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Stationary-Action Stochastic Control Representation of the Schrödinger Initial Value Problem

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Stationary-Action Stochastic Control Representation of the Schrödinger Initial Value Problem

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

Engineering Sciences (Mechanical Engineering)

by

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2019

The Dissertation of Ruobing Zhao is approved and is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California San Diego

2019

DEDICATION

To my parents.

EPIGRAPH

“George, I think we’ve outgrown full-time education... Time to test our talents in the real world, d’you reckon?”

Fred Weasley, Harry Potter and the Order of the Phoenix

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ABSTRACT OF THE DISSERTATION

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by

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Doctor of Philosophy in Engineering Sciences (Mechanical Engineering)

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Professor William M. McEneaney, Chair
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Hamilton-Jacobi partial differential equations (HJ PDEs) arise in many scientific fields and applications, especially in mechanics and optimal control. Solutions to second order Hamilton-Jacobi partial differential equations (HJ PDEs) have controlled diffusion process representations. Of particular interest is the diffusion representation of an action functional associated with the solution of Schrödinger initial value problems (IVPs). In existing work that connects stochastic control problems to Schrödinger IVPs, one searches for the minimum of an action functional which is the payoff of the control problem. Issues arise, however, as the action functional loses convexity over longer time durations. The time duration where such

representation is valid is in fact infinitesimal when the system dimension is infinite. In this dissertation, we present an approach inspired by the Principle of Stationary Action in physics that removes this limitation, which leads to exceptional computational benefits. Instead of searching for local minima of the action functional as in traditional optimal control problems, we look for its stationary values instead.

We introduce the "staticization" operator and stationary-action control problems. A stationary-action stochastic control representation for the dequantized Schrödinger IVP in a non-inertial frame where the potential field is a polynomial is given. A solution approximation as a series expansion in a small parameter is obtained through the use of complex-valued diffusion-process representations.

Following this, the staticization operator is studied in detail. A new approach to solving conservative dynamical systems (e.g. paths of objects in gravitational or Coulomb potential field) using stationary-action control-theoretic methods with promising computational benefits is introduced. Two examples of application, the N-body problem and the Schrödinger IVP, are given.

Lastly, we demonstrate the existence of strong solutions of a class of degenerate stochastic differential equations (SDEs) that arises in the stationary-action stochastic control representation of Schrödinger IVPs with the Coulomb potential, which is non-smooth and has branch cuts in the complex range. The SDEs we consider have drift terms that have discontinuities and singularities along some manifolds, and diffusion coefficients that are degenerate, and there is no previously existing results regarding the existence of strong solutions for such SDEs.

Chapter 1

Introduction

Hamilton-Jacobi partial differential equations (HJ PDEs) arise in many scientific fields and applications, especially in mechanics and optimal control. Solutions to second order Hamilton-Jacobi partial differential equations (HJ PDEs) have controlled diffusion process representations. Of particular interest is the diffusion representation of an action functional associated with the solution of Schrödinger initial value problems (IVPs) of the following form:

$$
\begin{aligned}
&0 = i\hbar\psi_t(s,x) + \tfrac{\hbar^2}{2m}\Delta_x\psi(s,x) - \psi(s,x)\bar{V}(x), \quad (s,x) \in [0,t) \times \mathbb{R}^n, \\
&\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^n.
\end{aligned}
$$

Namely, one defines the functional S from a logarithmic transform $\psi = \exp\{\frac{i}{\hbar}S\}$, where ψ is the solution of the Schrödinger IVP, i is the imaginary unit, and $\hbar$ is the Planck's constant. In existing work that connects stochastic control problems to Schrödinger IVPs, cf. [1, 5, 4, 12, 9, 30, 29], the action functional S defined above is the payoff function of the control problem, where one seeks the minimum of the action functional. Issues arise, however, as the action functional loses convexity over longer time durations. The time duration where such representation is valid is in fact infinitesimal when the system dimension is infinite. In this dissertation, we present an approach inspired by the Principle of Stationary Action in physics that removes this limitation, which leads to exceptional computational benefits. Instead of searching for local minima of the action functional as in traditional optimal control problems, we look for its stationary values instead. We introduce the stat operator in analogy to min and max, and define argstat in analogy to argmin and argmax. Let $\mathscr{F}$ denote either the real or complex field. Suppose $\mathscr{U}$ is a normed vector space (over $\mathscr{F}$) with $\mathscr{A} \subseteq \mathscr{U}$, and suppose $G : \mathscr{A} \to \mathscr{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathscr{A}} G(u) \doteq \operatorname{argstat}\{G(u) \,|\, u \in \mathscr{A}\}$ if $\bar{u} \in \mathscr{A}$ and either

$$\limsup_{u\to\bar{u}, u\in\mathscr{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0, \tag{1.1}$$

or there exists $\delta > 0$ such that $\mathscr{A} \cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). We shall refer to the search for stationary values as "staticization". Unlike minimization or maximization, staticization not only applies to functions taking values in $\mathbb{R}$ but also to those taking values in Banach spaces. This allows us to obtain a stochastic representation for the Schrödinger action functional using a diffusion process with complex-valued diffusion coefficient. Moreover, we find that the gravitational potential and the Coulomb potential, which both take the form $-V(x) = \mu/|x|$, where $x \in \mathbb{R}^n\setminus\{0\}$ and μ is a contant, may be written as the stationary value of a polynomial in x and a new variable α. That is, we have $-V(x) = (\frac{3}{2})^{3/2} \operatorname{stat}_{\alpha>0}[\alpha - \frac{\alpha^3|x|^2}{2}]$. This leads to a new approach to studying dynamical systems in a conservative field with promising computational benefits, where upon introducing a new time-varying process $\alpha_\cdot$, the stationary value of the action functional has an iterated stat form. In particular, for a Newtonian particle of mass m moving in a potential field V, we can define the action functional $J(t,x,u) \doteq \int_0^t \frac{1}{2}m|u_r|^2 - V(\xi_r)dr$, where $\xi_r \doteq x + \int_0^t u_s ds$ is the path of the particle with velocity u. By the Principle of Stationary Action, the true trajectory of the particle is one for which the action functional is stationary; in other words, one solves for the stationary-action control problem, $\operatorname{stat}_u J(t,x,u)$. After introducing the process $\alpha_\cdot$, the problem becomes $\operatorname{stat}_u \operatorname{stat}_\alpha \tilde{J}(t,x,u,\alpha)$, where $\tilde{J}(t,x,u,\alpha) \doteq \int_0^t \frac{1}{2}m|u_r|^2 + (\frac{3}{2})^{3/2}[\alpha_r - \frac{\alpha_r^3|\xi_r|^2}{2}]dr$. If one is able to reorder the stat operators, the inner stat functional can be obtained from solutions of α-indexed differential Riccati equations (DREs), which can be solved much like a linear quadratic (LQ) control problem, but again, one searches for the stationary value rather than a minimum or maximum.

In Chapter 2, a particular class of Schrödinger IVPs is considered, wherein a classical point mass rotates around a charged nucleus in the central field under electrostatic forces. The Schrödinger IVP is converted to an HJ PDE for the action functional S. We introduce a non-inertial frame centered at the rotating point mass and approximate the potential field in the vicinity of the point mass as a series expansion in a small parameter, $\hat{\varepsilon}$, similar to a quantum harmonic oscillator. A numerical method for computing the solution of the wave equation is developed, where a solution approximation as a series expansion in $\hat{\varepsilon}$ is obtained, and the approximate solutions have Feynman-Kac type representations in terms of a complex-valued diffusion process. The approximate solutions (as a series expansion up to a finite order) can be computed for arbitrarily long time duration using the finite moments of the diffusion process, and the computation is purely analytical. Under a smoothness assumption, the series of approximate solutions converges to the exact solution. The quantum harmonic oscillator is recovered from the approximate solutions up to quadratic terms.

In Chapter 3, we take a detour, and study the staticization operator and its properties. In particular, we are interested in the problem of iterated staticization. As mentioned earlier, certain HJ PDEs that arise in conservative dynamical systems problems may be solved like stationary-action LQ control problems by introcing a time-varying process $\alpha_{\cdot}$, and a key step in that is to be able to reorder the stat operators. A general condition under which the stat operators can be reordered is obtained. We also considered some specific cases including certain kinds of semi-quadratic functionals and uniformly Morse functionals. Some possible applications of these results in physics are given. One important application is the two-point boundary value problem (TPBVP) associated with the N-body problem in astrodynamics. One may also employ a similar technique to obtain fundamental solutions of the Schrödinger IVP associated with Coulomb potential.

In Chapter 4, we discuss a class of stochastic differential equations (SDEs) with degenerate diffusion coefficients that arises in the stationary-action stochastic control representation of Schrödinger IVPs with the Coulomb potential. Unlike the quadratic potential associated with the quantum harmonic oscillator, the Coulomb potential poses significant difficulty. To allow complex-valued state processes, the domain of the potential field $-V(x) = \mu/|x|$ is extended from $\mathbb{R}^n$ to $\mathbb{C}^n$. The extended domain potential field $-V(x) = \mu/\sqrt{x^T x}$, where $x \in \mathbb{C}^n$, has a branch cut in the complex range, which leads to discontinuity in the drift term of the underlying diffusion process on a manifold of codimension 1. In addition, the drift term of the diffusion process may have singularities along a manifold of codimension 2. In existing work on the existence of strong solutions of SDEs where the drift term may have singularities (cf. [26, 55]), the diffusion coefficient of diffusion processs is assumed to be non-degenerate. The underlying diffusion process of our control representation of Schrödinger IVPs, however, is a $2m$-dimensional state process with $2m \times m$ degenerate diffusion coefficient. In this chapter, we first demonstrate the existence of weak solutions of the SDE, which is obtained through passing to the limit of a sequence of diffusion processes with C^1 drifts. After that, the existence of a strong solution follows from pathwise uniqueness property of the process.

Chapter 2

Diffusion process representations for a scalar-field Schrödinger equation solution in rotating coordinates

2.1 Introduction

Diffusion representations have long been a useful tool in solution of second-order Hamilton-Jacobi partial differential equations (HJ PDEs), cf. [13, 21] among many others. The bulk of such results apply to real-valued HJ PDEs, that is, to HJ PDEs where the coefficients and solutions are real-valued. The Schrödinger equation is complex-valued, although generally defined over a real-valued space domain, which presents difficulties for the development of stochastic control representations. In [31, 32], a representation for the solution of a Schrödinger-equation initial value problem over a scalar field was obtained as a stationary value for a complex-valued diffusion process control problem. Although there is substantial existing work on the relation of stochastic processes to the Schrödinger equation (cf. [16, 25, 45, 46, 56]), the approach considered in [31, 32] is along a slightly different path, closer to [4, 5, 9, 12, 24, 30]. However, the representation in [31, 32] employs stationarity of the payoff [38] rather than optimization of the payoff, where stationarity can be used to overcome the limited-duration constraints of methods that use optimization of the payoff.

