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Anisotropy in Gravity and Holography

by

Charles Milton Melby-Thompson

A dissertation submitted in partial satisfaction of the
requirements for the degree of

Doctor of Philosophy

in

Physics

in the

Graduate Division
of the
University of California, Berkeley

Committee in charge:
Professor Petr Hořava, Chair
Professor Ori Ganor
Professor Denis Auroux

Spring 2012

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Charles Milton Melby-Thompson

Abstract

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University of California, Berkeley

Professor Petr Hořava, Chair

In this thesis, we examine the dynamical structure of Hořava-Lifshitz gravity, and investigate its relationship with holography for anisotropic systems.

Hořava-Lifshitz gravity refers to a broad class of gravitational models that incorporate anisotropy at a fundamental level. The idea behind Hořava-Lifshitz gravity is to utilize ideas from the theory of dynamical critical phenomena into gravity to produce a theory of dynamical spacetime that is power-counting renormalizable, and is thus a candidate renormalizable quantum field theory of gravity.

One of the most distinctive features of Hořava-Lifshitz gravity is that its group of symmetries consists not of the diffeomorphisms of spacetime, but instead of the group of diffeomorphisms that preserve a given foliation by spatial slices. As a result of having a smaller group of symmetries, HL gravity naturally has one more propagating degree of freedom than general relativity.

The extra mode presents two possible difficulties with the theory, one relating to consistency, and the second to its viability as a phenomenological model. (1) It may destabilize the theory. (2) Phenomenologically, there are severe constraints on the existence of an extra propagating graviton polarization, as well as strong experimental constraints on the value of a parameter appearing in the dispersion relation of the extra mode.

In the first part of this dissertation we show that the extra mode can be eliminated by introducing a new local symmetry which steps in and takes the place of general covariance in the anisotropic context. While the identification of the appropriate symmetry is quite subtle in the full non-linear theory, once the dust settles, the resulting theory has a spectrum which matches that of general relativity in the infrared. This goes a good way toward answering the question of how close Hořava-Lifshitz gravity can come to reproducing general relativity in the infrared regime.

In the second part of the thesis we pursue the relationship between Hořava-Lifshitz gravity and holographic duals for anisotropic systems. A holographic correspondence is one that posits an equivalence between a theory of gravity on a given spacetime background and a field theory living on the "boundary" of that spacetime, which resides at infinite spatial

separation from the interior. It is a non-trivial problem how to define this boundary, but in the case of relativistic boundary field theories, there is a well-known definition due to Penrose of the boundary which produces the geometric structure required to make sense of the correspondence. However, the proposed dual geometries to anisotropic quantum field theories have a Penrose boundary that is incompatible with the assumed correspondence. We generalize Penrose's approach, using concepts from Hořava-Lifshitz gravity, to spacetimes with anisotropic boundary conditions, thereby arriving at the concept of *anisotropic conformal infinity* that is compatible with the holographic correspondence in these spacetimes.

We then apply this work to understanding the structure of holography for anisotropic systems in more detail. In particular, we examine the structure of divergences of a certain theory of gravity on Lifshitz space. We find, using our construction of anisotropic conformal infinity, that the appropriate geometric structure of the boundary is that of a foliated spacetime with an anisotropic metric complex. We then perform holographic renormalization in these spacetimes, yielding a computation of the divergent part of the effective action, and find that it exhibits precisely the structure of a Hořava-Lifshitz action. Moreover, we find that, for dynamical exponent $z = 2$, the logarithmic divergence gives rise to a conformal anomaly in 2+1 dimensions, whose general form is precisely that of conformal Hořava-Lifshitz gravity with detailed balance.

To my parents, Jane and John,

my sister Kirsten, and her husband Aaron,

who have shown me undying patience,

love and support.

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Acknowledgments

I wish first and most of all to thank my advisor, Petr Hořava, for his expertise, advice, and support, and for teaching me how to venture outside the box—without landing beyond the pale. My gratitude goes also to the other members of my doctoral committee, professors Ori Ganor and Denis Auroux.

Throughout my time at Berkeley I received help in many forms, ranging from useful tidbits to vigorous debates. Without the help of the members, current and former, of the BCTP, I would never have learned as much as I have. My thanks goes out to everyone, including Mina Aganagic, Chris Beam, Raphael Bousso, Ori Ganor, Tom Griffin, Kevin Grosvenor, Cindy Keeler, Stefan Leichenauer, Ruža Markov, Shannon McCurdy, Yu Nakayama, Omid Saremi, Gabe Wong, and Patrick Zulkowski.

Chapter 1

Introduction: Anisotropy in Gravity

Several years ago, two papers by Hořava (1; 2) introduced a class of anisotropic gravitational theories. These theories, referred to in the literature as Hořava-Lifshitz gravity, are power-counting renormalizable as quantum field theories, and are thus candidates for ultra-violet complete quantum field theories of gravity. Hořava's theories are unusual in that have anisotropy built in at a fundamental level, a situation that has interesting consequences and offers a number of peculiar challenges.

The most immediate challenge arises because these theories have a smaller group of local symmetries than general relativity; as a result, they generically have extra propagating modes. Moreover, these modes can have a pathological dispersion relation, leading to an apparent instability in the theory. It is therefore relevent to ask how close the dynamics can approach that of general relativity, and even whether these theories can be quantized at a non-perturbative level at all.

This last question, of whether the theory is well-defined beyond perturbation theory,[1] is an especially difficult one. One possible approach is to embed it into string theory, which is widely accepted to be a consistent framework of quantum gravity. An interesting possibility is suggested by recent work applying the principle of holographic correspondence to condensed matter systems.[2] While many systems of interest in condensed matter have emergent Lorentz-invariance in the low-energy limit, others exhibit anisotropy in space, or between space and time. Recent research has proposed dualities between such anisotropic field theories and gravity on backgrounds with appropriate anisotropic boundary conditions. The appearance of gravity together with anisotropy leads to anisotropy arising in the geometric structure of the bulk, which naturally leads to the question: what kind of signature does the bulk gravity have on the anisotropic boundary theory?

In this dissertation, we make some first steps toward understanding and resolving both of these issues, thereby bringing us closer to a knowledge of the role of anisotropy in gravity

[1] While power counting suggests these theories are renormalizable, we are not aware of a proof of perturbative renormalizability at present.

[2] A review of many of these developments can be found in (3).

and holography.

1.1 The Problem of Quantum Gravity

For more than half a century, the primary goal of high energy theory has been the unification of the various observed forces into a single underlying framework based on quantum principles. By the 1970's, this goal had largely been achieved in the form of the standard model. This model stands as one of the most accurate scientific theories ever created; no particle experiment has been able to produce effects which it could not be made to accommodate.

The success of this grand project of unification[3] has been accompanied by a correspondingly grand omission: it has as yet proved unable to accommodate gravity.[4] Even though Einstein wrote down his general theory of relativity nearly a hundred years ago, the formulation of a quantum theory of general relativity—in all but the "trivial" case of a two-dimensional spacetime—has remained chimerical.

During the 1960's, at a time when physicists were struggling to understand the increasingly bewildering array of new nuclear particles showing up in experiments, a theory of strings was introduced to account for certain features of the particle spectrum. However, this string theory had some flaws, including the appearance of a massless spin two particle in its spectrum. The theory hit its stride in the 1980's after the observation that this particle could be interpreted as a graviton, and that the low energy regime of a consistent quantum theory of fundamental strings automatically incorporates gravity.[5] Subsequent research indicated that string theory can make sense of interactions at arbitrary energies. It therefore constitutes the strongest theoretical candidate for a well-defined quantum theory of gravity to date.

In spite of its theoretical strengths, however, string theory has struck two crucial barriers to being accepted as a viable phenomenological model of the universe.

1. *Predictivity:* Although as a theoretical structure string theory is essentially unique, its effective behavior as a description of low-energy physics (*i.e.*, what we expect to be measurable in the foreseeable future) is determined by the choice of vacuum around which to expand. Unfortunately, the number of possibilities is enormous. So, without a solid principle that can single out a small class of acceptable vacua, string theory can make no falsifiable predictions about the structure of low energy physics.

[3]Grand unification generally refers to the unification of the electroweak and the strong force—a further unification of the forces present in the standard model. Here we are referring merely to the separate—but related—issue of describing all the forces of nature within a single framework.

[4]Its inability to be reconciled with gravity is not the only theoretical deficiency of the standard model, but it is probably the most glaring.

[5]Crucial to this story was the discovery of vacua stabilized by supersymmetry, allowing the tachyonic instabilities present up to then to be removed.

2. *Complexity:* While the assumptions underlying string theory are simple, the theory brings with it a complex mathematical machinery including an infinite tower of massive excitations, differential form fields, moduli, a battalion of extended objects of various dimensions, and supersymmetry, none of which appears to be relevant to low energy dynamics.

Indeed, given that the machinery of string theory (if indeed it describes our universe) is invisible at low energies, it is only natural to look for a corner of the space of string theories in which the machinery decouples, and gravity becomes complete on its own. Finding such a corner may be difficult, but if one exists, it can be searched for from the opposite direction, using the machinery of quantum field theory.

1.2 Proposals for Ultra-violet Theories of Gravity

Historically there have been many proposals for how gravity might be defined as an ultraviolet field theory. Two options which are instructive in the current context are asymptotic safety and conformal gravity (4).

There are two known ways for quantum field theories with a finite number of physical parameters to be defined in the ultraviolet. The first, known as asymptotic freedom, is shared by non-abelian gauge theories (such as QCD) for certain types of field content, and is the statement that as energies are increased, quantum effects cause the coupling grow continuously weaker, so that the theory asymptotically becomes non-interacting.

The second possibility is to encounter an *ultraviolet fixed point*, that is, a point (or surface) in the space of physical parameters where the beta function vanishes, and toward which the theory tends to flow as energies are increased.

Asymptotic safety is the hypothesis that there is an ultraviolet fixed point for gravity in 3+1 dimensions, so that, while the theory does not become weakly coupled at high energies, it at least asymptotically approaches a safe, finite coupling. Unfortunately, even if such a fixed point exists, the coupling would be so strong as to make the extraction of predictions from the theory prohibitively difficult. Moreover, lattice simulations ("dynamical triangulations") have been unable to find such fixed point behavior, rendering the existence of a fixed point unlikely.

A second approach is to find an asymptotically free completion of gravity by improving the high energy behavior of the perturbation expansion. One simple way to accomplish this is to add to the action terms quadratic in the curvature tensor, which induce not only new interactions, but modifications of the propagator as well. Around flat space the momentum space propagator has the general form

$$G \sim \frac{1}{k^2 - G_N k^4}. \tag{1.1}$$

While its infrared behavior $G \sim 1/k^2$ is that of general relativity, in the ultraviolet the propagator is suppressed as $1/k^4$, leading to a larger suppression of ultraviolet. The effect is sufficient to render the theory (perturbatively) renormalizable.

A difficulty becomes apparent when the propagator is expressed in the form

$$\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - G_N^{-1}}. \tag{1.2}$$

While the first term represents a healthy propagating mode, the second violates positivity (or unitarity) requirements on propagators in a quantum field theory. As a result, while higher derivative gravity may be sensible as a statistical system, it cannot be interpreted as a quantum mechanical theory—at least at the level of perturbation theory.

It is worthwhile noting that the problem here can essentially be traced to the appearance of multiple time derivatives; it is not surprising that adding time derivatives is difficult to reconcile with quantum mechanics, because the canonical quantization procedure in which positivity is manifest is executed in the Hamiltonian formalism, which must be extensively modified in the presence of extra time derivatives.

We will see, however, that it is possible to sidestep the issue of unitarity while including higher derivatives when we allow anisotropy into our theory.

1.3 Hořava-Lifshitz Gravity

Each of these proposals for how to complete gravity comes at some cost: string theory has its machinery, asymptotic safety the difficulty of performing computations at strong coupling, and higher derivative gravity its problem with ghosts. The grail would be a theory with a sensible weak coupling expansion around a free fixed point—that is also manifestly unitary.

Inspired by dynamical critical phenomena, common in condensed matter theory, Hořava introduced (1; 2) theories of gravity intended to satisfy both of these criteria. His construction, too, comes at a cost: the loss of isotropy.

1.3.1 Anisotropy in Field Theories

The original motivation for Hořava's work lies in the theory of dynamical critical phenomena. The prototype is the Lifshitz scalar theory (5), which was originally proposed as a description of tricritical phenomena exhibiting a modulated phase. In $D+1$ dimensions the free Lifshitz scalar has action

$$S = \frac{1}{2}\int dt\, d^D x \left(\dot{\phi}^2 - (\nabla^2\phi)^2\right). \tag{1.3}$$

(Throughout this dissertation, dots will denote time derivatives, and ∇ spatial derivatives, unless otherwise specified.) The Lifshitz scalar in $D+1$ dimensions supports higher-order

interaction terms than a relativistic scalar in the same dimension. To see this, we assign mass dimension to quantities in units of spatial derivatives:

$$[\nabla_i] = 1 \qquad [\partial_t] = 2; \tag{1.4}$$

then the scalar has dimension $(D-1)/2$. The momentum-space propagator is of the form

$$G(\omega, k) \sim \frac{1}{\omega^2 - (\vec{k}^2)^2}, \tag{1.5}$$

which has dimension -4, twice that of a relativistic theory. However, the mass dimension of the momentum space volume element is $[d\omega\, d^D k] = D + 2$, which is less than twice that in the relativistic case. As a result, the total dimension of Feynman diagrams is reduced in the anisotropic case, loop divergences become more strongly suppressed than in relativistic theories, and new renormalizable interaction terms become possible.

The free theory has a scale invariance in which space and time scale differently. This is a particular example of a broader class of scale invariances which appear in the theory of dynamical critical phenomena. Such models have invariance under rescalings of the form

$$t \mapsto \lambda^z t \qquad x^i \mapsto \lambda x^i. \tag{1.6}$$

The number z is called the *dynamical critical exponent.*

Some examples of $z = 2$ field theories that are more convergent in this context are renormalizable sigma models in $2+1$ dimensions (6) and gauge theories in $4+1$ dimensions (7; 8).

The improvement in ultraviolet behavior of these models arose from increasing the number of derivatives in the action without increasing the number of *time* derivatives, which was only possible because we are working in an anisotropic, non-relativistic context. In order to apply similar strategies to gain control over gravity, we must determine how to incorporate anisotropy into gravity at a fundamental level, while still maintaining as many of the properties of general relativity as possible.

1.3.2 Anisotropy in Gravity

The introduction of anisotropy into gravity is more complex than for a scalar field theory. This is because general covariance, which is essentially the statement that every coordinate system is equally good, is fundamentally at odds with anisotropy. An anisotropic theory can only have symmetries compatible with its anisotropic structure.

We start by splitting the action as the difference $S = S_K - S_V$, where S_K denotes the kinetic parts of the action, involving time derivatives. To ensure compatibility with quantum mechanics, we require the action to be quadratic in time derivatives. This gives it the basic form

$$S_K \sim \frac{1}{2\kappa^2} \int dt\, d^D x \sqrt{g} \dot{g}_{ij} G^{ijk\ell} \dot{g}_{k\ell}, \tag{1.7}$$

where we have used a generalization (1; 2)

$$G^{ijk\ell} = \frac{1}{2}(g^{ik}g^{j\ell} + g^{jk}g^{i\ell}) - \lambda g^{ij}g^{k\ell}, \tag{1.8}$$

of the DeWitt metric from canonical general relativity. In the generally covariant setting, λ is fixed by the symmetries of the system to equal one, but in our general setting it becomes a dimensionless coupling constant. The potential S_V is a functional of the metric and its spatial derivatives only.

This theory has no local symmetries, and so in addition to the tensor mode of general relativity, it will also have vector and scalar polarizations. To obtain a theory more closely resembling general relativity, we can require the theory to be invariant under the diffeomorphisms preserving a foliation $\mathcal{F}$ of spacetime M by spatial hypersurfaces Σ. The group of diffeomorphisms preserving the folation $\mathcal{F}$ is denoted $\mathrm{Diff}(M, \mathcal{F})$.

A pseudo-Riemannian metric on M decomposes into components that transform simply under the foliation-preserving diffeomorphisms. In a coordinate system $y^\mu = (t, x^i)$ compatible with the foliation, this is accomplished via the ADM decomposition (9) of the metric G into (N, N_i, g_{ij}) according to

$$G = \begin{pmatrix} -N^2 + N_i N_j g^{ij} & N_i \\ N_j & g_{ij} \end{pmatrix}. \tag{1.9}$$

We will often refer to this triple as the *metric complex* or *metric multiplet*. Spatial indices, denoted by roman letters $(i, j, \ldots)$, are raised and lowered using the spatial metric g_{ij}.

Under an infinitesimal element of our symmetry group

$$\delta t = f(t) \qquad \delta x^i = \xi^i(t, \mathbf{x}) \tag{1.10}$$

the metric complex transforms according to the rules

$$\begin{aligned} \delta g_{ij} &= \xi^k \partial_k g_{ij} + \partial_i \xi^k g_{jk} + \partial_j \xi^j g_{ik} \\ \delta N_i &= f\dot{N}_i + \xi^j \partial_j N_i + f\dot{N}_i + \partial_i \xi^j N_j + \dot{\xi}^j g_{ij} \\ \delta N &= f\dot{N} + \xi^j \partial_j N + \dot{f} N. \end{aligned} \tag{1.11}$$

The lapse and shift N and N_i play the role of gauge fields of the foliation-preserving diffeomorphism symmetry. Indeed, (1.11) shows that N (or, more precisely, $\log N$), and N^i transform as gauge fields under the gauge transformations,

$$\frac{\delta N}{N} = \dot{f} + \ldots, \qquad \delta N^i = \dot{\xi}^i + \ldots. \tag{1.12}$$

(Here the "..." stand for the standard Lie-derivative terms in (1.11).) As a result, it is natural to assume that N and N_i inherit the same dependence on spacetime as the corresponding generators (1.10): while $N_i(t, \mathbf{x})$ is a spacetime field, $N(t)$ is only a function of

time, constant along the spatial slices. Making this assumption about the lapse function will lead to the minimal theory of gravity with anisotropic scaling.

The covariantization of our theory is accomplished by replacing the spatial volume element with its covariant version,

$$\sqrt{g} \to \sqrt{g}\, N, \tag{1.13}$$

and by trading the time derivative of the metric for the extrinsic curvature K_{ij} of the leaves of the foliation $\mathcal{F}$,

$$\dot{g}_{ij} \to 2K_{ij} \equiv \frac{1}{N}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right). \tag{1.14}$$

In this fashion, we obtain the minimal realization of the idea of anisotropic scaling in gravity. This minimal theory is sometimes referred to as "projectable", because the spacetime metric assembled from the ingredients g_{ij}, N_i and N satisfies the axioms of a "projectable metric" on $(M, \mathcal{F})$, as defined in the geometric theory of foliations. Later we will also consider the "non-projectable" case, where the lapse variable can depend on space as well as time.

The action of the minimal theory is

$$S = \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\, \sqrt{g}\, N \left(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}\right), \tag{1.15}$$

where $K = g^{ij}K_{ij}$. The potential term $\mathcal{V}$ is an arbitrary $\mathrm{Diff}(\Sigma)$-invariant local scalar functional built out of the spatial metric, its Riemann tensor and the spatial covariant derivatives, without the use of time derivatives.

The potential term is allowed to be any expression constructed out of $\mathrm{Diff}_{\mathcal{F}}(M)$ invariants which involves no time derivatives. In the projectable case the only invariants possible are local functionals of the Riemann tensor and its derivatives,

$$S_V = \int dt d^D x N \sqrt{g} V(R_{ijk\ell}); \tag{1.16}$$

when expanding around a fixed point with dynamical exponent z we require V to have at least $2z$ spatial derivatives of the metric.

In the presence of $2z$ spatial derivatives and dropping relevant terms, in the decoupling limit the theory has the scale invariance

$$t \mapsto \lambda^z t \qquad x^i \mapsto \lambda x^i \tag{1.17}$$

and so indeed has dynamical exponent z.

As in the case of a scalar theory, around a free-field fixed point with dynamical exponent z we will measure the scaling dimensions of fields in the units of the spatial momentum: $[\nabla_i] \equiv 1$. In these units, the volume element in the action is of dimension $[dt\, d^D\mathbf{x}] = -z - D$, suggesting the natural scaling dimensions for the field multiplet,

$$[g_{ij}] = 0, \qquad [N_i] = z - 1, \qquad [N] = 0. \tag{1.18}$$

This scaling further implies that $[\kappa^2] = z - D$.

If we wish for the theory to be power-counting renormalizable, it is natural to start the analysis of possible terms appearing in $\mathcal{V}$ at short distances. Around a hypothetical Gaussian fixed point, power-counting renormalizability in $D+1$ dimensions requires $\mathcal{V}$ to be dominated by terms with $2D$ spatial derivatives, implying in turn that the dynamical critical exponent should be equal to $z = D$. For example, in $3+1$ dimensions, there is a natural potential

$$\mathcal{V}_{\rm UV} = w^2 C_{ij} C^{ij} + \dots, \tag{1.19}$$

where w is a dimensionless coupling, and $C^{ij} = \varepsilon^{ik\ell}\nabla_k(R^j_\ell - \frac{1}{4}R\delta^j_\ell)$ is the Cotton tensor. In (1.19), we have indicated only the part of the potential that is dominant in the ultraviolet, with "..." denoting the relevant terms which contain fewer than six spatial derivatives and become important at longer distances.

1.3.3 Anisotropic Weyl Transformations

Under certain circumstances, we can impose additional gauge symmetries to further constrain the classical action of gravity with anisotropic scaling. When $z = D$, one can thus require invariance under a local version of the anisotropic scaling (4.1), which acts on the metric multiplet by anisotropic Weyl transformations

$$N \mapsto e^{z\omega} N \qquad N_i \mapsto e^{2\omega} N_i \qquad \gamma_{ij} \mapsto e^{2\omega}\gamma_{ij}, \tag{1.20}$$

with an arbitrary local function $\omega(t, \mathbf{x})$. We denote the group of anisotropic Weyl transformations (1.20) with dynamical exponent z by $\mathrm{Weyl}_z(M, \mathcal{F})$. Crucially, this group extends the group of foliation preserving diffeomorphisms into a semi-direct product (*c.f.*Chapter 3 and (1))

$$\mathrm{Weyl}_z(M, \mathcal{F}) \rtimes \mathrm{Diff}(M, \mathcal{F}). \tag{1.21}$$

Indeed, the commutator between an infinitesimal foliation-preserving diffeomorphism $\delta_{(f,\ \xi^i)}$ of (1.10) and an infinitesimal generator δ_ω of the anisotropic Weyl transformation (1.20) yields another infinitesimal anisotropic Weyl transformation,

$$[\delta_{(f,\ \xi^i)}, \delta_\omega] = \delta_{f\partial_t\omega + \xi^i\partial_i\omega}, \tag{1.22}$$

with the same fixed – but otherwise arbitrary – value of z. On the other hand, had we tried to extend $\mathrm{Diff}(M, \mathcal{F})$ into the full spacetime diffeomorphism group, the closure of the symmetries would have forced the relativistic scaling with $z = 1$. Thus, anisotropic Weyl symmetry is only possible when we relax the spacetime diffeomorphism symmetry to the symmetries of the preferred foliation $\mathcal{F}$.

Since $\mathrm{Weyl}_z(M, \mathcal{F})$ acts on N by a spacetime-dependent gauge transformation (1.20), N itself must be a spacetime-dependent field, hence it cannot satisfy the projectability

condition. This suggests that the natural environment for conformal gravity with anisotropic Weyl invariance is the non-projectable version of the theory.[6]

1.4 Anisotropy in Gravity and Holography

This dissertation is concerned with investigating two key questions about Hořava-Lifshitz gravity. The first part, consisting of Chapter 2, addresses the issue of stability and the problem of approaching general relativity in the infrared; while the second, consisting of Chapters 3 and 4, looks at the relationship between Hořava-Lifshitz gravity and the holographic correspondence for systems with anisotropy, forging a first link with string theory.

1.4.1 Anisotropic General Covariance

Hořava-Lifshitz gravity starts with the elimination of one of the symmetries of general relativity. As a result, the theory, as formulated in Section 1.3, has an extra local degree of freedom, which arises as a scalar polarization of the graviton. There are two potential problems arising from the presence of the extra mode:

1. *Stability.* Because of the properties of the scalar polarization, a theory that looks healthy in the ultraviolet may have an unstable dispersion relation in the infrared. In general this means that the wrong "vacuum" has been chosen.[7] However, even for backgrounds that are perturbatively stable, it is not clear that the mode does not induce non-perturbative instabilities.

2. *Comparison with experiment.* For the purpose of phenomenological applications, the scalar mode poses problems due to strong experimental constraints (10) on the λ parameter appearing in the generalized DeWitt metric (1.8). Reproducing the expected behavior in the infrared requires tuning the value of λ very close to unity; however, it has been suggested that the scalar mode has a strong coupling singularity at this value, and it is an unresolved question whether the dynamics around this point are well-behaved (11).

In Chapter 2, previously published as (12), we give an extension of Hořava-Lifshitz gravity which solves the difficulties associated with the scalar mode by eliminating it entirely. We do so by first observing that, since the presence of the scalar is due to a reduction in the symmetry group, the simplest and most natural way to eliminate the scalar mode is

[6] One might consider restricting the Weyl invariant combination $\widetilde{N} \equiv N/\sqrt{\gamma}$ to be a function of time only (1); we leave the study of such a "conformally projectable" theory outside of the scope of the present work.

[7] I am investigating alternative points of view on this problem in work with Petr Hořava, Kevin Grosvenor, and Patrick Zulkowski.

to impose a new symmetry that can take the place of general covariance in an anisotropic context.

We find that it is possible to introduce such a symmetry at the linear level by introducing an extra gauge field into the theory. At the non-linear level there appears to be an obstruction to this symmetry, but we show that the obstruction can be removed if we add a second auxiliary field, the "Newton pre-potential". Both of these fields function as constraints, and so introduce no new degrees of freedom. However, the additional gauge symmetry allows one of the degrees of freedom of the original theory to be "gauged away". The resulting theory has a symmetry group of the same size as the diffeomorphism group, and a spectrum that matches that of general relativity in the infrared regime.

1.4.2 Anisotropy and Holography

The second part of the thesis, consisting of Chapters 3 and 4, examines the role played by Hořava-Lifshitz gravity in the context of holography.

Research during the past decade has proposed holographic duals to various phenomena relevant to condensed matter (see *e.g.* (3)). The initial work in this direction considered only field theories that are relativistic. However, many important systems in condensed matter have some degree of anisotropy, and more recent work has proposed dual geometries appropriate to such systems (13). In these proposals the gravitational theory remains relativistic, and anisotropy is imposed by the asymptotic boundary conditions of the spacetime geometry. The presence of anisotropy in the boundary conditions, however, renders the identification of the appropriate structure of the boundary geometry more subtle than in the relativistic case.

Chapter 3, as published in (14), deals with the problem of defining boundary geometry in the presence of anisotropy in the dual field theory. In the original AdS/CFT correspondence, the boundary of spacetime used is the conformal infinity of Penrose (15; 16). In this formalism, the boundary of a metric space (M, g) is obtained via a conformal embedding in an auxiliary space $(\widetilde{M}, \widetilde{g})$, such that the closure of the image of M is compact. The boundary of M, denoted ∂M, is then defined to be the boundary of the image of M. It has a natural induced topology, and while it has no canonical metric, it comes equipped with a conformal structure—*i.e.*, an equivalence class of metrics under local rescalings.

For spaces appropriate to gravitational duals of anisotropic field theories in $D + 1$ dimensions, the Penrose definition of the boundary is singular: for example, in the case of a theory with dynamical critical exponent $z > 1$, the boundary is one-dimensional. However, the duality requires a conception of the boundary geometry that is $D + 1$-dimensional.

Inspired by Hořava-Lifshitz gravity, we propose a definition of the boundary geometry by requiring the space to be equipped with an *asymptotic foliation*. This allows the generalization of conformal embeddings to conformal rescalings that are anisotropic on the metric. While local anisotropic rescalings are not locally compatible with the group of reparametrizations, they are *asymptotically* compatible. As a result, the anisotropic confor-

mal embeddings are also asymptotically compatible, allowing *anisotropic conformal infinity* to be defined. In particular, in the most straightforward case of "Lifshitz space" (13) the appropriate boundary structure is precisely that of a metric complex (or rather, conformal class of complexes) familiar from Hořava-Lifshitz gravity.

Expanding on these ideas, Chapter 4, which appeared previously as (17), examines the properties of holographic renormalization for proposed gravitational duals to anisotropic systems. We start with the simplest theory giving Lifshitz space as a solution (18), Einstein gravity coupled to a massive vector field in $3 + 1$ dimensions, and analyze the renormalization of the on-shell action for arbitrary boundary metric. This is done in the Hamiltonian formulation of holographic renormalization (19; 20; 21).

We find that the counterterms take the form of allowed action terms in Hořava-Lifshitz gravity. Despite the odd dimension of the boundary, we find that generically this process gives rise to anisotropic Weyl anomalies. We classify anomalies in 2+1 dimensions, and we find that, up to gravitational counterterms, there are two independent contributions to the anomaly. Moreover, we find that in 2+1 dimensions the holographic Weyl anomaly in our setup takes precisely the form of conformal Hořava-Lifshitz gravity with detailed balance (2). The second possible anomaly term does not arise.

Finally, we conclude with comments on the origin of the detailed balance condition in our setup, and draw connections to the wavefunction of the universe in a certain model of gravity in 3+1 dimensions.

Chapter 2

General Covariance in Quantum Gravity at a Lifshitz Point

In the minimal formulation of gravity with Lifshitz-type anisotropic scaling, the gauge symmetries of the system are foliation-preserving diffeomorphisms of spacetime. Consequently, compared to general relativity, the spectrum contains an extra scalar graviton polarization. Here we investigate the possibility of extending the gauge group by a local $U(1)$ symmetry to "nonrelativistic general covariance." This extended gauge symmetry eliminates the scalar graviton, and forces the coupling constant λ in the kinetic term of the minimal formulation to take its relativistic value, $\lambda = 1$. The resulting theory exhibits anisotropic scaling at short distances, and reproduces many features of general relativity at long distances.

2.1 Introduction

The idea of gravity with anisotropic scaling (1; 2; 22) has attracted a lot of attention recently. There are two, somewhat distinct, motivations for developing this approach to gravity. The first is driven by the long-standing search for a theoretical framework in which the classical theory of gravity is reconciled with the laws of quantum mechanics. A successful outcome of this search would result in a mathematically self-consistent framework for quantum gravity, not necessarily subjected to experimental tests. Examples already exist – the ten-dimensional supersymmetric vacua of string theory belong to this category. The second motivation comes from a goal which is more narrow, and also much more ambitious: To find, within such a self-consistent quantum gravity framework, a theory that reproduces the observed gravitational phenomena in our universe of $3+1$ macroscopic dimensions.

Both of these motivations are relevant for the development of gravity with anisotropic scaling. For a large class of possible applications, it does not matter whether or not the theory matches general relativity at long distances, or conforms to the available experimental tests of gravity in $3+1$ dimensions. A mathematically consistent quantum gravity which lacks this phenomenological matching can still be useful in the context of AdS/CFT correspondence, and produce novel gravity duals for a broader class of field theories, of interest for example in condensed matter applications. It can also have interesting mathematical implications, given the close connection between the theory formulated in (1; 2) and the mathematical theory of the Ricci flow on Riemannian manifolds.

However, explaining the observed features of gravity in our universe of $3+1$ dimensions is still perhaps the leading motivation for developing a quantum theory of gravity. Therefore, it makes sense to ask how close we can get, in the new framework of gravity with anisotropic scaling, to reproducing general relativity in the range of scales where the laws of gravity have been experimentally tested.

The comparison to general relativity is facilitated by the fact that in the framework proposed in (1; 2; 22), gravity is also described simply as a field theory of the dynamical metric on spacetime. Unlike in general relativity, however, the spacetime manifold M (which we take to be of a general dimension $D+1$) is equipped with a preferred structure of a codimension-one foliation $\mathcal{F}$ by slices of constant time, $\Sigma(t)$.[1] In the minimal realization of the theory, given in Section 1.3, the gauge symmetries of the system are the foliation-preserving diffeomorphisms $\mathrm{Diff}(M,\mathcal{F})$. Since this symmetry contains one less gauge invariance per spacetime point compared to the full spacetime diffeomorphisms $\mathrm{Diff}(M)$, the spectrum of the linearized theory around flat spacetime contains one additional, scalar polarization of the graviton.

At short distances, the anisotropy between space and time is measured by a nontrivial dynamical critical exponent $z>1$, leading to an improved ultraviolet behavior of the theory. At long distances, on the other hand, the theory is driven to an infrared regime where it

[1] For simplicity, we will assume throughout this chapter that the leaves $\Sigma(t)$ of this foliation all have the same topology of a D-dimensional manifold Σ.

shares many features with general relativity. First of all, under the influence of relevant terms in the classical action, the scaling becomes naturally isotropic, with the relativistic value of $z = 1$. Moreover, the lowest-dimension terms that dominate the action in the infrared are exactly those that appear in the ADM decomposition (9) of the Einstein-Hilbert action: The scalar curvature term, which sets the value of the effective Newton constant, and the cosmological constant term.

Thus, in the low-energy regime, the action of the minimal theory with anisotropic scaling looks very similar to that of general relativity. However, this similarity has its limits, and the theories are clearly different even in the infrared. The differences can be understood in three related ways: As a difference in gauge symmetries, a difference in the graviton spectrum, and a difference in the number of independent coupling constants. First, in the theory with anisotropic scaling, the gauge symmetry is reduced to $\mathrm{Diff}(M, \mathcal{F})$, and the theory propagates an extra scalar polarization of the graviton. In addition, the kinetic term in the action allows for an additional coupling λ, which is undetermined by any symmetry of the minimal theory, and therefore expected to run with the scale in the quantum theory. In general relativity, the spacetime diffeomorphism symmetries force $\lambda = 1$ and protect this value from quantum corrections. Stringent experimental limits on the value of λ have been advocated in the literature (23; 10), suggesting that at least in this class of models, it must be very near its relativistic value $\lambda = 1$. However, in the regime near $\lambda = 1$, difficulties with the dynamics of the additional scalar graviton have been pointed out (see, for example, (11; 24; 25)).

