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GRAPHS WITH LARGE MINIMUM DEGREE AND NO SHORT ODD CYCLES ARE 3-COLOURABLE

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Abstract. Answering a question by Letzter and Snyder, we prove that for large enough k , any n -vertex graph G with minimum degree at least $\frac{1}{2k-1}n$ and without odd cycles of length less than $2k+1$ is 3-colourable. In fact, we prove a stronger result that works with a slightly smaller minimum degree.

Keywords. Chromatic profile, Minimum degree, Graph colouring

Mathematics Subject Classifications. 05C35, 05C15

1. Introduction

Determining the chromatic number of a graph is a difficult problem. This explains why a wealth of results in graph theory aims instead at determining meaningful upper bounds on this quantity, which can also be seen as bounds on the permitted structural complexity of the graphs under consideration. One natural question in this direction then is if the chromatic number of the family of $\mathcal{H}$ -free graphs is bounded for finite non-trivial $\mathcal{H}$. Here, a graph G is $\mathcal{H}$ -free for a set $\mathcal{H}$ of graphs, if G does not contain any member of $\mathcal{H}$ as a subgraph, we also write H -free when $\mathcal{H} = \{H\}$, and we say that $\mathcal{H}$ is *non-trivial* if none of the graphs in $\mathcal{H}$ is a forest. This question was answered negatively by Erdős [Erd59] in one of the early applications of the

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celebrated probabilistic method: For every finite non-trivial $\mathcal{H}$ and every positive integer c , there are $\mathcal{H}$ -free graphs with chromatic number at least c .

In another influential paper, Erdős and Simonovits [ES73] asked what happens if a minimum degree condition is also introduced. More precisely, they initiated the study of the so-called chromatic profile of $\mathcal{H}$. To define this, it is convenient to introduce some notation. We denote by $\mathcal{G}(\mathcal{H})$ the family of all $\mathcal{H}$ -free graphs, and by $\mathcal{G}(\mathcal{H}, \alpha)$ the set of graphs in $\mathcal{G}(\mathcal{H})$ of minimum degree at least $\alpha|V(G)|$. For $c \geq 2$ a positive integer, the *chromatic profile* of $\mathcal{H}$ as a function in c is

$$\delta_\chi(\mathcal{H}, c) = \inf\{\alpha \in [0, 1] : \forall G \in \mathcal{G}(\mathcal{H}, \alpha), \chi(G) \leq c\}.$$

This function measures how large the minimum degree needs to be in order to guarantee that an $\mathcal{H}$ -free graph has chromatic number at most c . Erdős and Simonovits [ES73] judged that in full generality this quantity seemed ‘too complicated’ to study. Despite considerable progress in the last few decades, this judgment still stands firm. The goal of this paper is to contribute to the understanding of the chromatic profile of the family of odd cycles up to a certain length.

But let us first summarise what is known. Soon after Erdős and Simonovits’s paper, Andrásfai, Erdős and Sós [AES74] proved that K_r -free graphs of minimum degree strictly larger than $\frac{3r-7}{3r-4}|V(G)|$ have chromatic number at most $r-1$. Moreover, in the same paper examples were given of K_r -free graphs of minimum degree $\frac{3r-7}{3r-4}|V(G)|$ whose chromatic number is exactly r . In other words, $\delta_\chi(\{K_r\}, r-1) = \frac{3r-7}{3r-4}$. Other known results include $\delta_\chi(\{K_3\}, 3) = \frac{10}{29}$ by Häggkvist [H82] and Jin [Jin95], and $\delta_\chi(\{K_3\}, c) = \frac{1}{3}$ for every $c \geq 4$ by Brandt and Thomassé [BT], where the lower bound is due to Hajnal (see [ES73]). On the other hand, Thomassen [Tho07] showed that, for any odd $k \geq 5$, we have $\delta_\chi(\{C_k\}, c) \leq \frac{6}{c}$ and more generally found a similar upper bound for $\delta_\chi(\{C_k\}, c)$; together with a result of Ma [Ma16] this implies for every fixed odd $k \geq 5$ that

$$\Omega\left((k+1)^{-4(c+1)}\right) = \delta_\chi(\{C_k\}, c) = O\left(\frac{k}{c}\right).$$

The next developments concerned two quantities related to the chromatic profile: the chromatic threshold and the homomorphism threshold. The *chromatic threshold* of $\mathcal{H}$ is

$$\delta_\chi(\mathcal{H}) = \inf\{\alpha \in [0, 1] : \exists K \text{ s.t. } \forall G \in \mathcal{G}(\mathcal{H}, \alpha), \chi(G) \leq K\},$$

and measures how large the minimum degree needs to be to guarantee that the chromatic number of $\mathcal{H}$ -free graphs is bounded by some constant. For example, the result by Brandt and Thomassé [BT] mentioned above shows that $\delta_\chi(\{C_3\}) = \frac{1}{3}$. The chromatic threshold is by now much better understood than the chromatic profile. Building on the work of Łuczak and Thomassé [LT10], and generalising various previous results, Allen, Böttcher, Griffiths, Kohayakawa, and Morris [ABG⁺13] determined the chromatic threshold of every finite family $\mathcal{H}$. For more details about the history of the study of the chromatic threshold see this paper and the references therein.

We remind here that a graph G is *homomorphic* to another graph F if there exists a not-necessarily injective map from $V(G)$ to $V(F)$ such that edges are preserved. To get an even better picture, one can consider the more restrictive notion of the *homomorphism threshold* $\delta_{\text{hom}}(\mathcal{H})$

of a family $\mathcal{H}$, which is a measure of the smallest minimum degree that guarantees that $\mathcal{H}$ -free graphs are homomorphic to a small $\mathcal{H}$ -free graph. For $\delta_{\text{hom}}(\mathcal{H})$ there are several equivalent definitions and, following Ebsen and Schacht [ES20], we define

$$\delta_{\text{hom}}(\mathcal{H}) = \inf\{\alpha \in [0, 1] : \exists F \in \mathcal{G}(\mathcal{H}) \text{ s.t. } \forall G \in \mathcal{G}(\mathcal{H}, \alpha), G \text{ is homomorphic to } F\}.$$

Note that $\delta_{\text{hom}}(\mathcal{H}) \geq \delta_{\chi}(\mathcal{H})$. Determining homomorphism thresholds is distinctively harder than determining chromatic thresholds. Łuczak [Łuc06] showed that for K_3 the homomorphism threshold equals the chromatic threshold, which as discussed above is $\frac{1}{3}$. Goddard and Lyle [GL11] and Nikiforov [Nik10] extended this to all cliques, showing $\delta_{\text{hom}}(\{K_k\}) = \delta_{\chi}(\{K_k\}) = \frac{2k-5}{2k-3}$. Letzter and Snyder [LS19] considered a generalisation to longer odd cycles instead. They proved $\delta_{\text{hom}}(\{C_5\}) \leq \frac{1}{5}$ and $\delta_{\text{hom}}(\mathcal{C}_5) = \frac{1}{5}$, where $\mathcal{C}_{2k-1} = \{C_3, \dots, C_{2k-1}\}$ is the family of odd cycles up to length $2k-1$. Extending this, Ebsen and Schacht [ES20] proved $\delta_{\text{hom}}(\{C_{2k-1}\}) \leq \frac{1}{2k-1}$ and $\delta_{\text{hom}}(\mathcal{C}_{2k-1}) = \frac{1}{2k-1}$ for all $k \geq 2$. Complementing the first of these results, Sankar [San22] recently proved that $\delta_{\text{hom}}(\{C_{2k-1}\}) > 0$ for all $k \geq 2$. This shows that, in contrast to cliques, the homomorphism threshold for odd cycles behaves differently than the chromatic threshold since $\delta_{\chi}(\mathcal{C}_{2k-1}) = 0$ for $k > 2$, see [Tho07].

Returning to the chromatic profile, what can be said about families of odd cycles? Already the methods by Andrásfai, Erdős and Sós [AES74] give $\delta_{\chi}(\mathcal{C}_{2k-1}, 2) = \frac{2}{2k+1}$, where the lower bound comes from a blow-up of C_{2k+1} . Moving on to 3-colourability, when establishing the homomorphism threshold for $\mathcal{C}_5$, Letzter and Snyder [LS19] showed that graphs in $\mathcal{G}(\mathcal{C}_5, \frac{1}{5} + \varepsilon)$ are in fact homomorphic to graphs of chromatic number 3, which implies $\delta_{\chi}(\mathcal{C}_5, 3) \leq \frac{1}{5}$. The best-known lower bound, on the other hand, is $\delta_{\chi}(\mathcal{C}_5, 3) \geq \frac{14}{73}$, which is given by an asymmetric blow-up of a $\mathcal{C}_5$ -free graph on 22 vertices (cf. the graph $G_{3,3}$ in Van Ngoc and Tuza [VNT95]). The homomorphisms constructed in Ebsen and Schacht's [ES20] generalisation, however, were not to 3-colourable graphs. Thus, their result does not imply an upper bound on $\delta_{\chi}(\mathcal{C}_{2k-1}, 3)$.

Providing such an upper bound is the main contribution of this paper. We prove that for k large enough the homomorphism threshold $\frac{1}{2k-1}$ is an upper bound on the chromatic profile $\delta_{\chi}(\mathcal{C}_{2k-1}, 3)$. This answers a question of Letzter and Snyder [LS19]. In fact, we can prove a slightly stronger upper bound, which shows that $\delta_{\chi}(\mathcal{C}_{2k-1}, 3)$ is strictly smaller than $\delta_{\text{hom}}(\mathcal{C}_{2k-1})$ for large k .

Theorem 1.1. *For any $t \in \mathbb{N}$ and any integer $k \geq k(t) = 1460 + 20t$ the following holds. Any $\mathcal{C}_{2k-1}$ -free graph G of minimum degree at least $\frac{1}{2k+t}|V(G)|$ is 3-colourable. In other words, for $k \geq k(t)$, we have $\delta_{\chi}(\mathcal{C}_{2k-1}, 3) \leq \frac{1}{2k+t}$.*

Since $k(t)$ is linear in t , we conclude that for any $0 < \varepsilon < \frac{1}{45}$ we have $\delta_{\chi}(\mathcal{C}_{2k-1}, 3) \leq \frac{1}{(2+\varepsilon)k}$ for large enough k . Concerning lower bounds for $\delta_{\chi}(\mathcal{C}_{2k-1}, 3)$, we only know of bounds that are much smaller, for example, the following. A balanced blow up of a generalised $\mathcal{C}_{2k-1}$ -free Mycielski graph of minimum degree 3 and chromatic number 4, as described for example in Van Ngoc and Tuza [VNT95], see Figure 1.1, gives a 4-chromatic $\mathcal{C}_{2k-1}$ -free graph G of minimum degree $\frac{3}{2k^2+k+1}|V(G)|$. As this lower bound and our upper leave a considerable gap, we make no further effort here in optimising the constant factor in either of them.

