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THE QUALITATIVE ANALYSIS OF A DYNAMICAL SYSTEM MODELING THE FORMATION OF MULTILAYER SCALES ON PURE METALS*

H. C. AKUEZUE†, R. L. BAKER‡, AND M. W. HIRSCH§

Abstract. Gesmundo and Viani modeled the growth rates of three-oxide scales by the system

$$\begin{aligned}\frac{dq_1}{dt} &= m \frac{K_1}{2q_1} - \frac{m-1}{m} \frac{K_2}{2q_2}, \\ \frac{dq_2}{dt} &= -m \frac{K_1}{2q_1} + \left(\frac{m-1}{m} + \frac{n}{m} \right) \frac{K_2}{2q_2} - \frac{n-1}{n} \frac{K_3}{2q_3}, \\ \frac{dq_3}{dt} &= -\frac{n}{m} \frac{K_2}{2q_2} + \frac{K_3}{2q_3}.\end{aligned}$$

The authors consider the more general n -dimensional dynamical system:

$$\frac{dq_i}{dt} = - \sum_{j=1}^n \frac{a_{ij}}{q_j}, \quad q_i(t) > 0, \quad i = 1, \dots, n.$$

Under mild algebraic conditions on the constant matrix $A = (a_{ij})$, it is shown that every solution $\mathbf{q}(t)$ extends to a solution defined for all $t \geq 0$, and $\lim_{t \rightarrow +\infty} q_i(t) = +\infty$. The difference between any two solutions is bounded as a function of t . When A is an irreducible tridiagonal matrix, then every solution is eventually increasing.

It is shown that when $m, n > 1$ the Gesmundo–Viani system admits a unique parabolic solution $q_i(t) = c_i \sqrt{t}$. The authors conjecture that this parabolic solution attracts all other solutions.

Key words. differential equations, dynamical system, nonlinear dynamical system, cooperative system

AMS subject classifications. 34C35, 70K05

1. Introduction. The parabolic growth of complex oxide scales containing two or three components of pure metals has been studied by Gesmundo and Viani [3]. They obtained the following nonlinear three-dimensional dynamical system as a model for the growth rates of three-oxide scales:

$$(1.1.1) \quad \frac{dq_1}{dt} = m \frac{K_1}{2q_1} - \frac{m-1}{m} \frac{K_2}{2q_2},$$

$$(1.1.2) \quad \frac{dq_2}{dt} = -m \frac{K_1}{2q_1} + \left(\frac{m-1}{m} + \frac{n}{m} \right) \frac{K_2}{2q_2} - \frac{n-1}{n} \frac{K_3}{2q_3},$$

$$(1.1.3) \quad \frac{dq_3}{dt} = -\frac{n}{m} \frac{K_2}{2q_2} + \frac{K_3}{2q_3}.$$

Here $K_i > 0$ ($i = 1, 2, 3$) are rate constants, $m > 0, n > 0$ are parameters, and $q_i > 0$ is the weight of oxygen contained in oxide i per unit area.

In the present paper we study (1.1.1)–(1.1.3) as a case of the more general n -dimensional system:

$$(1.2) \quad \frac{dq_i}{dt} = - \sum_{j=1}^n \frac{a_{ij}}{q_j}, \quad q_i(t) > 0, \quad i = 1, \dots, n.$$

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We show that under mild algebraic conditions on the $n \times n$ constant matrix $A = (a_{ij})$, in the long run the trajectories of (1.2) are well behaved in the sense that every solution $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$, $t \in [0, a]$, $0 < a < +\infty$, can be extended to a solution on $[0, +\infty)$, and $\lim_{t \rightarrow \infty} q_i(t) = +\infty$, $i = 1, \dots, n$. Moreover the difference between any two solutions is bounded as a function of t . Furthermore, if A is irreducible and tridiagonal then all solutions are eventually monotone increasing.

Finally, we give a partial qualitative analysis of (1.1) in the case where the parameters m, n have values in the interval $(1, +\infty)$: If $m, n > 1$ there is a unique parabolic solution $q_i(t) = c_i \sqrt{t}$, $c_i > 0$, $t > 0$. We conjecture that this parabolic solution attracts all other solutions of (1.1). See the conjecture of Gesmundo and Viani [3].

For the two-dimensional case of (1.1) analogous results were obtained by Baker and Akuezie in [1].

In obtaining our results we have made essential application of algebraic techniques: the Kamke–Müller comparison principle, and a theorem of Smillie concerning the long-run monotonicity of flows of cooperative tridiagonal dynamical systems.