Here we discuss a particular problem class, and use diffusion representations as a tool for approximate solution of the Schrödinger equation. We will consider a specific type of weak field problem. Suppose we have a particle in a scalar field centered at the origin, but in the special case where the particle is sufficiently far from the origin that the distribution associated to the corresponding Schrödinger equation has negligible density near the origin. More specifically, let the particle mass be denoted by m, and let $\hbar$ denote Planck's constant. The simplest scalar-field example, which can be instructive if only purely academic, is the quadratic-field case, generating the quantum harmonic oscillator. Of somewhat more interest is the case where one has the potential energy generated by the field interacting with the particle taking the form $\bar{V}(x) = -\bar{c}/|x|$. Let the solution of the Schrödinger equation at time, t, and position, x, be denoted by $\psi(t,x)$, and consider the associated distribution given by $\tilde{P}(t,x) \doteq [\psi^* \psi](t,x)$. Formally speaking, when $\hbar/m$ is sufficiently small, one expects that $\tilde{P}(t,\cdot)$ can be approximated in some sense by a Dirac-delta function centered at $\xi(t)$, where $m\ddot{\xi}(t) = -\nabla_x \bar{V}(\xi(t))$. We will consider a non-inertial frame where the origin will be centered at $\xi(t)$ for all t. In particular, we consider a case where $\xi(t)$ follows a circular orbit with constant angular velocity. That is, we consider $\xi(t) = \hat{\delta}\big(\cos(\omega t), \sin(\omega t)\big)$ where $\hat{\delta} \in (0,\infty)$. (In the interests of space and reduction of clutter, where it will not lead to confusion, we will often write (x_1,x_2) in place of $(x_1,x_2)^T$, etc.) Although such motion can be generated by a two-dimensional harmonic oscillator, we will focus mainly on the $\bar{V}(x) = -\bar{c}/|x|$ class, in which case $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$. We suppose that $\hat{\delta}$ is sufficiently large such that $\tilde{P}(t,x) \ll 1$ for $|x| < \hat{\delta}/2$, and thus that one may approximate $\bar{V}$ in the vicinity of $\xi(t)$ by a finite number of terms in a power series expansion centered at $\xi(t)$. We will use a set of complex-valued diffusion representations to obtain an approximation to the resulting Schrödinger equation solution. If the solution is holomorphic in x and a small parameter, then the approximate solution converges as the number of terms in the set of diffusion representations approaches infinity.

The analysis will be carried out only in the case of a holomorphic field approximation. As our motivation is the case where $\hat{\delta}$ is large relative to the associated position distribution, one expects that the case of a $-\bar{c}/|x|$ potential may be sufficiently well-modeled by a finite number of terms in a power series expansion. However, an analysis of the errors induced by such an approximation to a $-\bar{c}/|x|$ potential is beyond the scope of this already long paper, and may be addressed in a later effort; the focus here is restricted to the diffusion-representation based method of solution approximation method *given* such an approximation to the potential. We remark that in the case of a quadratic potential, we recover the quantum harmonic oscillator solution. Also, in the case of $\bar{V}(x) = -\bar{c}/|x|$, as $\hat{\delta} \to \infty$, the solution approaches that of the free particle case. The computations required for solution up to any finite polynomial-in-space order may be performed analytically.

In Section 2.2, we review the Schrödinger initial value problem, and the dequantized form of the problem. The solution to the dequantized form of the problem will be approximated through the use of diffusion representations; the solution to the originating Schrödinger initial value problem is recovered by a simple transformation. As it is used in Section 2.2, we briefly recall the stat operator in Section 2.3.1. In Section 2.3.2, the dequantized form will be converted into a form over a rotating and translating reference frame centered at the position of a classical particle following a circular trajectory generated by the central field. Then, in Section 2.3.3, we discuss equivalent forms over a complex space domain, and over a double-dimension real-valued domain. Classical existence, uniqueness and smoothness results will be applied to the problem in this last form. These will then be transferred to the original form as a complex-valued solution over a real space domain. In Section 2.4, we indicate the expansion of the solution in a small parameter related to the inverse of the distance to the origin of the field. A power series representation will be used, where this will be over both space and the small parameter. In particular, we will assume that at each time, the solution will be holomorphic over space and the small parameter. The functions in the expansion are solutions to corresponding HJ PDEs, where these are also indicated here. The HJ PDE for the first term, say $k = 0$, has a closed-form solution, and this is given in Section 2.5. Then, in Section 2.6, it is shown that for $k \geq 1$, given the solutions to the preceding terms, the HJ PDE for the k^{th} term takes a linear parabolic form, with a corresponding diffusion representation. It is shown that diffusion representation may be used to obtain the solution of the $k+1$ HJ PDE given the solutions to the k-and-lower HJ PDE solutions. The required computations may be performed analytically. In Section 2.7, this method is applied to obtain the next term in the expansion in the case of a cubic approximation of the classic $1/r$ type of potential, and additional terms may be obtained similarly.

2.2 Dequantization

We recall the Schrödinger initial value problem, given as

$$0 = i\hbar\psi_t(s,x) + \tfrac{\hbar^2}{2m}\Delta_x\psi(s,x) - \psi(s,x)\bar{V}(x), \quad (s,x) \in \mathscr{D}, \tag{2.1}$$

$$\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^n, \tag{2.2}$$

where initial condition ψ_0 takes values in $\mathbb{C}$, Δ_x denotes the Laplacian with respect to the space (second) variable, $\mathscr{D} \doteq (0,t) \times \mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We also let $\bar{\mathscr{D}} \doteq [0,t) \times \mathbb{R}^n$. We consider the Maslov dequantization of the solution of the Schrödinger equation (cf. [29]), which similar to a standard log transform, is $S : \bar{\mathscr{D}} \to \mathbb{C}$ given by $\psi(s,x) = \exp\{\frac{i}{\hbar}S(s,x)\}$. Note that $\psi_t = \frac{i}{\hbar}\psi S_t$, $\psi_x = \frac{i}{\hbar}\psi S_x$ and $\Delta_x\psi = \frac{i}{\hbar}\psi\Delta_x S - \frac{1}{\hbar^2}\psi|S_x|_c^2$ where for $y \in \mathbb{C}^n$, $|y|_c^2 \doteq \sum_{j=1}^n y_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We find that (2.1)–(2.2) become

$$0 = -S_t(s,x) + \tfrac{i\hbar}{2m}\Delta_x S(s,x) + H^0(x, S_x(s,x)), \quad (s,x) \in \mathscr{D}, \tag{2.3}$$

$$S(0,x) = \bar{\phi}(x), \quad x \in \mathbb{R}^n, \tag{2.4}$$

where $H : \mathbb{R}^n \times \mathbb{C}^n \to \mathbb{C}$ is the Hamiltonian given by

$$H^0(x,p) = -\left[\tfrac{1}{2m}|p|_c^2 + \bar{V}(x)\right] = \operatorname*{stat}_{v\in\mathbb{C}^n}\left\{v \cdot p + \tfrac{m}{2}|v|_c^2 - \bar{V}(x)\right\}, \tag{2.5}$$

and stat is defined in Section 2.3.1. We look for solutions in the space

$$\mathscr{S} \doteq \{S : \bar{\mathscr{D}} \to \mathbb{C} \,|\, S \in C_p^{1,2}(\mathscr{D}) \cap C(\bar{\mathscr{D}})\}, \tag{2.6}$$

where $C_p^{1,2}$ denotes the space of functions which are continuously differentiable once in time and twice in space, and which satisfy a polynomial-growth bound.

2.3 Preliminaries

In this section, we collect condensed discussions of relevant classical material as well as some recently obtained results and definitions.

2.3.1 Stationarity definitions

Recall that classical systems obey the stationary action principle, where the path taken by the system is that which is a stationary point of the action functional. For this and other reasons, as in the definition of the Hamiltonian given in (2.5), we find it useful to develop additional notation and nomenclature. Specifically, we will refer to the search for stationary points more succinctly as *staticization*, and we make the following definitions. Suppose $(\mathscr{Y}, |\cdot|)$ is a generic normed vector space over $\mathbb{C}$ with $\mathscr{G} \subseteq \mathscr{Y}$, and suppose $F : \mathscr{G} \to \mathbb{C}$. We say $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\}$ if $\bar{y} \in \mathscr{G}$ and either $\limsup_{y \to \bar{y}, y \in \mathscr{G} \setminus \{\bar{y}\}} |F(y) - F(\bar{y})|/|y - \bar{y}| = 0$, or there exists $\delta > 0$ such that $\mathscr{G} \cap B_\delta(\bar{y}) = \{\bar{y}\}$ (where $B_\delta(\bar{y})$ denotes the ball of radius δ around $\bar{y}$). If $\operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\} \neq \emptyset$, we define the possibly set-valued stat^s operator by

$$\operatorname*{stat}^s_{y \in \mathscr{G}} F(y) \doteq \operatorname{stat}^s\{F(y) \,|\, y \in \mathscr{G}\} \doteq \left\{F(\bar{y}) \,\middle|\, \bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\}\right\}.$$

If $\operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\} = \emptyset$, $\operatorname{stat}^s_{y \in \mathscr{G}} F(y)$ is undefined. We will also be interested in a single-valued stat operation. In particular, if there exists $a \in \mathbb{C}$ such that $\operatorname{stat}^s_{y \in \mathscr{G}} F(y) = \{a\}$, then $\operatorname{stat}_{y \in \mathscr{G}} F(y) \doteq a$; otherwise, $\operatorname{stat}_{y \in \mathscr{G}} F(y)$ is undefined. At times, we may abuse notation by writing $\bar{y} = \operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\}$ in the event that the argstat is the set $\{\bar{y}\}$. For further discussion, we refer the reader to [38]. The following is immediate from the above definitions.

Lemma 1. *Suppose $\mathscr{Y}$ is a Hilbert space, with open set $\mathscr{G} \subseteq \mathscr{Y}$, and that $F : \mathscr{G} \to \mathbb{C}$ is Fréchet differentiable at $\bar{y} \in \mathscr{G}$ with Riesz representation $F_y(\bar{y}) \in \mathscr{Y}$ Then, $\bar{y} \in \operatorname{argstat}\{F(y) \,|\, y \in \mathscr{G}\}$ if and only if $F_y(\bar{y}) = 0$.*

2.3.2 The non-inertial frame

As noted in the introduction, we suppose a central scalar field such that a particular solution for the motion of a classical particle in the field takes the form $\xi(t) = \hat{\delta}\big(\cos(\omega t), \sin(\omega t)\big)$ where $\hat{\delta}, \omega \in (0, \infty)$. In particular, we concentrate on the potential $\bar{V}(x) = -\bar{c}/|x|$, in which case $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$. We consider a two-dimensional space model and a non-inertial frame centered at $\xi(t)$ for all $t \in (0, \infty)$, with the first basis axis in the positive radial direction and the second basis vector in the direction of the velocity of the particle. Let positions in the non-inertial frame be denoted by $z \in \mathbb{R}^2$, where the transformation between frames at time $t \in \mathbb{R}$ is given by

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = G_{\omega t} x - \begin{pmatrix} \hat{\delta} \\ 0 \end{pmatrix} \doteq \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} \hat{\delta} \\ 0 \end{pmatrix}. \tag{2.7}$$

We will denote this transformation as $z = z^*(x)$, with its inverse denoted similarly as $x = x^*(z)$, where $x^*(z) = (G_{\omega t})^T (z + (\hat{\delta}, 0)^T)$.