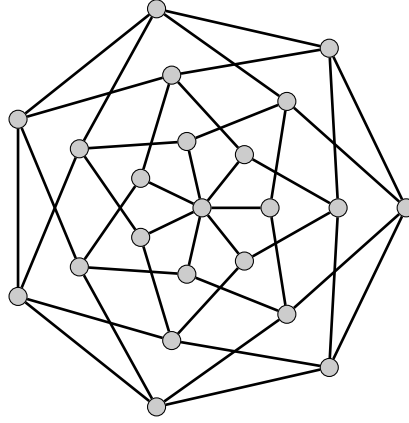


Figure 1.1: A 4-chromatic $\mathcal{C}_5$ -free graph on 22 vertices with minimum degree three.

It would be interesting to know if our upper bound or this lower bound provides the right order of magnitude for $\delta_\chi(\mathcal{C}_{2k-1}, 3)$.

Question 1.2. Is $\delta_\chi(\mathcal{C}_{2k-1}, 3) \geq \frac{c}{f(k)}$ with c constant and $f(k)$ linear in k ? Or is $\delta_\chi(\mathcal{C}_{2k-1}, 3) \leq \frac{c}{f(k)}$ with c constant and $f(k)$ quadratic in k ?

Similarly, we did not try to optimise our $k(t)$, since with our proof technique one probably cannot bring this down to a single digit when $t = 0$. Nevertheless, it would be interesting to know what happens for small k . In particular, our result motivates the following question.

Question 1.3. Is $\delta_\chi(\mathcal{C}_5, 3) < \frac{1}{5} = \delta_{\text{hom}}(\mathcal{C}_5)$?

Finally, we remark that with our argument it is easy to derive a more general upper bound $\delta_\chi(\mathcal{C}_{2k-1}, c) \leq \frac{1}{2k \lfloor c/3 \rfloor}$ for $c \geq 3$ and sufficiently large k . Observe that $\delta_\chi(\mathcal{C}_{2k-1}, c) \leq \delta_\chi(\{\mathcal{C}_{2k-1}\}, c)$ and thus this upper bound complements the bound $\delta_\chi(\{\mathcal{C}_k\}, c) = O\left(\frac{k}{c}\right)$ mentioned earlier in that it applies to the case of fixed c and large k , while the latter is meaningful for fixed k and large c . We will briefly explain how our general bound can be obtained at the end of Section 3.

Organisation

The remainder of this paper is organised as follows. We start in Section 2 with introducing some basic notation, explaining the strategy of our proof of Theorem 1.1, providing the setup used in this proof as well as the main lemmas we need for this, and outlining what further will be needed for the proof of these lemmas. In Section 3 we then prove Theorem 1.1. The proof of our main technical lemma (Lemma 2.3) is provided in Section 6. To prepare for this proof, we develop tools for finding bipartite subgraphs in a weighted graph in Section 4, and for lower bounding the neighbourhood size of certain cycles and paths in Section 5. We will explain as part of Section 2 how these tools are used.

2. Notation and overview of the proof

Before we explain the proof idea for our main theorem we review some (mostly) standard notation and transfer it in a natural way to graphs that are equipped with weights on their edges.

Notation

Let G be a graph and let $B \subseteq V(G)$ be a set of vertices. We occasionally also refer to a graph G and mean its set of vertices $V(G)$. We denote by $G[B]$ the subgraph of G induced by B . If $G[B]$ is connected, then we also say as a shorthand that B is *connected*. We write $G \setminus B$ for the graph $G[V(G) \setminus B]$. As usual, $N(v)$ denotes the (open) *neighbourhood* of a vertex v of G . For a set of vertices B , we denote with $\text{int}(B) = \{v \in B : N(v) \subseteq B\}$ the *interior* of B . We write B^c for the *complement* $V(G) \setminus B$ of B in G . A path in G is a sequence of vertices $v_1, \dots, v_t$ without repetition, such that $v_i v_{i+1}$ is an edge of G for $i = 1, \dots, t-1$, and its *length* is $t-1$. For a graph G the *distance* $d_G(x, y)$ of two vertices x, y in G is the minimum length of a path in G with end-vertices x and y . For two sets of vertices $A, B \subseteq V(G)$ the *distance* $d_G(A, B)$ is the minimum of $d_G(x, y)$ over all $x \in A$ and $y \in B$. For an integer $i \geq 0$ the (closed) *i-th neighbourhood* of B in G is given by

$$N_G^i[B] = \{x \in V(G) : \exists v \in B \text{ s.t. } d_G(x, v) \leq i\}.$$

Often we also omit the subscript G when it is clear from the context in which graph we are taking neighbourhoods.

We shall also work with the following type of auxiliary graphs with weights on their edges. For a graph H a *weight function* is a function of the form $\omega : E(H) \rightarrow \mathbb{N}$ and a graph endowed with such a function is called a *weighted graph*. All concepts defined above for unweighted graphs also apply to weighted graphs. The weight of a subgraph H' of H is $\omega(H') = \sum_{e \in E(H')} \omega(e)$. We say that H is *weighted bipartite* if there is no cycle in H of odd weight. We also say that $B \subseteq V(H)$ is *weighted bipartite* when $H[B]$ is and the graph is clear from the context.

In addition to the notion of unweighted distance defined above, for weighted graphs we shall also use a weighted version as follows. The *weighted distance* $d_{\omega, H}(x, y)$ of two vertices x, y in a weighted graph H is the minimum weight of a path from x to y . Moreover, for any vertex v and for an integer $i \geq 0$, we define the (closed) *weighted i-th neighbourhood* around v as

$$N_{\omega}^i[v] = \{x \in V(H) : d_{\omega}(x, v) \leq i\}.$$

Overview of the proof

The starting point of our proof of Theorem 1.1 is inspired by Thomassen's approach [Tho07] to establishing the chromatic threshold of C_5 . As in that approach, we start by fixing a maximal set of non-adjacent vertices $v_1, \dots, v_h$ with disjoint neighbourhoods $N(v_1), \dots, N(v_h)$, which leaves a set of remaining vertices $X = V(G) \setminus \bigcup_{i=1}^h N^1[v_i]$, and then analyse the structure of our graph based on the resulting vertex partition. However, our analysis uses different and new ideas and is substantially more complex as we work with a different setup.

It turns out that given any two of the vertices above, say v_i, v_j , the crucial information we need for this analysis is the length of a shortest path between $N(v_i)$ and $N(v_j)$ whose internal vertices lie in X . Moreover, we only care about this path if it is of length at most 3. Such a path of length at most 3 gives a v_i, v_j -path of length in $\{3, 4, 5\}$. Consequently, one main idea in our proof is to represent the structure of our graph by introducing an auxiliary weighted graph H on the vertex set $[h]$. In H we have an edge ij whenever such a v_i, v_j -path with length in $\{3, 4, 5\}$ exists; moreover, we assign as a weight to the edge ij the length of the path between v_i and v_j . Since our graph has no odd cycles of length smaller than $2k + 1$, it is easy to see that this auxiliary graph has no cycles of odd weight smaller than $2k + 1$. Moreover, by assuming that G is connected and by choosing the vertices $v_1, \dots, v_h$ carefully, we can guarantee that H has a spanning tree of edges of weight 3. This is the motivation for the definition of the following family of graphs.

Definition 2.1. For $k \in \mathbb{N}$ we denote by $\mathcal{H}(k)$ the family of graphs H with the following two properties:

- There is a weight function $\omega : E(H) \rightarrow \{3, 4, 5\}$ on the edges of H such that in H there are no cycles C such that $\omega(C)$ is odd and smaller than $2k + 1$.
- There is a tree T spanning H such that all edges of T have weight 3.

Furthermore, we denote by $\mathcal{H}(k, s)$ the graphs in $\mathcal{H}(k)$ on at most s vertices.

This auxiliary graph H encapsulates substantial structural information of G , which is essential in our proof of the 3-colourability of G . Our access to 3-colourability is by a partition of the vertex set of G as required by the following lemma.

Lemma 2.2. *Let G be a graph on vertex set V . If there is a set of vertices $A \subseteq V$ such that $G[A]$ is connected, $G[V \setminus A]$ is bipartite, and for all $v \in V \setminus A$ we have that $G[A \cup \{v\}]$ is bipartite, then $\chi(G) \leq 3$.*

Proof. First choose a proper colouring of A using colours $\{1, 2\}$, and a proper colouring of $V \setminus A$ using colours $\{3, 4\}$. Note that since A is connected, every neighbour of a vertex $v \in V \setminus A$ in A is of the same colour. Now we recolour vertices of colour 4 as follows. If a vertex $v \in V \setminus A$ of colour 4 is connected to a vertex of colour 1, recolour it with 2, otherwise recolour it with 1. $\square$

This criterion that guarantees 3-colourability motivates the following decomposition lemma of the auxiliary graph H , which is the heart of our proof of Theorem 1.1.

Lemma 2.3 (Main technical lemma). *For any $t \in \mathbb{N}$ and any integer $k \geq k(t) = 20t + 1460$ the following holds. For any $H \in \mathcal{H}(k, 2k + t)$ there exists a subset B of $V(H)$ such that $H[B]$ is connected, $H \setminus B$ is weighted bipartite, and $H[B \cup \{v\}]$ is weighted bipartite for all $v \in V(H) \setminus B$.*

Given this lemma the main task in proving Theorem 1.1 is to “translate” this partition of the auxiliary graph H into a partition of G with essentially the same properties. The proof of this lemma relies on a surgical analysis of the neighbourhood $N^1[C]$ of a cycle C of odd weight and a careful combination of paths to build B . We now provide the key ideas of the argument, before turning to the proof of the main theorem.

A strategy to approach Lemma 2.3

We briefly discuss here the main ideas behind our proof of Lemma 2.3, which is detailed in Sections 4, 5 and 6. First, we note that we prove a statement which is slightly stronger than Lemma 2.3. Indeed, when constructing the connected set B we ensure that both $N[B]$ and $H \setminus B$ are weighted bipartite.

In Section 4 we study how to guarantee that $N[B]$ is weighted bipartite. In particular, we are going to show that simple constructions like balls around a vertex and neighbourhoods of lightest paths are weighted bipartite. Moreover, we prove that for certain sets the property of being weighted bipartite passes to the union in a very precise way. This allows us to build larger weighted bipartite sets.

However, these results alone are not sufficient to obtain our goal. Indeed, once we get such a candidate set B , we need to prove that also $H \setminus B$ is weighted bipartite. The following lemma shows that for this it is sufficient that the interior of B is large enough.

Lemma 2.4. *Let $k \geq 8$ and t be natural numbers, and $H \in \mathcal{H}(k, 2k + t)$. Then for any $B \subseteq V(H)$ with $|\text{int}(B)| \geq \frac{4}{3}k + t$ we have that $H \setminus B$ is weighted bipartite.*

If the interior of B has size at least $\frac{4}{3}k + t$, its complement is of size at most $|V(H)| - |\text{int}(B)| \leq \frac{2k}{3}$ and we are going to show that this is not enough space to contain a cycle of odd weight. For example, it can not contain a cycle of odd weight with only edges of weight 3, because this cycle has at least $\frac{2k+1}{3}$ vertices, too much to fit into the complement of B . In general, the cycle might have edges of other weights, but the spanning tree of weight 3 then guarantees that we find additional vertices in the neighbourhood of the cycle. It turns out that also in general we get exactly the same bound as in the example above.

