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The combinatorics of h*-polynomials of rational polytopes

Author
Bajo, Esme

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The combinatorics of h^*-polynomials of rational polytopes

by

Esme Bajo

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Committee in charge:

Professor Matthias Beck, Co-chair
Professor Sylvie Corteel, Co-chair
Professor Nikhil Srivastava

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Abstract

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Esme Bajo

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Professor Matthias Beck, Co-chair

Professor Sylvie Corteel, Co-chair

The h^*-polynomial captures the enumeration of lattice points in dilates of rational polytopes. For various classes of polytopes, there are many potential properties of this polynomial–including nonnegativity, monotonicity, unimodality, real-rootedness, and palindromicity–that are of interest within geometric combinatorics. In this dissertation, we investigate these properties for three variations of the h^*-polynomial: the boundary h^*-polynomial, the weighted h^*-polynomial, and the local h^*-polynomial. We conclude with an application of h^*-polynomials to enumerating proper vertex colorings of graphs.

To Elise Blackwell & David Bajo

Several novel dedications later and I'm happy to finally (sort of) return the sentiment.
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Acknowledgments

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Chapter 1

Introduction

In this dissertation, we study several variations of h^*-polynomials of rational polytopes, including boundary h^*-polynomials, weighted h^*-polynomials, and local h^*-polynomials. Each of these polynomials captures the enumeration of lattice points in dilates of their associated polytopes, which are the integral solutions to systems of linear inequalities. Thus, this field of mathematics can be used to model a wide range of problems in other fields of mathematics and across the natural and social sciences.

In 1962, Eugene Ehrhart introduced Ehrhart polynomials of lattice polytopes [56]. Given a lattice polytope P, that is, the convex hull of finitely many points in $\mathbb{Z}^d$, the counting function $\mathrm{ehr}_P(n) = |nP \cap \mathbb{Z}^d|$ agrees with a polynomial in n of degree $\dim(P)$, called the *Ehrhart polynomial* of P [56]. It is helpful to study this polynomial via its generating function, called the *Ehrhart series* $\mathrm{Ehr}_P(z)$ of P. If P is a d-dimensional polytope, its Ehrhart series can be expressed in the form

$$\mathrm{Ehr}_P(z) := 1 + \sum_{n \geqslant 1} \mathrm{ehr}_P(n) z^n = \frac{h_P^*(z)}{(1-z)^{d+1}},$$

where $h_P^*(z)$ is a polynomial in z of degree at most $\dim(P)$, called the *h^*-polynomial* of P.

Ehrhart theory is an active area of research within combinatorics. Several classes of open problems arise, including:

1. **Classifications** – Can we classify which polynomials can be Ehrhart polynomials? What about which polynomials can be h^*-polynomials? Can we classify which polytopes have Ehrhart polynomials with positive coefficients? What about which polytopes have unimodal h^*-polynomials or real-rooted h^*-polynomials? Which polytopes have special properties, e.g., have unimodular triangulations or have the integer decomposition property? (See, e.g., [22, 39, 46, 73, 84, 85].)

2. **Generalizations** – Can we create various generalizations of Ehrhart theory (rational, weighted, etc.)? What classical Ehrhart-theoretic results generalize in these cases? Can we apply these generalized Ehrhart theories to other fields of mathematics (e.g.,

enumerative combinatorics and algebraic geometry) in the same ways we can apply classical Ehrhart theory? (See, e.g., [11, 20, 47, 88, 124].)

3. **Applications** – What types of questions in related fields of mathematics can be approached using discrete geometry? For instance, can we construct combinatorially-defined polytopes (order polytopes, lecture hall simplices, permutahedra, parking function polytopes, etc.) whose Ehrhart theory will have implications for their underlying combinatorial objects? (See, e.g., [7, 21, 22, 64, 101].)

In this dissertation, we will tackle problems within each of these classes, mostly by considering h^*-polynomials (as opposed to Ehrhart polynomials) from a combinatorial perspective. Stanley proved that the coefficients of the h^*-polynomial are nonnegative, originally via commutative algebra [115] and later using a shelling argument [111]. There is also a simple combinatorial proof by giving the coefficients of the h^*-polynomial a combinatorial interpretation in terms of the simplices in a *half-open triangulation* of the polytope (which we go through in detail in Chapter 2). This method gives us a natural approach to many lattice-point enumeration problems, which we use throughout this dissertation. We also explore how this interpretation holds up across the three variations of the h^*-polynomial, and study its implications in each case.

In Chapter 2, we formalize the notions of simplices, triangulations, and half-open triangulations. We also define the *integer-point transform* of a cone and connect this to Ehrhart theory in order to understand the h^*-polynomial and arrive at this interpretation of its coefficients. Much of this chapter follows the presentation of Ehrhart theory given in [25]. Furthermore, we extend the Ehrhart theory of lattice polytopes to *rational polytopes*, i.e., polytopes with rational vertices. Throughout this chapter, we use *order polytopes* as an example to demonstrate the potential of these techniques, and we will also revisit the Ehrhart theory of this particular class of polytopes in Chapter 6.

In Chapter 3, which is joint work with Matthias Beck, we present our first variation of the h^*-polynomial, the *boundary h^*-polynomial*; this work is published in [8]. We first define boundary h^*-polynomials and give a combinatorial interpretation of their coefficients via simplices in *half-open boundary triangulations* (mirroring the classical result). We then use these boundary h^*-polynomials to provide an "easy" proof of Stapledon's symmetric decomposition of classical h^*-polynomials [123] and Beck, Braun, and Vindas-Meléndez's rational generalization [19]:

Theorem 3.1.1. *Let P be a full-dimensional rational polytope with denominator q and let ℓ be the smallest positive integer such that ℓP contains an interior lattice point. Then*

$$\frac{1 + z + \cdots + z^{\ell-1}}{1 + z + \cdots + z^{q-1}} h_P^*(z) = a(z) + z^\ell b(z)\,,$$

where $a(z)$ and $b(z)$ are palindromic polynomials with nonnegative integer coefficients.

Moreover, we present the implications of this proof to *Gorenstein polytopes*, a special class of polytopes with connections to commutative algebra and algebraic geometry. In particular, we define a more general notion, that of *rational Gorenstein polytopes*, and prove that:

Theorem 3.5.4. *If P is a rational g-Gorenstein polytope with denominator q, then*

$$h_P^*(z) = \frac{1 + z + \cdots + z^{q-1}}{1 + z + \cdots + z^{g-1}}\, h_{\partial P}^*(z)\,,$$

where $h_{\partial P}^*(z)$ is the boundary h^*-polynomial of P.

Finally, we discuss the connection between our study of boundary h^*-polynomials and Beck, Elia, and Rehberg's *rational Ehrhart theory* (which studies rational dilates of polytopes, as opposed to just integer dilates) [20].

In Chapter 4, which is joint work with Robert Davis, Jesús A. De Loera, Alexey Garber, Sofía Garzón Mora, Katharina Jochemko, and Josephine Yu, we introduce our second variation, the *weighted h^*-polynomial*; this work is published in [11]. We place a weight function on our lattice points, generalize the Ehrhart counting functions to weighted versions, and explore which weights allow us to generalize Stanley's classical nonnegativity and monotonicity theorems. In particular, we define classes of weights R_P (sums of products of linear forms that are nonnegative on the polytope P) and S_P (sums of nonnegative products of linear forms on the polytope P), and prove the following nonnnegativity and monotonicity theorems:

Nonnegativity Theorem (Theorem 4.2.5). *Let P be a rational polytope and let $h_{P,w}^*(z)$ be its weighted h^*-polynomial.*

1. *If the weight w is a homogeneous element of R_P, then the coefficients of $h_{P,w}^*(z)$ are nonnegative.*

2. *If the weight w is a homogeneous element of S_P, then $h_{P,w}^*(z) \geqslant 0$ for $z \geqslant 0$.*

First Monotonicity Theorem (Theorem 4.2.7). *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in R_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim P+m+1} h_{P,w}^*(z) \preceq (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim Q+m+1} h_{Q,w}^*(z)$$

(where $\preceq$ indicates coefficient-wise inequality).

Second Monotonicity Theorem (Theorem 4.2.8). *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes of the same dimension $D = \dim P = \dim Q$, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in S_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{D+m+1} h_{P,w}^*(z) \leqslant (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{D+m+1} h_{Q,w}^*(z)$$

for all $z \geqslant 0$.

We also give some evidence for the tightness of these results by producing counterexamples when these restraints on the weight function w are relaxed.

In Chapter 5, which is joint work with Benjamin Braun, Giulia Codenotti, Johannes Hofscheier, and Andrés R. Vindas-Meléndez, we explore our third and final variation, the *local h^*-polynomial* (sometimes called the *box polynomial*) of a simplex; a preprint of this work can be found in [10]. Local h^*-polynomials have already been a point of interest within Ehrhart theory, specifically as they relate the unimodality of classical h^*-polynomials. (We call a polynomial with nonnegative coefficients *unimodal* if its vector of coefficients has one peak.) In this chapter, we restrict ourselves to the study of "one-row Hermite normal form simplices," that is, simplices whose Hermite normal form expression has only one nontrivial row, and characterize the conditions for the unimodality of the local h^*-polynomials of two subclasses: the "all ones" simplices and the "geometric sequence" simplices.

Theorem 5.3.1. *For the d-dimensional simplex S in one-row Hermite normal form with last row $(1, \ldots, 1, N)$, where $N = (d-2)q + r$ for some $0 \leqslant r \leqslant d-3$, the local h^*-polynomial is unimodal if and only if $r \in \{0, 1, 2, d-3\}$.*

Theorem 5.3.9. *For any integers $q \geqslant 2$ and $k \geqslant 2$, the simplex S with non-trivial row*

$$\left(1, q^{k-1}, \ldots, q, 1, q^k\right)$$

has a unimodal local h^*-polynomial.

Additionally, we look at the asymptotic behavior of the coefficients of the local h^*-polynomials of these one-row Hermite normal form simplices as the normal volume of the simplices tends to infinity.

In Chapter 6, which is joint work with Matthias Beck and Andrés R. Vindas-Meléndez, we conclude with an application of Ehrhart theory to graph colorings; a preprint of this work can be found in [9]. The *chromatic polynomial* of a graph can be expressed as a sum of Ehrhart polynomials of order polytopes, and more generally, the *principal specialization* of Stanley's *chromatic symmetric function* [108] can be expressed as a sum of Chapoton's *q-analog Ehrhart polynomials* [47] of order polytopes, which yields the following structural result:

Theorem 6.1.1. *For any graph $G = (V, E)$ and linear form $\lambda : V \to \mathbb{Z}_{>0}$, there exists a polynomial $\widetilde{\chi}_G^\lambda(q, x) \in \mathbb{Q}(q)[x]$ such that*

$$\widetilde{\chi}_G^\lambda(q, [n]_q) \ = \sum_c q^{\sum_{v \in V} \lambda(v) c(v)},$$

where the sum is over all proper n-colorings of V and $[n]_q := 1 + q + \cdots + q^{n-1}$.

We mainly let $\lambda = \mathbf{1}$, in which case we refer to $\widetilde{\chi}_G^\lambda(q, x)$ as the "q-analog chromatic polynomial." We use methods from Ehrhart theory (inside-out polytopes and h^*-bases) to obtain two expressions for the leading coefficient $c_T^{\mathbf{1}}(q)$ of the q-analog chromatic polynomial of a tree:

Corollary 6.4.2. *Given a tree $T = (V, E)$ on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q, n)$ equals*

$$\begin{aligned} c_T^{\mathbf{1}}(q) &= (q - q^2)^d \sum_{S \subseteq E} \prod_{C \in P(S)} \frac{1}{1 - q^{\Lambda_C}} \\ &= \frac{1}{[d]_q!} \sum_{(\varrho, \sigma)} q^{d + \operatorname{maj} \sigma}, \end{aligned}$$

where $P(S)$ denotes the collection of vertex sets of the connected components of the graph (V, S) and where the latter sum ranges over all pairs of acyclic orientations ϱ of T and linear extensions σ of the poset induced by ϱ.

From these expressions, we study Stanley's conjecture that the chromatic symmetric function $X_G(x_1, x_2, \ldots)$ distinguishes trees [108], as well as Loehr and Warrington's refinement that $X_G(1, q, \ldots, q^d, 0, 0, \ldots)$ already does [87], from a geometric perspective. We further conjecture that:

Conjecture 6.1.3. *The leading coefficient of the q-chromatic polynomial $\widetilde{\chi}_T^{\mathbf{1}}(x)$ distinguishes trees.*

Moreover, we generalize this principal specialization to a q, λ*-analog chromatic polynomial* (which does not align with Stanley's original chromatic symmetric function but which does arise naturally from our geometric perspective). This q, λ-analog chromatic polynomial comes with a *deletion-contraction formula*, which enables us to construct a recursive algorithm to compute it (see Figure 6.3).

Chapter 2

Background

2.1 Polytopes and faces

A **polytope** $P \subseteq \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$, equivalently a bounded intersection of finitely many hyperspaces (see, e.g., Theorem 1.1 in [128]). For example, the triangle T with vertices $(0,0), (3,0)$, and $(0,3)$ can be expressed with **vertex description**

$$T = \text{conv}\{(0,0),(3,0),(0,3)\}$$

or with **hyperplane description**

$$T = \left\{ (x_1, x_2) \in \mathbb{R}^2 : \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leqslant \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}, \tag{2.1}$$

as shown in Figure 2.1.

In order to define the **faces** of a polytope P, we need to define the notion of a **supporting hyperplane**. Given a polytope $P \subseteq \mathbb{R}^d$, we call a hyperplane

$$H = \{x \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d = b\}$$

supporting if P falls on one side of the hyperplane (including the hyperplane itself), that is, if

$$P \subseteq \{x \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d \leqslant b\} \text{ or } P \subseteq \{x \in \mathbb{R}^d : a_1x_1 + \cdots + a_dx_d \geqslant b\}.$$

The set of **faces** of P is the set of intersections of P with a supporting hyperplane. We also define the empty face $\varnothing$ and the polytope P to be faces. For example, the faces of the triangle T are: the empty face, the zero-dimensional faces $(0,0)$, $(3,0)$, and $(0,3)$, the three edges of the triangle, and T itself (see Figure 2.2 for an example of a supporting hyperplane for each nontrivial face). The zero-dimensional faces of a polytope are its **vertices** and the

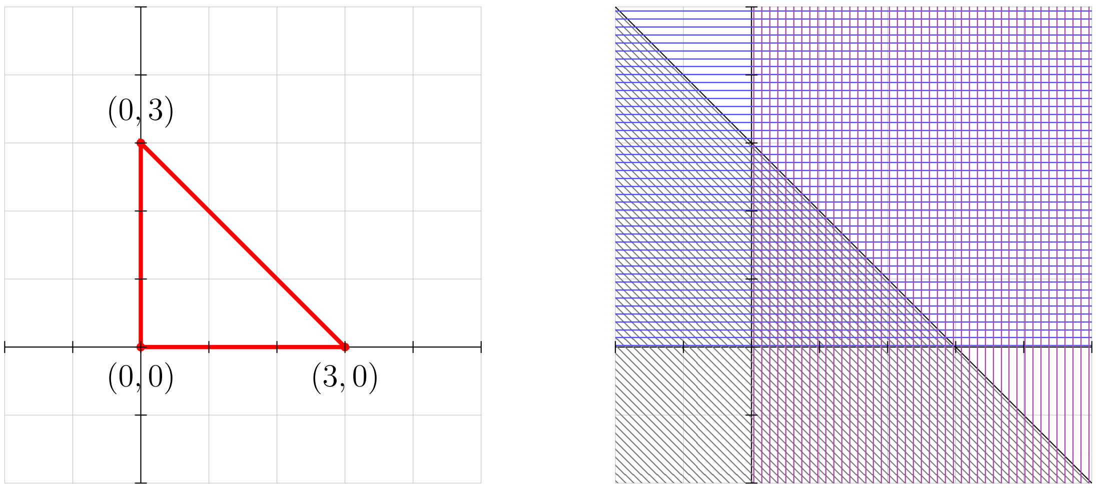


Figure 2.1: The vertex and hyperplane descriptions of the triangle

$$\operatorname{conv}\{(0,0),(0,3),(3,0)\} = \{x_1 \geqslant 0\} \cap \{x_2 \geqslant 0\} \cap \{x_1 + x_2 \leqslant 3\}$$

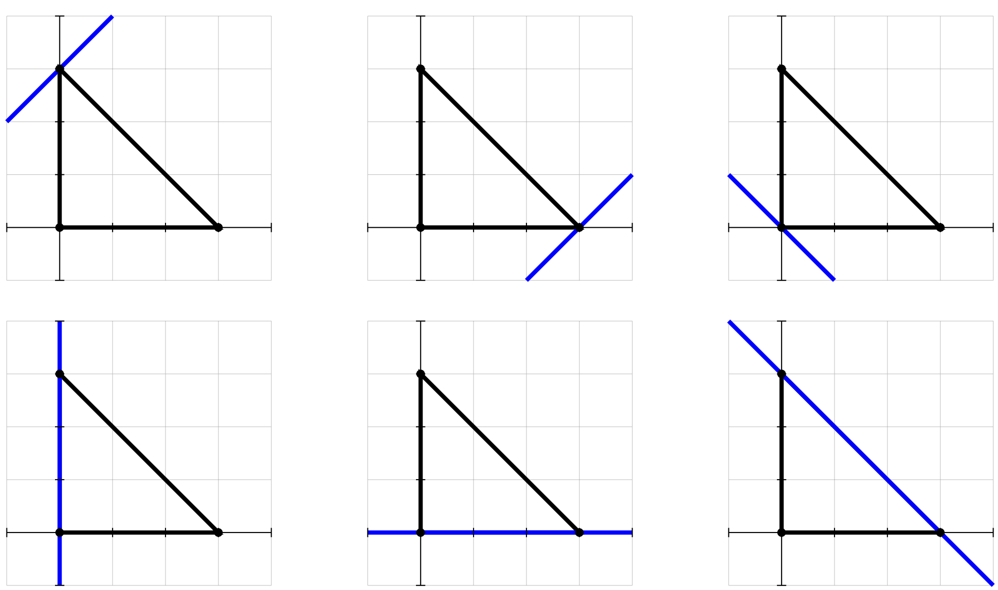


Figure 2.2: A supporting hyperplane for each nontrivial face of the triangle in Figure 2.1

$(d-1)$-dimensional faces (or codimension 1 faces) are its **facets**. The supporting hyperplanes that witness the facets of a polytope are called its **facet-defining hyperplanes**. The facet-defining hyperplanes constitute a minimal hyperplane description for P (see, e.g., equation 2.1) [24].

In this dissertation, we will primarily study **lattice polytopes**, i.e., polytopes whose vertices have integer coordinates, and **rational polytopes**, i.e., polytopes whose vertices have rational coordinates. An important quantity for a rational polytope P is its **denominator**, or the smallest positive integer $q \geqslant 1$ such that its qth dilate qP is a lattice polytope.

2.2 Simplices and triangulations

We will often study polytopes by decomposing them into simpler polytopes, usually **simplices**. A **simplex** is a d-dimensional polytope with $d+1$ vertices (equivalently, $d+1$ facets), e.g., a triangle, a tetrahedron, and so on.

Definition 2.2.1. $\mathcal{T}$ is a **triangulation** of a d-dimensional polytope P if $\mathcal{T}$ is a finite collection of d-dimensional simplices $\{\Delta_1, \ldots, \Delta_m\}$ such that

$$P = \bigcup_{i=1}^{m} \Delta_i$$

and such that for every $\Delta_i, \Delta_j \in \mathcal{T}$, $\Delta_i \cap \Delta_j$ is a face of both Δ_i and Δ_j.

It is a well-known (and extremely useful) theorem that every polytope admits a triangulation. Moreover, such a triangulation can be constructed using *no additional vertices*, implying that every lattice polytope has a triangulation into lattice simplices (sometimes called a **lattice triangulation**). Constructions of various such triangulations (including *pulling triangulations* and *pushing triangulations*) can be found in [25].

Moreover, we will frequently take advantage of the existence of **disjoint triangulations** of polytopes into **half-open simplices** (that is, simplices with zero or more facets removed). Such triangulations will allow us to avoid making inclusion-exclusion arguments.

Construction 2.2.2. Any triangulation $\mathcal{T}$ of $P \subseteq \mathbb{R}^d$ can be made into a disjoint triangulation using what is commonly referred to as a "visibility argument." Choose a point q in the interior of P that is *generic* relative to the triangulation $\mathcal{T}$, that is, a point q that is not contained in any of the facet-defining hyperplanes of any of the simplices in $\mathcal{T}$. Pass through each simplex $\Delta \in \mathcal{T}$ one at a time and do the following: for each facet F of Δ, if q is on the opposite side of F's defining hyperplane as Δ (i.e., if q's "view" of F is not blocked by Δ), remove the facet F from Δ. An example of this procedure is shown in Figure 2.3 and a more technical description via *tangent cones* is given in [25].

We will end this section with the construction of a very combinatorial disjoint triangulation of an important class of polytopes, called **order polytopes**.

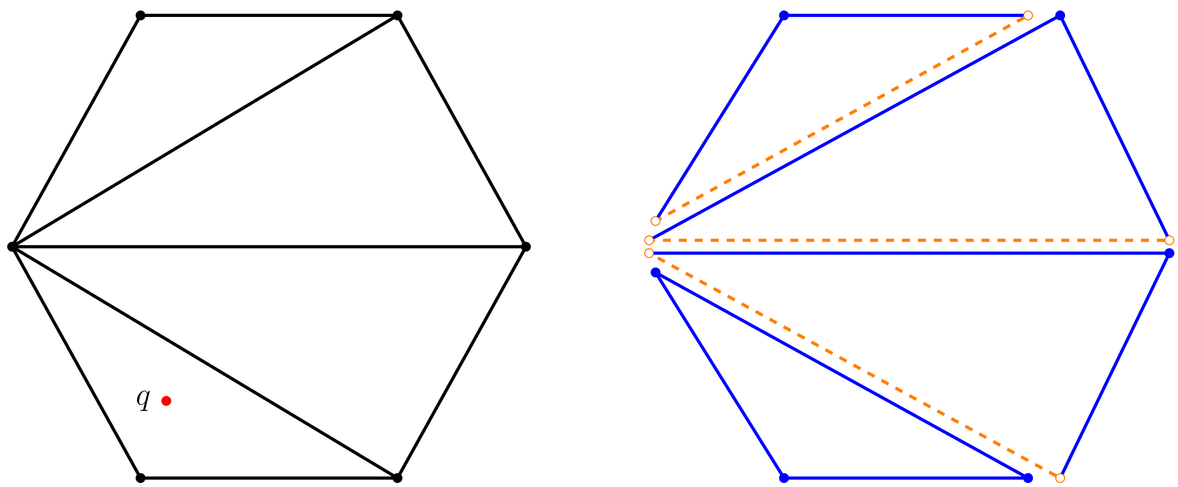


Figure 2.3: Making a triangulation disjoint via a visibility construction. Closed facets (or facets that are "blocked" from q by the simplex containing them) are shown in blue and open facets (or facets that are "visible" from q) are shown in orange.

Definition 2.2.3. Given a finite poset Π with base set $\{1, \ldots, d\}$ and order relation $\preceq$, the **order polytope** $\mathcal{O}(\Pi)$ is the polytope

$$\mathcal{O}(\Pi) := \{(x_1, \ldots, x_d) \in [0,1]^d : x_i \leqslant x_j \text{ if } i \preceq j\}.$$

For example, the order polytope of the antichain is the unit cube and the order polytope of a chain is a simplex. The order polytope $\mathcal{O}(\Pi)$ can be triangulated using the *linear extensions* of the underlying poset, in particular, into the simplices

$$\Delta_\sigma := \{(x_1, \ldots, x_d) \in [0,1]^d : 0 \leqslant x_{\sigma_1} \leqslant \ldots \leqslant x_{\sigma_d} \leqslant 1\},$$

where σ is a linear extension of Π. By a linear extension, we mean a permutation such that in its one-line notation $\sigma = \sigma_1 \ldots \sigma_d$, i appears before j if $i \prec j$. We will be assuming our underlying poset is *naturally labelled*, that is, we assume that the identity is a linear extension of Π.

We can make this triangulation of $\mathcal{O}(\Pi)$ disjoint by using the *descents* of the linear extensions to decide which simplices' facets to remove (this is equivalent to choosing the generic point q in Construction 2.2.2 to be in the simplex corresponding to the identity permutation). This yields the following disjoint lattice triangulation of the order polytope:

$$\mathcal{O}(\Pi) = \biguplus_{\sigma \in \mathcal{JH}(\Pi)} \left\{ (x_1, \ldots, x_d) \in \mathbb{R}^d : \begin{array}{l} 0 \leqslant x_{\sigma_1} \leqslant \ldots \leqslant x_{\sigma_d} \leqslant 1, \\ x_{\sigma_i} < x_{\sigma_{i+1}} \text{ if } \sigma_i > \sigma_{i+i} \end{array} \right\}, \tag{2.2}$$

where $\mathcal{JH}(\Pi)$ is used to denote the set of linear extensions of the poset. This triangulation is a special type of lattice triangulation called a *unimodular triangulation*, which will prove very useful for the Ehrhart theory in the following sections.

2.3 Integer-point transforms of half-open simplicial cones

Throughout the following chapters, we will be studying integer points in dilates of polytopes, and this will prove easiest to accomplish for simplices. This is because we can think of integer points in dilates of polytopes as integer points in a cone called the "homogenization" of the polytope. When the polytope is a simplex, this homogenization is a *simplicial cone*. The geometry of this type of cone makes organizing the integer points in the cone into an easily enumerable set much simpler.

In this section, we will define this cone and give the enumeration (called an integer-point transform) in the simplicial case, and later in this chapter, we will reconcile this with the Ehrhart theory of general polytopes.

Definition 2.3.1. Given a polytope $P \subseteq \mathbb{R}^d$, its **homogenization** (sometimes referred to as the **cone over** P) is

$$\hom(P) := \{(tp, t) : p \in P, t \geqslant 0\} \subseteq \mathbb{R}^{d+1}.$$

Equivalently, it is the cone generated by the vectors

$$\left\{ \begin{pmatrix} v \\ 1 \end{pmatrix} : v \text{ a vertex of } P \right\}.$$

Our goal in this section is to enumerate the lattice points in the homogenizations of half-open lattice (and rational) simplices. Specifically, we wish to find a rational function description of the generating function

$$\sigma_{\hom(\Delta)}(z_1, \ldots, z_{d+1}) := \sum_{x \in \hom(\Delta) \cap Z^{d+1}} z_1^{x_1} \cdots z_{d+1}^{x_{d+1}},$$

which is called the **integer-point transform** of $\hom(\Delta)$. In order to do this, we take advantage of the fact that $\hom(\Delta)$ is a *simplicial cone* when Δ is a simplex. That is, its generators are $d+1$ linearly independent vectors that form a basis (over $\mathbb{Z}$) for some sublattice of $\mathbb{Z}^{d+1}$. Therefore, we are able to uniquely express every lattice point in $\hom(\Delta)$ as a sum of an integral linear combination of the generators and a point in its **fundamental parallelepiped**:

Definition 2.3.2. Let $\Delta = \operatorname{conv}\{v_1, \ldots, v_{d+1}\} \subseteq \mathbb{R}^d$ be a d-dimensional half-open rational simplex with denominator q with the facets opposite $v_1, \ldots, v_r$ removed (for some $0 \leqslant r \leqslant$

$d+1$), i.e. suppose

$$\Delta = \left\{ \sum_{i=1}^{d+1} \lambda_i v_i \ : \ \begin{array}{l} \lambda_1 + \cdots + \lambda_{d+1} = 1, \\ \lambda_1, \ldots, \lambda_r > 0, \\ \lambda_{r+1}, \ldots, \lambda_{d+1} \geqslant 0 \end{array} \right\}.$$

The **half-open fundamental parallelepiped** of $\hom(\Delta)$ is

$$\square(\Delta) := \left\{ \sum_{i=1}^{d+1} \lambda_i \begin{pmatrix} qv_i \\ q \end{pmatrix} \ : \ \begin{array}{l} 0 < \lambda_1, \ldots, \lambda_r \leqslant 1, \\ 0 \leqslant \lambda_{r+1}, \ldots, \lambda_{d+1} < 1 \end{array} \right\}.$$

For example, Figure 2.4 shows the fundamental parallelepiped of the homogenization of the half-open integer line segment $(1,3]$ and the fundamental parallelepiped of the homogenization of the closed rational line segment $[0.5, 2.5]$ (with denominator $q = 2$).

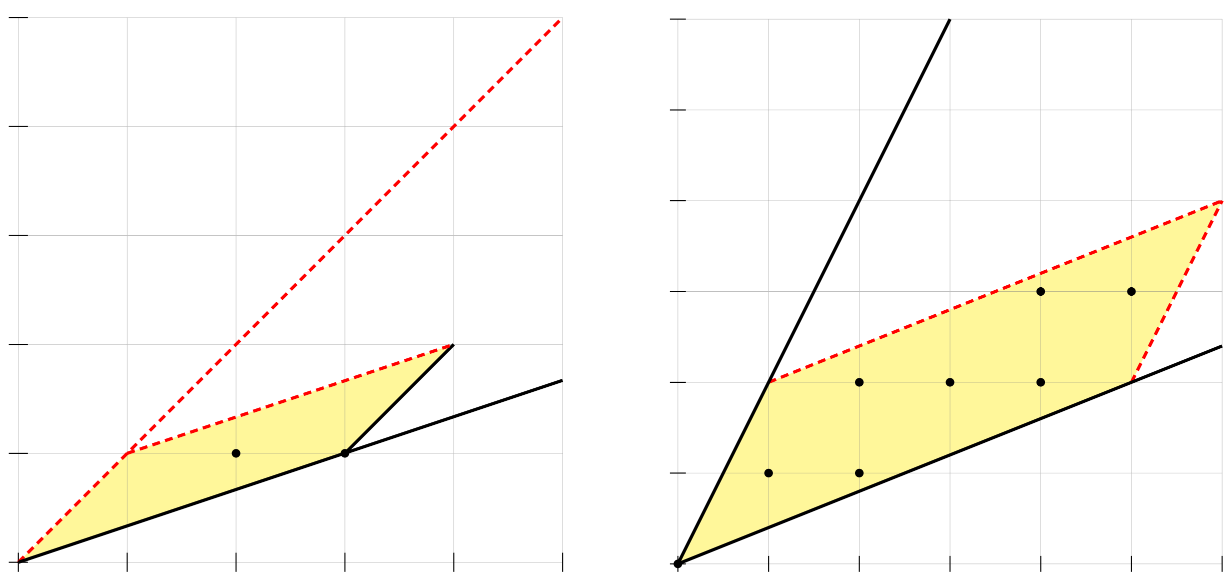


Figure 2.4: The fundamental parallelepipeds of the homogenizations of the line segments $(1,3]$ and $[0.5, 2.5]$

The half-open fundamental parallelepiped is defined so that the homogenization $\hom(\Delta)$ can be tiled with integral translates of $\square(\Delta)$, that is,

$$\hom(\Delta) = \biguplus_{k_1,\ldots,k_{d+1} \in \mathbb{Z}_{\geqslant 0}} \left(\square(\Delta) + \sum_{i=1}^{d+1} k_i \begin{pmatrix} qv_i \\ q \end{pmatrix} \right).$$

In other words, any point $x \in \hom(\Delta) \cap \mathbb{Z}^{d+1}$ can be uniquely written as the sum of an integer point in $\Box(\Delta)$ and a nonnegative integral linear combination of the generators of $\hom(\Delta)$. This allows us to compute the integer-point transform of $\hom(\Delta)$ (which enumerates infinitely many lattice points) just in terms of the (finitely many) integer points in the fundamental parallelepiped. We denote the coordinates of each v_i by $(v_{i,1}, \ldots, v_{i,d})$, and compute:

$$\begin{aligned}
\sigma_{\hom(\Delta)}(z_1, \ldots, z_{d+1}) &= \sum_{x \in \hom(\Delta) \cap \mathbb{Z}^{d+1}} z_1^{x_1} \cdots z_{d+1}^{x_{d+1}} \\
&= \sum_{y \in \Box(\Delta) \cap \mathbb{Z}^{d+1}} \left(\sum_{k_1, \ldots, k_{d+1} \in \mathbb{Z}_{\geqslant 0}} z_1^{y_1 + \sum_{i=1}^{d+1} k_i q v_{i,1}} \cdots z_d^{y_d + \sum_{i=1}^{d+1} k_i q v_{i,d}} z_{d+1}^{y_{d+1} + \sum_{i=1}^{d+1} k_i q} \right) \\
&= \sum_{y \in \Box(\Delta) \cap \mathbb{Z}^{d+1}} z_1^{y_1} \cdots z_{d+1}^{y_{d+1}} \sum_{k_1, \ldots, k_{d+1} \in \mathbb{Z}_{\geqslant 0}} \prod_{j=1}^{d+1} (z_1^{q v_{j,1}} \cdots z_d^{q v_{j,d}} z_{d+1}^{q})^{k_j} \\
&= \frac{\sum_{y \in \Box(\Delta) \cap \mathbb{Z}^{d+1}} z_1^{y_1} \cdots z_{d+1}^{y_{d+1}}}{(1 - z_1^{q v_{1,1}} \cdots z_d^{q v_{1,d}} z_{d+1}^{q}) \cdots (1 - z_1^{q v_{d+1,1}} \cdots z_d^{q v_{d+1,d}} z_{d+1}^{q})},
\end{aligned} \tag{2.3}$$

where the final equality comes from recognizing the inner sum as a product of $d + 1$ infinite geometric series. That is,

$$\sigma_{\hom(\Delta)}(z_1, \ldots, z_{d+1}) = \frac{\sum_{y \in \Box(\Delta) \cap \mathbb{Z}^{d+1}} z^y}{(1 - z^{w_1}) \cdots (1 - z^{w_{d+1}})}, \tag{2.4}$$

where w_i denotes the cone generator (qv_i, q) and z^x denotes the monomial $z_1^{x_1} \cdots z_{d+1}^{x_{d+1}}$. For example, using the fundamental parallelepipeds in Figure 2.4, we can compute

$$\begin{aligned}
\sigma_{\hom((1,3])}(z_1, z_2) &= \frac{z_1^2 z_2 + z_1^3 z_2}{(1 - z_1 z_2)(1 - z_1^3 z_2)} \text{ and} \\
\sigma_{\hom([0.5,2.5])}(z_1, z_2) &= \frac{1 + z_2(z_1 + z_1^2) + z_2^2(z_1^2 + z_1^3 + z_1^4) + z_2^3(z_1^4 + z_1^5)}{(1 - z_1 z_2^2)(1 - z_1^5 z_2^2)}.
\end{aligned}$$

2.4 Ehrhart theory of lattice polytopes

Classical Ehrhart theory is the study of lattice-point enumeration in dilates of lattice polytopes. In particular, given any d-dimensional polytope $P \subseteq \mathbb{R}^d$, define the counting function

$$\operatorname{ehr}_P(n) := |nP \cap \mathbb{Z}^d|,$$

where $nP = \{nx : x \in P\}$. For example, if P is the standard 2-simplex $\operatorname{conv}\{0, e_1, e_2\}$ shown in Figure 2.5, $\operatorname{ehr}_P(n)$ counts the number of nonnegative integer solutions to $x_1 + x_2 \leqslant n$, which is

$$\binom{n+2}{2} = \frac{1}{2}n^2 + \frac{3}{2}n + 1. \tag{2.5}$$

It is no coincidence that this is a polynomial of degree 2, in fact,

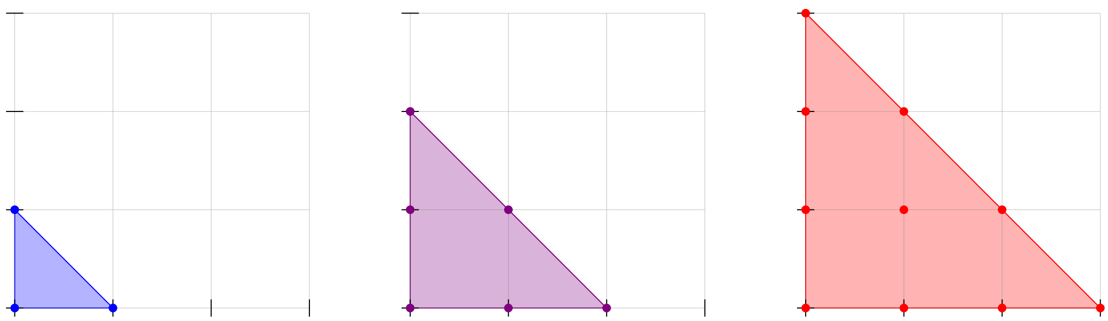


Figure 2.5: Dilates of the standard 2-simplex

Theorem 2.4.1 (Ehrhart, [56]). *If P is a lattice polytope,* $\mathrm{ehr}_P(n)$ *is a polynomial in n of degree* $\dim(P)$*, called the **Ehrhart polynomial** of P.*

One way to see this is to study its generating function

$$\mathrm{Ehr}_P(z) = \sum_{n \geqslant 0} \mathrm{ehr}_P(n) z^n,$$

called the **Ehrhart series** of P (we define $\mathrm{ehr}_P(0)$ to be 1 for closed polytopes and 0 otherwise). Choose any disjoint lattice triangulation $\mathcal{T}$ of P, as in Section 2.2, and express

$$\mathrm{Ehr}_P(z) = \sum_{\Delta \in \mathcal{T}} \mathrm{Ehr}_\Delta(z).$$

The homogenization $\mathrm{hom}(\Delta)$ is defined to exactly contain the nth dilate $n\Delta$ of Δ at "height" $x_{d+1} = n$ (see Figure 2.6), therefore the Ehrhart series enumerates all integer points x in the cone $\mathrm{hom}(\Delta)$, weighted by $z^{x_{d+1}}$. Thus, by equation 2.4,

$$\mathrm{Ehr}_\Delta(z) = \sigma_{\mathrm{hom}(\Delta)}(1, \ldots, 1, z) = \frac{\sum_{y \in \mathrm{hom}(\Delta) \cap \mathbb{Z}^{d+1}} z^{y_{d+1}}}{(1-z)^{d+1}}. \tag{2.6}$$

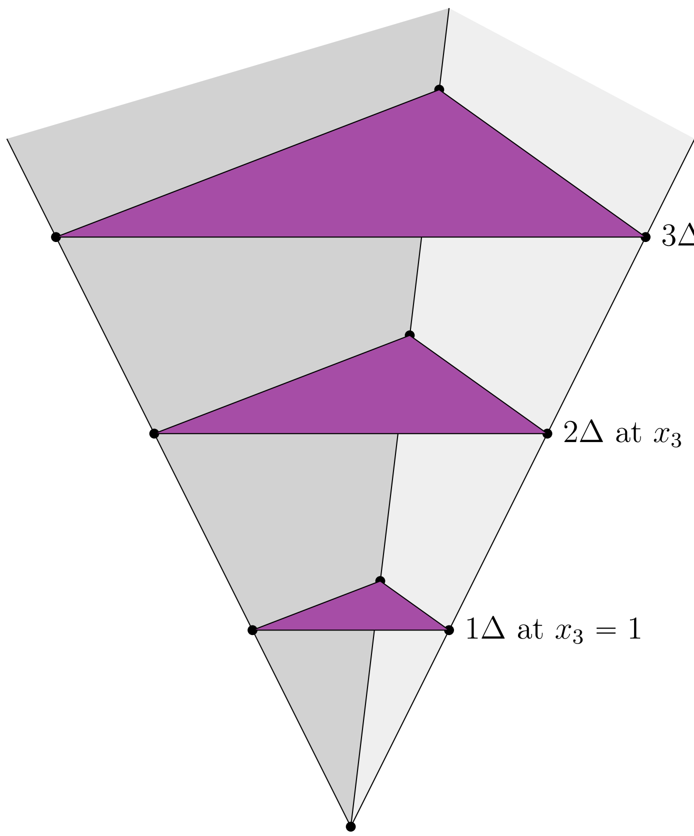


Figure 2.6: The nth dilate of a polytope is the slice of its homogenization at $x_{d+1} = n$, therefore the Ehrhart series is an enumeration of the integer points in the homogenization of a polytope.