For $z \in \mathbb{R}^2$, define $V(z) \doteq \bar{V}(x^*(z))$ and $\phi(z) \doteq \bar{\phi}(x^*(z))$. Then, $\tilde{S}^f : \bar{\mathscr{D}} \to \mathbb{C}$ defined by $\tilde{S}^f(s,z) \doteq \hat{\bar{S}}^f(s, x^*(z))$ is a solution of the forward-time dequantized HJ PDE problem given by

$$0 = -S_t(s,z) + \tfrac{i\hbar}{2m} \Delta_z S(s,z) - (A_0 z + b_0)^T S_z(s,z) - \tfrac{1}{2m} |S_z(s,z)|_c^2 - V(z), \;\; (s,z) \in \mathscr{D}, \tag{2.8}$$

$$S(0,z) = \phi(z), \quad z \in \mathbb{R}^2, \quad \text{where} \;\; A_0 \doteq \omega \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;\text{and}\; b_0 \doteq -\omega\hat{\delta} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.9}$$

if and only is $\hat{\hat{S}}^f$ is a solution of (2.3)–(2.4). (We remark that one may see [52] for further discussion of non-inertial frames in the context of the Schrödinger equation.) In order to apply the diffusion representations as an aid in solution, we will find it helpful to reverse the time variable, and hence we look instead, and equivalently, at the Hamilton-Jacobi partial differential equation (HJ PDE) problem given by

$$0 = S_t(s,z) + \tfrac{i\hbar}{2m}\Delta_z S(s,x) - (A_0 z + b_0)^T S_z(s,z) - \tfrac{1}{2m}|S_z(s,z)|_c^2 - V(z), \quad (s,z) \in \mathscr{D}, \tag{2.10}$$

$$S(t,z) = \phi(z), \quad z \in \mathbb{R}^n. \tag{2.11}$$

In this last form, we will fix $t \in (0,\infty)$, and allow s to vary in $(0,t]$.

2.3.3 Extensions to the complex domain

Various details of extensions to the complex domain must be considered prior to the development of the representation. This material is rather standard, but it is required for the main development. Models (2.1)–(2.2), (2.3)–(2.4) and (2.10)–(2.11) are typically given as HJ PDE problems over real space domains. However, as in Doss et al. [1, 4, 5], we will find it convenient to change the domain to one where the space components lie over the complex field. We also extend the domain of the potential to $\mathbb{C}^2$, i.e., $V : \mathbb{C}^2 \to \mathbb{C}$, and we will abuse notation by employing the same symbol for the extended-domain functions. Throughout, for $k \in \mathbb{N}$, and $z \in \mathbb{C}^k$ or $z \in \mathbb{R}^k$, we let $|z|$ denote the Euclidean norm. Let $\mathscr{D}_{\mathbb{C}} \doteq (0,t) \times \mathbb{C}^2$ and $\overline{\mathscr{D}}_{\mathbb{C}} = (0,t] \times \mathbb{C}^2$, and define

$$\begin{aligned} \mathscr{S}_{\mathbb{C}} \doteq \{S : \overline{\mathscr{D}}_{\mathbb{C}} \to \mathbb{C} \,|\, &S \text{ is continuous on } \overline{\mathscr{D}}_{\mathbb{C}}, \text{ continuously differentiable in time on} \\ &\mathscr{D}_{\mathbb{C}}, \text{ and holomorphic on } \mathbb{C}^2 \text{ for all } r \in (0,t]\}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \mathscr{S}_{\mathbb{C}}^p \doteq \{S \in \mathscr{S}_{\mathbb{C}} \,|\, &S \text{ satisfies a polynomial growth condition in space,} \\ &\text{uniformly on } (0,t]\}. \end{aligned} \tag{2.13}$$

The extended-domain form of problem (2.10)–(2.11) is

$$0 = S_t(s,z) + \tfrac{i\hbar}{2m}\Delta_z S(s,z) - (A_0 z + b_0)^T S_z(s,z) - \tfrac{1}{2m}|S_z(s,z)|_c^2 - V(z), \ \ (s,z) \in \mathscr{D}_{\mathbb{C}}, \quad (2.14)$$

$$S(t,z) = \phi(z), \quad z \in \mathbb{C}^2. \quad (2.15)$$

Remark 1. *We remark that a holomorphic function on $\mathbb{C}^2$ is uniquely defined by its values on the real part of its domain. In particular, $\tilde{S} : \bar{\mathscr{D}} \to \mathbb{C}$ uniquely defines its extension to a time-indexed holomorphic function over complex space, say $\bar{S} : \overline{\mathscr{D}_{\mathbb{C}}} \to \mathbb{C}$, if the latter exists. Consequently, although (2.10)–(2.11) form an HJ PDE problem for a complex-valued solution over real time and real space, (2.14)–(2.15) is an equivalent formulation, under the assumptions that a holomorphic solution exists and one has uniqueness for both.*

The following result by McEneaney (cf. [32]) provides a stochastic control verification theorem for the Schrödinger IVP.

Theorem 1. *The extended domain Schrödinger IVP is*

$$0 = S_t(t,x) + \tfrac{i\hbar}{2m}\Delta S(t,x) + \operatorname*{stat}_{v\in\mathbb{C}^n}\{v^T S_x(t,x) + \tfrac{m}{2}|v|_c^2 - V(x)\} \ (t,x) \in (0,t) \times \mathbb{C}^n$$

$$S(t,x) = \phi(x), \ x \in \mathbb{C}^n$$

Let $d\xi_t \doteq u_t dt + \sqrt{\tfrac{\hbar}{2m}}\tfrac{1+i}{\sqrt{2}} dB_t$. *Then* $S(t,x) = \operatorname{stat}_u \mathbb{E}\{\int_0^t \tfrac{m}{2}|u_r|_c^2 - V(\xi_r)dr + \phi(\xi_t)\}$.

Throughout the remainder, we will assume the following.

$$V, \phi : \mathbb{C}^2 \to \mathbb{C} \text{ are holomorphic on } \mathbb{C}^2. \quad (A.1)$$

Remark 2. *The assumption on V requires a remark. Recall that we are interested here in a class of problems where $\hat{\delta}$ is large in the sense that the distribution associated to the solution of the Schrödinger initial value problem has only very small probability mass outside a ball of radius less than $\hat{\delta}$. If $\bar{V}$ is of the $\bar{c}/|x|$ form, one would use only a finite number of terms in the power series expansion around $z = 0$. The focus here is on a diffusion-representation based method for* *approximate solution of the Schrödinger initial value problem* given *a holomorphic potential. The errors introduced by the use of a truncated power series for a $\bar{c}/|x|$-type potential for large $\hat{\delta}$ are outside the scope of the discussion.*

We will refer to a linear space over the complex [real] field as a complex [real] space. Although (2.14)–(2.15) form an HJ PDE problem for a complex-valued solution over real time and complex space, there is an equivalent formulation as a real-valued solution over real time and a double-dimension real space. We will find such formulations to be helpful in the analysis to follow. Further, although it is natural to work with complex-valued state processes in this problem domain, in order to easily apply many of the existing results regarding existence, uniqueness and moments, we will also find it handy to use a "vectorized" real-valued representation for the complex-valued state processes. We begin from the standard mapping of the complex plane into $\mathbb{R}^2$, denoted here by $\mathscr{V}_{00}:\mathbb{C}\to\mathbb{R}^2$, with $\mathscr{V}_{00}(z)\doteq(x,y)^T$, where $x=\mathbf{Re}(z)$ and $y=\mathbf{Im}(z)$. This immediately yields the mapping $\mathscr{V}_0:\mathbb{C}^2\to\mathbb{R}^{2n}$ given by $\mathscr{V}_0(x+iy)\doteq(x^T,y^T)^T$, where component-wise, $(x_j,y_j)^T=\mathscr{V}_{00}(x_j)$ for all $j\in]1,n[$, where throughout, for integer $a\le b$, we define $]a,b[=\{a,a+1,\dots b\}$. Also in the interests of a reduction of cumbersome notation, we will henceforth frequently abuse notation by writing (x,y) in place of $(x^T,y^T)^T$ when the meaning is clear. Lastly, we may decompose any function in $\mathscr{S}_{\mathbb{C}}$, say $F\in\mathscr{S}_{\mathbb{C}}$, as

$$(\bar{R}(r,\mathscr{V}_0(z)),\bar{T}(r,\mathscr{V}_0(z)))^T\doteq\mathscr{V}_{00}(F(r,z)),\tag{2.16}$$

where $\bar{R},\bar{T}:\overline{\mathscr{D}_2}\doteq(0,t]\times\mathbb{R}^{2n}\to\mathbb{R}$, and we also let $\mathscr{D}_2\doteq(0,t)\times\mathbb{R}^{2n}$. For later reference, it will be helpful to recall some standard relations between derivative components, which are induced by the Cauchy-Riemann equations. For all $(r,z)=(r,x+iy)\in(0,t)\times\mathbb{C}^2$ and all $j,k,\ell\in]1,n[$, and suppressing the arguments for reasons of space we have

$$\mathbf{Re}[F_{z_j,z_k}]=\bar{R}_{x_j,x_k}=-\bar{R}_{y_j,y_k}=\bar{T}_{y_j,x_k}=\bar{T}_{x_j,y_k},\tag{2.17}$$

$$\mathbf{Im}[F_{z_j,z_k}]=-\bar{R}_{x_j,y_k}=-\bar{R}_{y_j,x_k}=-\bar{T}_{y_j,y_k}=\bar{T}_{x_j,x_k}.\tag{2.18}$$

2.4 An expansion

We now reduce our problem class to the two-dimensional space case (i.e., $n = 2$). We will expand the desired solutions of our problems, and use these expansions as a means for approximation of the solution. First, we consider holomorphic V in the form of a finite or infinite power series. In the simple example case where $\bar{V}$ generates the quantum harmonic oscillator, one may take $\bar{V}(x) = \frac{\bar{c}^q}{2}[x_1^2 + x_2^2]$, in which case

$$V(z) = \frac{\bar{c}^q}{2}\hat{\delta}^2 + \bar{c}^q\hat{\delta}z_1 + \frac{\bar{c}^q}{2}[z_1^2 + z_2^2].$$

The scalar field of most interest takes the form $-\bar{V}(x) = \bar{c}/|x|$, yielding $-V(z) = \bar{c}/|z + (\hat{\delta}, 0)|$. In this case, recalling that this effort focuses on the case where $\hat{\delta}$ is large relative to the radius of the "non-negligible" portion of the probability distribution associated to the solution, we consider only a truncated power series, and let $\check{V}^K(z)$ denote the partial sum containing only terms up to order $K + 2 < \infty$ in z. We will be interested in the dependence of the potential and the resulting solutions in the parameter $\hat{\varepsilon} \doteq 1/\hat{\delta}$. We also recall from Section 2.3.2 that $\omega \doteq [\bar{c}/(m\hat{\delta}^3)]^{1/2}$, or $\bar{c} = m\omega^2\hat{\delta}^3$. We explicitly indicate the expansion up to the fourth-order term in z and the form of higher-order terms. One finds,

$$\begin{aligned}
-\check{V}^2(z) &= -\sum_{k=0}^{2} \hat{\varepsilon}^k \hat{V}^k(z), \qquad (2.19)\\
-\hat{V}^0(z) &= m\omega^2\big[\hat{\delta}^2 - \hat{\delta}z_1 + (z_1^2 - z_2^2/2)\big],\\
-\hat{V}^1(z) &= m\omega^2\big[-z_1^3 + 3z_1z_2^2/2\big], \quad -\hat{V}^2(z) = m\omega^2\big[z_1^4 - 3z_1^2z_2^2 + 3z_2^4/8\big],
\end{aligned}$$

and more generally, for $k > 1$, $-\hat{V}^k(z) = m\omega^2\Big[\sum_{j=0}^{k+2} c^V_{k+2,j} z_1^j z_2^{k-j}\Big]$, for proper choice of coefficients $c^V_{k,j}$.