Lemma 2.5. *Let $k \geq 8$ and $H \in \mathcal{H}(k)$. If C is a non-spanning cycle of odd weight, then $|N^1[C]| \geq \frac{2k+1}{3}$.*

We would like to emphasise that exactly this is the reason why we require the spanning tree of weight 3. We quickly give the details of how to obtain Lemma 2.4 from Lemma 2.5.

Proof of Lemma 2.4. By assumption on the interior, B is not empty. Assume $H \setminus B$ is not weighted bipartite, and let $S \subseteq B^c = V(H) \setminus B$ be a cycle of odd in $H \setminus B$. As B is not empty this cycle is not spanning in H . Hence, we can apply Lemma 2.5 to conclude that $|N^1[S]| \geq \frac{2k+1}{3}$. Since no vertex of $S \subseteq B^c$ can have a neighbour in $\text{int}(B)$, we have $N^1[S] \subseteq \text{int}(B)^c$ and so $|\text{int}(B)^c| \geq |N^1[S]| \geq \frac{2k+1}{3}$. However, we also have $|V(H)| = 2k + t$ and $|\text{int}(B)| \geq \frac{4}{3}k + t$ which gives $|\text{int}(B)^c| \leq 2k - \frac{4}{3}k < \frac{2k+1}{3}$, a contradiction. $\square$

In Section 5 we prove Lemma 2.5 and a useful corollary. Finally, in Section 6 we combine the results presented in Sections 4 and 5 to show the existence of a weighted bipartite set B with large interior to prove Lemma 2.3. As promised we now turn to the proof of the main theorem.

3. Proof of the main result

Proof of Theorem 1.1. Let $t \in \mathbb{N}$ and let $k \geq k(t) = 1460 + 20t$ be an integer. Let $G = (V, E)$ be an n -vertex graph with minimum degree $\delta(G) \geq n/(2k + t)$ that does not contain an odd cycle of length shorter than $2k + 1$. Since we want to show that the chromatic number of G is at most 3, we may assume that G is connected.

First, we construct an auxiliary graph H on $h \leq 2k + t$ vertices with weight function $w : E(H) \rightarrow \{3, 4, 5\}$ as follows. Let $v_1 \in V$ be any vertex, set $V_1 = N^1[v_1]$, and set the index i to $i = 2$. If possible, we pick a vertex $v_i \in V \setminus V_{i-1}$ such that $N^1[v_i]$ is disjoint from V_{i-1} and such that there is an edge between $N(v_i)$ and $N(v_j)$ for some j , $1 \leq j \leq i - 1$. We let $V_i = N^1[v_i] \cup V_{i-1}$, we increase the index i by one, and repeat the above. We stop this process when there is no vertex v_i with the above properties. We let $h \geq 1$ be the index of the last vertex we picked before the process stopped. Note that $h \leq 2k + t$ because $n \geq |V_h| > hn/(2k + t)$ by the minimum degree of G . Furthermore, we set $X = V \setminus V_h$.

Now let H be the graph with vertex set $[h]$ and with all edges $ij \in \binom{[h]}{2}$ such that $d(N(v_i), N(v_j)) \leq 3$. To every edge $e = ij \in E(H)$, we assign the weight

$$\omega(e) = d(N(v_i), N(v_j)) + 2 \in \{3, 4, 5\},$$

which is an upper bound on the distance between v_i and v_j in G . We thus obtain a graph H on $h \leq 2k + t$ vertices and with weight function $w : E(H) \rightarrow \{3, 4, 5\}$. We observe the following simple properties of H and the V_i .

Claim 3.1.

- (P1) *There is no cycle C in H whose weight is odd and less than $2k + 1$.*
- (P2) *Each vertex $x \in X$ has a neighbour in $N(v_i)$ for some $i \in [h]$.*
- (P3) *For every $i \in [h]$ the neighbourhood $N(v_i)$ is independent, if $k \geq 2$.*
- (P4) *For every $i \in [h]$ the set $\{u \in V(H) : d(u, v_i) = 2\}$ is independent, if $k \geq 3$.*
- (P5) *If for some $i, j \in [h]$ with $i \neq j$ there is a path of length 2 from $N(v_i)$ to $N(v_j)$ in G , then $\omega(ij) = 4$ in H , as long as $k \geq 4$.*
- (P6) *If for some $i, j \in [h]$ $i \neq j$ there is a path of length 3 from $N(v_i)$ to $N(v_j)$ in G , then $\omega(ij) \in \{3, 5\}$ in H , as long as $k \geq 5$.*

Proof. Property (P1) follows directly from our assumptions, because any cycle in H of odd weight less than $2k + 1$ would directly create an odd cycle of length less than $2k + 1$ in G . Indeed, as $d(v_i, v_j) \leq \omega(ij)$ for any $i \neq j$, a cycle C in H with an odd weight less than $2k + 1$ corresponds, by our construction, to a closed odd walk with fewer than $2k + 1$ edges in G , which in turn contains an odd cycle of length at most $2k + 1$.

To see (P2), observe that if this was not the case then $N^1[x]$ would be disjoint from V_h . Hence, a shortest path from x to V_h , which exists as G is connected, has length at least 2. But then the

antepenultimate vertex on this path could be chosen as v_{h+1} , contradicting our assumption that the selection process stopped.

Since an edge pq in $N(v_i)$ gives a triangle v_i, p, q in G we obtain (P3). For (P4), assume that $k \geq 3$ and there is an edge pq in $\{u \in V(H) : d(u, v_i) = 2\}$. Let p' be a neighbour of p in $N(v_i)$, and q' be a neighbour of q in $N(v_i)$. Then p, p', v_i, q', q is a closed walk of length 5, a contradiction.

Next we show (P5). Let p, x, q be a path of length 2 from $N(v_i)$ to $N(v_j)$. Assume that $\omega(ij) \neq 4$. Then $\omega(ij)$ must be 3, so there is an edge $p'q'$ between $N(v_i)$ and $N(v_j)$. But then $p, x, q, v_j, q', p', v_i$ is a closed walk of length 7, a contradiction if $k \geq 4$.

It remains to prove (P6). Let p, x, y, q be a path of length 3 from $N(v_i)$ to $N(v_j)$, and assume that $\omega(ij) \notin \{3, 5\}$. Then $\omega(ij)$ must be 4, so there is a path p', z, q' of length 2 between $N(v_i)$ and $N(v_j)$. But then $p, x, y, q, v_j, q', z, p', v_i$ is a closed walk of length 9, a contradiction if $k \geq 5$. $\square$

It follows from the construction of H that there is a spanning tree T in H with $\omega(e) = 3$ for all $e \in E(T)$. As also Property (P1) holds, $H \in \mathcal{H}(k, 2k + t)$. As $k \geq k(t)$, by Lemma 2.3, there exists a set $B \subseteq [h]$ such that $H[B]$ is connected, $H[[h] \setminus B]$ is weighted bipartite, and $H[B \cup \{u\}]$ is weighted bipartite for all $u \in [h]$.

Our goal is to use this set B to construct a set $A \subseteq V$ such that G and A satisfy the assumptions of Lemma 2.2, so that we can conclude that G is 3-colourable. This is the case if A satisfies the following properties.

- (A1) $G[A]$ is connected,
- (A2) $G[V \setminus A]$ is bipartite,
- (A3) $G[A \cup \{v\}]$ is bipartite for all $v \in V$.

We construct A as follows. Denote by A_0 the union of the sets $N^1[v_b]$ over all $b \in B$, let $X_0 \subseteq X = V \setminus V_h$ be the set of vertices that have a neighbour in A_0 , and set $A = A_0 \cup X_0$. It remains to verify that A satisfies conditions (A1)–(A3).

Since $H[B]$ is connected, we can deduce (A1): Indeed, it is easy to verify that $G[A]$ is connected if $H[B]$ is connected and if additionally for any edge bb' in $H[B]$ we have a path from v_b to $v_{b'}$ in $G[A]$. The latter, however, is the case because by definition of H if there is an edge bb' in $H[B]$, we have $d(N(v_b), N(v_{b'})) \leq 3$ and this implies that there is a path of length at most 3 between $N(v_b)$ and $N(v_{b'})$ in G . This path is actually in $G[A]$ because A contains all vertices at distance at most 2 from v_b or $v_{b'}$.

For proving that (A2) also holds, we shall use the following claim.

Claim 3.2. *Each vertex in $X \setminus X_0$ has a neighbour in some $N(v_i)$ with $i \in [h] \setminus B$.*

Proof. Any vertex in X has a neighbour in some $N(v_i)$ with $i \in [h]$ by (P2) of Claim 3.1. In addition, $X_0 \subseteq X$ contains all the vertices that have a neighbour in some $N(v_i)$ with $i \in B$. The claim follows. $\square$

This allows us to show (A2).

Claim 3.3. $G[V \setminus A]$ is bipartite.

Proof. Assume that $G[V \setminus A]$ is not bipartite, and fix an odd cycle C of shortest length. Recall that the set $V \setminus A$ consists of vertices in $N^1[v_i]$ with $i \in [h] \setminus B$, and the vertices in $X \setminus X_0$.

If $C \cap \{v_1, \dots, v_h\} = \emptyset$ we call this the *degenerate case* and let $Q'_1 = C$. We now assume this is not the case and start with the following operations. Removing from C all vertices in $C \cap \{v_1, \dots, v_h\}$ gives a collection $Q'_1, \dots, Q'_{\ell'}$ of pairwise vertex-disjoint paths. Observe that by definition of A each removed vertex v_j has $j \in [h] \setminus B$. In each Q'_i we now further identify all vertices in $\bigcup_{j \in [h] \setminus B} N(v_j)$ and split Q'_i along these vertices into (sub)paths. More precisely, for a fixed i let $Q'_i = q'_1, \dots, q'_{s'}$ and let $j_1 \leq \dots \leq j_\ell$ be all indices j such that $q'_j \in \bigcup_{j \in [h] \setminus B} N(v_j)$. Then Q'_i is split into the paths $q'_1, \dots, q'_{j_1}$ and $q'_{j_1}, \dots, q'_{j_2}$ and so on, up to $q'_{j_\ell}, \dots, q'_{s'}$. By performing this splitting for all Q'_i we obtain, in total, a collection $Q_1, \dots, Q_\ell$ of pairwise internally vertex-disjoint paths which has the following properties for each $i = 1, \dots, \ell$ by definition of A and Claim 3.2. All internal vertices of Q_i are contained in $X \setminus X_0$, and there is $j \in [h] \setminus B$ such that the first vertex of Q_i and the last vertex of Q_{i-1} (which might be the same) are both contained in $N(v_j)$, where $Q_0 = Q_\ell$. Again, we allow the degenerate case that we only have $Q_1 = C$ with all vertices internal.

Next, for each fixed $i \in [\ell]$, we construct a walk R_i in H corresponding to the path $Q_i = q_1, \dots, q_s$ whose weight has the same parity as the length $s - 1$ of Q_i . To this end let $r_1, r_s \in [h] \setminus B$ be such that $q_1 \in N(v_{r_1})$ and $q_s \in N(v_{r_s})$. Our walk R_i has end-vertices r_1 and r_s , which potentially could be the same. Recall that $q_2, \dots, q_{s-1} \in X \setminus X_0$. We distinguish three cases.