Therefore, for any d-dimensional polytope P and any disjoint triangulation $\mathcal{T}$ of P,

$$\mathrm{Ehr}_P(z) = \frac{\sum_{\Delta \in \mathcal{T}} \left(\sum_{y \in \mathrm{hom}(\Delta) \cap \mathbb{Z}^{d+1}} z^{y_{d+1}} \right)}{(1-z)^{d+1}}. \tag{2.7}$$

The numerator of this rational function (i.e., $(1-z)^{d+1}\,\mathrm{Ehr}_P(z)$) is called the h^***-polynomial** of P and is denoted $h^*_P(z)$. From the combinatorial interpretation of the coefficients of the h^*-polynomial in equation 2.7, the fact that its degree is at most $d+1$ (d for a closed polytope) is immediate, as are the following two classical results of Stanley:

Theorem 2.4.2 (Nonnegativity, [107])**.** *For any lattice polytope P, the coefficients of $h^*_P(z)$ are nonnegative integers.*

Theorem 2.4.3 (Monotonocity, [107]). *For any lattice polytopes P and Q such that $P \subseteq Q$, $h^*_P(z) \leqslant h^*_Q(z)$, coefficient-wise.*

Moreover, being able to compute the h^*-polynomial $h^*_P(z) = h^*_d z^d + \cdots + h^*_1 z + h^*_0$ from the integer points in the fundamental parallelepipeds of the simplices in a triangulation of P also enables us to explicitly compute the Ehrhart polynomial of P. By extracting the coefficient of z^n on both sides of the equation

$$\mathrm{Ehr}_P(z) = \frac{h^*_d z^d + \cdots + h^*_1 z + h^*_0}{(1-z)^{d+1}} = (h^*_d z^d + \cdots + h^*_1 z + h^*_0) \sum_{n \geqslant 0} \binom{n+d}{d} z^n, \tag{2.8}$$

we obtain

$$\mathrm{ehr}_P(n) = \sum_{i=0}^{d} h^*_i \binom{n+d-i}{d}.$$

Comparing this expression to the standard 2-simplex example in equation 2.5, we see that in this case we had a very simple h^*-polynomial of 1. This polytope is an example of a **unimodular simplex**, that is, a d-dimensional lattice simplex for which the generators of its homogenization form a lattice basis for $\mathbb{Z}^{d+1}$ (equivalently, a lattice simplex with volume $1/d!$). The fundamental parallelepiped of the cone of a unimodular simplex only contains 1 lattice point (in particular, the sum of the generators corresponding to the vertices that are opposite the removed facets), thus

Theorem 2.4.4. *If Δ is a d-dimensional unimodular simplex with k removed facets, its h^*-polynomial is $h^*_\Delta(z) = z^k$.*

When a polytope has a **unimodular triangulation**, or a triangulation into unimodular simplices, this makes computing its h^*-polynomial very quick. For example, every order polytope has a unimodular triangulation (namely, the one given in equation 2.2). The simplex corresponding to the linear extension σ has $\operatorname{des}\sigma$ many facets removed, therefore

$$h^*_{\mathcal{O}(\Pi)}(z) = \sum_{\sigma \in \mathcal{JH}(\Pi)} z^{\operatorname{des}\sigma}.$$

2.5 Ehrhart theory of rational polytopes

Since we can also express integer-point transforms of rational simplices in terms of their (slightly larger) fundamental parallelepipeds, this generalizes to an Ehrhart theory of rational polytopes. Given a rational polytope $Q \subseteq \mathbb{R}^d$ with denominator $q \geqslant 1$, define its **Ehrhart series** to be

$$\mathrm{Ehr}_Q(z) = \sum_{n \geqslant 0} |nQ \cap \mathbb{Z}^d| z^n.$$

Just as in the lattice case, $\mathrm{Ehr}_Q(z)$ enumerates all lattice points in the homogenization of Q, therefore

$$\mathrm{Ehr}_Q(z) = \sigma_{\mathrm{hom}(Q)}(1, \ldots, 1, z).$$

There exists a triangulation $\mathcal{T}$ of Q into rational simplices of denominator q [1], and by equation 2.4, the Ehrhart series of such a simplex Δ will be equal to

$$\mathrm{Ehr}_\Delta(z) = \frac{\sum\limits_{y \in \square(\Delta)\cap\mathbb{Z}^{d+1}} z^{y_{d+1}}}{(1-z^q)^{d+1}}$$

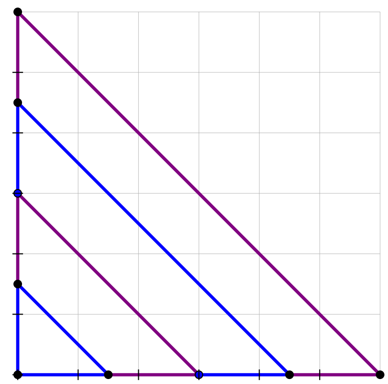
(for $\square(\Delta)$ its fundamental parallelepiped formed using the generators at last coordinate q). Now the last coordinate of an integer point in such a fundamental parallelepiped can be at most $q(d+1)$, thus the Ehrhart series of Q can be expressed in the form

$$\mathrm{Ehr}_Q(z) = \frac{h_Q^*(z)}{(1-z^q)^{d+1}}\,,$$

for $h_Q^*(z)$ a polynomial of degree at most $q(d+1)$ (and at most $q(d+1)-1$ if Q is closed). Again, $h_Q^*(z)$ is called the **h^*-polynomial of Q**. Note that in the rational case, we have to be slightly careful; we define the h^*-polynomial to be the numerator when the denominator of the Ehrhart series is $(1-z^q)^{d+1}$ (even if the rational function $\mathrm{Ehr}_Q(z)$ can be simplified). For this definition $h_Q^*(z) := (1-z^q)^{d+1}\,\mathrm{Ehr}_Q(z)$, Stanley's classical nonnegativity and monotonicity results generalize using the same combinatorial interpretation of the coefficients of the h^*-polynomial. These results will be stated and explored much further in Chapter 4.

As the Ehrhart series of a rational polytope is more complicated when $q > 1$, the counting function $|nQ \cap \mathbb{Z}^d|$ ends up being more complicated as well. The same method of expanding out the Ehrhart series (as in equation 2.8) and collecting the coefficient of z^n on both sides also reveals that $|nQ \cap \mathbb{Z}^d|$ agrees with a polynomial in n, but *which* polynomial it agrees with depends on the value of n mod q. See, for example, the rational triangle in Figure 2.7, which has denominator 2 and an *Ehrhart quasipolynomial* of period 2.

[1]Here, there is a subtlely in the denominator of the simplices. Because not all original vertices will be used in every simplex, some simplices may end up having a smaller denominator. But we can always choose to use a larger fundamental parallelepiped for such simplices, in particular the one with generators of last coordinate q, effectively forcing each simplex's Ehrhart series to still have denominator $(1-z^q)^{d+1}$.



$$\mathrm{ehr}_Q(n) = \begin{cases} \frac{9}{8}n^2 + \frac{9}{4}n + 1 & \text{if } n \equiv 0 \bmod 2 \\ \frac{9}{8}n^2 + \frac{3}{2}n + \frac{3}{8} & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

Figure 2.7: The Ehrhart quasipolynomial of the rational triangle $\mathrm{conv}\{(0,0),(0,1.5),(1.5,0)\}$

Theorem 2.5.1. *If $Q \subseteq \mathbb{R}^d$ is a rational d-dimensional polytope with denominator q, then $\mathrm{ehr}_Q(n) := |nQ \cap \mathbb{Z}^d|$ agrees with a **quasipolynomial** in n. That is, there exist polynomials $p_0(x), \ldots, p_{q-1}(x)$ of degree d such that*

$$\mathrm{ehr}_Q(n) = \begin{cases} p_0(n) & \text{if } n \equiv 0 \bmod q \\ p_1(n) & \text{if } n \equiv 1 \bmod q \\ \vdots & \vdots \\ p_{q-1}(n) & \text{if } n \equiv q-1 \bmod q. \end{cases}$$

(Note that a particular polytope's quasipolynomial might end up being periodic in some $k|q$ many polynomials. If $k < q$, this is referred to as "period collapse.")

Chapter 3

Boundary h^*-polynomials

In this chapter, we follow up Construction 2.2.2 with an analogous construction of a disjoint triangulation for the *boundary* of a polytope, which allows us to define the *boundary h^*-polynomial.* We then use the boundary h^*-polynomial to revisit a theorem of Stapledon [123] from a new lens and to develop new results for *Gorenstein polytopes* and *rational dilations* of polytopes.

3.1 Introduction

Stanley's theorem that h^*-polynomials of rational polytopes have nonnegative coefficients was refined by Stapledon [123] (for lattice polytopes, i.e., $q = 1$) and Beck–Braun–Vindas-Meléndez [19] (for rational polytopes), who showed that the h^*-polynomial can be decomposed using palindromic polynomials with nonnegative coefficients. A *palindromic polynomial* $f(z) = \sum f_i z^i$ has symmetric coefficients, that is, $f_i = f_{\deg(f)-i}$ for $i = 0, \ldots, \deg(f)$; equivalently, $z^{\deg(f)} f(\frac{1}{z}) = f(z)$.

Theorem 3.1.1. *Let P be a full-dimensional rational polytope with denominator q and let ℓ be the smallest positive integer such that ℓP contains an interior lattice point. Then*

$$\frac{1 + z + \cdots + z^{\ell-1}}{1 + z + \cdots + z^{q-1}} \, h_P^*(z) = a(z) + z^\ell \, b(z) \,,$$

where $a(z)$ and $b(z)$ are palindromic polynomials with nonnegative integer coefficients.

We remark that our assumption that P is full dimensional and our convention for $h_P^*(z)$ imply that $h_P^*(z)$ is divisible by $1 + z + \cdots + z^{q-1}$, because the leading coefficient of $\mathrm{ehr}_P(n)$ is constant (namely, the volume of P) and thus $z = 1$ is the unique pole with maximal order of the rational function $\mathrm{Ehr}_P(z)$; see, e.g., [113, Theorem 4.1.1]. Thus the decomposition of the polynomial $\frac{1+z+\cdots+z^{\ell-1}}{1+z+\cdots+z^{q-1}} \, h_P^*(z)$ into two palindromic polynomials as in Theorem 3.1.1 is unique and an easy exercise. The point of Theorem 3.1.1 is that $a(z)$ and $b(z)$ have nonnegative coefficients.

As was pointed out in [19], Theorem 3.1.1 immediately implies the inequalities

$$h_0^* + \cdots + h_{j+1}^* \geqslant h_{q(d+1)-1}^* + \cdots + h_{q(d+1)-1-j}^* \,, \qquad j = 0, \ldots, \left\lfloor \tfrac{q(d+1)-1}{2} \right\rfloor - 1 \,,$$
$$h_s^* + \cdots + h_{s-j}^* \geqslant h_0^* + \cdots + h_j^* \,, \qquad j = 0, \ldots, s \,,$$

where $s := \deg h_P^*(z)$. For the case $q = 1$ (i.e., lattice polytopes), they go back to Hibi [68, 71] and Stanley [116].

The case $q = \ell = 1$ of Theorem 3.1.1 was proved much earlier by Betke–McMullen [29]. Writing a (combinatorially defined) polynomial as a sum of two palindromic polynomials as in Theorem 3.1.1 is often referred to as a *symmetric decomposition* and has applications beyond a refinement of nonnegativity; we mention one representative (much more can be found, e.g., in [36]): $p(z) = a(z) + z\, b(z)$, with $a(z)$ and $b(z)$ palindromic, is alternatingly increasing (i.e., the coefficients of $p(z)$ satisfy $0 \leqslant p_0 \leqslant p_d \leqslant p_1 \leqslant p_{d-1} \leqslant \cdots$) if and only if the coefficients of both $a(z)$ and $b(z)$ are nonnegative and unimodal (i.e., the coefficients increase up to some index and then decrease).

Betke–McMullen's, Stapledon's, as well as Beck–Braun–Vindas-Meléndez's proofs use local h-vectors of a triangulation, their nonnegativity, and the Dehn–Sommerville relations. Our main goal is to give an "easy" conceptual proof of Theorem 3.1.1—in particular, one that is independent of local h-vectors (though they are hiding under the surface). Our *ansatz* is to study the h^*-polynomial of the *boundary* of a rational polytope, and our second goal is to exhibit that such a study is worthwhile, with the hope for further applications. The connection to Theorem 3.1.1 is that the h^*-polynomial of the boundary of P, defined via

$$\mathrm{Ehr}_{\partial P}(z) := 1 + \sum_{n \geqslant 1} \mathrm{ehr}_{\partial P}(n) z^n = \frac{h_{\partial P}^*(z)}{(1 - z^q)^d} \,, \tag{3.1}$$

equals $a(z)$, for any q and ℓ;[1] as we will see below, this equality follows from the uniqueness of the symmetric decomposition and the palindromicity (3.2) of $h_{\partial P}^*(z)$. We remark that, a priori, it is not clear that we can always represent the generating function of $\mathrm{ehr}_{\partial P}(n)$ in the form (3.1); in particular, we will see below (where we will show that this form always exists) that $h_{\partial P}^*(z)$ has degree qd, contrary to the degree of $h_P^*(z)$, and so the quasipolynomial $\mathrm{ehr}_{\partial P}(n)$ does not have constant term 1.

To illustrate the philosophy behind our approach, here is a do-it-yourself proof setup for Theorem 3.1.1 in the case $q = \ell = 1$:

- fix a (half-open) triangulation T of the boundary ∂P and extend T to a (half-open) triangulation of P by coning over an interior lattice point $\boldsymbol{x}$;
- convince yourself that $a(z) = h_{\partial P}^*(z)$ is palindromic with nonnegative (in fact, as we will show in Theorem 3.2.4, positive) coefficients;

[1] We suspect that this fact is well known to the experts, but we could not find it in the literature. We thank Katharina Jochemko for pointing it out to us.

- realize that the h^*-polynomial of each half-open simplex $\Delta \in T$ is coefficient-wise less than or equal to the h^*-polynomial of $\operatorname{conv}(\Delta, \boldsymbol{x})$, and thus $h_P^*(z) - a(z)$ has nonnegative coefficients.

It turns out that this philosophy works for general q and ℓ, with slight modifications (for example, for $\ell > 1$, the interior point $\boldsymbol{x}$ cannot be a lattice point). To state our *ansatz* from a different angle, we approach Theorem 3.1.1 by giving (1) a (positive, symmetric) interpretation of $a(z)$ and (2) an explicit construction which shows that $a(z) \leqslant \frac{1+z+\cdots+z^{\ell-1}}{1+z+\cdots+z^{q-1}}\, h_P^*(z)$.

This point of view has consequences beyond a (somewhat short) proof of Theorem 3.1.1. One of these consequences is an inequality among the two leading coefficients of an Ehrhart polynomial which seems to be novel.

Corollary 3.1.2. *Let P be a full-dimensional lattice d-polytope and let ℓ be the smallest positive integer such that ℓP contains an interior lattice point. Then the two leading coefficients of* $\operatorname{ehr}_P(n) = k_d\, n^d + k_{d-1}\, n^{d-1} + \cdots + k_0$ *satisfy*

$$\frac{\ell\, d}{2}\, k_d \geqslant k_{d-1}\,.$$

Betke–McMullen [29, Theorem 6] gave upper bounds for each k_j in terms of k_d and Stirling numbers of the first kind. For $j = d-1$ they yield $\binom{d+1}{2} k_d \geqslant k_{d-1}$, which Corollary 3.1.2 improves upon.[2]

The structure of this chapter is as follows. Section 3.2 serves as a point of departure for our study of $h_{\partial P}^*(z)$. We prove several inequalities for the coefficients of $h_{\partial P}^*(z)$, among them a lower bound result (Theorem 3.2.4), which in particular shows that $h_{\partial P}^*(z)$ has *positive* coefficients.

Section 3.3 gives a construction for certain half-open triangulations we will need for our proof of Theorem 3.1.1 in Section 3.4; this section also contains a proof of Corollary 3.1.2.

In Section 3.5 we discuss reflexive and Gorenstein polytopes, as well as their rational analogues, and implications for these polytopes from the viewpoint of $h_{\partial P}^*(z)$. Our main result in this section (Theorem 3.5.4) says that if P is a rational polytope with denominator q such that gP is reflexive, then

$$h_P^*(z) = \frac{1+z+\cdots+z^{q-1}}{1+z+\cdots+z^{g-1}}\, h_{\partial P}^*(z)\,.$$

In Section 3.6 we prove an analogue of Theorem 3.1.1 for rational (or, equivalently, real) Ehrhart dilations.

---

[2]We thank Martin Henk for reminding us about the Betke–McMullen inequalities and asking whether they could be improved using our setup.

3.2 h^*-polynomials of boundaries of polytopes

We start by addressing some of the subtleties in defining the Ehrhart series (and thus the h^*-polynomial) of ∂P, especially when we ultimately compute the Ehrhart series using a half-open triangulation of the boundary. The reason for the convention (3.1) that $\mathrm{Ehr}_{\partial P}(z)$ (and therefore also $h^*_{\partial P}(z)$) has constant term 1 is *Ehrhart–Macdonald reciprocity* (see, e.g., [25, Corollary 5.4.5]): it says that the rational generating functions $\mathrm{Ehr}_P(z)$ and

$$\mathrm{Ehr}_{P^\circ}(z) := \sum_{n \geqslant 1} \mathrm{ehr}_{P^\circ}(n) z^n = \frac{h^*_{P^\circ}(z)}{(1-z^q)^{d+1}}$$

are related via

$$\mathrm{Ehr}_{P^\circ}(\tfrac{1}{z}) = (-1)^{d+1}\,\mathrm{Ehr}_P(z)$$

or, equivalently,

$$z^{q(d+1)}\, h^*_{P^\circ}(\tfrac{1}{z}) = h^*_P(z)\,.$$

As $\mathrm{Ehr}_{\partial P}(z) = \mathrm{Ehr}_P(z) - \mathrm{Ehr}_{P^\circ}(z)$, we obtain

$$h^*_{\partial P}(z) = \frac{h^*_P(z) - h^*_{P^\circ}(z)}{1 - z^q}$$

and thus $h^*_{\partial P}(z)$ is palindromic, i.e.,

$$z^{qd}\, h^*_{\partial P}(\tfrac{1}{z}) = h^*_{\partial P}(z)\,. \tag{3.2}$$

(See [25, Proposition 5.6.3] for connections to more general self-reciprocal complexes and the Dehn–Sommerville relations.)

So h^*-polynomials of boundaries of polytopes are in a sense more restricted than h^*-polynomials of polytopes. Even further, as we will show in Theorem 3.2.4 below, $h^*_{\partial P}(z)$ has no internal zero coefficients, which is far from true for $h^*_P(z)$ (see, e.g., [75]).

On the other hand, there is the following alternative extension of nonnegativity of $h^*_P(z)$ due to Stanley [107] (we state only the version for lattice polytopes):

Theorem 3.2.1. *Let P and Q be lattice polytopes with $P \subseteq Q$. Then $h^*_P(z) \leqslant h^*_Q(z)$ coefficient-wise.*

Monotonicity does not hold for $h^*_{\partial P}(z)$, as the following example exhibits.

Example 3.2.2. Let

$$P = \mathrm{conv}\{(0,0),(0,2),(2,0),(2,2)\} \text{ and } Q = \mathrm{conv}\{(0,0),(0,2),(2,0),(3,3)\}.$$

Then

$$h^*_{\partial P}(z) = 1 + 6z + z^2 \qquad \text{and} \qquad h^*_{\partial Q}(z) = 1 + 4z + z^2.$$

Thus $P \subseteq Q$ but $h^*_{\partial P}(z) \geqslant h^*_{\partial Q}(z)$ coefficient-wise.

Moreover, even though $\partial P \subseteq P$, it is not always true that $h^*_{\partial P}(z) \leqslant h^*_P(z)$: while $h^*_{\partial P}(z)$ always has degree qd, there are many polytopes for which $h^*_P(z)$ has lower degree. On the other hand, the $\ell = q = 1$ case of Theorem 3.1.1 implies

$$h^*_P(z) = h^*_{\partial P}(z) + z\, b(z)\,,$$

thus when P is a lattice polytope containing an interior lattice point, it is true that $h^*_{\partial P}(z) \leqslant h^*_P(z)$. The rational version gives us a more general necessary condition in terms of ℓ and q for $h^*_{\partial P}(z) \leqslant h^*_P(z)$ to hold (again, assuming Theorem 3.1.1):

Corollary 3.2.3. *Let $P \subset \mathbb{R}^d$ be a rational polytope with denominator q, and let $\ell \geqslant 1$ be the smallest dilate of P that contains an interior lattice point. If $\ell \leqslant q$, then $h^*_{\partial P}(z) \leqslant h^*_P(z)$. (In particular, when P is a lattice polytope, $\ell = 1$ suffices.)*

Proof. By Theorem 3.1.1 (with the interpretation $a(z) = h^*_{\partial P}(z)$) and the assumption $\ell \leqslant q$,

$$h^*_{\partial P}(z) \leqslant (1 + z + \cdots + z^{\ell-1})\, \frac{h^*_P(z)}{1 + z + \cdots + z^{q-1}} \leqslant (1 + z + \cdots + z^{q-1})\, \frac{h^*_P(z)}{1 + z + \cdots + z^{q-1}}$$

coefficient-wise. □

The following lower-bound result is a restatement of a theorem of Stapledon [123, Theorem 2.14] (and a simplified proof) in our language:

Theorem 3.2.4. *If P is a lattice d-polytope with boundary h^*-polynomial $h^*_{\partial P}(z) = h^*_{\partial P,d} z^d + \cdots + h^*_{\partial P,1} z + h^*_{\partial P,0}$ then*

$$1 = h^*_{\partial P,0} \leqslant h^*_{\partial P,1} \leqslant h^*_{\partial P,j} \qquad \text{for } j = 2, \ldots, d-1.$$

In particular, $h^*_{\partial P}(z)$ has positive coefficients.

This result parallels a lower-bound theorem of Hibi [68], who proved that if P is a d-dimensional lattice polytope with h^*-polynomial $h^*_P(z) = h^*_{P,d} z^d + \cdots + h^*_{P,1} z + h^*_{P,0}$ of degree d (i.e., P contains an interior lattice point), then $1 = h^*_{P,0} \leqslant h^*_{P,1} \leqslant h^*_{P,j}$ for $j = 2, \ldots, d-1$. Our proof mirrors that of Hibi; it turns out that adapting it for $h^*_{\partial P}(z)$ simplifies the proof.

Proof of Theorem 3.2.4. Let T be a triangulation of ∂P that uses every lattice point in ∂P (i.e., every lattice point in ∂P is a vertex of a simplex in T), with h-vector $(h_0, \ldots, h_d)$ defined, as usual, via

$$h_d\, x^d + h_{d-1}\, x^{d-1} + \cdots + h_0 = \sum_{k=-1}^{d-1} f_k\, x^{k+1} (1-x)^{d-1-k}$$

where f_k denotes the number of k-simplices in T and $f_{-1} = 1$. The definitions of h and h^* imply

$$h_1 = f_0 - d = \left|\partial P \cap \mathbb{Z}^d\right| - d = h^*_{\partial P,1}\,. \tag{3.3}$$

(Stanley [110] proved much more about relations between h and h^*.) Barnette's famous lower bound theorem [15] (see also [69, Theorem 13.1]) says

$$h_1 \leqslant h_j \qquad \text{for } j = 2, \ldots, d-1. \tag{3.4}$$

Finally, we apply a theorem of Betke and McMullen [29, Theorem 2] to our situation: it yields

$$h_j \leqslant h^*_{\partial P, j} \qquad \text{for } j = 0, \ldots, d. \tag{3.5}$$

(Betke–McMullen [29] prove much more, giving a formula for h^* in terms of local h-vectors.) The inequalities (3.3), (3.4), and (3.5) line up to complete our proof. □

We finish this section by recalling, for the record, another set of inequalities for $h^*_{\partial P}(z)$ in the special (and important) case that P admits a regular unimodular boundary triangulation, once more due to Stapledon [123, Theorem 2.20]:

$$1 = h^*_{\partial P,0} \leqslant h^*_{\partial P,1} \leqslant \cdots \leqslant h^*_{\partial P, \lfloor \frac{d}{2} \rfloor} \qquad \text{and} \qquad h^*_{\partial P,j} \leqslant \binom{h^*_{\partial P,1} + j - 1}{j}.$$

3.3 Half-open boundary triangulations

Our approach is to triangulate ∂P into disjoint half-open simplices of dimension $d-1$, in order to avoid inclusion–exclusion arguments. There is a subtlety stemming from our convention that the h^*-polynomial of ∂P has constant term 1, which we need to address here.

Before introducing the half-open boundary triangulations we will use in the later proofs, we recall the Ehrhart series of a *half-open* simplex. Let $\Delta \subseteq \mathbb{R}^d$ be the simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1} \in \frac{1}{q}\mathbb{Z}^d$, where the facets opposite $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are missing. That is, let

$$\Delta = \left\{ \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_{d+1} \boldsymbol{v}_{d+1} : \begin{array}{c} \lambda_1, \ldots, \lambda_r > 0 \\ \lambda_{r+1}, \ldots, \lambda_{d+1} \geqslant 0 \\ \lambda_1 + \cdots + \lambda_{d+1} = 1 \end{array} \right\}.$$

Its Ehrhart series is

$$\mathrm{Ehr}_\Delta(z) := \sum_{n \geqslant 0} \mathrm{ehr}_\Delta(n) z^n,$$

with constant term 1 if and only if Δ is closed, therefore it is possible for its expression as a rational function to be improper. That is, as seen in the construction below, the Ehrhart series can still be expressed in the form

$$\mathrm{Ehr}_\Delta(z) = \frac{h^*_\Delta(z)}{(1-z^q)^{d+1}},$$

but it is possible for the h^*-polynomial to have degree equal to $q(d+1)$. The following tiling argument of the homogenization of a rational simplex will also be crucial to the proof of Theorem 3.1.1 in the next section.

Construction 3.3.1. We follow [25, Section 4.6] and define the *homogenization* of Δ to be the half-open cone

$$\mathrm{hom}(\Delta) := \sum_{j=1}^{r} \mathbb{R}_{>0} \begin{pmatrix} \boldsymbol{v}_j \\ 1 \end{pmatrix} + \sum_{j=r+1}^{d+1} \mathbb{R}_{\geqslant 0} \begin{pmatrix} \boldsymbol{v}_j \\ 1 \end{pmatrix} \subset \mathbb{R}^{d+1}.$$

Observe that the intersection of $\mathrm{hom}(\Delta)$ with the hyperplane $x_{d+1} = n$ gives precisely a copy of the nth dilate of Δ, whence

$$\mathrm{Ehr}_\Delta(z) = \sum_{\boldsymbol{x} \in \mathrm{hom}(\Delta) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}}.$$

Define the *fundamental parallelepiped* of Δ to be

$$\mathcal{F}_\Delta := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \sum_{j=r+1}^{d+1} [0,1) \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix}.$$

Because the generators of $\mathrm{hom}(\Delta)$ are linearly independent, $\mathrm{hom}(\Delta)$ can be tiled with translates of $\mathcal{F}_\Delta$. More precisely, every point in $\mathrm{hom}(\Delta)$ can be uniquely expressed as the sum of a nonnegative integral combination of the $\binom{q\boldsymbol{v}_j}{q}$ and an integral point in $\mathcal{F}_\Delta$. This yields

$$\mathrm{Ehr}_\Delta(z) = \sum_{k_1,\ldots,k_{d+1} \in \mathbb{Z}_{\geqslant 0}} \sum_{\boldsymbol{m} \in \mathcal{F}_\Delta \cap \mathbb{Z}^{d+1}} z^{qk_1 + \cdots + qk_{d+1} + m_{d+1}} = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}_\Delta \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^q)^{d+1}}.$$

Observe that we chose the fundamental parallelepiped to have generators each with last coordinate q, however it is not necessary for this to be the case. For example, suppose $q_1, \ldots, q_{d+1}$ are such that $\boldsymbol{v}_j \in \frac{1}{q_j}\mathbb{Z}^{d+1}$. Then we can tile $\mathrm{hom}(\Delta)$ with translates of the fundamental parallelepiped

$$\mathcal{F}'_\Delta := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q_j\boldsymbol{v}_j \\ q_j \end{pmatrix} + \sum_{j=r+1}^{d+1} [0,1) \begin{pmatrix} q_j\boldsymbol{v}_j \\ q_j \end{pmatrix}$$

to obtain the expression for the Ehrhart series

$$\mathrm{Ehr}_\Delta(z) = \sum_{k_1,\ldots,k_{d+1} \in \mathbb{Z}_{\geqslant 0}} \sum_{\boldsymbol{m} \in \mathcal{F}'_\Delta \cap \mathbb{Z}^{d+1}} z^{q_1k_1 + \cdots + q_{d+1}k_{d+1} + m_{d+1}} = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}'_\Delta \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^{q_1}) \cdots (1-z^{q_{d+1}})}.$$

Therefore, the denominator in our resulting expression of the Ehrhart series depends on our choice of fundamental parallelepiped, a subtlety that will be important in the following section.

Naturally, we would like to construct a triangulation of ∂P such that

$$h^*_{\partial P}(z) = \sum_{\Delta \in T} h^*_\Delta(z) \tag{3.6}$$

when T is a disjoint half-open triangulation of ∂P. We achieve this as follows:

- For a d-dimensional rational polytope P with denominator q, choose a rational triangulation T of ∂P into $(d-1)$-dimensional simplices with denominators dividing q. This can be accomplished, for example, by picking a triangulation that uses only the vertices of P.

- Use the convention to express the Ehrhart series as if each simplex $\Delta \in T$ has denominator q, i.e., define
$$h_\Delta^*(z) = (1 - z^q)^d \operatorname{Ehr}_\Delta(z) .$$
This guarantees that we will already have a common denominator when adding up the Ehrhart series of the simplices in the boundary triangulation.

- Apply Construction 3.3.2 below to turn T into a disjoint triangulation with exactly one closed simplex. This guarantees that the constant term of $\operatorname{Ehr}_{\partial P}(z)$ will match the constant term of the sum of the Ehrhart series of the simplices in T.

Once we are able to construct such a disjoint boundary triangulation T with exactly one closed simplex, we will obtain
$$\sum_{\Delta \in T} \frac{h_\Delta^*(z)}{(1-z^q)^d} = \sum_{n \geqslant 0} \sum_{\Delta \in T} \operatorname{ehr}_\Delta(n) z^n = 1 + \sum_{n \geqslant 1} \operatorname{ehr}_{\partial P}(n) = \frac{h_{\partial P}^*(z)}{(1-z^q)^d} ,$$
as desired.

Fortunately, the following construction gives a disjoint boundary triangulation T into half-open $(d-1)$-simplices, exactly one of which is closed; for such a T, we will have (3.6).

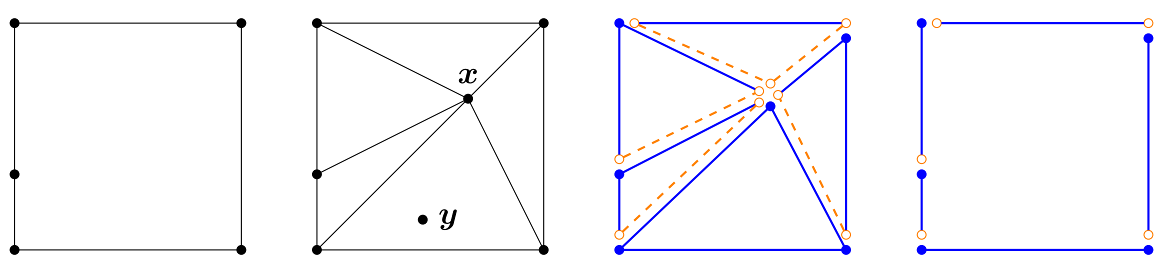


Figure 3.1: Making a boundary triangulation disjoint via a visibility construction. Closed facets are shown in blue and open facets are shown in orange.

Construction 3.3.2. Fix a rational polytope P and a rational triangulation T of ∂P into $(d-1)$-dimensional simplices. We will construct, from this given T, a *disjoint* boundary triangulation into half-open simplices. Choose any point $\boldsymbol{x} \in P^\circ$ and let T' be the triangulation of P given by coning over T, i.e.,
$$T' := \{\operatorname{Pyr}(\boldsymbol{x}, \Delta) : \Delta \in T\}$$

where for

$$\Delta = \left\{ \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_d \boldsymbol{v}_d : \begin{array}{l} \lambda_1, \ldots, \lambda_d \geqslant 0 \\ \lambda_1 + \cdots + \lambda_d = 1 \end{array} \right\}$$

we define

$$\operatorname{Pyr}(\boldsymbol{x}, \Delta) := \left\{ \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_d \boldsymbol{v}_d + \lambda_{d+1} \boldsymbol{x} : \begin{array}{l} \lambda_1, \ldots, \lambda_{d+1} \geqslant 0 \\ \lambda_1 + \cdots + \lambda_{d+1} = 1 \end{array} \right\}.$$

Choose a point $\boldsymbol{y} \in P$ that is *generic* with respect to T', i.e., $\boldsymbol{y}$ is not contained in any facet-defining hyperplane of any simplex in T'. For each simplex $\Delta' \in T'$, remove all facets of Δ' that are *visible* from $\boldsymbol{y}$, i.e., remove all facets F of Δ' such that $\boldsymbol{y}$ is not in the halfspace corresponding to F that defines Δ' (see, e.g., [25, Chapter 5] for more details about half-open triangulations using a visibility construction). This makes T' into a disjoint triangulation of P, which restricts to a disjoint triangulation of ∂P. Moreover, the only closed simplex in the disjoint triangulation of ∂P is the one corresponding to Δ' that contains $\boldsymbol{y}$. An example is shown in Figure 3.1.

Remark 1. We note that if T is a disjoint lattice triangulation with exactly one closed simplex, then $h^*_{\partial P,k}$ is at least the number of simplices in T with k missing faces (since the fundamental parallelepiped of such a simplex has a lattice point equal to the sum of the k generators opposite those missing faces). In the example in Figure 3.1, if T is a lattice triangulation, we learn that $h^*_{\partial P}(z) \geqslant 1 + 3z + z^2$, coefficient-wise.

3.4 Symmetric decompositions of h^*-polynomials

We will now prove Theorem 3.1.1 using a half-open triangulation of the boundary of the polytope. The geometric interpretation of the coefficients of the h^*-polynomial of a half-open simplex in Construction 3.3.1 allows us to compare h^*-polynomials of simplices via their fundamental parallelepipeds, which will ultimately allow us to compare $h^*_P(z)$ and $h^*_{\partial P}(z)$. In the lattice case, we will see the finite geometric series $1 + z + \cdots + z^{\ell-1}$ arise naturally as the correction between the denominator of the Ehrhart series of a lattice polytope and the denominator of the Ehrhart series of a rational polytope with all integral vertices except for one vertex with denominator ℓ. In the rational case, we will see the term $\frac{1+z+\cdots+z^{\ell-1}}{1+z+\cdots+z^{q-1}}$ arise as the correction between the denominator of the Ehrhart series of a rational polytope with denominator q and the denominator of the Ehrhart series of a rational polytope with all denominator-q vertices except for one vertex with denominator ℓ.

Proof of Theorem 3.1.1. Let T be a disjoint triangulation of ∂P into $(d-1)$-dimensional rational half-open simplices with denominator q. By Construction 3.3.2, we may assume that exactly one simplex in T is closed. Let $\boldsymbol{x} \in P^\circ$ be such that $\ell \boldsymbol{x} \in (\ell P)^\circ \cap \mathbb{Z}^d$; by the minimality of ℓ, we know that $\boldsymbol{x}$ has denominator ℓ. Given $\Delta \in T$, let $\operatorname{Pyr}(\boldsymbol{x}, \Delta)$ be the corresponding half-open d-simplex. These simplices give rise to the disjoint rational triangulation $T' := \{\operatorname{Pyr}(\boldsymbol{x}, \Delta) : \Delta \in T\}$ of P.

We first claim that for any $\Delta \in T$,

$$h^*_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}(z) = h^*_\Delta(z) + z^\ell \, b_\Delta(z) \tag{3.7}$$

for some polynomial $b_\Delta(z)$ with nonnegative coefficients. To see this, let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_d \in \mathbb{Q}^d$ be the vertices of Δ (so $q\boldsymbol{v}_1, \dots, q\boldsymbol{v}_d \in \mathbb{Z}^d$). Relabel the vertices such that the facets of Δ opposite $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ are missing and the facets of Δ opposite $\boldsymbol{v}_{r+1}, \dots, \boldsymbol{v}_d$ are present in Δ.

Now we apply Construction 3.3.1 with the half-open parallelepiped

$$\mathcal{F}_\Delta := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \sum_{j=r+1}^{d} [0,1) \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix}$$

yielding

$$\mathrm{Ehr}_\Delta(z) = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}_\Delta \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^q)^d} \, .$$

Meanwhile, $\mathrm{Pyr}(\boldsymbol{x}, \Delta)$ is a d-dimensional simplex with vertices $\boldsymbol{v}_1, \dots, \boldsymbol{v}_d, \boldsymbol{x}$, with missing facets opposite $\boldsymbol{v}_1, \dots, \boldsymbol{v}_r$ and included facets opposite $\boldsymbol{v}_{r+1}, \dots, \boldsymbol{v}_d, \boldsymbol{x}$. Then we again apply Construction 3.3.1 (noting the choice of fundamental parallelepiped) to obtain a tiling of

$$\mathrm{hom}(\mathrm{Pyr}(\boldsymbol{x}, \Delta)) = \sum_{j=1}^{r} \mathbb{R}_{>0} \begin{pmatrix} \boldsymbol{v}_j \\ 1 \end{pmatrix} + \sum_{j=r+1}^{d} \mathbb{R}_{\geqslant 0} \begin{pmatrix} \boldsymbol{v}_j \\ 1 \end{pmatrix} + \mathbb{R}_{\geqslant 0} \begin{pmatrix} \boldsymbol{x} \\ 1 \end{pmatrix}$$

with translates of the half-open parallelepiped

$$\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)} := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \sum_{j=r+1}^{d} [0,1) \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + [0,1) \begin{pmatrix} \ell\boldsymbol{x} \\ \ell \end{pmatrix}$$

giving

$$\mathrm{Ehr}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}(z) = \sum_{\substack{k_1, \dots, k_{d+1} \in \mathbb{Z}_{\geqslant 0}, \\ \boldsymbol{m} \in \mathcal{F}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)} \cap \mathbb{Z}^{d+1}}} z^{qk_1 + \cdots + qk_d + \ell k_{d+1} + m_{d+1}} = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)} \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^q)^d (1-z^\ell)} \, .$$

Now let $\boldsymbol{m}$ be a lattice point in $\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}$, say

$$\boldsymbol{m} = \alpha_1 \begin{pmatrix} q\boldsymbol{v}_1 \\ q \end{pmatrix} + \cdots + \alpha_d \begin{pmatrix} q\boldsymbol{v}_d \\ q \end{pmatrix} + \beta \begin{pmatrix} \ell\boldsymbol{x} \\ \ell \end{pmatrix} \tag{3.8}$$

with $0 < \alpha_1, \dots, \alpha_r \leqslant 1$ and $0 \leqslant \alpha_{r+1}, \dots, \alpha_d, \beta < 1$. If $\beta = 0$, then $\boldsymbol{m} \in \mathcal{F}_\Delta$. If $\beta > 0$, then the first d coordinates of $\boldsymbol{m}$ are given by

$$\alpha_1 q\boldsymbol{v}_1 + \cdots + \alpha_d q\boldsymbol{v}_d + \beta\ell\boldsymbol{x} \, ,$$

and since $\beta > 0$, this is a lattice point in $m_{d+1}(\mathrm{Pyr}(\boldsymbol{x},\Delta)\backslash\Delta)$, where

$$m_{d+1} = \alpha_1 q + \cdots + \alpha_d q + \beta\ell\,.$$

But by the minimality of ℓ, we know that $j(\mathrm{Pyr}(\boldsymbol{x},\Delta)\backslash\Delta)$ contains no lattice points for $j = 1, \ldots, \ell-1$; thus, $m_{d+1} \geqslant \ell$. Therefore,

$$\begin{aligned} h^*_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}(z) &= \sum_{\boldsymbol{m}\in\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}\cap\mathbb{Z}^{d+1}} z^{m_{d+1}} = \sum_{\boldsymbol{m}\in\mathcal{F}_{\Delta}\cap\mathbb{Z}^{d+1}} z^{m_{d+1}} + z^\ell\, b_\Delta(z) \\ &= h^*_\Delta(z) + z^\ell\, b_\Delta(z) \end{aligned}$$

for some polynomial $b_\Delta(z)$ with nonnegative integral coefficients, which proves claim (3.7).

Summing over all $\Delta \in T$ now yields

$$\mathrm{Ehr}_P(z) = \sum_{\Delta\in T} \mathrm{Ehr}_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}(z) = \frac{\sum_{\Delta\in T} h^*_{\mathrm{Pyr}(\boldsymbol{x},\Delta)}(z)}{(1-z^q)^d(1-z^\ell)} = \frac{\sum_{\Delta\in T}\left(h^*_\Delta(z) + z^\ell\, b_\Delta(z)\right)}{(1-z^q)^d(1-z^\ell)}\,.$$

Define $b_P(z) := \sum_{\Delta\in T} b_\Delta(z)$, which we know to be a polynomial with nonnegative integral coefficients. Thus

$$\mathrm{Ehr}_P(z) = \frac{h^*_P(z)}{(1-z^q)^{d+1}} = \frac{h^*_{\partial P}(z) + z^\ell\, b_P(z)}{(1-z^q)^d(1-z^\ell)}$$

and so

$$\frac{1+z+\cdots+z^{\ell-1}}{1+z+\cdots+z^{q-1}}\, h^*_P(z) = h^*_{\partial P}(z) + z^\ell\, b_P(z)\,.$$

By (3.2), $h^*_{\partial P}(z)$ is palindromic with $z^{qd}\, h^*_{\partial P}(\frac{1}{z}) = h^*_{\partial P}(z)$, and the palindromicity of

$$b_P(z) = \frac{\frac{1-z^\ell}{1-z^q}\, h^*_P(z) - h^*_{\partial P}(z)}{z^\ell}$$

follows from this, $h^*_P(z) = h^*_{P^\circ}(z) + (1-z^q)h^*_{\partial P}(z)$, and Ehrhart–Macdonald reciprocity:

$$\begin{aligned} z^{qd-\ell} b_P(\tfrac{1}{z}) &= z^{qd-\ell}\, \frac{\frac{1-\frac{1}{z^\ell}}{1-\frac{1}{z^q}}\, h^*_P(\frac{1}{z}) - h^*_{\partial P}(\frac{1}{z})}{\frac{1}{z^\ell}} \\ &= z^{qd}\left(\frac{1-\frac{1}{z^\ell}}{1-\frac{1}{z^q}}\left(h^*_{P^\circ}(\tfrac{1}{z}) + (1-\tfrac{1}{z^q})h^*_{\partial P}(\tfrac{1}{z})\right) - h^*_{\partial P}(\tfrac{1}{z})\right) \\ &= z^{qd}\left(\frac{1-\frac{1}{z^\ell}}{1-\frac{1}{z^q}}\, h^*_{P^\circ}(\tfrac{1}{z}) + (1-\tfrac{1}{z^\ell})h^*_{\partial P}(\tfrac{1}{z}) - h^*_{\partial P}(\tfrac{1}{z})\right) \\ &= \frac{1}{z^\ell}\left(\frac{z^\ell-1}{z^q-1}\, z^{q(d+1)}\, h^*_{P^\circ}(\tfrac{1}{z}) - z^{qd}\, h^*_{\partial P}(\tfrac{1}{z})\right) \\ &= \frac{1}{z^\ell}\left(\frac{1-z^\ell}{1-z^q}\, h^*_P(z) - h^*_{\partial P}(z)\right) \\ &= b_P(z)\,. \end{aligned}$$

□

Remark 2. If $P \subset \mathbb{R}^{d-1}$ is a lattice polytope, it is well known that $h_P^*(z)$ equals the h^*-polynomial of $\operatorname{Pyr}(\boldsymbol{e_d}, P) \subset \mathbb{R}^d$. Our above proof implies for more general pyramids

$$h_P^*(z) \leqslant h_{\operatorname{Pyr}(\boldsymbol{x},P)}^*(z)$$

coefficient-wise, for any $\boldsymbol{x} \in \mathbb{Z}^d$ not in the affine span of P. Moreover, the proof reveals a rational analogue:

Corollary 3.4.1. *Let $P \subset \mathbb{R}^{d-1}$ be a rational polytope with denominator q. Then*

$$h_P^*(z) = h_{\operatorname{Pyr}(\boldsymbol{e_d},P)}^*(z)\,,$$

where $h_P^*(z) = (1-z^q)^d \operatorname{Ehr}_P(z)$ and $h_{\operatorname{Pyr}(\boldsymbol{e_d},P)}^*(z) = (1-z^q)^d(1-z)\operatorname{Ehr}_{\operatorname{Pyr}(\boldsymbol{e_d},P)}(z)$. If $\boldsymbol{x} \in \frac{1}{r}\mathbb{Z}^d$, then

$$h_P^*(z) \leqslant h_{\operatorname{Pyr}(\boldsymbol{x},P)}^*(z)\,,$$

where $h_P^*(z) = (1-z^q)^d \operatorname{Ehr}_P(z)$ and $h_{\operatorname{Pyr}(\boldsymbol{x},P)}^*(z) = (1-z^q)^d(1-z^r)\operatorname{Ehr}_{\operatorname{Pyr}(\boldsymbol{x},P)}(z)$.