The following notation will be used:

$$\begin{aligned}\mathbf{R}_+^n &= \{(q_1, \dots, q_n) \in \mathbf{R}^n | q_i \geq 0, i = 1, \dots, n\}, \\ \mathbf{R}_{++}^n &= \{(q_1, \dots, q_n) \in \mathbf{R}^n | q_i > 0, i = 1, \dots, n\}. \\ \|\mathbf{x}\| &\text{ is the Euclidean norm of the vector } \mathbf{x}.\end{aligned}$$

Here are the main results:

THEOREM I. *Assume that the $n \times n$ matrix $A = (a_{ij})$ in (1.2) satisfies the following four conditions:*

- (a) $\det A \neq 0$ and $a_{ij} \geq 0$, for $i \neq j$;
- (b) A is irreducible;
- (c) for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}_+^n$, if $x_i \sum_{j=1}^n a_{ij} x_j = 0$ for $i = 1, \dots, n$, then $\mathbf{x} = 0$;
- (d) every real eigenvalue of A is negative.

Then every solution of (1.2) of the form

$$\mathbf{q} = (q_1, \dots, q_n) : [0, a] \rightarrow \mathbf{R}_{++}^n, \quad 0 < a < +\infty,$$

extends uniquely to a solution

$$\mathbf{q} : [0, +\infty) \rightarrow \mathbf{R}_{++}^n,$$

and

$$\lim_{t \rightarrow +\infty} q_i(t) = +\infty, \quad i = 1, \dots, n.$$

Moreover, if $\mathbf{r}(t) = (r_1(t), \dots, r_n(t))$, $t \in [0, +\infty)$, is any other solution of (1.2) in $\mathbf{R}_{++}^n$, then

$$\sup_{0 \leq t < +\infty} \|\mathbf{q}(t) - \mathbf{r}(t)\| < +\infty,$$

and hence

$$\lim_{t \rightarrow +\infty} \frac{q_i(t)}{r_i(t)} = 1, \quad i = 1, \dots, n.$$

Finally, if the matrix A is tridiagonal, then every solution $\mathbf{q}(t), t \in [0, +\infty)$, of (1.2) in $\mathbf{R}_{++}^n$ is eventually monotone increasing on $[0, +\infty)$.

We define a solution $\mathbf{q}(t)$ of (1.1) to be *parabolic* provided it is defined for all $t \geq 0$, takes values in $\mathbf{R}_{++}^3$, and has the following form:

$$\mathbf{q}(t) = (q_1(t), q_2(t), q_3(t)), \quad q_i(t) = c_i \sqrt{t}, \quad c_i > 0, \quad i = 1, 2, 3.$$

THEOREM II. Assume that $m, n > 1$ in the dynamical system (1.1). Then every solution $\mathbf{p} : [0, a] \rightarrow \mathbf{R}_{++}^3, 0 < a < +\infty$, extends uniquely to a solution $[0, +\infty) \rightarrow \mathbf{R}_{++}^3$ such that $\lim_{t \rightarrow +\infty} p_i(t) = +\infty, i = 1, 2, 3$, and this solution is eventually increasing. There exists a unique parabolic solution

$$\mathbf{q}(t) = (q_1(t), q_2(t), q_3(t)), \quad q_i(t) = c_i \sqrt{t}, \quad c_i > 0, \quad i = 1, 2, 3, \quad 0 \leq t < \infty.$$

If $\mathbf{p} : [0, +\infty) \rightarrow \mathbf{R}_{++}^3$ is any other solution, then

$$\sup_{0 \leq t < +\infty} \|\mathbf{p}(t) - \mathbf{q}(t)\| < +\infty;$$

therefore,

$$\lim_{t \rightarrow +\infty} \frac{p_i(t)}{q_i(t)} = 1, \quad \lim_{t \rightarrow +\infty} \frac{p_i(t)}{p_j(t)} = \frac{c_i}{c_j}, \quad 1 \leq i, j \leq n.$$

Based on numerical exploration of the system (1.1), we present the following conjecture.

CONJECTURE. If $\mathbf{q} : [0, +\infty) \rightarrow \mathbf{R}_{++}^3$ is the unique parabolic solution of system (1.1), and if $\mathbf{p} : [0, +\infty) \rightarrow \mathbf{R}_{++}^3$ is any other solution, then

$$\lim_{t \rightarrow +\infty} \|\mathbf{p}(t) - \mathbf{q}(t)\| = 0.$$

2. Preliminaries. In this section we present background material that we will use in the proofs of Theorems I and II. We also introduce a change of variable that transforms system (1.2) into a more convenient form.