- $s = 2$: In this case $r_1 \neq r_2$ by (P3) of Claim 3.1. In this case, for R_i we take the edge $e = r_1 r_2$, which has weight $\omega(e) = 3$ because $q_1 q_2$ is an edge between $N(v_{r_1})$ and $N(v_{r_2})$.
- $s = 3$: If $r_1 = r_3$, we can simply take the one vertex path $R_i = r_1$. Otherwise, namely if $r_1 \neq r_3$, we have by (P5) of Claim 3.1 that the edge $r_1 r_3$ has weight 4. We take this edge for R_i .
- $s > 3$: For $j = 3, \dots, s - 2$, we use Claim 3.2 to conclude there is $r_j \in [h] \setminus B$ such that q_j has a neighbour y_j in $N(v_{r_j})$. We set $r_2 = r_1$, $r_{s-1} = r_s$, and let $y_2 = q_1$, $y_{s-1} = q_s$. Note that with this q_j has a neighbour y_j in $N(v_{r_j})$ also for $j = 2$ and $j = s - 1$. Finally, we define R_i as $r_2, \dots, r_{s-1}$.

We now show that R_i is a walk from r_1 to r_s whose weight has the same parity as the length of Q_i also in this case. First, we observe that R_i starts at $r_2 = r_1$ and ends at $r_{s-1} = r_s$. Next we note that $r_j \neq r_{j+1}$ for $j = 2, \dots, s - 2$ by (P4) of Claim 3.1. Finally, (P6) of Claim 3.1 implies that $r_j r_{j+1}$ has weight 3 or 5, since by construction, there is a path of length 3 between $N(v_{r_j})$ and $N(v_{r_{j+1}})$ (namely $y_j, q_j, q_{j+1}, y_{j+1}$). Since the weight of each edge in R_i is odd, the weight of R_i has the same parity as $s - 3$ (the number of edges of R_i). Since Q_i has length $s - 1$, the weight of R_i and the length of Q_i have the same parity as desired.

Let us now consider again the degenerate case. In this case, we have $\ell = 1$ and that Q_1 is a cycle $q_1, q_2, \dots, q_s, q_{s+1} = q_1$ of odd length s . For $j = 1, \dots, s$, we let $r_j \in [h] \setminus B$ be such that q_j has a neighbour in $N(v_{r_j})$. As in the previous case, we conclude that $r_1, \dots, r_s, r_1$ is a walk with edges of weight 3 or 5, hence a closed odd walk.

This completes the construction of the walks R_i in H . As C was an odd cycle in G , the sum of the lengths of the Q_i is odd. Further, by construction, either we are in the degenerate case when we get one closed odd walk, or we are in the non-degenerate case and each walk R_i ends in the same vertex as R_{i+1} starts in (where indices are taken modulo ℓ). In either case, the union of the walks R_i thus is a closed walk of odd weight in $H[[h] \setminus B]$ which contains a cycle of odd weight. This is the desired contradiction and, therefore, $G[V \setminus A]$ is bipartite. $\square$

Our final claim shows that (A3) holds.

Claim 3.4. $G[A \cup \{v\}]$ is bipartite for every $v \in V \setminus A$.

Proof. Let us assume that, for some $v \in V \setminus A$, there is an odd cycle C in $G[A \cup \{v\}]$. There are three cases: either $v = v_w$ with $w \in [h] \setminus B$, or $v \in N(v_w)$ with $w \in [h] \setminus B$, or $v \in X \setminus X_0$. We start by ruling out the first. Indeed, if $v = v_w$ with $w \in [h] \setminus B$, then v cannot be contained in C because $N(v_w) \subseteq V \setminus A$, hence $C \subseteq A$. We conclude that in this case we can simply choose some new $v \in N(v_w)$ and continue the following argument with this v .

In the other two cases, we proceed as follows. If $v \in X \setminus X_0$, by Claim 3.2 we can fix a $w \in [h] \setminus B$ such that v has a neighbour in $N(v_w)$. Otherwise, we fix $w \in [h] \setminus B$ such that $v \in N(v_w)$. By assumption $H[B \cup \{w\}]$ is weighted bipartite.

Recall that A consists of $N^1[v_i]$ with $i \in B$ and the vertices in X_0 , and that every vertex in X_0 has a neighbour in some $N(v_i)$ with $i \in B$. We want to construct a cycle of odd weight in $H[B \cup \{w\}]$ to obtain a contradiction. We proceed almost exactly as in Claim 3.3 and we shall not repeat the details here, but only indicate the differences: First of all, the relevant indices are now chosen from $B \cup \{w\}$ instead of $[h] \setminus B$, and the internal vertices of the paths $Q_1, \dots, Q_t$ come from X_0 instead of $X \setminus X_0$. Moreover, if $v \in N(v_w)$ and v appears as an end-vertex of a path Q_i then we need to take w for the corresponding end-vertex of the path R_i . Similarly, in the case when $v \in X \setminus X_0$ and v appears as an internal vertex of a path Q_i , we take w as the corresponding vertex in the path R_i . The remaining arguments work as before, also in these two cases. $\square$

This completes the proof of Theorem 1.1. $\square$

For the general upper bound $\delta_X(\mathcal{C}_{2k-1}, c) \leq \frac{1}{2k \lfloor c/3 \rfloor}$ we let G be a $\mathcal{C}_{2k-1}$ -free graph of minimum degree at least $\frac{1}{2k \lfloor c/3 \rfloor} |V(G)|$ and obtain an auxiliary graph $H \in \mathcal{H}(k, 2k \lfloor c/3 \rfloor)$ in the same way. Then we can partition H into $\lfloor c/3 \rfloor$ parts of size at most $2k$ and apply Lemma 2.3 to each of them. Almost exactly as above we can then translate the partition of each part back to a 3-colouring of the corresponding part of G , while also taking care of the left-over vertices in X , to obtain a $3 \lfloor c/3 \rfloor$ -colouring of G .

4. Finding and combining weighted bipartite sets

In this section, we focus on finding sufficient conditions for a set to be weighted bipartite. We start with the following lemma, which states that certain balls around a vertex are weighted bipartite.

Lemma 4.1. *Let $k \geq 5$ be an integer and H be a weighted graph with edge weight $\omega: E(H) \rightarrow \{3, 4, 5\}$. If H contains no cycle of odd weight smaller than $2k + 1$, then for any $u \in V(H)$ we have that $N_{\omega}^{k-3}[u]$ is weighted bipartite.*

Proof of Lemma 4.1. For this proof, it is practical to return to the unweighted setting. Hence, let G be the (unweighted) graph obtained from H by replacing every edge of weight s by a path with s edges. By construction, all vertices of H are also vertices of G . Note further that any odd cycle C in G corresponds to a cycle in H whose weight is exactly the length of C and vice versa.

Let us now assume for contradiction that for some $u \in V(H)$ there exists a cycle C_H of odd weight in $N_{\omega}^{k-3}[u]$, and denote by C the corresponding odd cycle in G . We define for all non-negative integers j , the level sets $L_j = \{x \in V(G) : d_G(u, x) = j\} \subseteq V(G)$ to be the sets containing all vertices in G at distance exactly j from u , and the set $B = \bigcup_{j=0}^{k-1} L_j$. We claim that $V(C) \subseteq B$. Indeed, for $x \in V(C) \cap V(H) \subseteq V(G)$ we have $d_G(u, x) = d_{\omega, H}(u, x) \leq k-3$. For any $y \in V(C)$, there are $x, x' \in V(C) \cap V(H)$ such that y is on a path from x to x' of length at most 5. W.l.o.g. $d_G(x, y) \leq 2$ and, hence $d_G(u, y) \leq d_G(u, x) + 2 \leq k-1$.

Since C is an odd cycle, there must be an edge xy of C with x and y in the same level set L_j . Indeed, otherwise we could properly 2-colour the vertices of the odd cycle C by parity of the level of each vertex. We conclude that there are a u, x -path and a u, y -path each with exactly $j \leq k-1$ edges. The odd closed walk obtained from these two paths and the edge xy contains an odd cycle of length at most $2j + 1 \leq 2k-1$. But this corresponds to a cycle in H of weight odd and smaller than $2k + 1$, which contradicts our assumption. $\square$

Lemma 4.1 gives us a large family of sets that are weighted bipartite. This gives us access to many possible candidates for our set B . The additional advantage of Lemma 4.1 is that the sets it refers to are very simple, and this makes it easier to interpret our constructions later on. Our next lemma provides a similarly useful construction, allowing us to build weighted bipartite sets starting from a minimal weight path.

Lemma 4.2. *Let $i \geq 1$ be an integer and $k \geq 10i + 20$. Let H be a weighted graph with edge weight $\omega: E(H) \rightarrow \{3, 4, 5\}$ which contains no cycle of weight odd and smaller than $2k + 1$. If P is a path of minimal weight between its end-vertices, then $N^i[P]$ is weighted bipartite.*

Proof. Assume that there exists a path $P = p_1, \dots, p_{\ell}$, which is of minimal weight between its end-vertices and such that $N^i[P]$ is not weighted bipartite. Further, assume that P is minimal with this property, in particular for $P' = p_1, \dots, p_{\ell-1}$, we have that $N^i[P']$ is weighted bipartite. We label the vertices in $N^i[P] \setminus N^i[P']$ by $w_1, \dots, w_m$. Notice that for each $i = 1, \dots, m$ there is an edge between w_i and $N^i[P']$. Take h the minimal index such that $L_h = N^i[P'] \cup \{w_1, \dots, w_h\}$ is not weighted bipartite. This implies that in L_h there exists a cycle of odd weight. Let Q be a cycle of minimal odd weight in L_h . Note that Q has to pass through w_h , so we denote with x

and y the two neighbours of w_h in Q . Let x' and y' be the vertices in P' closest to x and y respectively.

Note that $d_{L_{h-1}}(x', x) \leq i + 1$ and $d_{L_h}(x, p_{\ell-1}) \leq d_{L_h}(w_h, p_{\ell-1}) + 1 \leq i + 2$. Where the second inequality comes from the fact that by definition $w_h \in N^i[P] \setminus N^i[P']$. As P is a path of minimal weight between its end-vertices, the sub-path between x' and $p_{\ell-1}$ is also of minimal weight. Therefore,

$$d_{\omega, P}(x', p_{\ell-1}) \leq d_{\omega, L_{h-1}}(x', x) + d_{\omega, L_h}(x, p_{\ell-1}) \leq 5(2i + 3)$$

and the analogous argument gives $d_{\omega, P}(y', p_{\ell-1}) \leq 5(2i + 3)$. This gives $d_{\omega, P}(x', y') \leq 5(2i + 3)$ because x', y' and $p_{\ell-1}$ are in the same path and $p_{\ell-1}$ is one of the two end-vertices. This also implies $d_{\omega, L_{h-1}}(x, y) \leq 10(2i + 3)$. We let $Q' \subseteq Q$ be the path in L_{h-1} with end-vertices x and y , i.e. $Q \setminus \{z\}$. The parity of $\omega(Q')$ and $d_{\omega, L_{h-1}}(x, y)$ has to be the same, as otherwise there would be a cycle of odd weight in L_{h-1} . But, as $\omega(xw_h) + \omega(yw_h)$ and $\omega(Q')$ have different parity, the parity of $d_{\omega, L_{h-1}}(x, y)$ is also different from the parity of $\omega(xw_h) + \omega(yw_h)$. Therefore, using that Q is the lightest cycle of odd weight, we get that $\omega(Q) \leq d_{\omega, L_{h-1}}(x, y) + 10 \leq 10(2i + 3) + 10$. This is less than $2k + 1$ by our choice of k and gives us the desired contradiction. $\square$

Now that we proved that the most basic sets (paths and balls) have our desired property, we are ready to start the construction of more complicated sets. In particular, the next Lemma shows how to combine two weighted bipartite sets. We need to point out that this combination is not always possible. It might be better to interpret the next result as a condition under which the property of being weighted bipartite is preserved under the union operation.

