GENERALISED HOWE DUALITY AND INJECTIVITY
OF INDUCTION: THE SYMPLECTIC CASE

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Abstract. We study the symplectic Howe duality using two new and independent combinatorial methods: via determinantal formulae on the one hand, and via (bi)crystals on the other hand. The first approach allows us to establish a generalised version where weight multiplicities are replaced by branching coefficients. In turn, this generalised Howe duality is used to prove the injectivity of induction for Levi branchings as previously conjectured by the last two authors.

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1. Introduction

Let n and m be two positive integers and let μ be a partition whose Young diagram is contained in the rectangle $n \times m$ with n rows and m columns. Then the conjugate partition μ' of μ can be written $\mu' = (\mu'_1, \ldots, \mu'_m)$. A classical result in the representation theory of linear Lie algebras states that

> The multiplicity of the irreducible $\mathfrak{gl}_n(\mathbb{C})$-module $V(\lambda)$ in the tensor product
>
> $$\Lambda^{\mu'}_{n,m} = \Lambda^{\mu'_1}(\mathbb{C}^n) \otimes \ldots \otimes \Lambda^{\mu'_m}(\mathbb{C}^n) \tag{1.1}$$
>
> equals the dimension of the μ'-weight space in the irreducible $\mathfrak{gl}_m(\mathbb{C})$-modules $V(\lambda')$.

There exist numerous proofs of this identity. For instance, it is a direct consequence of the Howe duality [How95, Theorem 4.1.1], which can itself be deduced from the well-known Schur–Weyl duality. Most proofs are based on computations on Schur functions or on purely combinatorial arguments using semistandard tableaux [FH91].

For the other classical Lie algebras (or classical Lie groups), there exist similar constructions due to Howe [How95]. In this paper, we restrict ourselves to the symplectic case (i.e. to the root systems of type C). Then, the symplectic Howe duality implies that

> The multiplicity of the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-module $V(\lambda)$ in the tensor product
>
> $$\Lambda_{2n,m}^{\mu'} = \Lambda^{\mu_1'}(\mathbb{C}^{2n}) \otimes \cdots \otimes \Lambda^{\mu_m'}(\mathbb{C}^{2n}) \tag{1.2}$$
>
> equals the dimension of the $\widehat{\mu}$-weight space in the irreducible $\mathfrak{sp}_{2m}(\mathbb{C})$-module $V(\widehat{\lambda})$.

In contrast with the type A case, the conjugate partitions μ' and λ' are here replaced by the partitions $\widehat{\mu}$ and $\widehat{\lambda}$ defined as the conjugates of the complements of μ and λ in the rectangle $n \times m$.

The goal of this paper is three-fold. Firstly, we give a simple combinatorial proof of (1.2) based on the determinantal formulae for Weyl characters (analogue to the Jacobi–Trudi formulae for the Schur polynomials). This approach is essential for establishing the generalisation of (1.2) in Section 7. More precisely, the tools and computations that we use in our proof generalise those developed in [Lec06a] and extend naturally to the case where the fundamental $\mathfrak{gl}_{2n}(\mathbb{C})$-modules appearing in the tensor products (1.2) are replaced by tensor products of simple $\mathfrak{gl}_{2n}(\mathbb{C})$-modules (restricted to $\mathfrak{sp}_{2n}(\mathbb{C})$) or simple $\mathfrak{sp}_{2n}(\mathbb{C})$- modules. We prove in Theorem 7.5 that the corresponding tensor product multiplicities are equal this time to branching coefficients corresponding to the restriction of the simple $\mathfrak{sp}_{2m}(\mathbb{C})$-modules to a block diagonal subalgebras $\mathfrak{s}$ of $\mathfrak{sp}_{2m}(\mathbb{C})$. This means that we need to consider restrictions to subalgebras $\mathfrak{s} \simeq \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where each $\mathfrak{g}_i$ is a Lie algebra isomorphic to $\mathfrak{sp}_{2m_i}$ or $\mathfrak{gl}_{m_i}$ with $m_1 + \cdots + m_r = m$. This generalises (1.2) which corresponds to the case $r = m$ and $m_i = 1$ for all i.

Secondly, the previous identities (1.1) and (1.2) can be generalised when μ' is replaced by any m-tuple β of nonnegative integers. Then the spaces

$$\Lambda_{n,m}^A = \bigoplus_{\substack{\beta=(\beta_1,\ldots,\beta_m)\in\mathbb{Z}_{\geqslant 0}^m \\ \max(\beta)\leqslant n}} \Lambda_{n,m}^\beta \quad \text{and} \quad \Lambda_{2n,m}^C = \bigoplus_{\substack{\beta=(\beta_1,\ldots,\beta_m)\in\mathbb{Z}_{\geqslant 0}^m \\ \max(\beta)\leqslant 2n}} \Lambda_{2n,m}^\beta$$

admit a structure of $\mathfrak{gl}_n \times \mathfrak{gl}_m$-bimodule and of $\mathfrak{sp}_{2n} \times \mathfrak{sp}_{2m}$-bimodule respectively. It is well known that a lot of information about simple modules associated to a simple Lie algebras is encoded by particular combinatorial structures studied by Lusztig, Kashiwara and Littelmann: their crystal graph. It then makes sense to look for bicrystal structures associated to $\Lambda_{n,m}^A$ and $\Lambda_{2n,m}^C$. The second main result of the paper is Theorem 5.11. It uses the combinatorial duality techniques developed in [GL20] to get a simple bijection between the highest weight vertices in the $\mathfrak{sp}_{2n}$-crystal $B(\Lambda_{2n,m}^C)$ associated to $\Lambda_{2n,m}^C$ and the King tableaux (a particular model of tableaux counting the weight multiplicities in type C_m, see [Kin76]). This is reminiscent of results by Lee [Lee19] and Heo and Kwon [HK22] expressed in a different combinatorial

language. In contrast to [Lee19], where a type $\mathfrak{sp}_{2n} \times \mathfrak{sp}_{2m}$-bicrystal structure is proposed, we then study the action of the type A_{2m-1}-crystal operators on $B(\Lambda^C_{2n,m})$. These are indeed the operators which are in connection with the charge statistics defined in [LL20]. In particular, the actions of the A_{2m-1}-crystal operators associated to nodes with unbarred label correspond to contraction operations on columns of type C_n in $\Lambda^C_{2n,m}$ whereas the action corresponding to nodes with barred label yield jeu de taquin operations on the positive or negative part of these columns. This leads to an intriguing statistics on tensor products of type C_n columns which does not coincide with the intrinsic energy defined from their affine crystal structure.

Finally, our third objective is to use the generalised identity Theorem 7.5 to prove a conjecture by the last two authors [GL16]. Consider a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{sp}_{2m}(\mathbb{C})$ and $\nu^{(1)}$, $\nu^{(2)}$ two dominant weights for $\mathfrak{l}$. The conjecture claims that the two $\mathfrak{sp}_{2m}(\mathbb{C})$-modules obtained by induction from $\nu^{(1)}$ and $\nu^{(2)}$ are isomorphic if and only if $\nu^{(1)}$ and $\nu^{(2)}$ coincide up to an automorphism of the Dynkin diagram associated to $\mathfrak{l}$ (or equivalently up to permutation of the components in $\nu^{(1)}$ and $\nu^{(2)}$ associated to isomorphic simple subalgebras). It was proved in [GL16] under restrictive conditions on $\nu^{(1)}$ and $\nu^{(2)}$. Here we prove this conjecture in full generality and obtain in fact a more general result in which $\mathfrak{l}$ can also be replaced by any direct sum $\mathfrak{s}$ of subalgebras of type C. This is Theorem 8.10. Its proof uses Theorem 7.5 and some elegant results by Rajan [Raj14] on the irreducibility of Weyl characters.

Most of the techniques and results developed in this paper can be extended to the orthogonal types. There are nevertheless complications due to the existence of the spin representations and the lack of a natural analogue of King tableaux in the duality context which is relevant for the paper. This will be addressed in future work. We also tried to make the paper more accessible by starting with proofs of some known results using methods that will be central for the generalisations that we propose here and in later work.

The paper is organised as follows. In Section 2, we review some well-known results on the combinatorics of root systems and Lie algebras, mainly to set up the notations that we will use. In Section 3, we use the Jacobi–Trudi formula for Schur functions to (re)prove the type A identity (1.1). This allows us to introduce the main tools and methods which will be reinvested in Section 4 where the type C identity (1.2) is derived similarly from determinantal identities for the Weyl characters of type C_n. We have tried to make Sections 2 to 4 easily readable with only a combinatorial background on symmetric functions and determinantal identities. In Section 5, we use crystals to define a natural combinatorial duality between highest weight vertices in $B(\Lambda^C_{2n,m})$ and King tableaux, thereby giving a bijective proof of (1.1). We then study in Section 6 the behavior of $B(\Lambda^C_{2n,m})$ under some crystal operators of type A_{2m-1}. Section 7 is devoted to establishing the the generalised version of (1.2) in Theorem 7.5 using determinantal techniques similar to that of Section 4. Finally, we prove the generalised version of the conjecture of [GL16] in Section 8.

2. Generalities and settings

Let $\mathfrak{g}$ be the complex Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ (type A) or $\mathfrak{sp}_{2n}(\mathbb{C})$ (type C) with triangular decomposition

$$\mathfrak{g} = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{-\alpha}.$$

The Cartan subalgebra $\mathfrak{h}$ is of dimension n with basis the family of $n \times n$ elementary matrices $E_{i,i}$ for $1 \leqslant i \leqslant n$ when $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and the family of $2n \times 2n$ matrices $E_{i,i} - E_{n+i,n+i}$ for $1 \leqslant i \leqslant n$ when $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$. Let $(\varepsilon_i)_{1 \leqslant i \leqslant n}$ be the dual basis of the basis of $\mathfrak{h}$ described above and let V be the real Euclidean space $\oplus \mathbb{R}\varepsilon_i$ with inner product denoted by $\langle \cdot, \cdot \rangle$. For all $k \in \{1, \ldots, n\}$ let $\omega_k = \sum_{i=1}^k \varepsilon_i$ be the k-th fundamental weight. The weight lattice of $\mathfrak{g}$ is

$$P = \bigoplus_{i=1}^n \mathbb{Z}\varepsilon_i = \bigoplus_{i=1}^n \mathbb{Z}\omega_i.$$

The positive root system R^+ of $\mathfrak{gl}_n(\mathbb{C})$ is of type A_{n-1} and is given by

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, n\}, i < j\}.$$

The positive root system R^+ of $\mathfrak{sp}_{2n}(\mathbb{C})$ is of type C_n and is given by

$$R^+ = \{\varepsilon_i - \varepsilon_j \mid i, j \in \{1, \ldots, n\}, i < j\} \cup \{2\varepsilon_i \mid i = 1, \ldots, n\}.$$

For $i \in \{1, \ldots, n-1\}$ we set $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\alpha_n = 2\varepsilon_n$. Then $\Delta = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ are simple systems in R^+ in type A and C, respectively. We denote by $R = R^+ \cup (-R^+)$ the full root system.

For $\alpha \in R$, let $\alpha^\vee = \dfrac{2\alpha}{\langle \alpha, \alpha \rangle}$ be the coroot associated to α. We then have

$$\langle \alpha_i^\vee, \omega_j \rangle = \delta_{i,j} \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

The Weyl group W of R is the group generated by the orthogonal reflections $(s_\alpha)_{\alpha \in R}$ with respect to the hyperplanes

$$H_\alpha := \{x \in V \mid \langle x, \alpha^\vee \rangle = 0\}.$$

The group W is a finite Coxeter group with distinguished set of generators $S = \{s_\alpha \mid \alpha \in \Delta\}$. The reflection $s_{\varepsilon_i - \varepsilon_j}$ acts on V by permuting the i-th and j-th coordinates. In type C, the reflection $s_{2\varepsilon_i}$ changes the sign of the i-th coordinate.

The closure of the connected components of the set $V \setminus \cup_{\alpha \in R} H_\alpha$ are called the Weyl chambers. The fundamental Weyl chamber is defined by

$$\mathcal{C}_0 = \{x \in V \mid \langle x, \alpha \rangle \geqslant 0 \text{ for all } \alpha \in R^+\}.$$

The Weyl group W acts simply transitively on the set of Weyl chambers.

Let $P^+ = P \cap \mathcal{C}_0$ be the set of dominant weights. Note that for all $\lambda \in P$, there exists $w \in W$ such that $w(\lambda) \in \mathcal{C}_0$. In type A, we have

$$P^+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i - \lambda_{i+1} \geqslant 0 \text{ for all } i = 1, \ldots, n-1\}$$

and any partition of length at most n can be seen as an element of P^+. In type C, we have

$$P^+ = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i - \lambda_{i+1} \geqslant 0 \text{ for all } i \text{ and } \lambda_n \geqslant 0\} = \bigoplus \mathbb{N}\omega_i$$

and there is a bijection between the set of partitions of length at most n and P^+.

We now turn to the representation theory of $\mathfrak{g}$. The *representation ring* of $\mathfrak{g}$ is the ring with basis indexed by the isomorphism classes $[V]$ of irreducible representations V of $\mathfrak{g}$ over $\mathbb{C}$. The addition is defined such that $[V] + [V'] = [V'']$ whenever $V'' \simeq V \oplus V'$ and the multiplication is defined by $[V] \times [W] = [V \otimes W]$. We will denote it by $\mathcal{R}(\mathfrak{g})$.

Let $\mathbb{Z}[P]$ be the integral group ring on the abelian group P. We will write x^λ for the element associated to $\lambda \in P$ so that we have $\mathsf{x}^\lambda \cdot \mathsf{x}^{\lambda'} = \mathsf{x}^{\lambda+\lambda'}$. The Weyl group W of R acts naturally on $\mathbb{Z}[P]$ by setting $w \cdot \mathsf{x}^\lambda = \mathsf{x}^{w(\lambda)}$. Let $x_i = \mathsf{x}^{\varepsilon_i}$ so that $\mathsf{x}^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$ where $\beta = (\beta_1, \ldots, \beta_n) \in P$. Then $\mathbb{Z}[P]$ can be seen as the ring of symmetric Laurent polynomials in n variables. We denote by $\mathbb{Z}[P]^W$ the set of fix points:

$$\mathbb{Z}[P]^W = \{f \in \mathbb{Z}[P] \mid w \cdot f = f\}.$$

Let char be the injective ring homomorphism from the representation ring of $\mathfrak{g}$ to $\mathbb{Z}[P]$ defined by

$$\mathrm{char}([V]) = \sum \dim(V_\mu)\mathsf{x}^\mu$$

where V_μ is the μ-weight space in V. For $\lambda \in P^+$, let $V(\lambda)$ be the irreducible module of highest weight λ in $\mathfrak{g}$.

Remark 2.1. In this paper, we will deal with modules for linear Lie algebras (type A) and symplectic Lie algebras (type C) of various rank. When necessary, we will add a superscript to the notation $V(\lambda)$ to indicate the type and the rank of the Lie algebra we are working with. For instance, we will write $V^C(\lambda)$ for the irreducible module of highest weight λ in a Lie algebra of type C and $V^{C_m}(\lambda)$ if we further want to indicate the rank m of the Lie algebra.

Theorem 2.2 ([FH91, Theorem 23.24])**.** *The homomorphism* char $: \mathcal{R}(\mathfrak{g}) \to \mathbb{Z}[P]^W$ *is an isomorphism.*

Let ρ be the half sum of positive roots. Then the Weyl character formula asserts that, for $\lambda \in P^+$, the character of $V(\lambda)$ is

$$\mathsf{s}_\lambda = \frac{a_{\lambda+\rho}}{a_\rho} \quad \text{where} \quad a_\lambda = \sum_{w \in W} \varepsilon(w)\mathsf{x}^{w\lambda}.$$

The set $\{\mathsf{s}_\lambda \mid \lambda \in P^+\}$ is a basis of $\mathbb{Z}[P]^W$. The Kostant partition function $\mathcal{P}$ is defined by the formula

$$\frac{1}{\prod_{\alpha \in R^+}(1 - \mathsf{x}^\alpha)} = \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}(\beta)\mathsf{x}^\beta.$$

The dot action of the Weyl group on P is defined by

$$w \circ \lambda = w(\lambda + \rho) - \rho = t_{-\rho} w t_\rho(\lambda),$$

where, for all $\gamma \in P$, t_γ denotes the translation by γ.

Theorem 2.3 (Kostant mutliplicity formula)**.** *Let $\lambda, \mu \in P^+$. Let $K_{\lambda,\mu}$ be the dimension of the μ-weight space in the irreducible representation $V(\lambda)$ of highest weight λ. We have*

$$K_{\lambda,\mu} = \sum_{w \in W} \varepsilon(w) \mathcal{P}(w \circ \lambda - \mu).$$

Example 2.4. Assume that $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Then we have $W = \mathfrak{S}_n$ and $w \cdot x_i = x_{w(i)}$ for all $w \in W$. As a consequence, we have

$$\mathbb{Z}[P]^W \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathfrak{S}_n}$$

where $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{\mathfrak{S}_n}$ is the ring of symmetric Laurent polynomials in n variables. The representations $V(\omega_k)$ for $k = 1, \ldots, n$ are isomorphic to $\Lambda^k(\mathbb{C}^n)$, the k-th exterior power of the natural representation $\mathbb{C}^n$ of $\mathfrak{g}$. Then we have $\mathrm{char}(V(\omega_k)) = \mathsf{e}_k(x_1, \ldots, x_n) \in \mathbb{Z}[P]^W$ where e_k is k-th elementary symmetric function in n variables:

$$\mathsf{e}_i(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant n} x_{i_1} \cdots x_{i_k}.$$

The polynomial representations of $\mathfrak{gl}_n(\mathbb{C})$ are the representations V such that $\mathrm{char}(V) \in \mathbb{Z}[x_1, \ldots, x_n]^W$ is a polynomial in the x_i's (and not a Laurent polynomial). The set of weights $\lambda \in P^+$ such that $V(\lambda)$ is polynomial is exactly the set $\mathcal{P}_n$ of partitions with at most n parts and $\{\mathsf{s}_\lambda \mid \lambda \in \mathcal{P}_n\}$ forms a basis of the $\mathbb{Z}[x_1, \ldots, x_n]^W$.

3. The duality in type A

In this section, we prove (Equation (1.1)) using the same methods that we will use to prove the (Equation (1.2)). The root system associated to $\mathfrak{gl}_n(\mathbb{C})$ is described in the previous section.

For all integers $n, m \in \mathbb{Z}_{\geqslant 1}$, we denote by $\mathcal{P}_n$ the set of partitions with at most n parts and by $\mathcal{P}_{n,m}$ the set of partitions in $\mathcal{P}_n$ such that $\lambda_1 \leqslant m$ (i.e. the Young diagram of a partition in $\mathcal{P}_{n,m}$ is included in a rectangle with sides $n \times m$). For any partition $\lambda \in \mathcal{P}_n$, we denote by $(\lambda_1, \ldots, \lambda_n)$ the parts of λ and by λ' the conjugate partition. Note that $\lambda \in \mathcal{P}_{n,m}$ if and only if $\lambda' \in \mathcal{P}_{m,n}$. Recall that any partition $\lambda \in \mathcal{P}_n$ is a dominant weight of $\mathfrak{gl}_n(\mathbb{C})$.