In order to get closer to general relativity, it is tempting to focus on the structure of gauge symmetries. However, this needs to be done cautiously, keeping in mind that gauge symmetries are just convenient redundancies in the description of a physical system, and therefore to some extent in the eye of the beholder. The more physical perspective is to focus on the spectrum of propagating degrees of freedom.[2] Thus, we will be interested in finding an extension of the gauge symmetry that will turn the extra, scalar polarization of the graviton into a gauge artifact. A second option would be to find a mechanism for generating a finite mass gap for the scalar graviton – in this chapter, we concentrate on the first possibility.

We will find such an extended gauge symmetry, with as many generators per spacetime point as in general relativity. This gauge symmetry can be viewed as representing "non-

[2]Of course, one can make the theory diffeomorphism invariant in a trivial way, by integrating in the gauge invariance without changing the number of physical degrees of freedom. This leads to a theory which is formally generally covariant, but effectively equivalent to the original model. Perhaps any sensible theory can be "parametrized" in this way (26), and formally rewritten as a generally covariant theory. This process of covariantizing a given theory is very closely connected to the Stückelberg mechanism used prominently in particle physics (see, *e.g.*, (27) for a review). It has been applied to the models of gravity with anisotropic scaling (1; 2) in (11; 28; 29). In this chapter, instead, we are interested in the nontrivial extension of gauge symmetry which actually reduces the number of physical degrees of freedom. Investigating how the resulting theory then responds to the Stückelberg trick is an interesting question, beyond the scope of this dissertation.

relativistic general covariance" in gravity with anisotropic scaling. The extended symmetry eliminates the scalar polarization of the graviton from the spectrum. As a bonus, we find that the extended gauge symmetry requires $\lambda = 1$, thereby reducing the kinetic term to coincide with that of general relativity. It is in fact important that our entire construction depends only on the form of the kinetic term, and therefore does not restrict the form of the potential term in the action. Hence, at short distances, the covariant theory can exhibit the same improved ultraviolet behavior associated with $z > 1$ in the minimal theory of (1; 2).

In this chapter, our perspective is that of (effective) quantum field theory, but we restrict our analysis to the leading tree-level, or classical, approximation. Quantum corrections are expected to modify the scaling behavior of our models, but they are beyond the scope of this work. In addition, our analysis will be strictly local: We freely integrate by parts and ignore total derivative terms. Boundary terms play a notoriously central role in relativistic gravity; it will be important to extend our analysis to include their precise structure and clarify their role in theories of gravity with anisotropic scaling. These are among the many interesting issues left for future work.

The main result of the chapter is the construction of the generally covariant gravity with anisotropic scaling, which we present in Section 2.4. Sections 2.1, 2.2 and 2.3 prepare the ground for a better understanding of the main results, and explore a few additional issues of interest.

2.1.1 General covariance

In order to explain what exactly we mean by "general covariance," we first consider two theories – general relativity, and the ultralocal theory of gravity (30; 31) – and illustrate our point using the Hamiltonian formulation of these two theories.

In fact, throughout this chapter we will often resort to the Hamiltonian formalism,[3] for a number of reasons. First of all, the time versus space split of the Hamiltonian formalism is particularly natural for gravity with anisotropic scaling. More importantly, the technology available in the Hamiltonian formalism allows us to get a precise count of the number of propagating degrees of freedom, and offers a better insight into the structure of the gauge symmetries of the theory. Indeed, one of the advantages of the Hamiltonian formulation is that one does not have to specify the gauge symmetries *a priori*. Instead, the structure of the Hamiltonian constraints provides an essentially algorithmic way in which the correct gauge symmetry structure is determined automatically (26). In the process, the consistency of the equations of motion is tied to the closure of the constraint algebra and the preservation of the constraints under the time evolution. Once the full system of constraints has been determined, the constraints are separated into first-class (whose commutators with other constraints vanish on the constraint surface) and second-class (whose commutators define a nondegenerate symplectic form). As an additional benefit, after determining the numbers

[3]For the canonical reference on Hamiltonian systems with constraints, see (26).

$\mathcal{C}_1$ of first-class and $\mathcal{C}_2$ of second-class constraints, the number of degrees of freedom $\mathcal{N}$ can be reliably evaluated by the standard formula (26)

$$\mathcal{N} = \frac{1}{2}\left(\dim \mathcal{P} - 2\mathcal{C}_1 - \mathcal{C}_2\right), \tag{2.1}$$

where $\dim \mathcal{P}$ is the number of fields in the canonical formulation (*i.e.*, the dimension of phase space). In local field theory, this formula can be interpreted per spacetime point, giving the number of local degrees of freedom. We will use this formula repeatedly throughout the chapter.

In canonical general relativity (32; 33; 9) on a spacetime manifold M with $D+1$ coordinates (x^i, t), the algebra of constraints contains the "superhamiltonian" $\mathcal{H}_\perp(\mathbf{x})$ and the "supermomentum" $\mathcal{H}_i(\mathbf{x})$, and the total Hamiltonian is just a sum of constraints:

$$H = \int d^D\mathbf{x}\left(N\mathcal{H}_\perp + N^i\mathcal{H}_i\right). \tag{2.2}$$

N and N^i are the lapse and shift variables of the metric, and $\mathcal{H}_\perp$ and $\mathcal{H}_i$ are functions of the spatial components g_{ij} of the metric and their canonically conjugate momenta π^{ij}. Since the constraints are all first-class, they generate gauge symmetries, whose generators are

$$\mathcal{H}(\xi^i) \equiv \int d^D\mathbf{x}\,\xi^i(\mathbf{x},t)\mathcal{H}_i(\mathbf{x},t), \qquad \mathcal{H}_\perp(\xi^0) \equiv \int d^D\mathbf{x}\,\xi^0(\mathbf{x},t)\mathcal{H}_\perp(\mathbf{x},t). \tag{2.3}$$

Their commutation relations are well-understood, even though they do not quite yield the naively expected spacetime diffeomorphism algebra. True, the commutator of two $\mathcal{H}_i$'s

$$[\mathcal{H}(\xi^i), \mathcal{H}(\zeta^j)] = \mathcal{H}(\xi^k\partial_k\zeta^i - \zeta^k\partial_k\xi^i) \tag{2.4}$$

reproduces the algebra of spatial diffeomorphisms $\mathrm{Diff}(\Sigma)$, and

$$[\mathcal{H}(\xi^i), \mathcal{H}_\perp(\zeta^0)] = \mathcal{H}_\perp(\xi^k\partial_k\zeta^0) \tag{2.5}$$

just states that $\mathcal{H}_\perp$ transforms correctly under $\mathrm{Diff}(\Sigma)$. However, the commutator of $\mathcal{H}_\perp$ with itself gives a field-dependent result,

$$[\mathcal{H}_\perp(\xi^0), \mathcal{H}_\perp(\zeta^0)] = -\sigma\mathcal{H}\left(g^{ij}(\xi^0\partial_j\zeta^0 - \zeta^0\partial_j\xi^0)\right). \tag{2.6}$$

Here σ denotes the signature of spacetime: $\sigma = -1$ for general relativity in Minkowski signature. This generalized, Dirac algebra is the Hamiltonian manifestation of the original diffeomorphism symmetry (and general covariance) of general relativity, with $D+1$ gauge symmetries per spacetime point.

Our second example is the ultralocal theory of gravity, which results from dropping the spatial scalar curvature term R in the action of general relativity. Of course, this step selects

a preferred foliation $\mathcal{F}$ of spacetime, and therefore violates spacetime diffeomorphism invariance. One might naively assume that the symmetry is reduced to the foliation-preserving diffeomorphisms $\mathrm{Diff}(M, \mathcal{F})$. However, the analysis of Hamiltonian constraints reveals a surprising fact (30; 31): The theory is still gauge invariant under as many gauge symmetries per spacetime point as general relativity. In contrast with general relativity, the ultralocal theory exhibits a contracted version of the Hamiltonian constraint algebra, with (2.6) replaced by its $\sigma \to 0$ limit:

$$[\mathcal{H}_\perp(\xi^0), \mathcal{H}(\zeta^0)] = 0, \tag{2.7}$$

while the remaining commutators (2.4) and (2.5) stay the same.

The theory is "generally covariant" – it has the same number $D + 1$ of (nontrivial) local gauge symmetries as general relativity, even though the algebra in the $\sigma \to 0$ limit still preserves the preferred spacetime foliation structure. Effectively, the spatial diffeomorphism symmetries have been kept intact, but the time reparametrization symmetry has been linearized, and its algebra contracted to a local $U(1)$ gauge symmetry.[4] Just as the Dirac algebra of Hamiltonian constraints (2.4), (2.5) and (2.6) in general relativity is associated with the Lagrangian symmetries described by the group of spacetime diffeomorphisms, $\mathrm{Diff}(M)$, the Teitelboim-Henneaux algebra (2.4), (2.5) and (2.7) can be associated with a Lagrangian symmetry group which takes the form of a semi-direct product,

$$U(1) \ltimes \mathrm{Diff}(M, \mathcal{F}). \tag{2.8}$$

It is natural to interpret (2.8) as the symmetry group of "nonrelativistic general covariance." This is the structure of gauge symmetries that we will try to implement in the case of gravity with anisotropic scaling for general values of z.

The restoration of general covariance characterized by (2.8) still maintains the special status of time, keeping it on a different footing from space. We view the fact that "time is different" as a virtue of this approach: Indeed, we are looking for possible concessions on the side of general relativity that would make it friendlier to the way in which time is treated in quantum mechanics, without changing too much of its elegant geometric nature.

2.1.2 Infrared Regime of Hořava-Lifshitz Gravity

In the theories of Section 1.3, the potential term was chosen so that in the ultraviolet the theory was dominated by contributions that are explicitly power-counter renormalizable. Under the influence of relevant terms, however, the theory will flow, until it is dominated at long distances by the most relevant terms. In this regime, it makes sense to reorganize the terms in $\mathcal{V}$ by focusing on those most dominant in the infrared:

$$\mathcal{V}_{\mathrm{IR}} = -\mu^2(R - 2\Lambda) + \dots. \tag{2.9}$$

[4] Throughout this chapter, we use the rather loose notation common in high-energy physics, and refer to any one-dimensional Abelian symmetry group as $U(1)$, regardless of whether or not it is actually compact. Moreover, we use the same notation also for the infinite-dimensional, gauge version of the $U(1)$ symmetry.

Here μ and Λ are dimensionful couplings of dimensions $[\mu] = z-1$ and $[\Lambda] = 2$, and the "..." now denote all the terms containing composite operators of *higher* dimension compared to the displayed, most dominant infrared terms.

It is useful to note that the algebra of gauge symmetries $\mathrm{Diff}(M, \mathcal{F})$, and their action on the fields, can be obtained simply by taking a nonrelativistic reduction of the fully relativistic spacetime diffeomorphism symmetry and its action on the relativistic metric $g_{\mu\nu}$. As we will see in Section 2.3.1, a natural extension of this procedure to subleading terms in $1/c$ leads to a natural geometric interpretation of the extended symmetries that are the focus of this chapter.

2.1.3 Comments on the nonprojectable case

The minimal, projectable theory can be rewritten in the Hamiltonian formalism, with the Hamiltonian similar to (2.2),

$$H = \int d^D\mathbf{x} \left(N\mathcal{H}_0 + N^i\mathcal{H}_i \right). \tag{2.10}$$

Here $\mathcal{H}_0$ and $\mathcal{H}_i$ are again functions of the spatial metric and its conjugate momenta. In fact, $\mathcal{H}_i$ takes the same form as in general relativity, $\mathcal{H}^i = -2\nabla_k \pi^{ik}$, and $\mathcal{H}_0$ depends on the choice of $\mathcal{V}$. The main conceptual difference compared to general relativity stems from the fact that because $N(t)$ is independent of the spatial coordinates x^i, it only gives rise to the integral constraint $\int d^D\mathbf{x}\, \mathcal{H}_0 = 0$. Consequently, compared to general relativity, the number of first-class constraints and hence gauge symmetries per spacetime point has been reduced by one.

The first, most naive temptation how to eliminate this discrepancy and get closer to general relativity is to restore the full dependence of the lapse function on space and time by hand. This option, often referred to in the literature as the "nonprojectable case" (1; 2), can be viewed at least from two different perspectives, which lead to different results.

First, one can follow the logic of effective field theory: Having postulated a multiplet of spacetime fields

$$g_{ij}(t, \mathbf{x}), \qquad N_i(t, \mathbf{x}), \qquad N(t, \mathbf{x}), \tag{2.11}$$

we postulate a list of global and gauge symmetries, and construct the most general action allowed. In the case at hand, the natural gauge symmetries are the foliation-preserving diffeomorphisms $\mathrm{Diff}(M, \mathcal{F})$. While (2.10) is invariant under $\mathrm{Diff}(M, \mathcal{F})$, *it is not the most general Hamiltonian compatible with these gauge symmetries.* As was pointed out already in (1; 2) and further elaborated in (24; 25), in this effective field theory approach to the nonprojectable theory, all terms compatible with the gauge symmetry should be allowed in the Lagrangian. Promoting the lapse function to a spacetime field gives a new ingredient for constructing gauge-invariant terms in the action,

$$\nabla_i N/N, \tag{2.12}$$

which transforms under $\mathrm{Diff}(M, \mathcal{F})$ as a spatial vector and a time scalar. Once terms with this new ingredient are allowed in the action, the Hamiltonian is no longer linear in N, but the algebra of constraints is well-behaved (34). The constraint implied by the variation of N is now second-class, and the expected number of propagating degrees of freedom is the same as in the minimal theory.

Another possible interpretation of the nonprojectable theory also starts by promoting N to a spacetime-dependent field $N(t, \mathbf{x})$. Instead of specifying *a priori* gauge symmetries, however, one can postulate that the Hamiltonian take the form (2.10), linear in N (35). This step must be followed by the analysis of the algebra of Hamiltonian constraints, which determines *a posteriori* whether this construction is consistent, and if so, what is the resulting structure of the gauge symmetries. Here the difficulty is in closing the constraint algebra (1; 36; 37; 35) (see also (38)): For general $\mathcal{V}$, the commutator of $\mathcal{H}_0(\mathbf{x})$ with $\mathcal{H}_0(\mathbf{y})$ is a complicated function of all variables, and the requirement of closure is difficult to implement. One interesting exception has been found in the infrared limit (39) (see also (35; 40)): Adding $\pi \equiv g_{ij}\pi^{ij}$ as another constraint closes the algebra, turning π and $\mathcal{H}_0$ into a pair of second-class constraints. This infrared theory can then be interpreted as general relativity whose gauge freedom has been partially fixed. This is a very appealing picture, but the problem is that it cannot be straightforwardly extended to the full theory beyond the infrared limit. However, as we will see below, the possibility of interpreting π as an additional constraint in the infrared regime as suggested in (39) will be echoed in the generally covariant theory which we present in Section 2.4.

In addition to the two perspectives just reviewed, there is another option how to close the constraint algebra in the nonprojectable theory, and interpret it as a topological field theory.[5] This is possible when the theory satisfies the detailed balance condition (1; 2), *i.e.*, when the potential $\mathcal{V}$ in (1.15) is of the special form

$$\mathcal{V} = \frac{1}{4}\mathcal{G}_{ijk\ell}\frac{\delta W}{\delta g_{ij}}\frac{\delta W}{\delta g_{k\ell}} \tag{2.13}$$

for some action functional $W(g_{ij})$ which depends only on g_{ij} and its spatial derivatives. In such cases, it is convenient to introduce a system of complex variables, defined as

$$a^{ij} = \mathrm{i}\pi^{ij} + \frac{1}{\kappa^2}\frac{\delta W}{\delta g_{ij}}, \qquad \overline{a}^{ij} = -\mathrm{i}\pi^{ij} + \frac{1}{\kappa^2}\frac{\delta W}{\delta g_{ij}}. \tag{2.14}$$

Under the Poisson bracket, these variables play essentially the role of a creation and annihilation pair, their only nonzero bracket being

$$[a^{ij}(\mathbf{x}), \overline{a}^{k\ell}(\mathbf{y})] = -\frac{2\mathrm{i}}{\kappa^2}\frac{\delta^2 W}{\delta g_{ij}(\mathbf{x})\delta g_{k\ell}(\mathbf{y})}. \tag{2.15}$$

[5] This option was pointed out by one of us in (36); see also Section 5.4 of (1).

The Hamiltonian constraints $\mathcal{H}_i$ and $\mathcal{H}_0$ can be expressed as simple functions of the complex variables,

$$\mathcal{H}^i = \mathrm{i}\nabla_j \left(a^{ij} - \overline{a}^{ij}\right), \qquad \mathcal{H}_0 = \frac{\kappa^2}{2} a^{ij}\mathcal{G}_{ijk\ell}\overline{a}^{k\ell}. \tag{2.16}$$

The problematic commutator of $\mathcal{H}_0(\mathbf{x})$ and $\mathcal{H}_0(\mathbf{y})$ is still rather complicated, but it clearly vanishes when a^{ij} or $\overline{a}^{ij}$ vanish. The constraint algebra can thus be closed by declaring $\mathcal{H}_i$, together with either a^{ij} or $\overline{a}^{ij}$ to be the primary constraints. This would then guarantee that the original Hamiltonian constraints $\mathcal{H}_i$ and $\mathcal{H}_0$, as well as all their commutators, are zero on the constraint surface.

This step can be made more precise as follows. Because a^{ij} and $\overline{a}^{ij}$ are complex conjugates of each other, it is not possible to declare only (say) a^{ij} to be first-class constraints, at least not without making a^{ij} and $\overline{a}^{ij}$ formally independent. Instead, we accomplish our goal by declaring both a^{ij} and $\overline{a}^{ij}$ as constraints. Because their commutator (2.15) is nonzero, these constraints are second-class and do not imply any additional gauge symmetry.

However, a pair of second-class constraints can often be interpreted as a first-class constraint, together with a gauge-fixing condition. We can interpret the theory with the second-class constraints a^{ij} and $\overline{a}^{ij}$ in this fashion: First, we choose $a^{ij} - \overline{a}^{ij}$ as the first-class constraint. The gauge symmetry generated by this constraint acts on g_{ij} via

$$\delta g_{ij} = \lambda_{ij}(t, \mathbf{x}), \tag{2.17}$$

with λ_{ij} an arbitrary spacetime-dependent symmetric two-tensor. This is just the topological gauge symmetry as introduced originally by Witten (41; 42), here acting on the spatial component of the metric. The theory is then fully specified by the choice of a gauge-fixing condition for the topological gauge symmetry. In our case, this choice should restore a^{ij} and $\overline{a}^{ij}$ as second-class constraints. Choosing $a^{ij} + \overline{a}^{ij}$ as the gauge fixing condition is certainly a consistent possibility; however, a more interesting scenario is available when we Wick-rotate the theory to imaginary time, $t = -i\tau$. This case is of particular interest because topological field theories are typically formulated in imaginary time. In this regime, a^{ij} and $\overline{a}^{ij}$ are now real, instead of being complex conjugates. We can then select an asymmetric gauge-fixing condition, for example

$$a^{ij} \equiv -\pi^{ij} + \frac{1}{\kappa^2}\frac{\delta W}{\delta g_{ij}} = 0. \tag{2.18}$$

This equation is a flow equation for g_{ij} as a function of the imaginary time τ, reminiscent of the Ricci flow equation and its cousins. We have thus obtained a topological field theory associated with the flow equations on Riemannian manifolds. Indeed, the number of topological gauge symmetries (2.17) is the same as the number of field components of g_{ij}: The theory has no local propagating degrees of freedom in the bulk.

The original $\mathrm{Diff}(M, \mathcal{F})$ symmetry can be viewed as a separate gauge symmetry in addition to the topological gauge symmetry (2.17). However, because the action (1.11) of $\mathrm{Diff}(M, \mathcal{F})$ on g_{ij} is a special case of (2.17), including $\mathrm{Diff}(M, \mathcal{F})$ explicitly leads to a

redundancy in gauge symmetries, and triggers the appearance of "ghost-for-ghosts" in the BRST formalism. In this respect, the structure of the gauge symmetries is very similar to the conventional topological field theories of the cohomological type (41; 42) such as topological Yang-Mills theory.

In this chapter, we are interested in gravity with bulk propagating degrees of freedom, whose spectrum contains the tensor polarizations of the graviton but not the scalar mode. Therefore, we do not pursue the nonprojectable theory further, and look for the missing gauge invariance elsewhere.

2.2 Global $U(1)_\Sigma$ Symmetry in the Minimal Theory at $\lambda = 1$

In general relativity, the value $\lambda = 1$ of the coupling in (1.15) is selected – and protected from renormalization – by the gauge symmetries $\mathrm{Diff}(M)$ of the theory. It is perhaps surprising that the case of $\lambda = 1$ plays a special role in the minimal theory with anisotropic scaling as well (1; 2). One can see this by examining the spectrum of the linarized fluctuations around the flat space solution.

For simplicity, we will now assume that the flat spacetime

$$g_{ij} = \delta_{ij}, \qquad N_i = 0, \qquad N = 1 \tag{2.19}$$

is a solution of the equations of motion, and expand the metric to linear order around this background,

$$g_{ij} = \delta_{ij} + \kappa h_{ij}, \qquad N_i = \kappa n_i, \qquad N = 1 + \kappa n. \tag{2.20}$$

Since n is not a spacetime field but only a function of time, its equation of motion gives one integral constraint, and does not affect the number of local degrees of freedom or their dispersion relations. Therefore, we only consider the equations of motion for the spacetime fields h_{ij} and n_i.

It will be convenient to further decompose the h_{ij} and n_i fluctuations into their irreducible components,

$$h_{ij} = s_{ij} + \partial_i w_j + \partial_j w_i + \left(\partial_i\partial_j - \frac{1}{D}\delta_{ij}\partial^2\right) B + \frac{1}{D}\delta_{ij} h, \tag{2.21}$$

where the scalar $h = h_{ii}$ is the trace part of h_{ij}, while s_{ij} is symmetric, traceless and transverse (*i.e.*, divergence-free: $\partial_i s_{ij} = 0$), and w_i is transverse; and similarly,

$$n_i = u_i + \partial_i C, \tag{2.22}$$

with u_i transverse, $\partial_i u_i = 0$. It is also useful to decompose the linearized gauge transformations,

$$\xi^i(\mathbf{x}, t) = \zeta_i(\mathbf{x}, t) + \partial_i \eta(\mathbf{x}, t). \tag{2.23}$$

In this decomposition, ζ^i satisfy $\partial_i \zeta_i = 0$, and therefore represent the generators of linearized volume-preserving spatial diffeomorphisms. The linearized gauge transformations act on the irreducible components of the fields via

$$\begin{aligned} \delta s_{ij} = 0, \qquad \delta w_i &= \zeta_i, \qquad \delta B = 2\eta, \qquad \delta h = 2\partial^2 \eta, \\ \delta u_i &= \dot{\zeta}_i, \qquad \delta C = \dot{\eta}. \end{aligned} \tag{2.24}$$

These rules suggest a few natural gauge-fixing conditions. For example, we can set $u_i = 0$ and $C = 0$, which leaves the residual symmetries with time-independent $\zeta_i(\mathbf{x})$ and $\eta(\mathbf{x})$, or $w_i = 0$ and $B = 0$, which fixes the gauge symmetries completely. In either gauge, the spectrum of linearized fluctuations around the flat background contains transverse traceless polarizations s_{ij} which all share the same dispersion relation (dependent on the details of $\mathcal{V}$), and a scalar whose dispersion is dependent on λ. In the vicinity of $\lambda = 1$, the dispersion relation of the scalar graviton exhibits a singular behavior,

$$\omega^2 = (\lambda - 1) F(\mathbf{k}^2, \lambda), \tag{2.25}$$

where $F(\mathbf{k}^2, \lambda)$ is a regular function of λ near $\lambda = 1$, whose details again depend on $\mathcal{V}$. Thus, the scalar dispersion relation degenerates to $\omega^2 = 0$ in the limit of $\lambda \to 1$ (1; 2).

2.2.1 Symmetries in the linearized approximation around flat spacetime

The spectrum of linear excitations around the flat spacetime shows that the relativistic value $\lambda = 1$ is special even in the nonrelativistic theory, as indicated by the dispersion relation of the scalar graviton mode (2.25) which degenerates as $\lambda \to 1$. This singular behavior was explained in (1): At $\lambda = 1$, the linearized theory with $\lambda = 1$ enjoys an interesting Abelian symmetry, which acts on the fields of the minimal theory via

$$\delta n_i = \partial_i \alpha, \qquad \delta h_{ij} = 0, \qquad \delta n = 0. \tag{2.26}$$

Here the parameter $\alpha(\mathbf{x})$ is an arbitrary smooth function of the spatial coordinates, constant in time:

$$\dot{\alpha} = 0. \tag{2.27}$$

Since the generator α is independent of time, it is natural to interpret this infinite-dimensional Abelian symmetry as a *global* symmetry: In the nonrelativistic setting, it is the hallmark of gauge symmetries in the Lagrangian formalism that their generators are arbitrary functions of time. In order to indicate that the Abelian symmetries generated by $\alpha(\mathbf{x})$ represent a collection of $U(1)$ symmetries parametrized by the spatial slice Σ of the spacetime foliation, we will refer to this infinite-dimensional symmetry by $U(1)_\Sigma$.

At this stage, it is interesting to note that the $U(1)_\Sigma$ symmetry looks very reminiscent of a residual gauge symmetry in a gauge theory, in which some sort of temporal gauge has

been chosen. As we will see in the rest of the chapter, this intuition is essentially correct, but the specific realization of this idea in the full nonlinear theory will be surprisingly subtle.

In order to see that $U(1)_\Sigma$ is indeed a symmetry of the linearized theory at $\lambda = 1$, it is instructive to restore temporarily λ, and evaluate the variation of the action under (2.26) in the linearized approximation (which we denote by "$\approx$"), while allowing α to be time dependent:

$$\delta_\alpha S \approx 2 \int dt\, d^D\mathbf{x} \left\{ \dot\alpha \left(\frac{D-1}{D} (\partial^2)^2 B + \frac{1-\lambda D}{D} \partial^2 h \right) - 2\alpha(\lambda - 1)(\partial^2)^2 C \right\}. \tag{2.28}$$

At $\lambda = 1$, the last term drops out, and the action is invariant under time-independent α. Note also that the term proportional to $\dot\alpha$ in (2.28) is not gauge invariant under (2.24), unless $\lambda = 1$ when it equals

$$\frac{D-1}{D} \left\{ (\partial^2)^2 B - \partial^2 h \right\} \approx R, \tag{2.29}$$

which we recognize as the linearized Ricci scalar of g_{ij}.

Given this global $U(1)_\Sigma$ symmetry, it is natural to ask whether it can be gauged. At $\lambda = 1$, this process can be easily completed in the linearized theory. We promote α to an arbitrary smooth function of $\mathbf{x}$ and t, introduce a gauge field $A(\mathbf{x}, t)$, and postulate its transformation rules under the gauge transformations,

$$\delta_\alpha A = \dot\alpha. \tag{2.30}$$

The gauging is accomplished by augmenting the action by a coupling of A to the linearized Ricci scalar,

$$S_A = -\frac{2(D-1)}{D} \int dt\, d^D\mathbf{x}\, A \left\{ (\partial^2)^2 B - \partial^2 h \right\}. \tag{2.31}$$

It is easy to see that in the gauged theory, the scalar mode of the graviton has been eliminated from the spectrum of physical excitations: With $A = 0$ as our gauge choice, the equations of motion are the same as in the original theory with the global $U(1)_\Sigma$ symmetry, plus the Gauss constraint

$$(\partial^2)^2 B - \partial^2 h = 0. \tag{2.32}$$

This Gauss constraint eliminates the scalar degree of freedom, leaving only the tensor modes of the graviton in the physical spectrum of the linearized theory.

2.2.2 The nonlinear theory

We would now like to extend the success of the $U(1)$ gauging from the linearized approximation to the full nonlinear theory. Before we can proceed with the gauging, however, we must first check whether $U(1)_\Sigma$ extends to a global symmetry of the nonlinear theory.

In the linearized theory before gauging, the parameter $\alpha(\mathbf{x})$ of the infinitesimal $U(1)_\Sigma$ transformation was independent of time, and consequently we interpreted $U(1)_\Sigma$ as a global symmetry. In the nonlinear theory, the linearized transformation of n_i in (2.26) simply becomes[6]

$$\delta_\alpha N_i = N\nabla_i\alpha. \tag{2.33}$$

However, the condition (2.27) expressing the time independence of α is not covariant under $\mathrm{Diff}(M,\mathcal{F})$. The correct covariant generalization takes the following modified form,

$$\dot\alpha - N^i\nabla_i\alpha = 0. \tag{2.34}$$

This condition of vanishing covariant time derivative of α is indeed invariant under the symmetry group $\mathrm{Diff}(M,\mathcal{F})$.

In the full nonlinear theory, the gauge field will transform as a spatial scalar and a time vector under $\mathrm{Diff}(M,\mathcal{F})$,

$$\delta A = \dot f A + f\dot A + \xi^i\partial_i A, \tag{2.35}$$

and the gauge transformation of the gauge field becomes

$$\delta_\alpha A = \dot\alpha - N^i\nabla_i\alpha. \tag{2.36}$$

It follows from (2.36) and (1.18) that the scaling dimensions of α and A are given by

$$[\alpha] = z-2, \qquad [A] = 2z-2. \tag{2.37}$$

In the process of evaluating the variation of the action under the general α transformation, we will encounter a particular combination of the second spatial derivatives of $\dot g_{ij}$, which can be expressed as the trace of the time derivative of the Ricci tensor:

$$g^{ij}\dot R_{ij} = \left(g^{ik}g^{j\ell} - g^{ij}g^{k\ell}\right)\nabla_i\nabla_j\dot g_{k\ell}. \tag{2.38}$$

This formula also implies

$$(\sqrt{g}R)^{\cdot} = -\sqrt{g}\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)\dot g_{ij} + \sqrt{g}\left(g^{ik}g^{j\ell} - g^{ij}g^{k\ell}\right)\nabla_i\nabla_j\dot g_{k\ell}. \tag{2.39}$$

It is now straightforward to see that there is an obstruction against extending the global symmetry to the full nonlinear theory, at least in dimensions greater than $2+1$. Indeed, for the variation of the action we get

$$\begin{aligned}\delta_\alpha S &= -\frac{1}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\left(\dot g_{ij} - \nabla_i N_j - \nabla_j N_i\right)\left(g^{ik}g^{j\ell} - g^{ij}g^{k\ell}\right)\left(\nabla_k\nabla_\ell\alpha + \nabla_\ell\nabla_k\alpha\right)\\ &= -\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,\alpha\left(g^{ij}\dot R_{ij} - 2G^{ijk\ell}\nabla_k\nabla_\ell\nabla_i N_j\right),\end{aligned} \tag{2.40}$$

[6]The explicit multiplicative factor of N in (2.33) is explained by the requirement of matching the tensorial properties of both sides in (2.33) under $\mathrm{Diff}(M,\mathcal{F})$. Thus, it is in fact $A_i \equiv N_i/N$ that transforms as the spatial projection of a spacetime vector field under $\mathrm{Diff}(M,\mathcal{F})$ and as the spatial part of a gauge potential under $U(1)$, with $\delta A_i = \nabla_i\alpha$.

where in the second line we have integrated by parts twice, dropped the corresponding spatial derivative terms, and used (2.38). The last, triple-derivative term in (2.40) can be simplified using

$$G^{ijk\ell}\nabla_k\nabla_\ell\nabla_i N_j = -\nabla^j[\nabla_j,\nabla_k]N^k + \frac{1}{2}[\nabla_k,\nabla_j]\nabla^j N^k = \nabla_j(R^{jk}N_k),$$

which yields

$$\delta_\alpha S = -\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,\alpha\left\{g^{ij}\dot{R}_{ij} - 2\nabla_j(R^{jk}N_k)\right\}. \tag{2.41}$$

Finally, after using the contracted Bianchi identity in the second term, integrating by parts in both terms, using (2.39) and dropping the total derivatives, we obtain

$$\begin{aligned}\delta_\alpha S &= \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\left(\dot{\alpha} - N^i\nabla_i\alpha\right)R \\ &\quad - \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,\alpha\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right).\end{aligned} \tag{2.42}$$

The first line in (2.42) vanishes for the covariantly time-independent α by virtue of (2.34), but the second line represents an obstruction against the invariance of S, even when α is restricted to be covariantly time-independent.

Another way of seeing the origin of the nonlinear obstruction against the $U(1)_\Sigma$ invariance of the minimal theory is the following. On the components N_i of the shift vector, the $U(1)_\Sigma$ transformations act as gauge transformations on the components of an Abelian connection. Define

$$F_{ij} = \partial_i N_j - \partial_j N_i. \tag{2.43}$$

Clearly, this is the field strength of N_i interpreted as a connection associated with $U(1)_\Sigma$. Thus, F_{ij} are invariant under $U(1)_\Sigma$, and transform as components of a two-form under $\mathrm{Diff}(\Sigma)$. However, F_{ij} do *not* transform as two-form components under time-dependent spatial diffeomorphisms.

In this new notation, the action with $\lambda = 1$ can be rewritten as

$$\begin{aligned}S &= \frac{1}{2\kappa^2}\int dt\, d^D\mathbf{x}\,\frac{\sqrt{g}}{N}\left\{\dot{g}_{ij}\left(g^{ik}g^{j\ell} - g^{ij}g^{k\ell}\right)\dot{g}_{k\ell} - F^{ij}F_{ij}\right. \\ &\quad \left. - 4R^{ij}N_iN_j + 4N^i\left(\nabla^j\dot{g}_{ij} - g^{jk}\nabla_i\dot{g}_{jk}\right)\right\} - \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,N\mathcal{V}.\end{aligned} \tag{2.44}$$

While the first two terms in (2.44) and the potential term are manifestly invariant under $U(1)_\Sigma$, the terms with explicit factors of N_i – which are required by the requirement of $\mathrm{Diff}(M,\mathcal{F})$ invariance – are not, and their variation reproduces (2.42). Intuitively, the obstruction can be related to the fact that N_i plays a dual role in the theory. First, as we have seen in (1.12), N_i is the gauge field of the time-dependent spatial diffeomorphisms along Σ. The second role is asked of N_i/N in our attempt to extend the gauge symmetry, and make N_i/N transform as the spatial components of a $U(1)$ gauge field.