Lemma 4.3. *Let $i \geq 1$ be an integer and let H be a weighted graph. Let B_1, B_2 and P be three sets of vertices in H such that $d(B_1, B_2) \geq 2i + 2$ and $H[P]$ is connected. If both $N^i[B_1 \cup P]$ and $N^i[B_2 \cup P]$ are weighted bipartite, then $N^i[B_1 \cup B_2 \cup P]$ is weighted bipartite.*

Proof. Let $K = B_1 \cup B_2 \cup P$. We want to show that $N^i[K]$ is weighted bipartite and let us assume for contradiction that it contains a cycle C of odd weight. Let us denote by B'_1 the set $N^i[B_1] \setminus N^i[P]$ and by B'_2 the set $N^i[B_2] \setminus N^i[P]$. Since both $N^i[B_1 \cup P]$ and $N^i[B_2 \cup P]$ are weighted bipartite, C must intersect both B'_1 and B'_2 . Let $y(C)$ be the number of connected components of C induced by $C \cap (B'_1 \cup B'_2)$ in H . In other words, $y(C)$ is the number of times that C leaves B'_1 or B'_2 . It is possible that C leaves B'_1 , continues in $N^i[P]$, but then returns to B'_1 (or the same with B'_2), so $y(C)$ does not need to be even, but it has to be at least 2. Assume that C is such that $y(C)$ is minimal.

Let w be any vertex in $C \cap B'_1$. Let q_1 and q_2 be the end-vertices of the maximal path in $C \cap N^{i+1}[B_1]$ containing w . That is, q_1 and q_2 are obtained by moving from w in both possible directions along C and then taking the first vertices that are outside $N^i[B_1]$. Because $q_1, q_2 \in N^{i+1}[B_1] \setminus N^i[B_1]$, $d(B_1, B_2) \geq 2i + 2$, and $C \cap B'_2 \neq \emptyset$ it follows that $q_1 \neq q_2$ and $q_1, q_2 \in N^i[P]$.

As P is connected, there is a path in $N^i[P]$ between q_1 and q_2 . Since this path is different from the two paths between q_1 and q_2 in C (as it cannot overlap with B'_1 and B'_2), we obtain from $C \cup P$ at least two cycles in $N^i[K]$, and at least one of them, let us call it C' , has odd weight. We have that $y(C') < y(C)$ since we substituted a path in C containing at least one component of $C \cap B'_1$ (and thus contributing at least one to $y(C)$) with a path in $N^i[P]$. This is a contradiction to the choice of C , which was picked with minimal value of $y(C)$. $\square$

5. Cycles of odd weight have large neighbourhoods

We dedicate this section to proving Lemma 2.5, restated here.

Lemma 2.5. *Let $k \geq 8$ and $H \in \mathcal{H}(k)$. If C is a non-spanning cycle of odd weight, then $|N^1[C]| \geq \frac{2k+1}{3}$.*

Note that if all edges of the cycle are of weight 3, the cycle itself has at least $\frac{2k+1}{3}$ vertices, but if edges have other weights it might have fewer vertices. To overcome this, we use the spanning tree of edges of weight 3. Indeed, each maximal path of weight 3 edges in the cycle has a neighbour outside of the cycle. Carefully analysing this situation gives the desired bound.

Before working with cycles, we prove an analogous result for paths, which we use to prove the former. Before proceeding, we also remark that lemmas in this section are not stated in terms of $\mathcal{H}(k)$ as we want to apply them in more generality.

Lemma 5.1. *Let F be a weighted graph with edge weight $\omega: E(F) \rightarrow \{3, 4, 5\}$. Assume that $F = T \cup P$, where T is a spanning tree in which all edges have weight 3 and P is a non-spanning path of weight ℓ with end-vertices x and y . If F has no cycles of weight 11 and P has minimal weight among all x, y -paths in $N_F^1[P]$, then $|N_F^1[P]| \geq \frac{\ell+5}{3}$.*

For convenience we nevertheless state the following immediate corollary.

Corollary 5.2. *Let $k \geq 6$ and $H \in \mathcal{H}(k)$. If P is a non-spanning path of minimal weight between its end-vertices, then $|N^1[P]| \geq \frac{\omega(P)+5}{3}$.*

Proof of Lemma 5.1. We write $P = Q_1, \dots, Q_s$ as a concatenation of (possibly trivial) sub-paths Q_i such that within each Q_i all edges have weight 3 and the edge e_i between Q_i and Q_{i+1} has weight $\omega(e_i) > 3$.

If $s = 1$ then each edge of P has weight 3 and we are done because $|N^1[P]| > |P| = \frac{\ell}{3} + 1$, where the strict inequality comes from the fact that F is connected, and hence P has a neighbour in $V(F) \setminus V(P)$ (which is not empty because P is not spanning).

Assume now that s is at least 2. Since T is a spanning tree in F , for each Q_i we can fix a vertex $z_i \in N^1[Q_i] \setminus P$ and a vertex x_i in Q_i such that $z_i x_i$ has weight 3.

For $i < j$ we have $z_i \neq z_j$ unless $j = i + 1$ and $e_i = x_i x_{i+1}$, because otherwise P would not be an x, y -path of minimal weight in its neighbourhood. If $z_i = z_{i+1}$, we say that $(i, i + 1)$ has a *hat*. In this case we also know that $e_i = x_i x_{i+1}$ has weight 4, as otherwise x_i, x_{i+1}, z_i would form a cycle of weight 11. Moreover, neither $(i - 1, i)$ nor $(i + 1, i + 2)$ has a hat (otherwise we would have without loss of generality that $z_{i-1} = z_{i+1}$ and we could replace the sub-path x_{i-1}, x_i, x_{i+1} of weight 8 with $x_{i-1}, z_{i+1}, x_{i+1}$ of weight 6).

Now, if $(i, i + 1)$ has a hat, we “merge” Q_i and Q_{i+1} : We rewrite $P = Q'_1, \dots, Q'_{s'}$ such that each Q'_j either is the concatenation Q_i, Q_{i+1} for some i such that $(i, i + 1)$ has a hat, or is Q_i for some i such that neither $(i - 1, i)$ nor $(i, i + 1)$ has a hat. In the former case, we say that Q'_j was formed by a hat. In both cases, we set $z'_j = z_i$. Observe that by construction $z'_j \neq z'_{j'}$ for $j \neq j'$. We thus conclude that we have

$$|N^1[P]| \geq \sum_{j \in [s']} (|Q'_j| + 1) = s' + |P|.$$

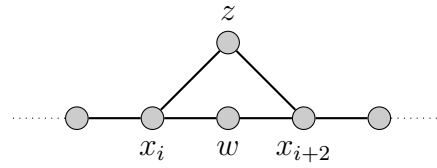


Figure 5.1: Shortcut in the case $d(C) = 2$, where $x_i z$ and $x_{i+2} z$ are edges of weight 3 and the sum of the weights of $x_i w$ and $x_{i+2} w$ is at least 7.

Moreover, since Q'_j and Q'_{j+1} are connected by an edge of weight at most 5, we have

$$\omega(P) \leq 5(s' - 1) + \sum_{j \in [s']} \omega(Q'_j).$$

If Q'_j was formed from a hat, then $\omega(Q'_j) = 3(|Q'_j| - 2) + 4 = 3|Q'_j| - 2$ and otherwise $\omega(Q'_j) = 3(|Q'_j| - 1) \leq 3|Q'_j| - 2$. Therefore,

$$\omega(P) \leq 5(s' - 1) + \sum_{j \in [s']} (3|Q'_j| - 2) = 3s' - 5 + 3|P|,$$

and hence $|N_F^1[P]| \geq |P| + s' \geq \frac{\omega(P)}{3} + \frac{5}{3}$ as desired. $\square$

We are now ready to present our proof of Lemma 2.5, which provides a similar lower bound on the size of the neighbourhood of a non-spanning cycle of odd weight in a graph $H \in \mathcal{H}(k)$.

Proof of Lemma 2.5. Let T be a spanning tree of edges of weight 3 associated to H . If C only has edges of weight 3, we are done because C has weight at least $2k + 1$. If not, we write C as a concatenation of maximal paths of edges of weight 3, this means that we write $C = Q_1, e_1, Q_2, e_2, \dots, Q_s, e_s$ where each Q_i is a (possibly trivial) sub-path of C only composed of edges of weight 3, and where $\omega(e_i) > 3$. We also call these sub-paths Q_i the *segments* of the cycle.

For each Q_i , we now want to choose vertices $x_i \in Q_i$ and $z_i \in N^1[Q_i] \setminus Q_i$ such that $x_i z_i$ is an edge of T and we call z_i the *pendant* of Q_i . This is possible, because T is a spanning tree of weight three. Given the notation $Z = \{z_i : i \in [s]\}$ and $Z' = Z \setminus C$, we select the z_i in such a way that $|Z'|$ is maximised. Further, among the options that maximise $|Z'|$, we select one that attains the maximum value for $|Z|$. Let $I(C) = V(C) \cup Z$. We also denote by $H(C)$ the graph with vertex set $I(C)$ and edge set $E(C) \cup \{x_i z_i : i \in [s]\}$.

We now could be tempted to immediately apply Lemma 5.1 to some spanning path P in C . However, this is not possible since P may not have minimal weight in $N_{H(C)}^1[P]$. Therefore, our goal is to “move” to a (possibly) different cycle C' in which we do not encounter this issue. For this procedure to work it is essential that we maximise the sizes of Z' and Z .

Claim 5.3. *There is a non-spanning cycle C' of odd weight in H such that C' has minimal weight among all cycles of odd weight in $H(C')$ (defined analogously as above) and such that $|I(C)| \geq |I(C')|$.*

Proof. We shall move through a sequence $C_1, C_2, \dots, C_{\tilde{\ell}}$ of odd weight cycles until we obtain a cycle $C_{\tilde{\ell}} = C'$ with the desired properties, where from one cycle C_ℓ to the next $C_{\ell+1}$ we do not increase the weight, we decrease the number s of segments, and we have $|I(C_\ell)| \geq |I(C_{\ell+1})|$. We terminate this process if only a unique odd weight cycle C_ℓ is left in $H(C_\ell)$. Observe that this happens eventually since we always decrease the number s of segments and if $s = 1$ for some cycle C_ℓ , then C_ℓ is the only cycle in $H(C_\ell)$. We set $C_1 = C$ and assume that we currently have a cycle C_ℓ with s segments and if it does not have minimal weight among all cycles of odd weight in $H(C_\ell)$, we move to a new cycle $C_{\ell+1}$ with the properties just specified. We will also keep track of the set Z_ℓ (and Z'_ℓ) of pendant vertices z_j (outside of C_ℓ), which is set to $Z_1 = Z$ (and $Z'_1 = Z'$) in the beginning. We distinguish two cases.