For $N \in \mathbb{Z}_{\geqslant 1}$ and $\gamma, \lambda \in \mathcal{P}_N$, let $K_{\lambda,\gamma}^{A_{N-1}}$ be the dimension of the γ-weight space in the irreducible $\mathfrak{gl}_N(\mathbb{C})$-module $V(\lambda)$ of highest weight λ. For any $\mathfrak{gl}_N(\mathbb{C})$-module M, let $[M : V(\lambda)]$ be the multiplicity of $V(\lambda)$ in M.

Theorem 3.1. *Let $\mu \in \mathcal{P}_{n,m}$ and let $\mu' = (\mu_1', \ldots, \mu_m')$. For all $\lambda \in \mathcal{P}_{n,m}$, we have*

$$\left[\Lambda^{\mu_1'}(\mathbb{C}^n) \otimes \cdots \otimes \Lambda^{\mu_m'}(\mathbb{C}^n) : V(\lambda) \right] = K_{\lambda',\mu'}^{A_{m-1}}$$

where $\Lambda^k(\mathbb{C}^n)$ is the k-th exterior power of the natural representation $\mathbb{C}^n$ of $\mathfrak{gl}_n(\mathbb{C})$.

The character of $\Lambda^k(\mathbb{C}^n)$ is $\mathsf{e}_k(x_1, \ldots, x_n)$ and so the character of the tensor product $\Lambda^{\mu'_1}(\mathbb{C}^n) \otimes \ldots \otimes \Lambda^{\mu'_m}(\mathbb{C}^n)$ is $\mathsf{e}_{\mu'_1} \ldots \mathsf{e}_{\mu'_m}$. If we define $u_{\lambda,\mu'}$ by the relation

$$\mathsf{e}_{\mu'_1} \ldots \mathsf{e}_{\mu'_m} = \sum_{\lambda \in \mathcal{P}_{n,m}} u_{\lambda,\mu'} \mathsf{s}_\lambda$$

the theorem states that $u_{\lambda,\mu'} = K^{A_{m-1}}_{\lambda',\mu'}$ for all $\lambda \in \mathcal{P}_{n,m}$.

For $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$ we define the matrix $\mathsf{V}(\beta)$ by

$$[\mathsf{V}(\beta)]_{i,j} = \mathsf{e}_{\beta_i+j-i}$$

where $\mathsf{e}_k = 0$ whenever $k \notin \{0, \ldots, n\}$. We set $\mathsf{v}_\beta = \det(\mathsf{V}(\beta))$. More explicitely we have:

$$\mathsf{v}_\beta = \begin{vmatrix} \mathsf{e}_{\beta_1} & \mathsf{e}_{\beta_1+1} & \cdots & \mathsf{e}_{\beta_1+m-1} \\ \mathsf{e}_{\beta_2-1} & \mathsf{e}_{\beta_2} & \cdots & \mathsf{e}_{\beta_2+m-2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathsf{e}_{\beta_m-m+1} & \mathsf{e}_{\beta_m-m+2} & \cdots & \mathsf{e}_{\beta_m} \end{vmatrix}.$$

The well-known Jacobi–Trudi formula [FH91, Appendix A] tells us that for all $\lambda \in \mathcal{P}_n$, we have $\mathsf{s}_\lambda = \mathsf{v}_{\lambda'}$.

Given $\beta \in \mathbb{Z}^m$ we set $\mathsf{x}^\beta = x_1^{\beta_1} \cdots x_m^{\beta_m}$ and $\mathsf{e}_\beta = \mathsf{e}_{\beta_1} \cdots \mathsf{e}_{\beta_m}$. We define E to be the linear map

$$\mathsf{E} : \begin{array}{ccc} \mathbb{Z}[P] & \to & \mathbb{Z}[P]^W \\ \mathsf{x}^\beta & \mapsto & \mathsf{e}_\beta \end{array}$$

where P is the weight lattice of $\mathfrak{gl}_m(\mathbb{C})$. This map can be extended to the set of formal series $\mathbb{Z}[[P]]$. Indeed, there are only finitely many $\beta \in \mathbb{Z}^m$ such that $\mathsf{e}_\beta \neq 0$. The map E satisfies the following useful lemma.

Lemma 3.2. *Let β and γ in $\mathbb{Z}^m$ be such that there exists an integer $1 \leqslant s \leqslant m$ with $\beta_i = 0$ for $s + 1 \leqslant i \leqslant m$ and $\gamma_i = 0$ for $1 \leqslant i \leqslant s$. Then* [1]

$$\mathsf{E}(\mathsf{x}^{\beta+\gamma}) = \mathsf{E}(\mathsf{x}^\beta \cdot \mathsf{x}^\gamma) = \mathsf{E}(\mathsf{x}^\beta) \cdot \mathsf{E}(\mathsf{x}^\gamma).$$

Proof. The equality $\mathsf{E}(\mathsf{x}^{\beta+\gamma}) = \mathsf{E}(\mathsf{x}^\beta \cdot \mathsf{x}^\gamma)$ is clear since we have $\mathsf{x}^{\beta+\gamma} = \mathsf{x}^\beta \cdot \mathsf{x}^\gamma$. On the one hand, we have

$$\mathsf{E}(\mathsf{x}^{\beta+\gamma}) = \mathsf{e}_{\beta_1} \cdots \mathsf{e}_{\beta_s} \cdot \mathsf{e}_{\gamma_{s+1}} \cdots \mathsf{e}_{\gamma_m}$$

by the hypothesis on β and γ. On the other hand, the same hypothesis implies that

$$\mathsf{E}(x^\beta) = \mathsf{e}_{\beta_1} \cdots \mathsf{e}_{\beta_s} \quad \text{and} \quad \mathsf{E}(\mathsf{x}^\gamma) = \mathsf{e}_{\gamma_{s+1}} \cdots \mathsf{e}_{\gamma_m}$$

since $\mathsf{e}_{\beta_i} = 1$ for any $s + 1 \leqslant i \leqslant m$ and $\mathsf{e}_{\gamma_i} = 1$ for any $1 \leqslant i \leqslant s$. This yields the desired equality. $\square$

---

[1] The map E is not a morphism of algebras, the equality only holds when the variables are separated in a given monomial.

We set

$$\Delta_m^A = \prod_{\alpha \in R^+} (1 - \mathsf{x}^\alpha) = \prod_{1 \leqslant i < j \leqslant m} (1 - \frac{x_i}{x_j}).$$

Proposition 3.3. *For all $\beta \in \mathbb{Z}^m$ we have $\mathsf{E}(\Delta_m^A \cdot \mathsf{x}^\beta) = \mathsf{v}_\beta$.*

Proof. Let $\beta = (\beta_1, \ldots, \beta_m)$. We have

$$
\begin{vmatrix}
x_1^{\beta_1} & x_1^{\beta_1+1} & \cdots & x_1^{\beta_1+m-1} \\
x_2^{\beta_2-1} & x_2^{\beta_2} & \cdots & x_2^{\beta_2+m-2} \\
\vdots & \vdots & \ddots & \vdots \\
x_m^{\beta_m-m+1} & x_m^{\beta_m-m+2} & \cdots & x_m^{\beta_m}
\end{vmatrix}
= x_1^{\beta_1} \ldots x_m^{\beta_m-m+1} \cdot
\begin{vmatrix}
1 & x_1 & \cdots & x_1^{m-1} \\
1 & x_2 & \cdots & x_2^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_m & \cdots & x_m^{m-1}
\end{vmatrix}
$$

$$
= x_1^{\beta_1} \ldots x_m^{\beta_m-m+1} \cdot \prod_{1 \leqslant i < j \leqslant m} (x_j - x_i)
$$

$$
= x_1^{\beta_1} \ldots x_m^{\beta_m} \cdot \prod_{1 \leqslant i < j \leqslant m} (1 - \frac{x_i}{x_j})
$$

$$
= \Delta_m^A \cdot \mathsf{x}^\beta.
$$

By expanding the determinant, we get

$$
\mathsf{E}(\Delta_m^A \cdot \mathsf{x}^\beta) = \mathsf{E}\left(\sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma) x_1^{\beta_1-(1-\sigma(1))} \ldots x_m^{\beta_m-(1-\sigma(m))} \right)
$$

$$
= \sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma) \mathsf{E}(x_1^{\beta_1-(1-\sigma(1))}) \ldots \mathsf{E}(x_m^{\beta_m-(m-\sigma(m))})
$$

$$
= \sum_{\sigma \in \mathfrak{S}_m} \varepsilon(\sigma) \mathsf{e}_{\beta_1-(1-\sigma(1))} \ldots \mathsf{e}_{\beta_m-(m-\sigma(m))}
$$

$$
= \mathsf{v}_\beta
$$

as required. □

Recall the definition of the dot action in Section 2.

Proposition 3.4. *Let $\beta \in \mathbb{Z}^m$. For all $w \in W$, we have $\mathsf{v}_{w \circ \beta} = \varepsilon(w) \mathsf{v}_\beta$. Further either $\mathsf{v}_\beta = 0$ or there exists a partition $\gamma \in \mathcal{P}_{m,n}$ and $w \in W$ such that $\mathsf{v}_\beta = \varepsilon(w) \mathsf{v}_\gamma$.*

Proof. We prove the first assertion. To do so, it is enough to show that for all $i \in \{1, \ldots, m-1\}$, we have $\mathsf{v}_{s_i \circ \beta} = -\mathsf{v}_\beta$. The matrix $\mathsf{V}(s_i \circ \beta)$ only differs from $\mathsf{V}(\beta)$ on rows i and $i+1$. Let $\beta = (\beta_1, \ldots, \beta_m)$. We have

$$
\begin{aligned}
[\mathsf{V}(s_i \circ \beta)]_{i,j} &= \mathsf{e}_{(s_i \circ \beta)_i - i + j} \\
&= \mathsf{e}_{\beta_{i+1}-1-i+j} \\
&= \mathsf{e}_{\beta_{i+1}-(i+1)+j} \\
&= [\mathsf{V}(\beta)]_{i+1,j}
\end{aligned}
\qquad
\begin{aligned}
[\mathsf{V}(s_i \circ \beta)]_{i+1,j} &= \mathsf{e}_{(s_i \circ \beta)_{i+1}-(i+1)+j} \\
&= \mathsf{e}_{\beta_i+1-(i+1)+j} \\
&= \mathsf{e}_{\beta_i+1-i+j} \\
&= [\mathsf{V}(\beta)]_{i,j}
\end{aligned}
$$

It follows that

$$\mathsf{v}_{s_i \circ \beta} = \det(\mathsf{V}(s_i \circ \beta)) = -\det(\mathsf{V}(\beta)) = -\mathsf{v}_\beta.$$

as required for the first assertion.

We prove the second assertion. Let $\beta \in \mathbb{Z}^m$ be such that $\mathsf{v}_\beta \neq 0$. Using the first relation and the fact that the dot action is simply the action of the Weyl group translated by $-\rho$, we see that there exists $w \in W$ such that $\gamma = w \circ \beta$ lies in the chamber $t_{-\rho}(\mathcal{C}_0)$. This means that for all $\alpha_i = \varepsilon_i - \varepsilon_{i+1} \in \Delta$, we have

$$\langle \gamma, \alpha_i \rangle \geqslant \langle -\rho, \alpha_i \rangle = -1.$$

If there exists α_i such that $\langle \gamma, \alpha_i \rangle = -1$, then $s_i \circ \gamma = \gamma$ and this forces v_γ and v_β to be 0. This is impossible since we assumed that $\mathsf{v}_\beta \neq 0$ hence we must have $\langle \gamma, \alpha_i \rangle \geqslant 0$ for all i, i.e. $\gamma_i \geqslant \gamma_{i+1}$. Next we must have $\gamma_1 \leqslant n$ otherwise the first row of $\mathsf{V}(\gamma)$ is 0 and $\gamma_m \geqslant 0$ otherwise the last row of $\mathsf{V}(\gamma)$ is 0. Finally we have $\gamma \in \mathcal{P}_{m,n}$ and $\mathsf{v}_\beta = \pm \mathsf{v}_\gamma$, as required. $\qquad\square$

We are now ready to prove the Schur duality.

Proof of Theorem 3.1. Let $\mu \in \mathcal{P}_{n,m}$. We have

$$\begin{aligned}
\mathsf{E}(\mathsf{x}^{\mu'}) &= \mathsf{E}\Big(\Delta_m^A \cdot \frac{1}{\Delta_m^A} \cdot \mathsf{x}^{\mu'}\Big) \\
&= \mathsf{E}\Big(\Delta_m^A \cdot \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}(\beta)\mathsf{x}^{\beta+\mu'}\Big) && \text{(by definition of } \mathcal{P}) \\
&= \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}(\beta)\mathsf{E}(\Delta_m^A \cdot \mathsf{x}^{\beta+\mu'}) \\
&= \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}(\beta)\mathsf{v}_{\beta+\mu'} && \text{(by Proposition 3.3)} \\
&= \sum_{\gamma \in \mathcal{P}_{m,n}} \sum_{w \in W} \varepsilon(w)\mathcal{P}(w \circ \gamma - \mu')\mathsf{v}_\gamma && \text{(by Proposition 3.4)} \\
&= \sum_{\gamma \in \mathcal{P}_{m,n}} K_{\gamma,\mu'}^{A_{m-1}}\mathsf{s}_{\gamma'} && \text{(by Theorem 2.3)} \\
&= \sum_{\lambda \in \mathcal{P}_{n,m}} K_{\lambda',\mu'}^{A_{m-1}}\mathsf{s}_\lambda.
\end{aligned}$$

Since $\mathsf{E}(\mathsf{x}^{\mu'}) = \mathsf{e}_{\mu'}$ we get the result. $\qquad\square$

4. The duality in type C

The root system associated to $\mathfrak{sp}_{2n}(\mathbb{C})$ is of type C_n and we keep the notation of Section 2. Recall that the set of dominant weights of $\mathfrak{sp}_{2n}(\mathbb{C})$ is in bijection with $\mathcal{P}_n$. We will freely identify those two sets.

Let V be the Euclidean space of dimension m with basis $(\varepsilon_1, \ldots, \varepsilon_m)$. We define the involution I on V by

$$\mathsf{I}(\beta_1, \ldots, \beta_m) = (-\beta_m, \ldots, -\beta_1).$$

Definition 4.1. Let $\beta \in \mathcal{P}_{n,m}$. We define the partition $\widehat{\beta} \in \mathcal{P}_{m,n}$ by

$$\widehat{\beta} := \mathsf{l}(\beta') + n \cdot (1, \ldots, 1).$$

The partition $\widehat{\beta}$ is obtained by taking the conjugate of the complement of β in the rectangle $n \times m$. Note that the map $\widehat{\cdot}$ depends on the integers n and m. It sends a weight of C_n to a weight of C_m.

Example 4.2. Let $\beta = (5, 4, 2, 1) \in 4 \times 5$. Then we have $\beta' = (4, 3, 2, 2, 1)$ and

$$\widehat{\beta} = \mathsf{l}(\beta') + 4 \cdot (1, 1, 1, 1, 1) = (3, 2, 2, 1, 0).$$

In the figure below, we represent the partition $\widehat{\beta}$ (in green) as the conjugate complement of the partition β (in white) in the rectangle 4×5:
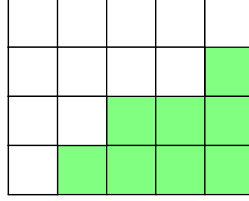
The map $\widehat{\cdot}$ will play a role analogue to the conjugation in Section 3. For $N \in \mathbb{Z}_{\geqslant 1}$ and $\gamma, \lambda \in \mathcal{P}_N$, we write $K_{\lambda,\gamma}^{C_N}$ for the dimension of the γ-weight space in the irreducible representation $V(\lambda) \in \mathfrak{sp}_{2N}(\mathbb{C})$ of highest weight λ.

The following result (stated as (1.2) in the introduction) is a consequence of the symplectic Howe duality [How95, Theorem 3.8.9.3]. We prove it using similar arguments to the proof of Theorem 3.1. This determinantal approach is crucial in our context, because we will see that it can easily be extended to Levi branchings (Theorem 7.5) without using any sophisticated algebraic construction.

Theorem 4.3. Let $\mu \in \mathcal{P}_{n,m}$ and let $\mu' = (\mu'_1, \ldots, \mu'_m)$. For all $\lambda \in \mathcal{P}_{n,m}$, we have

$$\left[\Lambda^{\mu'_1}(\mathbb{C}^{2n}) \otimes \ldots \otimes \Lambda^{\mu'_m}(\mathbb{C}^{2n}) : V(\lambda) \right] = K_{\widehat{\lambda},\widehat{\mu}}^{C_m}$$

where $\Lambda^k(\mathbb{C}^{2n})$ is the k-th exterior power of the natural representation $\mathbb{C}^{2n}$ of $\mathfrak{sp}_{2n}(\mathbb{C})$.

Following [FH91, Section 24.2], the character of $\Lambda^k(\mathbb{C}^{2n})$ is $\mathsf{e}_k(x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1})$. If we define the coefficients u by setting (we omit the variables)

$$\mathsf{e}_{\mu'_1} \ldots \mathsf{e}_{\mu'_m} = \sum_{\lambda \in \mathcal{P}_{n,m}} u_{\lambda,\mu'} \mathsf{s}_\lambda$$

then the theorem states that $u_{\lambda,\mu'} = K_{\widehat{\lambda},\widehat{\mu}}^{C_m}$ for all $\lambda \in \mathcal{P}_{n,m}$.

For $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{Z}^m$ we define the matrix $\mathsf{V}(\beta)$ by

$$[\mathsf{V}(\beta)]_{i,j} = \mathsf{e}_{\beta_i - (i-j)} - \mathsf{e}_{\beta_i - (i+j)}$$

where $e_k = 0$ whenever $k \notin \{0, \ldots, 2n\}$. We set $v_\beta = \det(V(\beta))$. More explicitely we have

$$
v_\beta = \begin{vmatrix}
e_{\beta_1} - e_{\beta_1-2} & e_{\beta_1+1} - e_{\beta_1-3} & \cdots & e_{\beta_1+m-1} - e_{\beta_1-m-1} \\
e_{\beta_2-1} - e_{\beta_2-3} & e_{\beta_2} - e_{\beta_2-4} & \cdots & e_{\beta_2+m-2} - e_{\beta_2-m-2} \\
\vdots & \vdots & \ddots & \vdots \\
e_{\beta_m-m+1} - e_{\beta_m-m-1} & e_{\beta_m-m+2} - e_{\beta_m-m-2} & \cdots & e_{\beta_m} - e_{\beta_m-2m}
\end{vmatrix}.
$$

The Jacobi–Trudi formula in type C_n [Mac95, Section I, Identity 3.5] states that for all $\lambda \in \mathcal{P}_{n,m}$, we have $s_\lambda = v_{\lambda'}$.

Given $\beta \in \mathbb{Z}^m$ we set $x^\beta = x_1^{\beta_1} \ldots x_m^{\beta_m}$ and $e_\beta = e_{\beta_1} \ldots e_{\beta_m}$. We then define E to be the linear application:

$$
\begin{array}{rccc}
E: & \mathbb{Z}[P] & \to & \mathbb{Z}[P]^W \\
& x^\beta & \mapsto & e_\beta
\end{array}
$$

where P is the weight lattice of $\mathfrak{sp}_{2m}(\mathbb{C})$. It is not hard to check that the map E can be extended to $\mathbb{Z}[[P]]$ and that it satisfies Lemma 3.2.