2.3 Gauging the $U(1)_\Sigma$ Symmetry: First Examples

Our intention is to gauge the action of the global $U(1)_\Sigma$ in the minimal theory with $\lambda = 1$. As we have seen in Section 2.4.1, such gauging is possible in the linearized approximation, and it has the desired effect of eliminating the scalar polarization of the graviton. However, in Section 2.2.2 we found an obstruction which prevents $U(1)_\Sigma$ from being a global symmetry of the minimal theory at the nonlinear level, and therefore precludes its straightforward gauging. More precisely, we found that the variation (2.42) of the action under an infinitesimal $U(1)$ gauge transformation $\alpha(t, \mathbf{x})$ consists of two parts,

$$\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\, \sqrt{g}\left(\dot{\alpha} - N^i\nabla_i\alpha\right) R \tag{2.45}$$

and

$$-\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\, \sqrt{g}\,\alpha\left(R^{ij} - \frac{1}{2}Rg^{ij}\right)\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right). \tag{2.46}$$

If $U(1)_\Sigma$ were a global symmetry, the first step of the Noether procedure would be to add a Noether coupling term to the action,

$$S_A = -\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\, \sqrt{g}\, AR. \tag{2.47}$$

The $U(1)$ variation (2.36) of A in (2.47) indeed cancels (2.45). However, at this stage the Noether procedure breaks down, and (2.46) represents the obstruction against gauge invariance.

In order to allow a straightforward gauging on $U(1)_\Sigma$, we will have to find a mechanism which eliminates this obstruction. Before proceeding with that, we first examine the geometric origin of $U(1)_\Sigma$ as a natural gauge symmetry, and consider several illustrative cases in which the gauging can be completed because the obstruction automatically vanishes. These include a generally covariant nonrelativistic gravity theory in $2+1$ dimensions, and an interacting Abelian theory of gravity in general dimensions.

2.3.1 Geometric interpretation of the $U(1)$ symmetry

The transformation rules (1.11) of $\mathrm{Diff}(M, \mathcal{F})$ on the gravity fields can be systematically derived (1; 2) from the action of relativistic diffeomorphisms $\mathrm{Diff}(M)$ on the spacetime metric $g_{\mu\nu}$, as the leading order in the nonrelativistic $1/c$ expansion.

It was already noted in (2) that the gauge field $A(t, \mathbf{x})$ and the $U(1)$ symmetry – which we introduced in a rather *ad hoc* fashion in Sections 2.2 and 2.3 – both acquire a natural geometric interpretation in the framework of the $1/c$ expansion: It turns out that A is simply the subleading term in the $1/c$ expansion of the relativistic lapse function, and $U(1)$ corresponds to the subleading, linearized spacetime-dependent time reparametrization

symmetry of the relativistic theory. In this section, now make these observations more precise.

In Section 2.1.3, we reviewed some of the difficulties faced in the attempts to promote the lapse function to a spacetime field,

$$N(t) \to N(t, \mathbf{x}). \tag{2.48}$$

In physical terms, the attempts to restore N as a spacetime field can be motivated by the desire to restore the information carried by the Newton potential in general relativity (for generic gauge choices). The geometric understanding of the gauge field A and the gauge symmetry $U(1)$ shows how the generally covariant theory with $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$ symmetry restores the Newton potential and avoid the difficulties of the nonprojectable theory: Instead of promoting $N(t)$ into a spacetime field as in (2.48), we keep $N(t)$ as the leading term of the lapse, and introduce the *subleading* term $A(t, \mathbf{x})$ in the $1/c$ expansion, at the order in which the Newton potential enters in the nonrelativistic approximation to general relativity:

$$N(t) \to N(t) - \frac{1}{c^2} A(t, \mathbf{x}). \tag{2.49}$$

Thus, it is only the subleading part of lapse that becomes a spacetime field.

It is useful to stress that the $1/c$ formalism of this section is just a trick, whose sole purpose is to provide a geometric explanation of the action of $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$, by taking the formal $c \to \infty$ limit of the relativistic $\mathrm{Diff}(M)$ symmetry. The "speed of light" c is a formal expansion parameter, and should not be confused with the physical speed of light which will be generated in our theory at large distances as a result of the relevant deformations.

The gauge field A and the Newton potential

In order to reproduce the field content of the theory, and the transformation rules under the gauge symmetries, we start with a relativistic spacetime metric, and expand it in the powers of $1/c$ as follows:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \dfrac{N_i N^i}{c^2} + \dfrac{2NA}{c^2} + \dots, & \dfrac{N_i}{c} + \dots \\ \\ \dfrac{N_i}{c} + \dots, & g_{ij} + \dots \end{pmatrix} \tag{2.50}$$

This step is complemented by a similar expansion of the relativistic spacetime diffeomorphisms with generators ζ^μ,

$$\zeta^0 = cf(\mathbf{x}, t) - \frac{1}{c} \frac{\alpha(\mathbf{x}, t)}{N} + \dots, \qquad \zeta^i = \xi^i(\mathbf{x}, t) + \dots. \tag{2.51}$$

In both cases, "..." refer to terms suppressed by $1/c^2$ compared to those displayed. In the transformation rules, the derivative with respect to the relativistic time coordinate is written as $\partial/\partial x^0 = (1/c)\partial/\partial t$; it is then the nonrelativistic time t which is held fixed as $c \to \infty$.

Taking the $c \to \infty$ limit first requires $\partial_i f = 0$, which means that the infinitesimal time reparametrizations $f(t)$ are restricted to be only functions of time. In addition, in accord with (2.49), we insist that N be only a function of time. The transformation rules (1.11) under the foliation-preserving diffeomorphisms then follow from the $c \to \infty$ limit of the spacetime diffeomorphism symmetry. In addition, we get

$$\begin{aligned} \delta_\alpha \left(\frac{N_i}{N}\right) &= \nabla_i \alpha, \\ \delta_\alpha A &= \dot{\alpha} - N^i \nabla_i \alpha, \end{aligned} \tag{2.52}$$

with all other fields invariant under δ_α. We see that the $U(1)$ gauge symmetry of interest is geometrically interpreted as the subleading part of time reparametrizations in the nonrelativistic limit of spacetime diffeomorphisms in general relativity.

This embedding of the gauge field A into the geometric framework of the $1/c$ expansion sheds additional light on the physical role of A in the theory. Recall that in the leading order of the Newtonian approximation to general relativity, the g_{00} component of the spacetime metric (in the natural gauge adapted to this approximation) is related to the Newton potential Φ via

$$g_{00} = -\left(1 + \frac{1}{c^2} 2\Phi + \dots\right). \tag{2.53}$$

Comparing this to (2.50), we find that our gauge field A effectively plays the role of the Newton potential,

$$A = -\Phi + \dots. \tag{2.54}$$

As we will see in Section (2.5.1), this relationship is corrected by higher order terms already at the next order in the post-Newtonian approximation.

Extending the $1/c$ expansion

We can also keep the subleading terms in the spatial metric, replacing

$$g_{ij} \to g_{ij} - \frac{1}{c^2} \frac{A_{ij}(\mathbf{x}, t)}{N} + \dots \tag{2.55}$$

in (2.50). Following the rules of transformation for the spatial metric to one higher order in $1/c^2$ than before, it turns out that A_{ij} also transforms under α,

$$\delta_\alpha A_{ij} = \alpha\, \dot{g}_{ij} + N_i \nabla_j \alpha + N_j \nabla_i \alpha. \tag{2.56}$$

This transformation property of A_{ij} is just what is needed to remedy the noninvariance of our action under the local $U(1)$ transformations, by introducing a new coupling

$$\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,A_{ij}\left(R^{ij}-\frac{1}{2}Rg^{ij}\right). \tag{2.57}$$

Interestingly, this term is also "accidentally" invariant under another Abelian gauge symmetry, which acts on the fields via

$$\delta A_{ij}=\nabla_i\varepsilon_j+\nabla_j\varepsilon_i. \tag{2.58}$$

The variation of all the other fields under the ε_i symmetry is zero. The total action is invariant under (2.58): The only term in the action which depends on A_{ij} is (2.57), and its invariance under (2.58) is a consequence of the Bianchi identity.

This new gauge symmetry (2.58) also has a natural geometrical origin: In the process of decomposing the relativistic symmetries in the powers of $1/c$, we could have also kept the subleading terms in spatial diffeomorphisms,

$$\xi^i=\zeta^i-\frac{1}{c^2}\frac{\varepsilon^i(t,\mathbf{x})}{N}+\dots. \tag{2.59}$$

The $c\to\infty$ limit of the relativistic diffeomorphisms then implies precisely the transformation rules (2.58).

It thus appears that by extending the gravity multiplet to include both the Newton-potential $A(t,\mathbf{x})$ and the field $A_{ij}(t,\mathbf{x})$, we succeeded in finding a formulation of gravity in which the $U(1)\ltimes\mathrm{Diff}(M,\mathcal{F})$ symmetry of "nonrelativistic general covariance" is realized in the full nonlinear theory without obstructions. In addition, we have also seen that this extended gravity multiplet has a clear and natural geometric interpretation in the context of the $1/c$ expansion. These features make this extended theory potentially attractive, but a closer inspection shows that the number of propagating degrees of freedom has been once again reduced to zero – the theory turns out to be effectively topological. Consequently, the spectrum of bulk gravitons in the low-energy limit will not match the prediction of low-energy general relativity.

In order to see this, and to count reliably the number of degrees of freedom, we turn once more to the Hamiltonian analysis (26). Because the subleading fields A and A_{ij} that we kept in the $1/c$ expansion appear in the action without time derivatives, they will all lead to constraints in the Hamiltonian formulation of the theory. The full phase space is parametrized by fields N_i, A, A_{ij}, g_{ij} and their canonical momenta P^i, P_A, P^{ij} and π^{ij}, implying that

$$\dim\mathcal{P}=2(D+1)^2 \tag{2.60}$$

per spatial point.[7] The vanishing of the momenta conjugate to A, A_{ij} and N_i represents $(D+2)(D+1)/2$ primary constraints. The condition that the primary constraints be

[7]There is also the canonical pair consisting of $N(t)$ and its conjugate momentum $P_0(t)$, which only yields an integral constraint and can be dropped for the purpose of counting the local degrees of freedom.

preserved in time yields secondary constraints: Insisting on $\dot{P}_A = 0$ requires the vanishing of R, and similarly $\dot{P}^{ij} = 0$ requires the vanishing of $R^{ij} - \frac{1}{2}Rg^{ij}$. In addition, as in general relativity, $\dot{P}^i = 0$ requires $\mathcal{H}_i = 0$.

Naively, there are thus $D(D+3)/2+1$ secondary constraints $\mathcal{H}_i$, R and $R^{ij} - \frac{1}{2}Rg^{ij}$. However, these are not all independent: $R^{ij} - \frac{1}{2}Rg^{ij}$ satisfies the Bianchi identity, and R is proportional to the trace of $R^{ij} - \frac{1}{2}Rg^{ij}$, leaving $D(D+1)/2$ independent secondary constraints.

All the primary and secondary constraints are first-class: Their commutators vanish on the constraint surface. As a result, we have the total number

$$\mathcal{C}_1 = (D+2)(D+1)/2 + D(D+1)/2 = (D+1)^2 \tag{2.61}$$

of first-class constraints. This implies, together with (2.60) and invoking (2.1), that the total number of local propagating degrees of freedom is

$$\mathcal{N} = \frac{1}{2}(\dim \mathcal{P} - 2\mathcal{C}_1) = 0. \tag{2.62}$$

The theory is effectively topological.

Since our primary interest in this chapter is to find a theory whose spectrum of gravitons matches general relativity at long distances, we will not pursue the extended theory in which the gravity multiplet contains the A_{ij} fields, and set $A_{ij} = 0$ from now on.

2.3.2 Generally covariant nonrelativistic gravity in $2+1$ dimensions

In $2+1$ dimensions, the Einstein tensor $R_{ij} - \frac{1}{2}Rg_{ij}$ of the spatial metric vanishes identically, which means that (2.46) is zero, and there is no obstruction against gauging the $U(1)_\Sigma$ symmetry in the full nonlinear theory. The Noether procedure terminates after one step and leads to the following action,

$$S = \frac{2}{\kappa^2}\int dt\, d^2\mathbf{x}\, \sqrt{g}\left\{N\left(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}\right) - AR\right\}. \tag{2.63}$$

This action exhibits the $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$ gauge symmetry of nonrelativistic general covariance in $2+1$ dimensions, for any choice of $\mathcal{V}$.

The extended gauge symmetry eliminates the scalar degree of freedom of the graviton. To see that, it is convenient to select $A = 0$ as the gauge choice. In this gauge, the equations of motion are the same as in the minimal model with $\lambda = 1$, with the addition of the Gauss constraint

$$R = 0. \tag{2.64}$$

It is this additional constraint which eliminates the scalar degree of freedom of the minimal theory. Moreover, since in $2+1$ dimensions the scalar graviton was the only local degree of

freedom, the generally covariant theory with the extended $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$ symmetry has no local propagating graviton polarizations. In this sense, it is akin to several other, much studied models of gravity in $2+1$ dimensions, such as standard general relativity or chiral gravity (43; 44).

Because of the absence of physical fluctuations, the geometry of classical solutions in this theory can be expected to be quite rigid, just as in the case of its relativistic cousins in $2+1$ dimensions. In particular, the Gauss constraint (2.64) forces the two-dimensional spatial slices to be flat. It is natural to look for deformations of this theory which would at least replace the Gauss constraint with the more general condition of constant spatial curvature, but there appear to be no consistent deformations that could modify the Gauss constraint to

$$R = 2\Omega, \tag{2.65}$$

with Ω a new coupling constant. However, once we learn in Section 2.4.2 how to gauge the $U(1)_\Sigma$ symmetry in the general case of $D+1$ dimensions, we will also find a mechanism for turning on this new coupling Ω.

2.3.3 Self-interacting Abelian gravity

Another way to eliminate the obstruction against gauging $U(1)_\Sigma$ is to linearize the gauge symmetries of the minimal theory. The fields in the theory with linearized gauge symmetries are h_{ij}, n_i and n. The gauge transformations $\xi_i(t, \mathbf{x})$ and $f(t)$ act via

$$\begin{aligned} \delta h_{ij} &= \partial_i \xi_j + \partial_j \xi_i, \\ \delta n_i &= \dot{\xi}_i, \\ \delta n &= \dot{f}, \end{aligned} \tag{2.66}$$

and represent the Abelian contraction of $\mathrm{Diff}(M, \mathcal{F})$.

The kinetic term (with $\lambda = 1$) takes the form

$$S_K = \frac{1}{2} \int dt\, d^D\mathbf{x} \left(\dot{h}_{ij} - \partial_i n_j - \partial_j n_i \right) \left(\delta_{ik}\delta_{j\ell} - \delta_{ij}\delta_{k\ell} \right) \left(\dot{h}_{k\ell} - \partial_k n_\ell - \partial_\ell n_k \right). \tag{2.67}$$

In this theory, the obstruction (2.46) against gauging vanishes identically, as was already established in our analysis of the linearized approximation to the minimal theory in Section 2.2.1.

At first glance, it would thus seem that keeping only the linearized gauge symmetries would reduce the model to the nonintreracting Gaussian theory studied in Section 2.2.1, but in fact it is not so. Even though the kinetic term takes the Gaussian form (2.67), the potential term need not be Gaussian.

Suitable terms in $\mathcal{V}$ are integrals of local operators, which are either invariant under (2.66), or invariant up to a total spatial derivative. The building blocks that can be used

to construct such operators are the linearized curvature tensor of the spatial metric, and its derivatives. We will denote the linearized Riemann tensor by

$$L_{ijk\ell} = \frac{1}{2}\left(\partial_j\partial_k h_{i\ell} - \partial_j\partial_\ell h_{ik} - \partial_i\partial_k h_{j\ell} + \partial_i\partial_\ell h_{jk}\right), \tag{2.68}$$

and similarly the linearized Ricci tensor by $L_{ij} \equiv L_{ikjk}$ and the Ricci scalar by $L \equiv L_{ii}$. Clearly, there is an infinite hierarchy of sutable operators, which reduces to a finite number if we limit the number of spatial derivatives by $2z$. For interesting values of $z > 1$, the general $\mathcal{V}$ built from such terms will not be purely Gaussian, leading to a self-interacting theory.

Thus, in the context of gravity with anisotropic scaling, linearizing the gauge symmetries does not necessarily make the theory noninteracting – we find a novel interacting theory of Abelian gravity instead. Curiously, a similar phenomenon has been observed in the case of general relativity 25 years ago by Wald (45), where it was shown that by taking the action to contain higher powers of the linearized curvature, one can construct a self-interacting theory of spin-two fields in flat spacetime with linearized spacetime diffeomorphisms as gauge symmetries. In the relativistic case studied in (45), this construction leads inevitably to higher time derivatives in the action, and therefore problems with ghosts in perturbation theory. In contrast, our nonrelativistic models of self-interacting Abelian gravity do not suffer from this problem – their self-interaction results from higher than quadratic terms in $\mathcal{V}$, with the kinetic term taking the Gaussian form (2.67). For suitable choices of the couplings in $\mathcal{V}$, the spectrum is free of both ghosts and tachyons.

For an arbitrary $\mathcal{V}$, the gauging of $U(1)_\Sigma$ is now accomplished by adding a new Gaussian term to the action,

$$-2\int dt\, d^D\mathbf{x}\, AL. \tag{2.69}$$

The theory is now gauge invariant under the linearized action of $U(1)$,

$$\delta A = \dot{\alpha}, \qquad \delta n_i = \partial_i\alpha. \tag{2.70}$$

Arguments identical to those in Section 2.2.1 show that the theory contains only the tensor graviton modes, eliminating the scalar.

At long distances, the dominant terms in $\mathcal{V}$ are those with the lowest number of spatial derivatives. Since the only suitable operator with just two derivatives is the quadratic part of the spatial Einstein-Hilbert term,

$$\int dt\, d^D\mathbf{x}\, \left(h_{ij}L_{ij} - \frac{1}{2}h_{ii}L\right), \tag{2.71}$$

the theory becomes automatically Gaussian at long distances, and approaches a free infrared fixed point with $z = 1$. This behavior can be avoided if we insist that all operators $\mathcal{V}$ are integrals of gauge-invariant operators: Since (2.71) is only invariant up to a total derivative,

it does not belong to this class. In these restricted theories, the infrared behavior will be controlled by Gaussian terms with $z \geq 2$. In fact, such self-interacting Abelian gravity theories, approaching free-field Lifshitz-type fixed points $z \geq 1$, have been encountered in the infrared regime of a family of condensed matter models on the rigid fcc lattice in (46).

While such self-interacting Abelian gravity models might be useful for describing new universality classes of bose liquids in condensed matter theory, they do not appear phenomenologically viable as candidates for describing the gravitational phenomena in the observed universe.

2.4 General Covariance at a Lifshitz Point

So far, we focused on the special cases in which the obstruction to the gauging of $U(1)_\Sigma$ is absent. However, none of the resulting theories of gravity with extended gauge symmetry discussed in Section 2.3 appear phenomenologically interesting as models of gravity in $3+1$ dimensions.

Here we change our perspective, and present a robust mechanism which allows $U(1)_\Sigma$ to be gauged in the general spacetime dimension $D+1$. This will lead to a theory of gravity with nonrelativistic general covariance which reproduces many properties of general relativity at long distances.

2.4.1 Repairing the global $U(1)_\Sigma$ symmetry

In Section 2.2.2, we found an obstruction that prevents the $U(1)_\Sigma$ from being a global symmetry of the full nonlinear theory for $D > 2$. Leaving aside the possibility that this obstruction could be cancelled by quantum effects (perhaps by a mechanism similar to (47; 48)), we look for a way to repair the $U(1)_\Sigma$ symmetry at the classical level.

The Newton prepotential

In order to eliminate the obstruction, we introduce an auxiliary scalar field ν, which transforms under $U(1)_\Sigma$ as

$$\delta_\alpha \nu = \alpha. \tag{2.72}$$

We will refer to this field as the "Newton prepotential." The scaling dimension of ν is the same as the dimension of α,

$$[\nu] = z - 2. \tag{2.73}$$

We can now repair the $U(1)_\Sigma$ symmetry by adding a new term to the action,

$$\begin{aligned} S_\nu &= \frac{2}{\kappa^2} \int dt\, d^D\mathbf{x}\, \sqrt{g}\, \nu \left(R^{ij} - \frac{1}{2} R g^{ij} \right) (\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i) \\ &\quad + \frac{2}{\kappa^2} \int dt\, d^D\mathbf{x}\, \sqrt{g}\, N\, \nu \left(R^{ij} - \frac{1}{2} R g^{ij} \right) \nabla_i \nabla_j \nu. \end{aligned} \tag{2.74}$$

The variation of ν in the linear term compensates for the noninvariance of the original action of the minimal theory. The term quadratic in ν is in turn required to cancel the variation of N_i in the term linear in ν.

Relevant deformations

We can check by linearizing around the flat background that the number of propagating degrees of freedom has not changed by the introduction of the Newton prepotential terms in the action. It is rather unconventional that in the expansion around the flat spacetime, the new field ν enters the action at the cubic order in small fields, *i.e.*, its presence does not affect the propagator. This issue is eliminated by noticing that a new term, of lower dimension and also invariant under the global $U(1)_\Sigma$ symmetry, can be added to the action:

$$S_\Omega = \frac{2\Omega}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,\nu g^{ij}\left(\dot{g}_{ij} - \nabla_i N_j - \nabla_j N_i\right) + \frac{2\Omega}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,N\,\nu\Delta\nu. \tag{2.75}$$

Here Ω is a coupling constant of dimension $[\Omega] = 2$. With the addition of this relevant term, the Newton prepotential enters the linearized theory, at the quadratic order in fields around the flat spacetime.

The Newton prepotential enters the action with global $U(1)_\Sigma$ symmetry quadratically, and can be integrated out by solving its equation of motion,

$$\Theta^{ij}\nabla_i\nabla_j\nu + \Theta^{ij}K_{ij} = 0, \tag{2.76}$$

where we have introduced

$$\Theta^{ij} = R^{ij} - \frac{1}{2}Rg^{ij} + \Omega g^{ij}. \tag{2.77}$$

Integrating out ν would result in a nonlocal action, because (2.76) is solved by

$$\nu_0(\mathbf{x},t) = -\int d^D\mathbf{x}'\frac{1}{\Theta^{ij}\nabla_i\nabla_j}(\mathbf{x},\mathbf{x}')(\Theta^{k\ell}K_{k\ell})(\mathbf{x}'). \tag{2.78}$$

Here we have assumed that the operator $\Theta^{ij}\nabla_i\nabla_j$ is invertible, and denoted its Green's function by $(\Theta^{ij}\nabla_i\nabla_j)^{-1}(\mathbf{x},\mathbf{x}')$. We will not try to determine the exact conditions under which this assumption is true. However, for example near the flat spacetime geometry (2.19), we have $\Theta^{ij}\nabla_i\nabla_j = \Omega\Delta + \mathcal{O}(h_{ij})$, where $\Delta = \partial^2$ is the flat-space Laplacian. This operator is invertible at least in perturbation theory, as long as we keep Ω nonzero.[8]

[8] Another interesting example, which will be important below when we gauge the $U(1)_\Sigma$ symmetry, is the case in which $\hat{g}_{ij}$ is the metric of a maximally symmetric space satisfying $\hat{R} = 2\Omega$. In this reference background, we have $\Theta^{ij}\nabla_i\nabla_j = 2\Omega\hat{\Delta}/D$ (with $\hat{\Delta}$ the Laplace operator of $\hat{g}_{ij}$), which is also invertible for $\Omega \neq 0$.

Note that the Green's function is still a local function in t. Note also that the expression (2.78) for ν_0 has the right form in order for the action to be $U(1)_\Sigma$ invariant after ν has been integrated out. In particular,

$$\begin{aligned}\delta_\alpha \nu_0(\mathbf{x},t) &= -\frac{1}{2}\int d^D\mathbf{x}' \frac{1}{\Theta^{ij}\nabla_i\nabla_j}(\mathbf{x},\mathbf{x}')\Theta^{k\ell}(-\nabla_k\nabla_\ell\alpha - \nabla_\ell\nabla_k\alpha)(\mathbf{x}') \\ &= \int d^D\mathbf{x}' \frac{1}{\Theta^{ij}\nabla_i\nabla_j}(\mathbf{x},\mathbf{x}')(\Theta^{k\ell}\nabla_k\nabla_\ell\alpha)(\mathbf{x}') = \alpha(\mathbf{x},t). \qquad (2.79)\end{aligned}$$

The nonlocality of the action obtained by integrating out ν is relatively mild: In particular, this nonlocality is purely spatial, along the leaves of the spacetime foliation $\mathcal{F}$. Such nonlocalities are quite common in rather conventional condensed matter systems. Nevertheless, in the rest of the chapter, we will keep the action manifestly local, by keeping the Newton prepotential ν as an independent field instead of integrating it out.

Linearized theory with global $U(1)_\Sigma$ around flat spacetime

First we will check that our repair of the global $U(1)_\Sigma$ symmetry has not changed the count of the number of degrees of freedom. Even with the Ω coupling turned on, the flat spacetime geometry

$$g_{ij} = \delta_{ij}, \qquad N_i = 0, \qquad N = 1, \qquad \nu = 0 \qquad (2.80)$$

is still a classical solution of the theory (with $\Lambda = 0$), and we can expand around it.

The ν equation of motion is

$$2\partial^2(\nu - C) + \dot{h} = 0. \qquad (2.81)$$

The momentum constraints give

$$\partial^2(\dot{w}_i - u_i) = 0 \qquad (2.82)$$

and

$$2\Omega\partial^2\nu + \frac{D-1}{D}\left\{(\partial^2)^2\dot{B} - \partial^2\dot{h}\right\} = 0. \qquad (2.83)$$

Setting $B = 0$ and $w_i = 0$ fixes gauge completely, and implies (with appropriate boundary conditions at infinity) that $u_i = 0$ and

$$2\Omega\nu = \frac{D-1}{D}\dot{h}. \qquad (2.84)$$

In this gauge, the remaining equations of motion are

$$-\ddot{s}_{ij} + \frac{D-1}{D}\delta_{ij}\ddot{h} + 2(\partial_i\partial_j - \delta_{ij}\partial^2)\dot{C} - 2\Omega\delta_{ij}\dot{\nu} - \frac{\delta\mathcal{V}_2}{\delta g_{ij}} = 0, \qquad (2.85)$$

where $\mathcal{V}_2$ denotes the quadratic part of $\mathcal{V}$ in the linearized theory. Using (2.84), we see that the $\dot{\nu}$ term cancels the $\ddot{h}$ term exactly, allowing one to determine h as a function of $\dot{C}$, and substitute back into (2.81). The resulting equation determines the dispersion relation of the scalar polarization of the graviton. For example, when we set the cosmological constant $\Lambda = 0$, the potential term will be dominated at long distances by $\mathcal{V} = -R$, and we get

$$\frac{\delta \mathcal{V}_2}{\delta g_{ij}} = -\partial^2 s_{ij} + \frac{D-2}{D}(\partial_i\partial_j - \delta_{ij}\partial^2)(\partial^2 B - h). \tag{2.86}$$

The metric equation of motion then implies that

$$h = \frac{2D}{2-D}\dot{C}, \tag{2.87}$$

and the ν equation of motion gives

$$\frac{D-1}{\Omega}\partial^2\ddot{C} + D\ddot{C} + (D-2)\partial^2 C = 0. \tag{2.88}$$

The spectrum thus contains the transverse, traceless polarizations s_{ij} with dispersion

$$\omega^2 = \mathbf{k}^2, \tag{2.89}$$

plus a scalar graviton (described in this gauge by C) which exhibits the dispersion relation implied by (2.88),

$$\omega^2 = -\frac{(D-2)\mathbf{k}^2}{D\left\{1 - \frac{D-1}{D\Omega}\mathbf{k}^2\right\}}. \tag{2.90}$$

Note that the scalar mode is inevitably tachyonic at low energies. This is implied by the choice of sign in $\mathcal{V}$, determined from the requirement that the tensor polarizations have the correct-sign dispersion relation (2.89). This tachyonic nature of the scalar mode is not a cause for any concern, because the model discussed here represents only an intermediate stage of our construction – our intention is to gauge the $U(1)_\Sigma$ symmetry, which will turn the scalar graviton into a gauge artifact.

Note also that taking the regulating dimensionful coupling Ω to zero reduces the scalar dispersion relation (2.90) correctly to the singular dispersion $\omega^2 = 0$, observed at $\lambda = 1$ in the minimal theory in (1) and in (2.25).

2.4.2 Gauging the $U(1)_\Sigma$ symmetry

Having repaired the global $U(1)_\Sigma$ symmetry, we can now gauge it. The Noether method closes after just one step; adding

$$S_{A,\Omega} = -\frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\,A\,(R - 2\Omega) \tag{2.91}$$

to the action makes the theory gauge invariant under the $U(1)$ symmetry. This procedure results in the following action of the generally covariant theory of gravity with anisotropic scaling,

$$S = \frac{2}{\kappa^2}\int dt\, d^D\mathbf{x}\,\sqrt{g}\left\{N\left[K_{ij}K^{ij} - K^2 - \mathcal{V} + \nu\,\Theta^{ij}\left(2K_{ij} + \nabla_i\nabla_j\nu\right)\right] - A\left(R - 2\Omega\right)\right\}, \tag{2.92}$$

with Θ^{ij} a short-hand notation for

$$\Theta^{ij} = R^{ij} - \frac{1}{2}g^{ij}R + \Omega g^{ij}. \tag{2.93}$$

Note that in the theory with the Newton prepotential ν, the issue about the possibility of adding the spatial cosmological constant Ω, raised in the generally covariant theory in $2+1$ dimensions at the end of Section 2.3.2, has been resolved by the introduction of the Newton prepotential.

Note also that in addition to the newly introduced gauge field A, the theory contains a composite

$$a = \dot{\nu} - N^i\nabla_i\nu + \frac{N}{2}\nabla^i\nu\nabla_i\nu \tag{2.94}$$

which also transforms as a gauge field under the $U(1)$ gauge transformations,

$$\delta_\alpha a = \dot{\alpha} - N^i\nabla_i\alpha. \tag{2.95}$$

Moreover, both A and the composite gauge field a share the same transformation properties under the rest of the gauge group,

$$\delta a = \xi^i\partial_i a + \dot{f}a + f\dot{a}. \tag{2.96}$$

As a result, the gauged action stays gauge invariant if we replace

$$A \to (1-\gamma)A + \gamma a, \tag{2.97}$$

with γ a real coefficient.

In fact, the composite field a already made its appearance in the theory with the global $U(1)_\Sigma$ symmetry presented in Section 2.4.1: Up to a total derivative, the relevant term (2.75) can be rewritten as

$$S_\Omega = -\frac{4\Omega}{\kappa^2}\int \sqrt{g}\,a. \tag{2.98}$$

Hence, the substitution (2.97) in the $\int dt\, d^D\mathbf{x}\,\sqrt{g}A$ term in (2.91) will just shift the effective value of Ω. Similarly, substituting (2.97) in the AR term in the Noether coupling (2.91) effectively shifts the coefficients in (2.74).

Hamiltonian formulation

The structure of gauge symmetries can be verified by analyzing the algebra of Hamiltonian constraints of the theory. In addition, this analysis will allow us to obtain the precise count of the number of propagating degrees of freedom, using formula (2.1). This approach to the count of the degrees of freedom is usually more accurate and more reliable than our previous analysis of the linearized spectrum around a fixed solution, for at least two reasons. First, it is less background-dependent, because it sidesteps the need to linearize the theory around a fixed solution. Secondly, because it is valid for the full nonlinear theory, it excludes the possible artifacts of the linearized approximation.

We will set $\kappa = 1$ to eliminate additional clutter, and denote the canonical momenta conjugate to the spatial metric by Π^{ij}:

$$\Pi^{ij} = \frac{\delta S}{\delta \dot{g}_{ij}} = 2\sqrt{g}\left(K^{ij} - g^{ij}K + \Theta^{ij}\nu\right) = \pi^{ij} + 2\sqrt{g}\,\Theta^{ij}\nu. \tag{2.99}$$

The lower-case π^{ij} are reserved for the standard expressions for the canonical momenta in general relativity,

$$\pi^{ij} \equiv 2\sqrt{g}\left(K^{ij} - g^{ij}K\right). \tag{2.100}$$

The remaining canonical momenta

$$P^i(\mathbf{x},t) \equiv \frac{\delta S}{\delta \dot{N}_i}, \qquad p_\nu(\mathbf{x},t) \equiv \frac{\delta S}{\delta \dot{\nu}}, \qquad P_A(\mathbf{x},t) \equiv \frac{\delta S}{\delta \dot{A}}, \qquad P_0(t) \equiv \frac{\delta S}{\delta \dot{N}} \tag{2.101}$$

all vanish, and represent the primary constraints. The Poisson brackets are

$$\begin{aligned}
[g_{ij}(\mathbf{x},t), \Pi^{k\ell}(\mathbf{y},t)] &= \frac{1}{2}\left(\delta_i^k\delta_j^\ell + \delta_i^\ell\delta_j^k\right)\delta(\mathbf{x}-\mathbf{y}), \\
[N_i(\mathbf{x},t), P^j(\mathbf{y},t)] &= \delta_i^j\delta(\mathbf{x}-\mathbf{y}), \qquad\qquad [N(t), P_0(t)] = 1, \\
[A(\mathbf{x},t), P_A(\mathbf{y},t)] &= \delta(\mathbf{x}-\mathbf{y}), \qquad\qquad [\nu(\mathbf{x},t), p_\nu(\mathbf{y},t)] = \delta(\mathbf{x}-\mathbf{y}),
\end{aligned}$$

and zero otherwise.

In the canonical variables, the Hamiltonian is given by

$$\begin{aligned}
H = \int d^D\mathbf{x}\, \Big\{ N \Big[&\frac{1}{2\sqrt{g}}\left(\Pi^{ij} - 2\sqrt{g}\,\Theta^{ij}\nu\right)\mathcal{G}_{ijk\ell}\left(\Pi^{k\ell} - 2\sqrt{g}\,\Theta^{k\ell}\nu\right) \\
&+ 2\sqrt{g}\,\Theta^{ij}\nabla_i\nu\nabla_j\nu + 2\sqrt{g}\,\mathcal{V}\Big] - 2N_i\nabla_j\Pi^{ij} + 2\sqrt{g}\,A\,(R - 2\Omega)\Big\}
\end{aligned} \tag{2.102}$$

where

$$\mathcal{G}_{ijk\ell} = \frac{1}{2}\left(g_{ik}g_{j\ell} + g_{i\ell}g_{jk}\right) - \frac{1}{D-1}g_{ij}g_{k\ell} \tag{2.103}$$

is the inverse of the De Witt metric $G^{ijk\ell}$ of (1.8) for $\lambda = 1$.