Case A. Assume C_ℓ has a chord in $H(C_\ell)$, that is, for some i we have $z_i \in C_\ell$. Let us denote by i' the index such that $z_i \in Q_{i'}$. Then this chord creates with C_ℓ two cycles and we let $C_{\ell+1}$ be the cycle of odd weight among the two. We divide $C_{\ell+1}$ into segments as before. All the segments of $C_{\ell+1}$ were also segments of C_ℓ with the exception of the segment $Q'_{i'}$ of $C_{\ell+1}$ containing $x_i z_i$. For the segments Q_j of $C_{\ell+1}$ (including $Q'_{i'}$), we find the vertices $x_j \in Q_j$ and $z_j \in N^1[Q_j] \setminus Q_j$ as above and again maximise the sizes of the sets

$$Z'_{\ell+1} = Z_{\ell+1} \setminus C_{\ell+1} \quad \text{and then} \quad Z_{\ell+1} = \{z_j : Q_j \text{ segment of } C_{\ell+1}\}.$$

Note that $C_{\ell+1}$ has fewer segments since the chord $z_i x_i$ that gave us $C_{\ell+1}$ joined two distinct segments as $z_i \notin Q_i$ by definition. Moreover $C_{\ell+1}$ has weight strictly smaller than the weight of C_ℓ because x_i and z_i are at unweighted distance at least 2 in C_ℓ while they are joined by an edge of weight 3 in $C_{\ell+1}$.

It remains to argue that $|I(C_{\ell+1})| \leq |I(C_\ell)|$. Note that all the vertices of $Z'_{\ell+1}$ were also available as vertices in Z'_ℓ , unless they are in $V(C_\ell) \setminus V(C_{\ell+1})$. Thus we have $|Z'_\ell| \geq |Z'_{\ell+1} \setminus V(C_\ell)|$. This implies

$$|I(C_\ell)| = |C_\ell| + |Z'_\ell| \geq |C_\ell| + |Z'_{\ell+1} \setminus V(C_\ell)| \geq |C_{\ell+1}| + |Z'_{\ell+1}| = |I(C_{\ell+1})|,$$

where for the second inequality we used $Z'_{\ell+1} \cap V(C_\ell) \subseteq V(C_\ell) \setminus V(C_{\ell+1})$.

Case B. Assume C_ℓ has no chords in $H(C_\ell)$, that is, $z_i \notin C_\ell$ for all i and, hence, $Z_\ell = Z'_\ell$. If there are no distinct x_i and x_j such that $z_i = z_j$, then $H(C_\ell)$ contains only one odd weight cycle and we are done. Otherwise, let $d(C_\ell)$ be the maximum unweighted distance on C_ℓ between x_i and x_j such that $z_i = z_j$ over all choices $i, j \in [s]$. If $d(C_\ell) \leq 1$ then $H(C_\ell)$ consists of a cycle where neighbouring vertices can have a common neighbour outside the cycle plus some pendent edges. Since the path $x_i z_i x_j$ always has larger weight than the edge $x_i x_j$, this implies that C_ℓ is of minimal weight in $H(C_\ell)$, and we are done.

We can therefore assume that $d(C_\ell) \geq 2$. Let x_i and x_j be vertices at distance at least 2 in C_ℓ such that $z_i = z_j$, and let P be the path between x_i and x_j in C_ℓ such that the cycle $C_{\ell+1} := P x_j z_i x_i$ has odd weight. This cycle has weight not larger than C_ℓ , as $C_\ell \setminus P$ has at least two edges and thus weight at least 6, and $x_j z_i x_i$ has weight exactly 6. Consider a choice of pendants that first maximises the size of $Z'_{\ell+1}$ and then the size of $Z_{\ell+1}$. Let z' be the pendant of the unique segment $\tilde{Q}$ containing $x_j z_j x_i$. Denote by $Z_{\ell+1,1}$ the set of pendants z

of other segments in $C_{\ell+1}$ such that $z \notin V(C_\ell)$, and by $Z_{\ell+1,2}$ the set of pendants z of other segments in $C_{\ell+1}$ such that $z \in V(C_\ell)$. Note that $Z_{\ell+1,1}$ and $Z_{\ell+1,2}$ are disjoint and z' could be in either of them or in none.

We may assume $Z_{\ell+1,1} \subseteq Z'_\ell$. Indeed, let $f : Z_{\ell+1,1} \rightarrow \mathbb{N}$ be such that z is a pendant of the segment $Q_{f(z)} \neq \bar{Q}$ of $C_{\ell+1}$ for $z \in Z_{\ell+1,1}$. Note that then $Q_{f(z)}$ is also a segment of C_ℓ . If $Z_{\ell+1,1} \not\subseteq Z'_\ell$, consider some $z \in Z_{\ell+1,1} \setminus Z'_\ell$, and replace the pendant of the segment $Q_{f(z)}$ in C_ℓ , which is in $Z_\ell = Z'_\ell$, by z . This either causes $|Z_{\ell+1,1} \setminus Z'_\ell|$ to decrease, or it causes $|\{z' \in Z_{\ell+1,1} \cap Z'_\ell : z' \text{ is not the pendant of } Q_{f(z')} \text{ in } C_\ell\}|$ to decrease while maintaining $|Z_{\ell+1,1} \setminus Z'_\ell|$. Indeed, let z'' be the pendant of $Q_{f(z)}$ in C_ℓ . Then by maximality of Z'_ℓ , if we replace z'' by z as a pendant in C_ℓ , then z'' gets removed from Z'_ℓ . If z'' is not in $Z_{\ell+1,1}$, then $|Z_{\ell+1,1} \setminus Z'_\ell|$ decreases. Otherwise, $|\{z' \in Z_{\ell+1,1} \cap Z'_\ell : z' \text{ is not the pendant of } Q_{f(z')} \text{ in } C_\ell\}|$ decreases, because before the change z'' was a pendant of only the segment $Q_{f(z)}$, so it was in this set, but after the change z is not in this set. Thus, repeating this step shows that we can assume $Z_{\ell+1,1} \subseteq Z'_\ell$.

If $z' \in Z'_\ell$, then $I(C_{\ell+1}) \setminus I(C_\ell)$ is empty, and hence $|I(C_\ell)| = |I(C_{\ell+1})|$. Thus assume $z' \notin Z'_\ell$ and, therefore, $|I(C_{\ell+1}) \setminus I(C_\ell)| \leq 1$. Now if $Z'_\ell \setminus (Z_{\ell+1,1} \cup \{z_i\})$ is non-empty, then $I(C_\ell) \setminus I(C_{\ell+1})$ contains a vertex from $Z'_\ell \setminus (Z_{\ell+1,1} \cup \{z_i\})$. Similarly, if $Z_{\ell+1,2}$ is empty, then $I(C_\ell) \setminus I(C_{\ell+1})$ contains a vertex from $V(C_\ell) \setminus V(C_{\ell+1})$. In either case, we have $|I(C_\ell) \setminus I(C_{\ell+1})| \geq 1$ and $|I(C_{\ell+1}) \setminus I(C_\ell)| \leq 1$ implying again $|I(C_\ell)| \geq |I(C_{\ell+1})|$. The only remaining case now is that $Z'_\ell = Z_{\ell+1,1} \cup \{z_i\}$ and $Z_{\ell+1,2}$ is not empty. In this case, let $z \in Z_{\ell+1,2}$, and let $Q_s \neq \bar{Q}$ be a segment in $C_{\ell+1}$ whose pendant is z . Note again that then Q_s is also a segment in C_ℓ . Then, we can set the pendant of Q_s (as a segment in C_ℓ) to be z without decreasing the size of Z'_ℓ (because for every $z'' \in Z'_\ell \setminus \{z_i\}$ we have a segment which is not Q_s with z'' as a pendant, and Q_i and Q_j have z_i as a pendant). Recalling that we are in the case where initially $Z_\ell = Z'_\ell$ and that $z \notin V(C_\ell)$, this increases the size of Z_ℓ , a contradiction to the maximality of Z_ℓ . We also note that the size of Z'_ℓ does not increase by this change. $\square$

Let C' be an odd cycle such as the one promised by this claim, let s' be the number of its segments, and let $z'_1, \dots, z'_{s'}$ be the neighbours of the segments. Our goal now is to argue that $|I(C')| \geq \frac{2k+1}{3}$, which proves the lemma since $|I(C')| \leq |I(C)|$. If $s' = 1$, then all but at most one edge of C' have weight 3 and thus $|I(C')| \geq |C'| + 1 \geq \frac{\omega(C')-2}{3} + 1 \geq \frac{2k+1}{3}$. For the first inequality, we used that there is a vertex in $I(C') \setminus C'$, which is true because T is connected and C' is not spanning in H . Hence, assume from now on that $s' \geq 2$.

We can fix one edge e in C' that is not in the spanning tree T . Removing e from C' , we obtain a path P of weight at least $(2k+1) - 5$. We let T' be the graph consisting of all edges of weight 3 in $H(C')$ except e and one additional auxiliary vertex v connected to each of $z'_1, \dots, z'_{s'}$ with an edge of weight 3. Observe that T' is a tree. Now, consider the graph $F = (T' \cup H(C')) \setminus \{e\}$. We have that $V(F) \setminus V(P) \neq \emptyset$ (since $v \in V(F) \setminus V(P)$), and P has minimal weight among all paths in $N_F^1[P]$ connecting its end-vertices since v is not contained in $N_F^1[P]$ and by the minimality of C' . Moreover, F has no cycles of weight 11 since any such cycle would need to include the auxiliary vertex which is only connected to the vertices $z'_1, \dots, z'_{s'}$ which form an independent set in F , and thus any cycle using the auxiliary vertex has weight at least 12. We conclude that we can apply Lemma 5.1 to P and T' to get $|I(C')| \geq N_F^1[P] \geq \frac{2k-4}{3} + \frac{5}{3} \geq \frac{2k+1}{3}$.

as required. $\square$

We end this section with a useful corollary of Lemma 2.5.

Corollary 5.4. *Let $\ell \geq 13$ be an odd integer. Let F be a weighted graph with edge weight $\omega: E(F) \rightarrow \{3, 4, 5\}$. Assume that $F = T \cup P$, where T is a spanning tree in which all edges have weight 3 and P is a non-spanning path with end-vertices x and y . If F has no cycles of odd weight below $\ell + 4$, and the minimal weight of an x, y -path in F is at least ℓ , then $|N_F^1[P]| \geq \frac{\ell+4}{3}$.*

Proof. We add the edge xy to F and define its weight to be $s \in \{4, 5\}$ such that $\omega(P) + s$ is odd. Let C in F be the cycle of odd weight consisting of P and the edge xy . Because any x, y -path is of weight at least ℓ , in F there is no cycle whose weight is odd and smaller than $\ell + 4$. Therefore, $F \in \mathcal{H}(\frac{1}{2}(\ell + 4 - 1))$. We can then apply Lemma 2.5 with $k = \frac{1}{2}(\ell + 4 - 1) \geq 8$ an integer (since ℓ is odd), to get $|N_F^1[P]| = |N_F^1[C]| \geq \frac{\ell+4}{3}$. $\square$

6. Proof of the main technical lemma

The main objective of this section is to prove our main technical lemma (Lemma 2.3). We need some further preparations. In the previous sections we first showed how to generate a candidate set B with weight bipartite neighbourhood, and then how to guarantee that $H \setminus B$ is weighted bipartite by analysing the size of $\text{int}(B)$. However, we did not yet combine results of these two types.