Here, we find it helpful to explicitly consider the dependence of $\tilde{S}$ and $\bar{S}$ (solutions of (2.10)–(2.11) and (2.14)–(2.15), respectively) on $\hat{\varepsilon}$, where for convenience of exposition, we also allow $\hat{\varepsilon}$ to take complex values. Abusing notation, we let $\tilde{S}:\, \bar{\mathscr{D}} \times \mathbb{C} \to \mathbb{C}$ and $\bar{S}:\, \overline{\mathscr{D}}_{\mathbb{C}} \times \mathbb{C} \to \mathbb{C}$, and denote the dependence on their arguments as $\tilde{S}(s,z,\hat{\varepsilon})$ and $\bar{S}(s,z,\hat{\varepsilon})$. We let $\breve{\mathscr{D}} \doteq \mathscr{D} \times \mathbb{C}$, $\breve{\bar{\mathscr{D}}} \doteq \bar{\mathscr{D}} \times \mathbb{C}$, $\breve{\mathscr{D}}_{\mathbb{C}} \doteq \mathscr{D}_{\mathbb{C}} \times \mathbb{C}$ and $\breve{\bar{\mathscr{D}}}_{\mathbb{C}} \doteq \overline{\mathscr{D}}_{\mathbb{C}} \times \mathbb{C}$, where we recall that the physical-space components are now restricted to the two-dimensional case. We also let

$$\begin{aligned} \breve{\mathscr{S}}_{\mathbb{C}} \doteq \{S : \breve{\bar{\mathscr{D}}}_{\mathbb{C}} \to \mathbb{C} \,|\, & S \text{ is continuous on } \breve{\bar{\mathscr{D}}}_{\mathbb{C}}, \text{ continuously differentiable in time on} \\ & \breve{\mathscr{D}}_{\mathbb{C}}, \text{ and } S(r,\cdot,\cdot) \text{ is holomorphic on } \mathbb{C}^2 \times \mathbb{C} \text{ for all } r \in (0,t]\,\}, \qquad (2.20) \\ \breve{\mathscr{S}}^{p}_{\mathbb{C}} \doteq \{S \in \breve{\mathscr{S}}_{\mathbb{C}} \,|\, & S \text{ satisfies a polynomial growth condition in space,} \\ & \text{uniformly on } (0,t]\,\}. \qquad (2.21) \end{aligned}$$

We will make the following assumption throughout the sequel.

$$\text{There exists a unique solution, } \bar{S} \in \breve{\mathscr{S}}_{\mathbb{C}}, \text{ to (2.14)–(2.15).} \qquad (A.2)$$

We also let the power series expansion for ϕ be arranged as

$$\phi(z) = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \phi^k(z) \doteq \phi^0(z) + \sum_{k=1}^{\infty} \hat{\varepsilon}^k \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} b^{\phi}_{k+2,l,j} z_1^j z_2^{l-j}, \qquad (2.22)$$

where $\phi^0(z)$ is quadratic in z. We consider the following terminal value problems. The zeroth-order problem is

$$0 = S^0_t + \tfrac{i\hbar}{2m}\Delta_z S^0 - \big(A_0 z + b_0\big)^T S^0_z - \tfrac{1}{2m}|S^0_z|^2_c - \hat{V}^0, \qquad (s,z) \in \mathscr{D}_{\mathbb{C}}, \qquad (2.23)$$

$$S^0(t,z) = \phi^0(z), \qquad z \in \mathbb{C}^2. \qquad (2.24)$$

For $k \geq 1$, the k^{th} terminal value problem is

$$0 = S^k_t + \tfrac{i\hbar}{2m}\Delta_z S^k - \big(A_0 z + b_0 + \tfrac{1}{m}S^0_z\big)^T S^k_z - \tfrac{1}{2m}\sum_{\kappa=1}^{k-1} \big(S^{\kappa}_z\big)^T S^{k-\kappa}_z - \hat{V}^k, \quad (s,z) \in \mathscr{D}_{\mathbb{C}}, \qquad (2.25)$$

$$S^k(t,z) = \phi^k(z), \qquad z \in \mathbb{C}^2. \qquad (2.26)$$

Note that for $k \geq 1$, given the $\hat{S}^\kappa$ for $\kappa < k$, (2.25) is a linear, parabolic, second-order PDE, while zeroth-order case (2.23) is a nonlinear, parabolic, second-order PDE. Also note that (2.23) is (2.25) in the case of $k = 0$, but as its form is different, it is worth breaking it out separately. It is also worth noting here that if the S^k are all polynomial in z of order up to k, then the right-hand side of (2.25) is polynomial in z of order up to k, as is the right-hand side of (2.26).

Theorem 2. *Assume there exists a unique solution, $\hat{S}^0$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (2.23)–(2.24), and that for each $k \geq 1$, there exists a unique solution, $\hat{S}^k$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (2.25)–(2.26). Then, $\bar{S} = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \hat{S}^k$.*

Remark 3. *It is worth noting here that if the S^k are all polynomial in z of order up to $k+2$, then for each k, the right-hand side of (2.25) is polynomial in z of order up to $k+2$, as is the right-hand side of (2.26). That is, with the expansion in powers of $\hat{\varepsilon} = \hat{\delta}^{-1}$, the resulting constituent HJ PDE problems indexed by k are such that one might hope for polynomial-in-z solutions of order $k+2$, and this hope will be realized further below.*

Proof. Let $\bar{\mathbb{N}} \doteq \mathbb{N} \cup \{0\}$. By Assumption $(A.2)$, $\bar{S}$ has a unique power series expansion on $\bar{\breve{\mathscr{D}}}_{\mathbb{C}}$, which we denote by

$$\bar{S}(s,z,\hat{\varepsilon}) = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \tilde{c}^k(s,z) \doteq \sum_{k=0}^{\infty} \hat{\varepsilon}^k \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \tilde{\tilde{c}}_{k,l,j}(s) z_1^j z_2^{l-j},$$

where the $\tilde{\tilde{c}}_{k,l,j}(\cdot) : (0,t] \to \mathbb{C}$ form a time-indexed set of coefficients, and obviously, the $\tilde{c}^k(\cdot,\cdot) : \overline{\mathscr{D}}_{\mathbb{C}} \to \mathbb{C}$ are given by $\tilde{c}^k(s,z) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \tilde{\tilde{c}}_{k,l,j}(s) z_1^j z_2^{l-j}$ for all $k \in \bar{\mathbb{N}}$. For all $k \in \bar{\mathbb{N}}$, define the notation $\tilde{c}^{-k}(\cdot,\cdot) \doteq \sum_{j=k+1}^{\infty} \hat{\varepsilon}^{j-(k+1)} \tilde{c}^j(\cdot,\cdot)$. Also define $V^{-k} \doteq \hat{\varepsilon}^{-(k+1)} \big[V - \sum_{j=0}^{k} \hat{\varepsilon}^j \hat{V}^j \big]$ and $\phi^{-k} \doteq \sum_{j=k+1}^{\infty} \hat{\varepsilon}^{j-(k+1)} \phi^j = \hat{\varepsilon}^{-(k+1)} \big[\phi - \sum_{j=0}^{k} \hat{\varepsilon}^j \phi^j \big]$ for all $k \in \bar{\mathbb{N}}$. Recall that $\bar{S}$ is the unique solution in $\breve{\mathscr{S}}_{\mathbb{C}}$ of (2.14)–(2.15). By (2.15),

$$\tilde{c}^k(t,z) = \phi^k(z) \;\text{ and }\; \tilde{c}^{-k}(t,z) = \phi^{-k}(z) \quad \forall z \in \mathbb{C}^2. \tag{2.27}$$

Separating the $\tilde{c}^0$ and $\tilde{c}^{-0}$ components of $\bar{S}$ in (2.14) yields

$$\begin{aligned} 0 =& \tilde{c}_t^0 + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^0 - (A_0 z + b_0)^T\tilde{c}_z^0 - \tfrac{1}{2m}|\tilde{c}_z^0|_c^2 - \hat{V}^0 \\ &+ \hat{\varepsilon}\Big\{\tilde{c}_t^{-0} + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^{-0} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^{-0} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-0}|_c^2 - V^{-0}\Big\}. \end{aligned} \tag{2.28}$$

Now, note that as $\bar{S}(s,\cdot,\cdot)$ is holomorphic for all $s \in (0,t]$, we have $\bar{S}_z(s,\cdot,\cdot)$ and $\Delta_z\bar{S}(s,\cdot,\cdot)$ holomorphic for all $s \in (0,t]$. Further, by standard results on the composition of holomorphic mappings, noting that $g : \mathbb{C}^2 \to \mathbb{C}$ given by $g(z) \doteq |z|_c^2 = z^T z$ is holomorphic, we see that $|\tilde{S}_z(s,\cdot,\cdot)|_c^2 = g\big(\tilde{S}_z(s,\cdot,\cdot)\big)$ is holomorphic for all $s \in (0,t]$. Combining these insights, we see that with $S = \bar{S}$ all terms on the right-hand side of (2.14), with the exception of S_t are holomorphic in $(z,\hat{\varepsilon})$, which implies that $\bar{S}_t(s,\cdot,\cdot)$ is holomorphic for all $s \in (0,t]$. Consequently, for any $s \in (0,t]$, the right-hand side of (2.14) with $S = \bar{S}$ has a unique power series expansion. This implies that, as (2.28) is satisfied for all $\hat{\varepsilon} \in \mathbb{C}$, we must have

$$0 = \tilde{c}_t^0 + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^0 - (A_0 z + b_0)^T\tilde{c}_z^0 - \tfrac{1}{2m}|\tilde{c}_z^0|_c^2 - \hat{V}^0, \tag{2.29}$$

$$0 = \tilde{c}_t^{-0} + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^{-0} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^{-0} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-0}|_c^2 - V^{-0}. \tag{2.30}$$

By (2.27), (2.29) and the assumptions, $\tilde{c}^0 = \hat{S}^0$.