Let $F : \mathbf{W} \rightarrow \mathbf{R}^n$ be a continuously differentiable vector field on an open set $\mathbf{W} \subseteq \mathbf{R}_+^n$, and consider the system

$$(2.1) \quad \frac{dx_i}{dt} = F_i(x_1, \dots, x_n), \quad \mathbf{x} = (x_1, \dots, x_n) \in \mathbf{W}, \quad i = 1, \dots, n.$$

DEFINITION 2.1. An $n \times n$ real matrix M is *irreducible* if for each distinct pair of indices i, j with $1 \leq i \neq j \leq n$, there exists a finite sequence $i = k_0, \dots, k_m = j$ such that $M_{k_{r-1}, k_r} \neq 0, r = 1, \dots, m$.

DEFINITION 2.2. Let $E \subseteq \mathbf{W}$ be any subset, where $\mathbf{W}$ is given in (2.1). System (2.1) is called *cooperative* in E if $\partial F_i / \partial x_j(\mathbf{a}) \geq 0$ for $i \neq j, \mathbf{a} \in E$; *irreducible* in E if each matrix $(\partial F_i / \partial x_j(\mathbf{a})), \mathbf{a} \in E$, is irreducible; and *tridiagonal* in E if $\partial F_i / \partial x_j(\mathbf{a}) = 0$ for $|i - j| > 1, \mathbf{a} \in E$.

The next lemma is the Kamke–Müller comparison principle (the last statement proved in Hirsch [5]). The following notation is used: For vectors $\mathbf{x}, \mathbf{y}$ we write $\mathbf{x} < \mathbf{y}$ to mean $x_i < y_i$ for all i . We write $\mathbf{x} \leq \mathbf{y}$ if $x_i \leq y_i$ for all i . If $\mathbf{x} \leq \mathbf{y}$ but $\mathbf{x} \neq \mathbf{y}$ we write $\mathbf{x} < \mathbf{y}$.

LEMMA 2.3. (Müller [8], Kamke [7]). Assume that the system (2.1) is cooperative in a convex subset $E \subseteq \mathbf{W}$ having nonempty interior. Let $\mathbf{x}(t), \mathbf{y}(t)$ be solutions in E

of (2.1) for $a \leq t \leq b$ where $a < b$. If $\mathbf{x}(a) \leq \mathbf{y}(a)$ then $\mathbf{x}(b) \leq \mathbf{y}(b)$. If $\mathbf{x}(a) < \mathbf{y}(a)$ then $\mathbf{x}(b) < \mathbf{y}(b)$. When the system is irreducible in E , if $\mathbf{x}(a) \prec \mathbf{y}(a)$ then $\mathbf{x}(b) \prec \mathbf{y}(b)$.

The next lemma, due to Smillie [10] with improvements by Smith [11], demonstrates long-run monotonicity of solutions to irreducible cooperative tridiagonal systems:

LEMMA 2.4. *Suppose that system (2.1) is cooperative, irreducible and tridiagonal in a convex subset $E \subseteq \mathbf{W}$ having nonempty interior. Let $\mathbf{x}(t)$ be a solution in E of (2.1) on a maximal interval of the form $[0, a)$, $0 < a \leq \infty$. Then each coordinate $x_i(t)$ is eventually monotone increasing or decreasing.*

We conclude this section by making the change of variable $p_i = 1/q_i$ in (1.2); this transforms (1.2) into a more convenient form:

$$(2.2) \quad \frac{dp_i}{dt} = p_i^2 \sum_{j=1}^n a_{ij} p_j = G_i(p_1, \dots, p_n), \quad p_i(t) > 0, \quad i = 1, \dots, n.$$

Notice that the vector field $(G_1, \dots, G_n)$ is defined in all of $\mathbf{R}^n$.

3. Proof of Theorem I. We begin with the following well-known consequence of the Perron–Frobenius theorem.

LEMMA 3.1. *Suppose that the $n \times n$ matrix $A = (a_{ij})$ satisfies (a) and (b) of Theorem I. Then A has an eigenvector $\mathbf{v} > 0$ and a simple real eigenvalue $\alpha \neq 0$ such that $A\mathbf{v} = \alpha\mathbf{v}$ and $\alpha > \operatorname{Re}\beta$ for all other eigenvalues β .*

The following standard property of cooperative systems is very useful in analyzing the geometry of phase portraits.