Observe that just taking a ball with Lemma 4.1 might only give a small set, while even cleverly removing a few vertices from a cycle of odd weight does not necessarily make it weighted bipartite. Therefore the first result of this section (and the last piece missing in order to prove Lemma 2.3), is a lemma combining these two ideas. Indeed, we show how we can create a candidate set which is weighted bipartite and with a lower bound on its size.

Lemma 6.1. *Let $i \geq 2$ and $k \geq 5i + 16$ be integers, and $H \in \mathcal{H}(k)$. For any odd cycle C in H and $p \in V(C)$ the following holds. There exists a path P such that $E(P) \subseteq E(C)$, $p \in V(P)$, $N^i_H[P]$ is weighted bipartite, and $|N^1_H[P]| \geq \frac{2}{3}k - \frac{10}{3}i - 5$.*

Proof. Let $x_0 = p$, fix an orientation of C , and denote by x_1° the clockwise neighbour of p , by x_2° the clockwise neighbour of x_1° and so on. Define analogously $x_1^\circ, x_2^\circ, \dots$. Now we can define the sequence $u_0 = x_0, u_1 = x_1^\circ, u_2 = x_1^\circ, u_3 = x_2^\circ, \dots$ that takes alternatively vertices from the clockwise and the anticlockwise side of p in C .

We take P to be the longest path of the form $C[\{u_0, \dots, u_j\}]$ such that $N^i_H[P]$ is weighted bipartite. Let us denote with x, y the two end-vertices of P and with u_x, u_y the two vertices in $C \setminus P$ adjacent in C to x and y respectively. Without loss of generality we assume $N^i_H[P \cup \{u_x\}]$ is not weighted bipartite.

In order to get the desired lower bound on $|N^1_H[P]|$, we want to use Corollary 5.4 with input $\ell = 2k - 10i - 19 \geq 13$. We do so in the following setting. First, we select a further subpath $P' = P_{x'y'}$ of P with endpoints x' and y' such that the weighted distance in $N^i_H[P]$ between x' and y' is at least ℓ . We then append to P' an auxiliary tree T' with the property that

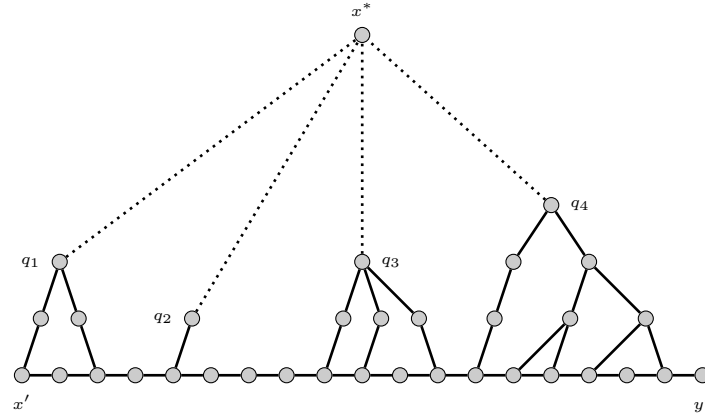


Figure 6.1: Construction of host graph F containing only P' and edges of a tree. To be used in Proof of Lemma 6.1.

there are no short paths with endpoints in P' that use edges of T' and such that $N_T^1[P] \subseteq N_{T'}^1[P]$, where T is the spanning tree of weight three associated with H . The host graph F we use to apply Corollary 5.4 will then be formed by edges in either P' or T' .

We start by selecting a subpath of P , the endpoints of which are given by the following claim.

Claim 6.2. *There are vertices x', y' in P which are at weighted distance at least ℓ in $N_H^i[P]$ and hence also in $N_H^i[P] \cap (P \cup T)$.*

Proof. By construction, we do not have any weighted odd cycles in $N_H^i[P]$ but we do have an odd cycle in $N_H^i[P \cup \{u_x\}]$ by choice of u_x . Let us fix an arbitrary order $z_1, \dots, z_m$ of the vertices of $N_H^i[\{u_x\}] \setminus N_H^i[P]$, let h be the minimum index such that $L_h = N_H^i[P] \cup \{z_1, \dots, z_h\}$ contains an odd cycle, and let Q be an odd cycle of minimal weight in L_h . By construction, we have that Q passes through z_h , and that there are no cycles of odd weight in L_{h-1} .

Let x'', y'' be the two neighbours of z_h in Q and let $x', y' \in P$ be vertices in P closest in L_{h-1} respectively to x'' and y'' . We claim that $d_{\omega, L_{h-1}}(x'', y'') \geq 2k - 9$ (weighted distance in L_{h-1} between x'' and y''). Indeed, assume that this is not true and let P' be the shortest path in L_{h-1} between x'' and y'' , and let P_Q be the path in $Q \setminus \{z_h\}$ with end-vertices x'' and y'' . We note that $P_Q \subseteq Q \setminus \{z_h\} = Q \cap L_{h-1}$, therefore the parity of $\omega(P')$ and $\omega(P_Q)$ can not be different, because then there would be a cycle of odd weight in L_{h-1} . But the parity of $\omega(x''z_h) + \omega(z_hy'') \leq 10$ is different from $\omega(P_Q)$ and, therefore, from the parity of $\omega(P')$. This means that P' (which is a path between x'' and y'') together with z_h gives a cycle of odd weight that is at most $2k$. This is a contradiction and proves the claimed bound. Since $d_{\omega, L_{h-1}}(x'', x'), d_{\omega, L_{h-1}}(y'', y') \leq 5 \cdot (i+1)$, we have that x' and y' are at weighted distance at least $2k - 9 - 2 \cdot 5 \cdot (i+1) = \ell$ in L_{h-1} (and therefore in $N_H^i[P]$ as $N_H^i[P] \subseteq L_{h-1}$), as desired. $\square$

Consider now the subpath $P' = P_{x'y'}$ of P with endpoints x' and y' given by our claim. We want to apply Corollary 5.4 with input $\ell = 2k - 10i - 19 \geq 13$ to a host graph F containing only P' and edges of a tree that we are going to build now (refer to Figure 6.1).

To build F , let $Q_1, \dots, Q_m$ be the connected components of the sub-forest $N_H^i[P'] \cap (T \setminus E(P'))$ of T on the vertices of $N_H^i[P']$ containing those edges of T which are not edges of P' . For each component Q_i of $N_H^i[P'] \cap (T \setminus E(P'))$ fix an arbitrary vertex $q_i \in Q_i \setminus P'$. Consider now a spider graph with m legs, each of length ℓ , and denote by x^* its central vertex and with $x_1, \dots, x_m$ its leaves (a graph obtained from the star $K_{1,m}$ by replacing each edge with a path of length ℓ). We denote by T' the tree obtained from this spider graph and the trees $Q_1, \dots, Q_m$ by identifying x_i with q_i and then let F be the (not disjoint) union of T' and P' .

Notice that in F we do not have any short cycles, moreover we have $N_F^1[P'] \subseteq N_H^1[P']$. Also, all paths in F with both endpoints in P' and not completely in $N_H^i[P'] \cap (P' \cup T)$ pass through x^* , and all paths with endpoints in P' passing through x^* have weight at least 2ℓ . In particular this gives us that the weighted distance in F between x' and y' is at least ℓ , which allows us to use Corollary 5.4 to obtain the desired lower bound on the size of $N_F^1[P']$ and hence also on the size of $N_H^1[P]$. $\square$

Now we are ready for proceeding to the main proof of this section. The strategy bringing all this together is to use results of Section 4 to combine together constructions such as the one in Corollary 5.2 and Lemma 6.1 whenever easier constructions, like balls, do not work. We restate and prove the main lemma.

Lemma 2.3 (Main technical lemma). *For any $t \in \mathbb{N}$ and any integer $k \geq k(t) = 20t + 1460$ the following holds. For any $H \in \mathcal{H}(k, 2k + t)$ there exists a subset B of $V(H)$ such that $H[B]$ is connected, $H \setminus B$ is weighted bipartite, and $H[B \cup \{v\}]$ is weighted bipartite for all $v \in V(H) \setminus B$.*

Proof of Lemma 2.3. Let us fix the functions $\ell_0 = \ell_0(t) = 5t + 310$ and $k(t) = 4\ell_0 + 220 = 20t + 1460$. Let t be a natural number and $k \geq k(t)$. In particular, note that since $k \geq 1460$, we have that k is large enough to apply Corollary 5.2, Lemma 6.1 with $i = 10$, and Lemma 4.3 with $i = 3$. Let $H \in \mathcal{H}(k, 2k + t)$ and distinguish two cases.

Case A. Assume that in H there are two cycles C_1 and C_2 of odd weight at weighted distance at least ℓ_0 from each other. Let P be a path of minimal weight between C_1 and C_2 (by assumption we have $\omega(P) \geq \ell_0$). Let p_1 and p_2 be the end-vertices of P in C_1 and C_2 , respectively. For $j = 1, 2$, let B_j be the path in C_j given by Lemma 6.1 with $i = 10$, and note that $d(B_1, B_2) \geq 8$. See Figure 6.2 for an illustration of the situation. We denote by B the set of vertices $N^2[B_1 \cup B_2 \cup P]$ and note that $H[B]$ is connected.

Claim 6.3. $N^1[B]$ is weighted bipartite

Proof. Let us first establish that $N^3[B_j \cup P]$ is weighted bipartite for $j = 1, 2$. Let P_j be the path on the first 8 vertices of P starting from B_j . Note that P_j is disjoint from B_{3-j} since $d(B_1, B_2) \geq 8$. Let P'_j be the subpath of P , starting right after the last vertex of P_j and ending in B_{3-j} , i.e. $P'_j = P \setminus V(P_j)$. Then P'_j is non-empty and since P_j contains 8 vertices we have $d(B_j, P'_j) \geq 8$.