The involution I on V induces a ring involution $\mathbb{Z}[P]$ that maps x^β to $x^{I(\beta)}$ for all $\beta \in P$. In particular it maps $x_i = x^{\varepsilon_i}$ to $x_{m-i}^{-1} = x^{-\varepsilon_{m-i}}$. We will still denote this involution by I. We set

$$
\Delta_m^C = I\left(\prod_{\alpha \in R^+} (1 - x^\alpha) \right) = \prod_{1 \leqslant i < j \leqslant m} \left(1 - \frac{x_i}{x_j} \right) \prod_{1 \leqslant i \leqslant j \leqslant m} \left(1 - \frac{1}{x_i x_j} \right).
$$

Proposition 4.4. *For all $\beta \in \mathbb{Z}^m$, we have* $E(\Delta_m^C \cdot x^\beta) = v_\beta$.

Proof. Let $\beta = (\beta_1, \ldots, \beta_m)$. We have

$$
\begin{aligned}
\det(x_i^{\beta_i-i+j} - x_i^{\beta_i-i-j}) &= \prod_{i=1}^{m} (x_i^{\beta_i-i+1} - x_i^{\beta_i-i-1}) \begin{vmatrix}
1 & x_1 + x_1^{-1} & \cdots & x_1^{m-1} + x_1^{-m+1} \\
1 & x_2 + x_2^{-1} & \cdots & x_2^{m-1} + x_2^{-m+1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_m + x_m^{-1} & \cdots & x_m^{m-1} + x_m^{-m+1}
\end{vmatrix} \\
&= \prod_{i=1}^{m} (x_i^{\beta_i-i+1} - x_i^{\beta_i-i-1}) \begin{vmatrix}
1 & x_1 + x_1^{-1} & \cdots & (x_1 + x_1^{-1})^{m-1} \\
1 & x_2 + x_2^{-1} & \cdots & (x_2 + x_2^{-1})^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_m + x_m^{-1} & \cdots & (x_m + x_m^{-1})^{m-1}
\end{vmatrix} \\
&= \prod_{i=1}^{m} (x_i^{\beta_i-i+1} - x_i^{\beta_i-i-1}) \prod_{1 \leqslant i < j \leqslant m} (x_j + x_j^{-1} - x_i - x_i^{-1}) \\
&= \prod_{i=1}^{m} (x_i^{\beta_i-i+1} - x_i^{\beta_i-i-1}) \prod_{1 \leqslant i < j \leqslant m} x_j (1 - \frac{x_i}{x_j})(1 - \frac{1}{x_i x_j}) \\
&= \prod_{i=1}^{m} (x_i^{\beta_i-i+1} - x_i^{\beta_i-i-1}) x_2 \ldots x_m^{m-1} \prod_{1 \leqslant i < j \leqslant m} (1 - \frac{x_i}{x_j})(1 - \frac{1}{x_i x_j})
\end{aligned}
$$

$$= \prod_{i=1}^{m}(x_i^{\beta_i} - x_i^{\beta_i-2}) \prod_{1\leqslant i<j\leqslant m}(1 - \frac{x_i}{x_j})(1 - \frac{1}{x_i x_j})$$

$$= \mathsf{x}^\beta \prod_{i=1}^{m}(1 - \frac{1}{x_i^2}) \prod_{1\leqslant i<j\leqslant m}(1 - \frac{x_i}{x_j})(1 - \frac{1}{x_i x_j})$$

$$= \Delta_m^C \cdot \mathsf{x}^\beta$$

as required. By expanding the determinant, we get

$$\mathsf{E}(\Delta_m^C \cdot \mathsf{x}^\beta) = \mathsf{E}\left(\sum_{\sigma\in\mathfrak{S}_m}\varepsilon(\sigma)\prod_{i=1}^{m}x_i^{\beta_i-(i-\sigma(i))} - x_i^{\beta_i-(i-\sigma(i))-2}\right)$$

$$= \sum_{\sigma\in\mathfrak{S}_m}\varepsilon(\sigma)\mathsf{E}\left(\prod_{i=1}^{m}x_i^{\beta_i-(i-\sigma(i))} - x_i^{\beta_i-(i-\sigma(i))-2}\right)$$

$$= \sum_{\sigma\in\mathfrak{S}_m}\varepsilon(\sigma)\prod_{i=1}^{m}\mathsf{E}\left(x_i^{\beta_i-(i-\sigma(i))} - x_i^{\beta_i-(i-\sigma(i))-2}\right) \qquad \text{(by Lemma 3.2)}$$

$$= \sum_{\sigma\in\mathfrak{S}_m}\varepsilon(\sigma)\prod_{i=1}^{m}\mathsf{e}_{\beta_i-(i-\sigma(i))} - \mathsf{e}_{\beta_i-(i-\sigma(i))-2}$$

$$= \mathsf{v}_\beta. \qquad\qquad \square$$

In the proof of Theorem 3.1, we used the trick

$$\mathsf{E}(\mathsf{x}^{\mu'}) = \mathsf{E}(\Delta_m^A \cdot \frac{1}{\Delta_m^A}\mathsf{x}^{\mu'})$$

and we expanded the inverse of Δ_m^A with the Kostant partition function. Using the same idea here involves some twisted Kostant partition function.

Definition 4.5. ([Lec06a, Lemma 3.3.1]) The element Δ_m^C is invertible in the ring of formal Laurent series in the variables $(x_i^{\pm 1})_{1\leqslant i\leqslant m}$ and we can define a partition function associated to Δ_m^C by setting

$$\frac{1}{\Delta_m^C} = \sum_{\beta\in\mathbb{Z}^m}\widetilde{\mathcal{P}}(\beta)\mathsf{x}^\beta.$$

Lemma 4.6. *We have* $\widetilde{\mathcal{P}}(\beta) = \mathcal{P}(\mathsf{I}(\beta))$ *where* $\mathcal{P}$ *is the usual Kostant partition function.*

Proof. Using the fact that I is an involution we get

$$\frac{1}{\Delta_m^C} = \mathsf{I}\left(\frac{1}{\prod_{\alpha\in R_+}(1-\mathsf{x}^\alpha))}\right) = \mathsf{I}(\sum_{\beta\in\mathbb{Z}^m}\mathcal{P}(\beta)\mathsf{x}^\beta) = \sum_{\beta\in\mathbb{Z}^m}\mathcal{P}(\beta)\mathsf{x}^{\mathsf{I}(\beta)} = \sum_{\beta\in\mathbb{Z}^m}\mathcal{P}(\mathsf{I}(\beta))\mathsf{x}^\beta$$

hence the result. $\qquad\qquad \square$

The next step in our proof of Theorem 3.1 was to show that the determinant v_β satisfied the relation $\mathsf{v}_{w \circ \beta} = \varepsilon(w)\mathsf{v}_\beta$ for all $w \in \mathfrak{S}_m$. In type C_m, we need to modify the dot action to obtain such a relation. More precisely, we are looking for $\delta \in P$ such that we have for all $w \in W$:

$$\mathsf{v}_{w \circ \beta} = \varepsilon(w)\mathsf{v}_\beta \quad \text{where} \quad w \circ \beta = w(\beta + \delta) - \delta.$$

It will turn out that there is a unique δ satisfying this relation. The proof is based on the following symmetry.

Proposition 4.7. *We have* $\mathsf{e}_{n+k} = \mathsf{e}_{n-k}$ *for all* $k \in \mathbb{Z}$ *where* $\mathsf{e}_k = 0$ *if* $k \notin \{1, \dots, 2n\}$.

Recall that the m-th fundamental weight in the root system of type C_m is $\omega_m = (1, \dots, 1)$ and that the half sum of positive roots is $\rho_m = (m, \dots, 1)$.

Theorem 4.8. *Let* $\delta = \mathsf{l}(\rho_m) - n \cdot \omega_m \in \mathbb{Z}^m$ *and define a "dot action" of* W *on* $\mathbb{Z}^m$ *by setting*

$$w \circ \beta = w(\beta + \delta) - \delta.$$

We have

(1) For all $\beta \in \mathbb{Z}^m$, we have $\mathsf{v}_{w \circ \beta} = \varepsilon(w)\mathsf{v}_\beta$.

(2) If $\mathsf{v}_\beta \neq 0$, there exists a partition $\gamma \in \mathcal{P}_{m,n}$ and $w \in W$ such that $w \circ \beta = \gamma$ and $\mathsf{v}_\beta = \varepsilon(w)\mathsf{v}_\gamma$.

Proof. We have $\delta = (-n - 1, \dots, -n - m)$. Recall that the Weyl group W is generated by the set $\{s_1, \dots, s_{m-1}, s_m\}$ where s_i acts on V by permuting the i and the $i + 1$-th coordinates and s_m acts by changing the sign of the m-th coordinate.

(1) To prove this assertion, it is enough to show that

$$\mathsf{v}_{s_i \circ \beta} = -\mathsf{v}_\beta \text{ for all } 1 \leqslant i \leqslant m - 1 \quad \text{and} \quad \mathsf{v}_{s_m \circ \beta} = -\mathsf{v}_\beta.$$

Let $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{Z}^m$ and $i \in \{1, \dots, m - 1\}$. We have

$$\begin{aligned}
s_i \circ \beta &= s_i(\beta + \delta) - \delta \\
&= (\beta_1 + \delta_1, \dots, \beta_{i+1} + \delta_{i+1}, \beta_i + \delta_i, \dots, \beta_m + \delta_m) - (\delta_1, \dots, \delta_m) \\
&= (\beta_1, \dots, \beta_{i+1} + \delta_{i+1} - \delta_i, \beta_i + \delta_i - \delta_{i+1}, \dots, \beta_m) \\
&= (\beta_1, \dots, \beta_{i+1} - 1, \beta_i + 1, \dots, \beta_m).
\end{aligned}$$

The matrix $\mathsf{V}(s_i \circ \beta)$ only differs from the matrix $\mathsf{V}(\beta)$ on rows i and $i + 1$. We have

$$\begin{aligned}
[\mathsf{V}(s_i \circ \beta)]_{i,j} &= \mathsf{e}_{[s_i \circ \beta]_i - (i+j)} - \mathsf{e}_{[s_i \circ \beta]_i - (i-j)} \\
&= \mathsf{e}_{\beta_{i+1} - 1 - (i+j)} - \mathsf{e}_{\beta_{i+1} - 1 - (i-j)} \\
&= \mathsf{e}_{\beta_{i+1} - ((i+1)+j)} - \mathsf{e}_{\beta_{i+1} - ((i+1)-j)} \\
&= [\mathsf{V}(\beta)]_{i+1,j}
\end{aligned}$$

$$[V(s_i \circ \beta)]_{i+1,j} = e_{[s_i \circ \beta]_{i+1} - (i+1+j)} - e_{[s_i \circ \beta]_{i+1} - (i+1-j)}$$
$$= e_{\beta_i + 1 - (i+1+j)} - e_{\beta_i + 1 - (i+1-j)}$$
$$= e_{\beta_i - (i+j)} - e_{\beta_i - (i-j)}$$
$$= [V(\beta)]_{i,j}$$

Hence the result on the determinant.

Next we have

$$s_m \circ \beta = s_m(\beta + \delta) - \delta$$
$$= (\beta_1 + \delta_1, \ldots, \beta_{m-1} + \delta_{m-1}, -\beta_m - \delta_m) - (\delta_1, \ldots, \delta_m)$$
$$= (\beta_1, \ldots, \beta_{m-1}, -\beta_m - 2\delta_m)$$
$$= (\beta_1, \ldots, \beta_{m-1}, -\beta_m + 2n + 2m).$$

The matrix $V(s_n \circ \beta)$ only differs from the matrix $V(\beta)$ on rows m and we have

$$[V(s_m \circ \beta)]_{m,j} = e_{[s_n \circ \beta]_m - (m+j)} - e_{[s_i \circ \beta]_m - (m-j)}$$
$$= e_{-\beta_m + 2n + 2m - (m+j)} - e_{-\beta_m + 2n + 2m - (m-j)}$$
$$= e_{n + (n - \beta_m + m - j)} - e_{n + (n - \beta_m + m + j)}$$
$$= e_{n - (n - \beta_m + m - j)} - e_{n - (n - \beta_m + m + j)} \qquad \text{(by the previous proposition)}$$
$$= e_{\beta_m - m + j} - e_{\beta_m - m - j}$$
$$= -[V(\beta)]_{m,j}$$

hence the result on the determinant.

(2) Let $\beta \in \mathbb{Z}^m$ be such that $v_\beta \neq 0$. The "dot action" is simply the action of W translated by $-\delta$. This action is transitive on the set of Weyl chambers centered at δ. The set of Weyl chambers centered at $-\delta$ is the set of Weyl chambers centered at 0 translated by $-\delta$. Now the set of Weyl chambers centered at 0 is parametrised by the set of simple system of the form $w\Delta$ where $w \in W$. More precisely, the Weyl chamber $w(\mathcal{C}_0)$ is

$$w(\mathcal{C}_0) = \{x \in V \mid \langle x, \alpha \rangle \geqslant 0 \text{ for all } \alpha \in w(\Delta)\}.$$

We have $I \in W$ and

$$I(\Delta) = \{I(\alpha_1), \ldots, I(\alpha_{m-1}), I(2\varepsilon_m)\} = \{\alpha_1, \ldots, \alpha_{m-1}, -2\varepsilon_1\}.$$

As a consequence the Weyl chamber $t_{-\delta}(I(\mathcal{C}_0))$ is defined by

$$x \in t_{-\delta}(I(\mathcal{C}_0)) \Leftrightarrow \langle x, \alpha^\vee \rangle \geqslant \langle -\delta, \alpha^\vee \rangle \text{ for all } \alpha \in I(\Delta).$$

By transitivity of the dot action, there exists $w \in W$ such that $\gamma = w \circ \beta \in t_{-\delta}(I(\mathcal{C}_0))$. Further, since we assumed that $v_\beta \neq 0$, γ cannot lie on the wall of the chamber $t_{-\delta}(I(\mathcal{C}_0))$.

The equations above for the root $\alpha_i \in \mathsf{I}(\Delta)$ and $-2\varepsilon_1 \in \mathsf{I}(\Delta)$ yield (using $\alpha_i^\vee = \alpha_i$ and $(-2\varepsilon_1)^\vee = -\varepsilon_1$)

$$\langle \gamma, \alpha_i \rangle > \langle -\delta, \alpha_i \rangle = -\delta_i + \delta_{i+1} = -1 \quad \text{and} \quad \langle \gamma, -\varepsilon_1 \rangle > \langle -\delta, -\varepsilon_1 \rangle = -(n+1)$$

in other words

$$\gamma_i - \gamma_{i+1} \geqslant 0 \quad \text{and} \quad \gamma_1 \leqslant n$$

which was what we were looking for, since $\mathsf{v}_\beta = \varepsilon(w)\mathsf{v}_\gamma$.

$\square$

We are now ready to prove Theorem 4.3.

Proof of Theorem 4.3. Let $\mu \in \mathcal{P}_{n,m}$. First we have

$$
\begin{aligned}
\mathsf{E}(\mathsf{x}^{\mu'}) &= \mathsf{E}\left(\Delta_m^C \cdot \frac{1}{\Delta_m^C} \cdot \mathsf{x}^{\mu'} \right) \\
&= \mathsf{E}\left(\Delta_m^C \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}(\beta) \mathsf{x}^{\beta+\mu'} \right) \\
&= \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}(\beta) \mathsf{E}\left(\Delta_m^C \mathsf{x}^{\beta+\mu'} \right) \\
&= \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}(\beta) \mathsf{v}_{\beta+\mu'} \\
&= \sum_{\gamma \in \mathcal{P}_{m,n}} \sum_{w \in W} \varepsilon(w) \widetilde{\mathcal{P}}(w \circ \gamma - \mu') \mathsf{v}_\gamma \\
&= \sum_{\gamma \in \mathcal{P}_{n,m}} \sum_{w \in W} \varepsilon(w) \widetilde{\mathcal{P}}(w \circ \lambda' - \mu') \mathsf{s}_\lambda.
\end{aligned}
$$

This shows that

$$u_{\lambda, \mu'} = \sum_{w \in W} \varepsilon(w) \widetilde{\mathcal{P}}(w \circ \lambda' - \mu').$$

Next, using the fact that $\mathsf{I}(\delta_{m,n}) = \rho_m + n \cdot \omega_m$ and the equality $\widetilde{\mathcal{P}}(\beta) = \mathcal{P}(\mathsf{I}(\beta))$ we get

$$
\begin{aligned}
\sum_{w \in W} \varepsilon(w) \widetilde{\mathcal{P}}(w \circ \lambda' - \mu') &= \sum_{w \in W} \varepsilon(w) \mathcal{P}(\mathsf{I}(w \circ \lambda') - \mathsf{I}(\mu')) \\
&= \sum_{w \in W} \varepsilon(w) \mathcal{P}(\mathsf{I}(w(\lambda' + \delta_{m,n}) - \delta_{m,n}) - \mathsf{I}(\mu')) \\
&= \sum_{w \in W} \varepsilon(w) \mathcal{P}(w(\mathsf{I}(\lambda') + \mathsf{I}(\delta_{m,n})) - \mathsf{I}(\delta_{m,n}) - \mathsf{I}(\mu')) \\
&= \sum_{w \in W} \varepsilon(w) \mathcal{P}(w(\mathsf{I}(\lambda') + \rho + n \cdot \omega_m) - \rho - n \cdot \omega_m - \mathsf{I}(\mu'))
\end{aligned}
$$

$$= \sum_{w \in W} \varepsilon(w) \mathcal{P}(w(\mathsf{I}(\lambda') + n \cdot \omega_m + \rho) - (\mathsf{I}(\mu') + n \cdot \omega_m) - \rho)$$

$$= \sum_{w \in W} \varepsilon(w) \mathcal{P}(w(\widehat{\lambda} + \rho) - \widehat{\mu} - \rho)$$

$$= K_{\widehat{\lambda},\widehat{\mu}}^{C_m}$$

as desired. $\square$

5. The combinatorial duality

The goal of this section is to give a bijective proof of Theorem 4.3. This is achieved by establishing in Theorem 5.11 a combinatorial duality between a certain set of tensor products of type C_n columns on the one hand, and a set of type C_m tableaux called *King tableaux* on the other hand. To make this duality consistent with the usual convention on the combinatorial objects that we shall need (tensor products of crystals and King tableaux) it will be convenient to use the following realisation of the root system of type C_n.

Let $n \in \mathbb{Z}_{\geqslant 1}$ and consider the type C_n alphabet

$$\mathscr{C}_n = \left\{ \overline{n} < \cdots < \overline{1} < 1 < \cdots < n \right\}.$$

This enables us to realise the root system of type C_n by setting

$$\begin{cases} \alpha_i = \varepsilon_{\overline{i}} - \varepsilon_{\overline{i+1}} & \text{for } i = 1, \ldots, n-1 \\ \alpha_0 = 2\varepsilon_{\overline{1}}. \end{cases}$$

for the simple roots, where $\varepsilon_{\overline{i}} = -\varepsilon_i$ for all $i = 1, \ldots, n$, and

$$\omega_i = \varepsilon_{\overline{n}} + \cdots + \varepsilon_{\overline{i+1}} \quad \text{for } i = 0, \ldots, n-1.$$

A column of height k on $\mathscr{C}_n$ (also called a column of type C_n) is a subset c of $\mathscr{C}_n$ of cardinality k, which we represent by the Young tableau of shape (1^k) filled by the elements of c, increasing from top to bottom. We will write $|c| = k$. For any column c on $\mathscr{C}_n$, and for all $i = 1, \ldots, n$ let

$$N_i(c) = \left| \left\{ x \in c \mid x \leqslant \overline{i} \text{ or } x \geqslant i \right\} \right|.$$

Definition 5.1. A column c on $\mathscr{C}_n$ is called n-*admissible* if $N_i(c) \leqslant n-i+1$ for all $i = 1, \ldots, n$.