LEMMA 3.2. *Suppose that the system (2.1) is cooperative in $\mathbf{R}_+^n$, and that $F(\mathbf{0}) = \mathbf{0}$. Let $\mathbf{x}(t)$, $t \in [0, a]$, $0 < a < +\infty$, be a solution of (2.1) in $\mathbf{R}_+^n$ such that $F(\mathbf{x}(0)) < \mathbf{0}$. Then this solution extends to a unique solution in $\mathbf{R}_{++}^n$ defined for $t \in [0, +\infty)$. Moreover, $\mathbf{x}(t)$ is strictly decreasing on $[0, +\infty)$ and converges to an equilibrium as $t \rightarrow \infty$.*

Outline of proof (compare Selgrade [9, Thm. 2.2] and Hirsch [5, Thm. 2.5]). It follows from the Kamke–Müller Comparison Theorem (Lemma 2.3) that $\mathbf{x}(t; \mathbf{x}_0)$ is strictly decreasing on $[0, a]$, and that $\mathbf{x}(0) \geq \mathbf{0}$. The usual compactness argument then implies that the solution extends over the whole positive half-line. Since each coordinate of the solution is decreasing and bounded below by 0, the solution converges, necessarily to an equilibrium. $\square$

Up to a change of variables (i.e., $p_i = 1/q_i$), the next lemma amounts to a special case of Theorem I.

LEMMA 3.3. *Suppose the matrix $A = (a_{ij})$ satisfies (a)–(d) of Theorem I. Let $\mathbf{p}(t)$ be a solution of (2.2) defined for $t \in [0, a]$, $0 < a < +\infty$, taking initial value $\mathbf{p}_0 \in \mathbf{R}_{++}^n$. Then $\mathbf{p}(t)$ uniquely extends to a solution defined for all $t \in [0, +\infty)$, and $\lim_{t \rightarrow +\infty} \mathbf{p}(t) = \mathbf{0}$. Moreover, if A is tridiagonal then $\mathbf{p}(t)$ is eventually decreasing on $[0, +\infty)$.*

Proof. By standard theory $\mathbf{p}(t)$ extends to a solution defined on a maximal interval $[0, b)$, $0 < b \leq \infty$. We need to prove $b = \infty$.

Let $\mathbf{v} > 0$ be the eigenvector of A given in Lemma 3.1 with simple real eigenvalue $\alpha \neq 0$, so that $A\mathbf{v} = \alpha\mathbf{v}$. Condition (d) of Theorem I implies that $\alpha < 0$. Define $c = 1 + \max_{1 \leq i \leq n} \{p_{0i}/v_i\}$, where $\mathbf{p}_0 = (p_{01}, \dots, p_{0n})$ and $\mathbf{v} = (v_1, \dots, v_n)$. Note that $c\mathbf{v} > \mathbf{p}_0$. With G_i given in (2.2), it is easy to see that $G_i(0) = 0$ and $G_i(c\mathbf{v}) = \alpha c^3 v_i^3 < 0$ for $i = 1, \dots, n$.

By Lemma 3.2 there is a solution $\mathbf{r}(t)$, $t \in [0, +\infty)$, of (2.2) in $\mathbf{R}_+^n$, having initial value $c\mathbf{p}_0$, which is strictly decreasing on $[0, +\infty)$. Hence $\lim_{t \rightarrow \infty} \mathbf{r}(t)$ exists as an equilibrium point of (2.2) in $\mathbf{R}_+^n$. Since condition (c) of Theorem I implies that $\mathbf{0}$ is the only equilibrium point in $\mathbf{R}_+^n$, therefore $\lim_{t \rightarrow \infty} \mathbf{r}(t) = \mathbf{0}$.

Condition (a) of Theorem I implies that (2.2) is cooperative in $\mathbf{R}_+^n$. We have $\mathbf{r}(0) = c\mathbf{v} > \mathbf{p}_0 = \mathbf{p}(0)$, hence the Comparison Theorem (Lemma 2.3) shows that $\mathbf{r}(t) > \mathbf{p}(t)$ for all $t \in [0, b)$.

Because the identically zero solution of (2.2) exists we have $\mathbf{p}(t) > \mathbf{0}$, and therefore $\mathbf{0} < \mathbf{p}(t) < \mathbf{r}(t) < c\mathbf{v}$, for all $t \in [0, b)$. The usual compactness argument now shows that $b = \infty$. Lemma 2.3 implies $\mathbf{r}(t) > \mathbf{p}(t) > \mathbf{0}$; therefore, $\lim_{t \rightarrow \infty} \mathbf{p}(t) = \mathbf{0}$.