Next, separating the $\tilde{c}^1$ and $\tilde{c}^{-1}$ components, (2.30) implies

$$\begin{aligned} 0 =& \tilde{c}_t^1 + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^1 - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^1 - \hat{V}^1 \\ &+ \hat{\varepsilon}\Big\{\tilde{c}_t^{-1} + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^{-1} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^{-1} - V^{-1} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-1}|_c^2\Big\}. \end{aligned} \tag{2.31}$$

Similar to the $k = 0$ case, as (2.31) is satisfied for all $\hat{\varepsilon} \in \mathbb{C}$, we have

$$0 = \tilde{c}_t^1 + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^1 - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^1 - \hat{V}^1, \tag{2.32}$$

$$0 = \tilde{c}_t^{-1} + \tfrac{i\hbar}{2m}\Delta_z\tilde{c}^{-1} - (A_0 z + b_0 + \tfrac{1}{m}\tilde{c}_z^0)^T\tilde{c}_z^{-1} - V^{-1} - \tfrac{1}{2m}\sum_{\kappa=1}^{0}\big(\tilde{c}_z^\kappa\big)^T\tilde{c}_z^{k-\kappa} - \tfrac{\hat{\varepsilon}}{2m}|\tilde{c}_z^{-1}|_c^2, \tag{2.33}$$

where the zero-valued penultimate term on the right-hand side of (2.33) is included because analogous terms will appear with non-zero value in higher-order expansion equations. By (2.27), (2.32) and the assumptions, $\tilde{c}^1 = \hat{S}^1$. Proceeding inductively, one finds $\tilde{c}^k = \hat{S}^k$ for all $k \in \bar{\mathbb{N}}$, which yields the assertion. $\square$

2.4.1 An alternate assumption

It may be worth noting the following reformulation and assumption. Let $\tilde{g}_{\hat{\delta}} : \mathbb{C}^2 \to \mathbb{C}^2$ and $\hat{g}_{\hat{\delta}} : \mathbb{R} \to \mathbb{R}$ be given by $\tilde{g}_{\hat{\delta}}(z) \doteq (1/\hat{\delta})z$ and $\hat{g}_{\hat{\delta}}(s) \doteq s/\hat{\delta}^2$. Let $\tilde{s} \doteq \hat{g}_{\hat{\delta}}(s) = s/\hat{\delta}^2$ and $\tilde{z} \doteq \tilde{g}_{\hat{\delta}}(z) = (1/\hat{\delta})z$. Note that under this change of variables, the angular rate becomes $\hat{\omega} = \frac{d\theta}{d\tilde{s}} = \frac{d\theta}{ds}\frac{ds}{d\tilde{s}} = \hat{\delta}^2\omega$, and where the units of $\hbar$ are such that the resulting scaling is the identity. Let $\tilde{\bar{S}}(\tilde{s},\tilde{z}) \doteq \bar{S}(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z})) = \bar{S}(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}), \hat{\varepsilon})$ for all $(s,z) \in \bar{\mathscr{D}}$, where we recall the abuse of notation regarding explicit inclusion of the third argument in $\bar{S}$. Note that $\tilde{\bar{S}}_{\tilde{s}}(\tilde{s},\tilde{z}) = \bar{S}_s(\hat{g}_{\hat{\delta}}^{-1}(\tilde{s}), \tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}))\frac{\hat{g}_{\hat{\delta}}^{-1}(\tilde{s})}{d\tilde{s}} = \hat{\delta}^2\bar{S}_s(s,z)$, with similar expressions for the space derivatives. The HJ PDE problem for $\tilde{\bar{S}}$, corresponding to (2.14)–(2.15) for $\bar{S}$, is

$$\begin{aligned} 0 = {} & S_{\tilde{s}}(\tilde{s},\tilde{z}) + \tfrac{i\hbar}{2m}\Delta_{\tilde{z}}S(\tilde{s},\tilde{z}) - \hat{\omega}(\bar{A}_0\tilde{z} + \bar{b}_0)^T S_{\tilde{z}}(\tilde{s},\tilde{z}) - \tfrac{1}{2m}|S_{\tilde{z}}(\tilde{s},\tilde{z})|_c^2 \\ & - \tilde{V}(\tilde{z}), \qquad (\tilde{s},\tilde{z}) \in (0,\tilde{t}) \times \mathbb{C}^2, \end{aligned} \tag{2.34}$$

$$S(\tilde{t},\tilde{z}) = \tilde{\phi}(\tilde{z}), \quad \tilde{z} \in \mathbb{C}^2, \qquad \bar{A}_0 \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \bar{b}_0 \doteq -\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.35}$$

$\tilde{t} = t/\hat{\delta}^2$, $\tilde{\phi}(\tilde{z}) \doteq \phi(\tilde{g}_{\hat{\delta}}^{-1}(\tilde{z}))$ and $\frac{1}{\hat{\delta}^2}\tilde{V}(\tilde{z}) \doteq V(\tilde{g}_{\hat{\delta}}^{-1}(\tilde{z})) = V(z)$.

Note that in the case of a truncated expansion of a potential of form $-\bar{V}(x) = \bar{c}/|x|$, one obtains $-\tilde{V}(\tilde{z}) = -\sum_{k=0}^K \hat{\tilde{V}}^k(\tilde{z})$ $-\hat{\tilde{V}}^0(\tilde{z}) = m\hat{\omega}^2\left[1 - \tilde{z}_1 + (\tilde{z}_1^2 - \tilde{z}_2^2/2)\right]$ and

$$-\hat{\tilde{V}}^k(\tilde{z}) = m\hat{\omega}^2 \sum_{j=0}^{k+2} c_{k+2,j}^V \tilde{z}_1^j \tilde{z}_2^{k+2-j} \quad \text{for } k \geq 1.$$

In particular, one should note that the change of variables leads to a lack of $\hat{\varepsilon}^k$ in the expansion of the potential. With this reformulation in hand, consider the following assumption, where we note that $\breve{\mathscr{S}}_{\mathbb{C}}$ in $(A.2)$ is replaced by $\mathscr{S}_{\mathbb{C}}$ in $(A.2')$.

$$\text{There exists a unique solution, } \tilde{\bar{S}} \in \mathscr{S}_{\mathbb{C}} \text{ to (2.34)–(2.35).} \qquad (A.2')$$

Corollary 1. *Assume $(A.2')$ in place of $(A.2)$. Assume there exists a unique solution, $\hat{S}^0$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (2.23)–(2.24), and that for each $k \geq 1$, there exists a unique solution, $\hat{S}^k$, in $\breve{\mathscr{S}}_{\mathbb{C}}$ to (2.25)–(2.26). Then, $\bar{S} = \sum_{k=0}^{\infty} \hat{\varepsilon}^k \hat{S}^k$.*

Proof. Let $\tilde{\bar{S}}$ satisfy $(A.2')$. Fix an arbitrary $\tilde{s} \in (0, \tilde{t})$, and let $D > 0$. Let $\mathscr{P}(D)$ denote the polydisc in $\mathbb{C}^2$ of multiradius $\bar{D} \doteq (D, D)$. By standard results (cf.[51]), for all $\tilde{z} \in \mathscr{P}(D)$,

$$\tilde{\bar{S}}(\tilde{s}, \tilde{z}) = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{\tilde{\bar{S}}_{z_1^j z_2^{l-j}}(\tilde{s}, 0)}{j!(l-j)!} \tilde{z}_1^j \tilde{z}_2^{l-j},$$

which through application of the Cauchy integral formula,

$$= \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{1}{(2\pi i)^2} \int_{\partial \mathscr{P}(D)} \frac{\tilde{\bar{S}}(\tilde{s}, \zeta_1, \zeta_2)}{\zeta_1^{j+1} \zeta_2^{l-j+1}} \, d\zeta_1 \, d\zeta_2 \tilde{z}_1^j \tilde{z}_2^{l-j} \quad \forall (\tilde{s}, \tilde{z}) \in (0, \infty) \times \mathbb{C}^2, \qquad (2.36)$$

where $\partial \mathscr{P}(D) \doteq \{\zeta \in \mathbb{C}^2 \,|\, |\zeta_1| = D, |\zeta_2| = D\}$. For each $\tilde{s} \in (0, \tilde{t})$, we may express the Taylor series representation for $\tilde{\bar{S}}$ as $\tilde{\bar{S}}(\tilde{s}, \tilde{z}) = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(\tilde{s}) \tilde{z}_1^j \tilde{z}_2^{l-j}$ for all $\tilde{z} \in \mathbb{C}^2$. Let $0 \leq j \leq l < \infty$. Then, by (2.36) and the uniqueness of the Taylor expansion, we see that

$$\tilde{\bar{c}}_{l,j}(\tilde{s}) = \frac{1}{(2\pi i)^2} \int_{\partial \mathscr{P}(D)} \frac{\tilde{\bar{S}}(\tilde{s}, \zeta_1, \zeta_2)}{\zeta_1^{j+1} \zeta_2^{l-j+1}} \, d\zeta_1 \, d\zeta_2,$$

and the right-hand side is independent of $D \in (0, \infty)$. Further, letting $\zeta_\kappa = De^{i\theta_k}$ for $\kappa \in \{1, 2\}$ and $\zeta \in \partial \mathscr{P}(D)$, this becomes

$$\tilde{\bar{c}}_{l,j}(\tilde{s}) = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}} \, d\theta_1 \, d\theta_2.$$

Let $\{\tilde{s}_n\} \subset (0, \tilde{t})$ be a sequence such that $\tilde{s}_n \to \tilde{s} \in (0, \tilde{t})$. By the Bounded Converegence Theorem, for any $0 \le j \le l < \infty$,

$$\lim_{n\to\infty} \tilde{\bar{c}}_{l,j}(\tilde{s}_n) = \frac{1}{(2\pi i)^2} \lim_{n\to\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s}_n, De^{i\theta_1}, De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}} \, d\theta_1 \, d\theta_2$$
$$= \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{-\tilde{\bar{S}}(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}} \, d\theta_1 \, d\theta_2 = \tilde{\bar{c}}_{l,j}(\tilde{s}),$$

and we see that each $\tilde{\bar{c}}_{l,j}(\cdot)$ is continuous.

Similarly, for $0 \le j \le l < \infty$,

$$\lim_{h\to 0} \frac{\tilde{\bar{c}}_{l,j}(\tilde{s}+h) - \tilde{\bar{c}}_{l,j}(\tilde{s})}{h} = \lim_{h\to 0} \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{-1}{D^l \exp\{i[(j+1)\theta_1 + (l-j+1)\theta_2]\}}$$
$$\cdot \frac{\tilde{\bar{S}}(\tilde{s}+h, De^{i\theta_1}, De^{i\theta_2}) - \tilde{\bar{S}}(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{h} \, d\theta_1 \, d\theta_2.$$

Recalling that $\tilde{\bar{S}}$ is continuously differentiable on $(0,\tilde{t})$, we find that the integrand is bounded, and another application of the Bounded Convergence Theorem yields

$$\lim_{h\to 0} \frac{\tilde{\bar{c}}_{l,j}(\tilde{s}+h) - \tilde{\bar{c}}_{l,j}(\tilde{s})}{h} = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{-\tilde{\bar{S}}_t(\tilde{s}, De^{i\theta_1}, De^{i\theta_2})}{D^l e^{i[(j+1)\theta_1 + (l-j+1)\theta_2]}} \, d\theta_1 \, d\theta_2,$$

and we see that $\tilde{\bar{c}}_{l,j} \in C^1(0,\tilde{t})$.