Example 5.2. The set $c = \{\overline{2}, \overline{1}, 1, 3\} = \begin{array}{|c|} \hline \overline{2} \\ \hline \overline{1} \\ \hline 1 \\ \hline 3 \\ \hline \end{array}$ is a column on $\mathscr{C}_n$ for all $n \geqslant 3$, and we have

$|c| = 4$. It is not 3-admissible since $N_1(c) = \left| \{3, \overline{2}, \overline{1}, 1\} \right| = 4 > 3 - 1 + 1$. However, c is n-admissible for $n \geqslant 4$.

Let us recall the crystal structure on the set of columns of a given height due to [KN94, Section 4.3]. First, columns of height 1 realise the crystal of the vector representation of type C_n as follows:

$$\boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}} \xrightarrow{0} \boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n}$$

where the arrow labeled by i denotes the action of the Kashiwara crystal operator f_i. In other terms, since

$$\text{wt}\left(\boxed{x}\right) = \sum_{i=1}^n a_i \varepsilon_{\overline{i}} \quad \text{where} \quad a_i = \begin{cases} 1 & \text{if } x = \overline{i} \\ -1 & \text{if } x = i \\ 0 & \text{otherwise,} \end{cases}$$

this realises the crystal $B(\omega_{n-1})$ of the fundamental representation $V(\omega_{n-1})$. To get the crystal structure on any tensor power $B(\omega_{n-1})^{\otimes \ell}$, we use Kashiwara's tensor product rule. It works as follows. Each vertex $b = \boxed{x_1} \otimes \boxed{x_2} \otimes \cdots \otimes \boxed{x_\ell} \in B(\omega_{n-1})^{\otimes \ell}$ can be identified with its word $w = x_1 \cdots x_\ell$ of length ℓ on $\mathscr{C}_n$. Now, label each letter of w in $\{\overline{i+1}, i\}$ (resp. in $\{\overline{i}, i+1\}$) with a symbol $+$ (resp. $-$) and ignore the others. Let w_i be the word in the symbols $+$ and $-$ so obtained. Bracket recursively all possible $+-$ in w_i (forgetting the previously bracketed symbols). Then $f_i w$ (resp. $e_i w$) is obtained by applying f_i (resp. e_i) to the letter of w which contributes as the leftmost unbracketed $+$ (resp. the rightmost unbracketed $-$). If this symbol does not exist, we set $f_i w = 0$ (resp. $e_i w = 0$). To compute $f_0 w$ (resp. $e_0 w$), one proceeds similarly but this time by encoding only the letters $\overline{n}$ in w by a $+$ and the letters n by a $-$.

Example 5.3. Let $n = 2$ and $w = \overline{1}2\overline{2}1221\overline{1}\overline{2}$. Then $w_1 = --++--+-+$ and we get the bracketing $w_1 = --(+(+-)-)(+-)+$. This gives $f_1 w = \overline{1}2\overline{2}1221\overline{1}\overline{1}$ and $e_1 w = \overline{1}1\overline{2}1221\overline{1}\overline{2}$.
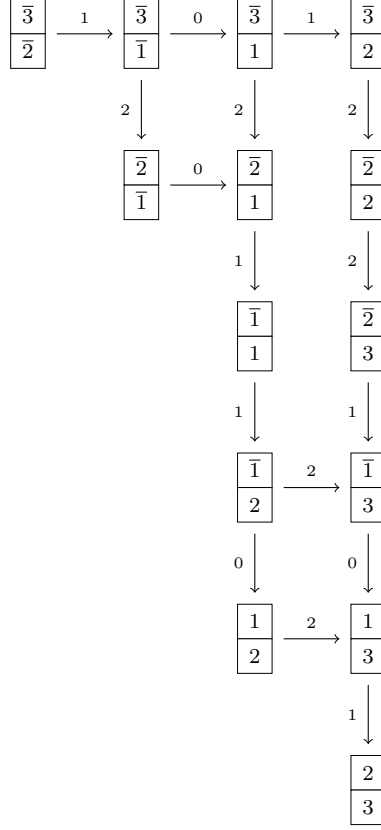
One can now interpret any column $c = \boxed{\begin{matrix} x_1 \\ \vdots \\ x_\ell \end{matrix}}$ of height ℓ as the element $\boxed{x_1} \otimes \cdots \otimes \boxed{x_\ell} \in$

$B(\omega_{n-1})^{\otimes \ell}$ In particular, for all $i = 0, \ldots, n-1$, the column $c = \boxed{\begin{matrix} \overline{n} \\ \vdots \\ \overline{n\text{-}i} \end{matrix}}$ is a highest weight vertex of

weight ω_{n-i-1}, and therefore generates the crystal $B(\omega_{n-i-1})$ of the fundamental representation $V(\omega_{n-i-1})$. Note that the shape of c is given by the partition (1^{i+1}), that is, the coordinates of $\text{wt}(c) = \omega_{n-i-1}$ in the basis $(\varepsilon_{\overline{n}}, \ldots, \varepsilon_{\overline{1}})$. The following result is due to [KN94, Section 4.5], see also [Lec07, Proposition 4.2.1] for the reformulation using Definition 5.1.

Theorem 5.4. *The vertices of $B(\omega_{n-i-1})$ are the n-admissible columns.*

Example 5.5. Take $n = 3$ and $i = 1$. We realise the fundamental crystal $B(\omega_1)$ by n-admissible columns of height 2 as follows. Note that only $\boxed{\begin{matrix} \overline{3} \\ 3 \end{matrix}}$ is not n-admissible.

$$
\boxed{\begin{array}{c}\overline{3}\\\overline{2}\end{array}} \xrightarrow{\ 1\ } \boxed{\begin{array}{c}\overline{3}\\\overline{1}\end{array}} \xrightarrow{\ 0\ } \boxed{\begin{array}{c}\overline{3}\\1\end{array}} \xrightarrow{\ 1\ } \boxed{\begin{array}{c}\overline{3}\\2\end{array}}
$$

$$
\Big\downarrow{\scriptstyle 2} \qquad \Big\downarrow{\scriptstyle 2} \qquad \Big\downarrow{\scriptstyle 2}
$$

$$
\boxed{\begin{array}{c}\overline{2}\\\overline{1}\end{array}} \xrightarrow{\ 0\ } \boxed{\begin{array}{c}\overline{2}\\1\end{array}} \qquad \boxed{\begin{array}{c}\overline{2}\\2\end{array}}
$$

$$
\Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 2}
$$

$$
\boxed{\begin{array}{c}\overline{1}\\1\end{array}} \qquad \boxed{\begin{array}{c}\overline{2}\\3\end{array}}
$$

$$
\Big\downarrow{\scriptstyle 1} \qquad \Big\downarrow{\scriptstyle 1}
$$

$$
\boxed{\begin{array}{c}\overline{1}\\2\end{array}} \xrightarrow{\ 2\ } \boxed{\begin{array}{c}\overline{1}\\3\end{array}}
$$

$$
\Big\downarrow{\scriptstyle 0} \qquad \Big\downarrow{\scriptstyle 0}
$$

$$
\boxed{\begin{array}{c}1\\2\end{array}} \xrightarrow{\ 2\ } \boxed{\begin{array}{c}1\\3\end{array}}
$$

$$
\Big\downarrow{\scriptstyle 1}
$$

$$
\boxed{\begin{array}{c}2\\3\end{array}}
$$

Let us now fix $n, m \in \mathbb{Z}_{\geqslant 1}$. In the spirit of [GL20, Section 2.2], we define a "combinatorial Fock space" which will naturally be endowed with a type C_n crystal structure.

Definition 5.6. The *Kashiwara–Nakashima (KN) Fock space* is the set

$$
\mathcal{F}_{n,m} = \{c_1 \otimes \cdots \otimes c_m \mid c_j \text{ is a column on } \mathscr{C}_n \text{ for all } j = 1, \ldots, m\}.
$$

We will be interested in a particular subset of the KN Fock space. Denote by $\mathcal{C}_{m,2n}$ the set of compositions $\mu' = (\mu'_1, \ldots, \mu'_m)$ such that $\max\{\mu'_j \,;\, 1 \leqslant j \leqslant m\} \leqslant 2n$. For $\mu' \in \mathcal{C}_{m,2n}$, consider the set

$$
B_{\mu'} = \{c_1 \otimes \cdots \otimes c_m \in \mathcal{F}_{n,m} \mid |c_j| = \mu'_j \text{ for all } j = 1, \ldots, m\}.
$$

By choosing to read the m columns of any vertex of $B_{\mu'}$ *first from left to right and next from top bottom*, we get an embedding of crystals $B_{\mu'} \hookrightarrow B(\omega_{n-1})^{\otimes |\mu|}$, and $B_{\mu'}$ realises the crystal of the representation

$$
\Lambda^{\mu'}_{2n,m} = \Lambda^{\mu'_1}(\mathbb{C}^{2n}) \otimes \cdots \otimes \Lambda^{\mu'_m}(\mathbb{C}^{2n})
$$

of $\mathfrak{sp}_{2n}(\mathbb{C})$ (which we have encountered in Section 4). Moreover, we have the decomposition as direct sum of crystals

$$
\mathcal{F}_{n,m} = \bigoplus_{\mu' \in \mathcal{C}_{m,2n}} B_{\mu'}.
$$

Note that in the basis $(\varepsilon_{\overline{n}}, \ldots, \varepsilon_{\overline{1}})$, we have $\mathrm{wt}(b) = (a_n, \ldots, a_1)$ where, for all $i = 1, \ldots, n$,

$$a_i = \# \text{ entries } \overline{i} \text{ in } b - \# \text{ entries } i \text{ in } b.$$

Inside $B_{\mu'}$, we consider the following two subsets

$$B_{\mu'}^{\mathrm{hw}} = \{b \in B_{\mu'} \mid e_i(b) = 0 \text{ for all } i = 0, \ldots, n-1\} \quad \text{and} \quad B_{\mu',\lambda}^{\mathrm{hw}} = \{b \in B_{\mu'}^{\mathrm{hw}} \mid \mathrm{wt}(b) = \lambda\}$$

for any $\lambda \in \mathcal{P}_{n,m}$. In other terms, $B_{\mu'}^{\mathrm{hw}}$ is the set of highest weight vertices in $B_{\mu'}$.

Let us recall briefly how to check that an element $b \in B_{\mu'}$ is in $B_{\mu'}^{\mathrm{hw}}$. Let w be the word obtained by reading b (as explained above). Define similarly the *weight* of a word on $\mathscr{C}_n$ to be the n-tuple whose i-th coordinate is the difference between the number of $\overline{i}$'s and the number of i's. Then b is a highest weight vertex if and only if the weight of each prefix of w is a partition.

Example 5.7. Let $n = 4$, $m = 3$, $\mu' = (2,3,1)$ and $b = \boxed{\begin{array}{c}\overline{4}\\\overline{3}\end{array}} \otimes \boxed{\begin{array}{c}\overline{2}\\\overline{1}\\1\end{array}} \otimes \boxed{4} \in B_{\mu'}$,

so that $\mathrm{w} = \overline{4}\,\overline{3}\,\overline{2}\,\overline{1}\,1\,4$. The prefixes of w are $\overline{4}, \overline{4}\overline{3}, \overline{4}\overline{3}\overline{2}, \overline{4}\overline{3}\overline{2}\overline{1}, \overline{4}\overline{3}\overline{2}\overline{1}1, \overline{4}\overline{3}\overline{2}\overline{1}1\overline{4}$ with respective weights $(1,0,0,0), (1,1,0,0), (1,1,1,0), (1,1,1,1), (1,1,1,0), (2,1,1,0)$, which are all partitions, therefore b is a highest weight vertex. More precisely, $b \in B_{(2,3,1),(2,1,1,0)}^{\mathrm{hw}}$.

Consider now the alternative alphabet $\mathscr{C}_m^* = \{1 < \overline{1} < \cdots < m < \overline{m}\}$. Similarly to Definition 5.6, we can consider columns on $\mathscr{C}_m^*$ and construct another combinatorial Fock space.

Definition 5.8. The *King Fock space* is the set

$$\dot{\mathcal{F}}_{m,n} = \{d_1 \otimes \cdots \otimes d_n \mid d_i \text{ is a column on } \mathscr{C}_m^* \text{ for all } i = 1, \ldots, n\}.$$

Remark 5.9. Unlike $\mathcal{F}_{n,m}$, there is no simple C_m-crystal structure on $\dot{\mathcal{F}}_{m,n}$. This will be discussed in more detail in Section 6.

We are ready to define a duality

$$\begin{array}{ccc} \mathcal{F}_{n,m} & \longrightarrow & \dot{\mathcal{F}}_{m,n} \\ b & \longmapsto & b^*. \end{array}$$

Let μ' be a composition as before and $b = c_1 \otimes \cdots \otimes c_m \in B_{\mu'}$. For each $j = 1, \ldots, m$, let

$$\begin{aligned} \tilde{c}_j &= \{1 \leqslant x \leqslant n \mid \overline{x} \notin c_j\} \quad \text{and} \\ \tilde{c}_{\overline{j}} &= \{1 \leqslant x \leqslant n \mid x \in c_j\} = \{1, \ldots, n\} \cap c_j, \end{aligned}$$

and set

$$\tilde{b} = \tilde{c}_1 \otimes \tilde{c}_{\overline{1}} \otimes \cdots \otimes \tilde{c}_m \otimes \tilde{c}_{\overline{m}}.$$

Then $\tilde{b}$ is a tensor product of $2m$ columns of type A_{n-1}. We can now apply the duality $*$ of [GL20] to the element $\tilde{b}$ to get an element $b^* \in \dot{\mathcal{F}}_{m,n}$. More precisely, we set

$$b^* = d_1 \otimes \cdots \otimes d_n \quad \text{where } d_i = \{x \in \mathscr{C}_m^* \mid i \in \tilde{c}_x\} \text{ for all } i = 1, \ldots, n.$$

Example 5.10. Let $n = 5$ and $b = c_1 \otimes c_2 = \begin{array}{c}\boxed{3}\\\boxed{2}\\\boxed{4}\\\boxed{5}\end{array} \otimes \begin{array}{c}\boxed{5}\\\boxed{2}\\\boxed{1}\\\boxed{1}\\\boxed{2}\\\boxed{4}\end{array}$. Then we have $\tilde{c}_1 = \begin{array}{c}\boxed{1}\\\boxed{4}\\\boxed{5}\end{array}$, $\tilde{c}_{\overline{1}} = \begin{array}{c}\boxed{4}\\\boxed{5}\end{array}$, $\tilde{c}_2 = \begin{array}{c}\boxed{3}\\\boxed{4}\end{array}$

and $\tilde{c}_{\overline{2}} = \begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{4}\end{array}$. We obtain the following product of columns of type A_4:

$$\tilde{b} = \tilde{c}_1 \otimes \tilde{c}_{\overline{1}} \otimes \tilde{c}_2 \otimes \tilde{c}_{\overline{2}} = \begin{array}{c}\boxed{1}\\\boxed{4}\\\boxed{5}\end{array} \otimes \begin{array}{c}\boxed{4}\\\boxed{5}\end{array} \otimes \begin{array}{c}\boxed{3}\\\boxed{4}\end{array} \otimes \begin{array}{c}\boxed{1}\\\boxed{2}\\\boxed{4}\end{array}.$$

Finally, we find

$$b^* = \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \otimes \boxed{\overline{2}} \otimes \boxed{2} \otimes \begin{array}{c}\boxed{1}\\\boxed{\overline{1}}\\\boxed{2}\\\boxed{\overline{2}}\end{array} \otimes \begin{array}{c}\boxed{1}\\\boxed{\overline{1}}\end{array}.$$

In particular, [GL20, Proposition 2.17] ensures that if $b \in B_{\mu'}^{\mathrm{hw}}$, then b^* is a semistandard tableau (on the alphabet $\mathscr{C}_m^*$), where we identify a tableau with the tensor product of its columns (from left to right with our convention). In fact, we have more. In order to state the following theorem, recall that a tableau with entries in $\mathscr{C}_m^*$ is called a *King tableau* if

(1) it is semistandard, and

(2) if each entry in row j is greater than or equal to j for every index j.

The *weight* of a tableau t with entries in $\mathscr{C}_m^*$ is the sequence $\mathrm{wt}(t) = (a_m, \ldots, a_1)$ where, for all $j = 1, \ldots, m$,

$$a_j = \# \text{ entries } j \text{ in } b - \# \text{ entries } \overline{j} \text{ in } b.$$

In order to state the main result of this section, recall the involutive map $\beta' \mapsto \widehat{\beta}$ defined in Section 4. Moreover, let us denote by $\mathcal{K}_{\widehat{\lambda}, \widehat{\mu}}$ the set of King tableaux of shape $\widehat{\lambda}$ and weight $\widehat{\mu}$.

Theorem 5.11. *We have a bijection*

$$\begin{array}{ccc} B_{\mu', \lambda}^{\mathrm{hw}} & \longrightarrow & \mathcal{K}_{\widehat{\lambda}, \widehat{\mu}} \\ b & \longmapsto & b^*. \end{array}$$

By general crystal theory, the cardinality of $B_{\mu', \lambda}^{\mathrm{hw}}$ equals the multiplicity of $V(\lambda)$ in $\Lambda_{2n, m}^{\mu'}$. Moreover, it is known that the cardinality of $\mathcal{K}_{\widehat{\lambda}, \widehat{\mu}}$ equals the weight multiplicity $K_{\widehat{\lambda}, \widehat{\mu}}^{C_m}$, see [Kin76]. Therefore, the bijection of Theorem 5.11 permits to recover (Theorem 4.3). In fact, we directly obtain the more general version where the heights of the columns in the tensor product are not necessarily decreasing (that is, using compositions instead of partitions).

Example 5.12. Let us go back to Example 5.7, where we had $b \in B_{\mu',\lambda}^{\mathrm{hw}}$ with $\mu' = (2,3,1)$ and $\lambda = (2,1,1,0)$. We compute $\tilde{b} = \young(1,2) \otimes \varnothing \otimes \young(3,4) \otimes \young(1) \otimes \young(1,2,3) \otimes \varnothing$, which yields $b^* = \young(1,\bar{2},3) \otimes \young(1,3) \otimes \young(2,3) \otimes \young(3)$,

represented by the tableau

$$\young(1123,\bar{2}33,3) \ ,$$

which is a King tableau of shape $(4,3,1) = \widehat{\lambda}$ and weight $(3,1,2) = \widehat{\mu}$.