At this stage, the primary constraints are included in the Hamiltonian with the use of Lagrange multipliers $U_i(\mathbf{x},t)$, $\mathcal{U}(\mathbf{x},t)$, $U_A(\mathbf{x},t)$, and $U_0(t)$,

$$H \to \hat{H} = H + \int d^D\mathbf{x} \left(U_i P^i + \mathcal{U} p_\nu + U_A P_A \right) + U_0 P_0. \tag{2.104}$$

The preservation of the primary constraints under the time evolution given by (2.104) requires that the commutators of the primary constraints with $\hat{H}$ vanish, yielding the following set of secondary constraints which are local in space,

$$\begin{aligned}
\mathcal{H}^i &\equiv [\hat{H}, P^i] = -2\nabla_j \Pi^{ij}, &(2.105)\\
\Phi &\equiv [\hat{H}, p_\nu] = -4\sqrt{g}\, N\, \Theta^{ij}\nabla_i\nabla_j\nu + 4\sqrt{g}\, N\, \Theta^{ij}\mathcal{G}_{ijk\ell}\Theta^{k\ell}\nu - 2N\,\Theta^{ij}\mathcal{G}_{ijk\ell}\Pi^{k\ell}, &(2.106)\\
\Psi &\equiv [\hat{H}, P_A] = 2\sqrt{g}(R - 2\Omega), &(2.107)
\end{aligned}$$

and one integral constraint

$$\begin{aligned}
\int d^D\mathbf{x}\, \mathcal{H}_0 \;\equiv\; [\hat{H}, P_0] = \int d^D\mathbf{x} \Big\{ &\frac{1}{2\sqrt{g}} \left(\Pi^{ij} - 2\sqrt{g}\,\Theta^{ij}\nu \right) \mathcal{G}_{ijk\ell} \left(\Pi^{k\ell} - 2\sqrt{g}\,\Theta^{k\ell}\nu \right) \\
&+ 2\sqrt{g}\,\Theta^{ij}\nabla_i\nu\nabla_j\nu + 2\sqrt{g}\,\mathcal{V} \Big\}. \qquad (2.108)
\end{aligned}$$

This integral constraint will not affect the number of local degrees of freedom. To avoid unnecessary clutter, we concentrate on the analysis of the local constraints, returning to (2.108) only at the end of this section.

Next, we need to ensure that the secondary constraints are preserved in time. The momentum constraints $\mathcal{H}^i$ take formally the same form as in general relativity or in the minimal theory of (1; 2). They are indeed preserved in time, albeit in a slightly more intricate way than in the minimal theory or in general relativity. In those cases (see the discussion in Section 4.4 of (1)), the commutator of $\mathcal{H}^i$ with H only gets a contribution from the $N_k\mathcal{H}^k$ terms in H. The rest of the commutator between the density of the Hamiltonian and $\mathcal{H}^i$ adds up to a total derivative, as a consequence of the transformation properties of a scalar density under spatial diffeomorphisms. Here, the argument is more subtle, and the commutator contains additional terms,

$$[\hat{H}, \mathcal{H}^i] = -\nabla_k(N^k\mathcal{H}_i) - (\nabla_i N^k)\mathcal{H}_k - (\nabla^i\nu)\Phi - (\nabla^i A)\Psi. \tag{2.109}$$

However, this expression vanishes on the constraint surface, and no tertiary constraints are produced at this stage.

The time preservation of the secondary constraint Φ requires the vanishing of

$$[\hat{H}, \Phi] \equiv 4\sqrt{g}\, N\, \Theta^{ij} \left(\nabla_i\nabla_j - \mathcal{G}_{ijk\ell}\Theta^{k\ell} \right) \mathcal{U} + [H, \Phi] - U_0 \frac{\Phi}{N} = 0. \tag{2.110}$$

Unlike the conditions for the time preservation of the primary constraints or the $\mathcal{H}^i$, condition (2.110) depends explicitly on one of the Lagrange multipliers, $\mathcal{U}$. Therefore, setting $[\hat{H}, \Phi] = 0$ yields an equation for $\mathcal{U}$, instead of producing an additional, tertiary constraint. Also, because the commutator

$$[p_\nu(\mathbf{x}), \Phi(\mathbf{y})] = 4\sqrt{g}\, N\, \Theta^{ij} \left(\nabla_i \nabla_j - \mathcal{G}_{ijk\ell}\Theta^{k\ell}\right) \delta(\mathbf{x} - \mathbf{y}) \tag{2.111}$$

does not vanish on the constraint surface, p_ν and Φ represent a pair of second-class constraints.

It now remains to check the condition for the preservation of Ψ in time. After a lengthy calculation, we get

$$[\hat{H}, \Psi] = +N\nabla_i \mathcal{H}^i - \Phi - \nabla_i \left(N^i \Psi\right). \tag{2.112}$$

This expression vanishes on the constraint surface. Again, no tertiary constraint is produced, and the process of generating the full list of constraints stops here.

One might be tempted to expect that Ψ is a first-class constraint, but that expectation is false: The commutator of $\Psi(\mathbf{x})$ and $\Phi(\mathbf{y})$ does not vanish on the constraint surface. Consequently, the first-class and second-class constraints are still entangled, and $\Psi(\mathbf{x})$ itself is a mixture of constraints of both classes. In order to disentangle the constraints, we must first evaluate

$$\begin{aligned} [\Psi(\mathbf{x}), \Phi(\mathbf{y})] &= -4\frac{\delta\left\{\sqrt{g}(R - 2\Omega)(\mathbf{x})\right\}}{\delta g_{ij}(\mathbf{y})} \left(N\, \mathcal{G}_{ijk\ell}\Theta^{k\ell}\right)(\mathbf{y}) \\ &= 4\sqrt{g}\, N\, \Theta^{ij} \left(\mathcal{G}_{ijk\ell}\Theta^{k\ell} - \nabla_i \nabla_j\right) \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{2.113}$$

This is equal, up to a sign, to the commutator of p_ν and Φ which we obtained in (2.111). Hence, it is natural to define

$$\mathcal{H}_A = \Psi + p_\nu. \tag{2.114}$$

$\mathcal{H}_A$ commutes both with Φ and with p_ν, and represents a first-class constraint.

Having identified $\mathcal{H}_A$ as the final first-class constraint, we can check that it generates the correct $U(1)$ gauge transformations on the fields. In the Hamiltonian formalism, the gauge transformation generated by a first-class constraint on an arbitrary phase-space variable ϕ is given by the commutator of ϕ with the corresponding constraint (26), for example

$$\delta_\alpha \phi(\mathbf{x}, t) = -[\int d^D\mathbf{y}\, \alpha(\mathbf{y}, t)\mathcal{H}_A, \phi(\mathbf{x}, t)]. \tag{2.115}$$

One can indeed use this Hamiltonian formula to check that the gauge symmetries implied by the first-class constraints reproduce those that we found in the Lagrangian formulation above.

Given our analysis of the constraints, we can now evaluate the number of degrees of freedom. Altogether, the theory has $\dim \mathcal{P} = D^2 + 3D + 4$ canonical variables per spacetime point. These variables are constrained by $\mathcal{C}_1 = 2D + 2$ first-class constraints (P^i, P_A, $\mathcal{H}^i$ and

$\mathcal{H}_A$), and $\mathcal{C}_2 = 2$ second-class constraints (p_ν and Φ). The number of degrees of freedom $\mathcal{N}$ per spacetime point is then given by formula (2.1),

$$\mathcal{N} = \frac{1}{2}\left(\dim \mathcal{P} - 2\mathcal{C}_1 - \mathcal{C}_2\right) = \frac{1}{2}(D+1)(D-2). \tag{2.116}$$

This correctly reproduces the number of tensor (*i.e.* transverse, traceless) polarizations of the graviton in $D+1$ spacetime dimensions.

Returning to the integral constraint (2.108), we note that its commutation relations with the rest of the constraint algebra can be read off from the commutators of H obtained above. This follows from the fact that, as in general relativity, the Hamiltonian can be written as a sum of constraints,

$$H = N \int d^D\mathbf{x}\, \mathcal{H}_\perp + \int d^D\mathbf{x}\left(N^i\mathcal{H}_i + A\Psi\right). \tag{2.117}$$

Actually, the role of the integral constraint (2.108) deserves to be investigated further. It is plausible that in the theory with nonrelativistic general covariance, where the $U(1)$ gauge symmetry mimics the role of relativistic time reparametrizations, one can choose not to impose the integral constraint on physical states. This would be eqiuvalent to the omission of nonrelativistic time reparametrizations $\delta t = f(t)$ from the gauge symmetries, effectively setting $N(t) = 1$. If consistent, this construction would lead to a theory of gravity with nonzero energy levels even in spacetimes with compact spatial slices Σ. In fact, this situation was already encountered on flat noncompact Σ in the context of Abelian gravity in (46). On noncompact Σ, the possibility of relaxing the integral Hamiltonian constraint will be closely tied to the structure of consistent boundary conditions at infinity in gravity with anisotropic scaling, whose study is initiated in Chapter 3.

Linearization around detailed balance

In principle, our result (2.116) for the number of degrees of freedom $\mathcal{N}$ can be checked by linearizing the theory around a chosen solution, and explicitly counting the number of propagating polarizations. However, in order to investigate the spectrum of the linearized theory after gauging, we cannot use the flat spacetime as a reference background, because it no longer solves the equations of motion if Ω is not zero.

This makes the analysis of the linearized approximation for general values of the couplings algebraically tedious, and we will not present it here in full generality. Instead, we content ourselves with testing (2.116) in the simpler case when the theory satisfies the detailed balance condition. Hence, we assume that the potential takes the special form

$$\mathcal{V} = \frac{1}{4}\mathcal{G}_{ijk\ell}\frac{\delta W}{\delta g_{ij}}\frac{\delta W}{\delta g_{k\ell}}, \tag{2.118}$$

and for concreteness we choose

$$W = \frac{1}{2\kappa_W^2}\int d^D\mathbf{x}\,\sqrt{g}(R - 2\Lambda_W). \tag{2.119}$$

Before the $U(1)_\Sigma$ is gauged, the theory in detailed balance admits a particularly simple static ground-state solution,

$$g_{ij} = \hat{g}_{ij}(\mathbf{x}), \qquad N = 1, \qquad N_i = 0, \qquad \nu = 0, \tag{2.120}$$

where $\hat{g}_{ij}$ is the maximally symmetric spatial metric which solves the equations of motion of W,

$$R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda_W g_{ij} = 0. \tag{2.121}$$

In order for this background to be a solution of the theory with the extended $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$ gauge symmetry, we must set the spatial cosmological constant Ω equal to

$$\Omega = \frac{D}{D-2}\Lambda_W. \tag{2.122}$$

For $\Omega > 0$, the ground-state geometry is the Einstein static universe, with spatial slices $\Sigma = S^D$. Conversely, when $\Omega < 0$, the ground state is the hyperbolic version of the Einstein static universe, with noncompact Σ. Its curvature tensor satisfies $\hat{R}_{ij} = \frac{2\Omega}{D}\hat{g}_{ij}$ and $\hat{R} = 2\Omega$.

We now determine the spectrum of linearized perturbations around this class of ground state solutions. The analysis closely parallels that of Sections 2.2.1 and 2.4.1, and we will be brief. We expand the metric, $g_{ij} = \hat{g}_{ij} + \kappa h_{ij}$, and decompose the linearized fluctuations as in (2.21) and (2.22):

$$\begin{aligned} h_{ij} &= s_{ij} + \hat{\nabla}_i w_j + \hat{\nabla}_j w_i + \left(\hat{\nabla}_i\hat{\nabla}_j - \frac{1}{D}\hat{g}_{ij}\hat{\Delta}\right)B + \frac{1}{D}h\,\hat{g}_{ij}, \\ n_i &= u_i + \hat{\nabla}_i C, \end{aligned} \tag{2.123}$$

with $\hat{\nabla}_i$ the covariant derivative of $\hat{g}_{ij}$. The ν equation of motion is

$$\frac{2\Omega}{D}\left(\hat{\Delta}\nu + \frac{1}{2}\dot{h} - \hat{\Delta}C\right) = 0, \tag{2.124}$$

and the momentum constraints give

$$\begin{aligned} \left(\hat{\Delta} + \frac{2\Omega}{D}\right)(\dot{w}_i - u_i) &= 0, \\ \hat{\nabla}_i\left(\frac{2\Omega}{D}\dot{B} + \frac{D-1}{D}(\hat{\Delta}B - \dot{h}) + \frac{4\Omega}{D}\nu\right) &= 0. \end{aligned} \tag{2.125}$$

To fix the Diff$(M, \mathcal{F})$ symmetries, we set $w_i = B = n = 0$. In this gauge, the momentum constraints reduce to

$$(\hat{\Delta} + 2\Omega/D)u_i = 0, \qquad \hat{\nabla}_i \left[4\Omega\,\nu - (D-1)\dot{h}\right] = 0, \tag{2.126}$$

which implies, with suitable boundary conditions, that u_i is not propagating, and that

$$\Omega\,\nu = \frac{D-1}{4}\dot{h}. \tag{2.127}$$

Plugging this back into (2.124) yields

$$\frac{D-1}{4\Omega}\hat{\Delta}\dot{h} + \frac{1}{2}\dot{h} - \hat{\Delta}C = 0. \tag{2.128}$$

Finally, there is the constraint $R - 2\Omega = 0$, which plays the role of the Gauss constraint in our gauge $A = 0$. Its linearization around our detailed balance background

$$R - \hat{R} \approx -\frac{1}{D}\left[(D-1)\hat{\Delta}h + 2\Omega h\right] = 0 \tag{2.129}$$

shows that h is not propagating. Combining (2.129) with (2.128) then implies that $\hat{\Delta}C = 0$. Hence, the only propagating modes are the transverse traceless polarizations of the graviton s_{ij}. In particular, the scalar graviton has been eliminated, and the number of physical degrees of freedom agrees with the result of our Hamiltonian analysis (2.116).

2.5 Conclusions

In this chapter, we have found a formulation of the theory of gravity with anisotropic scaling in which the gauge symmetry of foliation-preserving diffeomorphisms Diff$(M, \mathcal{F})$ is enhanced to the symmetry of "nonrelativistic general covariance," $U(1) \ltimes \text{Diff}(M, \mathcal{F})$.

The advantage of this construction is that it relies only on the structure of the kinetic term in the action (1.15) (and, in fact, forces it to take the general-relativistic form with $\lambda = 1$), while the form of the potential term $\mathcal{V}$ is left unconstrained. Therefore, we can consider the scenario proposed originally in (1; 2), in which the theory is defined at short distances by a $z > 1$ fixed point (with $\mathcal{V}$ dominated by higher-derivative terms), and is then expected to flow under the influence of relevant terms to $z = 1$ and isotropic scaling in the infrared. This classical scenario will of course receive quantum corrections, which could drive the theory outside the range of validity of the covariant action (2.92). In the rest of the chapter, we limit our attention to the possibility that the long-distance physics is still described by the same action (2.92), with $\mathcal{V}$ dominated by the most relevant terms.

The first good news is that, as a result of the extended gauge symmetry, the spectrum contains just the transverse-traceless (tensor) modes of the graviton. The scalar graviton

mode of the minimal theory has been eliminated. In $3+1$ spacetime dimensions, the elimination of the scalar mode has an interesting consequence in the short-distance regime of the theory. Recall that in the minimal theory with the potential dominated at short distances by the $z=3$ term (1.19), the scalar mode is the sole physical mode that does not get a contribution to its dispersion relation from (1.19), suggesting that terms with $z>3$ would be required to achieve a UV completion (2). In the theory with the extended gauge symmetry, the scalar mode is a gauge artifact, all physical modes aquire a $z=3$ dispersion relation at short distances from (1.19), and no terms with $z>3$ are needed.

The extended gauge symmetry of the theory with nonrelativistic general covariance has even more interesting consequences at long distances, because it improves the chances that the behavior of our theory can resemble general relativity in this observationally relevant regime. We conclude this chapter by previewing how our generally covariant theory compares to general relativity at long distances, focusing on the case of $3+1$ spacetime dimensions.

Note first that even before we take the long-distance limit, the elimination of the scalar mode of the graviton is certainly a good sign for the possible matching against general relativity at long distances, and so is the fact that the coupling constant λ in the kinetic term is now frozen by the symmetries of the generally covariant theory to take the relativistic value $\lambda=1$. As a result, the number and the tensor structure of the gravitational wave polarizations is the same as in general relativity.

In the infrared limit of our theory, the potential $\mathcal{V}$ is dominated by the scalar curvature and the cosmological constant term. In this regime, the natural scaling is isotropic, with dynamical exponent $z=1$. The low-energy physics is best represented in rescaled coordinates (x^0, x^i) and in terms of rescaled fields. First, the new time coordinate

$$x^0 = \mu t \tag{2.130}$$

is defined by absorbing the effective speed of light μ into the definition of time. Because $[\mu] = z-1$, this implies that $[x^0] = -1 = [x^i]$, in accord with the $z=1$ scaling. The rescaled fields are defined by

$$N_i^{\rm IR} = \frac{1}{\mu} N_i, \qquad A^{\rm IR} = \frac{1}{\mu^2} A. \tag{2.131}$$

This rescaling ensures (i) that $N_i^{\rm IR}$ carries the canonical dimension implied by $z=1$, and (ii) that the $U(1)$ gauge transformations are given by the standard relativistic formula

$$\delta N_i^{\rm IR} = \partial_i \alpha^{\rm IR}, \qquad \delta A^{\rm IR} = \partial_0 \alpha^{\rm IR}, \tag{2.132}$$

with $\alpha^{\rm IR} = \alpha/\mu$. In the rest of the chapter, we will drop the "IR" superscripts, and refer to the rescaled fields (2.131) in the infrared simply as N_i and A.

The action of the infrared theory in the infrared variables is

$$S_{\rm IR} = \frac{1}{16\pi G_N} \int dx^0\, d^D\mathbf{x}\, \sqrt{g}\, \Big\{ N \left(K_{ij}K^{ij} - K^2 + R - 2\Lambda \right) - A(R - 2\Omega) \Big\} + \dots, \tag{2.133}$$

where "..." denotes corrections due to higher dimension operators, as well as the ν-dependent terms in (2.92) which are unimportant for our arguments below. In (2.133), K_{ij} refers to the extrinsic curvature tensor in the infrared coordinates, of canonical scaling dimension equal to one; and the Newton constant is given by

$$G_N = \frac{\kappa^2}{32\pi\mu}. \tag{2.134}$$

In the remainder of this section, we comment on three issues: The structure of compact-object solutions (which will be relevant for solar system tests), the issue of Lorentz symmetry, and the nature of cosmological solutions in the infrared regime of our theory as described by (2.133).

2.5.1 Static compact-object solutions

To prepare the ground for solar system tests, consider the infrared limit (2.133) and set the cosmological constant to zero. Interestingly, as the Schwarzschild black hole turns out to be a solution of this infrared theory. In terms of our fields, this solution will be represented by

$$g_{ij}dx^i dx^j = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{2.135}$$

$$A = 1 - \left(1 - \frac{2M}{r}\right)^{1/2}, \qquad N = 1, \quad N_i = 0, \quad \nu = 0. \tag{2.136}$$

It is straightforward to see that this geometry satisfies the equations of motion of our theory for $\Omega = 0$, which is the appropriate choice if we are interested in asymptotically flat solutions. First, the equations of motion contain the condition $R = 2\Omega$. With $\Omega = 0$, this equation is indeed satisfied by the spatial slices (2.135) of the relativistic Schwarzschild metric in the Schwarzschild coordinate system. The ν and N_i equations of motion are also satisfied, because the extrinsic curvature K_{ij} vanishes for static backgrounds.

Finally, to show that the g_{ij} equation of motion are also satisfied, we use a simple but intriguing argument. Since the same argument generalizes in a useful way to the case of nonzero $\Omega = \Lambda$, and also of arbitrary dimension, we present this more general case. Start with static solutions with $K_{ij} = 0$, and observe that the equations of motion for g_{ij}, N and A are identical to the equations that follow from the following reduced action,

$$\int d^D\mathbf{x}\,\sqrt{g}(N - A)(R - 2\Omega). \tag{2.137}$$

Similarly, for static solutions with $K_{ij} = 0$ of general relativity in $D + 1$ dimensions, the corresponding equations of motion are those of the reduced Einstein-Hilbert action,

$$\int d^D\mathbf{x}\,\sqrt{g}\,\mathfrak{N}\,(R - 2\Lambda), \tag{2.138}$$

where $\mathfrak{N}$ is the general-relativistic lapse function. Consequently, if we identify the N and A fields with the lapse function $\mathfrak{N}$ of general relativity,

$$\mathfrak{N} = N - A, \tag{2.139}$$

we see that static solutions of general relativity are also solutions of our theory in the infrared limit. In retrospect, this mapping also explains the form of A in our representation of the Schwarzschild metric (2.136).

Note that the relationship (2.139) between the general-relativistic lapse function $\mathfrak{N}$ and the N and A variables of our theory reproduces exactly what we would have expected from the geometric interpretation of A as the subleading term in the expansion of the relativistic g_{00} in powers of $1/c$ as obtained in (2.50). Indeed, we start by expanding

$$g_{00} \equiv -\mathfrak{N}^2 = -(N-A)^2 \approx -N^2 + 2NA + \dots \tag{2.140}$$

Recall now that A in (2.140) is the infrared rescaled field (2.131), related to the microscopic gauge field by a rescaling factor $1/\mu^2$. Using the fact that μ plays the role of the speed of light (as we have seen in (2.130)), the two leading terms in (2.140) match exactly the leading two terms in the expansion (2.50).

These arguments prove that the Schwarzschild geometry in the Schwarzschild coordinates, with the indentification implied by (2.139), is a solution of the infrared limit of our theory, with $\Omega = \Lambda = 0$. However, in the parametrized post-Newtonian (PPN) formalism (49; 50) which is typically used in gravitational phenomenology, the compact-object solution is usually represented in the isotropic coordinates. In the case of general relativity, this is just a gauge choice, a fact which does not extend automatically to alternative approaches to gravity such as ours. Showing that the Schwarzschild geometry in the Schwarzschild coordinates is a solution of our theory does not imply that it will be a solution when represented in another coordinate system, because only those coordinate changes that belong to the gauge symmetry of our model will map a solution to a solution. However, because the transformation from the Schwarzschild coordinates to the isotropic ones only changes the radial coordinate,

$$r = \rho\left(1 + \frac{M}{2\rho}\right)^2, \tag{2.141}$$

while keeping t, θ, ϕ intact, it is a foliation-preserving diffeomorphism, a symmetry of the theory. Consequently, the Schwarzschild solution in the isotropic coordinates, represented by

$$g_{ij}dx^i dx^j = \left(1 + \frac{M}{2\rho}\right)^4 \left(d\rho^2 + \rho^2 d\Omega_2^2\right), \tag{2.142}$$

$$A = \left(1 + \frac{M}{2\rho}\right)^{-1} \frac{M}{\rho}, \qquad N = 1, \quad N_i = 0, \quad \nu = 0, \tag{2.143}$$

is also a solution of the infrared limit of our theory. Expanding this solution to the required order in the powers of M/ρ strongly suggests that in the infrared regime, the β and γ parameters of the PPN formalism (49; 50) will take the same values as in general relativity, $\beta = \gamma = 1$. This feature is favorable for the solar-system tests of the theory.

2.5.2 Lorentz symmetry

Perhaps the leading challenge in any attempt to make theories of gravity with anisotropic scaling phenomenologically viable in $3+1$ dimensions is the issue of restoring Lorentz symmetry, at least at the intermediate energies and distances where it has been so well tested experimentally. In particular, we need a mechanism ensuring that in the corresponding regime, all species of matter (including the gravitons) percieve the same lightcones and the same effective speed of light. In the minimal theory with anisotropic scaling, this issue arises already for pure gravity: At generic values of the couplings, the speeds of the tensor and scalar graviton polarizations are not related by any symmetry, and are generally different from each other already in the short-distance regime. In contrast, our generally covariant theory has only the tensor graviton polarizations, all sharing the same speed at all energies; however, the issue reemerges when pure gravity is coupled to non-gravitational matter. If the present theory is to be phenomenologically viable, its coupling to matter will have to be analyzed in detail. This analysis is beyond the scope of the present chapter; we only limit ourselves to one observation, which may be useful for the future analysis.

In general relativity, Lorentz symmetry is a global symmetry associated with the isometries of the Minkowski spacetime. In gravity with anisotropic scaling, we can adjust the couplings such that the flat spacetime geometry continues to be a solution. The global symmetries of this solution will then depend on the precise model of gravity with anisotropic scaling.

First consider the case of the minimal theory of Section (1.3), with the cosmological constant tuned to zero. The flat spacetime (2.19) is a solution, but it does not exhibit the full global Lorentz symmetry – the Lorentz boosts, generated by

$$\delta t = b_i x^i, \qquad \delta x^i = b_i t \tag{2.144}$$

(with b_i a constant vector), are not foliation-preserving diffeomorphisms. In this theory, the nonrelativistic analogs of the Killing symmetries of the flat spacetime solution correspond to spacetime translations and space rotations – the solution breaks all possible boost symmetries spontaneously, and defines a preferred rest frame.

In contrast, in our generally covariant theory, we can interpret the Lorentz transformation (2.144) as a generator of a transformation belonging to the extended symmetry group $U(1) \ltimes \mathrm{Diff}(M, \mathcal{F})$. More precisely, the Lorentz transformation (2.144) should be interpreted as a composition of an infinitesimal foliation-preserving diffeomorphism and an infinitesimal $U(1)$ transformation. Indeed, restoring the factors of c shows that the variation of t in (2.144) is suppressed by a factor of $1/c^2$ compared to the variation of x^i, and should

therefore be interpreted as an infinitesimal $U(1)$ transformation with $\alpha = b_i x^i$, accompanied in (2.144) by the infinitesimal foliation-preserving diffeomophism

$$\delta t = 0, \qquad \delta x^i = b_i t. \tag{2.145}$$

When interpreted in this way, the Lorentz transformation (2.144) is a symmetry of the flat spacetime geometry represented in our variables by $g_{ij} = \delta_{ij}$, $N = 1$ and $A = 0$. This does not yet imply that all preferred-frame effects are absent in this background: In particular, the Newton prepotential ν is not invariant under the Lorentz boosts, and defines a preferred frame for the flat spacetime, in which $\nu = 0$. The flat background is Lorentz invariant only to the extent that the effects of the Newton prepotential can be ignored.

2.5.3 Cosmological solutions

Moving beyond asymptotically flat spacetimes, it is natural to ask whether our theory has interesting cosmological solutions. One can start with a given spacetime geometry in general relativity, and investigate whether it satisfies the equations of motion of our theory. The answer to this question will again depend on the choice of spacetime foliation.

For example, arguments identical to those used above for the Schwarzschild metric show that the static patch of the de Sitter (or anti-de Sitter) spacetime, represented in our variables by

$$g_{ij}dx^i dx^j = \left(1 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega_2^2, \tag{2.146}$$

$$A = 1 - \left(1 - \frac{\Lambda r^2}{3}\right)^{1/2}, \qquad N = 1, \quad N_i = 0, \quad \nu = 0, \tag{2.147}$$

is a solution of our theory if we set $\Omega = \Lambda$.

It is encouraging to see that at least in the time-independent foliations, the de Sitter and anti-de Sitter spacetimes are solutions of our theory. In standard cosmological applications, however, the cosmological principle selects another natural foliation of spacetime, with homogeneous spatial slices and a time-dependent scale factor $a(t)$. On the face of it, it may appear difficult to obtain cosmological solutions of our theory with maximally symmetric and time-dependent spatial slices: The equation of motion for A plays the role of a Gauss constraint, and implies $R = 2\Omega$ in the vacuum. Assuming the standard FRW Ansatz

$$g_{ij} = a^2(t)\gamma_{ij}, \qquad N = 1, \qquad A = 0, \qquad N_i = 0, \qquad \nu = 0 \tag{2.148}$$

where γ_{ij} is a time-independent maximally symmetric spatial metric, the scalar curvature of g_{ij} has to be constant in time. Consequently, if the scalar curvature of γ_{ij} is nonzero, the cosmological scale factor must be independent of time.

Of course, if our spatial slices are flat, the Gauss constraint no longer restricts the time dependence of the cosmological scale factor. This requires $\Omega = 0$. The rest of the equations

of motion will be satisfied by the de Sitter spacetime in the inflationary coordinates, which in the FRW Ansatz (2.148) corresponds to

$$a(t) = e^{Ht}, \qquad \gamma_{ij} = \delta_{ij}. \tag{2.149}$$

The reason for this is again simple but illuminating: With $\Omega = 0$, the ν equation of motion is satisfied when the metric is flat. With ν and A both zero, the remaining equations are implied by Einstein's equations if we simply identify the relativistic lapse function with our $N(t)$, the relativistic cosmological constant with our Λ, and H with the Hubble constant.

Thus, we see that the same de Sitter spacetime in two different foliations is a solution of the infrared theory for two different choices of the coupling constant, one with $\Omega = \Lambda$ and the other with $\Omega = 0$ and nonzero Λ. Mapping out the general behavior of cosmological solutions as the coupling constants Ω and Λ are independently varied is one of questions left for future work.

In addition, there are at least two ways out of the potential difficulty with solving the Gauss constraint for cosmologically evolving spacetimes with maximally symmetric spatial slices of nonzero curvature. First, the equations of motion will change in the presence of matter. In the full system of equations for gravity and matter, the Gauss constraint is expected to be modified by a matter source, whose time dependence can then drive the time dependence of the scale factor in the spatial metric. The second possibility is related to the gauge freedom we have in describing cosmological solutions in general relativity: Instead of the standard FRW ansatz which leads to (2.148), one can choose coordinates in which the spatial metric is not only maximally symmetric but also constant in time. When we express the FRW geometry in such coordinates, the time-dependent scale factor appears in the dt^2 term in the metric, and non-zero components of the shift vector N_i are typically generated. In general relativity, this coordinate representation of FRW cosmologies is a legitimate albeit slightly unconventional gauge choice. In our theory, this parametrization of FRW universes has the advantage of being compatible with the vacuum Gauss constraint $R = 2\Omega$.

Chapter 3

Anisotropic Conformal Infinity

We generalize Penrose's notion of conformal infinity of spacetime, to situations with anisotropic scaling. This is relevant not only for Lifshitz-type anisotropic gravity models, but also in standard general relativity and string theory, for spacetimes exhibiting a natural *asymptotic* anisotropy. Examples include the Lifshitz and Schrödinger spaces (proposed as AdS/CFT duals of nonrelativistic field theories), warped AdS_3, and the near-horizon extreme Kerr geometry. The anisotropic conformal boundary appears crucial for resolving puzzles of holographic renormalization in such spacetimes.

3.1 Introduction

Recently, string theory and quantum gravity have begun to expand into territories traditionally associated with theoretical condensed matter physics. In the process, the apparent divide between relativistic and nonrelativistic systems is becoming significantly blurred. For example, relativistic gravity solutions have been proposed as duals of nonrelativistic quantum field theories (NRQFTs) (51; 52) characterized by anisotropic scaling of time and space,

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \tag{3.1}$$

with dynamical exponent $z \neq 1$. In another development, gravity models have been proposed (1; 2; 22) in which the gravitational field itself is subject to anisotropic scaling (3.1) at short spacetime distances, leading to an improved ultraviolet behavior.

At this new interface of condensed matter with quantum gravity, challenges and puzzles emerge. For example, extending the concept of holographic renormalization (see, *e.g.*, (20) for a review) to nonrelativistic QFTs has proven surprisingly difficult. In standard holographic renormalization, the counterterms in a relativistic field theory are constructed from the analysis of the asymptotic behavior of bulk gravity near the boundary of spacetime. Many of the difficulties with holographic renormalization of NRQFTs can be traced to the fact that the proposed gravity duals have a degenerate conformal boundary, as defined in the sense of Penrose (15; 16). This degenerate behavior indicates that Penrose's definition of conformal infinity is insufficient to handle holography in such spacetimes, and that it needs to be generalized to incorporate systems with anisotropic scaling.

In this chapter, we present such a generalization of conformal infinity of spacetime, which is based on concepts developed in the context of gravity at a Lifshitz point. Here we focus on the main idea of the construction, illustrated by a few examples.

3.2 Anisotropic Conformal Infinity: The Spatially Isotropic Case

One feature common to geometries dual to NRQFTs is that their asymptotic behavior "near the boundary" reflects the anisotropic scaling (3.1) of the dual NRQFT. This suggests that the correct notion of asymptopia and conformal infinity should reflect this anisotropy in the conformal transformations near the boundary. However, the idea of using anisotropic conformal transformations to define the boundary of spacetime immediately leads to apparent conflicts: The conformal boundary must be a geometric object, defined such that it is preserved by the symmetries of gravity; but spacetime diffeomorphisms only allow isotropic Weyl transformations, reducing us to Penrose's original definition.

3.2.1 The main idea

The observation crucial for resolving these conflicts was made (1) in gravity models with anisotropic scaling: Appropriately defined local anisotropic Weyl transformations are compatible with the restricted group $\mathrm{Diff}(M;\mathcal{F})$ of those diffeomorphisms of spacetime M that preserve a preferred foliation $\mathcal{F}$ of M by fixed time slices. This fact allows us to define the concept of *anisotropic conformal infinity*, which legitimizes the asymptopia of many spacetimes, including those that appeared as duals of NRQFTs.

In the anisotropic gravity models of (1; 2), the reduction of symmetries to $\mathrm{Diff}(M;\mathcal{F})$ is a consequence of the gauge symmetries of the system. However, our construction of anisotropic conformal infinity is valid beyond the context of (1; 2), and applies naturally to a large class of solutions of standard general relativity and string theory: It is sufficient that the symmetries reduce to $\mathrm{Diff}(M;\mathcal{F})$ *only asymptotically, near the spacetime boundary.* As we will see, this is indeed the behavior exhibited by the gravity duals of NRQFTs. This shows that the ideas of (1; 2) find meaningful applications beyond the context of anisotropic gravity models.