Now, let $\bar{S}(s,z) \doteq \tilde{\bar{S}}(\hat{g}_{\hat{\delta}}(s), \tilde{g}_{\hat{\delta}}(z))$ for all $(s,z) \in (0,t] \times \mathbb{C}^2$, and let $\hat{\varepsilon} = 1/\hat{\delta}$. By Theorem 2, it is sufficient to show that $\bar{S}$ satisfies Assumption $(A.2)$. We have $\bar{S}(s,z) = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(s) \hat{\varepsilon}^l z_1^j z_2^{l-j}$ for all $(s,z) \in (0,t] \times \mathbb{C}^2$. Letting $\hat{S}^l(s,z) \doteq \sum_{j=0}^{l} \tilde{\bar{c}}_{l,j}(s) z_1^j z_2^{l-j}$ for $l \in \mathbb{N}$, the smoothness assertions of the corollary then follow directly from the above and the composition of analytic functions. The existence and uniqueness are also easily demonstrated, and the steps are omitted. □

2.5 Periodic $\hat{S}^0$ solutions

In order to begin computation of the terms in the expansion of Theorem 2, we must obtain a solution of the complex-valued, second-order, nonlinear HJ PDE problem given by (2.23)–(2.24). We note that we continue to work with the case of dimension $n = 2$ here. We will choose the initial condition, ϕ^0, such that the resulting solution will be periodic with frequency that is an integer multiple of ω, where we include the case where the multiple is zero (i.e., the steady-state case). We also note that we are seeking periodic solutions, $\hat{S}^0$ that are themselves clearly physically meaningful.

Recall that the original, forward-time solution, $\tilde{S}^f$, of (2.8)-(2.9) is a solution of the dequantized version of the original Schrödinger equation. Let $\tilde{\psi}^f(s,z) \doteq \exp\left\{\frac{i}{\hbar}\tilde{S}^f\right\}$ for all $(s,z) \in \bar{\mathscr{D}}^f \doteq [0,t) \times \mathbb{R}^2$. Recall also that for physically meaningful solutions, at each $s \in [0,t)$, $\tilde{P}^f(s,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ given by $\tilde{P}(s,\cdot) \doteq \left[\psi^*\psi\right](s,\cdot)$ represents an unnormalized density associated to the particle at time s. Let $\tilde{R}^f, \tilde{T}^f : \bar{\mathscr{D}}^f \to \mathbb{R}$ be given by $\tilde{R}^f(s,z) \doteq \mathbf{Re}[\tilde{S}^f(s,z)]$ and $\tilde{T}^f(s,z) \doteq \mathbf{Im}[\tilde{S}^f(s,z)]$ for all $(s,z) \in \bar{\mathscr{D}}^f$. Then, $\tilde{P}^f(s,z) = \exp\left\{\frac{-2}{\hbar}\tilde{T}^f(s,z)\right\}$ for all $(s,z) \in \bar{\mathscr{D}}^f$. This suggests that we should seek $\tilde{S}^f$ such that $\exp\left\{\frac{-2}{\hbar}\tilde{T}^f(s,\cdot)\right\}$ represents an unnormalized probability density for all $s \in [0,t)$.

Although the goal in this section is to generate a set of physically meaningful periodic solutions to the zeroth-order term, we do not attempt a full catalog of all possible such solutions. Let $\hat{S}^{0,f}(s,z) \doteq \hat{S}^0(t-s,z)$ for all $(s,z) \in \bar{\mathscr{D}}^f$. As we seek $\hat{S}^0(t-s,\cdot)$ that are quadratic, we let the resulting time-dependent coefficients be defined by

$$\hat{S}^{0,f}(s,z) = \tfrac{1}{2}z^T Q(s) z + \Lambda^T(s) z + \rho(s). \tag{2.37}$$

It should be noted here that the condition that $\exp\left\{\frac{-\hbar}{2}\tilde{T}^f(s,\cdot)\right\}$ represent an unnormalized density implies that the imaginary part of $Q(s)$ should be nonnegative definite for all $s \in [0,t)$, which is a significant restriction on the set of allowable solutions.

As $\hat{S}^{0,f}(s,\cdot)$ is quadratic, its values over $\mathbb{C}^2$ are fully defined by its values over $\mathbb{R}^2$, and hence it is sufficient to solve the problem on the real domain. The forward-time version of (2.23)–(2.24), with domain restricted to $\bar{\mathscr{D}}^f$ is

$$0 = -S_t^{0,f} + \tfrac{i\hbar}{2m}\Delta_z S^{0,f} - \big(A_0 z + b_0\big)^T S_z^{0,f} - \tfrac{1}{2m}|S_z^{0,f}|_c^2 - \hat{V}^0, \quad (s,z) \in (0,t) \times \mathbb{R}^2, \tag{2.38}$$

$$S^{0,f}(0,z) = \phi^0(z) \quad \forall z \in \mathbb{R}^2. \tag{2.39}$$

Remark 4. *It is worth noting that any solution of form (2.37) to (2.38)–(2.39) is the unique solution in $\mathscr{S}_{\mathbb{C}}^p$, and in particular, where this uniqueness is obtained through a controlled-diffusion representation [31, 32].*

Substituting form (2.37) into (2.38), and collecting terms, yields the system of ordinary differential equations (ODEs) given as

$$\frac{d}{ds}Q(s) = -\big(A_0^T Q(s) + Q(s)A_0\big) - \tfrac{1}{m}Q^2(s) + m\omega^2 T^V, \tag{2.40}$$

$$\frac{d}{ds}\Lambda(s) = -\big(A_0^T + \tfrac{1}{m}Q(s)\big)\Lambda + \omega\hat{\delta}Q(s)u^2 - m\omega^2\hat{\delta}u^1, \tag{2.41}$$

$$\frac{d}{ds}\rho(s) = \tfrac{i\hbar}{2m}\operatorname{tr}[Q(s)] + \omega\hat{\delta}(u^2)^T\Lambda(s) - \tfrac{1}{2m}\Lambda^T(s)\Lambda(s) + m\omega^2\hat{\delta}^2, \tag{2.42}$$

$$T^V = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad u^1 \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u^2 \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.43}$$

where $Q : [0,t) \to \mathbb{C}^{2\times 2}$, $\Lambda : [0,t) \to \mathbb{C}^2$ and $\rho : [0,t) \to \mathbb{C}$. Throughout, we assume that $Q(s)$ is symmetric for all $s \in [0,t)$. Note that if $Q(s)$ is nonsingular for all $s \in [0,t)$, then (2.37) may also be written as

$$\hat{S}^{0,f}(s,z) = \tfrac{1}{2}\big(z + Q^{-1}(s)\Lambda(s)\big)^T Q(s)\big(z + Q^{-1}(s)\Lambda(s)\big) + \rho(s) - \Lambda^T(s)Q^{-1}(s)\Lambda(s),$$

where we see that $-Q^{-1}(s)\Lambda(s)$ may be interpreted as a mean of the associated distribution at each time s. Consequently, we look for solutions with $-Q^{-1}(s)\Lambda(s) \in \mathbb{R}^2$ for all s.

One may use a Bernoulli-type substitution as a means for seeking solutions of (2.40). That is, suppose $Q(s) = W(s)U^{-1}(s)$, where $U(s)$ is nonsingular for all $s \in [0,t)$. Then, without loss of generality, we may take $W(0) = Q(0)$, $U(0) = \mathscr{I}_{2\times 2}$. The resulting system of ODEs is

$$\frac{d}{ds}\begin{pmatrix} U \\ W \end{pmatrix} = \mathscr{B}\begin{pmatrix} U \\ W \end{pmatrix}, \qquad \mathscr{B} \doteq \begin{bmatrix} 0 & \omega & 1/m & 0 \\ -\omega & 0 & 0 & 1/m \\ 2m\omega^2 & 0 & 0 & \omega \\ 0 & -m\omega^2 & -\omega & 0 \end{bmatrix}.$$

Employing the Jordan canonical form, one obtains the solution as

$$\left(U(s)^T, W(s)^T\right)^T = RPe^{J\omega s}P^{-1}R^{-1}\left(\mathscr{I}_{2\times 2}, Q(0)\right)^T, \tag{2.44}$$

where

$$P = \begin{bmatrix} 0 & 2 & -i & i \\ -3 & 0 & 2 & 2 \\ 3 & 0 & -1 & -1 \\ 0 & -1 & i & -i \end{bmatrix}, \qquad P^{-1} = \begin{bmatrix} 0 & 1/3 & 2/3 & 0 \\ 1 & 0 & 0 & 1 \\ -i/2 & 1/2 & 1/2 & -i \\ i/2 & 1/2 & 1/2 & i \end{bmatrix},$$

$$e^{J\omega s} = \begin{bmatrix} 1 & \omega s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \exp\{i\omega s\} & 0 \\ 0 & 0 & 0 & \exp\{-i\omega s\} \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m\omega & 0 \\ 0 & 0 & 0 & m\omega \end{bmatrix}.$$

We remark that, as our goal here is the generation of periodic solutions that may be used as a basis for the expansion to follow, and as this work is already of substantial length, we will not discuss the question of stability of the above solutions, to perturbations within the class of physically meaningful $Q = U^{-1}W$.

Note that we seek solutions that generate periodic densities $\tilde{P}^f(s,\cdot)$, and that the $(1,2)$ entry of $e^{J\omega s}$ has secular behavior. Examining (2.44), we see that a sufficient condition for avoidance of secular growth/decay of Q, is that entries in the second row of $P^{-1}R^{-1}\left(\mathscr{I}_{2\times 2}, Q(0)\right)^T$ be zero. One easily sees that this corresponds to $Q_{2,1}(0) = -m\omega$ and $Q_{2,2}(0) = 0$, and considering here only symmetric Q, we take $Q_{1,2}(0) = -m\omega$. That is, we have

$$Q(0) = \begin{bmatrix} \bar{k}_0 im\omega & -m\omega \\ -m\omega & 0 \end{bmatrix}, \tag{2.45}$$

for some $\bar{k}_0 \in \mathbb{C}$. Propagating the resulting solutions, we find that the imaginary part of $\bar{k}_0$ being nonnegative is necessary and sufficient for satisfaction of the condition that $\mathbf{Im}[Q(s)]$ be nonnegative-definite for all s. We also note that with such initial condition, $Q_{1,2}, Q_{2,1}, Q_{2,2}$ remain constant for all s, while the real and imaginary parts of $Q_{1,1}$ are periodic. That is, $Q(s)$ takes the form

$$Q(s) = \begin{bmatrix} im\omega p(s) & -m\omega \\ -m\omega & 0 \end{bmatrix} \quad \forall s \in [0,t), \tag{2.46}$$

where $p(s) = [\bar{k}_1^+ e^{2i\omega s} + \bar{k}_1^-]/[\bar{k}_1^+ e^{2i\omega s} - \bar{k}_1^-]$ with $\bar{k}_1^+ \doteq \bar{k}_0 + 1$ and $\bar{k}_1^- \doteq \bar{k}_0 - 1$.