The eventual monotonicity of solutions follows from Lemma 2.4. $\square$

LEMMA 3.4. *Let the $n \times n$ matrix $A = (a_{ij})$ satisfy conditions (a), (b), and (d) of Theorem I. Let $\mathbf{q}(t)$ and $\mathbf{r}(t)$, $t \in [0, +\infty)$, be solutions of (1.2). Assume $\mathbf{q}(0) < \mathbf{r}(0)$. Then*

$$(3.1) \quad \sup_{0 \leq t < +\infty} \|\mathbf{q}(t) - \mathbf{r}(t)\| < +\infty.$$

Proof. For $t \in [0, +\infty)$ and $i = 1, \dots, n$, define

$$s_i(t) = r_i(t) - q_i(t), \quad u_i(t) = \frac{1}{q_i(t)}, \quad w_i(t) = \frac{1}{r_i(t)}.$$

By Lemma 2.3, $\mathbf{q}(t) < \mathbf{r}(t)$, and thus $s(t) > 0$, for $t \in [0, +\infty)$.

Next define $H_i(x_1, \dots, x_{3n})$, $i = 1, \dots, 3n$, by

$$H_i(x_1, \dots, x_{3n}) = \begin{cases} \sum_{j=1}^n a_{ij} x_j x_{j+n} x_{j+2n} & \text{if } 1 \leq i \leq n; \\ x_i^2 \sum_{j=1}^n a_{ij} x_{j+n} & \text{if } n+1 \leq i \leq 2n; \\ x_i^2 \sum_{j=1}^n a_{ij} x_{j+2n} & \text{if } 2n+1 \leq i \leq 3n. \end{cases}$$

If we identify $(\mathbf{s}, \mathbf{u}, \mathbf{w})$ with the vector $\mathbf{x} = (x_1, \dots, x_{3n})$, then the vector function $(\mathbf{s}(t), \mathbf{u}(t), \mathbf{w}(t))$, $t \in [0, +\infty)$ is a trajectory of the following cooperative dynamical system in $\mathbf{R}_{++}^{3n}$:

$$(3.2) \quad \frac{dx_i}{dt} = H_i(\mathbf{x}), \quad \mathbf{x} \in \mathbf{R}_{++}^{3n}, \quad i = 1, \dots, 3n.$$

Let $\mathbf{v} = (v_1, \dots, v_n) > 0$ be the positive eigenvector of A given in Lemma 3.1, and let $\alpha \neq 0$ be the simple real eigenvalue of A given in that lemma such that $A\mathbf{v} = \alpha\mathbf{v}$. By condition (d) of Theorem I, $\alpha < 0$. Define $\mathbf{z} = (v_1^{-1}, \dots, v_n^{-1}, v_1, \dots, v_n, v_1, \dots, v_n)$. Let $c > 0$ be arbitrary.

It is easy to check that $H_i(c\mathbf{z}) < 0$ and $H_i(0) = 0$, $i = 1, \dots, 3n$. It follows from Lemma 3.2 that there exists a solution $\mathbf{x}(t)$ to equation (3.2) defined for all $t \in [0, +\infty)$, having initial value $c\mathbf{z}$ and taking values in $\mathbf{R}_+^{3n}$, such that $\mathbf{x}(t)$ is monotone decreasing on $[0, +\infty)$.

Now fix $c > 0$ so large that $c\mathbf{z} > (\mathbf{s}(0), \mathbf{u}(0), \mathbf{w}(0))$. Then the comparison principle implies $\mathbf{x}(t) > (\mathbf{s}(t), \mathbf{u}(t), \mathbf{w}(t))$ for all t . Therefore, because $\mathbf{x}(t)$ is monotone decreasing with initial value $c\mathbf{z}$, we see that $\mathbf{0} < \mathbf{r}(t) - \mathbf{q}(t) = \mathbf{s}(t) < c(v_1^{-1}, \dots, v_n^{-1})$ for all $t \in [0, +\infty)$. This proves that (3.1) holds. $\square$

Proof of Theorem I. Suppose the matrix $A = (a_{ij})$ satisfies conditions (a)–(d) of Theorem I. Let $\mathbf{q}(t) = (q_1(t), \dots, q_n(t))$, $t \in [0, a]$, $0 < a < +\infty$, be a solution of (2.1) in $\mathbf{R}_{++}^n$. Define $\mathbf{p}(t) = (p_1(t), \dots, p_n(t))$, $t \in [0, a]$, by

$$p_i(t) = \frac{1}{q_i(t)}, \quad i = 1, \dots, n.$$

Then $\mathbf{p}(t)$, $t \in [0, a]$, is a solution of (2.2) in $\mathbf{R}_{++}^n$, and hence Lemma 3.3 implies that this solution can be uniquely extended to a solution $\mathbf{p}(t)$, $t \in [0, +\infty)$, of (2.2) in $\mathbf{R}_{++}^n$ such that $\lim_{t \rightarrow +\infty} \mathbf{p}(t) = 0$. Moreover, when A is tridiagonal then this extended solution is eventually monotone decreasing on $[0, +\infty)$.