Next we note that $N^3[B_j \cup P_j] \subseteq N^{10}[B_j]$ is weighted bipartite because we defined B_j according to Lemma 6.1 with $i = 10$. Also $N^3[P_j \cup P'_j] = N^3[P]$ is weighted bipartite by Lemma 4.2 as P is a shortest path between its end-vertices and $k \geq 50$. We can then apply Lemma 4.3 with $B_j \cup P_j$, $P_j \cup P'_j$, and P_j to deduce that $N^3[B_j \cup P_j \cup P'_j] = N^3[B_j \cup P]$ is

weighted bipartite. Another application of Lemma 4.3 with B_1 , B_2 , and P immediately gives that $N^1[B] = N^3[B_1 \cup B_2 \cup P]$ is weighted bipartite. $\square$

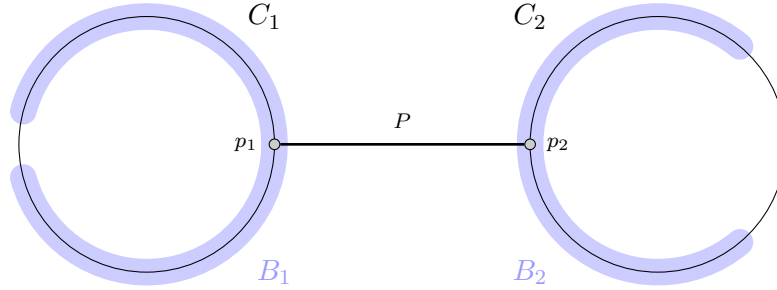


Figure 6.2: Two cycles C_1 and C_2 at weighted distance at least ℓ_0 and the construction of the weighted bipartite set.

Since $N^1[B]$ is weighted bipartite, it remains to show that $H \setminus B$ is weighted bipartite. In view of Lemma 2.4, it thus suffices to show that $|\text{int}(B)| \geq |N^1[B_1 \cup B_2 \cup P]| \geq \frac{4}{3}k + t$. As B_j is given by Lemma 6.1 with $i = 10$, we have $|N^1[B_j]| \geq \frac{2k}{3} - 29$. Therefore, as $N^1[B_1] \cap N^1[B_2] = \emptyset$ and $|P \cap N^1[B_j]| \leq 2$ for $j = 1, 2$, we get

$$|N^1[B_1 \cup B_2 \cup P]| \geq |N^1[B_1]| + |N^1[B_2]| + |P| - 4 > \frac{4}{3}k - 58 + \frac{1}{5}\ell_0 - 4 \geq \frac{4}{3}k + t,$$

where the last inequality uses the lower bound $\ell_0 = \ell_0(t) \geq 5 \cdot (62 + t) = 310 + 5t$.

Case B. In this case we have that between any two odd cycles there is a path of weight at most ℓ_0 . Let C be a cycle in H with minimal odd weight. Up to increasing the value of k in the statement, we may assume C has weight $2k + 1$. Fix an arbitrary vertex $x \in C$, and let $y \in C$ be a vertex of maximal weighted distance from x . Let P denote the path of minimal weight in C between x and y . Note that P is also a path of minimal weight between x and y in H , as a path of smaller weight not in C between x and y would create an odd cycle of weight smaller than C . Hence, $d_{\omega, H}(x, y) \geq (\omega(C) - 5)/2 = k - 2$.

Let $A_x = N^1[C] \cap N_{\omega}^{k/2-18}[x]$ be the set of vertices in $N^1[C]$ with weighted distance at most $\frac{k}{2} - 18$ from x and, similarly, let $A_y = N^1[C] \cap N_{\omega}^{k/2-18}[y]$. Denote by P' the path obtained by removing from P the vertices at unweighted distance at most 30 from its endpoints, x and y , which is non-empty as $\omega(P) - 2 \cdot 30 \cdot 5 \geq k - 302 > 0$.

Let C_x be a cycle of minimal odd weight in $H \setminus N_{\omega}^{k-8}[y]$. We might assume that such a C_x exists because otherwise we can take as set B the set $N_{\omega}^{k-8}[y]$ which respects our conditions by Lemma 4.1. Analogously, let C_y be a cycle of minimal odd weight in $H \setminus N_{\omega}^{k-8}[x]$. We can see C_y exists by the same argument used above. Let P_x and P_y be paths of minimal weight between C and C_x and C and C_y , respectively. Denote by p_x and p_y the weight of P_x and P_y , respectively, and note that $0 \leq p_x, p_y \leq \ell_0$ as we are in Case B.

We claim that all vertices of P_x are at weighted distance at most $2p_x + 11 \leq 2\ell_0 + 11$ from x and similarly the vertices of P_y are close to y . To see this let x' be the end-point of P_x on C and

note that $d_\omega(v, x') \leq p_x$ for $v \in V(P_x)$. As $C_x \cap N_\omega^{k-8}[y] = \emptyset$ implies $d_\omega(x', y) \geq k - 8 - p_x$ and $d_\omega(x, y) \geq k - 2$ implies that both paths in C with end-vertices x and y have weight at most $k + 3$, we get $d_\omega(x, x') \leq k + 3 - (k - 8 - p_x) \leq 11 + p_x$.

Now set $B_x = C_x \cap N_\omega^{k-38}[x]$, $B_y = C_y \cap N_\omega^{k-38}[y]$, $D_x = B_x \cup P_x \cup A_x$ and $D_y = B_y \cup P_y \cup A_y$ and view Figure 6.3 for an illustration. Our set B will essentially consists of D_x and D_y , which are both weighted bipartite by Lemma 4.1, connected by P' .

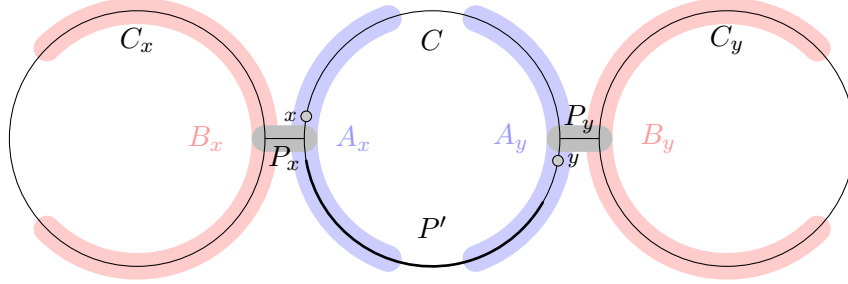


Figure 6.3: Three cycles C , C_x , and C_y together with the vertices x and y on C , the paths P' , P_x , and P_y , and the sets $A_x = N^1[C] \cap N_\omega^{k/2-18}[x]$, $A_y = N^1[C] \cap N_\omega^{k/2-18}[y]$, $B_x = C_x \cap N_\omega^{k-38}[x]$, and $B_y = C_y \cap N_\omega^{k-38}[y]$.

Claim 6.4. *The set $N^2[D_x \cup P' \cup D_y]$ is weighted bipartite.*

Proof. We show this by employing Lemma 4.3 with $i = 2$, with D_x as B_1 , with D_y as B_2 and with $V(P')$ as P .

Clearly P' is connected. We also note that $N^2[D_x \cup P']$ is contained in $N_\omega^{k-3}[x]$ and $N^2[D_y \cup P']$ is contained in $N_\omega^{k-3}[y]$, both of which are weighted bipartite.

It suffices now to show that the unweighted distance between D_x and D_y is at least 6, which we do by checking that D_x and D_y have weighted distance at least 30. Let us first note that the weighted distance $d_{\omega, H}(B_x, B_y)$ between B_x and B_y is at least 30, because $B_x \subseteq N_\omega^{k-38}[x]$ and $B_y \subseteq H \setminus N_\omega^{k-8}[x]$. Similarly, $d_{\omega, H}(A_x \cup P_x, B_y), d_{\omega, H}(A_y \cup P_y, B_x) \geq 30$. The weighted distance between P_x and A_y is at least $(k - 2) - (\frac{k}{2} - 18) - (2\ell_0 + 11) \geq 30$, because $d_\omega(x, P_x) \leq 2\ell_0 + 11$, $d_\omega(y, A_y) \leq k/2 - 18$, and $d_\omega(x, y) \geq k - 2$. Analogously, $d_\omega(P_y, A_x) \geq 30$. Finally, $d_\omega(A_x, A_y) \geq (k - 2) - 2 \cdot (\frac{k}{2} - 18) \geq 30$ by the definition of A_x and A_y and $d_\omega(x, y) \geq k - 2$.

Therefore, all the conditions for Lemma 4.3 are satisfied. $\square$

We now claim that $B = N^1[D_x \cup P' \cup D_y]$ satisfies the properties of Lemma 2.3. Note that $H[B]$ is connected, because $H[D_x]$ and $H[D_y]$ are connected and $V(P') \cap A_x \neq \emptyset$ and $V(P') \cap A_y \neq \emptyset$ as $\frac{k}{2} - 18 \geq 30$. By Claim 6.4, $N^1[B]$ is weighted bipartite. It remains to show that $H \setminus B$ is weighted bipartite. In order to do this, we prove that $|\text{int} B| \geq \frac{4}{3}k$, which suffices by Lemma 2.4.

Note that $\text{int} B$ contains the union of the following four pairwise disjoint sets: $A_x \cap N_\omega^{k-8}[y]$, $A_y \cap N_\omega^{k-8}[x]$, B_x , and B_y . To see that they are pairwise disjoint, note that $A_x \cap A_y = \emptyset$

as $d_\omega(x, y) \geq k - 2$, that $B_x \cap N_\omega^{k-8}[y] = \emptyset$ and $B_y \cap N_\omega^{k-8}[x] = \emptyset$ by definition, and that $B_x \subseteq N_\omega^{k-38}[x]$ and $B_y \subseteq N_\omega^{k-38}[y]$ by definition while $A_x \subseteq N_\omega^{k/2-18}[x]$ and $A_y \subseteq N_\omega^{k/2-18}[y]$.

The size of B_x is at least $\frac{2}{5}(k - 2p_x - 53) \geq \frac{2}{5}k - \frac{4}{5}\ell_0 - 22$. Indeed, as $d_\omega(x, C_x) \leq 2p_x + 11$ there are two paths of weight at least $k - 38 - 4 - 2p_x - 11$ in B_x that are disjoint. The same bound holds for the size for B_y .

For the set $A_x \cap N_\omega^{k-8}[y]$ we want to apply Corollary 5.2 to two different paths. We recall that Corollary 5.2 states that if a path of weighted length p is of minimal weight between its end-vertices, then the size of its 1-neighbourhood is at least $\frac{p+5}{3}$. Let x_1 and x_2 be the vertices closest to x in $C \cap N_\omega^{k-8}[y]$ and note that $d_\omega(x_i, x) \leq 15$ and $x_i \in A_x$ for $i = 1, 2$. Next let y_1 and y_2 be the vertices farthest from x in $C \cap N_\omega^{k/2-18}[x] \subseteq A_x$ and note that $d_\omega(y_i, x) \geq \frac{k}{2} - 22$ for $i = 1, 2$. Without loss of generality, the two paths induced in C from x_1 to y_1 and from x_2 to y_2 are pairwise disjoint. These paths have weight $\frac{k}{2} - 37$, are of minimal weight between their end-points, and are at unweighted distance at least three. Hence, we can apply Corollary 5.2 twice to obtain $|A_x \cap N_\omega^{k-8}[y]| \geq \frac{2}{3}(\frac{k}{2} - 32)$. We get the same bound for the size of $A_y \cap N_\omega^{k-8}[x]$ and in total obtain

$$|\text{int}B| \geq 2 \left(\frac{2}{5}k - \frac{4}{5}\ell_0 - 22 \right) + 2 \left(\frac{k}{3} - 22 \right) \geq \frac{22}{15}k - \frac{8}{5}\ell_0 - 88 \geq \frac{4}{3}k,$$

where the last inequality holds by our bound on k . □

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