Proof. Let us start by proving that the duality $*$ intertwines shape and weight as claimed. Write $\mathrm{wt}(b^*) = \overline{\mu} = (\overline{\mu}_m, \dots, \overline{\mu}_1)$. From the definition of $*$, we see that, for all $j = 1, \dots, m$,

$$\begin{aligned}
\overline{\mu}_j &= |\tilde{c}_{\bar{j}}| - |\tilde{c}_j| \\
&= \#\ \text{unbarred entries in } c_j - (n - \#\ \text{barred entries in } c_j) \\
&= n - |c_j| \\
&= n - \mu'_j.
\end{aligned}$$

Therefore, we have $\overline{\mu} = \widehat{\mu}$. Similarly, if we write $b^* = d_1 \otimes \cdots \otimes d_n$ and $\mathrm{sh}(b^*) = \overline{\lambda} = (\overline{\lambda}_1, \dots, \overline{\lambda}_m)$, we have, for all $i = 1, \dots, n$,

$$\begin{aligned}
\overline{\lambda}'_i &= |d_i| \\
&= \sum_{j=1}^{m} \left(\left(1 - \mathbb{1}_{c_j}(\bar{i})\right) + \mathbb{1}_{c_j}(i) \right) \\
&= m - \sum_{j=1}^{m} \left(\mathbb{1}_{c_j}(\bar{i}) - \mathbb{1}_{c_j}(i) \right) \\
&= m - \left(\#\ \text{entries } \bar{i} \text{ in } b - \#\ \text{entries } i \text{ in } b \right) \\
&= m - \lambda_i,
\end{aligned}$$

and taking the transpose yields $\overline{\lambda} = \widehat{\lambda}$.

In particular, if $b \in B_{\mu'}^{\mathrm{hw}}$ then $\widehat{\lambda}$ is a partition. In fact, $b \in B_{\mu'}^{\mathrm{hw}}$ is a highest weight vertex for the parabolic A_{n-1}-action if and only if b^* is semistandard on $\mathscr{C}_n^*$ by [GL20, Proposition 2.17]. The only thing that remains to be proved is that $e_0(b) = 0$ if and only if b^* satisfies the Condition (2) defining King tableaux. We prove this by induction on $|\mu|$.

If $|\mu| = 1$ then the only element of $B_{\mu'}^{\mathrm{hw}}$ is $b = \young(\bar{n})$. We have $b^* = \young(1) \otimes \cdots \otimes \young(1) \otimes \varnothing$, which is the only King tableau of weight $\widehat{\mu} = (n-1)$.

Fix $r \in \mathbb{Z}_{\geqslant 1}$ and assume that the claim holds for all $a \in B_\nu$ with $|\nu| = r$. Let $b \in B_{\mu'}$ with $|\mu| = r+1$. Let x be the bottommost entry of the rightmost column of b, and let a be the element obtained by deleting x from b. Assume first that a and b have the same number of non-trivial columns, say k. If $x = j \geqslant 1$, then by definition, b^* is obtained by adding a $\bar{k}$ in the j-th column of a^*. By the induction hypothesis applied to $a = a_1 \otimes \cdots \otimes a_n$, one sees that

Condition (2) holds for b^* unless $j = 1$ and $a_1 = \{1, 2, \ldots, k\}$. This is equivalent to saying that the contribution in $\varepsilon_{\bar{1}}$ in $\mathrm{wt}(b)$ is negative. If $x = \bar{j} \leqslant \bar{1}$, then b^* is obtained by adding a k in the j-th column of a^*. One sees that Condition (2) holds for b^* if and only if it holds for a^*, and that $e_0(b) = 0$ if and only if $e_0(a) = 0$, and we conclude using the induction hypothesis applied to a.

Finally, if a has k non-trivial columns and b has $k + 1$ non-trivial columns, one uses similar arguments to validate the induction step in this case too. $\square$

Remark 5.13. We conclude this section by mentioning two related recent results.

(1) In [Lee19, Theorem 2.7], Lee constructed a weight-preserving bijection between certain semistandard oscillating tableaux on the one hand and certain King tableaux on the other hand. In fact, the highest weight vertices of Theorem 5.11 correspond to certain semistandard oscillating tableaux (as illustrated in Example 5.7). Therefore, Theorem 4.3 gives a simple proof of [Lee19, Theorem 2.7] just based on the combinatorics of the columns of type C_n.

(2) Theorem 5.11 enables us to parametrise the highest weight vertices in the KN Fock space, and hence the connected components of the crystal, by appropriate King tableaux. On the other hand, it is well-known that the position of an element of the KN Fock space within its connected component is determined by its corresponding *Kashiwara–Nakashima (KN) tableau*, obtained by applying the symplectic jeu de taquin [Lec05]. This induces a bijection between $\mathcal{F}_{n,m}$ and pairs (P, Q) where P is a KN tableau and Q is a King tableau, with the property that λ is the shape of P if and only if $\widehat{\lambda}$ is the shape of Q (here, λ is the weight of the corresponding highest weight vertex). This is reminiscent of the crystal RSK correspondence in type A [GL20, Theorem 2.25]. In type C, Heo and Kwon introduced another analogue of the RSK correspondence (also thought of as a combinatorial symplectic Howe duality) [HK22]. They established a bijection which maps elements of the KN Fock space to pairs (P, Q) where P is an element of the so-called *spinor model* and Q is a King tableau [HK22, Theorem 7.7]. It turns out that P can be identified with a KN tableau via the bijection of [HK22, Proposition 4.7]; and λ is the shape of P if and only if $\widehat{\lambda}$ is the shape of Q. In particular, restricting to highest weight vertices in the KN Fock space yields the bijection in Theorem 5.11. Finally, it is natural to ask whether these symplectic RSK correspondences arise from a bicrystal structure on the KN Fock space in the spirit of [GL20, Theorem 2.25]. This will be briefly discussed in Remark 6.15(1) of the upcoming section.

6. Bicrystals and charge

As mentioned in Remark 5.9, $\dot{\mathcal{F}}_{m,n}$ does not come with a natural type C_m crystal structure. In fact, even on the subset of King tableaux, finding such a crystal structure is a challenging problem, see Remark 6.15. However, there is a natural type A_{2m-1} crystal structure on $\dot{\mathcal{F}}_{m,n}$ induced from the crystal of the vector representation below

$$\boxed{1} \xrightarrow{\ 1\ } \boxed{\bar{1}} \xrightarrow{\ \bar{1}\ } \boxed{2} \xrightarrow{\ 2\ } \boxed{\bar{2}} \xrightarrow{\ \bar{2}\ } \cdots \xrightarrow{\ \overline{m-1}\ } \boxed{m} \xrightarrow{\ m\ } \boxed{\bar{m}} \ .$$

To compute the A_{2m-1}-crystal structure on $\dot{\mathcal{F}}_{m,n}$, we use a different reading than the one used in Section 5. More precisely, we choose this time to read the columns of $b \in \dot{\mathcal{F}}_{m,n}$ *first from right to left and next from top to bottom*. The action of the crystal operators, denoted by $\dot{f}_j$, $1 \leqslant j \leqslant m$, and $\dot{f}_{\overline{j}}$, $1 \leqslant j \leqslant m-1$ (resp. $\dot{e}_j$ and $\dot{e}_{\overline{j}}$) on the resulting word is computed by using the same bracketing procedure as in Section 5 (illustrated in Example 5.3), where we encode this time j by $+$ and $\overline{j}$ by $-$ for $\dot{f}_j, \dot{e}_j$, and $\overline{j}$ by $+$ and $j+1$ by $-$ for $\dot{f}_{\overline{j}}, \dot{e}_{\overline{j}}$ (and we ignore the other letters). This is illustrated in the example below.

Example 6.1. Take $n = 3, m = 2$ and $b^* = \boxed{\begin{smallmatrix} \overline{1} \\ 2 \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \\ \overline{2} \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}$. Let us detail the computation of $\dot{f}_{\overline{1}}b^*$. Reading b^* yields the word $\mathrm{w}_{\overline{1}} = 121\overline{1}\,\overline{2}\,\overline{1}2$. Looking only at $\overline{1}$ and 2, we get $\mathrm{w}_{\overline{1}} = -++-$ and the bracketing yields $\mathrm{w}_{\overline{1}} = -+(+-)$. Thus we get $\dot{f}_{\overline{1}}b^* = \boxed{\begin{smallmatrix} \overline{1} \\ 2 \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \\ \overline{2} \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}$. Similarly, one checks that $\dot{e}_{\overline{1}}b^* = \boxed{\begin{smallmatrix} \overline{1} \\ 2 \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \\ \overline{2} \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \end{smallmatrix}}$, $\dot{f}_2 b^* = \boxed{\begin{smallmatrix} \overline{1} \\ \overline{2} \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \\ \overline{2} \end{smallmatrix}} \otimes \boxed{\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}$ and that $\dot{e}_1 b^* = \dot{e}_2 b^* = \dot{f}_1 b^* = 0$.

In the case $m = 1$, the duality $*$ intertwines the following two important properties.

Proposition 6.2. *Let c be a column on $\mathscr{C}_n$. Then c is n-admissible if and only if c^* is a highest weight vertex in the A_1-crystal.*

Proof. By definition of the A_1-crystal structure on $\dot{\mathcal{F}}_{m,n}$, the element $c^* = c_1^* \otimes \cdots \otimes c_n^*$ is a highest weight vertex if and only if for all $i = 1, \ldots, n$, $c_i^* \otimes \cdots \otimes c_n^*$ has at least as many entries 1 as $\overline{1}$. Now, recall that c is n-admissible if and only if $N_i(c) \leqslant n-i+1$ for all $i = 1, \ldots, n$ (Definition 5.1). We can compute $N_i(c)$ on c^* by using the definition of the duality: we get

$$N_i(c) = \big(\# \text{ entries } \overline{1} \text{ in } c_i^* \otimes \cdots \otimes c_n^* \big) + (n-i+1) - \big(\# \text{ entries } 1 \text{ in } c_i^* \otimes \cdots \otimes c_n^* \big).$$

Since $N_i(c) \leqslant n-i+1$ we get the desired characterisation. $\qquad\square$

Example 6.3. Let $n = 6$ and $c = \boxed{\begin{smallmatrix} 3 \\ \overline{1} \\ 1 \\ 3 \\ 4 \\ 6 \end{smallmatrix}}$, which is n-admissible. Now, we can compute $c^* =$

$\boxed{\overline{1}} \otimes \boxed{1} \otimes \boxed{\overline{1}} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \end{smallmatrix}} \otimes \boxed{1} \otimes \boxed{\begin{smallmatrix} 1 \\ \overline{1} \end{smallmatrix}}$, and we see that c^* verifies the expected property.

Remark 6.4. Similarly, there is a notion of coadmissibility for columns, which is easily characterised on the dual. More precisely, given a column c of type C_n, set, for all $i = 1, \ldots, n$,

$$M_i(c) = \big| \big\{ x \in c \mid \overline{i} \leqslant x \leqslant i \big\} \big|.$$

Then c is called n-*coadmissible* if $M_i(c) \leqslant i$ for all $i = 1, \ldots, n$. As in Proposition 6.2, we can prove that c is n-coadmissible if and only if c^* is a highest weight vertex with respect to the alternative A_1-crystal structure where we choose to read the n factors from left to right.

In general, we will now show some interesting relationships between the C_n-crystal structure on $\mathcal{F}_{n,m}$ and the A_{2m-1}-crystal structure on $\dot{\mathcal{F}}_{m,n}$.

For the next result, recall the type C_n plactic relation called *contraction* of a column defined in [Lec05, Remark (ii) p. 213], which is given by removing a certain pair $(\bar{k}, k)$ appearing in the column. It is easy to see that contraction has an inverse, which we call *dilatation*. Moreover, for $1 \leqslant j \leqslant m$ define $\kappa_j : \mathcal{F}_{n,m} \to \mathcal{F}_{n,m}$ by setting $a = \kappa_j b$ if and only if $a^* = \dot{e}_j b^*$ (i.e. there is an arrow $a^* \xrightarrow{j} b^*$).

Theorem 6.5. *For all $1 \leqslant j \leqslant m$, κ_j is the contraction of the j-th column.*

Proof. Let $b = c_1 \otimes \cdots \otimes c_m \in B_{\mu'}$ for some $\mu' \in \mathcal{C}_{m,2n}$. If $a^* = \dot{e}_j b^*$, then a^* is obtained from b^* by changing a $\bar{j}$ into a j, say in column k. That is, a is obtained from b by removing the pair $(\bar{k}, k)$ in column j. In fact, by construction, the contraction of the j-th column is the only C_n-crystal isomorphism which removes such a pair. Therefore, it suffices to prove that κ_j is a C_n-crystal isomorphism to deduce that κ_j is the contraction of the j-th column. So we will prove that

$$\kappa_j e_i b = e_i \kappa_j b \quad \text{for all } 0 \leqslant i \leqslant n - 1. \tag{6.1}$$

First of all, it is clear that (6.1) holds in the following two cases:

- e_i does not act on the j-th column of b,

- $i \notin \{k, k-1\}$.

Let us look at the remaining cases, that is, e_i acts on the j-th column of b and

(1) $i = k - 1$. In this case, since we already know that $\bar{k}, k \in c_j$, we are ensured that e_i acts non trivially on b if and only if $\overline{k-1} \in c_j$ and $k - 1 \notin c_j$. This means that we have the configuration

$$c_j = \begin{array}{|c|} \hline \vdots \\ \hline \bar{k} \\ \hline \overline{k\text{-}1} \\ \hline \vdots \\ \hline k \\ \hline \vdots \\ \hline \end{array}.$$

One checks that applying $\kappa_j e_i$ amounts to

(a) changing k into $k - 1$, followed by

(b) removing the pair $(\overline{k-1}, k-1)$

On the other hand, applying $e_i \kappa_j$ amounts to

(a) removing the pair $(\bar{k}, k)$, followed by

(b) changing $\overline{k-1}$ into $\bar{k}$

Both of these procedures yield the same result, namely, we get the j-th column

$$c_j = \boxed{\begin{array}{c} \vdots \\ \overline{k} \\ \vdots \end{array}}$$

(where $\overline{k-1}$ and k have been deleted), and we have (6.1) as expected.

(2) $i = k$. Similarly, e_i acts non trivially on b if and only if $\overline{k+1} \notin c_j$ and $k+1 \in c_j$, and this means that we have the configuration

$$c_j = \boxed{\begin{array}{c} \vdots \\ \overline{k} \\ \vdots \\ k \\ k+1 \\ \vdots \end{array}}.$$

In this case, one checks that (6.1) holds and that the resulting column is

$$c_j = \boxed{\begin{array}{c} \vdots \\ k \\ \vdots \end{array}}. \qquad \square$$

Example 6.6. We illustrate Case (1) of the previous proof by taking $n = 4$, $m = 1$ and $b = c_1 = \boxed{\begin{array}{c} \overline{4} \\ \overline{3} \\ 2 \\ 3 \end{array}}$. We have $j = 1$, we check that $k = 3$ and
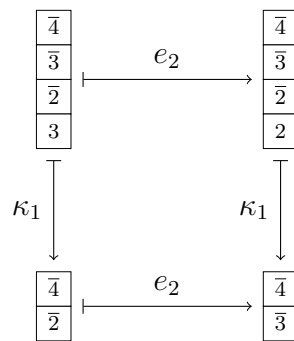
Therefore, we shall consider the $A_1 \times \cdots \times A_1$-crystal structure ($m$ factors) on $\dot{\mathcal{F}}_{m,n}$ induced by keeping only arrows of the form $\boxed{j} \xrightarrow{j} \boxed{\overline{j}}$. Theorem 6.5 directly implies the following corollary, which can be rephrased by saying that the combinatorial Fock space is endowed with a $(C_m \times A_1^m)$-bicrystal structure.

Corollary 6.7. *The duality $*$ intertwines the C_n-crystal on $\mathcal{F}_{n,m}$ and the $A_1 \times \cdots \times A_1$-crystal on $\dot{\mathcal{F}}_{m,n}$.*

Remark 6.8. Observe that Point (2) of [HK22, Lemma 4.9] is equivalent to Proposition 6.2, where the operator $\mathcal{E}$ coincides with κ_1.

We complete Theorem 6.5 by giving a characterisation of the dual $\kappa_{\bar{j}}$ of the Kashiwara operators $e_{\bar{j}}$ (corresponding to the arrows $\boxed{\bar{j}} \xrightarrow{\bar{j}} \boxed{j+1}$) in terms of jeu de taquin operators on $\mathcal{F}_{n,m}$. More precisely, if $b = c_1 \otimes \cdots \otimes c_m \in B_{\mu'}$, set for all $1 \leqslant j \leqslant m$,

$$\bar{c}_j = \{\bar{n} \leqslant \bar{x} \leqslant \bar{1} \mid \bar{x} \in c_j\} = \{\bar{n}, \ldots, \bar{1}\} \cap c_j \quad \text{and}$$
$$\bar{c}_{\bar{j}} = \{\bar{n} \leqslant \bar{x} \leqslant \bar{1} \mid x \notin c_j\},$$

so that each $\bar{c}_j$ is the complement of the column $\tilde{c}_j$ defined in Section 5. Also, we set

$$\bar{b} = \bar{c}_1 \otimes \bar{c}_{\bar{1}} \otimes \cdots \otimes \bar{c}_m \otimes \bar{c}_{\overline{m}}.$$

On elements of the form $\bar{b}$, consider for $1 \leqslant j \leqslant m-1$ the jeu de taquin operator $J_{\bar{j}}$ acting on columns $\bar{j}$ and $j+1$ as illustrated in Example 6.9 below.

Example 6.9. Let $n = 5, m = 2$ and $b = \boxed{\begin{matrix} 3 \\ 1 \\ 5 \end{matrix}} \otimes \boxed{\begin{matrix} 5 \\ 1 \\ 2 \\ 4 \\ 5 \end{matrix}}$ so that $\bar{b} = \bar{c}_1 \otimes \bar{c}_{\bar{1}} \otimes \bar{c}_2 \otimes \bar{c}_{\bar{2}} =$

$\boxed{3} \otimes \boxed{\begin{matrix} \bar{4} \\ \bar{3} \\ \bar{2} \end{matrix}} \otimes \boxed{\begin{matrix} \bar{5} \\ \bar{1} \end{matrix}} \otimes \boxed{\begin{matrix} \bar{3} \\ \bar{1} \end{matrix}}$. Let us perform one jeu de taquin operation between $\bar{c}_{\bar{1}} = \boxed{\begin{matrix} \bar{4} \\ \bar{3} \\ \bar{2} \end{matrix}}$ and $\bar{c}_2 = \boxed{\begin{matrix} \bar{5} \\ \bar{1} \end{matrix}}$.

We want to slide a box from $\bar{c}_{\bar{1}}$ to $\bar{c}_2$, so we reverse the order of the two columns, which yields the following jeu de taquin operation

$$\begin{matrix} & \boxed{\bar{4}} \\ \bullet & \boxed{\bar{3}} \\ \boxed{\bar{5}} & \boxed{\bar{2}} \\ \boxed{\bar{1}} & \end{matrix} \rightarrow \begin{matrix} & \boxed{\bar{4}} \\ \boxed{\bar{5}} & \boxed{\bar{3}} \\ \bullet & \boxed{\bar{2}} \\ \boxed{\bar{1}} & \end{matrix} \rightarrow \begin{matrix} & \boxed{\bar{4}} \\ \boxed{\bar{5}} & \boxed{\bar{3}} \\ \boxed{\bar{2}} & \bullet \\ \boxed{\bar{1}} & \end{matrix}.$$

Finally, we get

$$J_{\bar{j}}(\bar{b}) = \boxed{\bar{3}} \otimes \boxed{\begin{matrix} \bar{4} \\ \bar{3} \end{matrix}} \otimes \boxed{\begin{matrix} \bar{5} \\ \bar{2} \\ \bar{1} \end{matrix}} \otimes \boxed{\begin{matrix} \bar{3} \\ \bar{1} \end{matrix}}.$$

We extend the definition of $J_{\bar{j}}$ to $\mathcal{F}_{n,m}$ by setting $\overline{J_{\bar{j}}(b)} = J_{\bar{j}}(\bar{b})$. We are ready to state the desired result, which is a symplectic analogue of [GL20, Theorem 2.9].