We finish this section with proving Corollary 3.1.2.

Proof of Corollary 3.1.2. It is well known that $\frac{h_P^*(1)}{d!}$ equals the volume of P, which in turn equals k_d; this follows, e.g., by writing $\operatorname{ehr}_P(n)$ in terms of the binomial-coefficient basis $\binom{n}{d}$, $\binom{n+1}{d}$, …, $\binom{n+d}{d}$ (and then the coefficients are precisely the coefficients of $h_P^*(z)$; see, e.g., [24, Section 3.5]). By the same reasoning, $\frac{h_{\partial P}^*(1)}{(d-1)!}$ equals the sum of the volumes of the facets of P, each measures with respect to the sublattice in the affine span of the facet. This sum, in turn, is well known to equal $2\,k_{d-1}$. Putting it all together,

$$\ell\, d!\, k_d = \ell\, h_P^*(1) \geqslant h_{\partial P}^*(1) = 2(d-1)!\, k_{d-1}\,,$$

where the inequality follows from specializing Theorem 3.1.1 (with the interpretation $a(z) = h_{\partial P}^*(z)$) at $z = 1$. ☐

3.5 Symmetric decompositions of h^*-polynomials of Gorenstein polytopes

A d-dimensional polytope P is *reflexive* if it is a lattice polytope that contains the origin in its interior and one of the following (equivalent) statements holds:

1. the dual polytope of P, i.e.,

$$P^* := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \cdot \boldsymbol{y} \leqslant 1 \text{ for all } \boldsymbol{y} \in P\}\,,$$

   is a lattice polytope;

2. $P = \{\boldsymbol{x} \in \mathbb{R}^d : \mathbf{A}\boldsymbol{x} \leqslant \mathbf{1}\}$, where $\mathbf{A}$ is an integral matrix and $\mathbf{1}$ is a vector of all 1s.

Since h^*-polynomials are invariant under integral translation, we allow translates of reflexive polytopes. Now we need not require that $\mathbf{0} \in P^\circ$, and the following statements are all equivalent to being an integral translate of a reflexive polytope:

1. $|P^\circ \cap \mathbb{Z}^d| = 1$ and $(t+1)P^\circ \cap \mathbb{Z}^d = tP \cap \mathbb{Z}^d$ for all $t \in \mathbb{Z}_{>0}$;
2. P contains a unique interior lattice point that is lattice distance 1 away from each facet of P;
3. $z^d\, h_P^*(\frac{1}{z}) = h_P^*(z)$.

If P is a lattice polytope and there exists an integer $g \geqslant 1$ such that gP is reflexive, we say that P is *g-Gorenstein.*

By retracing the steps in the proof of Theorem 3.1.1 but observing that there is no $\beta > 0$ case if P is reflexive, we can obtain a relationship between $h_P^*(z)$ and $h_{\partial P}^*(z)$. Similarly, we can do this for Gorenstein polytopes. Both relationships quickly imply that $h_P^*(z)$ is palindromic since $h_{\partial P}^*(z)$ is. In this section we will see these results as corollaries of more general theorems for *rational* polytopes.

Fiset and Kasprzyk generalized the notion of being reflexive to rational polytopes in [57]. They define a polytope $P \subseteq \mathbb{R}^d$ to be *rational reflexive* if it is the convex hull of finitely many rational points in $\mathbb{Q}^d$, contains the origin in its interior, and has a lattice dual polytope. Equivalently, a rational polytope P is *rational reflexive* if it has a hyperplane description $P = \{\boldsymbol{x} \in \mathbb{R}^d : \mathbf{A}\boldsymbol{x} \leqslant \mathbf{1}\}$, where $\mathbf{A}$ is an integral matrix.

Fiset and Kasprzyk prove the following:

Theorem 3.5.1. *If P is a rational reflexive polytope, then $h_P^*(z)$ is palindromic.*

In their proof, they define the "h^*-polynomials" corresponding to each of the components of the Ehrhart quasipolynomial of the rational polytope, show (using reciprocity) that the ith such polynomial is equal to the $(q-1-i)$th polynomial with coefficients read backwards, and finally show that the h^*-polynomial of the rational polytope can be written in terms of these smaller polynomials. This is a generalized version of Hibi's proof [70] that lattice polytopes with lattice duals have symmetric h^*-polynomials. Our proof in the previous section shows the lattice version in an alternative way, and in this section, we will show the rational version. By (3.2), Fiset–Kasprzyk's theorem is an immediate corollary of the following:

Theorem 3.5.2. *If P is a rational reflexive polytope with denominator q, then*

$$h_P^*(z) = \left(1 + z + \cdots + z^{q-1}\right) h_{\partial P}^*(z) .$$

Proof. Fix a disjoint half-open triangulation T of ∂P into $(d-1)$-dimensional simplices using the vertices of P. Fix $\Delta \in T$, say $\Delta = \operatorname{conv}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d)$ with $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in \frac{1}{q}\mathbb{Z}^d$, where the facets of Δ opposite $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ are missing and the facets opposite $\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_d$ are included. We now follow the steps in our proof of Theorem 3.1.1 using the fundamental parallelepiped

$$\mathcal{F}_\Delta := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \sum_{j=r+1}^{d} [0,1) \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix}$$

yielding

$$\mathrm{Ehr}_\Delta(z) = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}_\Delta \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^q)^d}$$

and

$$\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)} := \sum_{j=1}^{r} (0,1] \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \sum_{j=r+1}^{d} [0,1) \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + [0,1) \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix}$$

giving

$$\mathrm{Ehr}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)}(z) = \frac{\sum_{\boldsymbol{m} \in \mathcal{F}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)} \cap \mathbb{Z}^{d+1}} z^{m_{d+1}}}{(1-z^q)^d (1-z)} \, .$$

Fix a point $\boldsymbol{p} \in \mathcal{F}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)}$, say

$$\boldsymbol{p} = \sum_{j=1}^{d} \alpha_j \begin{pmatrix} q\boldsymbol{v}_j \\ q \end{pmatrix} + \beta \begin{pmatrix} \boldsymbol{0} \\ 1 \end{pmatrix},$$

with $0 < \alpha_1, \ldots, \alpha_r \leqslant 1$ and $0 \leqslant \alpha_{r+1}, \ldots, \alpha_d, \beta < 1$. The simplex Δ is contained in some facet-defining hyperplane of P, say

$$H = \left\{ \boldsymbol{x} \in \mathbb{R}^d : a_1 x_1 + \cdots + a_d x_d = 1 \right\},$$

where $a_1, \ldots, a_d \in \mathbb{Z}$. Consider the value of $a_1 p_1 + \cdots + a_d p_d$ (where $p_1, \ldots, p_{d+1}$ are the coordinates of $\boldsymbol{p}$). If $v_{i,j}$ denotes the jth coordinate of $\boldsymbol{v}_i$, then

$$\begin{aligned} a_1 p_1 + \cdots + a_d p_d &= a_1(\alpha_1 q v_{1,1} + \cdots + \alpha_d q v_{d,1}) + \cdots + a_d(\alpha_1 q v_{d,1} + \cdots + \alpha_d q v_{d,d}) \\ &= q\alpha_1(a_1 v_{1,1} + \cdots + a_d v_{1,d}) + \cdots + q\alpha_d(a_1 v_{d,1} + \cdots + v_{d,d}) \, . \end{aligned}$$

Since each of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_d \in H$, this is equal to

$$q\alpha_1 + \cdots + q\alpha_d = p_{d+1} - \beta \, .$$

If $\boldsymbol{p}$ is a *lattice* point in $\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)}$, then both $a_1 p_1 + \cdots + a_d p_d$ and p_{d+1} are integers, so β must also be an integer. This forces $\beta = 0$ and thus

$$\mathcal{F}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)} \cap \mathbb{Z}^{d+1} = \mathcal{F}_\Delta \cap \mathbb{Z}^{d+1}. \tag{3.9}$$

Therefore,

$$\mathrm{Ehr}_{\mathrm{Pyr}(\boldsymbol{0},\Delta)}(z) = \frac{h^*_\Delta(z)}{(1-z^q)^d (1-z)} \, .$$

Summing over all Δ, we obtain

$$\mathrm{Ehr}_P(z) = \frac{h^*_{\partial P}(z)}{(1-z^q)^d(1-z)}$$

and so $h^*_P(z) = (1 + z + \cdots + z^{q-1})h^*_{\partial P}(z)$. □

Observe that a (lattice) reflexive polytope is a rational reflexive polytope that happens to be a lattice polytope, so letting $q = 1$ yields the following result:

Corollary 3.5.3. *If P is a (lattice) reflexive polytope, then $h^*_P(z) = h^*_{\partial P}(z)$.*

Remark 3. We note that for rational reflexive polytopes (as opposed to lattice reflexive polytopes) it is not necessarily the case that $P = \{\boldsymbol{x} \in \mathbb{R}^d : \mathbf{A}\boldsymbol{x} \leqslant \mathbf{1}\}$ implies that the origin is lattice distance 1 away from each facet-defining hyperplane of P. It is possible for a given facet-defining hyperplane to not contain any lattice points and thus be lattice distance less than 1 from the origin. However, the equality (3.9) between the sets of lattice points in the two fundamental parallelepipeds still holds.

We extend the definition of a rational reflexive to a rational Gorenstein polytope in a natural way. Let P be a rational polytope and let $g \geqslant 1$ be an integer. We say that P is *rational g-Gorenstein* if gP is an integral translate of a (lattice) reflexive polytope. We allow gP to be a translate of a reflexive polytope so that we do not force P to have an interior point. Also note that, if we have a rational g-Gorenstein polytope with denominator q, we must have $q|g$.

Theorem 3.5.4. *If P is a rational g-Gorenstein polytope with denominator q, then*

$$h^*_P(z) = \frac{1 + z + \cdots + z^{q-1}}{1 + z + \cdots + z^{g-1}}\, h^*_{\partial P}(z)\,.$$

Sketch of Proof. The unique interior lattice point of gP is lattice distance 1 away from the facet-defining hyperplanes of gP, so there is some polynomial $f(z)$ such that

$$\mathrm{Ehr}_{\partial P}(z) = \frac{f(z)}{(1-z^g)^d} \qquad \text{and} \qquad \mathrm{Ehr}_P(z) = \frac{f(z)}{(1-z^g)^{d+1}}\,.$$

On the other hand, $h^*_{\partial P}(z)$ and $h^*_P(z)$ are such that

$$\mathrm{Ehr}_{\partial P}(z) = \frac{h^*_{\partial P}(z)}{(1-z^q)^d} \qquad \text{and} \qquad \mathrm{Ehr}_P(z) = \frac{h^*_P(z)}{(1-z^q)^{d+1}}\,.$$

Therefore,

$$\begin{aligned} h^*_{\partial P}(z) &= \frac{(1-z^q)^d}{(1-z^g)^d}\, f(z) = \frac{(1-z^q)^d}{(1-z^g)^d}\,\frac{(1-z^g)^{d+1}}{(1-z^q)^{d+1}}\, h^*_P(z) \\ &= \frac{1-z^g}{1-z^q}\, h^*_P(z) = \frac{1 + z + \cdots + z^{g-1}}{1 + z + \cdots + z^{q-1}}\, h^*_P(z)\,. \end{aligned}$$

□

Observe that a (lattice) g-Gorenstein polytope is a rational g-Gorenstein polytope that happens to be lattice, so letting $q = 1$ yields the following generalization of Corollary 3.5.3.

Corollary 3.5.5. *If P is a (lattice) g-Gorenstein polytope, then*

$$h^*_{\partial P}(z) = (1 + z + \ldots + z^{g-1})h^*_P(z)\,.$$

Due to the palindromicity of $h^*_{\partial P}(z)$, these relations immediately imply that reflexive, Gorenstein, rational reflexive, and rational Gorenstein polytopes all have palindromic h^*-polynomials.

3.6 A generalization to rational Ehrhart theory

In the last decade, several works have been devoted to an Ehrhart theory of rational polytopes where we now allow *rational* (or, equivalently, real) dilation factors. The fundamental structure of the Ehrhart counting function in the rational parameter $\lambda > 0$ is that

$$\mathrm{ehr}_{\mathbb{Q}}(P;\lambda) := \left|\lambda P \cap \mathbb{Z}^d\right|$$

is a quasipolynomial in λ; this was first shown by Linke [83], who proved several other results about $\mathrm{ehr}_{\mathbb{Q}}(P;\lambda)$, including an analogue of Ehrhart–Macdonald reciprocity; see also [13, 99, 122, 124]. Our goal in this section is to prove an analogue of Theorem 3.1.1 in this setting.

To this end, we first recall *rational Ehrhart series*, which were introduced only recently [20]. Suppose the full-dimensional rational polytope $P \subset \mathbb{R}^d$ is given by the irredundant halfspace description

$$P \ = \ \left\{\boldsymbol{x} \in \mathbb{R}^d : \, \mathbf{A}\,\boldsymbol{x} \leqslant \boldsymbol{b}\right\},$$

where $\mathbf{A} \in \mathbb{Z}^{n\times d}$ and $\boldsymbol{b} \in \mathbb{Z}^n$ such that the greatest common divisor of b_j and the entries in the jth row of $\mathbf{A}$ equals 1, for every $j \in \{1,\ldots,n\}$. We define the *codenominator* r of P to be the least common multiple of the nonzero entries of $\boldsymbol{b}$. It turns out that $\mathrm{ehr}_{\mathbb{Q}}(P;\lambda)$ is fully determined by evaluations at rational numbers λ with denominator $2r$; if $\mathbf{0} \in P$ then we actually need to know only evaluations at rational numbers λ with denominator r [20, Corollary 5]. This motivates a study of the two generating functions

$$\mathrm{Ehr}_{\mathbb{Q}}\left(P;z\right) := 1 + \sum_{n\geqslant 1} \mathrm{ehr}_{\mathbb{Q}}(P;\tfrac{n}{r})\, z^{\frac{n}{r}}$$

and

$$\mathrm{Ehr}^{\mathsf{ref}}_{\mathbb{Q}}\left(P;z\right) := 1 + \sum_{n\geqslant 1} \mathrm{ehr}_{\mathbb{Q}}(P;\tfrac{n}{2r})\, z^{\frac{n}{2r}}\,,$$

which have the following rational form, as shown in [20, Theorem 12].

Theorem 3.6.1. *Let P be a rational d-polytope with codenominator r.*

(a) *Let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{r}P$ is a lattice polytope. Then*

$$\mathrm{Ehr}_{\mathbb{Q}}(P; z) = \frac{h^*_{\mathbb{Q}}(P; z; m)}{(1 - z^{\frac{m}{r}})^{d+1}}$$

where $h^*_{\mathbb{Q}}(P; z; m)$ is a polynomial in $\mathbb{Z}[z^{\frac{1}{r}}]$ of degree $< m(d+1)$ with nonnegative integral coefficients.

(b) *Let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{2r}P$ is a lattice polytope. Then*

$$\mathrm{Ehr}^{\mathit{ref}}_{\mathbb{Q}}(P; z) = \frac{h^{*\mathit{ref}}_{\mathbb{Q}}(P; z; m)}{(1 - z^{\frac{m}{2r}})^{d+1}}$$

where $h^{*\mathit{ref}}_{\mathbb{Q}}(P; z; m)$ is a polynomial in $\mathbb{Z}[z^{\frac{1}{2r}}]$ of degree $< m(d+1)$ with nonnegative integral coefficients.

Possibly more important than this theorem are the consequences one can derive from it, and [20] (re-)proved several previously-known and novel results in rational Ehrhart theory. The latter include the facts that $h^*_{\mathbb{Q}}(P; z; m)$ is palindromic if $\mathbf{0} \in P^\circ$ and that, if $r|m$, extracting the terms with integer exponents from $h^*_{\mathbb{Q}}(P; z; m)$ returns $h^*_P(z)$, which results in yet another proof of the Betke–McMullen version of Theorem 3.1.1 (the $\ell = 1$ case).

Theorem 3.6.1 is based on the observation that

$$h^*_{\mathbb{Q}}(P; z; m) = h^*_{\frac{1}{r}P}(z^{\frac{1}{r}}) \qquad \text{and} \qquad h^{*\mathrm{ref}}_{\mathbb{Q}}(P; z; m) = h^*_{\frac{1}{2r}P}(z^{\frac{1}{2r}});$$

note that m is implicitly included in the h^*-polynomial, as we really mean the numerator of the Ehrhart series of $\frac{1}{r}P$ (respectively, $\frac{1}{2r}P$) when the denominator is $(1 - z^m)^{d+1}$. The same observation yields the following variant of Theorem 3.1.1 for rational Ehrhart theory.

Corollary 3.6.2. *Let P be a rational polytope with codenominator r.*

(1) *If $\mathbf{0} \in P^\circ$, then $h^*_{\mathbb{Q}}(P; z; m)$ is palindromic.*

(2) *If $\mathbf{0} \in \partial P$, let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{r}P$ is a lattice polytope and let $\ell \geqslant 1$ be the smallest dilate of $\frac{1}{r}P$ that contains an interior point. Then*

$$\frac{1 - z^{\frac{\ell}{r}}}{1 - z^{\frac{m}{r}}}\, h^*_{\mathbb{Q}}(P; z; m) = a(z) + z^{\frac{\ell}{r}}\, b(z)$$

where $a(z) = h^*_{\mathbb{Q}}(\partial P; z; m)$ and $b(z)$ are palindromic polynomials in $\mathbb{Z}[z^{\frac{1}{r}}]$ with nonnegative integer coefficients.

(3) *If $\mathbf{0} \notin P$, let $m \in \mathbb{Z}_{>0}$ be such that $\frac{m}{2r}P$ is a lattice polytope and let $\ell \geqslant 1$ be the smallest dilate of $\frac{1}{2r}P$ that contains an interior point. Then*

$$\frac{1 - z^{\frac{\ell}{2r}}}{1 - z^{\frac{m}{2r}}} h_{\mathbb{Q}}^{*\mathsf{ref}}(P; z; m) = a(z) + z^{\frac{\ell}{2r}} b(z)$$

where $a(z) = h_{\mathbb{Q}}^{*\mathsf{ref}}(\partial P; z; m)$ and $b(z)$ are palindromic polynomials in $\mathbb{Z}[z^{\frac{1}{2r}}]$ with nonnegative integer coefficients.

Example 3.6.3. Let $P = \mathrm{conv}\{(0,0),(0,2),(5,2)\}$, alternatively the intersection of the halfspaces

$$\{x_1 \geqslant 0\} \cap \{x_2 \leqslant 2\} \cap \{5x_2 - 2x_1 \geqslant 0\}.$$

From this, we can see that P has codenominator $r = 2$. Since $\mathbf{0} \in \partial P$, we look at the rational dilate

$$\tfrac{1}{2}P = \mathrm{conv}\{(0,0),(0,1),(\tfrac{5}{2},1)\}.$$

The first dilate of $\frac{1}{2}P$ containing an interior lattice point is the $\ell = 2$nd dilate, which contains the points $(1,1)$ and $(2,1)$. We choose $m = r = 2$ minimal, and we compute the associated h^*-polynomial of $\frac{1}{2}P$:

$$h^*_{\frac{1}{2}P}(z) = (1-z^2)^3 \ \mathrm{Ehr}_{\frac{1}{2}P}(z) = 1 + 4z + 7z^2 + 6z^3 + 2z^4$$

and so

$$\begin{aligned} h^*_{\mathbb{Q}}(P; z; 2) &= 1 + 4z^{\frac{1}{2}} + 7z + 6z^{\frac{3}{2}} + 2z^2 \\ &= (1 + 4z^{\frac{1}{2}} + 6z + 4z^{\frac{3}{2}} + z^2) + z(1 + 2z^{\frac{1}{2}} + z)\,. \end{aligned}$$

We can check that the first polynomial in the decomposition is equal to

$$\begin{aligned} h^*_{\mathbb{Q}}(P; z; 2) = h^*(\partial \tfrac{1}{2}P; z^{\frac{1}{2}}) &= \frac{h^*(\frac{1}{2}P; z^{\frac{1}{2}}) - h^*(\frac{1}{2}P^\circ; z^{\frac{1}{2}})}{1 - (z^{\frac{1}{2}})^2} \\ &= 1 + 4z^{\frac{1}{2}} + 6z + 4z^{\frac{3}{2}} + z^2. \end{aligned}$$

Moreover, the power in front of the second polynomial is $z = z^{\frac{2}{2}} = z^{\frac{\ell}{r}}$ and both polynomials are palindromic.[3]

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[3]We thank Sophie Rehberg for suggesting this example and helping with computing it.

Chapter 4

Weighted h^*-polynomials

In this chapter, we explore the idea of adding *weights* to our lattice, and we generalize Ehrhart polynomials, Ehrhart series, and (mostly) h^*-polynomials to weighted versions. Of particular interest to us is classifying which weight functions preserve Stanley's classical nonnegativity and monotonicity results.

4.1 Introduction

Let $P \subseteq \mathbb{R}^d$ be a rational convex polytope, that is, a polytope with vertices in $\mathbb{Q}^d$, and let $w : \mathbb{R}^d \to \mathbb{R}$ be a polynomial function, often called a *weight function.* A computational problem arising throughout the mathematical sciences is to compute, or at least estimate, the sum of the values $w(x) := w(x_1, \ldots, x_d)$ over the set of integer points belonging to P, namely

$$\operatorname{ehr}(P, w) = \sum_{x \in P \cap \mathbb{Z}^d} w(x).$$

Weighted sums of the above type are also a classical topic in convex discrete geometry where they have been studied for a long time under the name *polynomial valuations* [2, 33, 91, 97]. They appear in the work of Brion and Vergne [44], who used weighting in the context of Euler-Maclaurin formulas. Other ideas of what it means to be weighted have been proposed later on, for instance, by Chapoton [47], who developed a related q-theory for the case when $w(x)$ is a linear form, also by Stapledon [125], who explored a grading with piece-wise linear functions, and by Ludwig and Silverstein [88], who introduced and studied Ehrhart tensor polynomials based on discrete moment tensors.

Important applications of such weighted problems appear, for instance, in enumerative combinatorics [6, 53], statistics [49, 55], non-linear optimization [52], and weighted lattice point sums, which have played a key role in the computation of volumes and integrals over polytopes [12].

The sum of the weighted integer points in the nth dilate of the rational polytope P for nonnegative integers $n \in \mathbb{N}$ is given by the *weighted Ehrhart function* $\operatorname{ehr}(nP, w)$. The main object of this chapter is the generating function

$$\mathrm{Ehr}(P, w; z) = \sum_{n \geqslant 0} \mathrm{ehr}(nP, w) z^n$$

called the *weighted Ehrhart series*. The fact that the weighted Ehrhart series is a rational function has been known for a long time, e.g., it has been used in computational software for at least ten years ([14, 45]). For our purposes we see in Proposition 4.2.4 why $\mathrm{Ehr}(P, w; z)$ is a rational function of the form

$$\mathrm{Ehr}(P, w; z) = \frac{h^*_{P,w}(z)}{(1 - z^q)^{r+m+1}}$$

whenever P is an r-dimensional rational polytope; here $m = \deg(w)$, $h^*_{P,w}(z)$ is a polynomial of degree at most $q(r + m + 1) - 1$, and q denotes the smallest positive integer such that qP has vertices in $\mathbb{Z}^d$, called the *denominator* of P.

We say that the empty polytope has denominator 1. We call $h^*_{P,w}(z)$ the *weighted h^*-polynomial of P* and its list of coefficients the *weighted h^*-vector of P* with respect to the weight w.

From the rationality of $\mathrm{Ehr}(P, w; z)$, it follows that the weighted Ehrhart function $\mathrm{ehr}(nP, w)$ is a quasipolynomial in n, that is, it has the form

$$\mathrm{ehr}(nP, w) = \sum_{i=0}^{d+m} E_i(n) n^i$$

where the coefficients $E_i : \mathbb{N} \to \mathbb{R}$ are periodic functions with periods dividing the denominator of P. The leading coefficient of the h^*-polynomial is equal to the integral of the weight w over the polytope P; these integrals were studied in [16], [18] and [12]. If $w = 1$, that is, if $\mathrm{ehr}(nP, w) = |nP \cap \mathbb{Z}^d|$, then we recover the classical Ehrhart theory counting lattice points in dilates of polytopes. Even in this case, it is an NP-hard problem to compute all of the coefficients E_i. See [17, 24] for excellent introductions to this topic.

In the classical case of $w = 1$, a fundamental theorem by Richard P. Stanley, often called *Stanley's nonnegativity theorem*, states that the h^*-polynomial of any rational polytope has only nonnegative integer coefficients [112]. Even stronger, for rational polytopes P and Q such that $P \subseteq Q$, Stanley proved $h^*_{P,1}(z) \preceq h^*_{Q,1}(z)$ where $\preceq$ denotes coefficient-wise comparison. This last property has been known as *h^* monotonicity property.* For details and proofs see e.g., [26, 107, 112].

Positivity and nonnegativity of coefficients is important in algebraic combinatorics (see e.g., [118] and its references), but we must stress that one nice aspect of our results is they connect to the nonnegativity of the associated h^*-polynomial as real-valued functions. This is a topic that goes back to the work on real algebraic geometry by Hilbert, Pólya, Artin and others (see [90, 96]), and it has seen renewed activity in the classical methods of moments, real algebraic geometry, and sums of squares decompositions for polynomials because it provides a natural approach for optimization algorithms (see [30, 90]).

Motivated by this prior work and context, this chapter discusses the nonnegativity and monotonicity properties of the coefficients of weighted h^*-polynomials, as well as their nonnegativity as real-valued functions.

Our Contributions

In contrast to its classical counterpart, the weighted h^*-polynomial may have negative coefficients, even when the weight function is nonnegative over the polytope and all of its nonnegative dilates. For example, when P is the line segment $[0,1] \subseteq \mathbb{R}$, one can calculate that

$$\mathrm{Ehr}(P, 1; z) = \frac{1}{(1-z)^2} \quad \text{and} \quad \mathrm{Ehr}(P, x^2; z) = \frac{z^2+z}{(1-z)^4},$$

and so their sum is

$$\mathrm{Ehr}(P, x^2+1; z) = \frac{2z^2-z+1}{(1-z)^4}.$$

As can be seen in this simple example, adding Ehrhart series corresponding to weights of different degrees may introduce negative coefficients to the h^*-polynomial since the rational functions have different denominators. We therefore focus on homogeneous polynomials as weight functions. For an investigation of how to deal with more general weights see [53].

We now consider the following, slightly more general setup, where the weight function w may depend not only on the coordinates of the points $nP \cap \mathbb{Z}^d$ but also on the scaling factor n. For any rational polytope $P \subseteq \mathbb{R}^d$, the *cone over* P (or, as defined in Chapter 2, the *homogenization* of P) is the rational polyhedral cone in $\mathbb{R}^{d+1}$

$$\mathrm{C}(P) := \mathrm{cone}(P \times \{1\}) = \{c(p,1) \mid c \geqslant 0, p \in P\}.$$

For any polynomial w in $d+1$ variables we consider the weighted Ehrhart series

$$\mathrm{Ehr}(P, w; z) = \sum_{x \in \mathrm{C}(P) \cap \mathbb{Z}^{d+1}} w(x) z^{x_{d+1}}.$$

Let $\mathrm{C}(P)^*$ be the cone consisting of the linear functions on $\mathbb{R}^{d+1}$ that are nonnegative on $\mathrm{C}(P)$. If the cone $\mathrm{C}(P)$ is defined by linear inequalities $\ell_1 \geqslant 0, \ldots, \ell_m \geqslant 0$, then $\mathrm{C}(P)^*$ is a polyhedral cone generated by nonnegative linear combinations of $\ell_1, \ldots, \ell_m$. We focus on the following two families of polynomials in $d+1$ variables as weights functions:

(i) the semiring R_P consisting of sums of products of linear forms in $\mathrm{C}(P)^*$. Each element of R_P has the form $c_1 \ell^{a_1} + \cdots + c_k \ell^{a_k}$ where $c_1, \ldots, c_k$ are positive real numbers and $\ell^{a_1}, \ldots, \ell^{a_k}$ are monomials in the generators $\ell_1, \ldots, \ell_m$ of $\mathrm{C}(P)^*$; and

(ii) the semiring S_P consisting of sums of nonnegative products of linear forms on P. If a product of linear forms is nonnegative on P, then each of the linear forms involved is either nonnegative on all of P or appears with an even power; otherwise the product would change sign across the hyperplane where the linear form vanishes. Therefore, an element of S_P has the form $s_1\ell^{a_1} + \cdots + s_k\ell^{a_k}$ where $s_1, \ldots, s_k$ are squares of products of any linear forms and $\ell^{a_1}, \ldots, \ell^{a_k}$ are monomials in the generators $\ell_1, \ldots, \ell_m$ of $\mathrm{C}(P)^*$.

In R_P each of the linear forms involved are nonnegative on P. In contrast, in S_P, each product is nonnegative but the individual linear forms may have negative values in P. Thus we have $R_P \subseteq S_P$. Both semirings are contained in the *preordering* generated by $\ell_1, \ldots, \ell_m$ consisting of elements of the form $s_1\ell^{a_1} + \cdots + s_k\ell^{a_k}$ where s_i are arbitrary squares of polynomials instead of just squares of products of linear forms. See, for example, [90].

The main results of this chapter are the following.

Nonnegativity Theorem. (Theorem 4.2.5). *Let P be a rational polytope, $\mathrm{C}(P)$ its cone, and $\mathrm{C}(P)^*$ the dual cone of linear functions on $\mathbb{R}^{d+1}$ which are nonnegative on $\mathrm{C}(P)$. Let R_P and S_P be, respectively, the semirings of sums of products of linear forms in $\mathrm{C}(P)^*$ and of sums of nonnegative products of linear forms on P.*

1. *If the weight w is a homogeneous element of R_P, then the coefficients of $h^*_{P,w}(z)$ are nonnegative.*

2. *If the weight w is a homogeneous element of S_P, then $h^*_{P,w}(z) \geqslant 0$ for $t \geqslant 0$.*

As we mentioned before Stanley also showed that the classical h^*-polynomials satisfy a monotonicity property: for lattice polytopes P and Q, of possibly different dimension, such that $P \subseteq Q$, we have $h^*_P(z) \preceq h^*_Q(z)$ where $\preceq$ denotes the coefficient-wise inequalities [107]. This can be seen as a generalization of the nonnegativity theorem when we set $P = \varnothing$ in which case the Ehrhart series and thus the h^*-polynomial is zero. Now we are able to prove the following:

First Monotonicity Theorem. (Theorem 4.2.7). *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in R_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim P+m+1} h^*_{P,w}(z) \preceq (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim Q+m+1} h^*_{Q,w}(z)\,.$$

In particular, if $P \subseteq Q$ are polytopes with the same denominator, then taking $g = \delta(P) = \delta(Q)$ gives

$$h^*_{P,w}(z) \preceq h^*_{Q,w}(z). \tag{4.1}$$

Second Monotonicity Theorem. (Theorem 4.2.8). *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes of the same dimension $D = \dim P = \dim Q$, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in S_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{D+m+1} h^*_{P,w}(z) \leqslant (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{D+m+1} h^*_{Q,w}(z)$$

for all $t \geqslant 0$. *In particular, if* $P \subseteq Q$ *are polytopes with the same denominator and dimension, then taking* $g = \delta(P) = \delta(Q)$ *gives*

$$h^*_{P,w}(z) \leqslant h^*_{Q,w}(z) \text{ for all } t \geqslant 0. \tag{4.2}$$

We wish to emphasize that while Theorem 4.2.5 is a generalization of Stanley's nonnegativity theorem, Theorem 4.2.7 is closer in spirit to *Pólya's theorem on positive polynomials* which says that if a homogeneous polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ is strictly positive on the standard simplex

$$\Delta_n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1, \ldots, x_n \geqslant 0, x_1 + \cdots + x_n = 1\},$$

then for sufficiently large N, all of the nonzero coefficients of $(X_1 + \cdots + X_n)^N f(X_1, \ldots, X_n)$ are strictly positive. Note also, the semiring R_P is a homogenized version of the semiring appearing in *Handelman's theorem* [65] which says that all polynomials strictly positive on a polytope P lie in the semiring generated by the linear forms which are nonnegative on the polytope. We remark that all homogeneous polynomials are sums of (unrestricted) products of linear forms and it is an important problem to find such decompositions (see [3, 92, 98] and references therein). Thus our restriction to R_P and S_P is a natural approach to understanding nonnegativity and bringing us close to the best possible result.

To study the limitations of our results we focus on the case when the weight is given by a single arbitrary linear form. In this case we strengthen our results for two dimensional lattice polygons.

Theorem 4.3.3. *For every (closed) convex lattice polygon* P *and every linear form* ℓ*, the* h^**-polynomial of* P *with respect to* $w(x) = \ell^2(x)$ *has only nonnegative coefficients.*

In particular, this shows that the weighted h^*-polynomial of any convex lattice polygon has nonnegative coefficients, even when the linear form takes negative values on the polygon. Furthermore, we provide examples that show that this result is no longer true if the assumptions on the polytope or weight are relaxed. In particular, we construct a 20-dimensional lattice simplex and a linear form such that the h^*-polynomial with respect to the square of the linear form has a negative coefficient (Example 4.3.7). These results have interpretations and implications in terms of generating functions of Ehrhart tensor polynomials. In particular, the example mentioned above gives a counterexample to a conjecture of Berg, Jochemko and Silverstein [28, Conjecture 6.1] on the positive semi-definiteness of h^2-tensor polynomials of lattice polytopes (Corollary 4.4.4).

Unlike the classical results of Stanley for $w = 1$, where techniques from commutative algebra can be applied since the Ehrhart series is actually the Hilbert series of a graded algebra, we do not see an obvious connection to commutative algebra methods. Instead, to prove Theorems 4.2.5 and 4.2.7 we consider the cone homogenization of polytopes and half-open decompositions and follow a variation of the triangulation ideas first outlined by Stanley in [112]. While this methodology has been used by many authors since then [8, 26, 79], we require a careful analysis of the properties of the semirings R_P and S_P. For this we consider multivariate generating functions for half-open cones and provide explicit combinatorial interpretations using generalized q-Eulerian polynomials [126]. The q-Eulerian polynomials and their relatives frequently appear in enumerative and geometric combinatorics [22, 34, 43, 63].

This chapter is organized as follows. In Section 4.2 we give an explicit formula for the weighted multivariate generating function for half-open simplicial cones (Lemma 4.2.2). This formula then allows us to show the rationality of the (univariate) weighted Ehrhart series (Proposition 4.2.4) as well as the first part of Theorem 4.2.5 by specialization and using half-open decompositions. The second part of Theorem 4.2.5 is obtained by considering subdivisions of the polytope induced by the linear forms involved in the weight function. A more refined analysis then also allows us to prove the monotonicity Theorems 4.2.7 and 4.2.8. In Section 4.3 we focus on the case when the weight function is given by a square of a single linear form and prove Theorem 4.3.3. We also show that the assumptions on convexity, denominator, dimension and degree are necessary by providing examples. In Section 4.4 we describe the connections and implications of our results to Ehrhart tensor polynomials. In particular, we show that weighted Ehrhart polynomials can be seen as certain evaluations of Ehrhart tensor polynomials (Proposition 4.4.1), and thus, positive semi-definiteness of h^2-tensor polynomials is equivalent to nonnegativity of weighted h^*-polynomials with respect to squares of linear forms (Proposition 4.4.2). In particular, Example 4.3.7 disproves [28, Conjecture 6.1] (Corollary 4.4.4).

4.2 Nonnegativity and monotonicity of weighted h^*-polynomials

Generating series

Let $P \subseteq \mathbb{R}^d$ be a rational polytope of dimension r with denominator q and let $w : \mathbb{R}^d \to \mathbb{R}$ be a polynomial of degree m. In this section we will see that $\mathrm{Ehr}(P, w; z)$ is a rational function of the form

$$\mathrm{Ehr}(P, w; z) = \frac{h^*_{P,w}(z)}{(1 - z^q)^{r+m+1}} ,$$

where $h^*_{P,w}(z)$ is a polynomial of degree at most $q(r + m + 1) - 1$. Our main goal is to study positivity properties of the numerator polynomial. Our approach uses general multivariate generating series of half-open simplicial cones and specializing to obtain the univariate generating function of the homogenization $\mathrm{C}(P)$ following ideas outlined in [112] but requiring careful analysis of the semirings R_P and S_P.

For a polynomial $w(x)$ in d variables, the multivariate weighted lattice point generating function of the cone C is $\sum_{x \in C \cap \mathbb{Z}^d} w(x) z^x$ where $z^x = z_1^{x_1} \cdots z_d^{x_d}$ is a monomial in d variables. We will now show that this generating function is a rational function and give an explicit formula when the weight is a product of linear forms.

Our expression uses the following parametrized generalization of Eulerian polynomials. For $\lambda \in [0,1]$, let $A_d^\lambda(z)$ be the polynomial defined by

$$\sum_{n \geqslant 0} (n+\lambda)^d z^n = \frac{A_d^\lambda(z)}{(1-z)^{d+1}}.$$

If $\lambda = 1$, then this is the usual Ehrhart series of a d-dimensional unit cube, and $A_d^1(z)$ is the Eulerian polynomial, all of whose roots are real and nonpositive. If $\lambda = \frac{1}{r}$ for some integer $r \geqslant 1$ then $r^d A_d^\lambda(z)$ equals the r-colored Eulerian polynomial [126]. For each $\lambda \in [0,1]$ the polynomial $A_d^\lambda(z)$ also has only real, nonpositive roots [42, Theorem 4.4.4]. In particular, all of its coefficients are nonnegative. We formally record this in a lemma.

Lemma 4.2.1 ([42, Theorem 4.4.4])**.** *For any integer $d \geqslant 1$ and real number $\lambda \in [0,1]$, the coefficients of $A_d^\lambda(z)$ are nonnegative.*

Our computations additionally use some concepts which we now introduce. For consistency, we assume that the polytopes live in the d-dimensional space $\mathbb{R}^d$ while cones live in the ambient space $\mathbb{R}^{d+1}$.

Let C be a *half-open* $(r+1)$-dimensional simplicial cone in $\mathbb{R}^{d+1}$ generated by nonzero integer vectors $v_1, \ldots, v_{r+1} \in \mathbb{Z}^{d+1}$ with the first k facets removed where $0 \leqslant k \leqslant r+1$. More precisely,

$$C = \{c_1 v_1 + \cdots + c_{r+1} v_{r+1} \mid c_1, \ldots, c_k > 0,\ c_{k+1}, \ldots, c_{r+1} \geqslant 0\}.$$

Since C is simplicial, every point $\alpha \in C$ can be written uniquely as

$$\alpha = x + s_1 v_1 + \cdots + s_{r+1} v_{r+1}$$

where $s_1, \ldots, s_{r+1}$ are nonnegative integers, and x is in the *half-open parallelepiped*

$$\Pi = \{\lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} \mid 0 < \lambda_1, \ldots, \lambda_k \leqslant 1,\ 0 \leqslant \lambda_{k+1}, \ldots, \lambda_{r+1} < 1\}.$$

We obtain the following explicit formula for the multivariate generating function of a half-open simplicial cone if the weight is a product of linear forms. Since every polynomial is a sum of product of linear forms, namely monomials, this gives a formula to compute the generating function for any polynomial weight.

Proposition 4.2.2. *Let C be an $(r+1)$-dimensional half-open simplicial cone in $\mathbb{R}^{d+1}$ with generators $v_1, \ldots, v_{r+1}$ in R_P. Let $w = \ell_1 \cdots \ell_m$ be a product of linear forms in $d+1$ variables. Then*

$$\sum_{\alpha \in C \cap \mathbb{Z}^{d+1}} w(\alpha) z^\alpha = \sum_{x \in \Pi \cap \mathbb{Z}^{d+1}} \left(z^x \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \prod_{i \in I_1} \ell_i(v_1) \cdots \prod_{i \in I_{r+1}} \ell_i(v_{r+1}) \prod_{j=1}^{r+1} \frac{A_{|I_j|}^{\lambda_j(x)}(z^{v_j})}{(1-z^{v_j})^{|I_j|+1}} \right) \tag{4.3}$$

where Π is the half-open parallelepiped as above and each $x \in \Pi$ is written $x = \lambda_1(x) v_1 + \cdots + \lambda_{r+1}(x) v_{r+1}$. The innermost sum runs over all the ordered partitions of $[m]$ into $r+1$ (possibly empty) parts and $I_1 \uplus \cdots \uplus I_{r+1}$ denotes the disjoint union of these parts.

Note that when $m = 0$ and the weight is constant, there is only the partition into empty sets where by definition the products are all 1 (empty products) and the Eulerian polynomials are all 1.