The group $\mathrm{Diff}(M;\mathcal{F})$ of foliation-preserving diffeomorphisms is generated by

$$\xi \equiv f(t)\partial_t + \xi^i(t,x^j)\partial_i. \tag{3.2}$$

In the canonical (ADM) parametrization of the metric,

$$ds^2 = -N^2dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{3.3}$$

the $\mathrm{Diff}(M;\mathcal{F})$ generators act via

$$\begin{aligned}
\delta_\xi N &= f\dot{N} + \dot{f}N + \xi^i\partial_i N,\\
\delta_\xi N_i &= f\dot{N}_i + \dot{f}N_i + \xi^j\partial_j N_i + \partial_i\xi^j N_j + \dot{\xi}^j g_{ij},\\
\delta_\xi g_{ij} &= f\dot{g}_{ij} + \xi^k\partial_k g_{ij} + \partial_i\xi^k g_{jk} + \partial_j\xi^k g_{ik}.
\end{aligned} \tag{3.4}$$

Using an arbitrary smooth nonzero scale factor $\Omega(t,x^i)$, we define the anisotropic Weyl transformations to be

$$\widetilde{N} = \Omega^z N, \qquad \widetilde{g}_{ij} = \Omega^2 g_{ij}, \qquad \widetilde{N}_i = \Omega^2 N_i. \tag{3.5}$$

It was observed in (1; 2) that the generators δ_ω of these anisotropic Weyl transformations form a closed algebra with the generators of $\mathrm{Diff}(M;\mathcal{F})$:

$$[\delta_\xi, \delta_\omega] = \delta_\varpi, \qquad \text{with} \quad \varpi = f\dot{\omega} + \xi^i\partial_i\omega. \tag{3.6}$$

Given (3.5), our definition of anisotropic conformal infinity of spacetime M with metric ds^2 is essentially the same as in the isotropic case: We map M by an anisotropic Weyl transformation Ω to an auxiliary spacetime $\widetilde{M}$ with a rescaled metric $\widetilde{ds^2}$, choosing Ω such that the region near infinity in M is mapped to points inside a compact region of $\widetilde{M}$. Under this map, the *ideal points at anisotropic conformal infinity of* M correspond to the boundary of the image of M inside $\widetilde{M}$, where the scale factor Ω vanishes while $d\Omega \neq 0$. We will denote the anisotropic conformal infinity of M by ∂M.

3.2.2 Asymptotic structure of the Lifshitz space

Our first example is the Lifshitz spacetime, with metric

$$ds^2 = -\frac{dt^2}{w^{2z}} + \frac{d\mathbf{x}^2 + dw^2}{w^2}. \tag{3.7}$$

This geometry was proposed in (13) as the gravity dual for NRQFTs with Lifshitz-type scaling without Galilean invariance. With the choice of $\Omega = w$, we find that the Lifshitz spacetime is anisotropically conformal to the portion of the flat spacetime with $w > 0$, with the standard metric

$$\widetilde{ds^2} = -dt^2 + d\mathbf{x}^2 + dw^2. \tag{3.8}$$

The anisotropic conformal boundary at infinity in (3.7) is mapped to $w = 0$ in (3.8). In this example, we see that

(a) the anisotropic boundary is of codimension one,

(b) the bulk metric induces an anisotropic conformal class of metrics in the boundary, and

(c) the action of the anisotropic conformal symmetry in the boundary is induced from the action of the bulk isometries.

Point (c) deserves a closer explanation: In analogy with the isotropic case, we define *anisotropic conformal transformations* of a fixed metric on ∂M to be those $\mathrm{Diff}(\partial M; \mathcal{F})$ transformations that map the metric to itself up to an anisotropic Weyl transformation. Here $-dt^2 + d\mathbf{x}^2$ is a representative of the anisotropic conformal class of metrics at the boundary of the Lifshitz space. The corresponding group of anisotropic conformal transformations is finite-dimensional, generated by time translations, spatial translations and rotations, and the anisotropic scaling transformation (3.1). It is this conformal symmetry group whose action on ∂M is induced from the bulk isometries of the Lifshitz space M.

3.3 Spatially Anisotropic Conformal Infinity

Our other examples require a more refined structure, with several dynamical exponents and with nested foliations of spacetime. We consider the case with scaling

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i, \qquad y^a \to \lambda^\zeta y^a, \tag{3.9}$$

and look for anisotropic Weyl transformations which reduce for a constant $\Omega \equiv \lambda$ to (3.9), and form a closed group with those spacetime diffeomorphisms $\mathrm{Diff}(M; \mathcal{F}_2)$ that preserve the structure of a nested foliation $\mathcal{F}_2$ of spacetime. $\mathrm{Diff}(M; \mathcal{F}_2)$ is generated by

$$\xi \equiv f(t)\partial_t + \xi^i(t, x^j)\partial_i + \eta^a(t, x^j, y^b)\partial_a. \tag{3.10}$$

One could first assume that g_{ab} is invertible, and simply iterate the logic from the single-foliation case. Examples with this behavior would include the obvious generalizations of the

Lifshitz spacetime, with an additional spatial anisotropy and scaling (3.9); the anisotropic conformal infinity of such spacetimes again exhibits the same features (a)-(c) as in the single-foliation Lifshitz space.

We will be interested in a different class of examples, in which g_{ab} is not necessarily invertible. The most interesting case corresponds to $\zeta = 0$; spatial dimensions with this scaling will be called "ultralocal." We specialize to the case of just one ultralocal dimension y, and parametrize the metric as

$$\begin{aligned} ds^2 &= g_{tt}dt^2 + 2g_{ty}dt\,dy + g_{yy}dy^2 \\ &\quad + g_{ij}[dx^i + g^{ik}(A_k dt + B_k dy)][dx^j + g^{j\ell}(A_\ell dt + B_\ell dy)]. \end{aligned} \tag{3.11}$$

Just as in the case of the single foliation (1; 2), the appropriate action of (3.10) on the fields of (3.11) can be obtained by taking a nonrelativistic scaling limit of full spacetime diffeomorphisms $\mathrm{Diff}(M)$, which results from substituting $A_i \to A_i/c$, $B_i \to cB_i$, parametrizing the generators of $\mathrm{Diff}(M)$ as $(cf, \xi^i, \eta/c)$, and taking $c \to \infty$. This process yields transformation rules for the metric components under the action of $\mathrm{Diff}(M; \mathcal{F}_2)$ which are compatible with the anisotropic Weyl transformations

$$\begin{aligned} &g_{tt} \to \Omega^{2z} g_{tt}, \qquad g_{ty} \to \Omega^2 g_{ty}, \quad g_{yy} \to g_{yy}, \\ &g_{ij} \to \Omega^2 g_{ij}, \qquad A_i \to \Omega^2 A_i, \quad B_i \to \Omega^2 B_i. \end{aligned} \tag{3.12}$$

As we now illustrate in a number of examples, this version of anisotropic Weyl transformations again leads to a natural notion of anisotropic conformal infinity.

3.3.1 Asymptotic structure of null warped AdS_3

Perhaps the simplest example is null warped AdS_3 (53),

$$ds^2 = -\frac{dt^2}{w^4} + \frac{2dt\,d\theta + dw^2}{w^2}. \tag{3.13}$$

We choose the global scaling of the coordinates to be

$$t \to \lambda^2 t, \qquad w \to \lambda w, \qquad \theta \to \theta. \tag{3.14}$$

This is an example of the scaling defined in (3.9). Using (3.12) with $\Omega = w$, the metric is mapped to

$$\widetilde{ds^2} = -dt^2 + 2dt\,d\theta + dw^2. \tag{3.15}$$

The anisotropic conformal boundary is again at $w = 0$, and satisfies properties (a)-(c) just like the Lifshitz space, with one novelty: The group of anisotropic conformal symmetries – defined again as those $\mathrm{Diff}(\partial M; \mathcal{F})$ elements that map the boundary metric $-dt^2 + 2dt\,d\theta$ to itself up to an anisotropic Weyl transformation – is now infinite dimensional, with generators

$$F(t)\partial_t + G(t)\partial_\theta, \tag{3.16}$$

with $F(t)$, $G(t)$ arbitrary. Their action on the conformal class of metrics in ∂M is induced by *asymptotic* $\mathrm{Diff}(M;\mathcal{F}_2)$ isometries of null warped AdS_3. In the quantum theory, (3.16) will give rise to a Virasoro algebra together with a $U(1)$ current algebra.

3.3.2 Asymptotic structure of the Schrödinger space

Our next example, the Schrödinger space

$$ds^2 = -\frac{dt^2}{w^{2z}} + \frac{2dt\,d\theta + d\mathbf{x}^2 + dw^2}{w^2}, \tag{3.17}$$

has been proposed (51; 52) as a gravity dual of Galilean-invariant nonrelativistic CFTs with dynamical exponent z. In order to get a well-behaved anisotropic conformal infinity, we use the scalings of (3.9), with $x^i \equiv (w, \mathbf{x})$ and with $y \equiv \theta$ an ultralocal dimension. Using (3.12) together with $\Omega = w$ yields

$$\widetilde{ds^2} = -dt^2 + 2dt\,d\theta + d\mathbf{x}^2 + dw^2, \tag{3.18}$$

with ∂M again at $w = 0$. Note an interesting feature: Because θ scales with conformal exponent $\zeta = 0$, this dimension is present *both in the bulk and in the boundary*, even if it is compactified; the conformal infinity is of codimension one. This interpretation of θ resolves some of the mysteries associated with this extra bulk dimension in holography of Schrödinger spaces.

The bulk isometries of (3.17) again induce the action of anisotropic conformal symmetries on the anisotropic conformal class $-dt^2 + 2dt\,d\theta + d\mathbf{x}^2$ of boundary metrics. For example, in the case of $z = 2$, this group of conformal transformations of ∂M induced from the bulk isometries is generated by

$$\begin{array}{llll} t^2\partial_t + tx^i\partial_i - \frac{1}{2}x^2\partial_\theta, & & t\partial_i + x^i\partial_\theta, & \\ t\partial_t + \frac{1}{2}x^i\partial_i, & \partial_i, & \partial_\theta, & x^i\partial_j - x^j\partial_i. \end{array} \tag{3.19}$$

These are the generators of the Schrödinger conformal group. Asymptotic bulk isometries formally extend this symmetry to an infinite-dimensional one (54; 55), analogous to (3.16).

The metric (3.17) describes the Schrödinger space in Poincaré-like coordinates. At least when $z = 2$, it can be analytically continued beyond the Poincaré patch, to global Schrödinger space (56)

$$ds^2 = -\left(1 + \frac{\hat{\mathbf{x}}^2}{\hat{w}^2} + \frac{1}{\hat{w}^4}\right) d\hat{t}^{\,2} + \frac{2d\hat{t}d\hat{\theta} + d\hat{\mathbf{x}}^2 + d\hat{w}^2}{\hat{w}^2}. \tag{3.20}$$

(3.17) and (3.20) are related by coordinate transformation

$$\hat{t} = \arctan t, \qquad \hat{\theta} = \theta + \frac{t}{2(1+t^2)}(\mathbf{x}^2 + w^2)$$

$$\hat{x}^i = \frac{x^i}{\sqrt{1+t^2}}, \qquad \hat{w} = \frac{w}{\sqrt{1+t^2}}. \tag{3.21}$$

It is reassuring that this transformation is a double-foliation preserving diffeomorphism, of the form (3.10). As a result, the anisotropic conformal boundary of global Schrödinger space can also be analyzed in our framework.

3.4 Holographic Renormalization and Anisotropic Conformal Infinity

In the few examples presented above, we simply determined the correct form of foliation-preserving diffeomorphisms and the correct anisotropic Weyl transformations by inspection. More complicated examples may involve multiple foliations and multiple anisotropies which obscure the precise details of the construction. It is therefore desirable to have an algorithmic tool for deriving the anisotropic asymptotic structure in more general cases.

We now outline how such rules can be systematically derived from considerations of holography in spacetimes with asymptotically anisotropic scaling.

3.4.1 The general prescription

The general prescription consists of the following steps:

(i) Identify consistent fall-off conditions on fields on M. (This step, which we take as an input, is a consequence of the precise definition of the dynamics, designed to identify a consistent phase space of the theory; see, *e.g.*, (57; 58; 59; 60). for examples).

(ii) Identify the maximal subgroup of diffeomorphism symmetries compatible with (i).

(iii) Identify the anisotropic Weyl transformations compatible with (ii).

Given (i)-(iii), one can then relax the asymptotic fall-off conditions on the background to allow for a generic boundary metric γ, and *derive* the action of the asymptotic symmetries from (ii) on γ. This action yields the appropriately scaled version of appropriate foliation-preserving diffeomorphisms of ∂M on the boundary metric γ. The anisotropic Weyl transformations on γ are determined simply from the reparametrizations of the radial coordinate. Finally, we use (iii) to construct the anisotropic conformal infinity of M.

We will apply these ideas to the holographic renormalization of Lifshitz space in Chapter 4, but here here we will illustrate this general prescription with spacelike warped AdS_3 as an example.

3.4.2 Asymptotic structure of spacelike warped AdS_3

The metric of the spacelike warped AdS_3 in global coordinates is (58)

$$ds^2 = -(1+r^2)du^2 + \frac{dr^2}{1+r^2} + \frac{4\nu^2}{\nu^2+3}(r\,du + dv)^2. \tag{3.22}$$

Following steps (i)-(iii) outlined above, we get:

(i) Fall-off conditions on the deviations $h_{\mu\nu}$ of the metric from the background (3.22) were proposed in (60),

$$\begin{aligned} h_{uu} &= \mathcal{O}(r), \qquad h_{vv} = \mathcal{O}\left(\frac{1}{r}\right), \qquad h_{rr} = \mathcal{O}\left(\frac{1}{r^3}\right), \\ h_{uv} &= \mathcal{O}(1), \qquad h_{ru} = \mathcal{O}\left(\frac{1}{r}\right), \qquad h_{rv} = \mathcal{O}\left(\frac{1}{r^2}\right). \end{aligned} \tag{3.23}$$

(ii) The group of diffeomorphisms preserving these fall-off conditions is generated by

$$\left[F(u) + \mathcal{O}\left(\frac{1}{r^2}\right)\right]\partial_u - \left[rF'(u) + \mathcal{O}\left(\frac{1}{r}\right)\right]\partial_r + \left[G(u) + \mathcal{O}\left(\frac{1}{r}\right)\right]\partial_v, \tag{3.24}$$

with $F(u)$, $G(u)$ arbitrary, and exhibits a natural asymptotic foliation structure.

(iii) Given (3.24), we choose the anisotropic Weyl transformations

$$g_{uu} \to \Omega^4 g_{uu}, \quad g_{uv} \to \Omega^2 g_{uv}, \quad g_{vv} \to g_{vv} \tag{3.25}$$

on the metric. These are of the form (3.12), with $z = 2$. Together with the asymptotic diffeomorphisms (3.24), the Weyl transformations form an algebra that closes up to subleading terms in $1/r$. With the choice of $\Omega = r^{-1/2}$, we obtain the anisotropic conformal infinity of spacelike warped AdS_3. The boundary at anisotropic infinity is two-dimensional, and carries an induced anisotropic conformal class of metrics represented by

$$-du^2 + \frac{4\nu^2}{\nu^2+3}(du+dv)^2. \tag{3.26}$$

The action of the correctly scaled form of $\mathrm{Diff}(\partial M; \mathcal{F})$ on the boundary metric $\gamma(u,v)$ can be determined by relaxing the background to

$$\begin{aligned} g_{uu} &= r^2\gamma_{uu}(u,v) + \mathcal{O}(r), \qquad g_{vv} = \gamma_{vv}(u,v) + \mathcal{O}\left(\frac{1}{r}\right), \\ g_{uv} &= r\gamma_{uv}(u,v) + \mathcal{O}(1), \qquad g_{ru} = \mathcal{O}\left(\frac{1}{r}\right), \\ g_{rr} &= \frac{1}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right), \qquad g_{rv} = \mathcal{O}\left(\frac{1}{r^2}\right), \end{aligned} \tag{3.27}$$

and acting with the group of bulk diffeomorphisms which preserve this asymptotic form of the metric,

$$\left[F(u) + \mathcal{O}\left(\frac{1}{r^2}\right)\right]\partial_u + \left[rH(u) + \mathcal{O}\left(\frac{1}{r}\right)\right]\partial_r + \left[G(u,v) + \mathcal{O}\left(\frac{1}{r}\right)\right]\partial_v. \tag{3.28}$$

The radial bulk diffeomorphisms generated by $rH(u)\partial_r$ induce the correct anisotropic Weyl transformation (3.25) on the boundary metric, with H as the generator. And the diffeomorphisms along u and v precisely reproduce the action of $\mathrm{Diff}(\partial M; \mathcal{F})$ on γ that we obtained from the $c \to \infty$ scaling below Eqn. (3.11)! We can use this action of $\mathrm{Diff}(\partial M; \mathcal{F})$ to define the group of anisotropic conformal transformations of the boundary metric. This group is found to be generated by

$$F(u)\partial_u + G(u)\partial_v. \tag{3.29}$$

For compact u, this reproduces the Virasoro algebra and the $U(1)$ current algebra found in (60).

A closely related example is the near-horizon extreme Kerr geometry (61; 59), which can be viewed as a family of spacelike warped AdS_3's, fibered over the polar coordinate θ. In this example, θ is an ultralocal dimension, analogous to θ of the Schrödinger space (3.17), but without translational invariance along θ.

3.5 Conclusions

The notion of anisotropic conformal infinity clarifies the asymptotic structure of vacuum spacetimes with asymptotically anisotropic scaling. As an application, we can now give a precise definition of black holes in spacetimes with anisotropic asymptopia: First, we define an event horizon in an asymptotically anisotropic spacetime as the boundary of the causal past of the anisotropic infinity, and define black holes as solutions with event horizons. Our definition of anisotropic conformal infinity naturally extends to the black holes themselves: For example, one can show that the spacelike warped AdS_3 black holes of (58) share the asymptotic structure of the spacelike warped AdS_3 vacuum determined above.

In relativistic gravity, the structure of conformal infinity is probed by null geodesics. Spacetimes with anisotropic scaling appearing in the context of (1; 2) can be similarly probed, by Lifshitz particles with a gapless nonrelativistic dispersion relation.

Results of this chapter illustrate that under pressure from the interface of quantum gravity with condensed matter, some of the central notions of general relativity must be revisited and adapted for the era in which quantum gravity is applied to systems with anisotropic scaling. In the process, is appears that we must disentangle two concepts which seemed so inseparable in the physics of the 20th century: gravity and relativity.

Chapter 4

Conformal Lifshitz Gravity from Holography

We show that holographic renormalization of relativistic gravity in asymptotically Lifshitz spacetimes naturally reproduces the structure of gravity with anisotropic scaling: The holographic counterterms induced near anisotropic infinity take the form of the action for gravity at a Lifshitz point, with the appropriate value of the dynamical critical exponent z. In the particular case of $3+1$ bulk dimensions and $z = 2$ asymptotic scaling near infinity, we find a logarithmic counterterm, related to anisotropic Weyl anomaly of the dual CFT, and show that this counterterm reproduces precisely the action of conformal gravity at a $z = 2$ Lifshitz point in $2 + 1$ dimensions, which enjoys anisotropic local Weyl invariance and satisfies the detailed balance condition. We explain how the detailed balance is a consequence of relations among holographic counterterms, and point out that a similar relation holds in the relativistic case of holography in AdS_5. Upon analytic continuation, analogous to the relativistic case studied recently by Maldacena, the action of conformal gravity at the $z = 2$ Lifshitz point features in the ground-state wavefunction of a gravitational system with an interesting type of spatial anisotropy.

4.1 Introduction

The possibility offered by Hořava-Lifshitz theories that gravity may exhibit multicritical behavior with Lifshitz-like anisotropic scaling at short distances (1; 2) has attracted a lot of attention recently (see, *e.g.*, (62; 63; 64) for some reviews). Such multicritical quantum gravity can be formulated as a field theory of the fluctuating spacetime metric, characterized by scaling which is anisotropic between time and space,

$$t \mapsto b^z t, \qquad \mathbf{x} \mapsto b\mathbf{x}, \tag{4.1}$$

with dynamical exponent z.

When z equals the number of spatial dimensions D, several interesting things happen: First, the theory becomes power-counting renormalizable, when we allow all terms compatible with the gauge symmetries in the action. In addition, the effective spectral dimension of spacetime flows to two at short distances, in accord with the lattice results first obtained in the causal dynamical triangulations approach to quantum gravity in (65; 66; 67), and independently confirmed recently in (68). Moreover, when $z = D$, one can further restrict the classical action by requiring its invariance under the local version of the rigid anisotropic scaling, which acts on the spacetime metric via anisotropic Weyl transformations. This leads to an anisotropic version of conformal gravity (1).

While such multicritical gravity models may not need string theory for a UV completion, it is still natural to ask whether they can be engineered from string theory. Indeed, it seems likely that any mathematically consistent theory of gravity should have a role to play in the bigger scheme of strings. Here we present one specific construction in which the action of multicritical gravity with anisotropic scaling appears naturally from string theory and AdS/CFT correspondence: The holographic renormalization of spacetimes which are asymptotically Lifshitz, or in other words, dual to non-relativistic field theories with Lifshitz scaling.

In recent years, the AdS/CFT correspondence has been extended to spacetimes which are asymptotically non-relativistic, with the hope of providing new techniques for understanding strongly coupled condensed matter systems using dualities originating in string theory (see, *e.g.*, (3; 69; 70; 71) for recent reviews of this program). Such asymptotically non-relativistic spacetimes fall into two classes: Either they approach Schrödinger symmetries (51; 52), or they exhibit Lifshitz-type scaling (13). In both cases, Penrose's standard definition of conformal infinity (see, *e.g.*, (16)) gives results which are puzzling and appear inconsistent with the expectations based on gauge-gravity duality. As we saw in Chapter 3, for spacetimes which carry an asymptotic foliation structure, a natural generalization of Penrose's notion of conformal infinity to asymptotically anisotropic spacetimes exists, and it reproduces features expected from holography.

This clearer picture of the asymptotic structure of Lifshitz spacetimes allows us to perform holographic renormalization, study the precise structure of holographic counterterms, and compare the results to the relativistic case. This is the goal of the present chapter. We

focus mainly on the case of $3+1$ bulk dimensions, in particular with the $z=2$ scaling. In this case, we find a logarithmic counterterm, which takes the form of the action of the $z=2$ conformal multicritical gravity in $2+1$ dimensions. In relativistic AdS/CFT, logarithmic gravitational counterterms appear only when the dimension d of the boundary is even. In that circumstance, they take the form of the Weyl anomaly (72) (see (73) for a review of the Weyl anomaly). For example, in the maximally supersymmetric case in $d=4$, the anomaly is given by the action of conformal supergravity (74) (see, *e.g.*, (75) for a review of the early history of conformal supergravity). In Lifshitz spacetimes, the logarithmic counterterms – if and when they appear – should be related to the little-studied nonrelativistic Weyl anomaly (see (76) for some early results on the Weyl anomaly at $z=3$ in $d=4$, and (77; 78) for a detailed discussion of axial anomalies in Lifshitz theories). In Appendix 4.8, we briefly discuss the cohomological structure of the $z=2$ anisotropic Weyl anomaly in $2+1$ dimensions, and (modulo total derivatives) find two independent terms that can appear in the action. However, perhaps surprisingly, the action that we obtain in the logarithmic counterterm turns out to satisfy the additional condition of detailed balance, which reduces the number of independent terms to one. We show how this condition is implied by the machinery of holographic renormalization, which relates the logarithmic counterterm to another, quadratic counterterm.

The techniques of holographic renormalization in asymptotically *AdS* spacetimes can also be usefully applied, upon appropriate Wick rotation, to asymptotically de Sitter geometries (20), leading to results about the ground-state wavefunction of the universe at superhorizon scales (79) (see also (80; 81), and the series of papers (82; 83; 84)). Since holography in asymptotically Lifshitz spacetimes parallels closely the case of *AdS*, it is natural to perform the corresponding Wick rotation, obtain a candidate ground-state wavefunction, and ask what kind of gravitational system this wavefunction describes. In the case of $z=2$ and bulk $3+1$ dimensions, we show that this wavefunction corresponds to a spatially anisotropic gravity theory with an interesting form of ultralocality.

Our discussion in the bulk of the chapter mostly focuses on the case of $3+1$ bulk dimensions, in particular with $z=2$. However, after summarizing our conventions and notation in Appendix 4.6, we present our detailed calculations also for general D and z in an extensive Appendix 4.7, with the hope that the inquisitive reader may find the results useful.

4.2 Conformal Gravity at a Lifshitz Point

The anisotropic gravity models of Section 1.3 generically have D local symmetries in $D+1$ spacetime dimensions. However, for certain choices of action the theory can become invariant under the anisotropic Weyl transformations (1.20) as well. Insisting on this additional symmetry implies that the coupling constant λ must take a fixed value, $\lambda = 1/D$. We will refer to it as the "conformal value" of λ. In this chatper, we will be mainly interested

in the case of $D = 2$, which requires $z = 2$ and the unique kinetic term

$$S_K = \frac{1}{\kappa^2} \int dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N \left(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2 \right). \tag{4.2}$$

One can easily check that this term is indeed invariant under (1.20).

The potential term $\mathcal{V}$ is also strongly constrained by the condition of anisotropic Weyl invariance (1.20). In $D = 2$, where the Riemann tensor of the spatial metric reduces to the Ricci scalar $\hat{R}$, there is only one term that can appear in $\mathcal{V}$:[1]

$$S_{\mathcal{V}} = \frac{1}{\kappa_{\mathcal{V}}^2} \int dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N \left\{ \hat{R} + \frac{\nabla^2 N}{N} - \left(\frac{\nabla N}{N} \right)^2 \right\}^2 . \tag{4.3}$$

This term is also invariant under (1.20), but it does not satisfy the detailed balance condition: There is no *local* action that would yield this term as the sum of squares of its equations of motion.[2] Thus, pure $z = 2$ conformal gravity in $2 + 1$ dimensions with detailed balance has no potential term.

This conformal $z = 2$ gravity in $2 + 1$ dimensions can be coupled to scalars $X^a(t, \mathbf{x})$. Anisotropic Weyl invariance of the classical action will be preserved when we assign scaling dimension zero to X^a. The kinetic term becomes

$$S_K = \frac{1}{\kappa^2} \int dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N \left(\hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2 \right) + \frac{1}{2} \int dt\, d^2\mathbf{x}\, \frac{\sqrt{\gamma}}{N} \left(\partial_t X^a - N^i \partial_i X^a \right)^2 . \tag{4.4}$$

Even under the condition of detailed balance, this coupled theory develops a nontrivial potential. There is a unique potential term compatible both with anisotropic conformal invariance and the detailed balance condition,

$$S_{\mathcal{V}} = \int dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N \left\{ (\nabla^2 X^a)^2 + \frac{\kappa^2}{2} \left(\partial_i X^a \partial_j X^a - \frac{1}{2}\gamma_{ij} \partial^k X^a \partial_k X^a \right)^2 \right\}. \tag{4.5}$$

This theory, of $z = 2$ conformal gravity coupled to scalars and satisfying the detailed balance condition, first appeared in (1) as the worldvolume action of "membranes at quantum criticality," whose ground-state wavefunction on Riemann surface Σ reproduces the partition function of the critical bosonic string on Σ. The Euclidean action in $D = 2$ dimensions which yields (4.5) via the detailed balance construction is simply given by the action familiar from the critical string,

$$\mathcal{W} = \frac{1}{2} \int d^2\mathbf{x}\, \sqrt{\gamma}\, \gamma^{ij} \partial_i X^a \partial_j X^a. \tag{4.6}$$

We recognize the first term in (4.5) as the square of the X^a equation of motion, and the second term as the square of the energy-momentum tensor obtained from the γ_{ij} variation of (4.6).

[1] Throughout the chapter, we use the compact notation $\nabla^2 N \equiv \nabla i \nabla^i N$ and $(\nabla N)^2 \equiv \nabla_i N \nabla^i N$.

[2] However, as was discussed in (1), one *can* get $\mathcal{V} \sim R^2$ by squaring the equation of motion of a nonlocal action: the Polyakov conformal anomaly action $\int d^2\mathbf{x} \sqrt{\gamma} R \frac{1}{\nabla^2} R$.

4.3 Holography in Asymptotically Lifshitz Spacetimes

The metric of the Lifshitz spacetime $\mathcal{M}$ in $D+2$ dimensions,

$$ds^2 = -r^{2z}dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2}, \tag{4.7}$$

is designed so that its isometries match the expected conformal symmetries of Lifshitz field theory with dynamical exponent z. This geometry, and its various cousins, appears as a solution in several effective theories, such as the theory considered in (85) in which bulk Einstein gravity is coupled to a massive vector, and more recently also in a variety of constructions obtained from string theory (28; 29; 86; 87).

In this section, we first discuss some general features of holography and asymptotic structure of Lifshitz spacetime, which are universal and independent of the precise model. Then, in Section 4.4, we work – for specificity – in the effective bulk theory of relativistic gravity coupled to a massive vector, first without additional matter, and then coupled to bulk scalars. Even though our detailed results will depend of the chosen effective theory, we believe that our conclusions are largely universal and generalizable straightforwardly to other embeddings of Lifshitz spacetimes.

4.3.1 Anisotropic Conformal Infinity

The notion of conformal infinity plays a central role in general relativity (88; 16). It is constructed by mapping the original metric[3] $G_{\mu\nu}$ on a manifold $\mathcal{M}$ via a smooth conformal Weyl transformation to

$$\widetilde{G}_{\mu\nu} = \Omega^2(x) G_{\mu\nu}, \tag{4.8}$$

such that the rescaled metric $\widetilde{G}_{\mu\nu}$ is extendible to a larger manifold $\widetilde{\mathcal{M}}$, which contains the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ as a closed submanifold. The idea is to define the conformal infinity of $\mathcal{M}$ to be the set $\overline{\mathcal{M}} \setminus \mathcal{M}$. The scaling factor Ω must extend to $\widetilde{\mathcal{M}}$ and satisfy certain regularity conditions at $\overline{\mathcal{M}} \setminus \mathcal{M}$ (the most essential being that it should have a single zero there and that its exterior derivative should be nonzero), but is otherwise arbitrary. A change from one permissible scaling factor to another is interpreted as a conformal transformation at $\overline{\mathcal{M}} \setminus \mathcal{M}$: Hence, conformal infinity carries a naturally defined preferred conformal structure.

This notion of conformal infinity allows one to define precisely, and in a coordinate-independent way, the notion of an event horizon (and hence the notion of black holes), as the boundary of the causal past of the future infinity. Moreover, it allows us to define precisely the concept of spacetimes which "asymptotically approach" a chosen vacuum solution "at infinity." In the example of anti-de Sitter spacetime, this picture is naturally compatible with the physical ideas of holography: The conformal infinity of AdS is of codimension one,

[3] We are using Penrose's "abstract index" notation.

and carries the natural conformal structure induced from the asymptotic isometries of the bulk, just as predicted by the holographic dictionary.

In contrast, as explained in Chapter 3, the intuition of holographic renormalization in Lifshitz and Schrödinger spacetimes clashes with this classic notion of conformal infinity as defined by Penrose: For the Lifshitz spacetime (4.7) with $z > 1$, the relativistic conformal infinity is of dimension one for any D, and it does not inherit the conformal structure expected from the symmetries of nonrelativistic field theory in $D + 1$ dimensions. To see that, it is useful to switch first to the radial coordinate $u = 1/r$, which stays finite as we approach the naive infinity at $r \to \infty$, with the metric now

$$ds^2 = -\frac{dt^2}{u^{2z}} + \frac{d\mathbf{x}^2 + du^2}{u^2}. \tag{4.9}$$

For $z > 1$, the dt^2 term is the most divergent one as we take $u \to 0$. In order to make the rescaled metric finite, we would like to use $\Omega = u^z$. However, this choice of Ω does not have a simple zero at $u = 0$ in this coordinate system. In order to fix this, we change coordinates once again, to $w = u^z$. The metric becomes

$$ds^2 = -\frac{dt^2}{w^2} + \frac{d\mathbf{x}^2}{w^{2/z}} + \frac{dw^2}{z^2 w^2}. \tag{4.10}$$

We can now use $\Omega = w$ to rescale the metric, but the resulting geometry

$$\widetilde{ds}^2 = -dt^2 + \frac{1}{z^2} dw^2 + w^{2(1-1/z)} d\mathbf{x}^2 \tag{4.11}$$

is degenerate at the purported infinity $w = 0$ when $z \neq 1$. As a consequence of this rather pathological behavior of the standard notion of conformal infinity of the Lifshitz spacetime, it is a priori unclear how to perform holographic renormalization, and even how to define precisely what we mean by black holes in the bulk.

We saw in Chapter 3 how to resolve this tension, for spacetimes carrying the additional structure of an asymptotic foliation, by generalizing Penrose's notion of conformal infinity to reflect the asymptotic anisotropy permitted by the foliation. The basic idea is simple: When $\mathcal{M}$ carries a preferred foliation at least near infinity, we can use the anisotropic Weyl transformation (1.20), instead of the relativistic rescaling (4.8), to map $\mathcal{M}$ inside a larger manifold $\widetilde{\mathcal{M}}$ such that $\overline{\mathcal{M}} \subset \widetilde{\mathcal{M}}$. Even in this case, the rescaling factor $\Omega = e^{\omega}$ must satisfy regularity conditions at $\overline{\mathcal{M}} \setminus \mathcal{M}$. In particular, Ω must have a simple zero there. With a judiciously chosen value of z, the *anisotropic conformal infinity* $\overline{\mathcal{M}} \setminus \mathcal{M}$ can be of codimension one. Moreover, it naturally inherits a preferred "anisotropic conformal structure," with conformal transformations given by those foliation-preserving diffeomorphisms that preserve the boundary metric up to an anisotropic Weyl rescaling.

Both Lifshitz and Schrödinger spacetimes belong to this class of asymptotically foliated geometries, and the resulting notion of anisotropic conformal infinity matches the

intuitive expectations from holography. In the case of the Lifshitz spacetime (4.7), we start again with the metric as given in (4.9). We interpret this geometry as carrying a natural codimension-one foliation by leaves of constant t, at least near $u \to 0$. As we saw in Section 1.3.3, this additional structure of an asymptotic foliation gives us the additional freedom to use anisotropic Weyl transformations 1.20 without violating the symmetries. Choosing the rescaling factor

$$\Omega = u \tag{4.12}$$

and applying the anisotropic Weyl transformation (1.20) maps the Lifshitz metric in the asymptotic regime of $u \to 0$ to the flat metric,

$$\widetilde{ds}^2 = -dt^2 + d\mathbf{x}^2 + du^2. \tag{4.13}$$

u can now be analytically extended from $u > 0$ to all real values. The anisotropic conformal infinity of the $(D+2)$-dimensional Lifshitz spacetime is at $u = 0$. Topologically, it is $\mathbf{R}^{D+1}$, and very similar to the conformal infinity of the Poincaré patch of AdS_{D+2}. However, even though the induced metric on anisotropic conformal infinity at $u = 0$ in (4.13) looks naively relativistic, one must remember that its natural symmetries are not relativistic: This conformal infinity carries a preferred foliation by leaves of constant t, and a natural anisotropic conformal structure characterized by dynamical exponent z. The natural symmetries are given by those foliation-preserving diffeomorphisms that preserve the metric up to an anisotropic Weyl transformation (*c.f.* Chapter 3). In addition to the spatial rotations and spacetime translations of $\mathbf{R}^{D+1}$, one can easily check that this symmetry group contains also the anisotropic scaling transformations (4.1). Thus, the conformal structure of anisotropic conformal infinity nicely matches the expected conformal symmetries of the dual field theory.