We uniquely extend $\mathbf{q}(t)$, $t \in [0, a]$, to all of $[0, +\infty)$ by the definition

$$q_i(t) = \frac{1}{p_i(t)}, \quad i = 1, \dots, n, \quad t \in [0, +\infty).$$

Then this extension is a solution of (2.1) in $\mathbf{R}_{++}^n$ such that $\lim_{t \rightarrow +\infty} q_i(t) = +\infty$, $i = 1, \dots, n$. If A is tridiagonal then the extended solution $\mathbf{q}(t)$ is easily seen to be eventually monotone increasing. This proves the first part of Theorem I.

To prove the second part of Theorem I let $\mathbf{r}(t)$, $t \in [0, +\infty)$, be an arbitrary solution of (2.1) in $\mathbf{R}_{++}^n$. Let $\mathbf{v} > 0$ be a positive eigenvector of A as given in Lemma 3.1. Select $c > 0$ so small that $c^{-1}(v_1^{-1}, \dots, v_n^{-1}) > \mathbf{q}(0), \mathbf{r}(0)$. Using the same argument that we used in the proof of Lemma 3.3, we see that (2.2) has a solution $\mathbf{p}(t)$ in $\mathbf{R}_{++}^n$ defined for $t \in [0, +\infty)$, with initial value $c\mathbf{v}$, which is strictly decreasing. Define $\mathbf{u}(t) = (u_1(t), \dots, u_n(t))$, $t \in [0, +\infty)$, by

$$u_i(t) = \frac{1}{p_i(t)}, \quad i = 1, \dots, n.$$

Then $\mathbf{u}(t)$, $t \in [0, +\infty)$, is a strictly increasing solution of (2.1) and $\mathbf{u}(0) > \mathbf{q}(0), \mathbf{r}(0)$. By Lemma 3.4 we have

$$\sup_{0 \leq t < +\infty} \|\mathbf{q}(t) - \mathbf{r}(t)\| < +\infty, \quad \sup_{0 \leq t < +\infty} \|\mathbf{r}(t) - \mathbf{u}(t)\| < +\infty.$$

This implies

$$\sup_{0 \leq t < +\infty} \|\mathbf{q}(t) - \mathbf{r}(t)\| < +\infty.$$

For $i = 1, \dots, n$, we have

$$\left| \frac{q_i(t)}{r_i(t)} - 1 \right| \leq \frac{\|\mathbf{q}(t) - \mathbf{r}(t)\|}{r_i(t)}, \quad t \in [0, +\infty).$$

By the first part of Theorem I, we have $\lim_{t \rightarrow +\infty} r_i(t) = +\infty$; it follows that

$$\lim_{t \rightarrow +\infty} \left| \frac{q_i(t)}{r_i(t)} - 1 \right| = 0, \quad i = 1, \dots, n.$$

This completes the proof of Theorem I. $\square$

4. Proof Of Theorem II. In this section we prove Theorem II. We first prove that under the hypothesis of Theorem II, there exists a unique parabolic solution of system (1.1). We prove the remainder of Theorem II by an application of Theorem I.

LEMMA 4.1. *Let $K_1, K_2, K_3 > 0$ and $m, n > 1$. Define the 3×3 matrix $A = (a_{ij})$ as follows:*

$$A = \begin{pmatrix} -\frac{mK_1}{2} & \frac{m-1}{m} \frac{K_2}{2} & 0 \\ \frac{mK_1}{2} & -\left[\frac{m-1}{m} + \frac{n}{m}\right] \frac{K_2}{2} & \frac{n-1}{n} \frac{K_3}{2} \\ 0 & \frac{n}{m} \frac{K_2}{2} & -\frac{K_3}{2} \end{pmatrix}.$$

Then A satisfies conditions (a)–(d) of Theorem I.

Proof. Condition (a) and (b) of Theorem I are easily verified. Because $A = (a_{ij})$ is tridiagonal and $a_{ij} \neq 0$ whenever $|i - j| = 1$, condition (b) is immediate. Condition (c) follows from $\det A \neq 0$ and the fact that all the principal minors of A are nonzero. Hence, to prove the lemma, it suffices to show that (d) holds.