Proof. Using that any $\alpha \in C$ is $\alpha = x + s_1 v_1 + \cdots s_{r+1} v_{r+1}$ for $x \in \Pi$, the generating function is

$$\begin{aligned}\sum_{\alpha \in C \cap \mathbb{Z}^{d+1}} w(\alpha) z^\alpha &= \sum_{\alpha \in C \cap \mathbb{Z}^{d+1}} \left(\prod_{i=1}^m \ell_i(\alpha) \right) z^\alpha \\ &= \sum_{x \in \Pi \cap \mathbb{Z}^{d+1}} \sum_{s_1 \geqslant 0} \cdots \sum_{s_{r+1} \geqslant 0} \underbrace{\prod_{i=1}^m \ell_i(x + s_1 v_1 + \cdots + s_{r+1} v_{r+1})}_{(*)} z^{x + s_1 v_1 + \cdots + s_{r+1} v_{r+1}}.\end{aligned}$$

Since $x \in \Pi$ is $x = \lambda_1(x) v_1 + \cdots + \lambda_{r+1}(x) v_{r+1}$, using linearity of each ℓ_i we can expand out

$$\begin{aligned}(*) &= \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \left[\prod_{i \in I_1} \ell_i((s_1 + \lambda_1(x)) v_1) \right] \cdots \left[\prod_{i \in I_{r+1}} \ell_i((s_{r+1} + \lambda_{r+1}(x)) v_{r+1}) \right] \\ &= \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \left[\prod_{i \in I_1} \ell_i(v_1) \right] \cdots \left[\prod_{i \in I_{r+1}} \ell_i(v_{r+1}) \right] (s_1 + \lambda_1(x))^{|I_1|} \cdots (s_{r+1} + \lambda_{r+1}(x))^{|I_{r+1}|}.\end{aligned}$$

where $i \in I_j$ represents the term $(s_j + \lambda_j(x)) v_j$ being chosen from ℓ_i when multiplying out. Placing this into our original series, we obtain

$$\sum_{\alpha \in C \cap \mathbb{Z}^{d+1}} w(\alpha) z^\alpha = \sum_{x \in \Pi \cap \mathbb{Z}^{d+1}} z^x \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \prod_{i \in I_1} \ell_i(v_1) \cdots \prod_{i \in I_{r+1}} \ell_i(v_{r+1}) \prod_{j=1}^{r+1} \left(\sum_{s_j \geqslant 0} (s_j + \lambda_j(x))^{|I_j|} z^{s_j v_j} \right). \quad (4.4)$$

For the innermost sum on the right, we can write for each j

$$\sum_{s_j \geqslant 0} (s_j + \lambda_j(x))^{|I_j|} (z^{v_j})^{s_j} = \frac{A_{|I_j|}^{\lambda_j(x)}(z^{v_j})}{(1 - z^{v_j})^{|I_j|+1}}. \quad (4.5)$$

This completes the proof. □

In order to show that $\mathrm{Ehr}(P, w; z)$ is a rational function for any rational polytope P we consider partitions into half-open simplices. Given affinely independent vectors $u_1, \ldots, u_{r+1} \in \mathbb{R}^d$, the half-open simplex with the first $k \in \{0, 1, \ldots, r+1\}$ facets removed is defined as

$$\Delta = \left\{ \sum_{i=1}^{r+1} c_i u_i \mid c_1, \ldots c_k > 0, c_{k+1}, \ldots, c_{r+1} \geqslant 0, \sum_{i=1}^{r+1} \lambda_i = 1 \right\},$$

and the homogenization of Δ is the half-open simplicial cone

$$\mathrm{C}(\Delta) = \{c_1 v_1 + \cdots + c_{r+1} v_{r+1} \mid c_1 > 0, \ldots, c_k > 0, c_{k+1} \geqslant 0, \ldots, c_{r+1} \geqslant 0\}$$

where $v_i = (u_i, 1)$ for all i.

Given an r-dimensional polytope P and a triangulation, we can partition P into half-open simplices in the following way. Let q be a generic point in the relative interior of P and let $S = \mathrm{conv}\{u_1, \ldots, u_{r+1}\}$ be a maximal cell in the triangulation where $\mathrm{conv}\{\cdot\}$ denotes the convex hull. We say that a point $p \in S$ is *visible* from q if $(p, q] \cap S = \varnothing$. A half-open simplex, denoted $H_q S$, is then obtained by removing all points that are visible from q, which can be seen to be equal to

$$H_q S = \{c_1 u_1 + \cdots + c_{r+1} u_{r+1} \in S \mid c_i > 0 \text{ for all } i \in I_q\}$$

where $I_q = \{i \in [r+1] \mid u_i \text{ not visible from } q\}$.

The following is a special case of a result of Köppe and Verdoolaege [81].

Theorem 4.2.3 ([81])**.** *Let P be a polytope, $q \in \mathrm{aff}\, P$ be a generic point and $S_1, \ldots, S_m$ be the maximal cells of a triangulation of P. Then*

$$P \;=\; H_q S_1 \uplus H_q S_2 \uplus \cdots \uplus H_q S_m$$

is a partition into half-open simplices.

With the notation as in the previous theorem, it follows that

$$\mathrm{C}(P) \;=\; \mathrm{C}(H_q S_1) \uplus \mathrm{C}(H_q S_2) \uplus \cdots \uplus \mathrm{C}(H_q S_m), \tag{4.6}$$

that is, the homogenization $\mathrm{C}(P)$ of P can be partitioned into half-open simplicial cones. This, together with Proposition 4.2.2, allows us to show rationality of $\mathrm{Ehr}(P, w; z)$.

Proposition 4.2.4. *For any rational polytope P of dimension r and any degree-m form w on $\mathrm{C}(P)$, the weighted Ehrhart series is a rational function of the form*

$$\mathrm{Ehr}(P, w; z) = \frac{h^*_{P,w}(z)}{(1 - z^q)^{r+m+1}}$$

where q is a positive integer such that qP has integer vertices and $h^*_{P,w}(z)$ is a polynomial of degree at most $q(r+m+1) - 1$.

Proof. Let $S_1, \ldots, S_m$ be the maximal cells of a triangulation of P using no new vertices, that is, for all i, the vertex set of S_i is contained in the vertex set of P. Let

$$P \;=\; H_q S_1 \uplus H_q S_2 \uplus \cdots \uplus H_q S_m$$

be a partition into half-open simplices, and let

$$\mathrm{Ehr}(H_qS_i, w; z) = \sum_{x \in C(H_qS_i) \cap \mathbb{Z}^{d+1}} w(x) z^{x_{d+1}}$$

for all i. By equation (4.6), we have

$$\mathrm{Ehr}(P, w; z) = \mathrm{Ehr}(H_qS_1, w; z) + \cdots + \mathrm{Ehr}(H_qS_m, w; z).$$

It thus suffices to prove the claimed rational form for all half-open simplices in the partition.

Let $\Delta = H_qS_i$ be a rational half-open simplex in the partition. Let $v_1, \ldots, v_{r+1} \in \mathbb{Z}^{d+1}$ be generators of the half-open simplical cone $\mathrm{C}(\Delta)$. Since the triangulation of P used only vertices of P, we can choose $v_1, \ldots, v_{r+1} \in \mathbb{Z}^{d+1}$ such that their last coordinates are all equal to q.

Since every degree-m form is a sum of monomials, each of which is a product of linear forms, it furthermore suffices to consider the case when w is a product of linear forms. The weighted Ehrhart series is obtained by substituting $z_1 = \cdots = z_d = 1$ and $z_{d+1} = t$ into the generating function in Proposition 4.2.2. Thus

$$\mathrm{Ehr}(\Delta, w; z) = \sum_{x \in \Pi \cap \mathbb{Z}^{d+1}} \left(z^{x_{d+1}} \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \prod_{i \in I_1} \ell_i(v_1) \cdots \prod_{i \in I_{r+1}} \ell_i(v_{r+1}) \prod_{j=1}^{r+1} \frac{A_{|I_j|}^{\lambda_j(x)}(z^q)}{(1-z^q)^{|I_j|+1}} \right).$$

where Π is the half-open parallelepiped in $\mathrm{C}(\Delta)$ and each $x \in \Pi$ is written $x = \lambda_1(x)v_1 + \cdots + \lambda_{r+1}(x)v_{r+1}$.

Since $|I_1| + \cdots |I_{r+1}| + r + 1 = m + r + 1$, we have

$$\prod_{j=1}^{r+1} \frac{1}{(1-z^q)^{|I_j|+1}} = \frac{1}{(1-z^q)^{m+r+1}}. \tag{4.7}$$

Then we have

$$h^*_{\Delta,w}(z) = \sum_{x \in \Pi \cap \mathbb{Z}^{d+1}} z^{x_{d+1}} \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \prod_{i \in I_1} \ell_i(v_1) \cdots \prod_{i \in I_{r+1}} \ell_i(v_{r+1}) \prod_{j=1}^{r+1} A_{|I_j|}^{\lambda_j(x)}(z^q). \tag{4.8}$$

Thus the claim follows with $h^*_{P,w}(z) = h^*_{H_qS_1,w}(z) + \cdots + h^*_{H_qS_m,w}(z)$. □

Remark 4. In the multivariate version of the weighted Ehrhart rational function, the denominators do not simplify nicely as in (4.7). When bringing all constituents of the multivariate generating function of $C(P)$ in a common denominator this affects the positivity of the numerator polynomial.

Nonnegativity

We are now ready to prove the main theorem stated in the introduction. Recall that R_P is the semiring consisting of sums of products of nonnegative linear forms on P and S_P is the semiring consisting of sums of nonnegative products of linear forms on P.

Theorem 4.2.5 (Nonnegativity Theorem)**.** *Let P be a rational polytope.*

1. *If the weight w is a homogeneous element of R_P, then the coefficients of $h^*_{P,w}(z)$ are nonnegative.*

2. *If the weight w is a homogeneous element of S_P, then $h^*_{P,w}(z) \geqslant 0$ for $t \geqslant 0$.*

Proof. Let P be a rational polytope of dimension r.

For (1), it suffices to prove the statement when the weight is a product of nonnegative linear forms on $\mathrm{C}(P)$. The proof follows from the argument given in the proof of Proposition 4.2.4 where $h^*_{P,w}(z)$ is expressed as a sum of polynomials $h^*_{\Delta,w}(z)$ as given in Equation 4.8 where Δ ranges over all half-open simplices in a half-open triangulation of P. Each of the vectors v_i in Equation 4.8 is a generator of $\mathrm{C}(P)$. Thus, if $w \in R_P$, $h^*_{\Delta,w}(z)$ has nonnegative coefficients and so does $h^*_{P,w}(z)$ as a sum of these polynomials.

For (2), let w be a product of linear forms $\ell_1, \dots, \ell_m$ on $\mathrm{C}(P)$, and assume w is nonnegative on P. First suppose $\ell_1, \dots, \ell_m$ all have rational coefficients. Subdivide P into rational polytopes using the hyperplanes $\ell_1 = 0, \dots, \ell_m = 0$. Let s be a positive integer such that sQ has integer coordinates for every r-dimensional polytope Q that is part of the subdivision. Then s is divisible by the denominator $q = \delta(P)$ of P. On each such polytope Q, each linear form ℓ_i is either entirely nonnegative or entirely nonpositive, and the number of nonpositive ones is even because their product w is nonnegative. Thus after changing the signs of an even number of the linear forms on Q, which does not change w, we can apply the part (1) result to obtain that

$$\mathrm{Ehr}(Q, w; z) = \frac{h_Q(z)}{(1-z^s)^{r+m+1}} = \frac{h^*_{Q,w}(z)(1 + z^{\delta(Q)} + \ldots + z^{s-\delta(Q)})^{r+m+1}}{(1-z^{\delta(Q)})^{r+m+1}(1 + z^{\delta(Q)} + \ldots + z^{s-\delta(Q)})^{r+m+1}}$$

where $h_Q(z)$ has nonnegative coefficients for every polytope Q in the subdivision since $h^*_{Q,w}(z)$ has nonnegative coefficients by part (1). The weight w is zero on the boundaries where the polytopes overlap in the subdivision, so the Ehrhart series of P is the sum of Ehrhart series of the r-dimensional polytopes in the subdivision. Summing them up gives

$$\mathrm{Ehr}(P, w; z) = \frac{h(z)}{(1-z^s)^{r+m+1}},$$

for some polynomial $h(z)$ with nonnegative coefficients. Since s is divisible by the denominator q of P, we have

$$\frac{h^*_{P,w}(z)}{(1-z^q)^{r+m+1}} = \frac{h(z)}{(1-z^s)^{r+m+1}} = \frac{h(z)}{((1-z^q)(1 + z^q + z^{2q} + \cdots + z^{s-q}))^{r+m+1}},$$

so

$$h^*_{P,w}(z)(1 + z^q + z^{2q} + \cdots + z^{s-q})^{r+m+1} = h(z).$$

The polynomial $h(z)$ has nonnegative coefficients, so $h(z) > 0$ for $t > 0$. It follows that $h^*_{P,w}(z) > 0$ for all $t > 0$. This proves part (2) when the linear forms have rational coefficients.

To deal with irrational coefficients, note that for a fixed polytope P, the map that sends a weight w to the corresponding h^*-polynomial $h^*_{P,w}(z)$ is a linear, hence continuous, map from the vector space of homogeneous degree m polynomials to the vector space of degree $\leqslant r + m$ polynomials. The set of polynomials h^* satisfying $h^*(z) \geqslant 0$ when $t \geqslant 0$ is a closed set. Thus we obtain the result (2) for linear forms with irrational coefficients as well. □

Monotonicity

In this subsection we generalize Stanley's monotonicity result for the h^*-polynomial for rational polytopes to a weighted version by proving Theorem 4.2.7. Our proof follows a similar structure as the proof of nonnegativity. We start by proving a version of the claim for pyramids over half-open simplices and then extend it to all rational polytopes. This will become useful when comparing h^*-polynomials of polytopes of different dimension in the general case.

Given a half-open r-dimensional rational simplex $F \subseteq \mathbb{R}^d$, say

$$F = \{\lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} \mid \lambda_1, \ldots, \lambda_k \geqslant 0, \lambda_{k+1}, \ldots, \lambda_{r+1} > 0, \lambda_1 + \cdots + \lambda_{r+1} = 1\},$$

and a rational point $u \in \mathbb{R}^d$ not in the affine span of F, we let the *pyramid of u over F* be

$$\begin{aligned}\mathrm{Pyr}(u, F) := \{&\mu u + \lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} \mid \\ &\mu, \lambda_1, \ldots, \lambda_k \geqslant 0, \lambda_{k+1}, \ldots, \lambda_{r+1} > 0, \mu + \lambda_1 + \cdots + \lambda_{r+1} = 1\}.\end{aligned}$$

We denote the s-fold pyramid of $u_1, \ldots, u_s \in \mathbb{Q}^d$ over F by

$$\mathrm{Pyr}^{(s)}(u_1, \ldots, u_s, F) := \mathrm{Pyr}(u_1, \mathrm{Pyr}(u_2, \ldots \mathrm{Pyr}(u_s, F))),$$

now a half-open simplex of dimension $s + r$.

Lemma 4.2.6. *Let $F \subseteq \mathbb{R}^d$ be a half-open r-dimensional rational simplex with denominator $\delta(F)$ and let Δ be an s-fold pyramid over F with denominator $\delta(\Delta)$. For all $g \geqslant 1$ divisible by $\delta(\Delta)$ and all $w = \ell_1 \cdots \ell_m \in R_\Delta$,*

$$(1 + z^{\delta(F)} + \cdots + z^{g-\delta(F)})^{r+m+1} h^*_{F,w}(z) \preceq (1 + z^{\delta(\Delta)} + \cdots + z^{g-\delta(\Delta)})^{s+r+m+1} h^*_{\Delta,w}(z).$$

Proof. Let $v_1, \ldots, v_{r+1} \in \frac{1}{\delta(F)}\mathbb{Z}^d$ be vertices of F, labeled such that

$$F = \{\lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} \mid \lambda_1, \ldots, \lambda_k \geqslant 0, \lambda_{k+1}, \ldots, \lambda_{r+1} > 0, \lambda_1 + \cdots + \lambda_{r+1} = 1\}.$$

Suppose $u_1, \dots, u_s \in \frac{1}{\delta(\Delta)}\mathbb{Z}^d$ are such that $\Delta = \operatorname{Pyr}^{(s)}(u_1, \dots, u_s, F)$, that is, suppose

$$\Delta = \{\mu_1 u_1 + \cdots + \mu_s u_s + \lambda_1 v_1 + \cdots + \lambda_{r+1} v_{r+1} \mid \\ \mu_1, \dots, \mu_s, \lambda_1, \dots, \lambda_k \geqslant 0, \lambda_{k+1}, \dots, \lambda_{r+1} > 0, \mu_1 + \cdots + \mu_s + \lambda_1 + \cdots + \lambda_{r+1} = 1\}.$$

Considering the cone $\mathrm{C}(F)$ with generators of last coordinate g and fundamental parallelepiped

$$\Pi_g(F) = \left\{\lambda_1 \begin{pmatrix} gv_1 \\ g \end{pmatrix} + \cdots + \lambda_{r+1} \begin{pmatrix} gv_{r+1} \\ g \end{pmatrix} \mid 0 \leqslant \lambda_1, \dots, \lambda_k < 1, 0 < \lambda_{k+1}, \dots, \lambda_{r+1} \leqslant 1 \right\},$$

we obtain by Proposition 4.2.2

$$\operatorname{Ehr}(F, w; z) = \frac{\displaystyle\sum_{x \in \Pi_g(F) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}} \sum_{I_1 \uplus \cdots \uplus I_{r+1} = [m]} \prod_{i \in I_1} \ell_i(gv_1) \cdots \prod_{i \in I_{r+1}} \ell_i(gv_{r+1}) \prod_{j=1}^{r+1} A_{|I_j|}^{\lambda_j(x)}(z^g)}{(1 - z^g)^{r+m+1}}. \tag{4.9}$$

Analogously, considering the cone $\mathrm{C}(\Delta)$ with generators of last coordinate g and fundamental parallelepiped

$$\Pi_g(\Delta) = \left\{\mu_1 \begin{pmatrix} gu_1 \\ g \end{pmatrix} + \cdots + \mu_s \begin{pmatrix} gu_s \\ g \end{pmatrix} + \lambda_1 \begin{pmatrix} gv_1 \\ g \end{pmatrix} + \cdots + \lambda_{r+1} \begin{pmatrix} gv_{r+1} \\ g \end{pmatrix} \mid \\ 0 \leqslant \mu_1, \dots, \mu_s, \lambda_1, \dots, \lambda_k < 1, 0 < \lambda_{k+1}, \dots, \lambda_{r+1} \leqslant 1 \right\},$$

we obtain by Proposition 4.2.2

$$\operatorname{Ehr}(\Delta, w; z) \tag{4.10}$$

$$= \frac{\displaystyle\sum_{x \in \Pi_g(\Delta) \cap \mathbb{Z}^{d+1}} z^{x_{d+1}} \sum_{I_1 \uplus \cdots \uplus I_{s+r+1} = [m]} \prod_{i \in I_1} \ell_i(gv_1) \cdots \prod_{i \in I_{r+1}} \ell_i(gv_{r+1}) \prod_{i \in I_{r+2}} \ell_i(gu_1) \cdots \prod_{i \in I_{s+r+1}} \ell_i(gu_s) \prod_{j=1}^{s+r+1} A_{|I_j|}^{\lambda_j(x)}(z^g)}{(1 - z^g)^{s+r+m+1}}.$$

Observe that $\Pi_g(F) \subseteq \Pi_g(\Delta)$. In particular, the points in $\Pi_g(F)$ are those in $\Pi_g(\Delta)$ with $\mu_1 = \cdots = \mu_s = 0$. Therefore, for every $x \in \Pi_g(F) \cap \mathbb{Z}^{d+1}$, each term in the inner sum of the numerator of (4.9) appears as a term of the numerator of (4.10) with $I_{r+2} = \cdots = I_{s+r+1} = \varnothing$ (where $\lambda_{r+1}(x) = \cdots = \lambda_{s+r+1}(x) = 0$). Thus, since $w \in R_\Delta$, the nonnegativity of the remaining terms implies that

$$(1 - z^g)^{r+m+1} \operatorname{Ehr}(F, w; z) \preceq (1 - z^g)^{s+r+m+1} \operatorname{Ehr}(\Delta, w; z).$$

Recalling that the denominators of the Ehrhart series $\operatorname{Ehr}(F, w; z)$ and $\operatorname{Ehr}(\Delta, w; z)$ are $(1 - z^{\delta(F)})^{r+m+1}$ and $(1 - z^{\delta(\Delta)})^{s+r+m+1}$, respectively, we cancel these denominators and get the desired claim

$$(1 + z^{\delta(F)} + \cdots + z^{g-\delta(F)})^{r+m+1} h^*_{F,w}(z) \preceq (1 + z^{\delta(\Delta)} + \cdots + z^{g-\delta(\Delta)})^{s+r+m+1} h^*_{\Delta,w}(z).$$

□

We are now ready to prove the monotonicity theorems stated in the introduction. Recall that R_Q is the semiring consisting of sums of products of nonnegative linear forms on Q.

Theorem 4.2.7 (First Monotonicity Theorem)**.** *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in R_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim P+m+1} h^*_{P,w}(z) \preceq (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim Q+m+1} h^*_{Q,w}(z)\,.$$

In particular, if $P \subseteq Q$ are polytopes with the same denominator, then taking $g = \delta(P) = \delta(Q)$ gives

$$h^*_{P,w}(z) \preceq h^*_{Q,w}(z) \tag{4.11}$$

Proof. If P is empty, then $h^*_{P,w}(z) = 0$, so the statement becomes part (1) of the Nonnegativity Theorem (Theorem 4.2.5) above. Now let us assume that P is nonempty. We can extend a half-open triangulation of P to a half-open triangulation of Q as follows.

Let T be a half-open triangulation of P into simplices of dimension $\dim P$ with denominators dividing $\delta(P)$. Choose $u_1, \ldots, u_s \in Q \cap \frac{1}{g}\mathbb{Z}^d$, where $s = \dim Q - \dim P$, so that for each $F \in T$ the s-fold pyramid $\Delta_F = \operatorname{Pyr}^{(s)}(u_1, \ldots, u_s, F) \subseteq Q$ is a half-open simplex of dimension $\dim Q$. This is always possible by, for example, starting with a triangulation of P using no new vertices and choosing $u_1, \ldots, u_s$ successively from the vertices of Q that do not lie on the affine hull of the previous ones together with P. Let $\operatorname{Pyr}^{(s)}(P)$ denote the union of the Δ_F which form a half-open triangulation. By Lemma 4.2.6, for every $F \in T$,

$$(1 + z^{\delta(F)} + \cdots + z^{g-\delta(F)})^{\dim P+m+1} h^*_{F,w}(z) \preceq (1 + z^{\delta(\Delta_F)} + \cdots + z^{g-\delta(\Delta_F)})^{\dim Q+m+1} h^*_{\Delta_F,w}(z). \tag{4.12}$$

The left-hand side of (4.12) is equal to $(1-z^g)^{\dim P+m+1}\operatorname{Ehr}(F, w; z)$ and the right-hand side of (4.12) is equal to $(1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(\Delta_F, w; z)$. Therefore, summing over all $F \in T$ yields

$$(1 - z^g)^{\dim P+m+1}\operatorname{Ehr}(P, w; z) \preceq (1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(\operatorname{Pyr}^{(s)}(P), w; z). \tag{4.13}$$

Next we extend the half-open triangulation of $\operatorname{Pyr}^{(s)}(P)$ to a half-open triangulation T' of the entire polytope Q. This can be done by using a sequence of pushings (or placings) of the vertices of Q that are not in P to extend the triangulation of $\operatorname{Pyr}^{(s)}(P)$ to Q; see page 96 and Section 4.3 of [54] for more details. Using a generic point in Theorem 4.2.3 to be in the interior of $\operatorname{Pyr}^{(s)}(P)$ the resulting triangulation of Q becomes half-open. Each half-open simplex in T' has dimension $\dim Q$ and denominator dividing g. By Proposition 4.2.4, for each $\Delta \in T'$, $(1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(\Delta, w; z)$ is a polynomial with nonnegative coefficients. Therefore,

$$(1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(\operatorname{Pyr}^{(s)}(P), w; z) \preceq (1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(Q, w; z). \tag{4.14}$$

From (4.13) and (4.14) it follows that

$$(1 - z^g)^{\dim P+m+1}\operatorname{Ehr}(P, w; z) \preceq (1 - z^g)^{\dim Q+m+1}\operatorname{Ehr}(Q, w; z).$$

Equivalently,

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim P+m+1} h^*_{P,w}(z) \preceq (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim Q+m+1} h^*_{Q,w}(z).$$

□

Theorem 4.2.8 (Second Monotonicity Theorem)**.** *Let $P, Q \subseteq \mathbb{R}^d$ be rational polytopes of the same dimension $D = \dim P = \dim Q$, $P \subseteq Q$, and let g be a common multiple of the denominators $\delta(P)$ of P and $\delta(Q)$ of Q. Then, for all weights $w \in S_Q$,*

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{D+m+1} h^*_{P,w}(z) \leqslant (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{D+m+1} h^*_{Q,w}(z)$$

for all $t \geqslant 0$. In particular, if $P \subseteq Q$ are polytopes with the same denominator and dimension, then taking $g = \delta(P) = \delta(Q)$ gives

$$h^*_{P,w}(z) \leqslant h^*_{Q,w}(z) \text{ for all } t \geqslant 0. \tag{4.15}$$

Proof. Let w be a product of linear forms $\ell_1, ..., \ell_m$ on the homogenization $\mathrm{C}(P)$ such that w is nonnegative on P and $\ell_1, \ldots, \ell_m$ have rational coefficients. Now, let us use the hyperplanes $\ell_1 = 0, ..., \ell_m = 0$, as in the proof of Theorem 4.2.5 (2), to subdivide P and Q into rational polytopes $P'_1, \ldots, P'_k$ and $Q'_1, \ldots, Q'_k$, $P'_i \subseteq Q'_i$, respectively. Note, if any of these polytopes in the subdivision has dimension smaller than D then it is included in one of the hyperplanes and thus its h^*-polynomial is zero. Thus, we can compute the Ehrhart series of P and Q by summing up the series of those subpolytopes P'_is and $Q'_i s$ where $\dim(P'_i) = \dim(Q') = D$, and we may assume that each P'_i in the subdivision of P that we consider is contained in a unique polytope Q'_i in the subdivision of Q.

As before, every linear form ℓ_i with $1 \leqslant i \leqslant m$ is either entirely nonpositive or entirely nonnegative on each such polytope $P'_i \subseteq Q'_i$. Hence, we can change the signs of an even number of linear forms on P'_i and Q'_i without changing the weight w since the product of these linear forms is nonnegative.

Let g' be a positive integer multiple of all the denominators of P'_is and $Q'_i s$ in the subdivisions that additionally is also a multiple of g. We may now apply Theorem 4.2.7 to all $P'_i \subseteq Q'_i$ and obtain that

$$(1 + z^{\delta(P'_i)} + \cdots + z^{g'-\delta(P'_i)})^{D+m+1} h^*_{P'_i,w}(z) \preceq (1 + z^{\delta(Q'_i)} + \cdots + z^{g'-\delta(Q'_i)})^{D+m+1} h^*_{Q'_i,w}(z)\,.$$

We can rewrite this as

$$(1 - z^{g'})^{D+m+1} \operatorname{Ehr}(P'_i, w; z) \preceq (1 - z^{g'})^{D+m+1} \operatorname{Ehr}(Q'_i, w; z)\,.$$

Since the weight w is zero on the boundaries of the subdivision given by the linear forms $\ell_1, \ldots, \ell_m$, we can add up the inequalities for all pairs of polytopes $P_i \subseteq Q_i$ obtaining the following

$$(1 - z^{g'})^{D+m+1} \operatorname{Ehr}(P, w; z) \preceq (1 - z^{g'})^{D+m+1} \operatorname{Ehr}(Q, w; z)\,. \tag{4.16}$$

The left hand side of the inequality (4.16) equals

$$(1 + z^g + \cdots + z^{g'-g})^{D+m+1}(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{D+m+1}h^*_{P,w}(z)$$

and similarly for Q. Thus, we obtain that the polynomial $(1+z^g+\cdots+z^{g'-g})^{D+m+1}$ multiplied with

$$(1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{D+m+1}h^*_{Q,w}(z) - (1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{D+m+1}h^*_{P,w}(z) \quad (4.17)$$

has only nonnegative coefficients. In particular, evaluations at $t \geqslant 0$ of the product are nonnegative. Since $(1 + z^g + \cdots + z^{g'-g})^{D+m+1} > 0$ the nonnegativity of the evaluation of the second factor at nonnegative reals follows.

For linear forms with irrational coefficients as well as for an arbitrary element of S_P, we can argue again by linearity and continuity of the coefficients of the h^*-polynomials as in the proof of Theorem 4.2.7. □

Unlike the unweighted case of Stanley [107] the following example shows that the monotonicity in (4.15) need not hold when the polytopes do not have the same dimension:

Example 4.2.9. Consider $w = \ell^2$ for $\ell(x) = 2x_1 + 3x_2$, $v_1 = (3, -2), v_2 = (2, -2), v_3 = (2, -1)$, $P = \text{conv}(v_1, v_2)$, $Q = \text{conv}(v_1, v_2, v_3)$. We have $\ell(v_1) = 0, \ell(v_2) = -2, \ell(v_3) = 1$. Both P and Q are unimodular simplices, thus there is only one lattice point in the fundamental parallelepiped, namely 0. Thus, by Lemma 4.3.1 with all $\lambda_i = 0$, we obtain

$$h^*_{Q,w}(z) = z^2(\ell(v_1) + \ell(v_2) + \ell(v_3))^2 + z(\ell(v_1)^2 + \ell(v_2)^2 + \ell(v_3)^2) = z^2 + 5t$$

$$h^*_{P,w}(z) = z^2(\ell(v_1) + \ell(v_2))^2 + z(\ell(v_1)^2 + \ell(v_2)^2) = 4z^2 + 4t$$

Thus, the coefficients of the h^* polynomials are not monotone, and neither are the values since $h^*_{Q,w}(1) = 6 < 8 = h^*_{P,w}(1)$. □

Remark 5. As was shown in Example 4.2.9, the monotonicity in (4.15) does not need to hold for rational polytopes $P, Q \subseteq \mathbb{R}^d$, $P \subseteq Q$, of different dimension. In this case, the same arguments as in the proof of Theorem 4.2.8 nevertheless yield the existence of an integer g divisible by $\delta(P)$ and $\delta(Q)$ such that

$$(1 + z^{\delta(P)} + \cdots + z^{g-\delta(P)})^{\dim P+m+1}h^*_{P,w}(z) \leqslant (1 + z^{\delta(Q)} + \cdots + z^{g-\delta(Q)})^{\dim Q+m+1}h^*_{Q,w}(z)$$

for all $t \geqslant 0$ if the linear forms involved in the weight function have rational coefficients. Here we are no longer able to choose any g divisible by $\delta(P)$ and $\delta(Q)$, as the integer g depends on linear forms involved.

4.3 Squares of arbitrary linear forms

In this section we focus on weights given as squares of arbitrary linear forms, not necessarily in R_P and h^*-polynomials of polygons in the plane, and strengthen Theorem 4.2.5 in this special case. We prove that if P is a convex lattice polygon and the weight $w(x) = \ell(x)^2$ is given by a square of a linear form $\ell(x)$ then the coefficients of $h^*_{P,w}(z)$ are nonnegative, regardless of whether $\ell(x)$ is nonnegative on P or not. This result is established in Theorem 4.3.3 below. This is a reformulation of results on the positivity of Ehrhart tensor polynomials of lattice polytopes considered in [28]. See Section 4.4 below. Here, we present a proof that is arguably more elementary. We also present examples that show the limitations of our results if the conditions on the degree, dimension, denominator or convexity are removed.

Lattice polygons

We begin by providing the following more concise version of Equation (4.8) in the case of the weight being given as a square of a linear form that holds in any dimension.

Lemma 4.3.1. *Let $\ell : \mathbb{R}^d \to \mathbb{R}$ be a linear form. The h^*-polynomial $h^*_{\Delta,w}(z)$ with respect to the weight $w = \ell^2$ of any rational simplex $\Delta = \operatorname{conv}\{u_0, \ldots, u_r\}$ with denominator q is given by the sum of the contributions*

$$q^2\left(\left(\sum(1-\lambda_i)\ell(u_i)\right)^2 z^{2q} + \left(\sum \ell^2(u_i) + \left(\sum \ell(u_i)\right)^2 - \left(\sum \lambda_i\ell(u_i)\right)^2 - \left(\sum(1-\lambda_i)\ell(u_i)\right)^2\right) z^q + \left(\sum \lambda_i \ell(u_i)\right)^2\right) z^{x_{d+1}} \quad (4.18)$$

of each lattice point $x = \sum \lambda_i(x)(qu_i, q) \in \Pi(\Delta) \cap \mathbb{Z}^{d+1}$ in the fundamental parallelepiped where all summations are taken for indices i from 0 to r.

Proof. If $w(x) = \ell(x)^2$ then the weight is a product of $m = 2$ linear forms and the contributions of each lattice point in the fundamental parallelepiped given in Equation (4.8) is a linear combination of products of $A_0^\lambda(z) = 1$,

$$A_2^\lambda(z) = (1-\lambda)^2 z^2 + (1 + 2\lambda - 2\lambda^2)z + \lambda^2 \qquad \text{and} \qquad A_1^\lambda(z) = (1-\lambda)z + \lambda$$

for $0 \leqslant \lambda \leqslant 1$. More precisely, we use the homogenized linear form ℓ' associated with ℓ that takes in account the scaling factor in Equation (4.8). Then $\ell'(qu_i, q) = q\ell'(u_i, 1) = q\ell(u_i)$ and we get that the contribution of any such point $x = \sum \lambda_i(qu_i, q)$ is

$$q^2\left(\sum_{0\leqslant i\leqslant r} A_2^{\lambda_i}(z^q)\ell^2(u_i) + 2\sum_{0\leqslant i<j\leqslant r} A_1^{\lambda_i}(z^q)A_1^{\lambda_j}(z^q)\ell(u_i)\ell(u_j)\right) z^{x_{d+1}},$$

where the first sum corresponds to the ordered partitions $[2] = I_0 \uplus I_1 \uplus \cdots \uplus I_r$ into $r+1$ parts where $|I_i| = 2$ for some i and the second sum corresponds to partitions for which $|I_i| = |I_j| = 1$ for some $i \neq j$.

The factor q^2 is present in both cases. The coefficients of z^{2q} and 1 (times $z^{x_{d+1}}$) of the polynomial above are easily seen. Indeed, the first sum contributes $\sum(1-\lambda_i)^2\ell^2(u_i)$ and the second sum contributes $2\sum(1-\lambda_i)(1-\lambda_j)\ell(u_i)\ell(u_i)$ to the coefficient of z^{2q}. Combining these, we obtain $\left(\sum(1-\lambda_i)\ell(u_i)\right)^2$ as claimed. Analogous arguments yield the coefficient of 1 of every contribution.

A similar analysis gives that the coefficient of z^q is equal to

$$\begin{aligned}
&\sum_i (1+2\lambda_i-2\lambda_i^2)\ell^2(u_i) + 2\sum_{i<j}\Big((1-\lambda_i)\lambda_j + (1-\lambda_j)\lambda_i\Big)\ell(u_i)\ell(u_j) \\
&\quad = \sum_i (1+2\lambda_i-2\lambda_i^2)\ell^2(u_i) + 2\left(\sum_i \lambda_i \ell(u_i)\right)\left(\sum_j (1-\lambda_j)\ell(u_j)\right) - 2\sum_i \lambda_i(1-\lambda_1)\ell^2(u_i) \\
&\quad = \sum_i \ell^2(u_i) + 2\left(\sum_i \lambda_i \ell(u_i)\right)\left(\sum_j (1-\lambda_j)\ell(u_j)\right).
\end{aligned}$$

By squaring both sides of the identity

$$\sum_i \ell(u_i) = \left(\sum_i \lambda_i \ell(u_i)\right) + \left(\sum_j (1-\lambda_j)\ell(u_j)\right)$$

we get the claimed coefficient of z^q. □

Lemma 4.3.2. *Let $\Delta \subseteq \mathbb{R}^2$ be a half-open triangle with vertices in $\mathbb{Z}^2$, let $\ell : \mathbb{R}^2 \to \mathbb{R}$ be a linear form and let $w(x) = \ell^2(x)$. If the h^*-polynomial $h^*_{\Delta,w}(z)$ of Δ with respect to $w(x) = \ell^2(x)$ has negative coefficients then the following two conditions must both be satisfied.*

(i) Δ is neither completely closed nor completely open, and

(ii) the line $\ker \ell$ intersects the relative interior of two sides of Δ that are either both "open" or both "closed".

Proof. Let u_0, u_1, u_2 be the vertices of Δ. We argue by induction over the area of Δ.

We begin by assuming that Δ has area $1/2$, the minimal area among all triangles with vertices in the integer lattice. In this case, the half-open fundamental parallelepiped $\Pi(\Delta)$ contains exactly one lattice point $x = \lambda_0(u_0, 1) + \lambda_1(u_1, 1) + \lambda_2(u_2, 1)$ where $\lambda_0, \lambda_1, \lambda_2 \in \{0, 1\}$.

If Δ is completely closed then $\lambda_0 = \lambda_1 = \lambda_2 = 0$ and by Lemma 4.3.1,

$$h^*_{\Delta,w}(z) = \left(\sum \ell(v_i)\right)^2 z^2 + \left(\sum \ell(v_i)^2\right) t\,.$$

Similarly, if Δ is completely open, then $\lambda_0, \lambda_1 = \lambda_2 = 1$ and

$$h^*_{\Delta,w}(z) = \left(\sum \ell(u_i)\right)^2 z^3 + \left(\sum \ell(u_i)^2\right) z^4$$

In particular, in both cases we see that the h^*-polynomial has only nonnegative coefficients. Thus, if a half-open lattice triangle has a negative coefficient condition (i) needs to be satisfied, that is, Δ is neither completely open nor closed. In this case, $\lambda_0, \lambda_1, \lambda_2$ are not all equal.

We consider the case $\lambda_0 = \lambda_1 = 0$ and $\lambda_2 = 1$. Then, by Lemma 4.3.1,

$$h^*_{\Delta,w}(z) =$$
$$(\ell(u_0) + \ell(u_1))^2 z^3 + \Big(\ell^2(u_0) + \ell^2(u_1) + \ell^2(u_2) + (\ell(u_0) + \ell(u_1) + \ell(u_2))^2 - \ell^2(u_2) - (\ell(u_0) + \ell(u_1))^2\Big) z^2 + \ell^2(u_2)t.$$

The first and last coefficient are squares and thus always nonnegative. The coefficient of z^2 can be simplified to

$$(\ell(u_0) + \ell(u_2))^2 + (\ell(u_1) + \ell(u_2))^2 - \ell^2(u_2)\,.$$

We observe that if $\ell(u_2)$ has the same sign as $\ell(u_i)$, $i = 0, 1$, then $(\ell(u_i) + \ell(u_2))^2 - \ell^2(u_2) \geqslant 0$ and thus the coefficient is nonnegative. It follows that $h_{\Delta,w}(z)$ can have a negative coefficient only if $\ell(u_2)$ has a different sign than both $\ell(u_0)$ and $\ell(u_1)$, that is, $\ker \ell$ separates u_2 from u_0 and u_1 as claimed. The case $\lambda_0 = \lambda_1 = 1$ and $\lambda_2 = 0$ follows analogously. This proves the claim if Δ has minimal area.

Now we assume that Δ has area greater than $1/2$ and that the result has already been proved for all Δ of smaller area. In order to prove the claim it suffices to show that if Δ does not satisfy at least one of the conditions (i) or (ii) then it can be partitioned into half-open triangles that have h^*-polynomials with only nonnegative coefficients; then, by additivity also the h^*-polynomial of Δ is nonnegative and the proof will follow.

If Δ has area greater than $1/2$ then it contains at least one lattice point aside of its vertices, either in the relative interior of a side or in the interior of the triangle. By coning over the sides in which this point is not contained we obtain a subdivision into two or three smaller lattice triangles. By induction hypothesis it suffices to show that this subdivision can be made half-open in such a way that the half-open triangles in the partition do not satisfy both condition (i) and (ii).

This is indeed always possible. In Figure 4.1 the case of an interior lattice point and a subdivision into three smaller triangles is considered. The first row shows how to partition a completely closed triangle into smaller triangles that violate conditions (i) or (ii), depending on the position of $\ker \ell$. If Δ is completely open, then open and closed sides are flipped. The second row shows how such a partition is established in case Δ is half-open but $\ker \ell$ intersects in an open and a closed side. The non-intersected side can be removed in the case that it is excluded.

The case of a partition into two triangles can be treated in a similar way. □

Theorem 4.3.3. *For every (closed) convex lattice polygon P and every linear form ℓ, the h^*-polynomial of P with respect to $w(x) = \ell^2(x)$ has only non-negative coefficients.*

Proof. If $\ker \ell$ does not intersect the interior of P, then the statement follows from Theorem 4.2.5. Otherwise, $\ker \ell$ intersects the boundary of P twice: either in two vertices, or in a vertex and the interior of a side, or the interior of two sides.

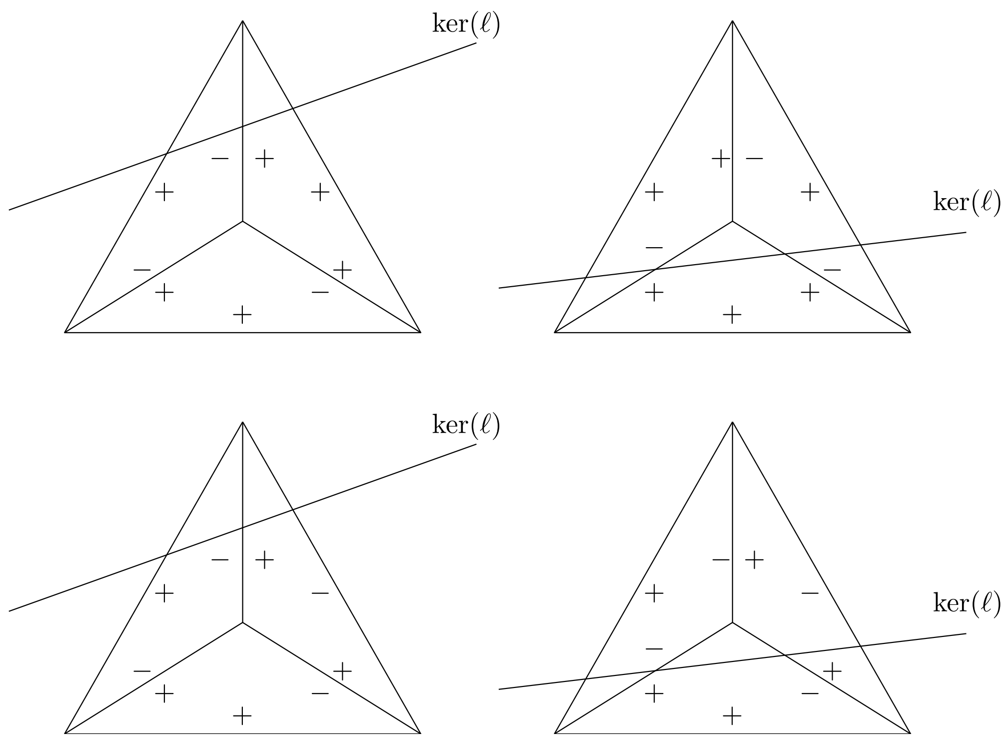


Figure 4.1: A subdivision of a triangle using an interior lattice point. Each edge is marked with + or − to indicate which simplex includes it; the simplex containing + contains the edge and the simplex containing − excludes it.