One may seek steady-state solutions by substitution of form (2.45) into the right-hand side of (2.40), and setting this to be zero. One easily finds that the unique steady state solution (among those with $\mathbf{Im}[\bar{k}_0] \geq 0$) is

$$Q(s) = \bar{Q}^0 \doteq \begin{bmatrix} im\omega & -m\omega \\ -m\omega & 0 \end{bmatrix} \quad \forall s \in [0,t). \tag{2.47}$$

Next we consider the linear term in $\hat{S}^{0,f}$, where this satisfies (2.41). We focus on the steady-state Q case of (2.47). Substituting (2.47) into (2.41) yields

$$\dot{\Lambda} = \begin{bmatrix} -i\omega & 2\omega \\ 0 & 0 \end{bmatrix} \Lambda - 2m\omega^2 \hat{\delta} u^1.$$

This has a steady-state solution in the case that $-i\Lambda_1(0) + 2\Lambda(0) = 2m\omega\hat{\delta}$, or equivalently, the one-parameter set of steady-state solutions given by $\Lambda(s) = \bar{\Lambda}^0 \doteq \left(id, m\omega\hat{\delta} - d/2\right)^T$ for $d \in \mathbb{C}$. This includes, in particular, the cases $\bar{\Lambda}^0 = (0, m\omega\hat{\delta})^T$ (i.e., $d = 0$) and $\bar{\Lambda}^0 = m\omega\hat{\delta}(-2i, 2)^T$ (i.e., $d = -2m\omega\hat{\delta}$), where this latter case is obtained if one requires $(\bar{Q}^0)^{-1}\bar{\Lambda}^0$ to be real valued. We also remark that more generally, the solution is given for all $s \in [0, t)$ by

$$\Lambda(s) = \begin{bmatrix} -i\exp\{-i\omega s\} & 2i[\exp\{-i\omega s\} - 1] \\ 0 & 0 \end{bmatrix} \Lambda(0) + 2i[1 - \exp\{-i\omega s\}] m\omega\hat{\delta} u^1.$$

Lastly, we turn to the zeroth-order term. Note that the one may allow secular growth in the real part of $\rho(\cdot)$ with no effect on the associated probability distribution, as is standard in solutions of the quantum harmonic oscillator. Continuing to focus on the steady-state solution, but allowing a real-valued secular zeroth-order term, we substitute the above steady-state quadratic and linear coefficients into (2.42). This yields

$$\dot{\rho} = \frac{-\hbar\omega}{2} + m\omega^2 \left[\frac{3\hat{\delta}^2}{2} + \frac{3d^2}{4(m\omega)^2} \right] \doteq \bar{c}_1(d),$$

and we see that this is purely real if and only if $d \in \mathbb{R}$, and we have

$$\Lambda(s) = \bar{\Lambda}^0 \doteq \left(id, m\omega\hat{\delta} - d/2\right)^T, \qquad \rho^0(s) = \rho^0(0) + \bar{c}_1(d)s \quad \forall s \in [0, t). \tag{2.48}$$

We will restrict ourselves to the simple, steady-state case (modulo the real part of ρ^0) given by (2.47),(2.48) with $\bar{k}_0 = 1$, $d = 0$, for our actual computations of succeeding terms in the expansion. However, the theory will be sufficiently general to encompass the periodic case as well.

2.6 Diffusion representations for succeeding terms

As noted above, we will use diffusion representations to obtain the solutions to the HJ PDEs (2.25)–(2.26) that define the succeeding terms in the expansion, i.e., to obtain the $\hat{S}^k$ for $k \in \mathbb{N}$. In order to achieve this goal, we need to define the complex-valued diffusion dynamics and the expected payoffs that will yield the $\hat{S}^k$. The representation result naturally employs the Itô integral rule. As the dynamics are complex-valued, we need an extension of the Itô rule to that process domain. In a similar fashion to that of Section 2.3.3, we use the Itô rule for the double-dimension real case to obtain the rule for the complex case. Once the Itô rule is established, the proof of the representation is straightforward. However, additional effort is require to generate the machinery by which the the actual solutions are computed, where the machinery relies mainly on computation of moments for the diffusion process.

2.6.1 The underlying stochastic dynamics

We let $(\Omega, \mathscr{F}, P)$ be a probability triple, where Ω denotes a sample space, $\mathscr{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega, \mathscr{F})$. Let $\{\mathscr{F}_s \,|\, s \in [0,t]\}$ denote a filtration on $(\Omega, \mathscr{F}, P)$, and let $B_\cdot$ denote an $\mathscr{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. We will be interested in diffusion processes given by the linear stochastic differential equation (SDE) in integral form

$$\begin{aligned}\zeta_r = \zeta_r^{(s,z)} &= z + \int_s^r -\left(A_0\zeta_\rho + b_0 + \tfrac{1}{m}\hat{S}_z^0(\rho,\zeta_\rho)\right) d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r dB_\rho \\ &\doteq z + \int_s^r \lambda(\rho,\zeta_\rho)\, d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}} B_r^\Delta \quad \forall r \in [s,t], \end{aligned} \tag{2.49}$$

where $z \in \mathbb{C}^2$, $s \in [0,t)$, $B_r^\Delta \doteq B_r - B_s$ for $r \in [s,t)$, and

$$\begin{aligned}\lambda(\rho,z) &\doteq -[A_0 z + b_0 + \tfrac{1}{m} S_z^0(\rho,z)] = -[A_0 z + b_0 + \tfrac{1}{m} Q(\rho) z + \tfrac{1}{m}\Lambda(\rho)] \\ &\doteq -A_{>0}(\rho) z - b_{>0}(\rho). \end{aligned} \tag{2.50}$$

Let $\bar{f} : [0,t] \times \mathbb{C}^2 \to \mathbb{C}^2$, and suppose there exists $K_{\bar{f}} < \infty$ such that $|\bar{f}(s,z^1) - \bar{f}(s,z^2)| \le K_{\bar{f}} |z^1 - z^2|$ for all $(s,z^1),(s,z^2) \in \overline{\mathscr{D}}_{\mathbb{C}}$. For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, consider the complex-valued diffusion, $\zeta_\cdot \in \mathscr{X}_s$, given by

$$\zeta_r = \zeta_r^{(s,z)} = z + \int_s^r \bar{f}(\rho,\zeta_\rho)\, d\rho + \int_s^r \tfrac{1+i}{\sqrt{2}} \sigma \, dB_\rho, \tag{2.51}$$

where $\sigma \in \mathbb{R}^{n\times n}$, and note that this is a slight generalization of (2.49). For $s \in (0,t]$, let

$$\begin{aligned}\mathscr{X}_s \doteq \{\zeta : [s,t] \times \Omega \to \mathbb{C}^2 \,|\, \zeta \text{ is } \mathscr{F}_\cdot\text{-adapted, right-continuous and such that} \\ \mathbb{E} \sup_{r\in[s,t]} |\zeta_r|^m < \infty \;\forall m \in \mathbb{N}\}. \end{aligned} \tag{2.52}$$

We supply $\mathscr{X}_s$ with the norm $\|\zeta\|_{\mathscr{X}_s} \doteq \max_{m\in]1,\bar{M}[} \left[\mathbb{E} \sup_{r\in[s,t]} |\zeta_r|^m\right]^{1/m}$. It is important to note here that complex-valued diffusions have been discussed elsewhere in the literature; see for example, [54] and the references therein.

We also define the isometric isomorphism, $\mathscr{V} : \mathscr{X}_s \to \mathscr{X}_s^\nu$ by $[\mathscr{V}(\zeta)]_r \doteq [\mathscr{V}(\xi + i\nu)]_r \doteq (\xi_r^T, \nu_r^T)^T$ for all $r \in [s,t]$ and $\omega \in \Omega$, where

$$\begin{aligned}\mathscr{X}_s^\nu \doteq \{(\xi,\nu) : [s,t] \times \Omega \to \mathbb{R}^{2n} \,|\, (\xi,\nu) \text{ is } \mathscr{F}_\cdot\text{-adapted, right-continuous and} \\ \text{such that } \mathbb{E} \sup_{r\in[s,t]} [|\xi_r|^m + |\nu_r|^m] < \infty \;\forall m \in \mathbb{N}\}, \end{aligned} \tag{2.53}$$

$$\|(\xi,\nu)\|_{\mathscr{X}_s^\nu} \doteq \max_{m\in]1,\bar{M}[} \left[\mathbb{E} \sup_{r\in[s,t]} (|\xi_r|^m + |\nu_r|^m)\right]^{1/m}. \tag{2.54}$$

Under transformation by $\mathscr{V}$, (2.51) becomes

$$\begin{pmatrix} \xi_r \\ \nu_r \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \int_s^r \hat{f}(\rho, \xi_\rho, \nu_\rho)\, d\rho + \int_s^r \tfrac{1}{\sqrt{2}} \hat{\sigma}\, dB_\rho \quad \forall r \in [s,t], \tag{2.55}$$

where $\hat{f}(\rho, \xi_\rho, \nu_\rho) \doteq \left((\mathbf{Re}[\bar{f}(\rho, \xi_\rho + i\nu_\rho)])^T, (\mathbf{Im}[\bar{f}(\rho, \xi_\rho + i\nu_\rho)])^T \right)^T$ and $\hat{\sigma} \doteq (1,1)^T$. Throughout, concerning both real and complex stochastic differential equations, typically given in integral form such as in (2.56), *solution* refers to a strong solution, unless specifically cited as a weak solution. The following are easily obtained from existing results; see [32, 48].

Lemma 2. *Let $s \in [0,t)$, $z \in \mathbb{C}^2$ and $(x,y) = \mathscr{V}_0(z)$. There exists a unique solution, $(\xi, \nu) \in \mathscr{X}_s^\nu$, to (2.55).*

Lemma 3. *Let $s \in [0,t)$, $z \in \mathbb{C}^2$ and $(x,y) = \mathscr{V}_0(z)$. $\zeta \in \mathscr{X}_s$ is a solution of (2.51) if and only if $\mathscr{V}(\zeta) \in \mathscr{X}_s^\nu$ is a solution of (2.55).*

Lemma 4. *Let $s \in [0,t)$ and $z \in \mathbb{C}^2$. There exists a unique solution, $\zeta \in \mathscr{X}_s$, to (2.51).*

We remark that one may apply Lemmas 2–4 to the specific case of (2.49) in order to establish existence and uniqueness. In particular, for the dynamics of (2.49), the corresponding process $(\xi, \nu) = \mathscr{V}(\zeta)$ satisfies

$$\begin{aligned} \begin{pmatrix} \xi_r \\ \nu_r \end{pmatrix} &= \begin{pmatrix} x \\ y \end{pmatrix} + \int_s^r - \left[\begin{pmatrix} A^r_{>0}(\rho) & -A^i_{>0}(\rho) \\ A^i_{>0}(\rho) & A^r_{>0}(\rho) \end{pmatrix} \begin{pmatrix} \xi_r \\ \nu_r \end{pmatrix} + \begin{pmatrix} b^r_{>0}(\rho) \\ b^i_{>0}(\rho) \end{pmatrix} \right] d\rho \\ &\quad + \sqrt{\tfrac{\hbar}{2m}} \begin{pmatrix} I_{n\times n} \\ I_{n\times n} \end{pmatrix} B^\Delta_r \\ &\doteq \begin{pmatrix} x \\ y \end{pmatrix} + \int_s^r -\bar{A}_{>0}(\rho) \begin{pmatrix} \xi_r \\ \nu_r \end{pmatrix} - \bar{b}_{>0}(\rho)\, d\rho + \sqrt{\tfrac{\hbar}{2m}}\, \bar{\mathscr{I}} B^\Delta_r \quad \forall r \in [s,t], \end{aligned} \tag{2.56}$$

where $A^r_{>0}(\rho) \doteq \mathbf{Re}(A_{>0}(\rho)), A^i_{>0}(\rho) \doteq \mathbf{Im}(A_{>0}(\rho)),\ \left((b^r_{>0}(\rho))^T, (b^i_{>0}(\rho))^T\right)^T \doteq \mathscr{V}_0(b_{>0}(\rho))$

for all $\rho \in [0,t)$.