Theorem 6.10. *For all $1 \leqslant j \leqslant m-1$, we have $\kappa_{\bar{j}} = J_{\bar{j}}$.*

Proof. We use a similar argument to the proof of Theorem 6.5. Namely, we first notice that if $a^* = e_{\bar{j}} b^*$, then a^* is obtained from b^* by changing a $j+1$ into a $\bar{j}$, say in column k. By definition of $*$, this means that. $\bar{a}$ is obtained from $\bar{b}$ by moving an entry $\bar{k}$ from column $\bar{j}$ to column $j+1$. Then we use the fact that $J_{\bar{j}}$ is the only map that verifies this property and that commutes with the C_n-crystal operators $e_1, \ldots, e_{n-1}$, and show by case analysis that this also holds for $\kappa_{\bar{j}}$. $\quad\square$

Remark 6.11. Unlike the operators κ_j, we do not have that $\kappa_{\bar{j}}$ commute with the type C_n Kashiwara operator e_0. For instance, if $n = 1, m = 2$, and $b = \varnothing \otimes \boxed{1}$, one checks that $\kappa_{\bar{1}} e_0 b = 0$ but $e_0 \kappa_{\bar{1}} b = \boxed{\bar{1}} \otimes \begin{array}{c}\boxed{\bar{1}}\\\boxed{1}\end{array}$. Therefore, we do not get an analogue of Corollary 6.7.

Example 6.12. Let $n = 5, m = 2, j = 1$ and

$$b^* = \begin{array}{c}\boxed{1}\\\boxed{\bar{1}}\\\boxed{\bar{2}}\end{array} \otimes \boxed{2} \otimes \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \otimes \begin{array}{c}\boxed{\bar{1}}\\\boxed{2}\end{array} \otimes \begin{array}{c}\boxed{1}\\\boxed{\bar{1}}\end{array} \quad \text{so that} \quad b = \begin{array}{c}\boxed{\bar{4}}\\\boxed{\bar{2}}\\\boxed{1}\\\boxed{4}\\\boxed{5}\end{array} \otimes \begin{array}{c}\boxed{\bar{5}}\\\boxed{\bar{1}}\\\boxed{1}\end{array}.$$

We get

$$e_{\bar{1}} b^* = \begin{array}{c}\boxed{1}\\\boxed{\bar{1}}\\\boxed{\bar{2}}\end{array} \otimes \boxed{\bar{1}} \otimes \begin{array}{c}\boxed{1}\\\boxed{2}\end{array} \otimes \begin{array}{c}\boxed{\bar{1}}\\\boxed{2}\end{array} \otimes \begin{array}{c}\boxed{1}\\\boxed{\bar{1}}\end{array}, \quad \text{so } k = 2 \text{ and } \kappa_{\bar{1}} b = \begin{array}{c}\boxed{\bar{4}}\\\boxed{\bar{2}}\\\boxed{1}\\\boxed{2}\\\boxed{4}\\\boxed{5}\end{array} \otimes \begin{array}{c}\boxed{\bar{5}}\\\boxed{\bar{2}}\\\boxed{\bar{1}}\\\boxed{1}\end{array}.$$

On the other hand, we have

$$\bar{b} = \begin{array}{c}\boxed{\bar{4}}\\\boxed{\bar{2}}\end{array} \otimes \begin{array}{c}\boxed{\bar{3}}\\\boxed{\bar{2}}\end{array} \otimes \begin{array}{c}\boxed{\bar{5}}\\\boxed{\bar{1}}\end{array} \otimes \begin{array}{c}\boxed{\bar{5}}\\\boxed{\bar{4}}\\\boxed{\bar{3}}\\\boxed{\bar{2}}\end{array}.$$

The jeu de taquin operation corresponding to $J_{\bar{1}}$ is

$$\begin{array}{cc}\bullet & \boxed{\bar{3}}\\\boxed{\bar{5}} & \boxed{\bar{2}}\\\boxed{\bar{1}} & \end{array} \rightarrow \begin{array}{cc}\boxed{\bar{5}} & \boxed{\bar{3}}\\\bullet & \boxed{\bar{2}}\\\boxed{\bar{1}} & \end{array} \rightarrow \begin{array}{cc}\boxed{\bar{5}} & \boxed{\bar{3}}\\\boxed{\bar{2}} & \bullet\\\boxed{\bar{1}} & \end{array},$$

and we check that this yields $J_{\bar{1}}(b) = \kappa_{\bar{1}}(b)$.

At this point, we make a small digression and explain briefly the relationship with the charge statistic defined in [LL20, Theorem 6.13] to compute q-weight zero multiplicities.

Let $b \in B_{\mu'}^{\mathrm{hw}}$ such that b^* is a King tableau (Theorem 5.11) of weight $\widehat{\mu} = 0$. In particular, we have $\mu' = (m^n)$, that is all the columns in μ have height n. Denote b_{low}^* the lowest weight vertex corresponding to b^* in the $A_1 \times \cdots \times A_1$-crystal. In other terms, b_{low}^* is obtained from b^* by applying all possible $\dot{f}_j$ to b^* recursively (they commute).

Definition 6.13. The *charge* of b^* is the nonnegative integer

$$\mathrm{ch}(b^*) = \sum_{j=1}^{m} (2(m-j)+1) \frac{\varepsilon_j(b_{\mathrm{low}}^*)}{2} + \sum_{j=1}^{m-1} 2(m-j) \left\lceil \frac{\varepsilon_{\bar{j}}(b_{\mathrm{low}}^*)}{2} \right\rceil$$

We can in fact construct a statistic D directly on $B_{\mu'}$ such that

$$D(b) = \mathrm{ch}(b^*).$$

To do this, starting from $b = c_1 \otimes \cdots \otimes c_m \in B_{\mu'}$. let $b_{\mathrm{dil}} = d_1 \otimes \cdots \otimes d_m$ where, for all $j = 1, \ldots, m$, d_j is the type C_n column obtained by dilating c_j recursively as much as possible. Then d_j can be contracted a certain number of times, say δ_j, until it becomes admissible. At this point, we have by Theorem 6.5

$$(b_{\mathrm{dil}})^* = b_{\mathrm{low}}^*,$$

and

$$\varepsilon_j(b_{\mathrm{low}}^*) = \delta_j.$$

In fact, we can easily compute δ_j. Let h denote the height of the admissible column corresponding to c_j (i.e. obtained from c_j by applying recursively as many contractions as possible).

Proposition 6.14. *We have then $\delta_j = n - h$. Moreover, δ_j is even.*

Proof. Note that since $|c_j| = n$, the difference of the sizes $n - h$ is even (since each contraction deletes 2 entries). Therefore, it suffices to show that $\delta_j = n - h$. Denote c_j^{adm} the admissible columns corresponding to c_j, so that $h = |c_j^{\mathrm{adm}}|$. Since contraction is a C_n-crystal isomorphism, we can compute δ_j by considering the highest weight column associated to c_j^{adm} in the C_n-crystal. Obviously, this column also has height h and is admissible (since c_j^{adm} is admissible), so it must be the column $\{\overline{n}, \overline{n-1}, \ldots, \overline{n-h+1}\}$. It is straightforward to see that it can be dilated at most $n - h$ times, resulting in the column $\{\overline{n}, \overline{n-1}, \ldots, \overline{n-h+1}, \overline{n-h}, \ldots, \overline{1}, 1, \ldots, n-h\}$ of height $2n - h$ (which is the highest weight column associated to d_j), that is, $\delta_j = n - h$. $\square$

It remains to express $\varepsilon_{\overline{j}}(b_{\mathrm{low}}^*)$ directly on b_{dil}. By Theorem 6.10, this equals the number γ_j of successive possible application of $J_{\overline{j}}$ to the pair $(\overline{d}_{\overline{j}}, \overline{d}_{j+1})$. In other terms, if the minimal skew tableau associated to $(\overline{d}_{\overline{j}}, \overline{d}_{j+1})$ has skew shape $\nu/(1^\ell)$, we have $\gamma_j = \ell$. To sum up, we set

$$D(b) = \sum_{j=1}^{m} (2(m-j) + 1) \frac{\delta_j}{2} + \sum_{j=1}^{m-1} 2(m-j) \left\lceil \frac{\gamma_j}{2} \right\rceil.$$

This can be seen as an analogue of the energy statistic for type C, naturally generalising [GL20, Theorem 2.51].

We end this section by three important remarks.

Remark 6.15.

(1) As mentioned in Remark 6.11, since the operators $\kappa_{\overline{j}}$ do not commute with e_0, we cannot establish a $(C_n \times A_{2m-1})$-bicrystal structure on $B_{\mu'}$. In another direction, we expect to obtain a $(C_n \times C_m)$-bicrystal structure by considering appropriate jeu de taquin operations and contraction on columns. In this case, it would be interesting to compare the resulting bicrystal structure to that obtained by Lee in [Lee19].

(2) Let us consider elements of $B_{\mu'}^{\mathrm{hw}}$ that are products of admissible columns. Combining Theorem 5.11 and Proposition 6.2, these are in duality with King tableaux of weight $\widehat{\mu}$ which are highest weight vertices in the A_1^m-crystal. This can be seen as a combinatorial version the new duality which will be proved in Theorem 7.5, in the special case $r = m$, $\boldsymbol{m} = (1, \dots, 1)$, $\boldsymbol{X} = (A, \dots, A)$ and $\boldsymbol{\mu} = ((1^{\mu'_1}), \dots, (1^{\mu'_m}))$ (columns of height μ'_j).

(3) The energy function D defined previously does not coincide with the intrinsic energy function on type $C_n^{(1)}$ tensor products of column Kirillov–Reshetikhin crystals. This reflects the fact that the q-weight multiplicities do not coincide with the one-dimensional sums beyond type A. Nevertheless, this suggests that other interesting statistics could exist on these particular affine crystals.

7. Branching coefficients and a new duality

The aim of this section is to extend the results of Section 4 to the case where the fundamental $\mathfrak{gl}_{2n}(\mathbb{C})$-modules appearing in the tensor products in Theorem 4.3 are replaced by tensor products of simple $\mathfrak{gl}_{2n}(\mathbb{C})$ or simple $\mathfrak{sp}_{2n}(\mathbb{C})$-modules. More precisely, let $\boldsymbol{X}$ be a sequence $(X_1, \dots, X_r)$ of symbols in $\{A, C\}^r$ and let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ be a sequence of partitions such that $\mu^{(j)} \in \mathcal{P}_{n \times m_j}$. Then one can associate to $(\boldsymbol{X}, \boldsymbol{\mu})$ the tensor product

$$V^{X_1}(\mu^{(1)}) \otimes V^{X_2}(\mu^{(2)}) \otimes \cdots \otimes V^{X_r}(\mu^{(r)})$$

where $V^C(\delta)$ ($\delta \in \mathcal{P}_n$) denotes the irreducible $\mathfrak{sp}_{2n}(\mathbb{C})$-module of highest weight δ and $V^A(\delta)$ denotes the restriction to $\mathfrak{sp}_{2n}(\mathbb{C})$ of the irreducible $\mathfrak{gl}_{2n}(\mathbb{C})$-module of highest weight δ. We show that the mutliplicity of the highest weight module of weight λ in the tensor product above is a branching coefficient of the form $[V^{C_m}(\widehat{\lambda}) : V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})]$ where $m = \sum m_j$ and $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$ is an irreducible highest weight module for a block diagonal subalgebra of $\mathfrak{sp}_{2m}(\mathbb{C})$ that depends on $\boldsymbol{X}, \boldsymbol{\mu}$ and $\boldsymbol{m}$.

Remark 7.1. The weights of $\mathfrak{gl}_{2n}(\mathbb{C})$ are in bijection with non-increasing sequences of integers in $\mathbb{Z}^n$. If two such sequences δ, δ' differ by a multiple of $(1, \dots, 1) \in \mathbb{Z}^n$, then the corresponding Schur functions s_δ and $\mathsf{s}_{\delta'}$ will differ by a power of $x_1 \cdots x_{2n}$. Since the characters of $V^A(\delta)$ and $V^A(\delta')$ are the specialisation of s_δ and $\mathsf{s}_{\delta'}$ at $(x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1})$, these two characters will be equal. As a consequence, it is enough in the tensor products defined above to restrict ourself to partitions.

Our first task in this section is to defined the module $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$. We start by extending the definition of the map $\widehat{\cdot}$ to r-tuples of partitions. Let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ be a sequence of partitions such that $\mu^{(j)} \in \mathcal{P}_{n \times m_j}$. We set

$$\widehat{\boldsymbol{\mu}} = \left(\widehat{\mu^{(r)}}, \dots, \widehat{\mu^{(1)}} \right)$$

where for each $1 \leqslant j \leqslant r$, the partition $\widehat{\mu^{(j)}}$ is defined with respect to the pair $(n \times m_j)$ and lies in $\mathcal{P}_{m_j \times n}$. Note that the definition of this map depends on the pair $(n, \boldsymbol{m})$.

Remark 7.2. (1) Let $\mu \in \mathcal{P}_{n,m}$ and write $\mu' = (\mu'_1, \ldots, \mu'_m)$. Let $\boldsymbol{m} = (m_1, \ldots, m_r)$ be such that $\sum m_j = m$. If we see μ as an r-tuple of partitions $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ as follows

$$\Big(\underbrace{\mu'_1, \ldots, \mu'_{m_1}}_{\mu^{(1)}}, \underbrace{\mu'_{m_1+1}, \ldots, \mu'_{m_2}}_{\mu^{(2)}}, \ldots, \underbrace{\mu'_{m-m_r+1}, \ldots, \mu'_{m_r}}_{\mu^{(r)}} \Big)$$

then the partition $\widehat{\boldsymbol{\mu}}$ computed with respect to the pair $(n, \boldsymbol{m})$ is equal, as an element of $\mathbb{Z}^m$, to the partition $\widehat{\mu}$ computed with respect to the pair (n, m).

(2) Let $\boldsymbol{\mu} = (\mu^{(1)}, \ldots, \mu^{(r)})$ be such that $\mu^{(j)} \in \mathcal{P}_{n,m_j}$ and let $\boldsymbol{\mu}' = (\mu^{(1)'}, \ldots, \mu^{(r)'})$. Let $\boldsymbol{m} = (m_1, \ldots, m_r)$ and $m = \sum m_j$. We have

$$\widehat{\boldsymbol{\mu}} = \mathsf{l}(\boldsymbol{\mu}') + n \cdot \omega_m$$

where $\omega_m = (1, \ldots, 1)$ and the equality is an equality in $\mathbb{Z}^m$.

Example 7.3. Consider the sequence of partitions

$$\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)}) = \left(\begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array} \right) \in \mathcal{P}_{3,2} \times \mathcal{P}_{3,2} \times \mathcal{P}_{3,3}.$$

Let $n = 3$ and $\boldsymbol{m} = (2, 2, 3)$. We compute the image of $\boldsymbol{\mu}$ under the map $\widehat{}$ associated to $(n, \boldsymbol{m})$ by first taking the complement of the partitions of $\boldsymbol{\mu}$ (in their respective rectangles), which yields

$$\left(\begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array} \right) \in \mathcal{P}_{3,2} \times \mathcal{P}_{3,2} \times \mathcal{P}_{3,3},$$

and then by taking the conjugate of each green partition and reversing the order of the triple, which yields

$$\widehat{\boldsymbol{\mu}} = \left(\begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array}, \begin{array}{c} \boxminus \end{array} \right) \in \mathcal{P}_{3,3} \times \mathcal{P}_{2,3} \times \mathcal{P}_{2,3}.$$

Finally, note that $m = 7$ and that

$$\begin{aligned}
\mathsf{l}(\boldsymbol{\mu}') + 3 \cdot \omega_7 &= \mathsf{l}(3, 1, 1, 1, 2, 2, 1) + 3 \cdot \omega_7 \\
&= (-1, -2, -2, -1, -1, -1, -3) + 3 \cdot \omega_7 \\
&= (2, 1, 1, 2, 2, 2, 0)
\end{aligned}$$

which is indeed equal to $\widehat{\boldsymbol{\mu}}$ as an element of $\mathbb{Z}^7$.

Next we need to define the module $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$. To simplify the exposition, we will explain how to construct the module $V_{\boldsymbol{k}}^{\boldsymbol{X}}(\boldsymbol{\nu})$ associated to

- a sequence $\boldsymbol{k} = (k_1, \ldots, k_r)$ of positive integers;

- a sequence $\boldsymbol{X} = (X_1, \ldots, X_r)$ of symbols in $\{A, C\}$;

- a sequence $\boldsymbol{\nu} = (\nu^{(1)}, \ldots, \nu^{(r)})$ of partitions such that $\nu^{(j)} \in \mathcal{P}_{k_j}$.

We set $m = \sum k_i$ and we define the integers $K_0, \dots, K_r$ by setting

$$K_0 = 0 \quad \text{and} \quad K_j = \sum_{i=1}^{j} k_i.$$

The algebra $\mathfrak{g}_{(\boldsymbol{X}, \boldsymbol{k})}$ is defined to be the subalgebra of $\mathfrak{sp}_{2k}$ associated to the root system

$$R_{(\boldsymbol{X}, \boldsymbol{k})} = \bigsqcup_{j=1}^{r} R_{[K_{j-1}+1, K_j]}^{X_j}$$

where we have set for any pair of integers (a, b):

$$R_{[a,b]}^A = \{\varepsilon_i - \varepsilon_j \mid a \leqslant i < j \leqslant b\} \text{ and}$$
$$R_{[a,b]}^C = \{\varepsilon_i - \varepsilon_j \mid a \leqslant i < j \leqslant b\} \cup \{\varepsilon_i + \varepsilon_j \mid a \leqslant i < j \leqslant b\} \cup \{2\varepsilon_i \mid a \leqslant i \leqslant b\}.$$

In other words, we have

$$\mathfrak{g}_{(\boldsymbol{X}, \boldsymbol{k})} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r \quad \text{where} \quad \mathfrak{g}_j = \begin{cases} \mathfrak{gl}_{k_j}(\mathbb{C}) & \text{if } X_j = A \\ \mathfrak{sp}_{2k_j}(\mathbb{C}) & \text{if } X_j = C \end{cases}$$

Let $P_+^{(\boldsymbol{X}, \boldsymbol{k})}$ be the set of dominant weights of $\mathfrak{g}_{(\boldsymbol{X}, \boldsymbol{k})}$. A weight $\boldsymbol{\nu}$ in $P_+^{(\boldsymbol{X}, \boldsymbol{k})}$ is a sequence $\boldsymbol{\nu} = (\nu^{(1)}, \dots, \nu^{(r)})$ of partitions such that $\nu^{(j)} \in \mathcal{P}_{k_j}$. We define $V_{\boldsymbol{k}}^{\boldsymbol{X}}(\boldsymbol{\nu})$ to be the $\mathfrak{g}_{(\boldsymbol{X}, \boldsymbol{k})}$-module of highest weight $\boldsymbol{\nu}$.