If $\ker \ell$ intersects the boundary of P in two vertices, then the h^*-polynomial of P is the sum of the h^*-polynomials of the two (closed) lattice polygons $\ker \ell$ divides P into. This is because lattice points in $\ker \ell$ are weighted with 0. The h^*-polynomial of both lattice polygons in the subdivision have only nonnegative coefficients by Theorem 4.2.5 and so does their sum.

In the other two cases, if $\ker \ell$ intersects in a vertex and the interior of a side, or in the interior of two sides, the polygon can be subdivided into half-open triangles that do not satisfy the conditions (i) and (ii) in Lemma 4.3.2 as depicted in Figure 4.2: if the convex hull of the corresponding vertex and side/the two sides is a triangle, we take this closed triangle and extend it to a half-open triangulation as shown in the picture; if the convex hull of the two intersected sides i a quadrilateral, we partition this quadrilateral into a closed triangle and a half-open one along its diagonal; the rest of the polygon is again subdivided into half-open triangles that do not intersect $\ker \ell$, as depicted.

In all cases, the half-open triangles used in the half-open triangulation violate the conditions given in Lemma 4.3.2. Thus their h^*-polynomial have only nonnegative coefficients and so does their sum. □

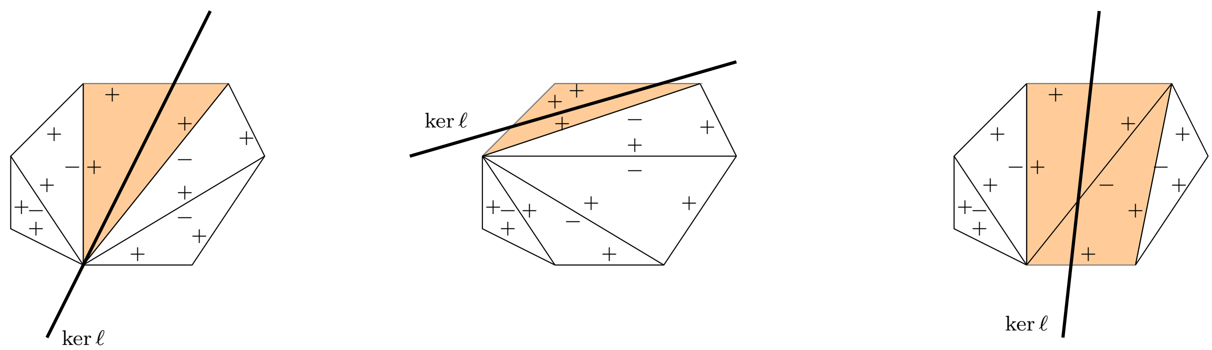


Figure 4.2: Half-open triangulations of a polygon in the cases where $\ker \ell$ intersects the boundary of the polygon in a vertex and the interior of a side (left) or two sides (middle/right). Removed/open faces are denoted by "$-$", closed/non-removed faces with "$+$". The convex hull of the corresponding vertex/sides is depicted in gray. All half-open triangles violate conditions (i) and (ii) of Lemma 4.3.2.

Negative examples

In this section we provide examples that show that most assumptions in Theorem 4.3.3 are necessary and cannot be further relaxed. Our examples are explicit and can be computed either by applying Equation (4.8) and/or by using `LattE` ([14]).

We begin with an example that shows that the nonnegativity of the h^*-polynomial for lattice polygons does not extend to weight functions that are squares of degree higher than 2.

Example 4.3.4. Let $w(x) = (2x_1 - x_2)^2(2x_2 - x_1)^2$ and P be the standard triangle with vertices $v_0 = (0,0), v_1 = (1,0)$, and $v_2 = (0,1)$. Then

$$h^*_{P,w}(z) = t(8 + 81z - 6z^2 + z^3).$$

While the classical Ehrhart theory deals with convex polytopes, in the two-dimensional case, Stanley's nonnegativity theorem and our Theorem 4.2.5 can be extended to non-convex polygons without holes as any such polygon can be dissected into (half-open) triangles. Next we give an example of a non-convex quadrilateral and weight given by a square of a linear form that shows that Theorem 4.3.3 does not extend to non-convex quadrilaterals.

Example 4.3.5. Let $w(x) = \ell(x)^2$ where $\ell(x) = x_1$ and $P = v_0v_1v_2v_3$ be the non-convex quadrilateral with vertices $v_0 = (1,0), v_1 = (-3,-1), v_2 = (2,0), v_3 = (-3,1)$ as depicted in Figure 4.3. Then

$$h^*_{P,w}(z) = t(23 - 4z + 9z^2).$$

Next, we note that Theorem 4.3.3 does not hold for rational polygons, not even in the case of "primitive" triangles as illustrated in the next example.

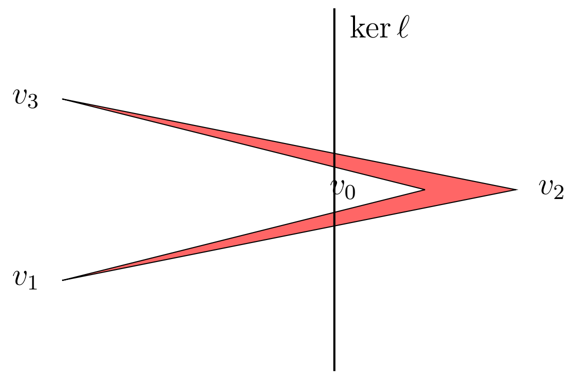


Figure 4.3: An example of a non-convex lattice quadrilateral that has an h^*-polynomial with negative coefficients with respect to the weight function $w(x) = x_1^2$

Example 4.3.6. For any integer $q \geqslant 1$, let $\Delta_q \subseteq \mathbb{R}^2$ be the rational triangle with vertices

$$u_0 = (1,1)\,, u_1 = \left(1, \frac{q-1}{q}\right) \text{ and } u_2 = \left(\frac{q+1}{q}, 1\right)$$

that has denominator q. Let $\ell_q \colon \mathbb{R}^2 \to \mathbb{R}$ be the linear form defined by $\ell_q(x) = 2q(1-q)x_1 + q(2q-1)x_2$. Then

$$\begin{aligned} \ell_q(u_0) &= q \\ \ell_q(u_1) &= 1-q \\ \ell_q(u_2) &= 2-q\,. \end{aligned}$$

The half-open fundamental parallelepiped spanned by $(qu_0, q), (qu_1, q), (qu_2, q)$ contains exactly q lattice points, namely

$$y_i = (i,i,i) \quad \text{for all } 0 \leqslant i \leqslant q-1\,.$$

By Lemma 4.3.1 we see that every non-zero coefficient of the h^*-polynomial of Δ with respect to $w_q(x) = \ell_q(x)^2$ arises from the contribution of exactly one of the y_is, namely y_i contributes to the coefficient of z^j if and only if $j \equiv i \mod q$. Thus, $h^*_{\Delta_q, w_q}(z)$ has a negative coefficient if and only if the contribution of one of the lattice points in the half-open parallelepiped has a negative coefficient.

We focus on

$$y_{q-1} = (q-1, q-1, q-1) = \frac{q-1}{q}(qu_0, q) + 0 \cdot (qu_1, q) + 0 \cdot (qu_2, q).$$

By Lemma 4.3.1, the second term in the contribution of y_{q-1}, and therefore the coefficient of z^{2q-1}, is equal to q^2 times

$$q^2 + (1-q)^2 + (2-q)^2 + (q + (1-q) + (2-q))^2 - \left(\tfrac{q-1}{q} q\right)^2 - \left(\tfrac{1}{q} q + (1-q) + (2-q)\right)^2$$

which is equal to $-q^4+6q^3-3q^2$. This evaluates to a negative number for all integers $q \geqslant 6$. As a consequence, the h^*-polynomial of Δ_q with respect to the weight $w_q(x) = \ell_q(x)^2$ has a negative coefficient in front of z^{2q-1} for all integers $q \geqslant 6$. For example, if $q = 6$ then $h^*_{\Delta_q,w_q}(z)$ equals

$$\begin{aligned}&2304z^{17} + 1764z^{16} + 1296z^{15} + 900z^{14} + 576z^{13} + 324z^{12} - 108z^{11} + 756z^{10}\\&+1476z^9 + 2052z^8 + 2484z^7 + 2772z^6 + 900z^5 + 576z^4 + 324z^3 + 144z^2 + 36z\end{aligned}$$

Last but not least, we show that the assumption on the dimension cannot be removed in Theorem 4.3.3 by providing an example of a 20-dimensional lattice simplex P and a linear form such that $h^*_{P,w}(x)$ has a negative coefficient where $w(x) = \ell(x)^2$. This also establishes a counterexample to a conjecture of Berg, Jochemko, Silverstein [28], see Section 4.4 below for details.

Example 4.3.7. We consider the 19-dimensional simplex $\Delta = \operatorname{conv}\{u_0, \ldots, u_{19}\}$ where u_0 is the origin, $u_1, \ldots, u_{18}$ are the standard basis vectors $e_1, \ldots, e_{18}$ and

$$\begin{aligned}u_{19} &= (1,1,1,1,1,1,1,1,1,-1,-1,-1,-1,-1,-1,-1,-1,-1,3)\\&= 3e_{19} + e_1 + \ldots + e_9 - e_{10} - \ldots - e_{18}.\end{aligned}$$

and the pyramid $\Delta' = \operatorname{conv}(0 \cup \Delta \times 1) \in \mathbb{R}^{20}$ which is a 20-dimensional simplex with vertices 0 and $v_i := (u_i, 1)$, $0 \leqslant i \leqslant 19$. Let $\ell : \mathbb{R}^{20} \to \mathbb{R}$ be the linear functional defined by

$$\ell(v_i) = \begin{cases} 1 & \text{if } 0 \leqslant i \leqslant 9 \\ -1 & \text{if } 10 \leqslant i \leqslant 19\,. \end{cases}$$

We claim that the h^*-polynomial of Δ' with respect to $w(x) = \ell(x)^2$ has a negative coefficient in front of z^{11}.

To see this, we observe that the determinant of the matrix with columns v_i, $0 \leqslant i \leqslant 19$ equals -3, that is, the normalized volume of Δ' is 3 and the half-open fundamental parallelepiped $\Pi(\Delta')$ contains exactly three lattice points. Those are $y_0 = 0$,

$$\begin{aligned}y_1 &= \frac{2}{3}\sum_{i=0}^{9}(v_i,1) + \frac{1}{3}\sum_{i=10}^{19}(v_i,1) = (1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,1,10,10)\,,\\ y_2 &= \frac{1}{3}\sum_{i=0}^{9}(v_i,1) + \frac{2}{3}\sum_{i=10}^{19}(v_i,1) = (1,1,1,1,1,1,1,1,1,0,0,0,0,0,0,0,0,0,2,10,10)\,.\end{aligned}$$

By Lemma 4.3.1, the coefficient of z^{11} in the contribution of y_j, $j = 1,2$, equals

$$\sum \ell^2(v_i) + \left(\sum \ell(v_i)\right)^2 - \left(\sum \lambda_i \ell(v_i)\right)^2 - \left(\sum (1-\lambda_i)\ell(v_i)\right)^2$$

where $\lambda_0 = \cdots = \lambda_9 = 2/3$ and $\lambda_{10} = \cdots = \lambda_{19} = 1/3$ for y_1, and for y_2 the values are flipped. In both cases, the term evaluates to

$$20 + (0)^2 - \left(\frac{2}{3} \cdot 10 + \frac{1}{3} \cdot (-10)\right)^2 - \left(\frac{1}{3} \cdot 10 + \frac{2}{3} \cdot (-10)\right)^2 = \frac{-20}{9}.$$

Note that $y_0 = 0$ does not contribute to the z^{11}-coefficient of the h^*-polynomial. In summary, the coefficient of z^{11} equals $2 \cdot \frac{-20}{9} < 0$ and is thus negative.

4.4 Ehrhart tensor polynomials

In this section we discuss the results of the previous section in relation to results and conjecture on Ehrhart tensor polynomials which were introduced by Ludwig and Silverstein [88].

For any integer $r \in \mathbb{N}$, let $\mathbb{T}^r$ be the vector space of symmetric tensors of rank r on $\mathbb{R}^d$. The *discrete moment tensor* of rank r of a lattice polytope $P \subset \mathbb{R}^d$ is defined as

$$L^r(P) = \sum_{x \in P \cap \mathbb{Z}^d} x^{\otimes r},$$

where $x^{\otimes r} = x \otimes \cdots \otimes x$ and $x^{\otimes 0} := 1$. Discrete moment tensors were introduced by Böröczky and Ludwig [33]. Note that for $r = 0$ we recover the number of lattice points in P, $|P \cap \mathbb{Z}^d|$. Ludwig and Silverstein [88, Theorem 1] showed that there exist maps L_i^r, $0 \leqslant i \leqslant d+1$, from the family of lattice polytopes to $\mathbb{T}^r$ such that

$$L^r(nP) = \sum_{i=0}^{d+r} L_i^r(P) n^i$$

for all integers $n \geqslant 0$, that is, the discrete moment tensor $L^r(nP)$ is given by a polynomial in the nonnegative integer dilation factor. The polynomial is called the *Ehrhart tensor polynomial*. Equivalently, if P is a d-dimensional lattice polytope,

$$\sum_{n \geqslant 0} L^r(nP) z^n = \frac{h_0^r(P) + h_1^r(P) z + \cdots + h_{d+r}^r(P) z^{r+d}}{(1-z)^{d+r+1}}$$

for tensors $h_0^r(P), h_1^r(P), \ldots, h_{r+d}^r(P) \in \mathbb{T}^r$. The numerator polynomial is called the *h^r-tensor polynomial* of P [28]. Observe that for $r = 0$ we recover the usual Ehrhart and h^*-polynomial of a lattice polytope.

The vector space of symmetric tensors $\mathbb{T}^r$ is isomorphic to the vector space of multilinear functionals $(\mathbb{R}^d)^r \to \mathbb{R}$ that are invariant under permutations of the arguments. In particular, for any $v_1, \ldots, v_r \in \mathbb{R}^d$,

$$L^r(P)(v_1, \ldots, v_r) = \sum_{x \in P \cap \mathbb{Z}^d} (x^T v_1) \cdots (x^T v_r).$$

Thus, weighted Ehrhart polynomials can be seen as evaluations of Ehrhart tensor polynomials in the following sense.

Proposition 4.4.1. *Let $w(x) = \ell_1(x) \cdots \ell_r(x)$ be a product of linear forms where each linear form $\ell_i : \mathbb{R}^d \to \mathbb{R}$ is given by $\ell_i(x) = x^T v_i$ for some $v_i \in \mathbb{R}^d$. Let P be a d-dimensional lattice polytope. Then*

$$\operatorname{ehr}(nP, w) = \sum_{i=0}^{d+r} L^r(P)(v_1, \ldots, v_r) n^i$$

and, equivalently,

$$h^*_{P,w}(z) = \sum_{i=0}^{d+r} h^r_i(P)(v_1, \ldots, v_r) z^i \,.$$

Proof. For any integer $n \geqslant 0$,

$$\operatorname{ehr}(nP, w) = \sum_{x \in nP \cap Z^d} x^T v_1 \cdots x^T v_r = L^r(nP)(v_1, \ldots, v_r) = \sum_{i=0}^{d+r} L^r_i(nP)(v_1, \ldots, v_r) n^i \,.$$

The claim for the h^*-polynomials follows similarly. □

In the case that $r = 2$, symmetric tensors can be identified with symmetric matrices via their values on pairs of standard vectors. Via this identification, a tensor is called positive semi-definite if the corresponding matrix is positive semi-definite. In particular, $L^2(P) = \sum_{x \in P \cap Z^d} xx^T$ is always positive semi-definite. However, the coefficients of the Ehrhart tensor polynomial and the h^2-tensor polynomial need not be in general [28], similarly as the coefficients of the usual Ehrhart polynomial are not positive in general. The following relation between the positivity of weighted h^*-polynomials and the positive semi-definiteness of the coefficients of the h^2-tensor polynomial is a consequence of Proposition 4.4.1.

Proposition 4.4.2. *For any lattice polytope $P \subset \mathbb{R}^d$, the h^2-tensor polynomial of P has only positive semi-definite coefficients if and only if $h^*_{P,w}(z)$ has only nonnegative coefficients for each weight that is a square of a linear form $w(x) = \ell^2(x)$.*

Proof. Let $M_i = h^2_i(P) \in \mathbb{R}^{2 \times 2}$ be the coefficients of the h^2-polynomial of P. By Proposition 4.4.1, for any linear form $\ell(x) = v^T x$ on $\mathbb{R}^d$, $h^*_{P,w}(z) = \sum_i v^T M_i v z^i$. Thus, $h^*_{P,w}(z)$ has only nonnegative coefficients for all weights $w(x) = \ell(x)^2$ if and only if the matrices M_i are all positive semi-definite. □

In [28] Berg, Jochemko and Silverstein investigated when h^2-tensor polynomials have only positive semi-definite coefficients. They proved that the coefficients are indeed positive semi-definite for lattice polygons [28, Theorem 5.2] and conjectured that this holds more general in arbitrary dimensions [28, Conjecture 6.1]. By Proposition 4.4.2, it follows that Theorem 4.3.3 is equivalent to [28, Theorem 5.2]; the proof given in Section 4.3 is arguably simpler.

Corollary 4.4.3 ([28, Theorem 5.2]). *The h^2-tensor polynomial of any lattice polygon has only positive semi-definite coefficients.*

Furthermore, Example 4.3.7 provides a 20-dimensional lattice polytope together with a weight $w(x) = \ell(x)^2$ that is a square of a linear form such that $h_{P,w}(z)$ has a negative coefficient. By Proposition 4.4.2 this establishes a counterexample to [28, Conjecture 6.1].

Corollary 4.4.4. *There exists a* 20*-dimensional lattice polytope whose* h^2*-tensor polynomial has a coefficient that is not positive semi-definite. In particular, this disproves [28, Conjecture 6.1]*

4.5 Open question

In Theorem 4.2.5 we have proved sufficient conditions on the homogeneous weight function that yield nonnegative coefficients of the h^*-polynomial. We also have shown our results are tight, in particular, in Section 4.3 we have seen that Theorem 4.2.5 can fail if the assumptions are relaxed, even in the simple case of a square of a single linear form.

We end this chapter posing a natural question.

Question 4.5.1. Can we precisely characterize the family of homogeneous weights that yield nonnegative coefficients of the h^*-polynomial?

Chapter 5

Local h^*-polynomials

In this chapter, we take interest in the connection between the unimodality of h^*-polynomials of lattice polytopes and the unimodality of the *local h^*-polynomials* of the simplices in their triangulations, and therefore try to classify some specific simplices that have unimodal local h^*-polynomials.

5.1 Introduction

Background

Given a lattice polytope P, there is another invariant related to counting the number of integer points in integer dilates of P: the *local h^*-polynomial*. Recently there has been renewed interest in the study of this polynomial, which is more complicated to define and has not been as extensively studied as the classical h^*-polynomial. This invariant, which we denote by $B(P; z)$, is the primary focus of this chapter.

The local h^*-polynomial has arisen in multiple contexts using different notation. For a detailed survey regarding local h^*-polynomials, see Section 2 of the recent paper by Borger, Kretschmer, and Nill [31]. The local h^*-polynomial was defined in substantial generality by Stanley in [119, Example 7.13], extending work first presented by Betke and McMullen [29]. Local h^*-polynomials were also studied by Borisov and Mavlyutov in connection to Calabi-Yau complete intersections in Gorenstein toric Fano varieties, where they were referred to as $\tilde{S}$-polynomials [32]. Local h^*-polynomials for simplices are sometimes referred to as *box polynomials*; these were studied by Gelfand, Kapranov, and Zelevinsky [60], who identified the importance of lattice simplices with vanishing local h^*-polynomials. Lattice polytopes with vanishing local h^*-polynomials are called *thin polytopes*, and these have recently been further investigated by Borger, Kretschmer, and Nill [31]. A key observation due to Nill and Schepers [31, 93] is that any lattice polytope admitting a regular unimodular triangulation has a unimodal local h^*-polynomial. This result was recently strengthened by Adiprasito, Papadakis, Petrotou, and Steinmeyer [1], who proved that lattice polytopes with the integer decomposition property (defined in Subsection 5.3) have unimodal local h^*-polynomials.

Unimodality of local h^*-polynomials also plays a role in the study of unimodality for h^*-polynomials. Schepers and Van Langenhoven [102] introduced the concept of a *box unimodal* triangulation, which is a lattice triangulation T of a lattice polytope for which every face of T has a unimodal local h^*-polynomial. The motivation for the term "box unimodal" is the "box polynomial" nomenclature used by Gelfand, Kapronov, and Zelevinsky. Schepers and Van Langenhoven proved that if a reflexive lattice polytope has a box unimodal triangulation, then it has a unimodal h^*-polynomial. This generated interest in determining which simplices have unimodal local h^*-polynomials, since these are the simplices appearing in box unimodal triangulations. As one example of results in this direction, Solus and Gustafsson proved that every s-lecture hall order polytope admits a box unimodal triangulation [62].

Lattice simplices and our contributions

Motivated by the above context, our focus in this work is to investigate unimodality of local h^*-polynomials for lattice simplices.

The local h^*-polynomial of lattice simplices has the following beautiful geometric interpretation, which for the rest of this chapter we take as the definition. Let $\{v_1, \ldots, v_{d+1}\}$ be the vertices of a lattice simplex S, and let

$$\Pi_S := \left\{ \sum_i \lambda_i (1, v_i) : 0 < \lambda_i < 1 \right\}$$

define the open parallelepiped for $\{1\} \times S$. Then the local h^*-polynomial is

$$B(S; z) := \sum_{(m_0, \ldots, m_d) \in \Pi_S} z^{m_0} ,$$

i.e., $B(S; z)$ encodes the distribution of lattice points through the open parallelepiped of $\{1\} \times S$ with respect to the 0-th coordinate, which we refer to as the *height* of the point. To emphasize this distributional perspective, and to allow us to consider the shape of coefficient vectors of different local h^*-polynomials, we primarily consider in this work the coefficients of

$$B(S; z)/B(S; 1),$$

which encode the probability distribution for lattice points in Π_S with respect to height.

From this perspective, it seems natural that the local h^*-polynomial might have unimodal coefficients, as the parallelepiped is "fatter" geometrically in the middle than on the ends. It seems plausible that unimodality might even be typical in this setting, even without assumptions such as admitting a regular unimodular triangulation or the integer decomposition property. However, as we will see in this work, it is not clear whether or not these intuitions are correct; for example, Figure 5.3 and Figure 5.5 suggest a variety of possible conjectures.

Because an arbitrary lattice simplex is arithmetically complicated, we restrict our attention to a set of simplices with a more manageable arithmetical structure. Lattice simplices are classified through their Hermite normal form; see Theorem 5.2.1 for the precise statement. In Section 5.2, we define Hermite normal form simplices and recall how to compute their h^*- and local h^*-polynomials. We also discuss the relationship between local h^*-polynomials and Stapledon a/b-decompositions for h^*-polynomials. The family of simplices that we study are one-row Hermite normal form simplices, which are those arising as the convex hull of the rows of an integer matrix as in (5.1), specified by parameters $a_1, \ldots, a_{d-1}, N$ with $0 \leqslant a_i < N$ for all i. Note that the normalized volume of these simplices is exactly the parameter N.

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{d-2} & a_{d-1} & N \end{bmatrix}. \tag{5.1}$$

After providing detailed background on Hermite normal form, local h^*-polynomials, and the relationship between local h^*-polynomials and Stapledon a/b-decompositions for h^*-polynomials, our first contribution is to investigate two special subfamilies of one-row Hermite form simplices that exhibit distinct behavior with regard to their local h^*-polynomials. These special families illustrate both the variety of behavior observed with local h^*-polynomials and the proof techniques that we will use throughout this chapter. This is the content of Section 5.3, where we provide a complete investigation of local h^*-polynomials for "all-ones" simplices ($a_i = 1$ for all i) and for "geometric sequence" simplices ($a_i = q^{d-i}$ for all i). In the all-ones case, we find that their local h^*-polynomials are either constant or nearly so, as exemplified by Figure 5.1. We characterize the all-ones simplices with unimodal local h^*-polynomials. For the geometric sequence simplices, we find that their local h^*-polynomials have a pronounced unimodal behavior, as exemplified by Figure 5.2. We prove that all the geometric sequence simplices have unimodal local h^*-polynomials, and further show that they do not have the integer decomposition property and thus do not fall within the scope of prior work [1, 31, 93].

To study the set of all one-row Hermite normal form simplices, there are several ways to proceed. One approach, which we do not take in this work, is to fix N and vary the values $a_1, \ldots, a_{d-1}$. Limited experimental data suggests that this process often results in simplices that do not have the integer decomposition property, but which do have unimodal local h^*-polynomials. For example, in a random sample of 100 simplices of dimension 11 with $N = 505$, none of these simplices have the integer decomposition property but all of them have unimodal local h^*-polynomials. The distributions for these polynomials are plotted in

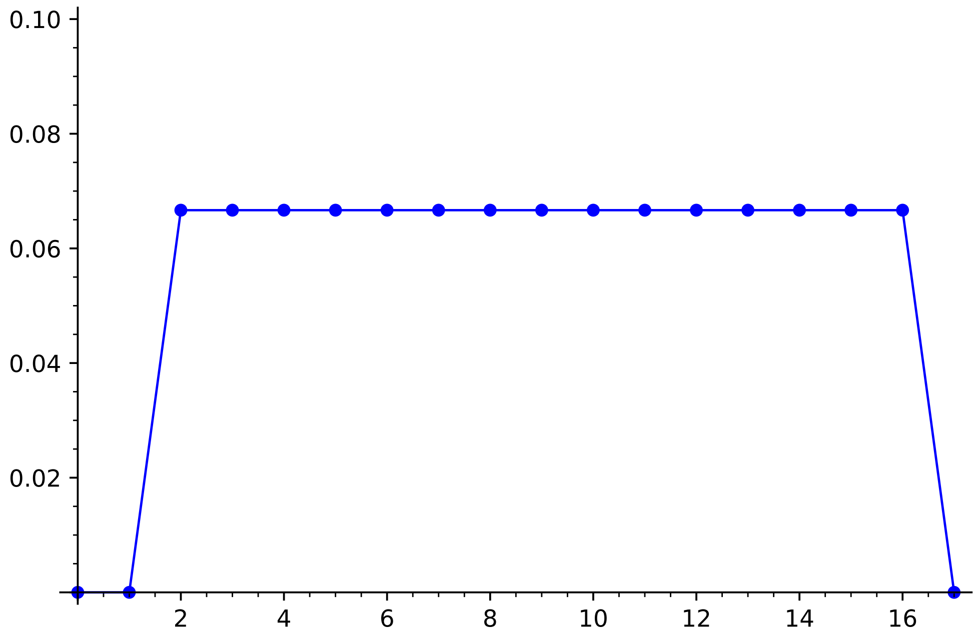


Figure 5.1: The distribution of the coefficients of $B(S; z)/B(S; 1)$ for the one-row Hermite normal form simplex S with non-trivial row $(1, 1, \ldots, 1, 331)$ in dimension 17. Note that $331 = 22 \cdot 15 + 1$ and $B(S; z) = 22 \cdot \sum_{i=2}^{16} z^i$.

Figure 5.3, which illustrates that the integer decomposition property does not fully explain local h^*-unimodality.

Another approach, which is the focus of our main result, is to consider one-row Hermite normal form simplices where all parameters a_i are fixed and analyze the asymptotic behavior of $B(S; z)/B(S; 1)$ as the normalized volume $N \to \infty$. In Section 5.5, we show that this asymptotic behavior is determined by the distribution of the coefficients for a relatively small normalized volume value. This demonstrates that in the one-row case, the arithmetical structure of the off-diagonal elements of a Hermite normal form matrix has a stronger influence on the local h^*-polynomial than the normalized volume. Our main result is Theorem 5.5.3, which can be summarized as follows:

Theorem 5.1.1. *Fix $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\geqslant 1}$ and let $M := \operatorname{lcm}(a_1, \ldots, a_{d-1}, -1 + \sum_{i=1}^{d-1} a_i)$. Consider the one row Hermite normal form simplices for these values of $a_1, \ldots, a_{d-1}$ and varying N. As $N \to \infty$, the distributions for the local h^*-polynomials converges to the distribution with $N = M + 1$.*

The results in this chapter are rather technical and thus the details of our definitions and arguments vary somewhat from the exposition provided so far. Since the proofs in Section 5.5 rely on several technical lemmas regarding floor and ceiling functions, these lemmas are

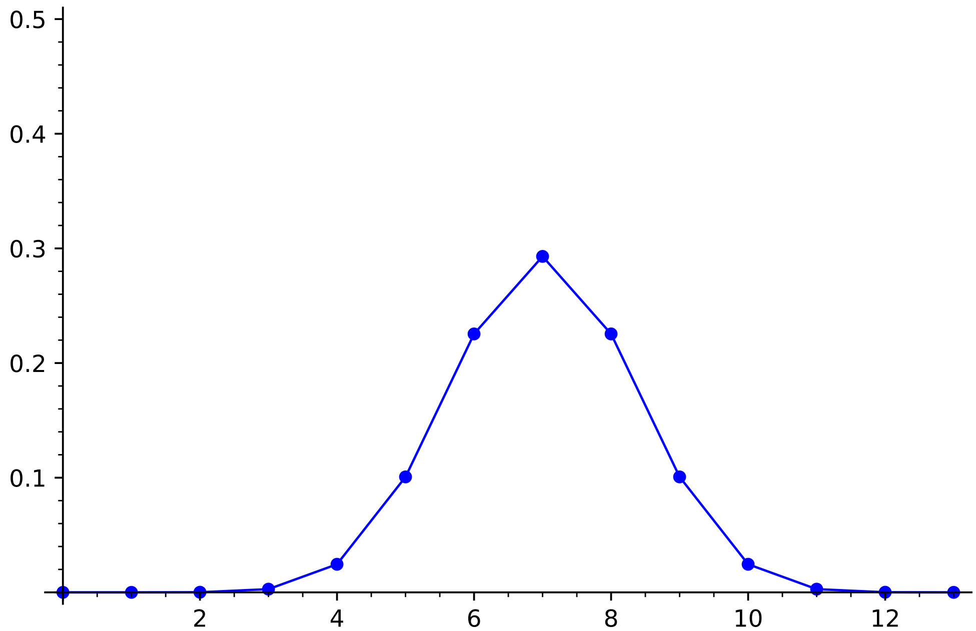


Figure 5.2: The distribution of the coefficients of $B(S;z)/B(S;1)$ for the one-row Hermite normal form simplex S with non-trivial row $(3^{11}, 3^{10}, 3^9, \ldots, 3^2, 3, 1, 3^{12})$ in dimension 13.

given in Section 5.4. We conclude the chapter in Section 5.6 with several further questions motivated by this work. All computations in this chapter were done using SageMath [100].

5.2 Properties of local h^*-polynomials

Hermite normal form simplices

It is well-known, that for every lattice simplex S, there exists a unique matrix H representing the vertices of S up to unimodular equivalence [103]. The matrix H, called the *Hermite normal form* of S, is described as follows.

Theorem 5.2.1. *Every d-dimensional lattice simplex in $\mathbb{R}^d$ is unimodularly equivalent to a simplex S arising as the convex hull of the rows of a $(d+1) \times d$ integer matrix H of the following form:*

- $a_{0,i} = 0$ *for* $i = 1, \ldots, d$
- $a_{i,i} \in \mathbb{Z}_{\geqslant 1}$ *for* $i = 1, \ldots, d$
- $0 \leqslant a_{i,j} < a_{i,i}$ *when* $j < i$ *for* $i = 1, \ldots, d$

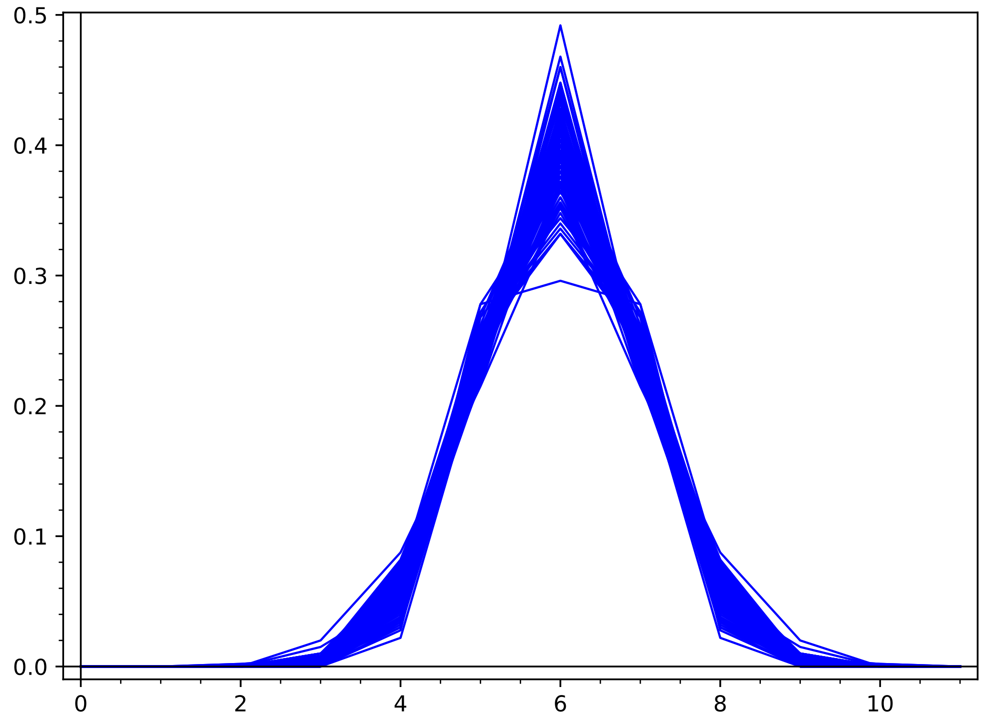


Figure 5.3: The distributions for 100 local h^*-polynomials of 11-dimensional one-row Hermite normal form simplices with normalized volume $N = 505$. None of these simplices have the Integer Decomposition Property.

- $a_{i,j} = 0$ *for* $j > i$ *for* $i = 1, \ldots, d$.

Note that we are using the convention that the *rows of* H *denote the vertices of* S, whereas some authors use column form. We denote by A the matrix H with an initial column of ones appended. This is equivalent to lifting the configuration of rows of H to height one in a space one dimension higher.

Example 5.2.2. An example of a Hermite normal form H of a simplex and the extended matrix A:

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 8 & 8 & 2 & 6 & 9 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 2 & 5 & 0 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 8 & 8 & 2 & 6 & 9 \end{bmatrix}.$$

Our focus in this work is on the following special class of Hermite normal form matrices, which have been the subject of extensive recent study [37, 38, 39, 40, 41, 50, 72, 74, 75, 76, 85, 95, 105, 106, 127].

Definition 5.2.3. For a simplex S in Hermite normal form, if $a_{i,i} = 1$ for $i = 1, \ldots, d-1$, then we say that H is of *one-row Hermite normal form.* For a one-row Hermite normal form matrix, we will often refer to this matrix by the values in the $(d+1)$-st row,

$$(a_1, a_2, \ldots, a_{d-1}, N),$$

where we write a_i for $a_{d,i}$ and N for $a_{d,d}$ as shown in (5.2).

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & a_1 & a_2 & a_3 & \cdots & a_{d-2} & a_{d-1} & N \end{bmatrix} \tag{5.2}$$

Example 5.2.4. An example of a simplex S in one-row Hermite normal form is given by

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \end{bmatrix}.$$

The following proposition shows that it is straightforward to determine the volume of S from the matrix A.

Proposition 5.2.5. *The normalized volume of a simplex given in Hermite normal form is* $\prod_{i=1}^{d} a_{i,i}$*, for* $a_{i,i} \in A$*. Thus, for a one-row Hermite normal form simplex, the normalized volume is* N*.*

h^*- and local h^*-polynomials

The arithmetic structure of the lattice points in the cone over S is captured by the lattice generated by A^{-1}. For a one-row Hermite normal form simplex with non-trivial row given by $(a_1, a_2, \ldots, a_{d-1}, N)$, it is straightforward to find A^{-1} as follows:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{-1+\sum_{i=1}^{d-1} a_i}{N} & -\frac{a_1}{N} & -\frac{a_2}{N} & -\frac{a_3}{N} & \cdots & -\frac{a_{d-2}}{N} & -\frac{a_{d-1}}{N} & \frac{1}{N} \end{bmatrix} \tag{5.3}$$

We study the lattice generated by the rows of A^{-1} using the following polynomials.

Definition 5.2.6. Define the *lattice for S* as $\Lambda = \Lambda(S) := \mathbb{Z}^{d+1} \cdot A^{-1}$ and the *parallelepiped group for S* as $\Gamma = \Gamma(S) := \Lambda(S)/\mathbb{Z}^{d+1}$. Define the *age* of an element $x = (x_0, \ldots, x_d) \in \Lambda(S)$ as $\text{age}(x) := \sum_{i=0}^{d} \{x_i\}$. Here $\{x\}$ denotes the *fractional part* of a real number $x \in \mathbb{R}$, i.e., $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Note that when S is a one-row Hermite normal form simplex, the elements of Γ are parameterized for $0 \leqslant \ell \leqslant N-1$ by

$$\left(\frac{\ell(-1+\sum_{i=1}^{d-1} a_i)}{N}, -\frac{\ell a_1}{N}, -\frac{\ell a_2}{N}, -\frac{\ell a_3}{N}, \ldots, -\frac{\ell a_{d-2}}{N}, -\frac{\ell a_{d-1}}{N}, \frac{\ell}{N} \right).$$

We will frequently use this parameterization of Γ throughout this work, and in particular to compute the following two polynomials associated to S.

Definition 5.2.7. Define the *h^*-polynomial* of S as

$$h^*(S; z) = \sum_{i=0}^{d} {h_i}^* z^i := \sum_{x \in \Gamma} z^{\text{age}(x)}$$

and the *local h^*-polynomial* for S as

$$B(S; z) = \sum_{i=1}^{d} b_i z^i := \sum_{x \in \Gamma \cap (0,1)^{d+1}} z^{\text{age}(x)}.$$

The coefficients of $h^*(S; z)$ and $B(S; z)$ form vectors called the *h^*-vector* and *local h^*-vector* of S, respectively. Local h^*-polynomials of simplices are also known as *box polynomials*.

Example 5.2.8. The simplex in Example 5.2.4 has $B(S; z) = z^2 + z^3 + z^4$ and $h^*(S; z) = 1 + 2z^2 + 2z^3 + z^4$.

It is straightforward to verify the following proposition.

Proposition 5.2.9. *For any lattice simplex S, the polynomial $B(S;z)/z$ has palindromic, i.e., symmetric, coefficients.*

The h^*-polynomial is a main object of study in Ehrhart theory for lattice polytopes. Local h^*-polynomials for simplices are a special case of local h^*-polynomials for arbitrary lattice polytopes. For a detailed discussion of the history of local h^*-polynomials in this general setting, see [31, Section 2.4]. It is common that in expositions of Ehrhart theory [24], these polynomials are defined using the fundamental parallelepiped of the cone over S, defined as follows. Let $\{r_0, \ldots, r_d\}$ denote the rows of the extended matrix A for S. Then the fundamental parallelepiped is

$$\Pi_S := \left\{ \sum_i \lambda_i r_i : 0 \leqslant \lambda_i < 1 \right\} . \tag{5.4}$$

Multiplication by A produces a bijection between the lattice $\Lambda(S)$ and $\mathbb{Z}^{d+1}$, and further this takes the half-open cube $[0,1)^{d+1}$ to Π_S. Thus, there is a bijection between the $\Lambda(S)$-points in $[0,1)^{d+1}$ and the $\mathbb{Z}^{d+1}$-points in Π_S. Further, the grading of $\mathbb{Z}^{d+1} \cap \Pi_S$ corresponding to the initial coordinate corresponds to the grading of $\Lambda(S) \cap [0,1)^{d+1}$ by the age function. Thus, the definitions of local-h^*- and h^*-polynomials given above agree with the usual definitions given in terms of the fundamental parallelepiped.

In the case of a one-row Hermite normal form simplex, certain number-theoretic conditions imply that the local h^*-polynomial and h^*-polynomial are essentially the same, as follows.

Theorem 5.2.10. *Let $M = \operatorname{lcm}(a_1, a_2, \ldots, a_{d-1}, -1 + \sum_i a_i)$, and let S be the simplex with non-trivial row $(a_1, \ldots, a_{d-1}, N)$. If $\gcd(M, N) = 1$, then*

$$h^*(S;z) = 1 + B(S;z) .$$

Proof. For a one-row Hermite normal form simplex, the parallelepiped group is generated by

$$\left(\frac{(-1 + \sum_{i=1}^{d-1} a_i)}{N}, -\frac{a_1}{N}, -\frac{a_2}{N}, -\frac{a_3}{N}, \ldots, -\frac{a_{d-2}}{N}, -\frac{a_{d-1}}{N}, \frac{1}{N} \right) .$$

The condition $h^*(S;z) = 1 + B(S;z)$ occurs if and only if every non-zero point in the parallelepiped group has all non-zero coordinates, since this is the criteria for all non-zero points to be in the open box $(0,1)^{d+1}$. This holds if and only if

$$\left(\frac{\ell(-1 + \sum_{i=1}^{d-1} a_i)}{N}, -\frac{\ell a_1}{N}, -\frac{\ell a_2}{N}, -\frac{\ell a_3}{N}, \ldots, -\frac{\ell a_{d-2}}{N}, -\frac{\ell a_{d-1}}{N}, \frac{\ell}{N} \right)$$

has all non-integer coordinates for every $\ell = 1, 2, \ldots, N-1$. This is equivalent to the gcd condition in the theorem. □

Remark 6. Note that the numerical conditions in Theorem 5.2.10 are identical to those identified by Hibi, Higashitani, and Li [72] as corresponding to "shifted symmetric" h^*-vectors. The symmetry that they observed is precisely the symmetry of the local h^*-polynomial from Proposition 5.2.9.