To verify (d) we apply Gershgorin's Circle Theorem (Golub and Van Loan [4]) to the transpose of A , concluding that the eigenvalues of A are contained in the union of the following three closed disks in the complex plane:

$$D_1 : \text{center} = -\frac{mK_1}{2}, \quad \text{radius} = \frac{mK_1}{2},$$

$$D_2 : \text{center} = -\frac{m+n-1}{m} \frac{K_2}{2}, \quad \text{radius} = \frac{|m-1| + n}{m} \frac{K_2}{2},$$

$$D_3 : \text{center} = -\frac{K_3}{2}, \quad \text{radius} = \frac{n-1}{n} \frac{K_3}{2}.$$

Because $m > 1, n > 1$ all three disks are in the closed left half plane, so all eigenvalues have nonpositive real parts. Because A is invertible, all eigenvalues have negative real parts. Thus real eigenvalues are negative. $\square$

For the remainder of this section we will assume that $K_i > 0, i = 1, 2, 3$ and that $m > 1, n > 1$.

The next lemma is a key to proving the existence of a parabolic solution of (1.1). The following notation is used. We denote by Δ^2 the *standard 2-simplex*, i.e., the set of all points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{R}_+^3$ such that $\sum_{i=1}^3 x_i = 1$; we denote the boundary of Δ^2 by $\partial\Delta^2$. Define $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$. For $1 \leq i, j \leq 3$, we let $[e_i, e_j]$ be the boundary simplex determined by the pair e_i, e_j , that is, $[e_i, e_j]$ is the convex hull of the pair e_i, e_j . Observe that $\partial\Delta^2$ is the union of all the boundaries $[e_i, e_j], i \neq j, 1 \leq i, j \leq 3$.

LEMMA 4.2. *Let $f : \Delta^2 \rightarrow \Delta^2$ be a continuous map which maps each vertex to itself and each edge into itself. Then $f(\Delta^2) = \Delta^2$.*

Proof. Standard theorems in algebraic topology show that any extension to the simplex, of a continuous map of the boundary of a simplex to itself having nonzero degree, must map onto the simplex. By looking at each edge it is easy to prove that the restriction of f to the boundary is a map of the boundary to itself which is

homotopic to the identity; it is well known that this implies degree 1. Therefore f is onto. $\square$

Remark. In fact, one can prove that f is a homeomorphism of $\partial\Delta^2$.

To introduce the notation used in the next lemma, define B to be the following matrix:

$$B = \begin{pmatrix} -m & \frac{m-1}{m} & 0 \\ m & -\left[\frac{m-1}{m} + \frac{n}{m}\right] & \frac{n-1}{n} \\ 0 & \frac{n}{m} & -1 \end{pmatrix}.$$

Then B is invertible, hence we may define the 3×3 matrix $P = (P_{ij})$ to be $-\frac{1}{2}B^{-1}$. The matrices A and B are related by the equation $A = B \operatorname{diag}(K_1, K_2, K_3)$. A calculation shows that $P_{ij} > 0$ for all i, j .

LEMMA 4.3. Define $f = (f_1, f_2, f_3) : \Delta^2 \rightarrow \Delta^2$ by

$$f_i(\mathbf{x}) = \left(x_i \sum_{j=1}^3 P_{ij} x_j \right) / \left(\sum_{k=1}^3 x_k \sum_{j=1}^3 P_{kj} x_j \right), \quad \mathbf{x} = (x_1, x_2, x_3) \in \Delta^2, \quad i = 1, 2, 3.$$

Then f maps Δ^2 onto itself.

Proof. It is easy to check that f is continuous and maps each edge of the simplex into itself. Therefore Lemma 4.2 implies f is onto. $\square$

LEMMA 4.4. There exists a unique parabolic solution of (1.1).

Proof. Uniqueness: Let $\mathbf{q}(t)$ and $\mathbf{r}(t)$ be two parabolic solutions of (1.1), with $q_i(t) = c_i \sqrt{t}$, $r_i(t) = d_i \sqrt{t}$, $c_i > 0$, $d_i > 0$, ($i = 1, 2, 3$). By Lemma 4.1, the matrix A in (1.1) satisfies conditions (a)–(d) of Theorem I, hence by that theorem we have $1 = \lim_{t \rightarrow +\infty} q_i(t)/r_i(t) = c_i/d_i$, ($i = 1, 2, 3$). Existence: Let K_i be as in (1.1) and define $\mathbf{y} \in \Delta^2$ by

$$y_i = K_i / \left(\sum_{j=1}^3 K_j \right), \quad i = 1, 2, 3.$$

Let $f : \Delta^2 \rightarrow \Delta^2$ be defined as in Lemma 4.3; then by that lemma there exists a point $\mathbf{u} \in \Delta^2$, such that $\mathbf{y} = f(\mathbf{u})$. Define ζ, η by

$$\zeta = \left(\sum_{j=1}^3 K_j \right)^{1/2}, \quad \eta = \left(\sum_{i=1}^3 u_i \sum_{j=1}^3 P_{ij} u_j \right)^{1/2}$$