4.3.2 Asymptotically Lifshitz Spacetimes

Equipped with the notion of anisotropic conformal infinity of spacetime, we can now give a precise definition of spacetime geometries that are "asymptotically Lifshitz." Simply put, given a value of z, a spacetime is said to be asymptotically Lifshitz if it exhibits the same anisotropic conformal infinity as the Lifshitz spacetime for that value of dynamical exponent z. This definition follows the logic that leads to the notions of asymptotic flatness and asymptotic AdS (88; 16), and extends such notions naturally to the case of anisotropic scaling.

As a part of their definition, the spacetimes which are asymptotically Lifshitz must carry an asymptotic foliation structure near their anisotropic conformal infinity. In the context of holographic renormalization, this condition translates into an important restriction on the form of the vielbein fall-off,

$$\frac{e^0_i}{r^z} \to 0 \qquad \text{as } r \to \infty. \tag{4.14}$$

This provides an answer to a question discussed in (21): Our definition of asymptotically Lifshitz spacetimes using the notion of anisotropic conformal infinity requires that the sources for the energy flux vanish.[4]

With the definition of "asymptotically Lifshitz" at hand, it is now possible to define precisely black holes and their event horizons in Lifshitz spacetimes, by referring to the properties of the anisotropic conformal infinity of spacetime just as in the more traditional spacetimes which have codimension-one isotropic conformal infinity. We refer the reader back to Chapter 3 for additional results, and to Appendix 4.6.6 for a summary of the asymptotic behavior of fields in the asymptotically Lifshitz spacetimes relevant for the rest of this chapter.

4.4 Holographic Renormalization in Asymptotically Lifshitz Spacetimes

Holographic duality in asymptotically AdS spacetimes[5] – or, by logical extension, in asymptotically Lifshitz spacetimes – relates the partition function of a bulk gravity system with Dirichlet boundary conditions at conformal infinity to the generating function of correlators in the appropriate dual quantum field theory. At low energies and to leading order, this correspondence gives the connected generating functional W with sources $f^{(0)}$ on the field theory side, in terms of the on-shell bulk gravity action evaluated with Dirichlet boundary conditions given by $f^{(0)}$:

$$W[f^{(0)}] = -S_{\text{on-shell}}[f^{(0)}]. \tag{4.15}$$

Both sides of this correspondence are divergent: Standard ultraviolet divergences appear on the field theory side, and they require conventional renormalization. This behavior is matched on the gravity side, where the divergences are infrared effects, due to the scales that diverge as we approach the spacetime boundary. Holographic renormalization (72; 91; 92; 93; 94; 19) (for reviews, see (95; 20; 96)) is the technology designed to perform the subtraction of infinities on the gravity side, in the form of divergent boundary terms in the on-shell action, and to make precise sense of (4.15).

Recent papers (21; 97; 98) have performed various steps of holographic renormalization in Lifshitz spacetime at the non-linear level, and this chapter builds on the results established there. Since we choose for our analysis the Hamiltonian approach to holographic renormalization (99; 100), our treatment is closest to that of (21).

[4]More precisely, it would be sufficient to impose $(\partial_i e^0_j - \partial_j e^0_i)/r^z \to 0$ at infinity, a constraint which also emerges naturally in the vielbein formulation of gravity with anisotropic scaling. In this chapter, we impose the stronger condition (4.14).

[5]For a pedagogical introduction, see the TASI lectures (89) and (90).

4.4.1 Hamiltonian Approach to Holographic Renormalization

The original analysis of holographic renormalization relied on properties of asymptotic expansions near the conformal infinity of spacetime (101; 102; 103). The Hamiltonian approach of (99; 100) aspires to give a somewhat more covariant picture, and the results of the earlier asymptotic expansion approach can be reproduced from it (99). Either way, we start by choosing a radial coordinate, r, in some neighborhood of the anisotropic conformal infinity of the Lifshitz spacetime $\mathcal{M}$, such that the hypersurfaces of constant r are diffeomorphic to the boundary $\partial\mathcal{M}$, and they equip $\mathcal{M}$ near $\partial\mathcal{M}$ with a codimension-one foliation structure.[6] This foliation should not be confused with the preferred folation of the anisotropic conformal boundary by leaves of constant t – the asymptotic regime of our spacetime carries a nested foliation structure, with leaves of constant radial coordinate r further foliated by leaves of constant t.

Our starting point is the theory of bulk gravity in $3+1$ dimensions[7] with negative Λ, coupled to some matter Φ to be specified later. The action is given by

$$S_{\text{bulk}} = \frac{1}{16\pi G_4}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\, dr\, \sqrt{-G}\,(\mathcal{R}-2\Lambda) + \frac{1}{8\pi G_4}\int_{\partial\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{-g}\,K + S_{\text{matter}}[\Phi, G]. \tag{4.16}$$

We will work throughout in the radial gauge, setting the radial lapse to $1/r$ and the radial shift to zero, in some neighborhood of the boundary (see Appendix 4.6 for a detailed summary of our notation and conventions).

Our task is to evaluate the on-shell action as a functional of the boudary fields, and perform the corresponding renormalization. Because of the infinite volume of Lifshitz space, the on-shell action diverges, and must first be regularized by inserting a cutoff at finite volume and indentifying terms that diverge in the asymptotic expansion in the cutoff, and then renormalized by and introducing appropriate counterterms to eliminate the divergences. The on-shell action is regulated by cutting the bulk spacetime off at some value $r < \infty$ of the radial coordinate. If $\mathcal{M}_r$ is the cut-off manifold, its boundary $\partial\mathcal{M}_r$ represents a regulated boundary of spacetime. The on-shell action is a function of the regulator r, and the boundary fields which include the metric multiplet N, N_i, γ_{ij} plus all sources associated with the bulk matter Φ, which we collectively denote by ϕ. From now on, we simply denote the on-shell action $S_{\text{on-shell}}$ – viewed as a functional of the boundary values of the fields – by S, and parametrize it as

$$S = \frac{1}{16\pi G_4}\int dt\, d^2\mathbf{x}\,\sqrt{\gamma}\, N\mathcal{L}. \tag{4.17}$$

Since the on-shell action S is a function of r and the boundary values of the fields, we can naturally interpret it as a solution to the Hamilton-Jacobi equation, regarding r as the

[6] In our conventions, $\partial\mathcal{M}$ is at $r=\infty$. The choice of $u = 1/r$ instead of r as a coordinate near $\partial\mathcal{M}$ would be more appropriate, since u is finite through $\partial\mathcal{M}$. In this section, we leave this more rigorous coordinate choice implicit.

[7] The case of general D and z is discussed in Appendix 4.6.

evolution parameter. This is the starting point for the Hamiltonian approach to holographic renormalization. The Hamilton-Jacobi theory implies that the first variation of the on-shell action with respect to the boundary fields gives the conjugate momenta. In the holographic dictionary, the boundary fields serve as sources, and their conjugate momenta are thus directly related to the one-point functions of the operators conjugate to the sources.

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to their scaling with r. More precisely, we define the dilatation operator by

$$\delta_{\mathcal{D}} = \int_{\partial\mathcal{M}_r} dt\, d^2\mathbf{x} \left(zN\frac{\delta}{\delta N} + 2N_i\frac{\delta}{\delta N_i} + 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}} - \sum_{\phi}\Delta_\phi\phi\frac{\delta}{\delta\phi}, \right), \tag{4.18}$$

where Δ_ϕ collectively denotes the asymptotic decay exponents of the bulk matter fields Φ. Quantities of interest can then be decomposed into a sum of terms with definite scaling dimension under $\delta_{\mathcal{D}}$. For example, the object of our central interest, $\mathcal{L}$, can be expanded as

$$\mathcal{L} = \sum_{\Delta}\mathcal{L}^{(\Delta)} + \widetilde{\mathcal{L}}^{(z+2)}\log r. \tag{4.19}$$

Throughout this chapter, superscripts in parentheses on any object $\mathcal{O}$ always denote the scaling dimension in the decomposition of $\mathcal{O}$ as a sum of terms of definite engineering scaling dimensions. For example, $T^{(0)}_{AB}$ is the constant part of the stress tensor, and $R^{(2)}$ is the dimension-two part of the scalar curvature.

The individual terms of the expansion (4.19) satisfy

$$\delta_{\mathcal{D}}\mathcal{L}^{(\Delta)} = -\Delta\,\mathcal{L}^{(\Delta)} \qquad \text{for} \quad \Delta \neq z+2. \tag{4.20}$$

When $\Delta = z+2$, the scaling behavior is anomalous,

$$\delta_{\mathcal{D}}\mathcal{L}^{(z+2)} = -(z+2)\mathcal{L}^{(z+2)} + \widetilde{\mathcal{L}}^{(z+2)}, \tag{4.21}$$

with the inhomogeneous term satisfying

$$\delta_{\mathcal{D}}\widetilde{\mathcal{L}}^{(z+2)} = -(z+2)\widetilde{\mathcal{L}}^{(z+2)}. \tag{4.22}$$

This logarithmic term in (4.19) reflects the possibility of an anisotropic Weyl anomaly.

The dynamical equations for the divergent part of $\mathcal{L}$ are determined as follows. Since the on-shell action satisfies the Hamilton-Jacobi equation, its radial derivative is determined in terms of the Hamiltonian. Because in our case the fields have fixed asymptotic behavior (see Appendix 4.6.6), in the asymptotic region the radial derivative is equivalent to the anisotropic scaling operator,

$$r\frac{d}{dr} \approx \delta_{\mathcal{D}}. \tag{4.23}$$

The Hamilton-Jacobi equation then relates the action of $\delta_{\mathcal{D}}$ on the on-shell action to the Hamiltonian of the system. In our case, with relativistic gravity in the bulk, the equation

of motion for the radial lapse gives the Hamiltonian constraint. Using the bulk equations of motion, one obtains a first-order differential equation for $\mathcal{L}$ in terms of the boundary values of the fields that can be solved iteratively in the expansion in eigenmodes of $\delta_{\mathcal{D}}$.

Equivalently, one can expand the Hamiltonian constraint in eigenmodes of $\delta_{\mathcal{D}}$. The structure of these equations allows for the momentum modes to be obtained recursively in terms of the boundary data. In this method, the dilatation operator acting on the on-shell action gives an expression linear in the canonical momenta, so that the values for the momenta obtained recursively from the Hamiltonian constraint give rise directly to the desired expression on-shell action. The resulting on-shell action will have divergent pieces that can be expressed as local functionals of the boundary data. These pieces can be subtracted, leading to the finite renormalized on-shell action.

Further technical details of the procedure for determining the coefficients $\mathcal{L}^{(\Delta)}$ and $\widetilde{\mathcal{L}}^{(z+2)}$ are summarized in Appendix 4.7.

4.4.2 Bulk Gravity with a Massive Vector

We begin with the theory of relativistic bulk gravity in $3+1$ dimensions, coupled to a bulk massive vector field $\mathcal{A}_\mu$. The action is

$$\begin{aligned} S_{\text{bulk}} &= \frac{1}{16\pi G_4}\int_{\mathcal{M}} dt\, d^2\mathbf{x}\, dr\, \sqrt{-G}\left(\mathcal{R}-2\Lambda-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}m^2\mathcal{A}_\mu\mathcal{A}^\mu\right) \\ &\quad +\frac{1}{8\pi G_4}\int_{\partial\mathcal{M}} dt\, d^2\mathbf{x}\,\sqrt{-g}\,K. \end{aligned} \tag{4.24}$$

As summarized in Appendix 4.6, this theory has the Lifshitz spacetime as a solution. In this theory, the boundary data we can specify reduce to the metric multiplet (N, N_i, γ_{ij}) – or, alternatively, their vielbein counterparts (see Appendix 4.6.3) – and a scalar source ψ for the massive vector.

Although our main interest will be in $z=2$, we start by considering general z. If we set $\psi=0$, the terms that will give rise to divergent contributions in the on-shell action for $z<4$ are $\mathcal{L}^{(0)}$, $\mathcal{L}^{(2)}$, $\mathcal{L}^{(2z)}$, and $\mathcal{L}^{(4)}$. The holographic renormalization equations, found in in (21) (and reviewed in Appendix 4.7), take the form

$$\mathcal{L}^{(0)} = 2(z+1), \tag{4.25}$$

$$z\mathcal{L}^{(2)} = R^{(2)}-\frac{1}{4}(F_{AB}F^{AB})^{(2)}, \tag{4.26}$$

$$(2-z)\mathcal{L}^{(2z)} = R^{(2z)}+\frac{1}{2m^2}\left((\nabla_A\pi^A)^{(z)}\right)^2 \tag{4.27}$$

$$(z-2)\mathcal{L}^{(4)} = K^{(2)}_{AB}\pi^{AB(2)}+\frac{1}{2}\pi^{A(2)}\pi_A{}^{(2)}. \tag{4.28}$$

With some effort these can be computed in terms of the boundary metric complex (N, N_i, γ_{ij}),

giving (up to total derivatives)

$$\mathcal{L}^{(0)} = 2(z+1), \tag{4.29}$$

$$z\mathcal{L}^{(2)} = \hat{R} + \frac{\alpha^2}{2}\left(\frac{\nabla N}{N}\right)^2, \tag{4.30}$$

$$(2-z)\mathcal{L}^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} + \frac{z-3}{2}\hat{K}^2, \tag{4.31}$$

$$\begin{aligned}(2-z)\mathcal{L}^{(4)} = {} & \frac{z-2}{2z^4(z+1)(z-2+\beta_z)^2}\left\{-4z(z-6+\beta_z)\left(\frac{\nabla^2 N}{N}\right)^2\right. \\ & + \left(12+36z-11z^2-2z^3+5z^4+\beta_z(z^3-7z-2)\right)\left[\left(\frac{\nabla N}{N}\right)^2\right]^2 \\ & - 2z\left(36-4z-7z^2+5z^3+\beta_z(z^2-z-6)\right)\frac{\nabla^2 N}{N}\left(\frac{\nabla N}{N}\right)^2 \\ & \left. + (z-6+\beta_z)\left[4z^2\left(\frac{\nabla N}{N}\right)^2\hat{R} - 4z^2\frac{\nabla^2 N}{N}\hat{R} - z^3\hat{R}^2\right]\right\}. \end{aligned} \tag{4.32}$$

When $z=2$ is approached, the divergent terms of dimension four become logarithmic, and the residue of the $\Delta=4$ (or $\Delta=2z$) terms at the $z=2$ pole give rise to $\widetilde{\mathcal{L}}^{(4)}$. Specifically, we get

$$\widetilde{\mathcal{L}}^{(4)} = \lim_{z\to 2}\left[(z-2)\mathcal{L}^{(4)} + (2-z)\mathcal{L}^{(2z)}\right]. \tag{4.33}$$

With this substitution, the $z=2$ divergent terms in the on-shell action are

$$\mathcal{L}^{(0)} = 2(z+1) = 6, \tag{4.34}$$

$$\mathcal{L}^{(2)} = \frac{1}{2}\hat{R} + \frac{1}{4}\left(\frac{\nabla N}{N}\right)^2, \tag{4.35}$$

$$\widetilde{\mathcal{L}}^{(4)} = \hat{K}_{ij}\hat{K}^{ij} - \frac{1}{2}\hat{K}^2. \tag{4.36}$$

The coefficient $\widetilde{\mathcal{L}}^{(4)}$ of the logarithmic divergence can be recognized as the unique kinetic term (4.2) for Lifshitz gravity with local conformal invariance in $2+1$ dimensions, invariant under the $z=2$ anisotropic Weyl transformations (1.20). This is one of the central results of this chapter.

The expression for the counterterms has no potential term – *i.e.*, the only derivatives that appear in the counterterm are the time derivatives. This is in spite of the fact that there exists a term with spatial derivatives, invariant under the local $z=2$ anisotropic Weyl transformations,

$$\int dt\, d^2\mathbf{x}\,\sqrt{g}\,N\left\{\hat{R} + \frac{\nabla^2 N}{N} - \left(\frac{\nabla N}{N}\right)^2\right\}^2, \tag{4.37}$$

which is not a total derivative.

It is surprising, at least at first sight, that such a potential term is not generated in the logarithmic counterterm of holographic renormalization in Lifshitz space. Indeed, as we show in Appendix 4.8, this term (4.37) represents a non-trivial cohomology class appropriate to appear as an anomaly. What would be a minimal generalization of our holographic setup, which would generate such a term in the anomaly? One might suspect that a different dynamical embedding of the Lifshitz space may perhaps produce a more general set of holographic counterterms, allowing (4.37) to appear. Even in the embedding considered here, we have not turned on the most general sources in the boundary, and one can ask whether allowing nonzero ψ generates new counterterms. However, a detailed calculation (see Appendix 4.7) reveals that turning on ψ also preserves detailed balance, and does not lead to the appearance of the second independent counterterm (4.37).

4.4.3 Gravity with a Massive Vector Coupled to Bulk Scalars

In order to probe further the structure of holographic counterterms in Lifshitz spacetime, it is useful to add additional matter fields in the bulk theory. The holographic renormalization procedure can be easily repeated with the inclusion of scalar fields in the bulk. We will see that for a marginal scalar at $z = 2$, there is a new logarithmically divergent counterterm, giving rise to a new, nongravitational contribution to the anisotropic Weyl anomaly. However, we will see that this new counterterm also satisfies the detailed balance condition: Even in the presence of the bulk scalars, the second gravitational counterterm (4.37) – which violates detailed balance – is not generated.

The bulk scalar action takes the standard relativistic form

$$S_{\text{bulk, } X} = -\frac{1}{2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-G}\left(G^{\mu\nu}\partial_\mu X^a\partial_\nu X^a + \mu^2 X^a X^a\right). \tag{4.38}$$

In this section, we set $d = 3$, and again follow the procedure of (21), with appropriate modifications to include the scalar fields. The holographic renormalization equations of (21) now become

$$(z+2-\Delta)\mathcal{L}^{(\Delta)} = \widetilde{\mathcal{Q}}^{(\Delta)} + \widetilde{\mathcal{S}}^{(\Delta)}, \tag{4.39}$$

where the quadratic and source terms $\mathcal{Q}$ and $\mathcal{S}$ are modified to

$$\widetilde{\mathcal{Q}}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + 8\pi G_4(\widetilde{\pi}^{a(\Delta/2)})^2 + 16\pi G_4 \sum_{s<\Delta/2;\ s\neq\widetilde{\Delta}_-} (\widetilde{\pi}^{a(s)}\widetilde{\pi}^{a(\Delta-s)}) \tag{4.40}$$

and

$$\widetilde{\mathcal{S}} = \mathcal{S} - 8\pi G_4(\partial_\alpha X^a\partial^\alpha X^a + \mu^2 X^a X^a). \tag{4.41}$$

In this expression, $\widetilde{\pi}^a = r\partial_r X^a$ is the scalar momentum and the scalars fall of asymptotically as $r^{-\widetilde{\Delta}_-}$, where

$$\mu^2 = \widetilde{\Delta}_-(\widetilde{\Delta}_- - 2 - z).$$

The additional source terms only contribute at orders $\Delta = 2\widetilde{\Delta}_-$, $2+2\widetilde{\Delta}_-$ and $2z+2\widetilde{\Delta}_-$:

$$\widetilde{\mathcal{S}}^{(2\widetilde{\Delta}_-)} = -8\pi G_4 \mu^2 X^a X^a, \tag{4.42}$$

$$\widetilde{\mathcal{S}}^{(2+2\widetilde{\Delta}_-)} = -\left[8\pi G_4 \partial_\alpha X^a \partial^\alpha X^a\right]^{(2+2\widetilde{\Delta}_-)} = -8\pi G_4 \partial_i X^a \partial^i X^a, \tag{4.43}$$

$$\widetilde{\mathcal{S}}^{(2z+2\widetilde{\Delta}_-)} = -\left[8\pi G_4 \partial_\alpha X^a \partial^\alpha X^a\right]^{(2z+2\widetilde{\Delta}_-)} = \frac{8\pi G_4}{N^2}(\partial_t X^a - N^i \partial_i X^a)^2. \tag{4.44}$$

We now specialize to the case of a marginal scalar, that is, a scalar which has $\widetilde{\Delta}_- = 0$. Note that this also means that the scalar is massless since $\mu^2 = \widetilde{\Delta}_-(\widetilde{\Delta}_- - 2 - z) = 0$. We are interested in calculating the contribution to the anisotropic Weyl anomaly in the case $z = 2$. The divergent pieces of the on-shell action that appear at orders $\Delta = 2 + 2\widetilde{\Delta}_-$ and $\Delta = 2z + 2\widetilde{\Delta}_-$ are straightforward to calculate as they only receive contributions from the source terms,

$$(z - 2\widetilde{\Delta}_-)\mathcal{L}^{(2+2\widetilde{\Delta}_-)} = -8\pi G_4 \partial_i X^a \partial^i X^a, \tag{4.45}$$

$$(2 - z - 2\widetilde{\Delta}_-)\mathcal{L}^{(2z+2\widetilde{\Delta}_-)} = \frac{8\pi G_4}{N^2}(\partial_t X^a - N^i \partial_i X^a)^2. \tag{4.46}$$

By taking the functional derivative of this term in the on-shell action with respect to the metric, the contribution to the boundary stress energy tensor can be calculated. For example, for $\Delta = 2 + 2\widetilde{\Delta}_-$

$$(z - 2\widetilde{\Delta}_-)T_{00}^{(2+2\widetilde{\Delta}_-)} = -8\pi G_4 \partial_i X^a \partial^i X^a, \tag{4.47}$$

$$(z - 2\widetilde{\Delta}_-)T_{IJ}^{(2+2\widetilde{\Delta}_-)} = -16\pi G_4 \partial_I X^a \partial_J X^a + 8\pi G_4 \partial_i X^a \partial^i X^a \delta_{IJ}, \tag{4.48}$$

$$(z - 2\widetilde{\Delta}_-)T_{0I}^{(3-z+2\widetilde{\Delta}_-)} = 0. \tag{4.49}$$

In addition, by taking the functional derivative with respect to the scalar, the boundary scalar momentum can be calculated, via

$$\widetilde{\pi}^a = -\frac{1}{N\sqrt{\gamma}}\frac{\delta S}{\delta X^a}.$$

For example, one gets

$$(z - 2\widetilde{\Delta}_-)\widetilde{\pi}^{a(2+\widetilde{\Delta}_-)} = -\frac{1}{N}\nabla^i(N\nabla_i X^a) = -\nabla^2 X^a - \frac{\nabla^i N \nabla_i X^a}{N}. \tag{4.50}$$

The higher order counterterms are more involved because they receive contributions from the quadratic piece. For example,

$$\begin{aligned}(z - \Delta_- - 2\widetilde{\Delta}_-)\mathcal{L}^{(2+\Delta_-+2\widetilde{\Delta}_-)} &= K_{AB}^{(\Delta_-)} T^{AB(2+2\widetilde{\Delta}_-)} \\ &= -\frac{\alpha\psi}{2(z+1)}\left[z(3z - \Delta_-)T^{00(2+2\widetilde{\Delta}_-)} + z(2z - 1 - \Delta_-)T_I^{I(2+2\widetilde{\Delta}_-)}\right]\end{aligned}$$

$$= -\frac{\alpha\psi}{2(z+1)}\left[z(3z-\Delta_-)T^{00(2+2\widetilde{\Delta}_-)}\right], \tag{4.51}$$

using the fact that $T_I^{I(2+2\widetilde{\Delta}_-)}=0$, as calculated above. Note that for $z=2$ this becomes $\mathcal{L}^{(2+\Delta_-+2\widetilde{\Delta}_-)}=-\psi T^{00(2+2\widetilde{\Delta}_-)}$. The calculation of this term is useful even when the source for the massive vector ψ is set to zero. This is because we can determine $\pi_\psi^{(2+2\widetilde{\Delta}_-)}$ by taking the functional derivative with respect to ψ:

$$(z-\Delta_--2\widetilde{\Delta}_-)\pi_\psi^{(2+2\widetilde{\Delta}_-)}=\frac{\alpha}{2(z+1)}\left[z(3z-\Delta_-)T^{00(2+2\widetilde{\Delta}_-)}\right]. \tag{4.52}$$

The following terms also receive contributions from the quadratic piece:

$$(z-2-2\widetilde{\Delta}_-)\mathcal{L}^{(4+2\widetilde{\Delta}_-)} = 2K_{AB}^{(2)}T^{AB(2+2\widetilde{\Delta}_-)}+\pi_A^{(2)}\pi^{A(2+2\widetilde{\Delta}_-)}+8\pi G_4\left(\widetilde{\pi}^{a(2+\widetilde{\Delta}_-)}\right)^2, \tag{4.53}$$

$$(z-2-4\widetilde{\Delta}_-)\mathcal{L}^{(4+4\widetilde{\Delta}_-)} = K_{AB}^{(2+2\widetilde{\Delta}_-)}T^{AB(2+2\widetilde{\Delta}_-)}+\frac{1}{2}\pi_A^{(2+2\widetilde{\Delta}_-)}\pi^{A(2+2\widetilde{\Delta}_-)}. \tag{4.54}$$

These are the terms that will contribute to the scaling anomaly when $z=2$. After a lengthy calculation of the right hand sides for $z=2$, the following result is obtained (up to total derivatives):

$$(z-2-2\widetilde{\Delta}_-)\mathcal{L}^{(4+2\widetilde{\Delta}_-)} = 2\pi G_4(\Delta X^a)^2, \tag{4.55}$$

$$\begin{aligned}(z-2-4\widetilde{\Delta}_-)\mathcal{L}^{(4+4\widetilde{\Delta}_-)} &= \frac{1}{4}T_{IJ}^{(2+2\widetilde{\Delta}_-)}T^{IJ(2+2\widetilde{\Delta}_-)},\\ &= 16\pi^2G_4^2\left(\partial_iX^a\partial_jX^a\partial^iX^b\partial^jX^b-\frac{1}{2}(\partial_iX^a\partial^iX^a)^2\right).\end{aligned} \tag{4.56}$$

By combining all these results, the contribution of the massless scalars to the logarithmically divergent counterterm when $z=2$ is (by equation (4.120)):

$$\begin{aligned}\widetilde{\mathcal{L}}_X^{(4)} &= \lim_{z\to2}\left((2-z)\mathcal{L}^{(2z+2\widetilde{\Delta}_-)}+(z-2)\mathcal{L}^{(4+2\widetilde{\Delta}_-)}+(z-2)\mathcal{L}^{(4+4\widetilde{\Delta}_-)}\right)\\ &= \frac{8\pi G_4}{N^2}(\partial_tX^a-N^i\partial_iX^a)^2+2\pi G_4(\nabla^2X^a)^2\\ &\quad+16\pi^2G_4^2\left(\partial_iX^a\partial_jX^a\partial^iX^b\partial^jX^b-\frac{1}{2}(\partial_iX^a\partial^iX^a)^2\right).\end{aligned} \tag{4.57}$$

Together with the gravitational counterterms from the previous section, the total counterterm action for $z=2$ is given by

$$\begin{aligned}S_{ct} = -\int_{\partial\mathcal{M}}dt\,d^2\mathbf{x}\sqrt{\gamma}N\Bigg\{&\frac{1}{16\pi G_4}\left[6+\frac{1}{2}\hat{R}+\frac{1}{4}\left(\frac{\nabla N}{N}\right)^2\right]-\frac{1}{4}\partial_iX^a\partial^iX^a\\ &-\log\epsilon\left[\frac{1}{16\pi G_4}(\hat{K}_{ij}\hat{K}^{ij}-\frac{1}{2}\hat{K}^2)+\frac{1}{2N^2}(\partial_tX^a-N^i\partial_iX^a)^2+\frac{1}{8}(\nabla^2X^a)^2\right.\end{aligned}$$

$$+ \pi G_4 \left(\partial_i X^a \partial_j X^a \partial^i X^b \partial^j X^b - \frac{1}{2} (\partial_i X^a \partial^i X^a)^2 \right) \Big] \Big\} . \tag{4.58}$$

Interestingly, this logarithmically divergent counterterm takes the form identical to the action written down in (1), describing the coupling of $z = 2$ gravity and $z = 2$ Lifshitz matter in $2 + 1$ dimensions. This action is invariant under $z = 2$ anisotropic Weyl transformations, with the scalars transforming with weight zero, and satisfies the detailed balance condition. As a result, it was shown in (1) that the ground-state wavefunction of this membrane action on a spatial surface Σ is given by the bosonic string partition function on Σ. We see that the property of detailed balance, satisfied by the logarithmic counterterms in the absence of extra matter, persists in the presence of the marginal scalar fields.

Two additional comments are worth making:

(1) The relative sign between the potential terms and the kinetic term in the logarithmic counterterm is opposite to the sign one would expect from the action of a unitary theory with $z = 2$ scaling in real time. This is not very surprising, and corresponds to the fact already appreciated in the relativistic case: The holographic counterterms do not have to reproduce the action of a unitary theory, as is clear from the appearance of the higher-derivative conformal gravity action in the holographic counterterms in AdS_5.

(2) In the classical theories with Lifshitz scaling, the coupling constants in front of the individual contributions to the potential term are not related by any symmetry to the kinetic terms. Therefore, they represent classically marginal couplings. In the structure of our counterterms, we find this freedom realized only partially: A uniform overall rescaling of all the couplings in the potential can be accomplished by a shift in r, but it appears that the interaction with the bulk relativistic system eliminates the apparent freedom of the relative rescaling between different contributions to the potential from species unrelated by any symmetry in the boundary theory. This mechanism deserves further study.

4.4.4 Explaining Detailed Balance

Now that we have accumulated some evidence suggesting that the appearance of the detailed balance condition in the structure of the counterterms is rather generic, it would be desirable to obtain a more systematic explanation of this fact. It would be interesting to see why this principle should be naturally satisfied in the context of holographic renormalization.

A closer look at the structure of the holographic renormalization equations (summarized in Appendix 4.7) reveals a simple answer: In the procedure we followed in $3 + 1$ bulk dimensions, the potential terms in the counterterm at order four are generated by quadratic terms in the stress-energy tensor and field momenta at order two. These momenta arise from the functional differentiation of the counterterm at order two. Consider the counterterm appearing above at order two:

$$S_{ct}^{(2)} = - \int_{\partial \mathcal{M}} dt \, d^2\mathbf{x} \, \sqrt{\gamma} N \left\{ \frac{1}{32\pi G_4} \left[\hat{R} + \frac{1}{2} \left(\frac{\nabla N}{N} \right)^2 \right] - \frac{1}{4} \partial_i X^a \partial^i X^a \right\} . \tag{4.59}$$

This Lagrangian is exactly the one used in the detailed balance condition in (1), in the case where N does not depend upon spatial coordinates.[8] Hence, the detailed balance relation, as reviewed in Section 2.1.3, is simply a consequence of the relationship between two counterterms implied by the holographic renormalization in asymptotically Lifshitz spacetime.

It should be noted that in the above procedure, the presence of the massive vector complicates the equations and make the detailed-balance-like relation between the two actions less transparent. But the logarithmic counterterm potential terms (with scaling dimension four) are nonetheless directly derivable from the counterterms with scaling dimension two.

In fact, an analogous result also holds in the relativistic case of holographic renormalization in AdS_5, where the second order counterterm is simply the Einstein-Hilbert action and the conformal anomaly is the action $S_{\rm conf}$ of conformal gravity in $3+1$ dimensions: It turns out that $S_{\rm conf}$ is obtained by squaring the functional derivative of the Einstein-Hilbert action. The reason behind this relationship is the same: $S_{\rm conf}$ and the Einstein-Hilbert action appear as two counterterms, linked via the holographic renormalization procedure into a condition reminiscent of detailed balance.

A closer look also reveals that the holographic justification for the detailed balance condition being satisfied by the logarithmic conterterm quickly ceases to be valid with increasing spacetime dimension. However, this property does not disappear completely: Instead, the holographic renormalization machinery implies a more complex relation between the logarithmic counterterm and the variational derivatives of the entire hierarchy of the power-law counterterms.

4.4.5 Analytic Continuation to the de Sitter-like Regime

In the relativistic AdS/CFT correspondence, the Hamilton-Jacobi formulation of holographic renormalization – with the radial direction r as the evolution parameter – can be easily continued analytically to de Sitter space. Upon this continuation, the evolution parameter r becomes the real time η, and the analytic continuation of the counterterms gives useful information about the wavefunction Ψ of the Universe on superhorizon scales (80; 81; 79). In particular, in the case of AdS_5 continued analytically to dS_5, the exponential of the logarithmic counterterm (known to take the form of the relativistic conformal gravity action $S_{\rm conf}$ in $3+1$ dimensions) is related to the wavefunction via

$$|\Psi|^2 = e^{-S_{\rm conf}}. \tag{4.60}$$

In this chapter, we have analyzed holographic counterterms in the Lifshitz space background, and in the case of $z=2$ and $3+1$ bulk dimensions, we also found a logarithmic counterterm in the form of a $z=2$ multicritical conformal gravity action. It is natural to ask whether an analytic continuation exists, similar to the one studied in (80; 81; 79), so that the $z=2$

[8] Detailed balance in the nonprojectable theory has been discussed recently in (104).

anisotropic conformal gravity action similarly produces the square of the wavefunction of the dual system. The answer appears to be yes, and the dual system is a gravity theory with an interesting kind of spatial anisotropy.