For each polynomial $f(z)$ with non-negative coefficients, we can associate to f a discrete probability distribution.

Definition 5.2.11. Given $f(z) \in \mathbb{R}_{\geqslant 0}[z]$, we define the *distribution associated to* f to be the distribution defined by the coefficients of $f(z)/f(1)$.

We will be interested in these distributions in the case of the h^*- and local h^*-polynomials for S.

Example 5.2.12. The local h^*-and h^*-polynomials in Example 5.2.8 yield the distributions

$$(0, 0, 1/3, 1/3, 1/3)$$

and

$$(1/6, 0, 1/3, 1/3, 1/6)\,,$$

respectively.

Relationship to Stapledon decompositions

Stapledon established [123] a decomposition of the h^*-polynomial of a lattice polytope into its boundary h^*-polynomial and its "b-polynomial." Therefore, the b-polynomial of a simplex S captures information about the interior of the cone over S, as does the local h^*-polynomial. In this section, we explore the relationship between these two polynomials. Decomposition of polynomial invariants has been widely studied and, in many settings, builds on the Stapledon decomposition (e.g., [19, 23, 35, 78, 82]).

The h^*-polynomial of an arbitrary d-dimensional lattice polytope $P \in \mathbb{R}^d$ is

$$h^*(P; z) := (1 - z)^{d+1} \left(1 + \sum_{n \geqslant 1} |nP \cap \mathbb{Z}^d| z^n \right),$$

and can be computed via the h^*-polynomials of the simplices in a triangulation of P. In particular, if $\mathcal{T}$ is a disjoint triangulation of P into d-dimensional half-open simplices, then

$$h^*(P; z) = \sum_{S \in \mathcal{T}} h^*(S; z).$$

Here, the h^*-polynomial of a half-open simplex is computed similarly to that of a closed simplex, but with a modification to the fundamental parallelepiped. Suppose $S \subseteq \mathbb{R}^d$ is a d-dimensional simplex with vertices $v_0, \dots, v_d \in \mathbb{Z}^d$, where the facets opposite the first r vertices are missing, for some $0 \leqslant r \leqslant d+1$. That is, suppose

$$S = \{\lambda_0 v_0 + \cdots + \lambda_d v_d : \lambda_0 + \cdots + \lambda_d = 1, \lambda_0, \dots, \lambda_{r-1} > 0, \lambda_r, \dots, \lambda_d \geqslant 0\}.$$

We define the fundamental parallelepiped of S to be

$$\Pi_S = \left\{ \sum_i \lambda_i r_i : 0 < \lambda_0, \dots, \lambda_{r-1} \leqslant 1, 0 \leqslant \lambda_r, \dots, \lambda_d < 1 \right\} \tag{5.5}$$

where $r_0, \dots, r_d$ are the rows of the extended matrix A for S (that is, $r_i = (1, v_i)$), and it then holds that

$$h^*(P; z) = \sum_{S \in \mathcal{T}} h^*(S; z) = \sum_{S \in \mathcal{T}} \sum_{\substack{(x_0, \dots, x_d) \\ \in \Pi_S \cap \mathbb{Z}^{d+1}}} z^{x_0}.$$

Theorem 5.2.13 (Stapledon, [123])**.** *If $P \in \mathbb{R}^d$ is a lattice polytope and $\ell \geqslant 1$ is the smallest integer such that $\ell P^\circ \cap \mathbb{Z}^d$ is nonempty, then there exist unique polynomials $a(P; z)$ and $b(P; z)$ such that*

$$(1 + z + \cdots + z^{\ell-1}) h^*(P; z) = a(P; z) + z^\ell b(P; z),$$

where $a(P; z)$ and $b(P; z)$ are palindromic polynomials with nonnegative integer coefficients.

Moreover, $a(P; z)$ is actually equal to the h^*-polynomial of the *boundary* ∂P of P, defined in [8] as

$$h^*(\partial P; z) := \frac{h^*(P; z) - h^*(P^\circ; z)}{1 - z}.$$

The h^*-polynomial of the boundary of P can also be expressed in terms of h^*-polynomials of half-open simplices. There is a disjoint triangulation T of ∂P into half-open $(d-1)$-dimensional lattice simplices (where exactly one simplex is closed), and for such a triangulation,

$$h^*(\partial P; z) = \sum_{S' \in T} h^*(S'; z).$$

In the case where the polytope is a simplex S and where S contains an interior lattice point (i.e. $\ell = 1$),

$$b(S; z) = \frac{h^*(S; z) - h^*(\partial S; z)}{z}$$

captures information about the interior of the cone over S. The local h^*-polynomial $B(S; z)$ also captures information about the interior of this cone, and in this case, we are able to directly compare these two polynomials:

Proposition 5.2.14. *Let S be a lattice simplex with an interior lattice point, and let*

$$b(S;z) = \frac{h^*(S;z) - h^*(\partial S;z)}{z}$$

be its b-polynomial (as in Stapledon's decomposition). Then $zb(z)$ is bounded above by the local h^*-polynomial $B(S;z)$ of S, coefficient-wise.

Proof. Use the facets of S as the maximal faces in a triangulation T of ∂S. Arrange the facets of S into disjoint half-open simplices such that exactly one simplex is closed (e.g., using a visibility construction as in [25, Chapter 5] or [8]). Let

$$\Pi_T = \bigcup_{S' \in T} \Pi_{S'}$$

be the union over the fundamental parallelepipeds of the half-open simplices in T. Then using (5.5) we obtain

$$\begin{aligned} zb(S;z) = h^*(S;z) - h^*(\partial S,z) &= \left(\sum_{z \in \Pi_S \cap \mathbb{Z}^{d+1}} z^{x_0} \right) - \left(\sum_{x \in \Pi_T \cap \mathbb{Z}^{d+1}} z^{x_0} \right) \\ &= \left(\sum_{z \in (\Pi_S \setminus \Pi_T) \cap \mathbb{Z}^{d+1}} z^{x_0} \right) - \left(\sum_{z \in (\Pi_T \setminus \Pi_S) \cap \mathbb{Z}^{d+1}} z^{x_0} \right). \end{aligned}$$

Note that Π_T depends on the specific choice of half-open simplices in T, but the polynomial

$$\sum_{x \in \Pi_T \cap \mathbb{Z}^{d+1}} z^{x_0}$$

is independent of this choice. Observe that Π_T contains all points in Π_S where some coefficient λ_i as in (5.4) is 0 (namely, all coefficients of the row vectors of A corresponding to vertices that are opposite a boundary facet containing the point) and Π_T contains no points in Π_S where each coefficient is positive. Therefore, the lattice points in $\Pi_S \setminus \Pi_T$ are precisely those with no coefficient equal to 0, that is, the points counted by the local h^*-polynomial $B(S;z)$. Thus,

$$zb(S;z) = B(S;z) - \left(\sum_{z \in (\Pi_T \setminus \Pi_S) \cap \mathbb{Z}^{d+1}} z^{x_0} \right).$$

In particular, $zb(S;z)$ is upper-bounded by $B(S;z)$, coefficient-wise. □

We remark that the same result does not follow if S does not contain an interior point, as now there is a factor in front of $h^*(S;z)$ in Stapledon's decomposition:

$$z^\ell b(S;z) = (1 + z + \ldots + z^{\ell-1})h^*(S;z) - h^*(\partial S;z).$$

For example, $S = \operatorname{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,4)\}$ has local h^*-polynomial

$$B(S;z) = 4z^2$$

and b-polynomial

$$b(S;z) = 2 + 2z \text{ (and } z^\ell b(S;z) = 2z^2 + 2z^3\text{)}.$$

We conclude this subsection with an observation about the implication of Stapledon's decomposition for *shifted symmetric polytopes*, as explored by Hibi, Higashitani, and Li in [72].

Definition 5.2.15. The h^*-polynomial $h^*_P(z) = h^*_0 + h^*_1 z + \dots h^*_d z^d$ of a lattice polytope P is *shifted symmetric* if $h^*_i = h^*_{d+1-i}$ for $i = 1, \dots, d$.

Hibi, Higashitani, and Li [72] proved that for a one-row Hermite normal form simplex S satisfying the conditions of Theorem 5.2.10, the h^*-polynomial of S is shifted symmetric. The following proposition due to Higashitani uses Stapledon decompositions to show that any lattice polytope with a shifted symmetric h^*-polynomial is a simplex with a similar geometric property to those in Theorem 5.2.10.

Proposition 5.2.16 (Higashitani [77]). *If P is a lattice polytope such that $h^*_P(z)$ is shifted symmetric, then P is a simplex with unimodular facets. Thus, $h^*(P;z) = 1 + B(P;z)$.*

5.3 Two illustrative examples

In this section, we provide a deep exploration of two examples of one-row Hermite normal form simplices that illustrate important phenomena. In particular, we focus on the cases where we take the non-trivial row to be $(1, \dots, 1, N)$ and $(q^{k-1}, \dots, q, 1, q^k)$. These results are independently interesting, and they also serve to motivate the results in Section 5.5.

Non-trivial row $(1, \dots, 1, N)$

In this subsection, let $N \in \mathbb{Z}$ be a fixed integer with $N > 1$. We consider a d-simplex S in one-row Hermite normal form with last row $(1, \dots, 1, N)$. By (5.3) we have that

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & N \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{(d-2)}{N} & -\frac{1}{N} & -\frac{1}{N} & -\frac{1}{N} & \cdots & -\frac{1}{N} & -\frac{1}{N} & \frac{1}{N} \end{bmatrix}.$$

Hence, the parallelepiped group of S is given by

$$\Gamma = \left(\mathbb{Z}^{d+1} + \mathbb{Z}\left(\frac{(d-2)}{N}, -\frac{1}{N}, \ldots, -\frac{1}{N}, \frac{1}{N}\right)^t\right)/\mathbb{Z}^{d+1}$$

Recall from above that the h^**-vector* $h^* = (h_0^*, h_1^*, \ldots, h_d^*)$ of S is given by

$$h_i^* = \#\{x \in \Gamma : \operatorname{age}(x) = i\}\,.$$

We can compute the local h^*-polynomial $B(S;z) = \sum_{i=0}^d b_i z^i$ as follows, parameterizing the points in Γ using $0 \leqslant k \leqslant N-1$:

$$\begin{aligned} b_i &= \left|\left\{x = \left(\frac{-k(d-2)}{N}, \frac{k}{N}, \ldots, \frac{k}{N}, -\frac{k}{N}\right) + \mathbb{Z}^{d+1} \in \Gamma : \operatorname{age}(x) = i, \frac{-k(d-2)}{N} \notin \mathbb{Z}\right\}\right| \\ &= \left|\left\{k = 1, \ldots, N-1 : i = 1 + \left\{\frac{-k(d-2)}{N}\right\} + \frac{k(d-2)}{N}, \left\{\frac{-k(d-2)}{N}\right\} \neq 0\right\}\right| \\ &= \left|\left\{k = 1, \ldots, N-1 : i = 1 - \left\lfloor\frac{-k(d-2)}{N}\right\rfloor, N \nmid k(d-2)\right\}\right| \\ &= \left|\left\{k = 1, \ldots, N-1 : i = 1 + \left\lceil\frac{k(d-2)}{N}\right\rceil, N \nmid k(d-2)\right\}\right|. \end{aligned}$$

Our goal in this subsection is to prove the following.

Theorem 5.3.1. *For the d-simplex S in one-row Hermite normal form with final row $(1, \ldots, 1, N)$, the local h^*-polynomial of S has every non-zero coefficient in $\{q, q+1\}$ where $N = (d-2)q + r$ for some $0 \leqslant r \leqslant d-3$. Further, the local h^*-polynomial is unimodal if and only if $r \in \{0, 1, 2, d-3\}$ and it has constant coefficients if and only if $r \in \{0, 1\}$.*

Example 5.3.2. Figure 5.1 displays the local h^*-polynomial distribution for the all ones non-trivial row with $d = 17$ and $N = 22 \cdot 15 + 1$. As claimed by Theorem 5.3.1, since $r = 1$, the local h^*-polynomial is both constant and unimodal.

The remainder of this subsection is devoted to a proof of Theorem 5.3.1. To simplify the notation, let $a, N \in \mathbb{N}$ be two natural numbers. Note that in our above setup, we have $a = d-2$. Throughout the remainder of this section, we will divide N by a with a remainder, i.e., $N = a \cdot q + r$ for some $q, r \in \mathbb{N}$ with $0 \leqslant r < a$. We define a vector $\alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{Z}^a$ as follows

$$\alpha_i = \#\left\{k = 1, \ldots, N-1 : \left\lceil\frac{ak}{N}\right\rceil = i, N \nmid ak\right\}.$$

Thus, when $a = d-2$, we have that the local h^*-polynomial coefficient is

$$b_i = \alpha_{i-1}$$

and hence, the coefficients of the local h^*-polynomial are a shift of the vector α. We will focus on the case that a and N are coprime and hence assume from now on until the end of the section that $\gcd(a, N) = 1$. This assumption simplifies the situation in that we do not need to consider the condition $N \nmid ak$.

Let us begin with an example to illustrate our general strategy.

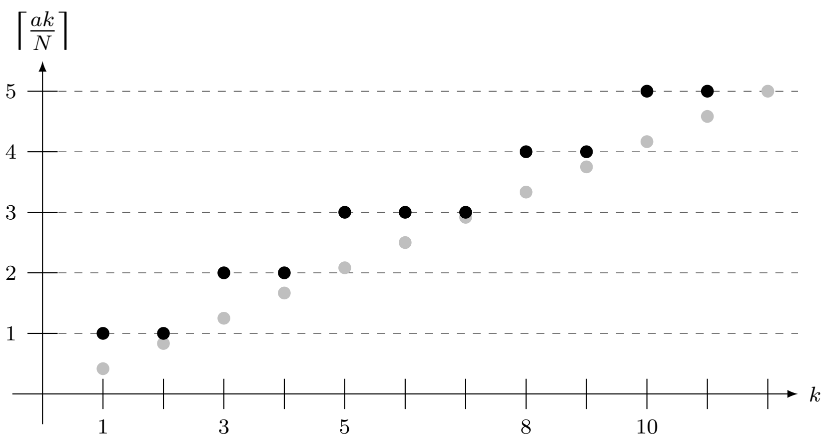


Figure 5.4: The distribution of the values $\lceil ak/N \rceil$ for $k = 1, \dots, N$

Example 5.3.3. Suppose that $a = 5$ and $N = 12$. Thus, we have $N = q \cdot a + r$ where $q = 2$ and $r = 2$. From the definition, we see that the α_i's depend on the distribution of the values $\lceil ak/N \rceil$ for $k = 1, \dots, N-1$. Figure 5.4 illustrates this distribution with respect to the multiples k. From Figure 5.4, we deduce $\alpha = (2, 2, 3, 2, 2)$. Notice that the fraction $12 \cdot a/N$ does not contribute to the α-vector since $N = 12$ and thus $N \mid 12 \cdot a$. Furthermore, observe that fractions ak/N contributing to the same entry α_i (for some fixed $i = 1, \dots, a$) appear consecutively and the possible values for α_i are either $q = 2$ or $q + 1 = 3$.

Our proof strategy will rely on studying the values i for which $\alpha_i = q$ (respectively $\alpha_i = q + 1$). Fix $i = 0, \dots, a-1$ and consider the smallest $k = 0, \dots, N-1$ such that $\lfloor ak/N \rfloor = i$. Notice that for our example with $a = 5$ and $N = 12$, we have that

$$ak = 5k \equiv \overline{-i \cdot 2} \pmod{12} \equiv \overline{-ir} \pmod{N}$$

where $\overline{\cdot}$ denotes smallest residue mod a. Furthermore, observe that for our example we have

$$\alpha_{i+1} = \begin{cases} q+1 & \text{if } 1 \leqslant a - \overline{ir} \leqslant r-1 \\ q & \text{if } r \leqslant a - \overline{ir} \leqslant a. \end{cases}$$

This is true in general and will be proven in Proposition 5.3.4.

Since a and N are coprime, it follows that $\gcd(a, r) = 1$, and thus $\mathbb{Z}/a\mathbb{Z} = \{\overline{ir} : i = 0, \dots a-1\}$. As $r - 1 = 2 - 1 = 1$, it follows that exactly one entry of the vector α is equal to $q+1$, and thus α is unimodal. With similar arguments, one can identify exactly when the vector α is unimodal in general. This will be shown in Theorem 5.3.6.

Proposition 5.3.4. *Let $i = 0, \dots, a-1$ and let $k = 0, \dots, N-1$ be the smallest multiple with $\lfloor ak/N \rfloor = i$. Then $ak \equiv \overline{-ir} \pmod N$ and*

$$\alpha_{i+1} = \begin{cases} q+1 & \text{if } 1 \leqslant a - \overline{ir} \leqslant r-1 \\ q & \text{if } r \leqslant a - \overline{ir} \leqslant a \end{cases}$$

where $\overline{\cdot}$ denotes smallest residue mod a.

Proof. The equation $ak \equiv \overline{-ir} \pmod N$ can be straightforwardly verified for $i = 0$. Suppose $i = 1, \dots, a-1$. Then the smallest $k = 0, \dots, N-1$ with $\lfloor ak/N \rfloor = i$ satisfies $a(k-1) < iN \leqslant ak$ and thus $0 < iN - a(k-1) < a$. Observe that $iN = iaq + ir = a(k-1) + x$ for some $x \in \mathbb{Z}$ with $x \equiv ir \pmod a$. With the condition $0 < iN - a(k-1) < a$, we deduce that $x = \overline{ir}$. Thus, $iN + (a - \overline{ir}) = a(k-1) + \overline{ir} + (a - \overline{ir}) = ak$, which yields $ak \equiv \overline{-ir} \pmod N$.

Notice that for $i = 1, \dots, a-1$, we have that

$$\alpha_{i+1} = \#\left\{ l \in \mathbb{Z} : i + \frac{a - \overline{ir}}{N} \leqslant i + \frac{al + (a - \overline{ir})}{N} < i+1 \right\}.$$

Furthermore, for $i = 0$ it follows that $\alpha_1 = \{l \in \mathbb{Z} : a/N \leqslant (al+a)/N < 1\}$. Hence, the second statement of our proposition is equivalent to

$$\#\left\{ l \in \mathbb{Z} : i + \frac{a - \overline{ir}}{N} \leqslant i + \frac{al + (a - \overline{ir})}{N} < i+1 \right\} = \begin{cases} q+1 & \text{if } \ 1 \leqslant a - \overline{ir} \leqslant r-1 \\ q & \text{if } \ r \leqslant a - \overline{ir} \leqslant a. \end{cases}$$

We begin with the case $1 \leqslant a - \overline{ir} \leqslant r-1$. Clearly, we have that

$$i + \frac{a - \overline{ir}}{N} \leqslant \underbrace{i + \frac{a - \overline{ir}}{N}, i + \frac{a + (a - \overline{ir})}{N}, \dots, i + \frac{a \cdot q + (a - \overline{ir})}{N}}_{q+1 \text{ many}} < i+1.$$

Furthermore, notice that $i + (a \cdot (q+1) + (a - \overline{ir}))/N > i+1$. Hence, the first case follows.

Next suppose that $r \leqslant a - \overline{ir} \leqslant a-1$. We have that

$$i + \frac{a - \overline{ir}}{N} \leqslant \underbrace{i + \frac{a - \overline{ir}}{N}, i + \frac{a + (a - \overline{ir})}{N}, \dots, i + \frac{a \cdot (q-1) + (a - \overline{ir})}{N}}_{q \text{ many}} < i+1.$$

Furthermore, notice that $i + (a \cdot q + (a - \overline{ir}))/N > i+1$. Hence, the second case follows. ☐

Corollary 5.3.5. *For a and N such that $\gcd(a, N) = 1$, we have $\alpha_i \in \{q, q+1\}$.*

Note that the coefficients of the local h^*-polynomial are a simple shift of the α-vector. Given Corollary 5.3.5, we can deduce that the local h^*-polynomial coefficients are unimodal precisely when the values of $q+1$ occur in a consecutive sequence in the vector. The following theorem characterizes when this occurs.

Theorem 5.3.6. *Given a and N with $N = aq + r$, the α-vector is unimodal if and only if $r \in \{0, 1, 2, a-1\}$. Furthermore, the α-vector is constant if and only if $r \in \{0, 1\}$.*

Proof. Since N and a are coprime, so are a and r, thus $\{\overline{ir} | i \in [a-1]\} = [a-1]$. Thus, the α-vector has exactly $r-1$ entries equal to $q+1$. It is enough to show that these $r-1$ entries equal to $q+1$ cannot all be consecutive. Equivalently, that the values of $a - \overline{ir}$ between 1 and $r-1$ cannot be consecutive in the sequence

$$a, a - \overline{r}, a - \overline{2r}, \ldots, a - \overline{(a-1)r}. \tag{5.6}$$

We note that an element of this sequence is obtained from the previous one by subtracting r if the result of the subtraction is positive and by adding $a - r$ otherwise.

To prove the "if" part of the theorem, we note that the α-vector in the case $r = 0$ is trivial. In the case $r = 1$ by the above formula all entries are equal to q. In the case $r = 2$ it has exactly one $q+1$ entry, not the first, and thus in all these cases the α-vector is unimodal. For $r = a-1$, the sequence (5.6) is simply $a, 1, 2, \ldots, a-1$, so there are exactly two q entries in the α-vector, the first and the last, making the vector unimodal.

Now for the "only if" direction: let $3 \leqslant r \leqslant a-2$. There are $r-1$ entries equal to $q+1$, so at least two and at most $a-3$ (i.e., at least three q entries). It is enough to show that these $r-1$ entries equal to $q+1$ cannot all be consecutive. That is, that in the sequence (5.6) the values between 1 and $r-1$ cannot all be consecutive. We distinguish two cases:

- If $r > a-r$ (i.e. $r > \frac{a}{2}$); suppose $a = k(a-r)+t$, with $0 \leqslant t < a-r$. Then the sequence (5.6) begins with $a, a-r, \ldots, k(a-r)$, where the second entry already corresponds to a $q+1$ entry in the α-vector, since $a - r < r$. Since $r > \frac{a}{2}$ and $a - r \geqslant 2$, it follows that $r \cdot (a-r) > a$, and thus $k \leqslant r-1$. Furthermore, notice that $k \cdot (a-r) = a - t > r$. We conclude that $\alpha_2 = q+1$, $\alpha_k = q$, and $k \leqslant r-1$. Hence, there exists $i > k$ with $\alpha_i = q+1$, which means that the sequence is not unimodal.

- If $r < a-r$ (i.e., $r < \frac{a}{2}$), we proceed similarly. Let us write $a = \ell \cdot r + s$ with $0 \leqslant s < r$. Then the sequence (5.6) begins with $a, a-r, \ldots, a-\ell r$, where the last entry satisfies $a - \ell r = s < r$, i.e., $\alpha_\ell = q+1$. In other words, the sequence (5.6) starts with consecutive q's followed by one or more $(q+1)$'s. Notice that $(\ell+1)r > \ell r + s = a$, so that $\overline{(\ell+1)r} = (\ell+1)r - a = r - s$. Hence, $a - \overline{(\ell+1)r} = a - r + s$. Since $r < \frac{a}{2}$, it follows that $a - \overline{(\ell+1)r} = a - r + s > \frac{a}{2} > r$, i.e., $\alpha_{\ell+1} = q$. Together with the symmetry $\alpha_i = \alpha_{a+1-i}$ for all $i = 1, \ldots, a$, we conclude this sequence is not unimodal.

□

Using our result for the case where $\gcd(a, N) = 1$, we can now extend our result to the general setting, which implies Theorem 5.3.1.

Corollary 5.3.7. *Let $a, N \in \mathbb{N}$ and let $b = \gcd(a, N) \neq 1$. Divide N by a with remainder, i.e., write $N = aq + r$ for some $q, r \in \mathbb{N}$ with $0 \leqslant r < a$. Then the α-vector is unimodal if and only if $r = 0$ or $r = b$.*

Proof. Let us write $a = a' \cdot b$ and $N = N' \cdot b$ for some $a', N' \in \mathbb{N}$. Firstly, observe that α is a b-fold concatenation of the vector α' that corresponds to the numbers a' and N'. Therefore, α is unimodal if and only if α' is constant, and hence it remains to check when this is the case.

Let us divide N' by a' with remainder, i.e., $N' = a' \cdot q' + r'$ for some $q', r' \in \mathbb{N}$ with $0 \leqslant r' < a'$. By Theorem 5.3.6, α' is constant if and only if $r' \in \{0, 1\}$. With $N = b \cdot N' = q' \cdot (a'b) + r' \cdot b = q' \cdot a + (r' \cdot b)$, the statement follows. □

Non-trivial row $\left(q^{k-1}, \dots, q, 1, q^k\right)$

In this subsection, let $k \in \mathbb{Z}$ be a fixed positive integer and $q \in \mathbb{Z}_{\geqslant 2}$. Consider the one-row Hermite normal form simplex S with final row $\left(q^{k-1}, \dots, q, 1, q^k\right)$. We study this as a test case for one-row Hermite normal form simplices where the final row has "well-spaced" entries.

To begin, in the following proposition we show that any S of this form does not have the Integer Decomposition Property (IDP). Recall that a simplex S has the IDP if every lattice point in the non-negative cone generated by the colums of A is a sum of lattice points in the cone having last coordinate equal to 1. It is known that polytopes with a unimodular triangulation have the IDP, and thus these simplices do not have a regular unimodular triangulation. This is noteworthy because, as we will see in Theorem 5.3.9 below, the local h^*-polynomial of every such S is unimodal, as illustrated in Figure 5.2. This family of simplices demonstrates that the existence of regular unimodular triangulations is not the only source of unimodularity for local h^*-polynomials of one-row Hermite normal form simplices.

Proposition 5.3.8. *The simplex S does not have the IDP.*

Proof. We begin by observing that

$$1 < \frac{1}{q} + \cdots + \frac{1}{q^k} + \left(1 - \frac{1}{q^k}\right) < 2.$$

Let $\lambda \in (0, 1)$ such that $\lambda + \sum_{i=1}^{k} \frac{1}{q^i} + \left(1 - \frac{1}{q^k}\right) = 2$, so that the convex combination

$$b = (q^{k-1}, \dots, q, 1, q^k - 1) = \lambda v_0 + \sum_{i=1}^{k} \frac{1}{q^i} v_i + \left(1 - \frac{1}{q^k}\right) v_{k+1} \in 2S.$$

Assume towards a contradiction that $b = b' + b''$ for two elements $b', b'' \in S \cap \mathbb{Z}^{k+1}$. Let us write $b' = (b'_0, \ldots, b'_{k+1})$ and $b'' = (b''_0, \ldots, b''_{k+1})$ for $b'_i, b''_j \in \mathbb{Z}_{\geqslant 0}$. Then the k-th coordinate of b' or b'' has to vanish, say $b'_k = 0$. We can write $b' = \sum_{i=0}^{k+1} \mu'_i v_i$, with $\mu'_i \in [0,1]$ such that $\sum_{i=0}^{k+1} \mu'_i = 1$. Since $0 = \mu'_{k+1} + \mu'_k$, it follows that $\mu'_{k+1} = \mu'_k = 0$. Thus, $b''_{k+1} = q^k - 1$.

Let us write $b'' = \sum_{i=0}^{k+1} \mu''_i v_i$ for $\mu''_i \in [0,1]$ with $\sum_{i=0}^{k+1} \mu''_i = 1$. Since b is in the interior of $2S$ and b' is on the boundary of S, it follows that b'' is in the interior of S, or in other words $\mu''_i \in (0,1)$. From $b''_{k+1} = q^k - 1$ we conclude that $\mu'_{k+1} = \frac{q^k-1}{q^k}$. Observe that the entry b''_{k+1} completely determines the vector b''. Indeed, for $i = 1, \ldots, k$, since $b''_i = \mu''_{k+1} q^{k-i} + \mu''_i$ is an integer and $\mu''_i \in (0,1)$, we have that $\mu''_i = \frac{1}{q^i}$, which also fixes b''_i. Note that $\sum_{i=1}^{k+1} \mu''_i > 1$ - a contradiction. Hence, the element $b \in 2S \cap \mathbb{Z}^{k+1}$ is not a sum of two elements in $S \cap \mathbb{Z}^{k+1}$. $\square$

Theorem 5.3.9. *For any integers $q \geqslant 2$ and $k \geqslant 2$, the simplex S with non-trivial row*

$$\left(1, q^{k-1}, \ldots, q, 1, q^k\right)$$

has a unimodal local h^*-polynomial.

The remainder of this subsection is devoted to a proof of Theorem 5.3.9. We begin by computing the parallelepiped group. Recall that this can be straightforwardly done, by lifting the vertices $v_0, v_1, \ldots, v_{k+1}$ on height one and use these vectors as the rows of a matrix A. Let us denote the rows of the inverse matrix A^{-1} by $r_1(A), \ldots, r_{k+2}(A)$. Then the parallelepiped group is $\Gamma = \mathbb{Z}^{k+2} + \sum_{i=1}^{k+2} \mathbb{Z} r_i(A)$. In our case, it is straightforward to verify that the inverse matrix A^{-1} is given by:

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ \frac{q^{k-2}+q^{k-3}+\cdots+q+1}{q^{k-1}} & -\frac{1}{q} & -\frac{1}{q^2} & -\frac{1}{q^3} & \cdots & -\frac{1}{q^{k-1}} & -\frac{1}{q^k} & \frac{1}{q^k} \end{pmatrix}$$

Hence, the parallelepiped group of S is given by

$$\Gamma = \left(\mathbb{Z}^{k+2} + \mathbb{Z}\left((-1) \cdot \frac{q^{k-2} + q^{k-3} + \cdots + q + 1}{q^{k-1}}, \frac{1}{q}, \frac{1}{q^2}, \ldots, \frac{1}{q^k}, (-1) \cdot \frac{1}{q^k}\right)\right) / \mathbb{Z}^{k+2}.$$

Hence, the coefficients of the local h^*-polynomial $B(S; z) = \sum_{i=0}^{k+2} b_i z^i$ satisfy:

$$b_i = \left|\left\{x = (x_1, \ldots, x_{k+2}) + \mathbb{Z}^{k+2} \in \Gamma : \operatorname{age}(x) = i, x_i \notin \mathbb{Z} \text{ for all } i\right\}\right|$$

In particular, we see from this that $\sum_{i=0}^{d} b_i = (q-1)\cdot q^{k-1}$. In order to show that the local h^*-polynomial coefficients form a unimodal sequence, it suffices to show that the sequence $(\delta'_1, \delta'_2, \ldots, \delta'_{k-1})$ is a unimodal sequence where

$$\begin{aligned}\delta'_i &= \left|\left\{\ell = 1,\ldots,q^k : q \nmid \ell, \sum_{i=1}^{k-1}\left\{\frac{\ell}{q^i}\right\} + \left\{-\sum_{i=1}^{k-1}\frac{\ell}{q^i}\right\} = i\right\}\right| \\ &= \left|\left\{\ell = 1,\ldots,q^k, : q \nmid \ell, \left\lceil \sum_{i=1}^{k-1}\left\{\frac{\ell}{q^i}\right\}\right\rceil = i\right\}\right|.\end{aligned}$$

We can further simplify this by defining the sequence $(\delta_1, \ldots, \delta_{k-1})$ where

$$\delta_i = \left|\left\{\ell = 1,\ldots,q^{k-1} : q \nmid \ell, \underbrace{\left\lceil \sum_{i=1}^{k-1}\left\{\frac{\ell}{q^i}\right\}\right\rceil}_{=:\mathrm{age}(\ell)} = i\right\}\right|$$

and observing that $\delta'_i = q\cdot\delta_i$, which is obtained by writing ℓ in base q. Thus, unimodality of the local h^*-polynomial is equivalent to unimodality of the sequence δ_i.

Let us next express $\ell = 1,\ldots,q^{k-1}$ in base q, i.e., we write $\ell = \sum_{i=0}^{k-2} c_i q^i$ for natural numbers $c_i \in \{0,1,\ldots,q-1\}$. Then $\mathrm{age}(\ell)$ can be rewritten as follows

$$\begin{aligned}\mathrm{age}(\ell) &= \left\lceil \sum_{i=1}^{k-1}\left\{\frac{\ell}{q^i}\right\}\right\rceil = \left\lceil \sum_{i=1}^{k-1}\left\{\frac{\sum_{j=0}^{k-2} c_j q^j}{q^i}\right\}\right\rceil = \left\lceil \sum_{i=1}^{k-1}\left\{\sum_{j=0}^{i-1} c_j q^{j-i}\right\}\right\rceil \\ &= \left\lceil \sum_{i=1}^{k-1}\sum_{j=0}^{i-1} c_j q^{j-i}\right\rceil = \left\lceil \sum_{j=0}^{k-2} c_j \sum_{i=1}^{k-1-j} q^{-i}\right\rceil = \left\lceil \sum_{j=0}^{k-2} c_j \frac{1-q^{-k+1+j}}{q-1}\right\rceil \\ &= \left\lceil \frac{c_0+c_1+\cdots+c_{k-2}}{q-1} - \frac{\ell}{q^{k-1}(q-1)}\right\rceil.\end{aligned}$$

The following proposition considerably simplifies the computation of $\mathrm{age}(\ell)$.

Proposition 5.3.10. *For all $\ell = 1,\ldots,q^{k-1}$, we have that*

$$\mathrm{age}(\ell) = \left\lceil \frac{c_0+c_1+\cdots+c_{k-2}}{q-1}\right\rceil,$$

where $\ell = c_0 + c_1 q + c_2 q^2 + \cdots + c_{k-2}q^{k-2}$ is the expression of ℓ to base q.

Proof. We begin by observing that (recall that $q \geqslant 2$)

$$\frac{\ell}{q^{k-1}(q-1)} < \frac{1}{q-1} \leqslant 1.$$

Let us write the fraction $(c_0 + \cdots + c_{k-2})/(q-1) = r + \delta/(q-1)$ for some natural numbers $r, \delta \in \mathbb{Z}$ where $\delta = 0, 1, \ldots, q-2$. We distinguish two cases.

Suppose $\delta = 0$. Since $\ell/(q^{k-1}(q-1)) < 1$, it follows that

$$\left\lceil \frac{c_0 + \cdots + c_{k-2}}{q-1} \right\rceil = \lceil r \rceil = \left\lceil r - \underbrace{\frac{\ell}{q^{k-1}(q-1)}}_{<1} \right\rceil = \operatorname{age}(\ell)..$$

Next suppose that $\delta > 0$. Since $\ell/(q^{k-1}(q-1)) < 1/(q-1)$, it follows that

$$\left\lceil \frac{c_0 + \cdots + c_{k-2}}{q-1} \right\rceil = \left\lceil r + \frac{\delta}{q-1} \right\rceil = \left\lceil r + \frac{\delta}{q-1} - \underbrace{\frac{\ell}{q^{k-1}(q-1)}}_{<1/(q-1)} \right\rceil = \operatorname{age}(\ell). \qquad \square$$

With the previous proposition, it straightforwardly follows that

$$\delta_i = \left| \bigsqcup_{j=(i-1)(q-1)+1}^{i(q-1)} \left\{ (c_0, \ldots, c_{k-2}) \in [1, q-1] \times [0, q-1]^{k-2} \cap \mathbb{Z}^{k-1} \colon \sum_{m=0}^{k-2} c_m = j \right\} \right|.$$

Clearly, the number of tuples $(c_0, \ldots, c_{k-2}) \in [1, q-1] \times [0, q-1]^{k-2} \cap \mathbb{Z}^{k-1}$ which sum up to the integer j coincides with the j-th coefficient of the following polynomial

$$f = (t + t^2 + \cdots + t^{q-1}) \cdot (1 + t + t^2 + \cdots + t^{q-1})^{k-2} = \sum_{m=0}^{(k-1)(q-1)} \alpha_m t^m \in \mathbb{Z}[t].$$

We get $\delta_i = \sum_{j=(i-1)(q-1)+1}^{i(q-1)} \alpha_j$. Since the sequence $(\delta_1, \ldots, \delta_{k-1})$ is palindromic, it suffices to show that $\delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_{\lfloor (k-1)/2 \rfloor}$. Since the polynomial f is a product of palindromic unimodal polynomials, it is palindromic and unimodal too. In particular, the coefficients α_m are monotonically increasing until the middle (i.e., up to the coefficient $\alpha_{\lfloor (k-1)(q-1)/2 \rfloor}$).

Since the sequence $(\delta_1, \ldots, \delta_{k-1})$ can be identified with disjoint successive partial sums of the sequence $(\alpha_0, \ldots, \alpha_{\lfloor (k-1)/2 \rfloor (q-1)})$ which is monotonically increasing as $\lfloor (k-1)/2 \rfloor (q-1) \leqslant \lfloor (k-1)(q-1)/2 \rfloor$, it follows that $(\delta_1, \ldots, \delta_{k-1})$ is unimodal. This completes the proof of Theorem 5.3.9.

5.4 Number theoretical results on floor and ceiling functions

In this section we prove a variety of results regarding floor and ceiling functions that we will use in Section 5.5.

Proposition 5.4.1. *Let k, m, n be positive integers and set $\varepsilon := \frac{1}{n}$. Then we have*

$$\left\lfloor \frac{k \cdot i + 1}{k \cdot m + \varepsilon} \right\rfloor = \left\lfloor \frac{k \cdot i + \delta + 1}{k \cdot m + \varepsilon} \right\rfloor$$

and

$$\left\lceil \frac{k \cdot i + 1}{k \cdot m + \varepsilon} \right\rceil = \left\lceil \frac{k \cdot i + \delta + 1}{k \cdot m + \varepsilon} \right\rceil$$

for all $i = 0, 1, \ldots, mn - 1$ and all $\delta = 0, 1, \ldots, k - 1$.

To prove this statement we consider the function

$$f \colon \mathbb{R}^2 \to \mathbb{R}; (x, y) \mapsto f(x, y) := k \cdot x + y + 1.$$

We first notice that if we fix $x_0 \in \mathbb{R}$ then the function $f_{x_0}(y) = f(x_0, y)$ is strictly monotonically increasing on $\mathbb{R}$. Similarly, we have for fixed $y_0 \in \mathbb{R}$ that the function $f_{y_0}(x) = f(x, y_0)$ is strictly monotonically increasing on $\mathbb{R}$.

Lemma 5.4.2. *For each $\ell \in [0, n - 1]$ we have that*

$$\ell \cdot (km + \varepsilon) < f(x, y) < (\ell + 1) \cdot (km + \varepsilon)$$

where $(x, y) \in [\ell m, (\ell + 1)m - 1] \times [0, k - 1]$.

Proof. Let us consider the restriction of $f(x, y)$ to the region $[\ell m, (\ell + 1)m - 1] \times [0, k - 1]$ which we denote by $g(x, y)$.

Notice that the domain of $g(x, y)$ is a rectangle with left-bottom vertex $(\ell m, 0)$ and top-right vertex $((\ell+1)m-1, k-1)$. With the above observed monotonicity of $f(x, y)$, it suffices to show that

$$\ell \cdot (km + \varepsilon) < f(\ell \cdot m, 0) \qquad \text{and} \qquad f((\ell + 1) \cdot m - 1, k - 1) < (\ell + 1) \cdot (km + \varepsilon)\,.$$

Although it's straightforward to verify these two inequalities, we want to briefly discuss them here to show where the assumptions are needed. Clearly, we have

$$\ell \cdot (km + \varepsilon) = k \cdot \ell \cdot m + \underbrace{\ell \cdot \varepsilon}_{<1} < k \cdot \ell \cdot m + 1 = f(\ell \cdot m, 0)\,.$$

Noticed that the assumption $\ell < n$ implies that $\ell \cdot \varepsilon < 1$. The verification of the second inequality is straightforward (and doesn't need the assumption $\ell < n$). □

Proof of Proposition 5.4.1. Notice that we can partition the integers in the interval $[0, mn - 1]$ into the following pairwise disjoint subsets:

$$[0, mn - 1] \cap \mathbb{Z} = \bigcup_{\ell=0}^{n-1} ([\ell m, (\ell + 1)m - 1] \cap \mathbb{Z})\,.$$

Let $i \in [\ell m, (\ell+1)m-1]$ for some $\ell = 0,1,\ldots,n-1$ and let $\delta = 0,1,\ldots,k-1$. By Lemma 5.4.2, we have that

$$\ell \cdot (k \cdot m + \varepsilon) < k \cdot i + \delta + 1 < (\ell+1) \cdot (k \cdot m + \varepsilon),$$

and thus

$$\ell < \frac{k \cdot i + \delta + 1}{k \cdot m + \varepsilon} < \ell + 1\,.$$

It follows that

$$\left\lfloor \frac{k \cdot i + \delta + 1}{k \cdot m + \varepsilon} \right\rfloor = \ell \qquad \text{and} \qquad \left\lceil \frac{k \cdot i + \delta + 1}{k \cdot m + \varepsilon} \right\rceil = \ell + 1$$

for each $\delta = 0,1,\ldots,k-1$. □

The following statement is straightforward to show:

Lemma 5.4.3. *For $\ell \in [0, n-1]$, we have that*

$$\ell \cdot (m + \varepsilon) < x + 1 < (\ell+1) \cdot (m + \varepsilon)$$

for all $x \in [\ell m, (\ell+1)m-1]$.

Corollary 5.4.4. *Let k, m, n be positive integers and set $\varepsilon := \frac{1}{n}$. Then we have*

$$\left\lfloor \frac{k \cdot q + 1}{k \cdot m + \varepsilon} \right\rfloor = \left\lfloor \frac{q+1}{m + \varepsilon} \right\rfloor \qquad \textit{and} \qquad \left\lceil \frac{k \cdot q + 1}{k \cdot m + \varepsilon} \right\rceil = \left\lceil \frac{q+1}{m + \varepsilon} \right\rceil$$

for all $q = 0,1,\ldots,mn-1$.

