (see e.g. [6]).

SU as applied to the constitutive model (1) essentially amounts to solving the modified equation

$$\nabla \cdot (K \nabla T_v) + \lambda \nabla T_v + T_v = 2 \mu_v D. \quad (8)$$

Here the tensor  $K$ , the anisotropic artificial diffusivity tensor, is given by

$$K = k \frac{u u}{u \cdot u}, \quad (9)$$

where  $k$  is a scalar on the order of element size.

SUPG solves the original constitutive equation (1) but uses a weight function differing from the basis function :

$$\langle w_i; \lambda \nabla T_v + T_v - 2 \mu_v D \rangle = 0, \quad (10)$$

$$w_i = \phi_i + c u \cdot \nabla \phi_i, \quad (11)$$

where  $c$  is also a scalar on the order of element size.

In 2-D steady-state flows, the streamlines are the characteristic lines for the hyperbolic constitutive model (1). Streamline integration (SI) is a method by which the constitutive equations are integrated along the particle paths [7]. First, streamlines are computed on the basis of a known velocity field. A second order A-stable adaptive method is then used to numerically integrate (1) along a discrete number of streamlines. The integration stepsize along a particle path is adjusted such that an estimate of the relative discretization error does not exceed some predefined tolerance.

### 3.2. Solution of the Full Set of Equations

Following [7], SI is used within a decoupled method [1] to predict viscoelastic extra-stress and kinematics for the Oldroyd-B fluid. The iterative method, henceforth referred to as SLE, is as follows :

- (1) Determine Newtonian kinematics via GFEM,
- (2) For each Gauss-point on a  $N \times N$  sub-mesh, determine new values for  $T_v$  via SI on the basis of current kinematics,
- (3) Determine kinematics based upon  $T_v$  via the Galerkin equations

$$\begin{aligned} & \langle \nabla \psi_j^T ; 2(\mu_a + \mu_N) D^{N+1} - p^{N+1} I \rangle \\ & = \langle \nabla \psi_j^T ; 2 \mu_a D^N - T_v^N \rangle + \langle \langle \psi_j ; t \rangle \rangle + \langle \psi_j ; f \rangle, \end{aligned} \quad (12)$$

$$\langle \pi_k ; \nabla \cdot u^{N+1} \rangle = 0, \quad (13)$$

where the superscripts denote the iteration index.

- (4) Check for convergence. Return to step (2) if necessary.

An "arbitrary" viscosity  $\mu_a$  is added to both sides of (12) to aid convergence of the iterative process. Following [7], the finite elements are made to conform with the streamlines computed at each iteration, so that families of Gauss-points lie on common paths.

Convergence of SLE is expected to be first order at best. Unlike for Newton-Raphson iterations, convergence criteria for SLE are not altogether obvious. *Maximum relative changes* (MRC's) are defined by

$$\frac{\max_g |T_{vg}^{N+1} - T_{vg}^N|}{\max_g |T_{vg}^{N+1}|}, \quad \frac{\max_j |u_j^{N+1} - u_j^N|}{\max_j |u_j^{N+1}|}, \quad (14)$$

where the subscript  $g$  refers to Gauss-points. Similar MRC's are also defined for pressure and streamfunction. *Residuals* are defined by

$$R = \left\| \langle \nabla \psi_j^T ; 2 \mu_N D^{N+1} - p^{N+1} I + T_v^{N+1} \rangle - \langle \langle \psi_j ; t \rangle \rangle - \langle \psi_j ; f \rangle \right\|_2. \quad (15)$$

Finally, *distances* are defined by

$$\|T_{vg}^{N+1} - T_{vg}^0\|_2, \|u_j^{N+1} - u_j^0\|_2, \quad (16)$$

again where similar quantities are defined for pressure and streamfunction. The superscript 0 in (16) refers to a particular (fixed) iterate. *Convergence of the iterative process* is defined here as the situation when all MRC's remain below 0.05 and all residuals and distances do not diverge over prolonged iterations.

#### 4. Results for Newtonian Kinematics

The planar stickslip problem [1] is chosen to test the various techniques of solving the differential constitutive equation (1). The Newtonian kinematics are determined via GFEM. GFEM, SU, SUPG, and SI are then used to predict the UCM extra-stress for  $We=3$  on the basis of the computed Newtonian kinematics.

In the full report [3], the predictions of these various techniques are compared along several streamlines for  $1 \times 1$ ,  $2 \times 2$ , and  $4 \times 4$  sub-meshes for the extra-stress computation. In summary, Galerkin, SU, and SUPG are inaccurate in the exit region where a steep stress boundary layer is predicted in the direction transverse to the flow. Away from the exit region, SU and SUPG give accurate results when a  $4 \times 4$  sub-mesh is used. SI results are the most stable and accurate; they do converge with increased resolution even near the exit.

#### 5. Results with the Decoupled Method

Performance of SLE is tested on the stickslip flow problem of an Oldroyd-B fluid with viscosity ratio  $\mu_N/\mu_v=1/8$ . Iterations for a  $1 \times 1$  sub-mesh begin at  $We=0$  and continue until convergence is lost slightly above  $We=1/2$ . SLE with  $4 \times 4$  sub-mesh refinement is used in a similar test. At  $We=1/2$ , convergence has already been lost (without catastrophic divergence) with the MRC in stresses being on the order of unity. The  $4 \times 4$  sub-refinement requires evaluation of stress much closer to the singularity than does its  $1 \times 1$  counterpart. An MRC in velocity on the order of 0.01 can produce an MRC in stress on the order of 1 in the elements close to the singularity.

In this work, the arbitrary viscosity  $\mu_a$  is chosen by trial and error. If  $\mu_a$  is too low, the iterations diverge wildly. If  $\mu_a$  is chosen too high, the computed MRC's are deceptively low, but the solution nevertheless can diverge with prolonged iterations. Typically, convergence (when it does occur) is slow and oscillatory. SLE on the  $1 \times 1$  sub-mesh takes on the order of 200 iterations at the total cost of about 8 CPU minutes on a CRAY X-MP (single processor) to obtain converged results for  $We=1/2$ .

#### 6. Conclusions

Current boundary conditions and constitutive models used in viscoelastic flow modeling often result in physically unreasonable stress and stress gradients. In the predicted boundary layers, small changes in kinematics can produce large changes in the computed stress field. Finite element upwinding techniques for viscoelastic extra-stress calculation have been shown to be an improvement over standard Galerkin methods. They do not, however, produce accurate results close to singularities. The streamline integration of the UCM constitutive model leads to stable and accurate stress predictions even near the singularity. A decoupled technique based on streamline integration does not, however, enjoy good convergence properties and fail to converge at rather low values of  $We$ . This work also shows that convergence criteria for decoupled viscoelastic iterations must be carefully defined.



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