Let us now recall how to compute the branching coefficient $[V(\kappa) : V_{\boldsymbol{k}}^{\boldsymbol{X}}(\boldsymbol{\nu})]$ where $\kappa \in \mathcal{P}_m$. To do so, we need the partition function $\mathcal{P}_{(\boldsymbol{X}, \boldsymbol{k})}$ defined by the expansion

$$\prod_{\alpha \in R^+ \setminus R_{(\boldsymbol{X}, \boldsymbol{k})}} \frac{1}{1 - \mathsf{x}^\alpha} = \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}_{(\boldsymbol{X}, \boldsymbol{k})}(\beta) \mathsf{x}^\beta.$$

The following proposition is a consequence of the Weyl character formula. Let W_m be the Weyl group of type C_m and ρ_m be the half sum of positive roots, that is $\rho_m = (m, \dots, 1)$.

Proposition 7.4 ([GW98, Theorem 8.2.1]). *With the previous notation, we have*

$$\left[V(\kappa) : V_{\boldsymbol{k}}^{\boldsymbol{X}}(\boldsymbol{\nu}) \right] = \sum_{w \in W_m} \varepsilon(w) \mathcal{P}_{(\boldsymbol{X}, \boldsymbol{k})}(w(\kappa + \rho_m) - (\boldsymbol{\nu} + \rho_m)).$$

We are now ready to state the main theorem of this section. For any tuple $\boldsymbol{a} = (a_1, \dots, a_k)$ we write $\boldsymbol{a}^*$ for the inverse tuple $(a_k, \dots, a_1)$.

Theorem 7.5. *Let $\boldsymbol{\mu} = (\mu^{(1)}, \dots, \mu^{(r)})$ be a sequence of partitions such that $\mu^{(j)} \in \mathcal{P}_{n \times m_j}$. Let $\boldsymbol{X} = (X_1, \dots, X_r)$ be a sequence of symbols in $\{A, C\}$ and $\boldsymbol{m} = (m_1, \dots, m_r) \in \mathbb{Z}_{\geqslant 1}^r$. For all $\lambda \in \mathcal{P}_n$, we have*

$$\left[V^{X_1}(\mu^{(1)}) \otimes \cdots \otimes V^{X_r}(\mu^{(r)}) : V(\lambda) \right] = \left[V(\widehat{\lambda}) : V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}}) \right].$$

The rest of this section is devoted to the proof of the theorem. From now on and until the end of the section, we fix $\boldsymbol{\mu}$, $\boldsymbol{X}$ and $\boldsymbol{m}$ as in the theorem. Further, we set $m = \sum m_i$ and we define the integers $M_0, \ldots, M_r$ by the relations

$$M_0 = 0 \quad \text{and} \quad M_j = \sum_{i=1}^{j} M_i.$$

For any $j = 1, \ldots, r$, and any partition $\delta \in \mathcal{P}_n$ write

- s_δ^C for the character of the irreducible module $V(\delta)$ of highest weight δ in $\mathfrak{sp}_{2n}(\mathbb{C})$

- s_δ^A for the function $\mathsf{s}_\delta(x_1, \ldots, x_n, \frac{1}{x_n}, \ldots, \frac{1}{x_1})$ where s_δ is the Schur function of type A in $2n$ variables.

The character of $V^{X_1}(\mu^{(1)}) \otimes \cdots \otimes V^{X_r}(\mu^{(r)})$ is

$$\mathsf{s}_{\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} := \prod_{j=1}^{r} \mathsf{s}_{\mu^{(j)}}^{X_j} \in \mathrm{char}(\mathfrak{sp}_{2n}).$$

Thus, if we define the coefficients $m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})}$ by decomposing the above character in the basis of irreducible characters in type C

$$\mathsf{s}_{\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} = \sum_{\lambda \in \mathcal{P}_n} m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} \mathsf{s}_\lambda^{C_n}$$

then the theorem states that

$$m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} = \left[V(\widehat{\lambda}) : V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}}) \right] \quad \text{for all } \lambda \in \mathcal{P}_n.$$

Let

$$\triangle_{(\boldsymbol{X},\boldsymbol{m})} = \prod_{\alpha \in R_{(\boldsymbol{X},\boldsymbol{m})}} (1 - x^{[\alpha]}) \text{ where } x^{[\alpha]} = \begin{cases} \frac{x_i}{x_j} \text{ if } \alpha = \varepsilon_i - \varepsilon_j \text{ where } i < j \\ \frac{1}{x_i x_j} \text{ if } \alpha = \varepsilon_i + \varepsilon_j \text{ where } i \leqslant j \end{cases}$$

Let $\boldsymbol{\beta} = (\beta^{(1)}, \ldots, \beta^{(r)})$ be an r-tuple of partitions such that $\beta^{(j)} = (\beta_1^{(j)}, \ldots, \beta_{m_j}^{(j)}) \in \mathcal{P}_{m_j}$. Then since $\sum m_i = m$, $\boldsymbol{\beta}$ can be seen as an element of $\mathbb{Z}^m$. We can then define $\mathsf{x}^{\boldsymbol{\beta}}$ and the map E that sends $\mathsf{x}^{\boldsymbol{\beta}}$ to $\mathsf{e}_{\boldsymbol{\beta}}$ as in Section 4.

Lemma 7.6. *We have* $\mathsf{s}_{\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} = \mathsf{E}(\triangle_{(\boldsymbol{X},\boldsymbol{m})} \mathsf{x}^{\boldsymbol{\mu}'})$.

Proof. We have

$$R_{(\boldsymbol{X},\boldsymbol{m})} = \bigsqcup_{j=1}^{r} R_{[M_{j-1}+1, M_j]}^{X_j}$$

and

$$\triangle_{(\boldsymbol{X},\boldsymbol{m})}\mathsf{x}^{\boldsymbol{\mu}'} = \prod_{j=1}^{r}\left(\prod_{\alpha\in R^{X_j}_{[M_{j-1}+1,M_j]}}(1-x^{[\alpha]})\right)\mathsf{x}^{\boldsymbol{\mu}'}.$$

Let $\boldsymbol{\mu}' = (\mu^{(1)'},\ldots,\mu^{(r)'})$ and $\mu^{(j)'} = (\mu^{(j)'}_1,\ldots,\mu^{(j)'}_{m_j}) \in \mathcal{P}_{m_j}$. For all $j = 1,\ldots,r$, we set

$$\mathsf{x}^{\mu^{(j)'}} = x^{\mu^{(j)'}_1}_{M_{j-1}+1}\cdots x^{\mu^{(j)'}_{m_j}}_{M_j}$$

Then we obtain

$$\triangle_{(\boldsymbol{X},\boldsymbol{m})}\mathsf{x}^{\boldsymbol{\mu}'} = \prod_{j=1}^{r}\left(\prod_{\alpha\in R^{X_j}_{[M_{j-1}+1,M_j]}}(1-x^{[\alpha]})\mathsf{x}^{\mu^{(j)'}}\right).$$

Observe that the variables appearing in each parenthesised expression are separated. Therefore, we can apply Lemma 3.2 recursively and get

$$\mathsf{E}(\triangle_{(\boldsymbol{X},\boldsymbol{m})}\mathsf{x}^{\boldsymbol{\mu}'}) = \prod_{j=1}^{r}\mathsf{E}\left(\prod_{\alpha\in R^{X_j}_{[M_{j-1}+1,M_j]}}(1-x^{[\alpha]})\mathsf{x}^{\mu^{(j)'}}\right).$$

But for any $j = 1,\ldots,r$, we have by Propositon 4.4 and its analogue for the ordinary Jacobi–Trudi formula

$$\mathsf{s}^{C}_{\mu^{(j)}} = \mathsf{E}\left(\prod_{\alpha\in R^{X_j}_{[M_{j-1}+1,M_j]}}(1-x^{[\alpha]})\mathsf{x}^{\mu^{(j)'}}\right) \quad \text{when} \quad X_j = C$$

and

$$\mathsf{s}^{A}_{\mu^{(j)}} = \mathsf{E}\left(\prod_{\alpha\in R^{X_j}_{[M_{j-1}+1,M_j]}}(1-x^{[\alpha]})\mathsf{x}^{\mu^{(j)'}}\right) \quad \text{when} \quad X_j = A.$$

Finally, we get $\mathsf{s}^{(\boldsymbol{X},\boldsymbol{m})}_{\boldsymbol{\mu}} = \mathsf{E}(\triangle_{(\boldsymbol{X},\boldsymbol{m})}\mathsf{x}^{\boldsymbol{\mu}'})$ as desired. $\square$

Define the partition function $\widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}$ by

$$\prod_{\alpha\in R^+\backslash R_{(\boldsymbol{X},\boldsymbol{m})}}\frac{1}{1-x^{[\alpha]}} = \sum_{\beta\in\mathbb{Z}^m}\widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta)\mathsf{x}^{\beta}.$$

Lemma 7.7. *We have*

$$\widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta) = \mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}(\mathsf{I}(\beta)) \text{ for any } \beta \in \mathbb{Z}^m.$$

Proof. The involution I sends the set $R_{(\boldsymbol{X},\boldsymbol{m})}$ to $R_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}$. Therefore we have

$$\mathsf{I}\left(\prod_{\alpha \in R^+ \backslash R_{(\boldsymbol{X},\boldsymbol{m})}} \frac{1}{1 - x^{[\alpha]}}\right) = \prod_{\alpha \in R^+ \backslash R_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}} \frac{1}{1 - x^{\alpha}} = \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}(\beta) \mathsf{x}^{\beta}$$

which gives

$$\sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta) \mathsf{x}^{\beta} = \prod_{\alpha \in R^+ \backslash R_{(\boldsymbol{X},\boldsymbol{m})}} \frac{1}{1 - x^{[\alpha]}} = \sum_{\beta \in \mathbb{Z}^m} \mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}(\mathsf{I}(\beta)) \mathsf{x}^{\beta}$$

hence the result. $\qquad\square$

Proposition 7.8. *For all $\lambda \in \mathcal{P}_n$, we have*

$$m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} = \sum_{w \in W_m} \varepsilon(w) \widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(w(\lambda' + \delta_{m,n}) - (\boldsymbol{\mu}' + \delta_{m,n}).$$

Proof. By definition of Δ_m^C (see Section 4) we have

$$\frac{\triangle_{(\boldsymbol{X},\boldsymbol{m})}}{\Delta_m^C} = \prod_{\alpha \in R^+ \backslash R_{(\boldsymbol{X},\boldsymbol{m})}} \frac{1}{1 - x^{[\alpha]}}.$$

We obtain

$$\begin{aligned}
s_{\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} &= \mathsf{E}(\triangle_{(\boldsymbol{X},\boldsymbol{m})} \mathsf{x}^{\boldsymbol{\mu}'}) && \text{(Lemma 7.6)} \\
&= \mathsf{E}(\Delta_m^C \cdot \frac{1}{\Delta_m^C} \triangle_{(\boldsymbol{X},\boldsymbol{m})} \mathsf{x}^{\boldsymbol{\mu}'}) \\
&= \mathsf{E}(\Delta_m^C \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta) \mathsf{x}^{\beta + \boldsymbol{\mu}'}) \\
&= \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta) \mathsf{E}(\Delta_m \mathsf{x}^{\beta + \boldsymbol{\mu}'}) \\
&= \sum_{\beta \in \mathbb{Z}^m} \widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}(\beta) \mathsf{v}_{\beta + \boldsymbol{\mu}'}.
\end{aligned}$$

We conclude using Theorem 4.8. $\qquad\square$

We are now ready to complete the proof of the main theorem.

Proof of Theorem 7.5. We have

$$
\begin{aligned}
m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{m})} &= \sum_{w\in W_m} \varepsilon(w)\widetilde{\mathcal{P}}_{(\boldsymbol{X},\boldsymbol{m})}\left(w(\lambda'+\delta_{m,n})-(\boldsymbol{\mu}'+\delta_{m,n})\right) \\
&= \sum_{w\in W_m} \varepsilon(w)\mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}\left(\mathsf{I}(w(\lambda'+\delta_{m,n}))-\mathsf{I}(\boldsymbol{\mu}'+\delta_{m,n})\right) \\
&= \sum_{w\in W_m} \varepsilon(w)\mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}\left(\mathsf{I}(w(\lambda'+\delta_{m,n}))-\mathsf{I}(\boldsymbol{\mu}')-n\cdot\omega_m-\rho_m\right).
\end{aligned}
$$

By setting $w'=|w|$ in the last sum and using Proposition 7.4, this yields

$$
\begin{aligned}
m_{\lambda,\boldsymbol{\mu}}^{(\boldsymbol{X},\boldsymbol{a})} &= \sum_{w'\in W_m} \varepsilon(w')\mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}\left(w'(\mathsf{I}(\lambda')+n\omega_m)+\rho_m)-(\mathsf{I}(\boldsymbol{\mu}')+n\omega_m)-\rho_m\right) \\
&= \sum_{w'\in W_m} \varepsilon(w')\mathcal{P}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}\left(w'(\widehat{\lambda}+\rho_m)-(\widehat{\boldsymbol{\mu}}-\rho_m)\right) \\
&= \left[V(\widehat{\lambda}):V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\boldsymbol{\mu})\right]
\end{aligned}
$$

as required. $\square$

8. Injectivity of the induction functor

Theorem 7.5 permits to express tensor multiplicities of $\mathfrak{sp}_{2n}(\mathbb{C})$-modules in terms of branching coefficients in irreducible $\mathfrak{sp}_{2m}$-modules. We started from the tensor product of $\mathfrak{sp}_{2n}(\mathbb{C})$-modules

$$
V^{X_1}(\mu^{(1)})\otimes V^{X_2}(\mu^{(2)})\otimes\cdots\otimes V^{X_r}(\mu^{(r)})
$$

associated to the sequence $\boldsymbol{\mu}=(\mu^{(1)},\dots,\mu^{(r)})$ of partitions such that $\mu^{(j)}\in\mathcal{P}_{n\times m_j}$ and to the sequence $\boldsymbol{X}=(X_1,\dots,X_r)$ of symbols in $\{A,C\}$. This determined the dominant weight $\widehat{\boldsymbol{\mu}}=\mathsf{I}(\boldsymbol{\mu}')+n\cdot\omega_m$ of the subalgebra $\mathfrak{g}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}$ of $\mathfrak{sp}_{2m}(\mathbb{C})$ where $m=\sum m_i$ and $\boldsymbol{m}=(m_1,\dots,m_r)$.

Conversely, we can start from a dominant weight $\widehat{\boldsymbol{\mu}}$ of $\mathfrak{g}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}\subset\mathfrak{sp}_{2m}(\mathbb{C})$ and realise the associated branching coefficient as a tensor product multiplicity for $\mathfrak{sp}_{2n}(\mathbb{C})$-modules. But here, one has to keep in mind that the datum of $\widehat{\boldsymbol{\mu}}$ and $\mathfrak{g}_{(\boldsymbol{X}^*,\boldsymbol{m}^*)}$ does not determine the integer n. One can only say that n is at least $n_{\widehat{\boldsymbol{\mu}}}$, the greatest part in the partitions $\widehat{\boldsymbol{\mu}}_{(j)}$ where $\widehat{\boldsymbol{\mu}}=(\widehat{\boldsymbol{\mu}}_{(1)},\dots,\widehat{\boldsymbol{\mu}}_{(r)})$. This means that for any integer $n\geqslant n_{\widehat{\boldsymbol{\mu}}}$, we have an r-tuple of partitions $\boldsymbol{\mu}[n]=(\mu^{(1)}[n],\dots,\mu^{(r)}[n])$ such that $\widehat{\boldsymbol{\mu}[n]}=\widehat{\boldsymbol{\mu}}$ (where the map $\widehat{\cdot}$ is defined with respect to the pair $(n,\boldsymbol{m})$). Observe in particular that for any $j=1,\dots,r$, we have

$$
\mu^{(j)}[a+n_{\widehat{\boldsymbol{\mu}}}]=\big(\underbrace{m_j,\dots,m_j}_{a\text{ times}},\mu^{(j)}[n_{\widehat{\boldsymbol{\mu}}}]\big)\text{ for any integer } a\geqslant 0 \tag{8.1}
$$

or equivalently, $\mu^{(j)}[n+1]$ is obtained by adding a part m_j to $\mu^{(j)}[n]$ for any $n\geqslant n_{\widehat{\boldsymbol{\mu}}}$.

Example 8.1. Consider the following sequence of partition:

$$\widehat{\boldsymbol{\mu}} = \left(\begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array} \right)$$

so that $n_{\widehat{\mu}} = 3$. We have seen in Example 7.3 that

$$\boldsymbol{\mu}[3] = \left(\begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array} \right).$$

Now setting $n = 5$ and adding 2 parts of respective size 2, 2 and 3 to $\boldsymbol{\mu}[3]$ we get

$$\boldsymbol{\mu}[5] = \left(\begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array} \right).$$

Then taking the complements in the rectangles of respective size 3×5, 2×5 and 2×5 we obtain

$$\left(\begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array}, \begin{array}{c} \square \end{array} \right),$$

and taking the conjugate of each green partition and reversing, we see that we obtain $\widehat{\boldsymbol{\mu}[5]} = \widehat{\boldsymbol{\mu}}$ (here $\widehat{\boldsymbol{\mu}[5]}$ is computed with respect to the pair $(5, (2, 2, 3))$).

Then, one can apply Theorem 7.5 and get for all $n \geqslant n_{\widehat{\mu}}$ and all partitions $\lambda \in P_n$ the equality

$$m_{\lambda, \boldsymbol{\mu}[n]}^{(\boldsymbol{X}, \boldsymbol{m})} = [V(\widehat{\lambda}) : V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})].$$

In the relation above, the partition $\widehat{\lambda}$ is defined from the action of the map $\widehat{\cdot}$ corresponding to the pair (n, m) and the r-partition $\widehat{\boldsymbol{\mu}}$ is defined similarly with respect to the pair $(n, \boldsymbol{m})$ (in both cases, the same n as in $\boldsymbol{\mu}[n]$).

In the following, we fix two dominant weights $\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\nu}}$ of the algebra $\mathfrak{g}_{(\boldsymbol{X}^*, \boldsymbol{m}^*)} \subset \mathfrak{sp}_{2m}$. From the definition, we see that $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ are sequences of partitions in which the j-th components lies in $\mathcal{P}_{m_{r-j+1}}$. We set

$$\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_{(1)}, \dots, \widehat{\mu}_{(r)}) \quad \text{and} \quad \widehat{\boldsymbol{\nu}} = (\widehat{\nu}_{(1)}, \dots, \widehat{\nu}_{(r)}).$$

Recall that $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$ and $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\nu}})$ are the associated highest weight $\mathfrak{g}_{(\boldsymbol{X}^*, \boldsymbol{m}^*)}$-modules. We consider the following problem.

Problem 8.1. Assume that the induction of $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$ and $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\nu}})$ to $\mathfrak{sp}_{2m}$ are isomorphic. What can we say about $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$?