Proof. We again partition the integers in the interval $[0, mn-1]$ into disjoint subsets:

$$([0, mn-1] \cap \mathbb{Z}) = \bigcup_{\ell=0}^{n-1} ([\ell m, (\ell+1)m-1] \cap \mathbb{Z})\,.$$

Suppose $q \in [\ell m, (\ell+1)m-1]$ for some $\ell = 0,1,\ldots,n-1$. By the Lemmas 5.4.2 and 5.4.3, we have that

$$\ell \cdot (km + \varepsilon) < k \cdot q + 1 < (\ell+1) \cdot (km + \varepsilon) \quad \text{and} \quad \ell \cdot (m + \varepsilon) < q + 1 < (\ell+1) \cdot (m + \varepsilon)\,.$$

Hence, we get that

$$\ell < \frac{k \cdot q + 1}{km + \varepsilon} < \ell + 1 \quad \text{and} \quad \ell < \frac{q+1}{m + \varepsilon} < \ell + 1\,.$$

From this the statement follows straightforwardly. □

Lemma 5.4.5. *Let k, n, m be positive integers, and let $i = 0, 1, \ldots, mn-1$ and $\delta = 0, 1, \ldots, k-1$. If $\delta \geqslant (n-1)m+1$ and $k \geqslant n$, then*

$$\left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \left(1+\frac{1}{k}\right) < \frac{ki+\delta+1}{km+\frac{1}{n}}. \tag{5.7}$$

Proof. Let $t = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, m-1$. Observe that since $\delta \geqslant (n-1)m+1$, we have

$$0 < nk(ks + \underbrace{\delta - tm}_{\geqslant 1}).$$

Since $0 \leqslant t \leqslant n-1$ and $k \geqslant n$, it follows that

$$0 < nk(ks+\delta+1-tm) - t(k+1) = nk(ks+\delta-tm) + \underbrace{nk - t(k+1)}_{\geqslant 1}.$$

Expanding and rearranging gives us

$$tkmn + t < nk^2s + nk\delta + nk - tk$$

from which it follows that

$$\frac{t}{k} < \frac{kns+n\delta+n-t}{kmn+1}.$$

Adding t to both sides gives

$$t\left(1+\frac{1}{k}\right) < t + \frac{kns+n\delta+n-t}{kmn+1}. \tag{5.8}$$

Because $0 \leqslant s \leqslant m-1$, $0 \leqslant \delta \leqslant k-1$, and $0 \leqslant t \leqslant n-1$, we have that

$$0 \leqslant \frac{kns+n\delta+n-t}{kmn+1} < 1 \quad \Rightarrow \quad t = \left\lfloor t + \frac{kns+n\delta+n-t}{kmn+1} \right\rfloor.$$

By substituting this expression for t in the left-hand-side of Equation (5.8), we obtain

$$\left\lfloor t + \frac{kns+n\delta+n-t}{kmn+1} \right\rfloor \left(1+\frac{1}{k}\right) < t + \frac{kns+n\delta+n-t}{kmn+1}.$$

Straightforward algebraic calculations then yield

$$\left\lfloor \frac{k(tm+s)+\delta+1}{km+\frac{1}{n}} \right\rfloor \left(1+\frac{1}{k}\right) < \frac{k(tm+s)+\delta+1}{km+\frac{1}{n}}.$$

Note that with our range of $t \in [0, n-1]$ and $s \in [0, m-1]$, the values of $i = tm+s$ exactly parameterize $i \in [0, mn-1]$, and thus our proof is complete. □

Theorem 5.4.6. *Let k, m, n, r be positive integers with $1 \leqslant r \leqslant mn$. For $k \geqslant mn$, $\delta = mn, mn+1, \ldots, k-1$, and $i = 0, 1, \ldots, mn-1$, the following hold:*

$$\left\lfloor \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rfloor = \left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \tag{5.9}$$

and

$$\left\lceil \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rceil = \left\lceil \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rceil . \tag{5.10}$$

Proof. Note that $1 \leqslant r$ implies that

$$\left\lfloor \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rfloor \leqslant \left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor$$

and

$$\left\lceil \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rceil \leqslant \left\lceil \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rceil .$$

Thus, our result will follow once we prove that the following holds for all $\delta \geqslant mn$:

$$\left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \leqslant \frac{ki+\delta+1}{\left(km+\frac{r}{n}\right)} \leqslant \frac{ki+\delta+1}{\left(km+\frac{1}{n}\right)} \leqslant \left\lceil \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rceil . \tag{5.11}$$

Note that the middle inequality above is a consequence of $1 \leqslant r$. It is a strict inequality if $1 < r$. To show the left-most inequality in (5.11), use $\delta \geqslant mn$ and apply Lemma 5.4.5 to obtain

$$\begin{aligned}\left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \left(\frac{km+\frac{r}{n}}{km+\frac{1}{n}}\right) = \left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \left(1+\frac{r-1}{knm+1}\right) &< \left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor \left(1+\frac{1}{k}\right) \\ &< \frac{ki+\delta+1}{km+\frac{1}{n}} .\end{aligned}$$

From this, we conclude that if $\delta \geqslant mn$, the following inequality holds:

$$\left\lfloor \frac{ki+\delta+1}{km+\frac{1}{n}} \right\rfloor < \frac{ki+\delta+1}{\left(km+\frac{r}{n}\right)} .$$

To show the right-most inequality in (5.11), we assume $\delta \geqslant mn$ and consider two cases. We write $i = tm + s$ where $t = 0, 1, \ldots, n-1$ and $s = 0, 1, \ldots, m-1$.

Our first case is when $s = 0$. Since $\delta \leqslant k-1$ and $m \geqslant 1$, we have

$$\delta \leqslant km + \frac{t+1}{n} - 1 = \left(km + \frac{1}{n}\right)(t+1) - ktm - 1 .$$

Combining the above inequality with $mn \leqslant \delta \leqslant k-1$ and $0 \leqslant t \leqslant n-1$, we obtain

$$\frac{ki+\delta+1}{km+\frac{1}{n}} = \frac{ktm+\delta+1}{km+\frac{1}{n}} = \frac{ktm+\frac{t}{n}+\delta+\frac{n-t}{n}}{km+\frac{1}{n}} = t + \frac{n\delta+n-t}{kmn+1} < t+1 =$$
$$= \left\lceil t + \underbrace{\frac{\delta+1-tm}{(k+1)m}}_{\in(0,1)} \right\rceil = \left\lceil \frac{ktm+\delta+1}{(k+1)m} \right\rceil \leqslant \left\lceil \frac{ktm+\delta+1}{km+\frac{r}{n}} \right\rceil .$$

Our second case is when $1 \leqslant s \leqslant m-1$. In this case, $\delta \leqslant k-1$ implies that

$$\delta \leqslant k(m-s)-1 \leqslant \frac{t}{n} + \frac{1}{n} + k(m-s) - 1 = \left(km+\frac{1}{n}\right)(t+1) - kmt - ks - 1$$

which implies since $\delta \geqslant mn$ and $i \leqslant mn-1$ that

$$\frac{ki+\delta+1}{km+\frac{1}{n}} = \frac{k(tm+s)+\delta+1}{km+\frac{1}{n}} < t+1 = \left\lceil \frac{i}{m} \right\rceil <$$
$$< \left\lceil \frac{(k+1)i-i+\delta+1}{km+m} \right\rceil \leqslant \left\lceil \frac{ki+\delta+1}{km+\frac{r}{n}} \right\rceil .$$

This shows that (5.11) holds for all $mn \leqslant \delta \leqslant k-1$. □

5.5 Asymptotic properties of local h^*-polynomials

In this section we consider one-row Hermite normal form simplices as in (5.2) with fixed $a_1, \ldots, a_{d-1}$ and increasing normalized volume N. Specifically, we establish that the general behavior of the local h^*-polynomial for these simplices is determined by the simplex with $N = M+1$, where $M = \operatorname{lcm}(a_1, \ldots, a_{d-1}, -1 + \sum_i a_i)$. The following theorem shows that there is a close connection between the local h^*-polynomials for $N = M+1$ and $N = kM+1$.

Theorem 5.5.1. *Fix $a_1, \ldots, a_{d-1} \in \mathbb{Z}$ and let $M := \operatorname{lcm}(a_1, \ldots, a_{d-1}, \sum_{i=1}^{d-1} a_i - 1)$. Denoting by S_{kM+1} the simplex given in (5.2) with normalized volume $N = k \cdot M + 1$ for some $k \in \mathbb{Z}_{>0}$, we have that*

$$B(S_{kM+1}; z) = k \cdot B(S_{M+1}; z) .$$

Thus, $B(S_{M+1}; z)$ is unimodal if and only if $B(S_{kM+1}; z)$ is unimodal for all positive integers k.

We will first establish Lemma 5.5.2, which will be needed for the proof of Theorem 5.5.1. Let $(a_1, \ldots, a_{d-1}) \in \mathbb{Z}^{d-1}$ be the vector of the non-trivial-row-entries and let $N \in \mathbb{Z}_{>0}$ be the normalized volume of the simplex $S \subseteq \mathbb{R}^d$. Set $M := \operatorname{lcm}\left(a_1, \ldots, a_{d-1}, \sum_{i=1}^{d-1} a_i - 1\right) \in \mathbb{Z}_{\geqslant 0}$.

Suppose $N = N^{(k)} = k \cdot M + 1$ for some $k \in \mathbb{Z}_{>0}$. Recall that the parallelepiped group is generated by the following element:

$$v_0^{(k)} := \left(\frac{1 - \sum_{i=1}^{d-1} a_i}{N}, \frac{a_1}{N}, \frac{a_2}{N}, \ldots, \frac{a_{d-1}}{N}, -\frac{1}{N} \right) \in \Gamma.$$

We observe that each element $x = \ell \cdot v_0^{(k)}$ for $\ell = 1, \ldots, N-1$ has age

$$\begin{aligned} \operatorname{age}(x) &= 1 + \left\lceil \frac{\ell \cdot \left(\left(\sum_{i=1}^{d-1} a_i \right) - 1 \right)}{N} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{\ell \cdot a_i}{N} \right\rfloor \\ &= 1 + \left\lceil \frac{\ell}{N_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{\ell}{N_i} \right\rfloor, \end{aligned}$$

where we have set

$$N_0 := \frac{N}{\sum_{i=1}^{d-1} a_i - 1} \qquad \text{and} \qquad N_i := \frac{N}{a_i} \quad \text{for } i = 1, \ldots, d-1.$$

We notice that N_i for $i = 0, \ldots, d-1$ is of the form

$$N_i = \frac{k \cdot M + 1}{D_i} = k \cdot \frac{M}{D_i} + \varepsilon_i = k \cdot M_i + \varepsilon_i$$

where $M_i := M/D_i$ is an integer and $\varepsilon_i = 1/D_i$. Substituting this into the formula for the age, we get

$$\operatorname{age}(x) = 1 + \left\lceil \frac{\ell}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{\ell}{k \cdot M_i + \varepsilon_i} \right\rfloor.$$

We now divide ℓ by k, say $\ell = k \cdot q + \delta + 1$ for some $\delta = 0, 1, \ldots, k-1$ and $q = 0, 1, \ldots, M-1$. By Proposition 5.4.1, it follows that

$$\begin{aligned} \operatorname{age}\left(\ell \cdot v_0^{(k)} \right) &= 1 + \left\lceil \frac{k \cdot q + \delta + 1}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{k \cdot q + \delta + 1}{k \cdot M_i + \varepsilon_i} \right\rfloor \\ &= 1 + \left\lceil \frac{k \cdot q + 1}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{k \cdot q + 1}{k \cdot M_i + \varepsilon_i} \right\rfloor = \operatorname{age}\left((k \cdot q + 1) \cdot v_0^{(k)} \right). \end{aligned}$$

Here, we used that $q = 0, 1, \ldots, M_i \cdot D_i - 1$ for each $i = 1, \ldots, d-1$ (notice $M_i \cdot D_i = M$). This proves the following statement.

Lemma 5.5.2. *Given the assumptions of Theorem 5.5.1, for all $q = 0, 1, \ldots, M-1$ and all $\delta = 0, 1, \ldots, k-1$, we have*

$$\operatorname{age}\left((k \cdot q + 1) v_0^{(k)} \right) = \operatorname{age}\left((k \cdot q + \delta + 1) v_0^{(k)} \right).$$

Proof of Theorem 5.5.1. We want to compare the box-polynomials of the two simplices where the a_i are fixed and only the normalized volume changes, namely the general case N_k with the 'initial case' N_1. Like before let $\ell = k \cdot q + \delta + 1$ for some $\delta = 0, 1, \ldots, k-1$ and $q = 0, 1, \ldots, M-1$. We continue to use the notation from above and notice that by Lemma 5.5.2

$$\operatorname{age}\Big((k \cdot q + \delta + 1)v_0^{(k)}\Big) = \operatorname{age}\Big((k \cdot q + 1)v_0^{(k)}\Big) = 1 + \left\lceil \frac{k \cdot q + 1}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{k \cdot q + 1}{k \cdot M_i + \varepsilon_i} \right\rfloor$$
$$= 1 + \left\lceil \frac{q + 1}{M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{q + 1}{M_i + \varepsilon_i} \right\rfloor = \operatorname{age}\Big((q+1)v_0^{(1)}\Big)$$

Here, we used Lemma 5.4.4 to justify the step from the first to the second line. From this the theorem follows. □

Next, we extend Theorem 5.5.1 to the setting of arbitrary normalized volume.

Theorem 5.5.3. *Fix $a_1, \ldots, a_{d-1} \in \mathbb{Z}_{\geqslant 1}$ and let $M := \operatorname{lcm}(a_1, \ldots, a_{d-1}, -1 + \sum_{i=1}^{d-1} a_i)$. We continue to use the notation from Section 5.2 and let S_N denote the simplex given there with normalized volume N. Let k be a positive integer and $0 \leqslant r \leqslant M - 1$. Then we have that*

$$\lim_{k \to \infty} B(S_{kM+r}; z)/B(S_{kM+r}; 1) = B(S_{M+1}; z)/B(S_{M+1}; 1).$$

It follows that if $B(S_{M+1}; z)$ is strictly unimodal, i.e., if the coefficients are unimodal with strict increases and strict decreases, then $B(S_{kM+r}; z)$ is strictly unimodal for all sufficiently large k.

Proof. We fix $r = 0, 1, \ldots, M-1$ and set $N(k, r) = kM + r$ and $a_0 = \sum_{i=1}^{d-1} a_i - 1$. Consider the generator of the parallelepiped group from above

$$v_0(k, r) = \left(-\frac{a_0}{N(k,r)}, \frac{a_1}{N(k,r)}, \frac{a_2}{N(k,r)}, \ldots, \frac{a_{d-1}}{N(k,r)}, -\frac{1}{N(k,r)} \right) \in \Gamma.$$

For $\ell = 1, 2, \ldots, N(k,r) - 1$, we write $\ell = k \cdot q + \delta + 1$ for $q = 0, 1, \ldots, M-1$ and $\delta = 0, 1, \ldots, k-1$, $M_i := M/a_i$ and $\varepsilon_i := r/a_i$ for $i = 0, 1, \ldots, d-1$. With this notation, the age of $\ell \cdot v_0(k, r)$ is expressed as follows:

$$\operatorname{age}(\ell \cdot v_0(k,r)) = 1 + \left\lceil \frac{\ell}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{\ell}{k \cdot M_i + \varepsilon_i} \right\rfloor$$
$$= 1 + \left\lceil \frac{k \cdot q + \delta + 1}{k \cdot M_0 + \varepsilon_0} \right\rceil - \sum_{i=1}^{d-1} \left\lfloor \frac{k \cdot q + \delta + 1}{k \cdot M_i + \varepsilon_i} \right\rfloor$$

We observe that Theorem 5.4.6 shows that

$$\left\lfloor \frac{k \cdot q + \delta + 1}{k \cdot M_i + \varepsilon_i} \right\rfloor = \left\lfloor \frac{k \cdot q + \delta + 1}{k \cdot M_i + 1} \right\rfloor \quad \text{and} \quad \left\lceil \frac{k \cdot q + \delta + 1}{k \cdot M_0 + \varepsilon_0} \right\rceil = \left\lceil \frac{k \cdot q + \delta + 1}{k \cdot M_0 + 1} \right\rceil$$

for all $q = 0, 1, \ldots, M-1$ and $\delta = M, M+1, \ldots, k-1$ (provided that k is large enough).

Hence, it follows for all $\ell = k \cdot q + \delta + 1$ with $q = 0, 1, \ldots, M-1$ and $\delta = M, M+1, \ldots, k-1$ that

$$\mathrm{age}(\ell \cdot v_0(k, r)) = \mathrm{age}(\ell \cdot v_0(k, 1)).$$

In particular, among the $kM + r - 1$ values of ℓ parameterizing the parallelepiped group for S_{kM+r}, the number of ℓ-values with ages differing from S_{kM+1} is bounded above by $M^2 + r$, which is constant with respect to k. In addition to this, in the case that $\gcd(M, kM+r) > 1$, it is possible that some of the ℓ-values for S_{kM+r} might not yield points in the open box $(0, 1)^{d+1}$. The maximum possible number of such points is $\sum_{i=0}^{d-1} a_i$, which is again constant with respect to k.

Thus, as $k \to \infty$, the number of $\ell = 1, 2, \ldots, kM + r - 1$ such that $\mathrm{age}(\ell \cdot v_0(k, r)) = \mathrm{age}(\ell \cdot v_0(k, 1))$ is at least $kM + r - 1 - (M^2 + r + \sum_i a_i)$. Hence, the fraction of such ℓ-values is

$$\frac{kM + r - 1 - (M^2 + r + \sum_i a_i)}{kM + r - 1}$$

which goes to 1 as $k \to \infty$. Hence, we have that

$$\lim_{k \to \infty} B(S_{kM+r}; z)/B(S_{kM+r}; 1) = \lim_{k \to \infty} B(S_{kM+1}; z)/B(S_{kM+1}; 1) = B(S_{M+1}; z)/B(S_{M+1}; 1).$$

□

For certain values of $kM + r$, Theorem 5.5.3 has implications for h^*-polynomials as well.

Corollary 5.5.4. *Using the notation from Theorem 5.5.3, if* $\gcd(M, r) = 1$*, then*

$$\lim_{k \to \infty} h^*(S_{kM+r}; z)/h^*(S_{kM+r}; 1) = B(S_{M+1}; z)/B(S_{M+1}; 1).$$

Thus, if S_{M+1} has a strictly unimodal local h^*-polynomial with a positive linear coefficient, then $h^*(S_{kM+r}; z)$ is unimodal for all sufficiently large k.

Proof. This is a straightforward consequence of Theorems 5.2.10 and 5.5.3. □

5.6 Further questions

Theorem 5.5.3 shows that for a one-row Hermite normal form simplex, if the values of $a_1, \ldots, a_{d-1}$ are fixed, the distribution of the local h^*-polynomial coefficients for arbitrary normalized volumes is to a large extent determined by a single normalized volume of $M + 1$. This suggests several questions leading to directions for further study.

Question 5.6.1. Which sequences $a_1, \ldots, a_{d-1}$ yield a unimodal (or strictly unimodal) local h^*-polynomial for S_{M+1}? How common is it for such a simplex to admit a regular unimodular triangulation?

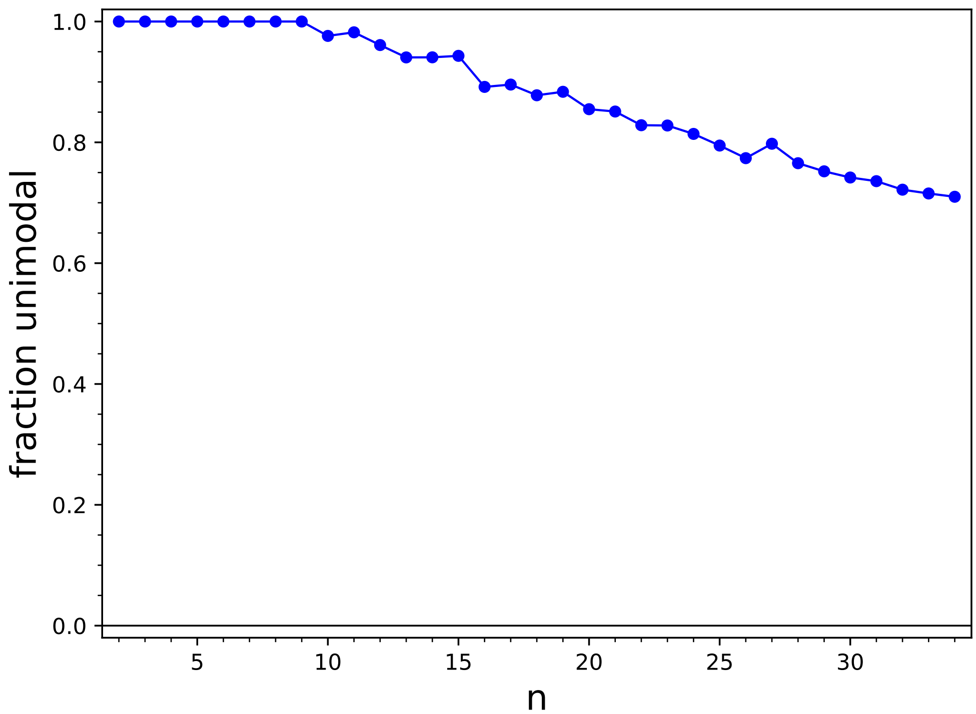


Figure 5.5: For each n, the fraction of unimodal local h^*-polynomials for one-row Hermite normal form simplices S_{M+1} whose one nontrivial row is given by a partition of n

There are several ways to approach Question 5.6.1. One approach is to fix a positive integer n and consider the set of all one row Hermite normal form matrices with final row formed by a partition of n and normalized volume $M+1$. One can ask what fraction of these have unimodal local h^*-polynomials. A plot of these fraction values is given in Figure 5.5 for $n \leqslant 34$. It is not clear what the long-term behavior of this sequence is.

Another approach, which seems more promising, is motivated by the observation that if the values $a_1, \ldots, a_{d-1}$ are distinct integers, it appears that this leads to local h^*-polynomial unimodality. For example, every partition of $n \leqslant 34$ with distinct parts yields a one row Hermite normal form simplex with a unimodal local h^*-polynomial, leading to the following question.

Question 5.6.2. If $(a_1, a_2, \ldots, a_{d-1})$ is a list of $d-1$ distinct positive integers, does the corresponding simplex S_{M+1} have a unimodal local h^*-polynomial?

Experimentally, it seems that having a list of completely distinct positive integers is a stronger condition than needed for unimodality. For example, we generated 121 examples in the following manner. Begin with a constant vector $a_i = k$ and fixed d, then randomly add

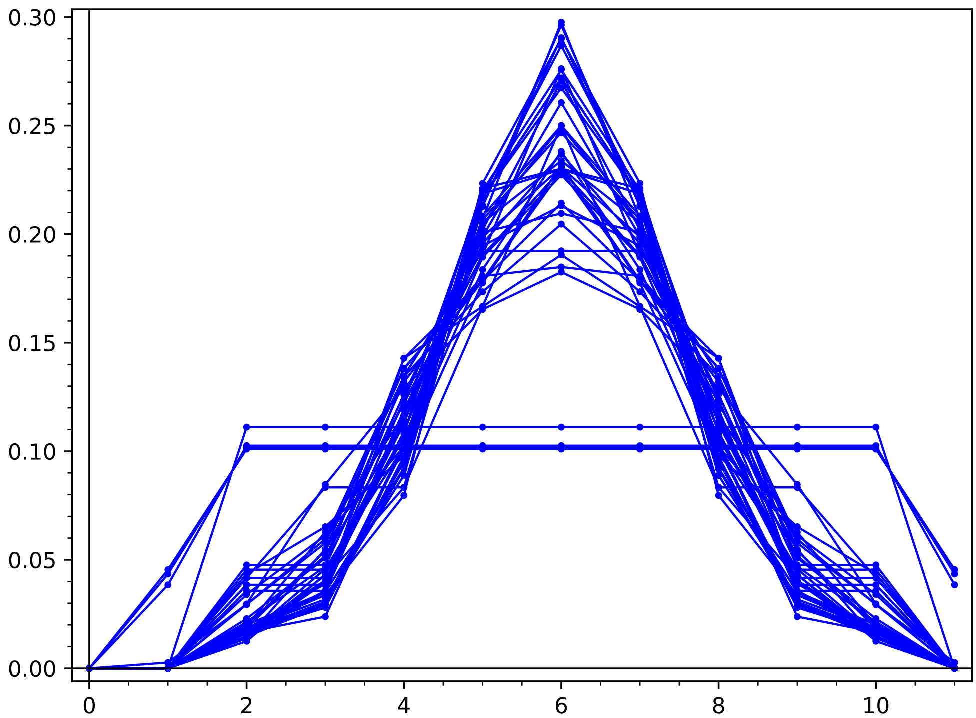


Figure 5.6: The distributions for 40 local h^*-polynomials of 11-dimensional simplices S_{M+1} generated by small random perturbations from constant rows

a value from $\{0, 1, 2, 3, 4\}$ to each entry. We considered the pairs d, k from

$$\{(8,1),(8,4),(8,7),(8,10),(11,1),(11,4),(11,7),(11,10),(14,1),(14,4),(14,7)\}$$

and generated eleven samples each, consisting of the constant row case and ten random perturbations. In all cases, we constructed the simplex S_{M+1}, based on its role in Theorem 5.5.3. Of this sample, 95.87% of the S_{M+1} had unimodal local h^*-polynomials. A plot of the distributions of the unimodal local h^*-polynomials for the 11-dimensional simplices in this sample is given in Figure 5.6. The distributions that are constant or nearly constant arise from the constant row values of $k = 1, 4, 7, 10$. The other distributions arise from the random perturbations, and these all have a pronounced unimodal behavior.

The five non-unimodal examples from our sample are given in Table 5.1, and each of the corresponding a-vectors have a single value appearing in around half of the entries. Note also that unimodality for these examples fails only in the central coefficients. Thus, it is reasonable to conjecture that if the multiplicity of each distinct entry in the row is sufficiently small relative to the dimension, the local h^*-polynomial is unimodal.

These observations lead to the following more general question.

a-vector	local h^*-vector
$[1, 4, 2, 2, 2, 1, 2, 1, 2, 1]$	$[0, 0, 4, 6, 8, 11, 10, 11, 8, 6, 4, 0]$
$[2, 3, 3, 2, 3, 4, 3, 4, 3, 3]$	$[0, 0, 12, 27, 54, 57, 48, 57, 54, 27, 12, 0]$
$[7, 7, 6, 7, 7, 7, 4, 4, 7, 5]$	$[0, 0, 11, 28, 59, 77, 70, 77, 59, 28, 11, 0]$
$[11, 10, 13, 13, 10, 10, 12, 11, 10, 10]$	$[0, 0, 12738, 45859, 139946, 185372, 167390, 185372, 139946, 45859, 12738, 0]$
$[6, 5, 4, 7, 6, 5, 4, 4, 4, 4, 4, 6, 7]$	$[0, 0, 84, 126, 213, 533, 888, 886, 886, 888, 533, 213, 126, 84, 0]$

Table 5.1: The five non-unimodal examples from the sample depicted in Figure 5.6

Question 5.6.3. For fixed d and N, consider the set of all one-row Hermite normal form simplices with normalized volume N and dimension d. What is the "typical" behavior of the local h^*-polynomial distributions for simplices in this set? What is the shape of the space of distributions associated to $B(S; z)$ for all S in this set?

Regarding Question 5.6.3, computational experiments suggest that when the values of $a_1, \ldots, a_{d-1}$ are "sufficiently" distinct, the distribution is more similar to Figure 5.2 than Figure 5.1, as was observed in Figure 5.6. However, it is not clear at this time how to translate these observations into a precise conjecture. Finally, it would be interesting to consider the more general family of Hermite normal form simplices.

Question 5.6.4. Is there an analogue of Theorem 5.5.3 for Hermite normal form simplices with more than one non-trivial row?

Data

The experimental data reported in Section 5.6 is available at:
`https://doi.org/10.17605/OSF.IO/XH58C`.

Chapter 6

q-analog chromatic polynomials

In this chapter, we conclude with an argument in favor of applying Ehrhart theory to other problems in mathematics. We discuss the connection between graph colorings and lattice points in polytopes, and we approach Stanley's conjecture that the chromatic symmetric function distinguishes trees [108] using *q-analog chromatic polynomials* (which we define using Chapoton's *q-analog Ehrhart polynomials* [47]).

6.1 Introduction

The *chromatic polynomial* of a graph $G = (V, E)$,

$$\chi_G(n) \ := \ \#\{c : V \to [n] \ : \ c(v) \neq c(w) \text{ if } vw \in E\} \,,$$

where $[n] := \{1, 2, \ldots, n\}$, is a famous and much-studied enumerative invariant of G. We introduce and study the following refinement: given $\lambda := (\lambda_1, \lambda_2, \ldots, \lambda_{|V|}) \in \mathbb{Z}^V$, let

$$\chi_G^\lambda(q, n) \ := \sum_{\substack{\text{proper colorings} \\ c\,:\,V \to [n]}} q^{\sum_{v \in V} \lambda_v c(v)} .$$

Naturally, $\chi_G^\lambda(1, n) = \chi_G(n)$. On the other hand, consider Stanley's *chromatic symmetric function* [108]

$$X_G(x_1, x_2, \ldots) \ := \sum_{\substack{\text{proper colorings} \\ c:V \to \mathbb{Z}_{>0}}} x_1^{\#c^{-1}(1)} x_2^{\#c^{-1}(2)} \cdots$$

(so that $X_G(1, 1, \ldots, 1, 0, 0, \ldots) = \chi_G(n)$). Its *principal evaluation* (sometimes referred to as the *principal specialization*)

$$X_G(q, q^2, \ldots, q^n, 0, 0, \ldots) \ = \sum_{\substack{\text{proper colorings} \\ c:V \to [n]}} q^{\sum_{v \in V} c(v)} \ = \ \chi_G^{\mathbf{1}}(q, n) \tag{6.1}$$

is the special case $\lambda = \mathbf{1} \in \mathbb{Z}^V$, i.e., λ is a vector all of whose entries are 1. In fact, $\chi_G^{\mathbf{1}}(q,n)$ was also the subject of [86]. We think of $X_G(x_1, x_2, \ldots)$ and $\chi_G^\lambda(q,n)$ as (quite) different generalizations of the chromatic polynomial, which meet in (6.1) and still generalize $\chi_G(n)$.

Our first result says that $\chi_G^\lambda(q,n)$ has a polynomial structure whose coefficients are rational functions in q, in the following sense:

Theorem 6.1.1. *There exists a polynomial* $\widetilde{\chi}_G^\lambda(q,x) \in \mathbb{Q}(q)[x]$ *such that*

$$\widetilde{\chi}_G^\lambda(q,[n]_q) \;=\; \chi_G^\lambda(q,n),$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$.

We thus call $\widetilde{\chi}_G^\lambda(q,x)$ (and sometimes, by a slight abuse of nomenclature, $\chi_G^\lambda(q,n)$) the *q-chromatic polynomial* of G with respect to λ. Our main goal is to initiate the study of this polynomial.

Example 6.1.2. Consider the path P_2 with 2 vertices. The following table shows $\widetilde{\chi}_G^\lambda(q,x)$ and $\chi_G^\lambda(q,n)$ for $\lambda = (1,1)$ and $(1,2)$.

λ	$\widetilde{\chi}_{P_2}^\lambda(q,x)$	$\chi_{P_2}^\lambda(q,n)$
(1,1)	$\frac{2q^2}{q+1}x^2 + \frac{-2q^2}{q+1}x$	$q^2\left(\left(\frac{1-q^n}{1-q}\right)^2 - \left(\frac{1-q^{2n}}{1-q^2}\right)\right)$
(1,2)	$\frac{q^5+q^4-2q^3}{q^3+2q^2+q+1}x^3 + \frac{-q^5+2q^4+5q^3}{q^3+2q^2+q+1}x^2 + \frac{-3q^3}{q^2+q+1}x$	$q^3\left(\frac{1-q^n}{1-q}\,\frac{1-q^{2n}}{1-q^2} - \frac{1-q^{3n}}{1-q^3}\right)$

Note that the chromatic polynomial $\chi_{P_2}(n) = n^2 - n$ appears for $q = 1$.

There are several motivations to study $\chi_G^\lambda(q,n)$ and $\widetilde{\chi}_G^\lambda(q,x)$. Their definition and basic structure mirror Chapoton's study of q-Ehrhart polynomials [47] and, in fact, Theorem 6.1.1 follows from Chapoton's work and the interplay of chromatic and order polynomials, as we will show in Section 6.2 below. On the graph-theoretic side, Stanley famously conjectured that $X_G(x_1, x_2, \ldots)$ distinguishes trees; this conjecture has been checked for trees with $\leqslant 29$ vertices [67], but remains open in general. For recent progress on this conjecture, see, e.g., [4, 5] and the references therein. The literature contains several variations of Stanley's chromatic symmetric function; some references on those different variations include [5, 59, 66, 94, 104]. We particularly point out recent work of Loehr and Warrington [87] who conjectured, more strongly, that the principal evaluation (6.1) distinguishes trees; they confirmed this conjecture for all trees with $\leqslant 17$ vertices. We offer the following further strengthening, which we have checked for all trees with $\leqslant 16$ vertices.

Conjecture 6.1.3. *The leading coefficient of the q-chromatic polynomial* $\widetilde{\chi}_G^{\mathbf{1}}(x)$ *distinguishes trees.*

Section 6.3 of this chapter contains several further structural results for q-chromatic polynomials: deletion–contraction musings (Theorems 6.3.1 and 6.3.2), a combinatorial reciprocity theorem (Theorem 6.3.3), and a formula for $\widetilde{\chi}_G^\lambda(q,x)$ in terms of the Möbius function of the flats of the given graph (Theorem 6.3.5). We mostly concentrate on results on the polynomial $\widetilde{\chi}_G^\lambda(q,x)$; there are further structural results on the enumeration function $\chi_G^\lambda(q,n)$ that are direct consequences of their counterparts on the (weighted) chromatic symmetric function side.

In Section 6.4 we give several formulas for $\widetilde{\chi}_G^{\mathbf{1}}(q,x)$. One of them naturally suggests an analogue of Stanley's P-partitions [117], moving from posets to graphs: we introduce and study G-partitions in Section 6.5 and show that Conjecture 6.1.3 is equivalent to saying that G-partitions distinguish trees.

6.2 q-Ehrhart polynomials

Chapoton [47] introduced a weighted generalization of the Ehrhart polynomial of a *lattice polytope* $P \subseteq \mathbb{R}^d$ (i.e., P is the convex hull of finitely many integer lattice points in $\mathbb{Z}^d$). We briefly sketch this theory and its application to order polytopes, which in turn allows us to exhibit a connection to q-chromatic polynomials.

Let $\lambda : \mathbb{Z}^d \to \mathbb{Z}$ be a linear form that is nonnegative on the vertices of P and distinguishes the vertices of any edge of P, and define

$$\mathrm{ehr}_P^\lambda(q,n) \ := \sum_{m \in nP \cap \mathbb{Z}^d} q^{\lambda(m)}.$$

The classical Ehrhart polynomial [56] is the specialization $\mathrm{ehr}_P^\lambda(1,n)$. Chapoton proved that there is a polynomial $\widetilde{\mathrm{ehr}}_P^\lambda(x) \in \mathbb{Q}(q)[x]$, such that

$$\widetilde{\mathrm{ehr}}_P^\lambda([n]_q) \ = \ \mathrm{ehr}_P^\lambda(q,n)\,. \tag{6.2}$$

We refer to $\widetilde{\mathrm{ehr}}_P^\lambda(x)$ as the *q-Ehrhart polynomial* with respect to λ. We often denote the linear form λ as a vector $(\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, where $\lambda_j = \lambda(e_j)$.

Parallel to the classical case, structural results for $\mathrm{ehr}_P^\lambda(q,n)$ follow from studying the *q-Ehrhart series*

$$\mathrm{Ehr}_P^\lambda(q,z) \ := \sum_{n \geqslant 0} \mathrm{ehr}_P^\lambda(q,n)\, z^n. \tag{6.3}$$

Chapoton [47] showed that (6.3) can be written as a rational function whose denominator consists of factors $1 - q^j z$, where $j = \lambda(v)$ for a vertex v of P. Furthermore, Chapoton proved the reciprocity theorem

$$(-1)^{\dim(P)}\, \widetilde{\mathrm{ehr}}_P^\lambda(q,[-n]_q) \ = \ \widetilde{\mathrm{ehr}}_{P^\circ}^\lambda\left(\tfrac{1}{q},[n]_{\frac{1}{q}}\right), \tag{6.4}$$

where P° denotes the (relative) interior of P and

$$[-n]_q \ := \ \frac{1-q^{-n}}{1-q} \ = \ -\left(q^{-n}+q^{-(n-1)}+\cdots+q^{-1}\right).$$

The case $q=1$ in (6.4) recovers the classical Ehrhart–Macdonald reciprocity theorem [25, 89].

Given a poset $\Pi = ([d], \preceq)$, the *order polytope* $\mathcal{O}(\Pi)$ is the lattice polytope

$$\mathcal{O}(\Pi) \ := \ \left\{(x_1,\ldots,x_d) \in [0,1]^d : \, x_i \leqslant x_j \text{ if } i \preceq j\right\}.$$

Order polytopes were introduced by Stanley [120]; they contain much information about a given poset and have provided important examples in polyhedral geometry.

Since all vertices of $\mathcal{O}(\Pi)$ are 0/1-vectors, $\mathrm{Ehr}^{\lambda}_{\mathcal{O}(\Pi)}(q,z)$ can be written as a rational function with factors $1-q^j z$ in the denominator where j is a sum of some of the entries of λ. Chapoton's genericity condition means for order polytopes that the coordinates of λ are positive, which we assume from now on. We also note the following corollary, which we record for future purposes.

Lemma 6.2.1. *Let* $\Lambda := \lambda_1+\lambda_2+\cdots+\lambda_d$*. The coefficients of* $[\Lambda]_q!\,\widetilde{\mathrm{ehr}}^{\lambda}_{\mathcal{O}(\Pi)}(x)$ *are polynomials in* q*.*

Kim and Stanton [80, Corollary 9.7] gave the following (equivalent) formulas for the case when $\lambda = \mathbf{1}$:

$$\begin{aligned} \mathrm{Ehr}^{\mathbf{1}}_{\mathcal{O}(\Pi)}(q,z) \ &= \ \frac{\displaystyle\sum_{\sigma\in\mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} z^{\mathrm{des}(\sigma)}}{(1-z)(1-qz)\cdots(1-q^d z)} \\ \mathrm{ehr}^{\mathbf{1}}_{\mathcal{O}(\Pi)}(q,n) \ &= \ \sum_{\sigma\in\mathcal{L}(\Pi)} q^{\mathrm{comaj}(\sigma)} \begin{bmatrix} n+d-\mathrm{des}(\sigma) \\ d \end{bmatrix}_q , \end{aligned} \tag{6.5}$$

where $\mathcal{L}(\Pi)$ is the set of linear extensions of Π and, writing a given linear extension σ as a permutation of $[d]$, and [1]

$$\mathrm{Des}(\sigma) := \{j : \, \sigma(j+1) < \sigma(j)\},$$

$$\mathrm{des}(\sigma) := |\,\mathrm{Des}(\sigma)|, \text{ and}$$

$$\mathrm{comaj}(\sigma) := \sum_{j\in\mathrm{Des}(\sigma)} (d-j).$$

[1]Here we fix a natural labeling of Π, i.e., an order-preserving bijection $\Pi \to [d]$. The permutation corresponding to a given linear extension σ can be read off from this labeling. Unfortunately, there are two different (and conflicting) definitions of the comajor index in the literature: the one we use here, and the sum of the ascent positions.

See [25, Chapter 6] for details on the interplay of linear extensions of a poset, their descent statistics, and the arithmetic of order polytopes.

Given a graph G, let $\mathcal{A}(G)$ denote the set of acyclic orientations of G; each acyclic orientation ϱ naturally induces a poset, which we denote Π_ϱ. There is a well-known connection (essentially going back to [109]) between the chromatic polynomial of a given graph G and the Ehrhart polynomials of the order polytopes of the acyclic orientations of G. In the language of q-chromatic polynomials and q-Ehrhart polynomials, it reads as follows.

Lemma 6.2.2. *The q-chromatic polynomial with respect to λ equals*

$$\chi_G^\lambda(q,n) \;=\; \sum_{\varrho \in \mathcal{A}(G)} \operatorname{ehr}^\lambda_{\mathcal{O}(\Pi_\varrho)^\circ}(q, n+1)\,.$$

Proof. We follow the philosophy of inside-out polytopes [27]. Let $d = |V|$. We may interpret each n-coloring of the vertices of G is a lattice point in the $(n+1)$st dilate of the open unit cube $(0,1)^d$ (where the jth coordinate is the color of vertex j). Furthermore, every proper n-coloring of $[d]$ is a lattice point that is not contained in the graphical hyperplane arrangement

$$\mathcal{H}_G \;:=\; \{x_i = x_j \,:\, ij \in E\} \tag{6.6}$$

(see, for example, Figure 6.1). The regions of $(0,1)^d \backslash \mathcal{H}_G$ are precisely the open order polytopes $\mathcal{O}(\Pi_\varrho)^\circ$ for ϱ an acyclic orientation of G. That is, each proper coloring c of G induces an acyclic orientation ϱ_c of G, where the edge ij is oriented from i to j if $c(i) < c(j)$ and from j to i if $c(j) < c(i)$. Therefore,

$$\begin{aligned}\chi_G^\lambda(q,n) \;&=\; \sum_{\substack{\text{proper colorings}\\ c:[d]\to[n]}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)} \;=\; \sum_{\varrho\in\mathcal{A}(G)} \sum_{\mathbf{c}\in(n+1)\mathcal{O}(\Pi_\varrho)^\circ\cap\mathbb{Z}^d} q^{\lambda_1 c_1+\cdots+\lambda_d c_d}\\ &=\; \sum_{\varrho\in\mathcal{A}(G)} \operatorname{ehr}^\lambda_{\mathcal{O}(\Pi_\varrho)^\circ}(q,n+1)\,.\end{aligned}$$

□

Lemma 6.2.3. *Suppose $f(x)$ and $g(x)$ are polynomials with coefficients that are rational functions in q such that*

$$f([n]_q) \;=\; g([n]_q) \qquad \text{for all } \; n \in \mathbb{Z}_{>0}\,.$$

Then $f(x) = g(x)$.