Reintroducing the length scale L_r in the spacetime metric of the Lifshitz space at $z = 2$,

$$ds^2 = L_r^2 \left(-r^4 dt^2 + r^2 d\mathbf{x}^2 + \frac{dr^2}{r^2} \right), \tag{4.61}$$

we can analytically continue our results by taking $r = i\eta$ and $L_r = -iL_\eta$ and relabeling $t = y$, which leads to the following spacetime:

$$ds^2 = L_\eta^2 \left(\eta^4 dy^2 + \eta^2 d\mathbf{x}^2 - \frac{d\eta^2}{\eta^2} \right). \tag{4.62}$$

This spacetime can be viewed as a spatially anisotropic, "multicritical" version of de Sitter space. We found the on-shell action for asymptotically Lifshitz space to be (with the cutoff at $r = 1/\epsilon_r$):

$$\begin{aligned} S &= \frac{L_r^2}{16\pi G_4} \int_{\partial \mathcal{M}_{1/\epsilon_r}} dt\, d^2\mathbf{x}\, \sqrt{\gamma}\, N (\mathcal{L}^{(0)} + \mathcal{L}^{(2)} + \mathcal{L}^{(4)} - \widetilde{\mathcal{L}}^{(4)} \log \epsilon_r) && (4.63) \\ &= \frac{L_r^2}{16\pi G_4} \int_{\partial \mathcal{M}_{1/\epsilon_r}} dt\, d^2\mathbf{x}\, \sqrt{\gamma_{\text{fin}}} N_{\text{fin}} \left\{ \frac{\mathcal{L}^{(0)}_{\text{fin}}}{\epsilon_r^4} + \frac{\mathcal{L}^{(2)}_{\text{fin}}}{\epsilon_r^2} + \mathcal{L}^{(4)}_{\text{fin}} - \widetilde{\mathcal{L}}^{(4)}_{\text{fin}} \log \epsilon_r \right\}, && (4.64) \end{aligned}$$

where the quantities with fins are defined to be finite as $r \to \infty$ (that is, $\mathcal{O}^{(\Delta)} = \mathcal{O}^{(\Delta)}_{\text{fin}} \epsilon_r^\Delta$). The analytic continuation implies that the cutoff changes to $\epsilon_r = -i\epsilon_\eta$, where $\epsilon_\eta < 0$. Note that all terms in the on-shell action remain real after the analytic continuation, except for the logarithm, which now has an imaginary part since $\log \epsilon_r = \log(-\epsilon_\eta) + i\pi/2$. Thus, after this analytic continuation, the square of the ground-state wavefunction for the spatially anisotropic version of de Sitter space is given solely by the coefficient of the logarithmic counterterm,

$$|\Psi|^2 = |e^{iS}|^2 = \exp\left\{ -\frac{L_\eta^2}{16 G_4} \int_{\partial \mathcal{M}} d^2\mathbf{x}\, dy\, \sqrt{\gamma}\, N\, \widetilde{\mathcal{L}}^{(4)} \right\}. \tag{4.65}$$

In the case of the theory studied in Section 4.4.2, we found that $\widetilde{\mathcal{L}}^{(4)}$ is the action of $z = 2$ conformal Lifshitz gravity in detailed balance. It depends only on the y derivatives but not the $\mathbf{x}$ derivatives of the metric. Thus, the ground-state wavefunction (4.65) represents a theory with spatial anisotropy, ultralocal along all but one spatial dimension, similar to the theory discussed in (105; 106).

In the theory with bulk scalars studied in Section 4.4.3, $\widetilde{\mathcal{L}}^{(4)}$ was found to be the action of $z = 2$ conformal Lifshitz gravity coupled to $z = 2$ scalars, still satisfying the detailed

balance condition. This action has a nontrivial potential term, of fourth order in the $\mathbf{x}$ derivatives of the scalars. Notably, the sign of this potential term, which we commented on at the end of Section 4.4.3, is such that the analytically continued $\widetilde{\mathcal{L}}^{(4)}$ appearing in (4.65) is positive definite.

4.5 Conclusions

The theory of gravity with anisotropic scaling introduced in (1; 2) has already been found to play a variety of roles in condensed matter. For example, linearized multicritical gravity with $z = 2$ and $z = 3$ emerges in the infrared regime of various bosonic lattice models, on a rigid lattice (46): Gravitons with the nonrelativistic dispersion relation represent low-energy collective excitations of the lattice degrees of freedom. Dynamical gravity with anisotropic scaling also emerges naturally from fermionic condensed matter systems when the fundamental fermions are integrated out (107). In the present chapter, we have added another role to this list: Multicritical gravity naturally appears in the process of holographic renormalization of relativistic systems in spacetimes which are asymptotically anisotropic and describe holographic duals of nonrelativistic field theories. In the process, for the special case of bulk $2+1$ dimensions with $z = 2$, we found that holographic renormalization imposes the condition of detailed balance on the action of $z = 2$ conformal gravity coupled to matter, and gives a new rationale for this – otherwise somewhat obscure – condition.

Clearly, various interesting open questions remain. First of all, our analysis of holographic renormalization in the simplest anisotropic example, of $z = 2$ in $3 + 1$ bulk dimensions, should generalize straightforwardly to higher integer values of z. Some calculations relevant for this task are reported in Appendix 4.7. In particular, at $z = 3$ in $4 + 1$ bulk dimensions, we expect the appearance of logarithmic counterterms taking the form of the action for $z = 3$ multicritical conformal gravity in $3+1$ dimensions, introduced in (2). Moreover, now that we have seen that the classical action of multicritical gravity appears from string-inspired holography, it would also be interesting to see whether the full *dynamics* of multicritical gravity can also be engineered from string theory, perhaps by taking judicious scaling limits of backgrounds without Lorentz invariance. Finally, it would also be natural to extend the study of nonrelativistic holography to the more general case, in which the bulk gravity itself exhibits spacetime anisotropies and multicriticality.[9] Such constructions could extend the list of nonrelativistic field theories amenable to a holographic description to a broader class, in which those nonrelativistic theories that have a relativistic bulk dual may well be only a minority.

[9]Some early steps in that direction were suggested in (1; 2).

4.6 Appendix: Notation and Conventions

We use the following bulk metric:

$$\begin{aligned} ds^2 &= G_{\mu\nu}dx^\mu dx^\nu = g_{\alpha\beta}dx^\alpha dx^\beta + \frac{dr^2}{r^2} \\ &= -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt) + \frac{dr^2}{r^2}. \end{aligned} \tag{4.66}$$

The boundary is at $r = \infty$. D is the number of spatial dimensions on the boundary and so there are $D + 2$ spacetime dimensions in the bulk and $d \equiv D + 1$ spacetime dimensions on the boundary. For coordinate indices, i, j are used for the D spatial boundary indices (x^i), whereas α, β are used for the $D + 1$ spacetime boundary indices (t, x^i) and μ, ν are used for the $D + 2$ bulk dimensions (t, x^i, r). Note that in (4.66), the bulk diffeomorphisms have been gauge fixed by setting the the bulk shift vector $\mathcal{N}_\alpha$ (defined as $\mathcal{N}_\alpha = G_{r\alpha}$) to $\mathcal{N}_\alpha = 0$, and the bulk lapse function (defined via $G_{rr} = \mathcal{N}^2 + g^{\alpha\beta}\mathcal{N}_\alpha\mathcal{N}_\beta$) to $\mathcal{N} = 1/r$. This *radial gauge* is adopted throughout the chapter. Moreover, in order to distinguish the lapse and shift variables in the bulk from those of the ADM decomposition on the boundary, we refer to the bulk variables $\mathcal{N}$ and $\mathcal{N}_i$ as the "radial lapse" and "radial shift."

It is often convenient to work in terms of vielbeins, which we define via

$$\begin{aligned} ds^2 &= \eta_{MN}E^M_\mu E^N_\nu dx^\mu dx^\nu = \eta_{AB}e^A_\alpha e^B_\beta dx^\alpha dx^\beta + \frac{dr^2}{r^2} \\ &= -N^2dt^2 + \delta_{IJ}\hat{e}^I_i\hat{e}^J_j(dx^i + N^i dt)(dx^j + N^j dt) + \frac{dr^2}{r^2}. \end{aligned} \tag{4.67}$$

For the internal frame indices, $M, N = 0, 1, ..., D+1$ are used for the $D+2$ bulk dimensions, $A, B = 0, 1, ..., D$ are used for the $D + 1$ spacetime boundary indices and $I, J = 1, ..., D$ are used for the D spatial boundary indices. The vielbeins allow coordinate indices to be changed to frame indices and vice versa, for example $F^{AB} = e^A_\alpha e^B_\beta F^{\alpha\beta}$. Also note that the vielbeins are related to the extrinsic curvature by $K_{\alpha\beta} = r(e^A_\alpha\partial_r e_{A\beta} + e^A_\beta\partial_r e_{A\alpha})/2$.

In order to distinguish the Riemann tensor and the extrinsic curvature tensor of the three different metrics $G_{\mu\nu}$, $g_{\alpha\beta}$ and γ_{ij}, we use the notation wherein $(D + 2)$-dimensional quantities are written in curly letters (for example, $\mathcal{R}$ for the Ricci scalar), $(D + 1)$-dimensional quantities are written in standard italics and D-dimensional quantities are written with hats.

4.6.1 The bulk action

The bulk spacetime relativistic action is:

$$S_{\text{bulk}} = \frac{1}{16\pi G_{D+2}}\int_{\mathcal{M}} dt\, d^D\mathbf{x}\, dr\, \sqrt{-G}\left(\mathcal{R} - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2\mathcal{A}_\mu\mathcal{A}^\mu\right)$$

$$+\frac{1}{8\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{-g}\,K. \tag{4.68}$$

Note that in order for the Lifshitz spacetime (4.7) to be a classical solution, we set

$$m^2 = Dz \quad \text{and} \quad \Lambda = -\frac{1}{2}\left(z^2 + (D-1)z + D^2\right). \tag{4.69}$$

The Lifshitz metric is sourced by a non-zero condensate of the vector field, and we denote by ψ the deviation away from this non-zero background:

$$\mathcal{A}_A = (\alpha + \psi)\delta^0_A, \tag{4.70}$$

with

$$\alpha^2 = \frac{2(z-1)}{z}. \tag{4.71}$$

The leading order behavior of the vector at the boundary is $\psi \sim r^{-\Delta_-}$, where we use the notation of (18; 108):

$$\Delta_- = \frac{1}{2}(z + D - \beta_z) \tag{4.72}$$

and

$$\beta_z = \sqrt{(z+D)^2 + 8(z-1)(z-D)}. \tag{4.73}$$

When the action (4.68) is evaluated as a function of the boundary fields we write it as:

$$S = \frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\mathcal{L}. \tag{4.74}$$

4.6.2 ADM decomposition in the metric formalism

In our calculations, we decompose the metric $g_{\alpha\beta}$ on the $(D+1)$-dimensional boundary of spacetime into the ADM decomposition

$$\begin{aligned} g_{tt} &= -N^2 + N^i N_i, \qquad g_{ij} = \gamma_{ij}, \qquad g_{ti} = N_i, \\ g^{tt} &= -\frac{1}{N^2}, \qquad g^{ij} = \gamma^{ij} - \frac{N^i N^j}{N^2}, \qquad g^{ti} = \frac{N^i}{N^2}. \end{aligned}$$

This metric leads to the following Christoffel symbols:

$$\begin{aligned}
\Gamma^t_{tt} &= \frac{\partial_t N}{N} + \frac{N^j \nabla_j N}{N} + \frac{N^i N^j \hat{K}_{ij}}{N}, \\
\Gamma^i_{tt} &= \gamma^{ij} N \nabla_j N + N\gamma^{ij}\partial_t\left(\frac{N_j}{N}\right) - \frac{N^i N^j \nabla_j N}{N} - \gamma^{ij} N^k \nabla_j N_k - \frac{N^i N^j N^k \hat{K}_{jk}}{N}, \\
\Gamma^t_{ti} &= \frac{\nabla_i N}{N} + \frac{N^j \hat{K}_{ij}}{N}, \\
\Gamma^j_{ti} &= N\gamma^{jk}\hat{K}_{ik} + N\nabla_i\left(\frac{N^j}{N}\right) - \frac{N^j N^k \hat{K}_{ik}}{N}, \\
\Gamma^t_{ij} &= \frac{\hat{K}_{ij}}{N}, \\
\Gamma^k_{ij} &= \hat{\Gamma}^k_{ij} - \frac{\hat{K}_{ij} N^k}{N},
\end{aligned}$$

where $\hat{K}_{ij} = \frac{1}{2N}(\partial_t \gamma_{ij} - \nabla_i N_j - \nabla_j N_i)$ is the D-dimensional extrinsic curvature.

These result in the following $(D+1)$-dimensional Ricci scalar R for the metric $g_{\alpha\beta}$ in terms of $\hat{R}$, the D-dimensional Ricci scalar for the metric γ_{ij}:

$$R = \hat{R} - \frac{2\nabla^2 N}{N} + \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{\partial_t Z}{N\sqrt{\gamma}} + \frac{\nabla^i Y_i}{N}, \tag{4.75}$$

where:

$$Z \equiv \gamma^{ij}\sqrt{\gamma}\nabla_i\left(\frac{N_j}{N}\right) + 2\hat{K}\sqrt{\gamma}, \tag{4.76}$$

$$Y_i \equiv -\partial_t\left(\frac{N_i}{N}\right) + \frac{N^j \nabla_i N_j}{N} + 2N^j \hat{K}_{ij} - 3N_i \hat{K} + \frac{N^j \nabla_j N_i}{N} - \frac{N_i \nabla_j N^j}{N}. \tag{4.77}$$

4.6.3 ADM decomposition in the vielbein formalism

The vielbeins are defined by $g_{\alpha\beta} = e^A_\alpha e^B_\beta \eta_{AB}$ and $\gamma_{ij} = \hat{e}^I_i \hat{e}^J_j \delta_{IJ}$. The $(D+1)$ dimensional boundary has vielbeins e^A given by:

$$e^0 = N dt, \qquad e^I = \hat{e}^I_i (N^i dt + dx^i) = N^I dt + \hat{e}^I. \tag{4.78}$$

The Ricci rotation coefficients are defined by $de^C = \Omega_{AB}{}^C e^A \wedge e^B$,

$$de^0 = \nabla_i N dx^i \wedge dt = \frac{\nabla_I N}{N} e^I \wedge e^0, \tag{4.79}$$

$$de^I = \left(\frac{\nabla_J N^I}{N} - \frac{\hat{e}^j_J \partial_t \hat{e}^I_j}{N}\right) e^J \wedge e^0 + \hat{\Omega}_{JK}{}^I e^J \wedge e^K. \tag{4.80}$$

This means that:

$$\begin{aligned}
\Omega_{0I}{}^{0} &= \frac{\nabla_I N}{2N}, && (4.81)\\
\Omega_{IJ}{}^{0} &= 0, && (4.82)\\
\Omega_{0J}{}^{I} &= -\frac{\nabla_J N^I}{2N} + \frac{\hat{e}^j_J \partial_t \hat{e}^I_j}{2N}, && (4.83)\\
\Omega_{JK}{}^{I} &= \hat{\Omega}_{JK}{}^{I}, && (4.84)\\
\Omega_{0I}{}^{I} &= -\frac{\nabla_I N^I}{2N} + \frac{\hat{e}^j_I \partial_t \hat{e}^I_j}{2N} = -\frac{\nabla_I N^I}{2N} + \frac{\gamma^{ij}\partial_t \gamma_{ij}}{4N} = \frac{\hat{K}}{2}. && (4.85)
\end{aligned}$$

Note that by definition $\Omega_{AB}{}^{C} = -\Omega_{BA}{}^{C}$. The covariant derivative is then given by:

$$\nabla_\alpha V_B = \partial_\alpha V_B - \omega_{\alpha B}{}^{C} V_C, \tag{4.86}$$

where $\omega_{ABC} = -\Omega_{ABC} + \Omega_{ACB} + \Omega_{BCA}$. Note that $\omega_{ABC} = -\omega_{ACB}$. Also $\omega_{[AB]}{}^{C} = -\Omega_{AB}{}^{C}$ and $\omega_{CD}{}^{C} = 2\Omega_{DC}{}^{C}$.

4.6.4 The massive vector

We take the massive vector to be $\mathcal{A}_A = (\alpha + \psi)\delta^0_A$ where $\alpha^2 = 2(z-1)/z$. Also, the massive vector has a non-zero component in the r direction, which the equation of motion for $\mathcal{A}_r$ gives as $\mathcal{A}_r = -\dfrac{\nabla^\alpha F_{r\alpha}}{m^2} = -\dfrac{\nabla^A \pi_A}{m^2 r}$. Then:

$$\mathcal{A}_\alpha = e^A_\alpha \mathcal{A}_A = e^0_\alpha(\alpha+\psi) = N(\alpha+\psi)\delta^t_\alpha. \tag{4.87}$$

The only non-zero component of $F_{\alpha\beta}$ is

$$F_{it} = -F_{ti} = \partial_i \mathcal{A}_t = \alpha \nabla_i N + \nabla_i(N\psi). \tag{4.88}$$

The non-zero components of $F^{\alpha\beta}$ are

$$F^{jt} = -F^{tj} = -\frac{\gamma^{ij}F_{it}}{N^2}, \qquad F^{jk} = \frac{(\gamma^{ij}N^k - \gamma^{ik}N^j)F_{it}}{N^2}. \tag{4.89}$$

Therefore we have that:

$$\begin{aligned}
F_{AB}F^{AB} = F_{\alpha\beta}F^{\alpha\beta} &= -\frac{2(\alpha\nabla_i N + \nabla_i(N\psi))(\alpha\nabla^i N + \nabla^i(N\psi))}{N^2}\\
&= -2\alpha^2\left(\frac{\nabla N}{N}\right)^2 - 4\alpha\frac{\nabla_i(N\psi)\nabla^i N}{N^2} - \frac{2\nabla_i(N\psi)\nabla^i(N\psi)}{N^2}. && (4.90)
\end{aligned}$$

4.6.5 Functional derivatives and the stress tensor

We define the momenta corresponding to the metric $g_{\alpha\beta}$ and vector field $\mathcal{A}_\alpha$ by $\pi_{\alpha\beta} = K_{\alpha\beta} - g_{\alpha\beta}K$ and $\pi_\alpha = rF_{r\alpha}$ respectively,[10] where $K_{\alpha\beta} = r\partial_r g_{\alpha\beta}/2$. As in the standard Hamilton-Jacobi theory, the momenta can also be obtained by functional differentiation of the on-shell action:

$$\pi^{\alpha\beta} = -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta g_{\alpha\beta}}, \qquad \pi^\alpha = -\frac{16\pi G_{D+2}}{\sqrt{-g}}\frac{\delta S}{\delta \mathcal{A}_\alpha}. \tag{4.91}$$

Equivalently, the variation of the on-shell action is:

$$\delta S = -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} d^dx\sqrt{-g}\left[\pi^{\alpha\beta}\delta g_{\alpha\beta} + \pi^\alpha\delta\mathcal{A}_\alpha\right] \tag{4.92}$$

$$= -\frac{1}{16\pi G_{D+2}}\int_{\partial\mathcal{M}} d^dx\sqrt{-g}\left[(2\pi^\alpha{}_\beta + \pi^\alpha\mathcal{A}_\beta)e^\beta_B\delta e^B_\alpha + \pi^A\delta\mathcal{A}_A\right]. \tag{4.93}$$

The boundary stress tensor $T^\alpha{}_B$, however, is defined by functional differentiation of the on-shell action with respect to the vielbeins e^B_α, while holding the vector field with *frame indices* ($\mathcal{A}_A$) fixed. Note that $\mathcal{A}_0 = \alpha + \psi$ is the only non-zero component of $\mathcal{A}_A$. Therefore, the variation of the on-shell action can also be written as:

$$\delta S = -\frac{1}{16\pi G_{D+2}}\int d^dx\sqrt{\gamma}N\left[T^\alpha{}_B\delta e^B_\alpha + \pi_\psi\delta\psi\right], \tag{4.94}$$

where

$$T^A{}_B = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}N}e^A_\alpha\frac{\delta S}{\delta e^B_\alpha} = -\frac{1}{\sqrt{\gamma}N}e^A_\alpha\frac{\delta}{\delta e^B_\alpha}\int dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\mathcal{L}, \tag{4.95}$$

$$\pi_\psi = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}N}\frac{\delta S}{\delta\psi} = -\frac{1}{\sqrt{\gamma}N}\frac{\delta}{\delta\psi}\int dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\mathcal{L}. \tag{4.96}$$

By comparing equations (4.93) and (4.94) we get the following relations:

$$T_{\alpha B} = (2\pi_{\alpha\beta} + \pi_\alpha\mathcal{A}_\beta)e^\beta_B, \qquad \pi_\psi = \pi^0. \tag{4.97}$$

Rearranging these expressions we have

$$\pi_{AB} = \frac{1}{2}(T_{AB} - \pi_A\mathcal{A}_B), \qquad \pi_I\mathcal{A}_0 = T_{I0} - T_{0I}. \tag{4.98}$$

[10] This differs from the usual canonical momenta by a factor of $\sqrt{-g}(16\pi G_{D+2})^{-1}$ in order to simplify some of the subsequent equations.

Finally, by using the expressions for the vielbeins derived in Appendix 4.6.3, we can write the stress tensor as:

$$T^0{}_0 = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}}\frac{\delta S}{\delta N}, \tag{4.99}$$

$$T^0{}_I = -\frac{16\pi G_{D+2}}{\sqrt{\gamma}}\frac{\delta S}{\delta N^I}, \tag{4.100}$$

$$\begin{aligned} T^I{}_J &= -16\pi G_{D+2}\left(\frac{N^I}{\sqrt{\gamma}N}\frac{\delta S}{\delta N^J}+\frac{1}{\sqrt{\gamma}N}\hat{e}^I_i\frac{\delta S}{\delta \hat{e}^J_i}\right) \\ &= -16\pi G_{D+2}\left(\frac{N^I}{\sqrt{\gamma}N}\frac{\delta S}{\delta N^J}+\frac{2}{\sqrt{\gamma}N}\hat{e}^I_i\hat{e}_{Jj}\frac{\delta S}{\delta \gamma_{ij}}\right). \end{aligned} \tag{4.101}$$

We will use these expressions to determine the stress tensor and vector momentum from the on-shell action

4.6.6 Boundary source fields and asymptotic scaling

The boundary conditions are specified by fixing the sources for the various field theory operators on the boundary. Our boundary conditions involve the following finite fixed sources as $r \to \infty$ (denoting each source with a bar):

$$\overline{e}^0_\alpha = \frac{e^0_\alpha}{r^z}, \qquad \overline{e}^I_\alpha = \frac{e^I_\alpha}{r}, \qquad \overline{\psi} = \frac{\psi}{r^{-\Delta_-}}. \tag{4.102}$$

In order to have a foliation on the boundary, it is necessary to set $\overline{e}^0_i$ (the source for the energy flux $\mathcal{E}^i$) equal to zero (21). For all of this chapter, we have set $\overline{e}^0_i = 0$.

Note that $T^\alpha{}_A$ is the vacuum expectation value of the operator sourced by $\overline{e}^A_\alpha$. In other words, $\overline{e}^0_\alpha$ is the source for the energy density $\mathcal{E}$ and the energy flux $\mathcal{E}^i$, whereas $\overline{e}^I_\alpha$ is the source for the momentum density $\mathcal{P}_i$ and the stress tensor $\Pi^i{}_j$. $\overline{\psi}$ is the source for $\mathcal{O}_\psi$, the operator dual to the massive vector ψ. The operator $\mathcal{O}_\psi$ is relevant for $z < D$ and irrelevant for $z > D$. Therefore, for $z > D$, we must take $\overline{\psi} = 0$ in order to preserve the asymptotic boundary conditions above. In the case $z = D$, the operator is classically marginal, and there is some evidence suggesting that it becomes marginally relevant in the case of $D = 2$ (109).

Note also that the scaling dimensions discussed here are the classical scaling dimensions, consistent with the fact that we perform our analysis near the ultraviolet fixed point with fixed z. In the bulk, this corresponds to the asymptotic analysis in the vicinity of the spacetime boundary at conformal infinity. Hence, in our analysis we systematically ignore most of the possible nontrivial infrared dynamics, such as the flow – generically expected of Lifshitz-type theories – towards lower values of z under the influence of relevant operators.

The above scaling behavior allows us to determine the scaling behavior of other quantities near the boundary. Any boundary quantity can be written in terms of the source fields

$\overline{e}^A_\alpha$ and $\overline{\psi}$ and then the scaling behavior can be read off from the resulting exponents of r. Consider a general object $\mathcal{O}$. When written in terms of the boundary source fields, we say that the term in $\mathcal{O}$ scaling as $r^{-\Delta}$ is of "order Δ" and denote it by $\mathcal{O}^{(\Delta)}$. For example, e^0_α has order $-z$, e^I_α has order -1 and ψ has order Δ_-. This means that N has order $-z$, N_i has order -2, γ_{ij} has order -2 and γ^{ij} has order 2.

From equation (4.75), R has components of order 2 and $2z$ given by:

$$R^{(2)} = \hat{R} - \frac{2\nabla^i\nabla_i N}{N}, \tag{4.103}$$

$$R^{(2z)} = \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{\partial_t Z}{N\sqrt{\gamma}} + \frac{\nabla^i Y_i}{N}. \tag{4.104}$$

From equation (4.90), $F_{AB}F^{AB}$ has components of order 2, $2+\Delta_-$ and $2+2\Delta_-$ given by:

$$(F_{AB}F^{AB})^{(2)} = -\frac{2\alpha^2\nabla_i N\nabla^i N}{N^2}, \tag{4.105}$$

$$(F_{AB}F^{AB})^{(2+\Delta_-)} = -\frac{4\alpha\nabla_i(N\psi)\nabla^i N}{N^2}, \tag{4.106}$$

$$(F_{AB}F^{AB})^{(2+2\Delta_-)} = -\frac{2\nabla_i(N\psi)\nabla^i(N\psi)}{N^2}. \tag{4.107}$$

Also, $\mathcal{A}_A\mathcal{A}^A = -(\alpha+\psi)^2$ has components of dimension $0, \Delta_-, 2\Delta_-$:

$$(\mathcal{A}_A\mathcal{A}^A)^{(0)} = -\alpha^2, \tag{4.108}$$

$$(\mathcal{A}_A\mathcal{A}^A)^{(\Delta_-)} = -2\alpha\psi, \tag{4.109}$$

$$(\mathcal{A}_A\mathcal{A}^A)^{(2\Delta_-)} = -\psi^2. \tag{4.110}$$

Note also that equations (4.95) and (4.96) imply that terms $\mathcal{L}^{(\Delta)}$ in the on-shell action determine $T^0{}_0{}^{(\Delta)}$, $T^0{}_I{}^{(\Delta+1-z)}$, $T^I{}_0{}^{(\Delta+z-1)}$, $T^I{}_J{}^{(\Delta)}$ and $\pi_\psi{}^{(\Delta-\Delta_-)}$.

4.7 Appendix: Holographic Renormalization Equations

The on-shell action is a function of the boundary fields and is written as

$$S = \frac{1}{16\pi G_{D+2}}\int dt\, d^D\mathbf{x}\,\sqrt{\gamma}\,N\mathcal{L}. \tag{4.111}$$

A convenient way of computing the divergent part of $\mathcal{L}$ is to organize the terms with respect to how they scale with r. More precisely, we define the dilatation operator by:

$$\delta_{\mathcal{D}} = \int dt\, d^D\mathbf{x}\,\left(ze^0_\mu\frac{\delta}{\delta e^0_\mu} + e^I_\mu\frac{\delta}{\delta e^I_\mu} - \Delta_-\psi\frac{\delta}{\delta\psi}\right). \tag{4.112}$$

This operator asymptotically represents $r\dfrac{\partial}{\partial r}$.

$\mathcal{L}$ can then be decomposed into a sum of terms as follows:

$$\mathcal{L} = \sum_{\Delta \geq 0} \mathcal{L}^{(\Delta)} + \widetilde{\mathcal{L}}^{(z+D)} \log r. \tag{4.113}$$

Note that we include a logarithmic term at order $z + D$ due to the possibility of a Weyl scaling anomaly. The individual terms of the expansion (4.113) satisfy

$$\delta_{\mathcal{D}} \mathcal{L}^{(\Delta)} = -\Delta\, \mathcal{L}^{(\Delta)} \qquad \text{for } \ \Delta \neq z + D, \tag{4.114}$$

$$\delta_{\mathcal{D}} \mathcal{L}^{(z+D)} = -(z + D)\mathcal{L}^{(z+D)} + \widetilde{\mathcal{L}}^{(z+D)}, \tag{4.115}$$

$$\delta_{\mathcal{D}} \widetilde{\mathcal{L}}^{(z+D)} = -(z + D)\widetilde{\mathcal{L}}^{(z+D)}. \tag{4.116}$$

Applying $\delta_{\mathcal{D}}$ to the on-shell action (4.111) and using equations (4.95) and (4.96) then yields:

$$(z + D + \delta_{\mathcal{D}})\mathcal{L} = -z T^0{}_0 - T^I{}_I + \Delta_- \psi \pi_\psi. \tag{4.117}$$

Expanding this at each order then results in:

$$(z + D - \Delta)\mathcal{L}^{(\Delta)} = -z T^0{}_0{}^{(\Delta)} - T^I{}_I{}^{(\Delta)} + \Delta_- \psi \pi_\psi^{(\Delta - \Delta_-)} \tag{4.118}$$

except for $\Delta = z + D$, when this becomes

$$\widetilde{\mathcal{L}}^{(z+D)} = -z T^0{}_0{}^{(z+D)} - T^I{}_I{}^{(z+D)} + \Delta_- \psi \pi_\psi^{(z+D-\Delta_-)}. \tag{4.119}$$

This allows us to solve for the anomaly. The above equations imply that the anomaly term can also be found by:

$$\widetilde{\mathcal{L}}^{(\Delta)} = \lim_{\Delta \to z+D} \left((z + D - \Delta)\mathcal{L}^{(\Delta)} \right). \tag{4.120}$$

Note that the value of $\mathcal{L}^{(z+D)}$ cannot be found by this asymptotic analysis.

We now move on to finding an explicit expression for these divergent terms in the onshell action $\mathcal{L}^{(\Delta)}$. The variation of the bulk action (4.68) with respect to $\mathcal{N}$ produces the Hamiltonian constraint equation,

$$K^2 - K_{AB}K^{AB} - \frac{1}{2}\pi_A \pi^A - \frac{1}{2m^2}(\nabla^A \pi_A)^2 = R - 2\Lambda - \frac{1}{4}F_{AB}F^{AB} - \frac{1}{2}m^2 \mathcal{A}_A \mathcal{A}^A. \tag{4.121}$$

Expanding this equation in its dilatation eigenvalues (utilizing equations (4.98), (4.137), (4.139), (4.140)) and then substituting it into equation (4.118) yields an expression for $\mathcal{L}^{(\Delta)}$ (see (21) for more details). Explicitly, the terms in the on-shell action are given for $\Delta \neq 0$, Δ_- and $2\Delta_-$ by:

$$(z + D - \Delta)\mathcal{L}^{(\Delta)} = \mathcal{Q}^{(\Delta)} + \mathcal{S}^{(\Delta)}, \tag{4.122}$$

where the quadratic term $\mathcal{Q}^{(\Delta)}$ is given by

$$\begin{aligned}\mathcal{Q}^{(\Delta)} &= \sum_{0<s<\Delta/2;s\neq\Delta_-}\left[2K^{(s)}_{AB}\pi^{AB(\Delta-s)}+\pi^{(s)}_A\pi^{A(\Delta-s)}+\frac{1}{m^2}(\nabla_A\pi^A)^{(s)}(\nabla_A\pi^A)^{(\Delta-s)}\right]\\ &+\left[K^{(\Delta_-)}_{AB}T^{AB(\Delta-\Delta_-)}+K^{(\Delta_-)}_{00}\pi^{0(\Delta-2\Delta_-)}\psi+\pi^{(\Delta_-)}_I\pi^{I(\Delta-\Delta_-)}\right]\\ &+\left[K^{(\Delta/2)}_{AB}\pi^{AB(\Delta/2)}+\frac{1}{2}\pi^{(\Delta/2)}_A\pi^{A(\Delta/2)}+\frac{1}{2m^2}(\nabla_A\pi^A)^{(\Delta/2)2}\right] \quad (4.123)\end{aligned}$$

and the source $\mathcal{S}$ is

$$\mathcal{S}=R-2\Lambda-\frac{1}{4}F_{AB}F^{AB}-\frac{1}{2}m^2\mathcal{A}_A\mathcal{A}^A. \quad (4.124)$$

We also have the following exceptions to the above formula:

$$\begin{aligned}(z+D)\mathcal{L}^{(0)} &= 2\mathcal{S}^{(0)}, &(4.125)\\ (z+D-\Delta_-)\mathcal{L}^{(\Delta_-)} &= (\Delta_--z)\psi\pi^{(0)}_\psi+\mathcal{S}^{(\Delta_-)}, &(4.126)\\ (z+D-2\Delta_-)\mathcal{L}^{(2\Delta_-)} &= (\Delta_--z)\psi\pi^{(\Delta_-)}_\psi+K^{(\Delta_-)}_{AB}\pi^{AB(\Delta_-)}\\ &\quad+\frac{1}{2}\pi^{(\Delta_-)}_A\pi^{A(\Delta_-)}+\mathcal{S}^{(2\Delta_-)}. &(4.127)\end{aligned}$$

$\mathcal{S}$ needs to be calculated at each order. The calculation in Appendix 4.6.6 shows that R has components of order 2 and $2z$, $F_{AB}F^{AB}$ has components of order $2, 2+\Delta_-, 2+2\Delta_-$ and $\mathcal{A}_A\mathcal{A}^A$ has components of order $0, \Delta_-, 2\Delta_-$, resulting in:

$$\begin{aligned}\mathcal{S}^{(0)} &= -2\Lambda+\frac{1}{2}m^2\alpha^2=(z+D)(z+D-1), &(4.128)\\ \mathcal{S}^{(\Delta_-)} &= m^2\alpha\psi=Dz\alpha\psi, &(4.129)\\ \mathcal{S}^{(2\Delta_-)} &= \frac{1}{2}m^2\psi^2=\frac{Dz}{2}\psi^2, &(4.130)\\ \mathcal{S}^{(2)} &= R^{(2)}-\frac{1}{4}(F_{AB}F^{AB})^{(2)}=\hat{R}-2\frac{\nabla^2N}{N}+\frac{\alpha^2}{2}\left(\frac{\nabla N}{N}\right)^2, &(4.131)\\ \mathcal{S}^{(2+\Delta_-)} &= -\frac{1}{4}(F_{AB}F^{AB})^{(2+\Delta_-)}=\frac{\alpha\nabla_iN\nabla^i(N\psi)}{N^2}, &(4.132)\\ \mathcal{S}^{(2+2\Delta_-)} &= -\frac{1}{4}(F_{AB}F^{AB})^{(2+2\Delta_-)}=\frac{\nabla^i(N\psi)\nabla_i(N\psi)}{2N^2}, &(4.133)\\ \mathcal{S}^{(2z)} &= R^{(2z)}=\hat{K}_{ij}\hat{K}^{ij}-\hat{K}^2+\text{total derivatives}. &(4.134)\end{aligned}$$

We now proceed to use these formulae to calculate the divergent terms in the on-shell action at each order. Once these divergent terms have been calculated, counterterms must be added to the action in order to subtract these divergences. With a boundary cutoff at $r=\frac{1}{\epsilon}$), the

counterterms are

$$S_{ct} = -\frac{1}{16\pi G_{D+2}} \int dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N \left(\sum_{0\le\Delta<z+D} \mathcal{L}^{(\Delta)} - \widetilde{\mathcal{L}}^{(z+D)} \log\epsilon \right). \tag{4.135}$$

4.7.1 Non-derivative counterterms

At order 0, we have:

$$\mathcal{L}^{(0)} = \frac{2\mathcal{S}^{(0)}}{z+D} = 2(z+D-1). \tag{4.136}$$

This yields $T^A{}_B{}^{(0)} = -2(z+D-1)\delta^A{}_B$.