We are going to prove that in the case where all the components in $\boldsymbol{X}^* = (X_r, \ldots, X_1)$ are of type C or when $\boldsymbol{X}^*$ is a parabolic Dynkin subdiagram (i.e. X_1 is the unique component of type C) then the sequences $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ coincide up to permutations of their parts. Further, in the second case we will show that $\widehat{\mu}_{(r)} = \widehat{\nu}_{(r)}$. In particular, this proves the main conjecture of [GL16] in type C: in the parabolic case, the two previous induced modules are isomorphic if and only if the dominant weights $\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\nu}}$ coincide up to an automorphism of the underlying parabolic Dynkin diagram. In [GL16], this conjecture was proved in any finite types but only when $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ are far enough from the walls of Weyl chambers.

To do this, recall first that the character ring of type C_n can be regarded as the ring $\mathbb{Z}^{W_n}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ of Laurent polynomials fixed by permutations of the variables x_i and the inversions $x_i \mapsto \frac{1}{x_i}$. Also, we have a total order on the monomials in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ defined by, for all $\beta, \gamma \in \mathbb{Z}^n$,

$$\mathsf{x}^\beta \leqslant \mathsf{x}^\gamma \iff \beta \leqslant_{\mathrm{lex}} \gamma$$

where $\leqslant_{\mathrm{lex}}$ is the lexicographic order on $\mathbb{Z}^n$. This enables us to define, for any $P \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ the monomial $\max(P)$ as the maximal monomial appearing in P. Given A and B in $\mathbb{Z}^{W_n}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we have $\max(AB) = \max(A) \times \max(B)$. This implies the following useful lemma.

Lemma 8.2. *Consider P and Q in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ such that Q divides P. Then $\max(Q) \leqslant \max(P)$.*

For any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}^n$, set $|\beta| = \beta_1 + \cdots + \beta_n$. Given $P = \sum_{\beta \in \mathbb{Z}^n} c_\beta x^\beta$ (all but a finite number of coefficients c_β are equal to zero) in $\mathbb{Z}^{W_n}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, define

$$\mathrm{head}(P) = \sum_{\substack{\beta \in \mathbb{Z}^n \\ |\beta| = |\max(P)|}} c_\beta \mathsf{x}^\beta.$$

We have the following easy lemma.

Lemma 8.3. *Let $P_1, \ldots, P_r$ be a sequence of polynomials in $\mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then*

$$\mathrm{head}(P_1 \times \cdots \times P_r) = \mathrm{head}(P_1) \times \cdots \times \mathrm{head}(P_r).$$

Now we need a result by Rajan [Raj14] on the irreducibility of the characters for $\mathfrak{sp}_{2n}$. In fact, we do not need Rajan's result in its full generality and we will only state a weaker version, sufficient for our purposes. Recall that $\rho_n = (n, n-1, \ldots, 1)$ and set $\widetilde{\rho}_n = (2n-1, 2n-3, \ldots, 1)$.

For any partition $\delta \in \mathcal{P}_n$, let

- s_δ^C be the character of the irreducible module $V(\delta)$ of highest weight δ in $\mathfrak{sp}_{2n}$,

- $\mathsf{s}_\delta(X)$ be the usual Schur function associated to δ (i.e. the character of the irreducible module $V(\delta)$ of highest weight δ in $\mathfrak{gl}_n$) thus a symmetric polynomial in the set of variables $x_1, \ldots, x_n$,

- $\mathsf{s}_\delta(X^{\pm 1})$ be the Schur function $\mathsf{s}_\delta(X^{\pm 1}) = \mathsf{s}_\delta(x_1, \ldots, x_n, \frac{1}{x_n}, \ldots, \frac{1}{x_1})$ where s_δ is the Schur function in $2n$ variables.

Theorem 8.4. *Let $\delta \in \mathcal{P}_n$ be a dominant weight for $\mathfrak{sp}_{2n}$ regarded as an element of $\mathbb{Z}^n$. Assume that the coordinates of $\delta + \rho_n$ are relatively prime (i.e. they have no trivial common divisor). Then the character s_δ^C is irreducible in $\mathbb{C}^{W_n}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.*

Remark 8.5. In fact the irreducibility property proved in [Raj14] is more general. Let $d(\delta)$ be the greatest common divisor for the coordinates of $\delta + \rho_n$ on the basis of the fundamental weights $\omega_1, \ldots, \omega_{n-1}, \omega_n$. Equivalently, $d(\delta)$ is the gcd of the coordinates of $\delta + \rho_n$ in the usual basis of $\mathbb{Z}^{n}$[2]. Define also $\widetilde{d}(\delta)$ as the greatest common divisor for the coordinates of $\delta + \rho_n$ on the weight $\mathbb{Q}$-basis $2\omega_1, \ldots, 2\omega_{n-1}, \omega_n$ (with the convention $\widetilde{d}(\delta) = 1$ as soon as we have a non integer coordinate). For any weight β, recall that $a_\beta = \sum_{w \in W} \varepsilon(w) \mathsf{x}^{w(\beta + \rho_n)}$. Then set

$$D(\delta) = \gcd(a_{d(\delta)\rho_n}, a_{\widetilde{d}(\delta)\rho_n})$$

where the greatest common divisor is here considered in $\mathbb{C}^{W_n}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. It is proved in [Raj14] that

$$S_\delta^C := \frac{a_{\delta + \rho_n}}{D(\delta)}$$

is irreducible as soon as $\delta + \rho_n$ is not a multiple of ρ_n or $\widetilde{\rho}_n$. By our assumption in the previous theorem, the coordinates of $\delta + \rho_n$ are relatively prime, therefore $d(\delta) = 1$ and $\delta + \rho_n$ is not a multiple of ρ_n. Now, observe that a_{ρ_n} divides $a_{k\rho_n}$ for any integer $k \geqslant 1$ because $\frac{a_{k\rho_n}}{a_{\rho_n}} = \mathsf{s}_{(k-1)\rho_n}^C$. This implies that $D(\delta) = \gcd(a_{\rho_n}, a_{\widetilde{d}(\delta)\rho_n}) = a_{\rho_n}$ and $\mathsf{s}_\delta^C = S_\delta^C$ is irreducible, as claimed.

Rajan's irreducibility result [Raj14] holds for any finite root systems. It can be used to prove the second result by Rajan that we shall need. Again, it holds for any root system but we shall only need it for the Schur functions.

Theorem 8.6. *Let $\lambda^{(1)}, \ldots, \lambda^{(r)}$ and $\mu^{(1)}, \ldots, \mu^{(r)}$ be two sequences of partitions in $\mathcal{P}_n$ such that*

$$\mathsf{s}_{\lambda^{(1)}}(X) \times \cdots \times \mathsf{s}_{\lambda^{(r)}}(X) = \mathsf{s}_{\mu^{(1)}}(X) \times \cdots \times \mathsf{s}_{\mu^{(r)}}(X).$$

Then we have the multiset equality $\{\lambda^{(1)}, \ldots, \lambda^{(r)}\} = \{\mu^{(1)}, \ldots, \mu^{(r)}\}$.

One can observe that this result on Schur functions easily implies its analogue on Weyl characters thanks to Lemma 8.3 and the simple observation

$$\mathrm{head}(\mathsf{s}_\nu^C) = \mathsf{s}_\nu(X) \text{ for all } \nu \in \mathcal{P}_n.$$

Corollary 8.7. *Consider $\lambda^{(1)}, \ldots, \lambda^{(r)}$ and $\mu^{(1)}, \ldots, \mu^{(r)}$ two sequences of partitions in $\mathcal{P}_n$ such that*

$$\mathsf{s}_{\lambda^{(1)}}^C \times \cdots \times \mathsf{s}_{\lambda^{(r)}}^C = \mathsf{s}_{\mu^{(1)}}^C \times \cdots \times \mathsf{s}_{\mu^{(r)}}^C.$$

Then we have the multiset equality $\{\lambda^{(1)}, \ldots, \lambda^{(r)}\} = \{\mu^{(1)}, \ldots, \mu^{(r)}\}$.

Now recall the branching formula for the decomposition of a Schur function $\mathsf{s}_\nu(X^{\pm 1})$ on the basis of the irreducible characters for $\mathfrak{sp}_{2n}$.

---

[2]For any $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{Z}$, we indeed have $\beta = \sum_{i=1}^n a_i \omega_i$ where $a_i = \beta_i - \beta_{i+1}$ for $1 \leqslant i \leqslant n-1$ and $\beta_n = a_n$.

Theorem 8.8 ([FH91], Appendix A]). *Let $\nu \in \mathcal{P}_n$. We have*

$$\mathsf{s}_\nu(X^{\pm 1}) = \sum_{\lambda \in \mathcal{P}_n, \delta \in \mathcal{P}_n^{(1,1)}} c_{\lambda,\delta}^\nu \mathsf{s}_\lambda^C.$$

where $\mathcal{P}_n^{(1,1)}$ is the subset of $\mathcal{P}_n$ of partitions which can be tiled in vertical dominoes and $c_{\lambda,\delta}^\nu$ is the usual Littlewood–Richardson coefficient.

Remark 8.9. Observe that $\mathsf{s}_\nu(X^{\pm 1}) \neq \mathsf{s}_\nu^C$ as soon as ν has at least two rows since $c_{\lambda,(1,1)}^\nu > 0$ for any λ obtained from ν by removing one box in two different rows. The previous decomposition can then be written

$$\mathsf{s}_\nu(X^{\pm 1}) = \mathsf{s}_\nu^C + \sum_{\lambda \lhd \nu} a_{\nu,\lambda} \mathsf{s}_\lambda^C$$

where the coefficients $a_{\nu,\lambda}$ belong to $\mathbb{N}$ and $\unlhd$ is the dominant order on partitions. Assume that s_ν^C divides $\mathsf{s}_\nu(X^{\pm 1})$ in $\mathbb{Z}[X^{\pm 1}]$. Then s_ν^C divides $\sum_{\lambda \lhd \nu} a_{\nu,\lambda} \mathsf{s}_\lambda^C$. But $\max(\mathsf{s}_\nu^C) = \mathsf{x}^\nu$ and $\max(\sum_{\lambda \lhd \nu} a_{\nu,\lambda} \mathsf{s}_\lambda^C) = \mathsf{x}^{\nu^\flat}$ where $\nu^\flat$ is the partition obtained by decreasing by 1 the two lowest nonzero parts of ν. Since $\mathsf{x}^\nu > \mathsf{x}^{\nu^\flat}$, this contradicts Lemma 8.2. This shows that s_ν^C divides $\mathsf{s}_\nu(X^{\pm 1})$ in $\mathbb{Z}[X^{\pm 1}]$ if and only if $\mathsf{s}_\nu^C = \mathsf{s}_\nu(X^{\pm 1})$.

Now let us come back to Problem 8.1 and assume that the induction of $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\mu}})$ and $V_{\boldsymbol{m}^*}^{\boldsymbol{X}^*}(\widehat{\boldsymbol{\nu}})$ to $\mathfrak{sp}_{2m}$ are isomorphic. This can be rewritten, thanks to our main theorem in the last section as:

$$\mathsf{s}_{\boldsymbol{\mu}[n]}^{\boldsymbol{X},\boldsymbol{m}} = \mathsf{s}_{\boldsymbol{\nu}[n]}^{(\boldsymbol{X},\boldsymbol{m})} \quad \text{for all } n \geqslant \max(n_{\widehat{\boldsymbol{\mu}}}, n_{\widehat{\boldsymbol{\nu}}}).$$

Assume first that all the components of $\boldsymbol{X}$ are of type C. We get

$$\mathsf{s}_{\mu^{(1)}[n]}^C \times \cdots \times \mathsf{s}_{\mu^{(r)}[n]}^C = \mathsf{s}_{\nu^{(1)}[n]}^C \times \cdots \times \mathsf{s}_{\nu^{(r)}[n]}^C \tag{8.2}$$

where for any $j = 1, \ldots, r$, the partitions $\mu^{(j)}[n]$ and $\nu^{(j)}[n]$ have at most m_j-columns. Here we can apply Corollary 8.7 and deduce the multiset equality $\{\mu^{(1)}[n], \ldots, \mu^{(r)}[n]\} = \{\nu^{(1)}[n], \ldots, \nu^{(r)}[n]\}$.

Alternatively, we can choose n so that each partition $\mu^{(j)}[n]$ and $\nu^{(j)}[n]$ starts with two parts equal to m_j. Then, the two first components in $\mu^{(j)}[n] + \rho_n$ or $\nu^{(j)}[n] + \rho_n$ are equal to $m_j + n$ and $m_j + n - 1$, respectively. In particular, this implies that the coordinates of $\mu^{(j)}[n] + \rho_n$ and $\nu^{(j)}[n] + \rho_n$ are relatively prime. We thus get by Theorem 8.4 that each character $\mathsf{s}_{\mu^{(j)}[n]}^C$ or $\mathsf{s}_{\nu^{(j)}[n]}^C$ are irreducible. Thus we recover the multiset equality $\{\mu^{(1)}[n], \ldots, \mu^{(r)}[n]\} = \{\nu^{(1)}[n], \ldots, \nu^{(r)}[n]\}$. It implies that the r-partitions $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ coincide up to permutation of their partitions.

Next, assume $\boldsymbol{X} = (C, A, \ldots, A)$. This time we get

$$\mathsf{s}_{\mu^{(1)}[n]}^C \times \mathsf{s}_{\mu^{(2)}[n]}(X^{\pm 1}) \times \cdots \times \mathsf{s}_{\mu^{(r)}[n]}(X^{\pm 1}) = \mathsf{s}_{\nu^{(1)}[n]}^C \times \mathsf{s}_{\nu^{(2)}[n]}(X^{\pm 1}) \times \cdots \times \mathsf{s}_{\nu^{(r)}[n]}(X^{\pm 1}). \tag{8.3}$$

Write $N = |\mu^{(1)}[n]| + \cdots + |\mu^{(r)}[n]|$ the sum of all parts in the partitions $\mu^{(j)}[n], j = 1, \ldots, r$ (that is the rank of the r-partition $\boldsymbol{\mu}[n]$). For any partition $\kappa \in \mathcal{P}_n$, we have

$$\max(\mathsf{s}_\kappa^C) = \max(\mathsf{s}_\kappa(X)) = \mathsf{x}^\kappa \quad \text{and} \quad \mathrm{head}(\mathsf{s}_\kappa^C) = \mathsf{s}_\kappa(X).$$

Then Lemma 8.3 implies that for any partition ν of rank N, the coordinates of

$$\mathsf{s}^C_{\mu^{(1)}(n)} \times \mathsf{s}_{\mu^{(2)}(n)}(X^{\pm 1}) \times \cdots \times \mathsf{s}_{\mu^{(r)}(n)}(X^{\pm 1})$$

on the irreducible character s^C_ν coincides with the coordinates of

$$\mathsf{s}_{\mu^{(1)}(n)}(X) \times \mathsf{s}_{\mu^{(2)}(n)}(X) \times \cdots \times \mathsf{s}_{\mu^{(r)}(n)}(X)$$

on the Schur function $\mathsf{s}_\nu(X)$. Equation (8.3) then gives

$$\mathsf{s}_{\mu^{(1)}(n)}(X) \times \mathsf{s}_{\mu^{(2)}(n)}(X) \times \cdots \times \mathsf{s}_{\mu^{(r)}(n)}(X) = \mathsf{s}_{\nu^{(1)}(n)}(X) \times \mathsf{s}_{\nu^{(2)}(n)}(X) \times \cdots \times \mathsf{s}_{\nu^{(r)}(n)}(X)$$

and we obtain $\{\mu^{(1)}[n], \ldots, \mu^{(r)}[n]\} = \{\nu^{(1)}[n], \ldots, \nu^{(r)}[n]\}$ by Theorem 8.6.

Having in hand the multiset equality $\{\mu^{(1)}[n], \ldots, \mu^{(r)}[n]\} = \{\nu^{(1)}[n], \ldots, \nu^{(r)}[n]\}$ and equation (8.3), we deduce that

(1) Either $\mu^{(1)}[n] = \nu^{(1)}[n]$ and $\{\mu^{(2)}[n], \ldots, \mu^{(r)}[n]\} = \{\nu^{(2)}[n], \ldots, \nu^{(r)}[n]\}$,

(2) or $\mu^{(1)}[n] = \nu^{(p)}[n]$, $\mu^{(q)}[n] = \nu^{(1)}[n]$ with $p > 1$ and $q > 1$ and

$$\{\mu^{(2)}[n], \ldots, \mu^{(r)}[n]\} \backslash \{\mu^{(q)}[n]\} = \{\nu^{(2)}[n], \ldots, \nu^{(r)}[n]\} \backslash \{\nu^{(p)}[n]\}.$$

In Case (1), since

$$\widehat{\boldsymbol{\mu}[n]} = \widehat{\boldsymbol{\mu}} \quad \text{and} \quad \begin{cases} \widehat{\boldsymbol{\mu}[n]} = (\widehat{\mu^{(r)}[n]}, \ldots, \widehat{\mu^{(1)}[n]}) \\ \widehat{\boldsymbol{\mu}} = (\widehat{\mu}_{(1)}, \ldots, \widehat{\mu}_{(r)}) \end{cases}$$

we get that $\widehat{\mu}_{(r)} = \widehat{\nu}_{(r)}$ and $\{\widehat{\mu}^{(1)}, \ldots, \widehat{\mu}^{(r-1)}\} = \{\widehat{\nu}^{(1)}, \ldots, \widehat{\nu}^{(r-1)}\}$.

In Case (2), by simplifying the identical factors in (8.3), we get

$$\mathsf{s}^C_{\mu^{(1)}[n]} \times \mathsf{s}_{\mu^{(q)}[n]}(X^{\pm 1}) = \mathsf{s}^C_{\mu^{(q)}[n]} \times \mathsf{s}_{\mu^{(1)}[n]}(X^{\pm 1}).$$

As before, we can choose the integer n so that $\mathsf{s}^C_{\mu^{(1)}[n]}$ and $\mathsf{s}^C_{\nu^{(1)}[n]} = \mathsf{s}^C_{\mu^{(q)}[n]}$ are irreducible polynomials, and the partitions $\mu^{(1)}[n]$ and $\nu^{(1)}[n]$ have their two first parts equal to m_1. This implies that $\mathsf{s}^C_{\mu^{(1)}[n]}$ divides $\mathsf{s}_{\mu^{(1)}[n]}(X^{\pm 1})$. But, by using Remak 8.9, this is only possible when $\mathsf{s}^C_{\mu^{(1)}[n]} = \mathsf{s}_{\mu^{(1)}[n]}(X^{\pm 1})$, that is when $\mu^{(1)}[n]$ is a row. We thus get a contradiction since $\mu^{(1)}[n]$ has at least two parts equal to m_1.

Finally, we have proved the following theorem.

Theorem 8.10. *Assume that the induction of $V^{X^*}_{m^*}(\widehat{\boldsymbol{\mu}})$ and $V^{X^*}_{m^*}(\widehat{\boldsymbol{\nu}})$ to $\mathfrak{sp}_{2m}$ are isomorphic. Then the following holds.*

(1) The r-partitions $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ coincide up to permutation of their partitions when all the components in $\boldsymbol{X}$ are of type C

(2) We have $\widehat{\mu}^{(r)} = \widehat{\nu}^{(r)}$ and $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\nu}}$ coincide up to permutation of their other partitions when $\boldsymbol{X}$ is a parabolic Dynkin subdiagram (i.e. X_1 is the unique component equal to C)

Acknowledgements

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