Proof. By our assumptions, the polynomial $f(x) - g(x) \in \mathbb{Q}(q)[x]$ has infinitely many zeros and so must by the zero polynomial. □

Together with Chapoton's result (6.2), Lemmas 6.2.2 and 6.2.3 prove Theorem 6.1.1. In fact, we can see more, namely, that the analogue of Lemma 6.2.1 holds also for q-chromatic polynomials.

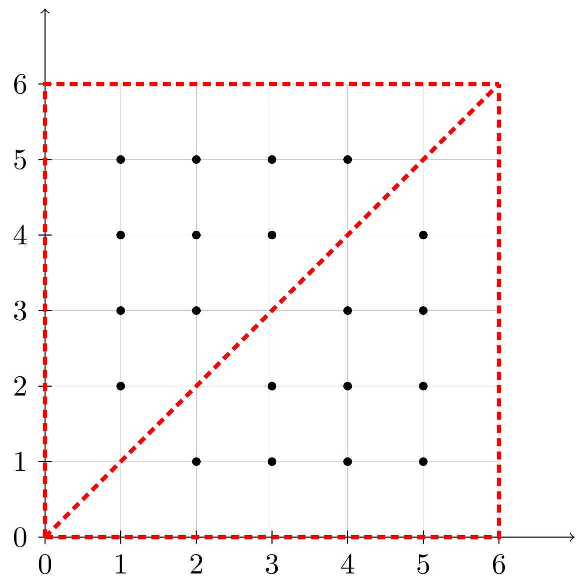


Figure 6.1: The proper 5-colorings of P_2 as points in the 6th dilates of the open order polytopes corresponding the acyclic orientations of the graph. The order polytope Δ_{12} contains colorings $c : [2] \to [5]$ with $c(1) < c(2)$ and the order polytope Δ_{21} contains colorings with $c(2) < c(1)$.

Corollary 6.2.4. *Let $\Lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_d$. The coefficients of $[\Lambda]_q!\, \widetilde{\chi}_G^{\lambda}(x)$ are polynomials in q.*

The case $\lambda = \mathbf{1}$ is particularly nice because we can employ (6.5).

Corollary 6.2.5. *For any graph $G = (V, E)$,*

$$\chi_G^{\mathbf{1}}(q, n) \;=\; \sum_{\varrho \in \mathcal{A}(G)} \; \sum_{\sigma \in \mathcal{L}(\Pi_\varrho)} q^{\binom{d+1}{2} - \operatorname{comaj} \sigma} \begin{bmatrix} n + \operatorname{des} \sigma \\ d \end{bmatrix}_q .$$

Proof. We apply (6.4) and (6.5):

$$\operatorname{ehr}^{\mathbf{1}}_{\mathcal{O}(\Pi_\varrho)^\circ}(q, n) \;=\; (-1)^d \operatorname{ehr}^{\mathbf{1}}_{\mathcal{O}(\Pi_\varrho)}(\tfrac{1}{q}, -n) \;=\; \sum_{\sigma \in \mathcal{L}(\Pi_\varrho)} q^{\binom{d+1}{2} - \operatorname{comaj} \sigma} \begin{bmatrix} n + \operatorname{des} \sigma - 1 \\ d \end{bmatrix}_q$$

and so Lemma 6.2.2 finishes the proof. □

Figure 6.2 shows the computation of $\chi_{P_3}^{\mathbf{1}}(q, n)$ via Corollary 6.2.5. We also record a few consequences of the corollary.

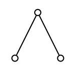
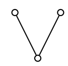
Acyclic Orientation ϱ	Induced Poset Π_ϱ	Linear Extensions $\mathcal{L}(\Pi_\varrho)$
		123
		123, $\underline{2}13$
		123, $1\underline{3}2$
		123

Figure 6.2: Computing the q-chromatic polynomial of P_3 from the linear extensions of the induced posets of its acyclic orientations:

$$\chi_{P_3}(q,n) = 4q^6 \begin{bmatrix} n \\ 3 \end{bmatrix}_q + (q^5+q^4) \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q$$

Corollary 6.2.6. *Let $G = ([d], E)$ and express $\chi_G^{\mathbf{1}}(q,n)$ in the form*

$$\chi_G^{\mathbf{1}}(n) \;=\; \sum_{j \geqslant 0} \beta_j(q) \begin{bmatrix} n+j \\ d \end{bmatrix}_q .$$

(a) Each $\beta_i(q)$ is a polynomial in q with nonnegative coefficients.

(b) $\beta_0(q) = |\mathcal{A}(G)|\, q^{\binom{d+1}{2}}$; in particular, if G is a tree then $\beta_0(q) = 2^{d-1} q^{\binom{d+1}{2}}$.

(c) The largest value i for which $\beta_i(q) \neq 0$ is $d - \xi$ where ξ is the chromatic number of G. Moreover,

$$\beta_{d-\xi}(q) \;=\; \sum_{\substack{\text{proper colorings} \\ c:V\to\xi}} q^{\sum_{v\in V} c(v)}.$$

Remark 7. Corollary 6.2.5 gives another way of realizing the largest value j for which $\beta_j(q) \neq 0$, namely, as the maximal number m of descents in a linear extension of a poset induced by an acyclic orientation of G. Therefore, the chromatic number of G is equal to $d - m$, which is one more than the minimal number of ascents in a linear extension of a poset induced by an acyclic orientation of G. This fact is known as the Gallai–Hasse–Roy–Vitaver Theorem (see, e.g., [48, Theorem 7.17]).

We also remark that $\beta_{d-\xi}(q)$ distinguishes between some trees, as the next example illustrates.

Example 6.2.7. Let T_1 be the path of length 3 and let T_2 be the star with degree sequence $(3,1,1,1)$. We compute

$$\begin{aligned}\chi^{\mathbf{1}}_{T_1}(q,n) &= 8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (4q^9+6q^8+4q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + 2q^6\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q \\ \chi^{\mathbf{1}}_{T_2}(q,n) &= 8q^{10}\begin{bmatrix} n \\ 4 \end{bmatrix}_q + (5q^9+4q^8+5q^7)\begin{bmatrix} n+1 \\ 4 \end{bmatrix}_q + (q^7+q^5)\begin{bmatrix} n+2 \\ 4 \end{bmatrix}_q .\end{aligned}$$

In particular, $\chi^{\mathbf{1}}_{T_1}(q,2) = 2q^6$ while $\chi^{\mathbf{1}}_{T_2}(q,2) = q^7+q^5$. However, the coefficient $\beta_{d-2}(q)$ is not enough to distinguish all non-isomorphic trees on d vertices.

6.3 The structure of q-chromatic polynomials

As with the classic chromatic polynomial, the q-chromatic polynomial satisfies a deletion–contraction relation. Naturally, this strongly relates to the deletion–contraction formula for Crew–Spirkl's weighted version of the chromatic symmetric function [51, Lemma 2].

Theorem 6.3.1. *Suppose $G = ([d], E)$ is a graph, $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, and $e = 12 \in E$. Then*

$$\chi^{\lambda}_{G}(q,n) = \chi^{\lambda}_{G\backslash e}(q,n) - \chi^{(\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_d)}_{G/e}(q,n) .$$

Proof. As usual, we observe that the proper n-colorings of G are precisely the proper n-colorings c of $G\backslash e$ that satisfy the additional condition $c(1) \neq c(2)$. Therefore, we may count them by counting all proper n-colorings c of $G\backslash e$ and then removing all such colorings c for which $c(1) = c(2)$:

$$\begin{aligned}\chi^{\lambda}_{G}(q,n) &= \sum_{\substack{\text{proper colorings}\\ c:[d]\to[n] \text{ of } G}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)} \\ &= \sum_{\substack{\text{proper colorings}\\ c:[d]\to[n] \text{ of } G\backslash e}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)} - \sum_{\substack{\text{proper colorings}\\ c:[d]\to[n] \text{ of } G\backslash e \\ \text{where } c(1)=c(2)}} q^{\lambda_1 c(1)+\cdots+\lambda_d c(d)} \\ &= \chi^{\lambda}_{G\backslash e}(q,n) - \chi^{(\lambda_1+\lambda_2,\lambda_3,\dots,\lambda_d)}_{G/e}(q,n) .\end{aligned}$$

□

We observe that a similar computation enables us to express any q-chromatic polynomial (for general λ with positive entries) as a linear combination of q-chromatic polynomials with $\lambda = \mathbf{1}$, via a repeated expansion–addition process as follows. If $G = ([d], E)$ is a graph and $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d_{\geqslant 0}$ with $\lambda_1 \geqslant 2$, split the vertex 1 into two vertices $1'$ and $1''$ with weights $\lambda_1 - 1$ and 1, respectively. Create the *expansion graph* $\exp(G, e)$ *of* G *at* 1 with vertex set $\{1', 1'', 2, \dots, d\}$ and edge set

$$\{1'i, 1''i : i \in \{2, \dots, d\} \text{ such that } 1i \in E\} \cup \{ij : i, j \in \{2, \dots, d\} \text{ such that } ij \in E\} ,$$

and let the *addition graph* $\operatorname{add}(G,e)$ *of* G *at* 1 be $\exp(G,e)$ with an edge added between the new vertices $1'$ and $1''$. Then

$$\chi_G^\lambda(q,n) \;=\; \chi_{\exp(G,e)}^{(\lambda_1-1,1,\lambda_2,\ldots,\lambda_d)}(q,n) - \chi_{\operatorname{add}(G,e)}^{(\lambda_1-1,1,\lambda_2,\ldots,\lambda_d)}(q,n)\,.$$

By repeatedly applying this process, we obtain the following result:

Theorem 6.3.2. *If $G = ([d],E)$ is a graph and $\lambda = (\lambda_1,\ldots,\lambda_d) \in \mathbb{Z}_{\geqslant 0}^d$, then there exist graphs $H_1,\ldots,H_\ell$ on $\lambda_1+\cdots+\lambda_d$ vertices and integers $k_1,\ldots,k_\ell$ such that*

$$\chi_G^\lambda(q,n) \;=\; \sum_{i=1}^{\ell} k_i\, \chi_{H_i}^{\mathbf{1}}(q,n)\,.$$

Our next result extends Stanley's famous reciprocity theorem for the chromatic polynomials to the q-setting. A (not necessarily proper) coloring c of a graph G is *compatible* with an acyclic orientation ϱ of G if c (weakly) increases along oriented edges. Stanley [109] proved that $|\chi_G(-n)|$ equals the number of pairs of an n-coloring and a compatible acyclic orientation of G. In particular, $|\chi_G(-1)|$ equals the number of acyclic orientations of G. This generalizes as follows.

Theorem 6.3.3. *Given a graph $G = (V,E)$ and $\lambda \in \mathbb{Z}^V$, let $\Lambda := \sum_{v\in V}\lambda_v$. Then*

$$(-1)^{|V|} q^{\Lambda}\, \widetilde{\chi}_G^\lambda\left(\tfrac{1}{q}, [-n]_{\frac{1}{q}}\right) \;=\; \sum_{(c,\varrho)} q^{\sum_{v\in V(G)}\lambda_v c(v)},$$

where the sum is over all pairs of an n-coloring c and a compatible acyclic orientation ϱ.

Example 6.3.4. For $\lambda = \mathbf{1}$, the path on 2 vertices has q-chromatic polynomial

$$\widetilde{\chi}_{P_2}^{\mathbf{1}}(q,x) \;=\; \frac{2q^2x^2 - 2q^2x}{1+q}\,.$$

Therefore,

$$(-q)^2\, \widetilde{\chi}_{P_2}^{\mathbf{1}}(\tfrac{1}{q},x) \;=\; q^2\,\frac{2q^{-2}x^2 - 2q^{-2}x}{1+q^{-1}} \;=\; \frac{2qx^2-2qx}{1+q}$$

and so, e.g.,

$$(-q)^2\, \widetilde{\chi}_{P_2}^{\mathbf{1}}(\tfrac{1}{q},-q-q^2) \;=\; \frac{2q(-q-q^2)^2 - 2q(-q-q^2)}{1+q} \;=\; 2q^4+2q^3+2q^2.$$

Indeed, this sums $q^{\sum c(v)}$ for the six pairs of 2-colorings and compatible acyclic orientations.

Proof of Theorem 6.3.3. Let $d := |V|$. We apply Chapoton's reciprocity theorem (6.4) to Lemma 6.2.2:

$$\begin{aligned}(-1)^d \, \widetilde{\chi}_G^{\lambda}(q, [-n]_q) &= \sum_{\varrho \in A(G)} (-1)^d \, \widetilde{\mathrm{ehr}}^{\lambda}_{\mathcal{O}(\Pi_\varrho)^\circ}(q, [-n+1]_q) \\ &= \sum_{\varrho \in A(G)} (-1)^d \, \widetilde{\mathrm{ehr}}^{\lambda}_{\mathcal{O}(\Pi_\varrho)} \left(\tfrac{1}{q}, [n-1]_{\frac{1}{q}} \right).\end{aligned}$$

Therefore,

$$\begin{aligned}(-1)^d \, \widetilde{\chi}_G^{\lambda} \left(\tfrac{1}{q}, [-n]_{\frac{1}{q}} \right) &= \sum_{\varrho \in A(G)} (-1)^d \, \widetilde{\mathrm{ehr}}^{\lambda}_{\mathcal{O}(\Pi_\varrho)}(q, [n-1]_q) \\ &= \sum_{\varrho \in A(G)} \mathrm{ehr}^{\lambda}_{\mathcal{O}(\Pi_\varrho)}(q, n-1) \, . \qquad (6.7)\end{aligned}$$

The integer lattice points in $(n-1)\mathcal{O}(\Pi_\varrho)$ can be interpreted as colorings of G using the color set $\{0, 1, \ldots, n-1\}$ that are compatible with ϱ, and so (6.7) equals

$$\sum_{(c,\varrho)} q^{\sum_{v \in V(G)} \lambda_v (c(v)-1)}. \qquad \square$$

We conclude this section with one more way of computing q-chromatic polynomials. A *flat* of a given graph $G = (V, E)$ is a subset $S \subseteq E$ such that for any edge $e \notin S$, the subgraph (V, S) has strictly more connected components than $(V, S \cup \{e\})$. Geometrically, the intersection H_S of the hyperplanes of the graphical arrangement $\mathcal{H}_G$ in (6.6) corresponding to S form a flat of $\mathcal{H}_G$. Let $P(S)$ be the collection of vertex sets of the connected components induced by S, and for $W \subseteq V$ and $\lambda \in \mathbb{Z}^V_{\geqslant 0}$, let

$$\Lambda_W := \sum_{v \in W} \lambda_v \, .$$

The flats of G form a poset (in fact, a lattice), whose Möbius function helps us compute, again via inside-out polytopes [27] (see also [25, Chapter 7]), that

$$\begin{aligned}\chi_G^{\lambda}(q, n) &= \sum_{\text{flats } S \subseteq E} \mu(\varnothing, S) \, \mathrm{ehr}^{\lambda}_{(0,1)^V \cap H_S}(n+1) = \sum_{\text{flats } S \subseteq E} \mu(\varnothing, S) \prod_{C \in P(S)} q^{\Lambda_C} [n]_{q^{\Lambda_C}} \\ &= q^{\Lambda_V} \sum_{\text{flats } S \subseteq E} \mu(\varnothing, S) \prod_{C \in P(S)} [n]_{q^{\Lambda_C}} \, .\end{aligned}$$

In particular, for a tree $T = (V, E)$,

$$\chi_T^{\lambda}(q, n) = q^{\Lambda_V} \sum_{S \subseteq E} (-1)^{|S|} \prod_{C \in P(S)} [n]_{q^{\Lambda_C}} \, .$$

These formulas can be viewed as analogues of [108, Theorem 2.5], where Stanley proves an expression for the chromatic symmetric function in the power sum basis.

Next, we employ the following trick from [47]: for integers $n \geqslant 0$ and $k \geqslant 1$,

$$\left.\frac{1-(1+qx-x)^k}{1-q^k}\right|_{x=[n]_q} = [n]_{q^k}\,.$$

This yields the following formulas for q-chromatic polynomials.

Theorem 6.3.5. *Given a graph $G = (V, E)$ and $\lambda \in \mathbb{Z}^V$,*

$$\widetilde{\chi}_G^\lambda(q,n) = q^{\Lambda_V} \sum_{\textit{flats } S\subseteq E} \mu(\varnothing, S) \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}\,.$$

In particular, for a tree $T = (V, E)$,

$$\widetilde{\chi}_T^\lambda(q,n) = q^{\Lambda_V} \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{\Lambda_C}}{1-q^{\Lambda_C}}\,.$$

Remark 8. In the following section, we will study the leading coefficient of this polynomial and see that it appears to distinguish trees. This is certainly not true of all other coefficients. For example, we can see that any tree on d vertices with the same total vertex weight Λ_V has the same linear coefficient (and the same constant 0, like the ordinary chromatic polynomial). Since

$$\frac{1-(1+(q-1)x)^{\Lambda_C}}{1-q^{\Lambda_C}} = \frac{-\Lambda_C(q-1)x-\binom{\Lambda_C}{2}(q-1)^2x^2-\cdots}{1-q_C^{\Lambda}}\,,$$

the only linear terms of $\widetilde{\chi}$ come from edge subsets S that result in 1 connected component; for trees T, the only such set is $S = E$. Thus, for a tree, the linear coefficient is determined only by d and Λ_V.

Example 6.3.6. Theorem 6.3.5 suggests highly structured formulas for certain families of graphs; we exercise this for the path P_k on k vertices when $\lambda = \mathbf{1}$, in analogy with the chromatic symmetric function [114, Exercise 7.47(k)].

$$\widetilde{\chi}_{P_k}^{\mathbf{1}}(q,x) = q^k \sum_{S\subseteq E} (-1)^{|S|} \prod_{C\in P(S)} \frac{1-(1+qx-x)^{|C|}}{1-q^{|C|}} = (-q)^k \sum_{S\subseteq E} \prod_{C\in P(S)} \Phi(q,x,|C|),$$

where

$$\Phi(q,x,j) := -\frac{1-(1+qx-x)^j}{1-q^j}$$

and we used the fact that (for a tree) $|S| + |P(S)| = |E| + 1$. The subsets of E (for the path P_k) are in one-to-one correspondence with the compositions (i.e., ordered partitions) of k, with parts given by the sizes of the sets in $P(S)$. Thus

$$\sum_{k\geqslant 1} \widetilde{\chi}_{P_k}^{\mathbf{1}}(q,x)\,t^k = \sum_{k\geqslant 1}\sum_{S\subseteq E}\prod_{C\in P(S)} \Phi(q,x,|C|)\,(-qt)^k = \sum_{\mu}\prod_{m\in P(\mu)} \Phi(q,x,m)\,(-qt)^{|\mu|},$$

where the sum is over all compositions μ, we collect the parts of μ in the multiset $P(\mu)$, and $|\mu|$ is the sum of the parts of μ.

Example 6.3.7. The analogous computation for the star S_{k+1} on $k+1$ vertices gives

$$\widetilde{\chi}^{\mathbf{1}}_{S_{k+1}}(q,x) \;=\; (-q)^{k+1} \sum_{S\subseteq E} \prod_{C\in P(S)} \Phi(q,x,|C|) \;=\; (-q)^{k+1} \sum_{j=0}^{k} \binom{k}{j} \Phi(q,x,j+1)\,(-x)^{k-j}$$

and so

$$\begin{aligned}\sum_{k\geqslant 0} \widetilde{\chi}^{\mathbf{1}}_{S_{k+1}}(q,x)\, t^{k+1} &= \sum_{k\geqslant 0}\sum_{j=0}^{k} \binom{k}{j} \Phi(q,x,j+1)\,(-x)^{k-j}\,(-qt)^{k+1} \\ &= -qt \sum_{j\geqslant 0} \Phi(q,x,j+1)\,(-x)^{-j} \sum_{k\geqslant j} \binom{k}{j} (xqt)^k \\ &= -qt \sum_{j\geqslant 0} \Phi(q,x,j+1)\,(-x)^{-j}\, \frac{(xqt)^j}{(1-xqt)^{j+1}} \\ &= \sum_{j\geqslant 0} (-1)^{j+1} \Phi(q,x,j+1) \left(\frac{qt}{1-xqt}\right)^{j+1} .\end{aligned}$$

6.4 The leading coefficient of a q-chromatic polynomial

We now focus our attention on the leading coefficient $c^{\lambda}_{T}(q)$ of $\widetilde{\chi}^{\lambda}_{T}(q,n)$ stemming from Theorem 6.3.5.

Corollary 6.4.1. *Given a tree $T=(V,E)$ and $\lambda\in\mathbb{Z}^V$, the leading coefficient of $\widetilde{\chi}^{\lambda}_{T}(q,n)$ equals*

$$c^{\lambda}_{T}(q) \;=\; (-1)^{|V|}(q^2-q)^{\Lambda_V} \sum_{S\subseteq E} \prod_{C\in P(S)} \frac{1}{1-q^{\Lambda_C}} \,.$$

In particular,

$$[\Lambda_V]_q!\; c^{\lambda}_{T}(q) \;=\; q^{\Lambda_V} (-1)^{|V|+\Lambda_V} \sum_{S\subseteq E} (1-q)^{\Lambda_V-\kappa(S)} \frac{[\Lambda_V]_q!}{\prod_{C\in P(S)} [\Lambda_C]_q}$$

(where $\kappa(S)$ denotes the number of components of the subgraph induced by S) is visibly a polynomial in q, as the fraction is a q-multinomial coefficient times a polynomial.

Remark 9. Deletion–contraction extends to $c^{\lambda}_{T}(q)$, and we provide a formula here which might be helpful for computations. Let l be a leaf of T and

$$\begin{aligned} A &:= \{S\subset E : l\notin e \text{ for all } e\in S\} \\ B &:= \{S\subset E : l\in e \text{ for for some } e\in S\} \,. \end{aligned}$$

Let $T' = (V', E')$ be the tree with l deleted; we will denote the number of connected components induced by $S \subseteq E'$ by $\kappa'(S)$. We further define λ' to be the vector λ with lth entry removed, and λ^+ to stem from λ where we add λ_l to the entry corresponding to the neighbor of l, with corresponding notation Λ_W^+ for $W \subseteq E'$. Then

$$\sum_{S \in A} (1-q)^{\Lambda_V - \kappa(S)} \frac{[\Lambda_V]_q!}{\prod_{C \in P(S)} [\Lambda_C]_q} = (1-q)^{\lambda_l - 1} \frac{[\Lambda_V]_q!}{[\lambda_l]_q \, [\Lambda_{V'}]_q!} \sum_{S \subseteq E'} (1-q)^{\Lambda_{V'} - \kappa'(S)} \frac{[\Lambda_{V'}]_q!}{\prod_{C \in P(S)} [\Lambda_C]_q}$$

and

$$\sum_{S \in B} (1-q)^{\Lambda_V - \kappa(S)} \frac{[\Lambda_V]_q!}{\prod_{C \in P(S)} [\Lambda_C]_q} = \sum_{S \subseteq E'} (1-q)^{\Lambda^+_{V'} - \kappa'(S)} \frac{[\Lambda^+_{V'}]_q!}{\prod_{C \in P(S)} [\Lambda^+_C]_q} .$$

Thus,

$$\begin{aligned} [\Lambda_V]_q! \; c_T^\lambda(q) &= q^{\Lambda_V} (-1)^{|V| + \Lambda_V} (1-q)^{\lambda_l - 1} \frac{[\Lambda_V]_q!}{[\lambda_l]_q \, [\Lambda_{V'}]_q!} \sum_{S \subseteq E'} (1-q)^{\Lambda_{V'} - \kappa'(S)} \frac{[\Lambda_{V'}]_q!}{\prod_{C \in P(S)} [\Lambda_C]_q} \\ &\quad + q^{\Lambda_V} (-1)^{|V| + \Lambda_V} \sum_{S \subseteq E'} (1-q)^{\Lambda^+_{V'} - \kappa'(S)} \frac{[\Lambda^+_{V'}]_q!}{\prod_{C \in P(S)} [\Lambda^+_C]_q} \\ &= q^{\lambda_l} (q-1)^{\lambda_l - 1} \frac{[\Lambda_V]_q!}{[\lambda_l]_q \, [\Lambda_{V'}]_q!} \left([\Lambda_{V'}]_q! \; c_{T'}^{\lambda'}(q) \right) - \left([\Lambda^+_{V'}]_q! \; c_{T'}^{\lambda^+}(q) \right) . \end{aligned}$$

Again the fraction is a polynomial (via a q-binomial coefficient). This formula allows us to implement a recursive algorithm to compute the leading coefficient of $\widetilde{\chi}$ for trees, as given in Figure 6.3. See, for example, the leading coefficients (for $\lambda = \mathbf{1}$) for all trees on $d = 6$ vertices computed in Figure 6.4.

We now further focus on the case $\lambda = \mathbf{1}$. Corollaries 6.2.5 and 6.4.1 give the following two (quite different) expressions for the leading coefficient.

Corollary 6.4.2. *Given a tree $T = (V, E)$ on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q, n)$ equals*

$$\begin{aligned} c_T^{\mathbf{1}}(q) &= (q - q^2)^d \sum_{S \subseteq E} \prod_{C \in P(S)} \frac{1}{1 - q^{\Lambda_C}} \\ &= \frac{1}{[d]_q!} \sum_{(\varrho, \sigma)} q^{d + \operatorname{maj} \sigma}, \end{aligned}$$

where the sum ranges over all pairs of acyclic orientations ϱ of T and linear extensions σ of the poset induced by ϱ.

Proof. The first formula follows directly from Corollary 6.4.1. The second follows from Corollary 6.2.5. The q-binomial coefficient in Corollary 6.2.5 can be expressed in terms of

function LEADING_COEFFICIENT(T, $(\lambda_1, \ldots, \lambda_d)$)
  $n \leftarrow \#$ of vertices of T
  if $n == 1$ **then**
    $i \leftarrow$ the sole vertex of T
    return $\dfrac{(q-1)^{\lambda_i}}{(q^{\lambda_i}-1)}$
  else
    $i \leftarrow$ a leaf of T
    $j \leftarrow i$'s unique neighbor
    $T_{\text{del}} \leftarrow T$ with vertex i and edge $\{i, j\}$ removed
    $\lambda_{\text{del}} \leftarrow (\lambda_1, \ldots, \lambda_d)$ with $\lambda_i = 0$
    $c_{\text{del}} \leftarrow$ LEADING_COEFFICIENT($T_{\text{del}}, \lambda_{\text{del}}$)
    $T_{\text{con}} \leftarrow T$ with edge $\{i, j\}$ contracted
    $\lambda_{\text{con}} \leftarrow (\lambda_1, \ldots, \lambda_d)$ with $\lambda_j = \lambda_j + \lambda_i$ and $\lambda_i = 0$
    $c_{\text{con}} \leftarrow$ LEADING_COEFFICIENT($T_{\text{con}}, \lambda_{\text{con}}$)
    return $\dfrac{(q-1)^{\lambda_i}}{(q^{\lambda_i}-1)} \cdot c_{\text{del}} - c_{\text{con}}$
  end if
end function
return $[d]_q! \cdot$ LEADING_COEFFICIENT($S, (\mu_1, \ldots, \mu_d)$)

Figure 6.3: A recursive algorithm that computes the leading coefficient of $\widetilde{\chi}_S^{(\mu_1,\ldots,\mu_d)}(q,x)$ of a tree S on d vertices via deletion-contraction

the q-integers via

$$\begin{bmatrix} n + \operatorname{des}\sigma \\ d \end{bmatrix}_q = \frac{[n+\operatorname{des}\sigma]_q \cdots [n]_q \cdots [n+\operatorname{des}\sigma-(d-1)]_q}{[d]_q!}$$

$$= \frac{1}{[d]_q!}\left(q^{\operatorname{des}\sigma}[n]_q + [\operatorname{des}\sigma]_q\right) \cdots [n]_q \cdots \left(\frac{[n]_q - [\operatorname{asc}\sigma]_q}{q^{\operatorname{asc}\sigma}}\right)$$

(since $(d-1) - \operatorname{des}\sigma = \operatorname{asc}\sigma$, the number of ascents of σ), which gives

$$c_T^{\mathbf{1}}(q) = \frac{1}{[d]_q!} \sum_{\varrho \in \mathcal{A}(G)} \sum_{\sigma \in \mathcal{L}(\Pi_\varrho)} q^{\binom{d+1}{2} + \binom{\operatorname{des}\sigma+1}{2} - \binom{\operatorname{asc}\sigma+1}{2} - \operatorname{comaj}\sigma}.$$

Using the relation $\operatorname{asc}\sigma + \operatorname{des}\sigma = d-1$ again, the exponent simplifies to

$$\binom{d+1}{2} + \binom{\operatorname{des}\sigma+1}{2} - \binom{\operatorname{asc}\sigma+1}{2} - \operatorname{comaj}\sigma = d + \operatorname{maj}\sigma. \qquad \square$$

In Corollary 6.4.2, the latter expression for $c_T^{\mathbf{1}}(q)$ illustrates that $\frac{1}{q^d}[d]_q!\, c_T^{\mathbf{1}}(q)$ is a polynomial in q with nonnegative coefficients. We provide this expression for all non-isomorphic trees on $d = 6$ vertices in Figure 6.4. We note that this expression also implies that $c_T^{\mathbf{1}}(q)$ is (up to a factor of q^d) the sum over the orientations ϱ of T of the *major index q-analogue of the poset* Π_ϱ, as studied, e.g., in [58].

Example 6.4.3. Continuing Example 6.3.6, we return to the path P_k on k vertices. Corollary 6.4.1 gives

$$c_{P_k}^{\mathbf{1}}(q) \;=\; (q-q^2)^k \sum_{S\subseteq E} \prod_{C\in P(S)} \frac{1}{1-q^{|C|}}$$

and thus

$$\sum_{k\geqslant 1} c_{P_k}^{\mathbf{1}}(q)\, t^k \;=\; \sum_{k\geqslant 1}\sum_{S\subseteq E} \prod_{C\in P(S)} \frac{1}{1-q^{|C|}} \left((q-q^2)\,t\right)^k \;=\; \sum_{\mu} \prod_{m\in P(\mu)} \frac{1}{1-q^m} \left((q-q^2)\,t\right)^{|\mu|},$$

where again the sum is over all compositions μ.

Example 6.4.4. Continuing Example 6.3.7 along similar lines, we compute for the star

$$c_{S_{k+1}}^{\mathbf{1}}(q) \;=\; (q-q^2)^{k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{1}{(1-q^{j+1})(1-q)^{k-j}} \;=\; q^{k+1} \sum_{j=0}^{k} \binom{k}{j} \frac{(1-q)^{j+1}}{1-q^{j+1}}$$

and so

$$\sum_{k\geqslant 0} c_{S_{k+1}}^{\mathbf{1}}(q)\, t^{k+1} \;=\; \sum_{j\geqslant 0} \frac{(1-q)^{j+1}}{1-q^{j+1}} \sum_{k\geqslant j} \binom{k}{j} (qt)^{k+1} \;=\; \sum_{j\geqslant 0} \frac{1}{1-q^{j+1}} \left(\frac{qt(1-q)}{1-qt}\right)^{j+1}$$

is a classical Lambert series.

Remark 10. Corollary 6.4.1 immediately distinguishes stars from all other trees: the largest possible major index one can obtain from a tree is from the linear extension $[1, d, d-1, \ldots, 3, 2$ and the only tree that realizes this is the star (with acyclic orientation where all edges point out from center). Consequently, the degree of $[d]_q!\, c_T^{\mathbf{1}}(q)$ for a star is strictly larger than that of any other tree with the same number of vertices.

6.5 G-partitions

The second formula in Corollary 6.4.2 is reminiscent of Stanley's P-partitions [117] and organically suggests an extension of that concept to graphs. We first review the part of Stanley's theory that we will need.

Given a poset $\Pi = ([d], \preceq)$, a *strict* Π-*partition* of $n \in \mathbb{Z}_{>0}$ is a tuple $\boldsymbol{m} \in \mathbb{Z}_{>0}^d$ such that[2]

$$\sum_{j=1}^{d} m_j = n \qquad \text{and} \qquad m_j < m_k \;\text{ whenever }\; j \prec k\,.$$

[2]Our definition differs from Stanley's inequalities, but the methodology is the same.

Let $p_\Pi(n)$ denote the number of strict Π-partitions of n, with accompanying generating function

$$P_\Pi(q) := \sum_{n>0} p_\Pi(n)\, q^n .$$

Then by [25, Exercise 6.23],

$$P_\Pi(q) \;=\; \frac{q^d \sum_{\sigma\in\mathcal{L}(\Pi)} \prod_{j\in\operatorname{Asc}\sigma} q^{d-j}}{(1-q)(1-q^2)\cdots(1-q^d)} \;=\; \frac{q^d \sum_{\sigma\in\mathcal{L}(\Pi)} q^{\operatorname{maj}\sigma^{op}}}{(1-q)(1-q^2)\cdots(1-q^d)}, \tag{6.8}$$

where $\operatorname{Asc}\sigma$ denotes the ascent set of σ, and we define $\sigma^{\mathrm{op}}(j) := \sigma(d+1-j)$. Note that we compute ascents and descents as in Section 6.2: we fix some natural labeling of Π, i.e., an order-preserving bijection $\Pi \to [d]$. The permutation corresponding to a given linear extension σ can be read off from this labeling. Viewing a poset as an (acyclic) directed graph, the following definition gives the natural analogue for an undirected graph.

Let $G = (V, E)$ be a graph. A *G-partition*[3] of $n \in \mathbb{Z}_{>0}$ is a tuple $\boldsymbol{m} \in \mathbb{Z}_{>0}^V$ such that

$$\sum_{v\in V} m_v = n \qquad \text{and} \qquad m_v \neq m_w \;\text{ whenever }\; vw \in E\,.$$

Let $p_G(n)$ denote the number of G-partitions of n, with accompanying generating function $P_G(q) := \sum_{n>0} p_G(n)\, q^n$.

Theorem 6.5.1. *Let G be a graph on d vertices. Then*

$$\begin{aligned} P_G(q) &= \frac{q^d \sum_{(\varrho,\sigma)} q^{\operatorname{maj}\sigma^{op}}}{(1-q)(1-q^2)\cdots(1-q^d)} \\ &= \frac{q^{\binom{d+1}{2}} \sum_{(\varrho,\sigma)} q^{-\operatorname{maj}\sigma}}{(1-q)(1-q^2)\cdots(1-q^d)}\,, \end{aligned}$$

where each sum ranges over all pairs of acyclic orientations ϱ of G and linear extensions σ of the poset induced by ϱ.

Proof. Since every G-partition is a Π_ϱ-partition for exactly one acyclic orientation ϱ of G (and, conversely, every Π_ϱ-partition is a G-partition),

$$p_G(n) \;=\; \sum_{\varrho\in\mathcal{A}(G)} p_{\Pi_\varrho}(n)$$

and so (6.8) gives the first formula:

$$P_G(q) \;=\; \frac{q^d \sum_{\varrho\in\mathcal{A}(G)} \sum_{\sigma\in\mathcal{L}(\Pi_\varrho)} q^{\operatorname{maj}\sigma^{op}}}{(1-q)(1-q^2)\cdots(1-q^d)}\,.$$

[3] We follow the (somewhat misleading) nomenclature of Stanley—in general, neither P- nor G-partitions are partitions, rather they are *compositions*, i.e., ordered partition of a given integer n.

To see the second formula, we note that each $\varrho \in \mathcal{A}(G)$ has a partner orientation $\overline{\varrho} \in \mathcal{A}(G)$ in which the direction of each edge is reversed. A linear extension $\sigma \in \mathcal{L}(\Pi_\varrho)$ has the corresponding linear extension $\overline{\sigma} \in \mathcal{L}(\Pi_{\overline{\varrho}})$ defined via

$$\overline{\sigma}(j) \;:=\; d+1-\sigma(d+1-j) \;=\; d+1-\sigma^{\mathrm{op}}(j)\,.$$

In particular, $j \in \mathrm{Des}(\sigma^{\mathrm{op}})$ if and only if $j \in \mathrm{Asc}(\overline{\sigma})$, and so

$$\sum_{(\varrho,\sigma)} q^{\mathrm{maj}\,\sigma^{op}} \;=\; \sum_{(\overline{\varrho},\overline{\sigma})} q^{\binom{d}{2}-\mathrm{maj}\,\overline{\sigma}}\,. \qquad \square$$

This yields a third equation that can be added to the ones in Corollary 6.4.2.

Corollary 6.5.2. *Given a tree $T=(V,E)$ on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals*

$$c_T^{\mathbf{1}}(q) \;=\; (-1)^d q^d\, P_G\left(\tfrac{1}{q}\right).$$

We can now see Remark 10 through this new lens: the star graph on d vertices is unique with $p_G(d+1)=1$. More generally, Corollary 6.5.2 implies that Conjecture 6.1.3 is equivalent to the following.

Conjecture 6.5.3. *The G-partition function $p_G(n)$ distinguishes trees.*

We conclude by making note of the connection between G-partitions and the *stable principal evaluation* $X_G(q,q^2,q^3,\ldots)$ of the chromatic symmetric function. Namely, from first principles we can see that

$$P_G(q) \;=\; X_G(q,q^2,q^3,\ldots)\,.$$

This yields one final equation for the leading coefficient that can be added to the ones in Corollary 6.4.2.

Corollary 6.5.4. *Given a tree $T=(V,E)$ on d vertices, the leading coefficient of $\widetilde{\chi}_T^{\mathbf{1}}(q,n)$ equals*

$$c_T^{\mathbf{1}}(q) \;=\; (-1)^d q^d\, X_G\left(\tfrac{1}{q},\tfrac{1}{q^2},\tfrac{1}{q^3},\ldots\right).$$

That is, when we express the principal evaluation of the chromatic symmetric function as a polynomial in the q-integers, the stable principal evaluation appears in its leading coefficient.

6.6 Open questions

From our construction of $\widetilde{\chi}_T^{\lambda}(q,n)$ for general λ, a natural weakening of Conjecture 6.1.3 (that is perhaps easier to prove) arises.

Conjecture 6.6.1. *For any pair of non-isomorphic trees S and T on d vertices, there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}_{>0}$ such that $\chi_S^\lambda(q,n) \neq \chi_T^\lambda(q,n)$.*

Another line of open questions emerges concerning the coefficients of the q-chromatic polynomial. The classical chromatic polynomial $\chi_G(n)$ is very well studied, and many of its coefficients have nice combinatorial interpretations. Can we generalize these to $\widetilde{\chi}_G^{\mathbf{1}}(q,n)$? For example:

1. The second coefficient of $\chi_G(n)$ is (negative) the number of edges of G. Can we refine this to a q version, i.e., does the second coefficient of $\widetilde{\chi}_G^{\mathbf{1}}(q,n)$ count the number of edges of G, but graded by some property of the edges?
2. Can the same be done for the linear coefficient which, in the classical case, counts the number of acyclic orientations with a unique sink at one fixed vertex? (This is not interesting for trees by Remark 8, but could be interesting for general graphs.)
3. The coefficients of $\chi_G(n)$ are alternating. Can we show that the coefficients of $\widetilde{\chi}_G^{\mathbf{1}}(q,n)$ are "strongly alternating," in the sense that the coefficient of x^j in $[d]_q! \cdot \widetilde{\chi}_G^{\mathbf{1}}(q,x)$ is a polynomial in q with either all positive or all negative coefficients (depending on the parity of $d-j$)?

Finally, as we mentioned in the introduction, there are further structural results and questions that stem from viewing $\chi_G^\lambda(q,n)$ as an evaluation of a (weighted) chromatic symmetric function. It is then natural to ask if there is anything to be gained by zeroing in on the polynomial $\widetilde{\chi}_G^\lambda(q,x)$; for example:

(4) Is there some (interesting) variant of the $(3+1)$-free Conjecture of Stanley and Stembridge [121] for $\widetilde{\chi}_G^\lambda(q,x)$?

(5) There exist variants of Whitney's Broken-Circuit Theorem for weighted chromatic symmetric functions; see [51, Lemma 3] and [61, Theorem 6.8]. Do they give rise to a meaningful broken-circuit result for the coefficients of $\widetilde{\chi}_G^\lambda(q,x)$?

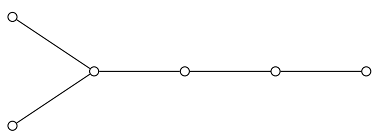
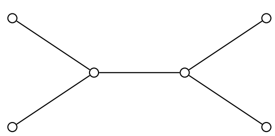
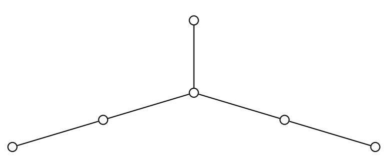
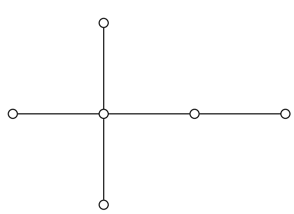
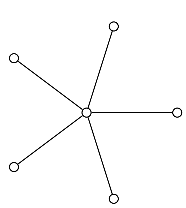
T	$\frac{[d]_q!}{q^d} \cdot c^{\mathbf{1}}_T(q)$
	$2q^{12} + 8q^{11} + 18q^{10} + 36q^9 + 62q^8 + 78q^7 + 102q^6 + 102q^5 + 106q^4 + 80q^3 + 62q^2 + 32q + 32$
	$q^{13} + q^{12} + 10q^{11} + 16q^{10} + 41q^9 + 57q^8 + 81q^7 + 95q^6 + 108q^5 + 100q^4 + 83q^3 + 59q^2 + 36q + 32$
	$4q^{12} + 8q^{11} + 18q^{10} + 42q^9 + 58q^8 + 78q^7 + 92q^6 + 110q^5 + 98q^4 + 82q^3 + 58q^2 + 40q + 32$
	$2q^{12} + 9q^{11} + 20q^{10} + 34q^9 + 65q^8 + 77q^7 + 96q^6 + 104q^5 + 107q^4 + 76q^3 + 62q^2 + 36q + 32$
	$q^{13} + 3q^{12} + 11q^{11} + 18q^{10} + 39q^9 + 60q^8 + 78q^7 + 87q^6 + 110q^5 + 101q^4 + 79q^3 + 59q^2 + 42q + 32$
	$q^{14} + 9q^{12} + 9q^{11} + 20q^{10} + 39q^9 + 60q^8 + 72q^7 + 81q^6 + 112q^5 + 99q^4 + 79q^3 + 58q^2 + 49q + 32$

Figure 6.4: A table of the leading coefficients of the q-chromatic polynomials of the non-isomorphic trees on $d = 6$ vertices

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