Let $\mathbf{c} = (\zeta/\eta)\mathbf{x}$. Then $\mathbf{y} = f(\mathbf{u})$ implies $K_i = c_i \sum_{j=1}^3 P_{ij} c_j$ for $i = 1, 2, 3$. This last set of equations is equivalent to

$$\frac{1}{2} c_i = - \sum_{j=1}^3 a_{ij} \left(\frac{1}{c_j} \right), \quad i = 1, 2, 3.$$

Define $\mathbf{q}(t)$, $t \in (0, +\infty)$, by $q_i(t) = c_i \sqrt{t}$, $i = 1, 2, 3$. The preceding equations imply that $\mathbf{q}(t)$ is a parabolic solution of (1.1). This proves the lemma. $\square$

Proof of Theorem II. To prove Theorem II, let $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t))$, $t \in [0, a]$, $0 < a < +\infty$, be a solution of (1.1) in $\mathbf{R}_{++}^3$. By Lemma 4.1, we may apply Theorem I to the system (1.1), hence there exists a unique extension of $\mathbf{p}(t)$, $t \in [0, a]$, to a solution $\mathbf{p}(t) = (p_1(t), p_2(t), p_3(t))$, $t \in [0, +\infty)$, of (1.1) in $\mathbf{R}_{++}^3$ such that $\lim_{t \rightarrow +\infty} p_i(t) = +\infty$, $i = 1, 2, 3$. Because the matrix A of the system (1.1) is tridiagonal, Theorem I implies that this extended solution is eventually monotone increasing on $[0, +\infty)$. By Lemma 4.4, there exists a unique parabolic solution $\mathbf{q}(t) = (c_1\sqrt{t}, c_2\sqrt{t}, c_3\sqrt{t})$, $(c_1, c_2, c_3) > 0$, $t \in (0, +\infty)$, of (1.1) in $\mathbf{R}_{++}^3$; and by Theorem I, we have

$$\sup_{0 \leq t < +\infty} \|\mathbf{p}(t) - \mathbf{q}(t)\| < +\infty,$$

and hence

$$\lim_{t \rightarrow +\infty} \frac{p_i(t)}{q_i(t)} = 1, \quad i = 1, 2, 3.$$

Therefore, if $1 \leq i, j \leq 3$, then

$$\lim_{t \rightarrow +\infty} \frac{p_i(t)}{p_j(t)} = \left(\frac{c_i}{c_j} \right) \lim_{t \rightarrow +\infty} \left(\frac{p_i(t)/q_i(t)}{p_j(t)/q_j(t)} \right) = \frac{c_i}{c_j}.$$

this completes the proof of Theorem II. $\square$

REFERENCES

- [1] R. BAKER, *The qualitative analysis of a system of differential equations arising from the study of two-layer scales on pure metals*, Pro. Amer. Math. Soc., to appear.
- [2] A. DOLD, *Algebraic Topology*, Springer-Verlag, New York, 1980, p. 56.
- [3] F. GESMUNDO AND F. VIANI, *The formation of multilayer scales in the parabolic oxidation of pure metals*, J. Corrosion Sci., 18 (1978), pp. 217–230.
- [4] G. H. GOLUB AND C. F. VAN LOAN, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, 1983.
- [5] M. W. HIRSCH, *Systems of differential equations that are competitive or cooperative II: Convergence almost everywhere*, SIAM J. Math. Anal., 16 (1985), pp. 423–439.
- [6] M. W. HIRSCH AND S. SMALE, *Differential Equations, Dynamical Systems, and Linear Algebra*, Academic Press, New York, 1974.
- [7] E. KAMKE, *Zur Theorie der System gewöhnlicher Differentialgleichungen II*, Acta Math., 58 (1932), pp. 57–85.
- [8] M. MÜLLER, *Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen*, Math. Z., 26 (1926), pp. 619–645.
- [9] J. SELGRADE, *Asymptotic behavior of solutions to single loop positive feedback systems*, J. Differential Equations, 38 (1980), pp. 80–103.
- [10] J. SMILLIE, *Competitive and cooperative tridiagonal systems of differential equations*, SIAM J. Math. Anal., 5 (1984), pp. 530–534.
- [11] H. L. SMITH, *Periodic tridiagonal competitive and cooperative systems of differential equations*, SIAM J. Math. Anal., 22 (1991), pp. 1102–1109.