To evaluate the order Δ_- and $2\Delta_-$ counterterms we need some additional information. From the asymptotic expansions given in (21), it is clear that:

$$K^0{}_0{}^{(0)} = z, \qquad K^I{}_J{}^{(0)} = \delta^I{}_J. \tag{4.137}$$

Also, the zero-component of the vector momentum is given by:

$$\pi_0 = rF_{r0} = r\partial_r\mathcal{A}_0 + \mathcal{A}_0 K^0{}_0 - r\partial_0\mathcal{A}_r. \tag{4.138}$$

This gives:

$$\pi_0^{(0)} = \alpha K^0{}_0{}^{(0)} = \alpha z \tag{4.139}$$

$$\pi_0^{(\Delta_-)} = r\partial_r\psi + \alpha K^0{}_0{}^{(\Delta_-)} + \psi K^0{}_0{}^{(0)} = \alpha K^0{}_0{}^{(\Delta_-)} + (z-\Delta_-)\psi \tag{4.140}$$

Note that $\pi_\psi \equiv \pi^0$.

Then:

$$(z+D-\Delta_-)\mathcal{L}^{(\Delta_-)} = -(z-\Delta_-)\psi\pi_\psi^{(0)} + \mathcal{S}^{(\Delta_-)} = (z-\Delta_-)\psi\alpha z + Dz\alpha\psi \tag{4.141}$$

$$\mathcal{L}^{(\Delta_-)} = z\alpha\psi \tag{4.142}$$

which yields $T^A{}_B{}^{(\Delta_-)} = -z\alpha\psi\delta^A{}_B$.

Note that $\pi^A{}_B = \frac{1}{2}(T^A{}_B - \pi^A\mathcal{A}_B) = K^A{}_B - K\delta^A{}_B$ and this means that:

$$\pi^0{}_0{}^{(\Delta_-)} = \frac{1}{2}(T^0{}_0{}^{(\Delta_-)} - \alpha\pi_\psi^{(\Delta_-)} - \psi\pi_\psi^{(0)}) = -\frac{1}{2}\alpha\pi_\psi^{(\Delta_-)} \tag{4.143}$$

$$\pi^I{}_J{}^{(\Delta_-)} = \frac{1}{2}T^I{}_J{}^{(\Delta_-)} = -\frac{z\alpha\psi}{2}\delta^I{}_J \tag{4.144}$$

$$K^{(\Delta_-)} = -\frac{\pi^A{}_A{}^{(\Delta_-)}}{D} = \frac{\alpha\pi_\psi^{(\Delta_-)}}{2D} + \frac{z\alpha\psi}{2} \tag{4.145}$$

$$K^0{}_0{}^{(\Delta_-)} = \pi^0{}_0{}^{(\Delta_-)} + K^{(\Delta_-)} = -\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2} \tag{4.146}$$

$$K^I{}_J{}^{(\Delta_-)} = \pi^I{}_J{}^{(\Delta_-)} + K^{(\Delta_-)}\delta^I{}_J = \frac{\alpha\pi_\psi^{(\Delta_-)}}{2D}\delta^I{}_J \tag{4.147}$$

Substituting this into the expression $\pi_0^{(\Delta_-)} = \alpha K^0{}_0{}^{(\Delta_-)} + (z-\Delta_-)\psi$ derived above gives:

$$\pi_0^{(\Delta_-)} = \alpha(-\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2}) + (z-\Delta_-)\psi \tag{4.148}$$

$$\pi_0^{(\Delta_-)} = \frac{2D(2z-1-\Delta_-)}{2D-\alpha^2(D-1)}\psi = \frac{Dz(2z-1-\Delta_-)}{z+D-1}\psi \tag{4.149}$$

$$\tag{4.150}$$

Therefore, using this result for $\pi_0^{(\Delta_-)}$:

$$K^{(\Delta_-)} = -\frac{\alpha z(2z-1-\Delta_-)}{2(z+D-1)}\psi + \frac{z\alpha\psi}{2} = -\frac{\alpha z(z-D-\Delta_-)}{2(z+D-1)}\psi \tag{4.151}$$

$$K^0{}_0{}^{(\Delta_-)} = \frac{\alpha z(D-1)(2z-1-\Delta_-)}{2(z+D-1)}\psi + \frac{z\alpha\psi}{2} \tag{4.152}$$

$$= \frac{\alpha z((2D-1)z-(D-1)\Delta_-)}{2(z+D-1)}\psi \tag{4.153}$$

$$K^I{}_J{}^{(\Delta_-)} = -\frac{\alpha z(2z-1-\Delta_-)}{2(z+D-1)}\psi\delta^I{}_J \tag{4.154}$$

Then:

$$\begin{aligned}(z+D-2\Delta_-)\mathcal{L}^{(2\Delta_-)} &= -(z-\Delta_-)\psi\pi_\psi^{(\Delta_-)} + K_{AB}^{(\Delta_-)}\pi^{AB(\Delta_-)} + \frac{1}{2}\pi_A^{(\Delta_-)}\pi^{A(\Delta_-)} + \mathcal{S}^{(2\Delta_-)} \\ &= -(z-\Delta_-)\psi\pi_\psi^{(\Delta_-)} + (-\frac{\alpha\pi_\psi^{(\Delta_-)}(D-1)}{2D} + \frac{z\alpha\psi}{2})(-\frac{1}{2}\alpha\pi_\psi^{(\Delta_-)}) \\ &\quad +(\frac{\alpha\pi_\psi^{(\Delta_-)}}{2})(-\frac{z\alpha\psi}{2}) + \frac{1}{2}(\pi_\psi^{(\Delta_-)})^2 - \frac{Dz}{2}\psi^2 \\ &= \frac{Dz\psi^2(4z^2-4z-4z\Delta_-+1+2\Delta_-+\Delta_-^2+z+D-1)}{2(z+D-1)} \\ &= \frac{Dz\psi^2(z+D-2\Delta_-)(2z-1-\Delta_-)}{2(z+D-1)}\end{aligned} \tag{4.155}$$

$$\mathcal{L}^{(2\Delta_-)} = \frac{Dz\psi^2(2z-1-\Delta_-)}{2(z+D-1)} \tag{4.156}$$

where $\Delta_- = \frac{1}{2}(z+D-\beta_z)$ and $\beta_z = \sqrt{(z+D)^2+8(z-1)(z-D)}$ has been used.

This result yields $T^A{}_B{}^{(2\Delta_-)} = -\frac{Dz\psi^2(2z-1-\Delta_-)}{2(z+D-1)}\delta^A{}_B$. Next we can calculate:

$$\begin{aligned}(z+D-3\Delta_-)\mathcal{L}^{(3\Delta_-)} &= K_{AB}^{(\Delta_-)}T^{AB(2\Delta_-)} + K_{00}^{(\Delta_-)}\pi^{0(0)}\psi \\ &= \frac{D\alpha z^2(2z-1-\Delta_-)(z-D-\Delta_-)}{4(z+D-1)^2}\psi^3\end{aligned}$$

$$
\begin{aligned}
&+\frac{D\alpha z^2((2D-1)z-(D-1)\Delta_-)(2z-1-\Delta_-)}{2(z+D-1)^2}\psi^3\\
&= \frac{D\alpha z^2(2z-1-\Delta_-)(-D+(4D-1)z-(2D-1)\Delta_-)}{4(z+D-1)^2}\psi^3
\end{aligned}
$$

which yields $\pi_\psi^{(2\Delta_-)} = -\dfrac{3D\alpha z^2(2z-1-\Delta_-)(-D+(4D-1)z-(2D-1)\Delta_-)}{4(z+D-3\Delta_-)(z+D-1)^2}\psi^2$.

This allows us to calculate $K_{AB}^{(2\Delta_-)}$:

$$
\begin{aligned}
\pi^0{}_0{}^{(2\Delta_-)} &= \frac{1}{2}(T^0{}_0{}^{(2\Delta_-)} - \alpha\pi_\psi^{(2\Delta_-)} - \psi\pi_\psi^{(\Delta_-)}) = -\frac{1}{2}\alpha\pi_\psi^{(2\Delta_-)} - \frac{1}{4}\psi\pi_\psi^{(\Delta_-)} && (4.157)\\
\pi^I{}_J{}^{(2\Delta_-)} &= \frac{1}{2}T^I{}_J{}^{(2\Delta_-)} = \frac{1}{4}\psi\pi_\psi^{(\Delta_-)}\delta^I{}_J && (4.158)\\
K^{(2\Delta_-)} &= -\frac{\pi^A{}_A{}^{(\Delta_-)}}{D} = \frac{1}{2D}\alpha\pi_\psi^{(2\Delta_-)} - \frac{(D-1)}{4D}\psi\pi_\psi^{(\Delta_-)} && (4.159)\\
K^0{}_0{}^{(2\Delta_-)} &= \pi^0{}_0{}^{(2\Delta_-)} + K^{(2\Delta_-)} = -\frac{\alpha\pi_\psi^{(2\Delta_-)}(D-1)}{2D} - \frac{(2D-1)}{4D}\psi\pi_\psi^{(\Delta_-)} && (4.160)\\
K^I{}_J{}^{(2\Delta_-)} &= \pi^I{}_J{}^{(2\Delta_-)} + K^{(2\Delta_-)}\delta^I{}_J = \left(\frac{1}{2D}\alpha\pi_\psi^{(2\Delta_-)} + \frac{1}{4D}\psi\pi_\psi^{(\Delta_-)}\right)\delta^I{}_J && (4.161)
\end{aligned}
$$

Higher order non-derivative terms can be calculated in a similar manner.

4.7.2 Two-derivative counterterms with $\psi = 0$

Up to total derivatives, the divergent term in the on-shell action of order 2 is:

$$
(z+D-2)\mathcal{L}^{(2)} = \mathcal{S}^{(2)} = \hat{R} - \frac{2\nabla^i\nabla_i N}{N} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2} = \hat{R} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2} \quad (4.162)
$$

This gives the following contribution to the stress tensor (see Appendix 4.6.5):

$$
\begin{aligned}
(z+D-2)T_{00}^{(2)} &= \hat{R} - \frac{\alpha^2\nabla^i\nabla_i N}{N} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2},\\
(z+D-2)T_{0I}^{(3-z)} &= 0,\\
(z+D-2)T_{IJ}^{(2)} &= 2\hat{R}_{IJ} - \frac{2\nabla_I\nabla_J N}{N} + \frac{\alpha^2\nabla_I N\nabla_J N}{N^2},\\
&\qquad + \delta_{IJ}\left(-\hat{R} + \frac{2\nabla^i\nabla_i N}{N} - \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}\right),\\
(z+D-2)T^I{}_I{}^{(2)} &= -(D-2)\hat{R} + \frac{2(D-1)\nabla^i\nabla_i N}{N} - \frac{\alpha^2(D-2)\nabla^i N\nabla_i N}{2N^2}.
\end{aligned}
$$

At order $2z$ there is a contribution from the quadratic term $\dfrac{1}{2m^2}(\nabla_A\pi^A)^{(z)2}$. Note that:

$$
(\nabla_A\pi^A)^{(z)} = (\partial_A\pi^A - \omega_A{}^{AB}\pi_B)^{(z)} = (\partial_A\pi^A - 2\Omega_A{}^{BA}\pi_B)^{(z)}
$$

$$\begin{aligned} &= (\partial_0(\pi^{0(0)}) - 2\Omega_I{}^{0I}\pi_0^{(0)}) = (\partial_0(-z\alpha) - 2\Omega_I{}^{0I}z\alpha) \\ &= -\alpha z\hat{K}, \end{aligned} \tag{4.163}$$

where expressions from Appendix 4.6.3 have been used. Therefore, up to total derivatives:

$$\begin{aligned} (z+D-2z)\mathcal{L}^{(2z)} &= \mathcal{S}^{(2z)} + \frac{1}{2m^2}(\nabla_A\pi^A)^{(z)2} \\ &= \hat{K}_{ij}\hat{K}^{ij} - \hat{K}^2 + \frac{1}{2m^2}(-\alpha z\hat{K})^2 \\ &= \hat{K}_{ij}\hat{K}^{ij} - \frac{(1+D-z)}{D}\hat{K}^2 \end{aligned} \tag{4.164}$$

4.7.3 Two-derivative counterterms involving ψ

We can also calculate various divergent terms involving ψ, for example:

$$\begin{aligned} &(z+D-2-\Delta_-)\mathcal{L}^{(2+\Delta_-)} = K_{AB}^{(\Delta_-)}T^{AB(2)} + \frac{\alpha\nabla_i N\nabla^i(N\psi)}{N^2} \\ &= -\frac{\alpha z\psi}{2(z+D-1)}\left[((2D-1)z-(D-1)\Delta_-)T^{00(2)} + (2z-1-\Delta_-)T^I{}_I{}^{(2)}\right] \\ &\qquad -\frac{\alpha\nabla^i\nabla_i N\psi}{N} + \frac{\alpha\nabla_i N\nabla^i N\psi}{N^2} \\ &= -\frac{\alpha z\psi}{2(z+D-1)(z+D-2)}\times \\ &\qquad \left[((2D-1)z-(D-1)\Delta_-)\left(\hat{R} - \frac{\alpha^2\nabla^i\nabla_i N}{N} + \frac{\alpha^2\nabla^i N\nabla_i N}{2N^2}\right)\right. \\ &\qquad \left. + (2z-1-\Delta_-)\left(-(D-2)\hat{R} + \frac{2(D-1)\nabla^i\nabla_i N}{N} - \frac{\alpha^2(D-2)\nabla^i N\nabla_i N}{2N^2}\right)\right] \\ &\quad -\frac{\alpha\nabla^i\nabla_i N\psi}{N} + \frac{\alpha\nabla_i N\nabla^i N\psi}{N^2}. \end{aligned} \tag{4.165}$$

Or, by defining some constants:

$$\mathcal{L}^{(2+\Delta_-)} = -\psi\left(c_1\hat{R} + c_2\frac{\nabla^i\nabla_i N}{N} + c_3\frac{\nabla_i N\nabla^i N}{N^2}\right) \tag{4.166}$$

where:

$$\begin{aligned} c_1 &= \frac{\alpha z(-2+D-\Delta_-+3z)}{2(D-2+z)(D-1+z)(z+D-2-\Delta_-)} \\ c_2 &= \frac{\alpha(4+2D^2-(4+(\alpha^2-2)\Delta_-)z+(\alpha^2-2)z^2)}{2(D-2+z)(D-1+z)(z+D-2-\Delta_-)} \\ &\quad + \frac{\alpha D((2+(\alpha^2-2)\Delta_-)z-2(\alpha^2-2)z^2-6)}{2(D-2+z)(D-1+z)(z+D-2-\Delta_-)} \end{aligned}$$

$$c_3 = \frac{\alpha(-8-4D^2-(-12+\alpha^2(2+\Delta_-))z+(3\alpha^2-4)z^2+D(12+(\alpha^2-8)z))}{4(D-2+z)(-1+D+z)(-2+D-\Delta_-+z)}$$

(Note that for $z=D=2$ we have $c_1=\frac{1}{2}$, $c_2=\frac{1}{2}$ and $c_3=-\frac{1}{4}$.)

This results in:

$$\pi_\psi^{(2)} = c_1\hat{R}+c_2\frac{\nabla^i\nabla_i N}{N}+c_3\frac{\nabla_i N\nabla^i N}{N^2} \tag{4.167}$$

and also:

$$\begin{aligned}
T_{00}^{(2+\Delta_-)} &= -c_1\psi\hat{R}-c_2\nabla^i\nabla_i\psi-c_3\frac{\nabla_i N\nabla^i N\psi}{N^2} \\
&\quad +2c_3\frac{\nabla_i\nabla^i N\psi}{N}+2c_3\frac{\nabla^i N\nabla_i\psi}{N} && (4.168)\\
T_{0I}^{(3-z+\Delta_-)} &= 0 && (4.169)\\
T_{IJ}^{(2+\Delta_-)} &= \delta_{IJ}\left(c_1\psi\hat{R}-c_2\frac{\nabla_i N\nabla^i\psi}{N}+c_3\frac{\nabla_i N\nabla^i N\psi}{N^2}\right) \\
&\quad -2c_1\psi\hat{R}_{IJ}+c_2\frac{\nabla_I N\nabla_J\psi}{N}+c_2\frac{\nabla_J N\nabla_I\psi}{N}-2c_3\frac{\nabla_I N\nabla_J N\psi}{N^2} \\
&\quad -2\delta_{IJ}c_1\frac{\nabla^i\nabla_i(N\psi)}{N}+2c_1\frac{\nabla_I\nabla_J(N\psi)}{N} && (4.170)\\
T^I{}_I{}^{(2+\Delta_-)} &= (D-2)\left(c_1\psi\hat{R}-c_2\frac{\nabla_i N\nabla^i\psi}{N}+c_3\frac{\nabla_i N\nabla^i N\psi}{N^2}\right)-2c_1\frac{\nabla^i\nabla_i(N\psi)}{N} && (4.171)
\end{aligned}$$

There are many more two-derivative terms involving ψ. For example:

$$\begin{aligned}
(z+D-2-2\Delta_-)\mathcal{L}^{(2+2\Delta_-)} &= 2K_{AB}^{(2\Delta_-)}\pi^{AB(2)}+\pi_A^{(2\Delta_-)}\pi^{A(2)} \\
&+ K_{AB}^{(\Delta_-)}T^{AB(2+\Delta_-)}+K_{00}^{(\Delta_-)}\pi^{0(2)}\psi+\mathcal{S}^{(2+2\Delta_-)} && (4.172)
\end{aligned}$$

This has been calculated explicitly in the case $D=z=2$:

$$\mathcal{L}^{(2+2\Delta_-)} = \psi^2\left(\frac{\nabla^i\nabla_i N}{8N}-\frac{\nabla^i N\nabla_i N}{2N^2}\right)+\frac{3}{4}\psi\nabla^i\nabla_i\psi \tag{4.173}$$

4.7.4 Four-derivative counterterms with $\psi = 0$

At fourth order we have:

$$\begin{aligned}
(z+D-4)\mathcal{L}^{(4)} &= K_{AB}^{(2)}\pi^{AB(2)}+\frac{1}{2}\pi_A^{(2)}\pi^{A(2)} \\
&= \frac{1}{a_0}\Big[a_1\Big(\frac{\nabla_i N\nabla^i N}{N^2}\Big)^2+a_2\frac{\nabla_i N\nabla^i N}{N^2}\hat{R}+a_3\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2} \\
&\quad +a_4\frac{\nabla^i N\nabla^j N}{N^2}\hat{R}_{ij}+a_5\frac{\nabla^i\nabla_i N}{N}\hat{R}+a_6\Big(\frac{\nabla^i\nabla_i N}{N}\Big)^2+a_7\hat{R}_{ij}^2+a_8\hat{R}^2\Big] && (4.174)
\end{aligned}$$

where:

$$
\begin{aligned}
a_0 &= -2Dz^2(-2+D+z)^2(-1+D+z)(-4+\beta+D+z)^2 && (4.175)\\
a_1 &= 32(z-1)^3 + D^4(-11+z(6+z)) \\
&\quad +D^3(52-3\beta_z - z(77+2\beta_z-(34+\beta_z)z+z^2)) \\
&\quad +D^2(16(-8+\beta_z)+z(2(116+\beta_z)+z(-145-8\beta_z+2(9+\beta_z)z+3z^2))) \\
&\quad +D(z-1)(16(-8+\beta_z)+z(184+z(-68-5\beta_z+z(-13+\beta_z+5z)))) && (4.176)\\
a_2 &= 2z(z-1)(D^4+D^3(\beta_z-z)+16(z-1)z+D^2(8-4\beta_z+z(-16+2\beta_z+3z)) \\
&\quad +Dz(-4(-8+\beta_z)+z(-24+\beta_z+5z))) && (4.177)\\
a_3 &= -2z(D^4(z-4)+32(z-1)^2+D^3(-2(7+2\beta_z)+(21+\beta_z-z)z) \\
&\quad +D^2(32+18\beta_z+z(-60-11\beta_z+z(10+2\beta_z+3z))) \\
&\quad +D(z-1)(8(8+\beta_z)+z(-40-6\beta_z+z(-18+\beta_z+5z)))) && (4.178)\\
a_4 &= -4zD(z-2)(D-1+z)(8+D^2-8z-2Dz+5z^2+\beta_z(-4+D+z)) && (4.179)\\
a_5 &= -4z^2(D^3+D^2(\beta_z-2z)+8(z-1)z+D(8+\beta_z(z-4)-3z^2)) && (4.180)\\
a_6 &= -4z(D-1)(D^3+D^2(\beta_z-2z)+8(z-1)z+D(8+\beta_z(z-4)-3z^2)) && (4.181)\\
a_7 &= -4z^2D(D-1+z)(8+D^2-8z-2Dz+5z^2+\beta_z(D-4+z)) && (4.182)\\
a_8 &= z^2(D^4+D^3(\beta_z-z)+8(z-1)z^2+D^2(8-4\beta_z+z(-8+2\beta_z+3z)) \\
&\quad +Dz(8-4\beta_z+z(-8+\beta_z+5z))) && (4.183)
\end{aligned}
$$

In the above expression we have used the following identities for terms in the action (up to total derivatives):

$$
\begin{aligned}
\frac{\nabla_i N\nabla_j N\nabla^i\nabla^j N}{N^3} &\sim \left(\frac{\nabla_i N\nabla^i N}{N^2}\right)^2 - \frac{\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3} \\
\frac{\nabla_i\nabla_j N\nabla^i\nabla^j N}{N^2} &\sim \left(\frac{\nabla_i N\nabla^i N}{N^2}\right)^2 - \frac{3\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3} + \left(\frac{\nabla^i\nabla_i N}{N}\right)^2 - \frac{\nabla^i N\nabla^j N R_{ij}}{N^2} \\
\frac{\nabla^i\nabla^j N R_{ij}}{N} &\sim \frac{\nabla^i\nabla_i N R}{2N}
\end{aligned}
$$

For $D=2$ we have further simplifications because $R_{ij}=\frac{R}{2}\delta_{ij}$ and so:

$$
\begin{aligned}
(z-2)\mathcal{L}^{(4)} &= \frac{(z-2)}{b_0}\left[b_1\left(\frac{\nabla_i N\nabla^i N}{N^2}\right)^2 + b_2\frac{\nabla_i N\nabla^i N}{N^2}\hat{R} + b_3\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2}\right. && (4.184)\\
&\quad \left. + b_4\frac{\nabla^i\nabla_i N}{N}\hat{R} + b_5\left(\frac{\nabla^i\nabla_i N}{N}\right)^2 + b_6\hat{R}^2\right], && (4.185)
\end{aligned}
$$

where:

$$
\begin{aligned}
b_0 &= -2z^4(z+1)(z-2+\beta_z)^2 && (4.186)\\
b_1 &= 12+36z-11z^2-2z^3+5z^4+\beta_z(-2-7z+z^3) && (4.187)\\
b_2 &= 4z^2(z-6+\beta_z) && (4.188)\\
b_3 &= -2z(36-4z-7z^2+5z^3+\beta_z(z^2-z-6)) && (4.189)\\
b_4 &= -4z^2(z-6+\beta_z) && (4.190)\\
b_5 &= -4z(z-6+\beta_z) && (4.191)\\
b_6 &= -z^3(z-6+\beta_z) && (4.192)
\end{aligned}
$$

Note that in the important case where $z \to 2$ (and still $D = 2$):

$$
(z-2)\mathcal{L}^{(4)} = \frac{(2-z)}{64}\left[3\left(\frac{\nabla_i N\nabla^i N}{N^2}\right)^2 - 4\frac{\nabla^i\nabla_i N}{N}\frac{\nabla_j N\nabla^j N}{N^2}\right]. \qquad (4.193)
$$

A useful check is for $z = 1$, which is the usual relativistic AdS case. The standard known result (93) is that the 4th order term involving only spatial derivative is (up to total derivatives):

$$
\begin{aligned}
\mathcal{L}^{(4)} &= \left[\frac{1}{(D-3)(D-1)^2}\left(R_{\alpha\beta}R^{\alpha\beta}-\frac{D+1}{4D}R^2\right)\right]^{(4)}\\
&= \frac{1}{(D-3)(D-1)^2}\left[\left(\frac{\nabla^i\nabla_i N}{N}\right)^2+\left(\hat{R}_{ij}-\frac{\nabla_i\nabla_j N}{N}\right)\left(\hat{R}^{ij}-\frac{\nabla^i\nabla^j N}{N}\right)\right.\\
&\qquad \left.-\frac{D+1}{4D}\left(\hat{R}-\frac{2\nabla^i\nabla_i N}{N}\right)^2\right] && (4.194)\\
&= -\frac{1}{(D-3)(D-1)^2}\left[-\left(\frac{\nabla_i N\nabla^i N}{N^2}\right)^2+\frac{3\nabla_i N\nabla^i N\nabla^j\nabla_j N}{2N^3}+\frac{\nabla^i N\nabla^j N\hat{R}_{ij}}{N^2}\right.\\
&\qquad \left.-\frac{1}{D}\frac{\nabla_i\nabla^i N}{N}\hat{R}-\frac{D-1}{D}\left(\frac{\nabla^i\nabla_i N}{N}\right)^2-\hat{R}_{ij}\hat{R}^{ij}+\frac{D+1}{4D}\hat{R}^2\right] && (4.195)
\end{aligned}
$$

This agrees exactly with the general result above. Of course, for $z = 1$ there will also be contributing terms at this order which come from the $4z$ and $2+2z$ order terms (these will involve time derivatives).

An easily computable case is $D = 1$ (for which $\hat{R} = 0$). The above expressions yield $(z-3)\mathcal{L}^{(4)} = \dfrac{(z-3)\nabla_i N\nabla^i N}{12z^3N^2}$. For $z = 3$, which is when this would possibly generate a scaling anomaly, this expression vanishes.

4.7.5 Four-derivative counterterms involving ψ

There are many possible four-derivative counterterms involving ψ, for example:

$$(z+D-4-\Delta_-)\mathcal{L}^{(4+\Delta_-)} = 2K^{(2)}_{AB}\pi^{AB(2+\Delta_-)} + \pi^{(2)}_{A}\pi^{A(2+\Delta_-)} + K^{(\Delta_-)}_{AB}T^{AB(4)}. \tag{4.196}$$

The right hand-side has been explicitly calculated and found to be zero in the case where $z=2$ and $D=2$.

4.8 Appendix: Anisotropic Weyl Anomaly in $2+1$ Dimensions

Just as in the relativistic case, a theory which has a classical symmetry under anisotropic Weyl transformations can develop an anomaly in this symmetry at the quantum level. Under the transformations

$$\delta_\omega N = zN\delta\omega, \qquad \delta_\omega N_i = 2N_i\delta\omega, \qquad \delta_\omega \gamma_{ij} = 2\gamma_{ij}\delta\omega, \tag{4.197}$$

the anomaly will show up as a nonvanishing variation of the partition function $\mathcal{Z}[N, N_i, \gamma_{ij}]$, of the general form

$$\delta_\omega \log \mathcal{Z}[N, N_i, \gamma_{ij}] = -\int dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N\, \mathfrak{A}\, \delta\omega, \tag{4.198}$$

where $\mathfrak{A}$ is now a function of N, N_i, and γ_{ij}.

We wish to determine what terms can arise in $\mathfrak{A}$. As in the relativistic case, this question is cohomological in nature.[11] We introduce a nilpotent BRST operator Q, which acts on the metric multiplet via the infinitesimal anisotropic Weyl transformations (4.197), with $\delta\omega$ replaced by a Grassmann parameter c of ghost number one. We can represent this operator as

$$Q = c\left(zN\frac{\delta}{\delta N} + 2N_i\frac{\delta}{\delta N_i} + 2\gamma_{ij}\frac{\delta}{\delta\gamma_{ij}}\right). \tag{4.199}$$

Since Q is nilpotent, the variation of the anomaly vanishes:

$$Q\int dt\, d^D\mathbf{x}\, \sqrt{\gamma}\, N\, \mathfrak{A}\, c = -Q^2 \log \mathcal{Z} = 0. \tag{4.200}$$

This puts a constraint on the terms that can arise as $\mathfrak{A}$.

As usual, some of these terms can be removed by including appropriate counterterms. If a term in the anomaly can be expressed as the variation some local counterterm, this (gravitational) counterterm can be subtracted from the action, thereby eliminating the associated

[11] The cohomological approach to the relativistic Weyl anomaly was developed in (110; 111; 112); see (113), Chapter 22, for a general review of this approach.

anomaly. Therefore the physical anomaly can be considered to lie in the cohomology of Q, at ghost number one. The number of possible independent terms (*i.e.*, generalized central charges) in the anomaly will be determined by the dimension of this cohomology.

In the case of $2+1$ dimensions with $z=2$, the anomaly must be – on dimensional grounds – a sum of terms of dimension four, lying in the cohomology of Q. The list of possible terms is rather large; however, all but two are cohomologically trivial and can therefore be eliminated using local counterterms. The only ones that cannot be removed are:

$$\hat{K}_{ij}\hat{K}^{ij}-\frac{1}{2}\hat{K}^2, \qquad \left\{\hat{R}-\left(\frac{\nabla N}{N}\right)^2+\frac{\nabla^2 N}{N}\right\}^2 . \tag{4.201}$$

We have seen in Section 4.4.2 that the first cohomology class in (4.201), quadratic in the extrinsic curvature $\hat{K}_{ij}$, indeed arises in the holographic computation of the anisotropic Weyl anomaly, but the second one does not. However, this term $\sim \hat{R}^2+\ldots$ is also non-trivial in the cohomology of Q, because $\sqrt{\gamma}\,N\,cR^2+\ldots$ cannot be obtained as the variation of another term (essentially because variations of all available terms give rise to derivatives). Hence, both classes should be expected to appear in the anomaly of generic $z=2$ field theories in $2+1$ dimensions.

In addition, we list $\mathfrak{A}$ for the five independent cocycles that contain only spatial derivatives, but are cohomologically trivial and can be eliminated by local counterterms:

$$\frac{1}{N}\nabla^2\left[N\left(\hat{R}+\frac{\nabla^2 N}{N}-\left(\frac{\nabla N}{N}\right)^2\right)\right], \tag{4.202}$$

$$\frac{1}{N}\nabla_i\left(\left[\hat{R}+\frac{\nabla^2 N}{N}-\left(\frac{\nabla N}{N}\right)^2\right]\nabla^i N\right), \tag{4.203}$$

$$\frac{1}{N}\nabla^2\left[\nabla^2 N-\frac{(\nabla N)^2}{N}\right], \tag{4.204}$$

$$\frac{1}{N}\nabla_i\left(\nabla^i N\left[\frac{\nabla^2 N}{N}-\left(\frac{\nabla N}{N}\right)^2\right]-\frac{1}{2}\nabla^i\frac{(\nabla N)^2}{N}\right), \tag{4.205}$$

$$\frac{1}{N}\nabla_i\left[\nabla^i N\left(\frac{\nabla N}{N}\right)^2\right]. \tag{4.206}$$

This classification can be easily extended to include terms with time derivatives as well.

As usual, this cohomology analysis only reveals the complete list of terms which *may* in principle occur in the anomaly. Whether or not such terms are generated in a particular theory is a dynamical question, which requires an additional calculation.

Chapter 5

Conclusions

The emergence of holographic dualities as a tool for understanding strongly-coupled field theories has spurred a growing exchange of ideas between quantum gravity and other areas of physics, particularly condensed matter. String theory is now being used successfully in qualitative analyses of strongly-coupled field theory phenomena, while at the same time the embedding of known phenomena within string theory has pushed the boundaries of our understanding of string theory.

With the advent of anisotropic theories of gravity, the synthesis of condensed matter and quantum gravity was expanded to obtain what is possibly its most striking form: renormalizable completions of gravity.

Because of the difference in symmetry structure between general relativity and Hořava-Lifshitz gravity, it is non-trivial to ascertain whether the two theories can coincide in the infrared. The most obvious difficulty in the infrared is the presence of an extra propagating mode, whose presence can be traced to the lower degree of symmetry. We proposed a simple solution to this problem, in which the missing gauge symmetry of general relativity is replaced by a new symmetry principle, for which we introduce a new gauge field into the model. While straightforward at the linear level, obstructions to the symmetry arise at the non-linear level. We find that the symmetry can be extended to the non-linear level, provided we introduce a new field, the "Newton pre-potential", which in the Hamiltonian formulation enforces a second class constraint. In the presence of this symmetry, the number of propagating modes is found to match that of general relativity, using both a linearized analysis around symmetric backgrounds and a more general Hamiltonian analysis. In addition, the *static* solutions of this theory precisely match those of general relativity. Thus the theory matches general relativity at the static level, and at the linearized level at low energies.

In the context of holography, we first addressed the problem of identifying the boundary of bulk geometries dual to anisotropic field theories by introducing the concept of anisotropic conformal infinity, a natural generalization of Penrose's conformal infinity.

Building on this idea, we extended work of others (18; 21; 97) on the holographic renor-

malization of gravity in Lifshitz space to show that (1) the gravitational counterterms have the form of Hořava-Lifshitz gravitational actions, and (2) there exist conformal anomalies, which have the structure of conformal Hořava-Lifshitz gravity living on the boundary. For the model we considered, the action is precisely that of conformal Hořava-Lifshitz gravity in 2+1 dimensions with detailed balance.

Our work constitutes a first step forward in understanding the relationship between Hořava-Lifshitz gravity and string theory, and the geometric role of anisotropy in gravity and holography.

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