2.6.2 Itô's rule

The representation results will rely on a minor generalization of Itô's rule to the specific complex-diffusion dynamics of interest here. It might be worthwhile to note that the complex-valued diffusions considered here belong to a very small subclass of complex-valued diffusions, and this is somehow related to the specific nature of the complex aspect of the Schrödinger equation. The following complex-case Itô rule is similar to existing results (cf., [54]).

Lemma 5. *Let $\bar{g} \in \mathscr{S}_{\mathbb{C}}$ and $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, and suppose diffusion process $\zeta_\cdot$ is given by (2.51). Then, for all $r \in [s,t]$,*

$$\begin{aligned}\bar{g}(r,\zeta_r) =& \bar{g}(s,z) + \int_s^r \bar{g}_t(\rho,\zeta_\rho) + \bar{g}_z^T(\rho,\zeta_\rho)\bar{f}(\rho,\zeta_\rho)\,d\rho + \int_s^r \tfrac{1+i}{\sqrt{2}}\bar{g}_z^T(\rho,\zeta_\rho)\sigma\,dB_\rho \\ &+ \tfrac{1}{2}\int_s^r \operatorname{tr}\left[\bar{g}_{zz}(\rho,\zeta_\rho)(\sigma\sigma^T)\right]d\rho. \end{aligned} \tag{2.57}$$

Proof. Let $(g^r(s,x,y), g^i(s,x,y)) \doteq \mathscr{V}_{00}\big(\bar{g}(s,\mathscr{V}_0^{-1}(x,y))\big)$, $(f^r(s,x,y), f^i(s,x,y)) \doteq \mathscr{V}_0\big(\bar{f}(s,\mathscr{V}_0^{-1}(x,y))\big)$ for all $(s,x,y) \in \overline{\mathscr{D}}_2$, and note that it is trivial to show that $\bar{g}_t(r,z) = g_t^r(r,x,y) + ig_t^i(r,x,y)$, for all $(x,y) = \mathscr{V}_0(z)$, $(r,z) \in \mathscr{D}_{\mathbb{C}}$. Also, using the Cauchy-Riemann equations,

$$\bar{g}_z^T(r,z)\bar{f}(r,z) = \left[(g_x^r)^T f^r + (g_y^r)^T f^i\right](r,x,y) + i\left[(g_x^i)^T f^r + (g_y^i)^T f^i\right](r,x,y),$$

for all $(x,y) = \mathscr{V}_0(z)$, $(r,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$. Defining the derivative notation $g_{x^2}^r(s,x,y) \doteq \big((g_x^r)^T, (g_y^r)^T\big)^T(r,x,y)$ and vector notation $\hat{f}(r,x,z) \doteq \big((f^r)^T, (f^i)^T\big)(r,x,y)$ for all $(r,x,y) \in \overline{\mathscr{D}}_2$, this becomes

$$\bar{g}_z^T(r,z)\bar{f}(r,z) = \big(g_{x^2}^r(r,x,y)\big)^T\hat{f}(r,x,y) + i\big(g_{x^2}^i(r,x,y)\big)^T\hat{f}(r,x,y), \tag{2.58}$$

for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$. Similarly, letting $\hat{\sigma}\doteq\big(\sigma^T,\sigma^T\big)^T$,

$$\bar{g}_z^T(r,z)\tfrac{1+i}{\sqrt{2}}\sigma=\tfrac{1}{\sqrt{2}}\big[\big(g^r_{x^2}(r,x,y)\big)^T\hat{\sigma}+i\big(g^i_{x^2}(r,x,y)\big)^T\hat{\sigma}\big], \tag{2.59}$$

for all $(x,y)=\mathscr{V}_0(z)$, $(r,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$.

Next, let $\bar{a}\doteq\big(\frac{1+i}{\sqrt{2}}\big)^2\sigma\sigma^T$ and $(a^r_{j,l},a^i_{j,l})\doteq\mathscr{V}_{00}(\bar{a}_{j,l})$ for all $j,l\in]1,n[$. Using (2.17),(2.18), we find

$$\bar{g}_{z_j,z_l}\bar{a}_{j,l}=g^i_{x_j,y_l}a^r_{j,l}+g^r_{x_j,y_l}a^i_{j,l}+i\big[-g^r_{x_j,y_l}a^r_{j,l}+g^i_{x_j,y_l}a^i_{j,l}\big]\quad\forall\, j,l\in]1,n[. \tag{2.60}$$

Also, by the definition of $\bar{a}$, we see that $a^r=0$ and $a^i=\sigma\sigma^T$. Applying these in (2.60) yields

$$\bar{g}_{z_j,z_l}\bar{a}_{j,l}=g^r_{x_j,y_l}[\sigma\sigma^T]_{j,l}+ig^i_{x_j,y_l}[\sigma\sigma^T]_{j,l}\quad\forall\, j,l\in]1,n[. \tag{2.61}$$

Considering (2.58), (2.59) and (2.61), and letting $(\xi_r,\nu_r)\doteq\mathscr{V}_0(\zeta_r)$ for all $r\in(0,t]$, a.e. $\omega\in\Omega$ we see that (2.57) is equivalent to a pair of equations for the real and imaginary parts, where the real-part equation is

$$\begin{aligned}g^r(r,\xi_r,\nu_r)=&g^r(s,x,y)+\int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho)+(g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho)\\&+\tfrac{1}{2}\sum_{j,l=1}^n g^r_{x_j,y_l}(\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho+\tfrac{1}{\sqrt{2}}\int_s^r(g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho,\end{aligned} \tag{2.62}$$

with an analogous equation corresponding to the imaginary part.

On the other hand, applying Itô's rule to real functions g^r and g^i separately, and then applying (2.17), (2.18), we find

$$\begin{aligned}g^r(r,\xi_r,\nu_r)=&g^r(s,x,y)+\int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho)+(g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho)\\&+\tfrac{1}{4}\sum_{j,l=1}^n\big[g^r_{x_j,x_l}+g^r_{x_j,y_l}+g^r_{y_j,x_l}+g^r_{y_j,y_l}\big](\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{\sqrt{2}}\int_s^r (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho,\\
=&g^r(s,x,y)+\int_s^r g^r_t(\rho,\xi_\rho,\nu_\rho)+(g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{f}(\rho,\xi_\rho,\nu_\rho) \qquad (2.63)\\
&+\tfrac{1}{2}\sum_{j,l=1}^{n} g^r_{x_j,y_l}(\rho,\xi_\rho,\nu_\rho)[\sigma\sigma^T]_{j,l}\,d\rho+\tfrac{1}{\sqrt{2}}\int_s^r (g^r_{x^2})^T(\rho,\xi_\rho,\nu_\rho)\hat{\sigma}\,dB_\rho,
\end{aligned}
$$

with a similar equation for the imaginary part. Comparing (2.63) with (2.62), and similarly for the imaginary parts, one obtains the result. □

We apply this result to the particular case of interest here.

Lemma 6. *Let $\hat{S}\in\mathscr{S}_{\mathbb{C}}$ and $(s,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$, and suppose ζ satisfies (2.49). Then, for all $r\in(s,t]$,*

$$
\begin{aligned}
\hat{S}(r,\zeta_r)=&\hat{S}(s,z)+\int_s^r \hat{S}_t(\rho,\zeta_\rho)-\hat{S}_z^T(\rho,\zeta_\rho)\left[A_{>0}(\rho)\zeta_\rho+b_{>0}(\rho)\right]+\tfrac{i\hbar}{2m}\Delta_z\hat{S}(\rho,\zeta_\rho)\,d\rho\\
&+\sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r \hat{S}_z^T(\rho,\zeta_\rho)\,dB_\rho. \qquad (2.64)
\end{aligned}
$$

Proof. Dynamics (2.49) have form (2.51) with $f(r,z)=A_{>0}(r)z+b_{>0}(r)$ and $\sigma=\sqrt{\frac{\hbar}{m}}\mathscr{I}_{n\times n}$. In this case, $\frac{1}{2}\operatorname{tr}\left[\hat{S}_{zz}(r,z)(\sigma\sigma^T)\right]=\frac{i\hbar}{2m}\Delta_z\hat{S}(r,z)$ for all $(r,z)\in\mathscr{S}_{\mathbb{C}}$, which yields the result. □

Theorem 3. *Let $k\in\mathbb{N}$. Let $\hat{S}^\kappa\in\mathscr{S}^p_{\mathbb{C}}$ satisfy (2.25)–(2.26) for all $\kappa\in]1,k[$. Let $(s,z)\in\overline{\mathscr{D}}_{\mathbb{C}}$, and let $\zeta\in\mathscr{X}_s$ satisfy (2.49). Then,*

$$
\hat{S}^k(s,z)=\mathbb{E}\left\{\int_s^t -\tfrac{1}{2m}\sum_{\kappa=1}^{k-1}\left[S_z^\kappa(r,\zeta_r)\right]^T S_z^{k-\kappa}(r,\zeta_r)-\hat{V}^k(\zeta_r)\,dr+\phi^k(\zeta_t)\right\}.
$$

Proof. Taking expectations in (2.64), and using the martingale property (cf., [11, 15]), we have

$$
\begin{aligned}
\hat{S}^k(s,z)=\mathbb{E}\Big\{&-\int_s^t \hat{S}_t(r,\zeta_r)-\hat{S}_z^T(r,\zeta_r)\left[A_{>0}(r)\zeta_r+b_{>0}(r)\right]+\tfrac{i\hbar}{2m}\Delta_z\hat{S}(r,\zeta_r)\,dr\\
&+\hat{S}^k(t,\zeta_t)\Big\}.
\end{aligned}
$$

Combining this with (2.25)–(2.26) yields the result. □

2.6.3 Moments and Iteration

Note that Theorem 3 yields an expression for the k^{th} term in our expansion for $\bar{S}$, $\hat{S}^k$, from the previous terms, $\hat{S}^\kappa$ for $\kappa < k$. We now examine how this generates a computationally tractable scheme. It is heuristically helpful to examine the first two iterates. For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, we have

$$\begin{aligned}
\hat{S}^1(s,z) &= \mathbb{E}\left\{ \int_s^t -\hat{V}^1(\zeta_r)\,dr + \phi^1(\zeta_t) \right\} \\
&= \mathbb{E}\left\{ \int_s^t m\omega^2\Big(-[\zeta_r]_1^3 + (3/2)[\zeta_r]_1[\zeta_r]_2^2 \Big)\,dr + \sum_{l=0}^{3}\sum_{j=0}^{l} b^{\phi}_{3,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j} \right\}, \qquad (2.65)\\
\hat{S}^2(s,z) &= \mathbb{E}\left\{ \int_s^t -\tfrac{1}{2m}\left|\hat{S}^1_z(r,\zeta_r)\right|_c^2 - \hat{V}^2(\zeta_r)\,dr + \phi^2(\zeta_t) \right\} \\
&= \mathbb{E}\left\{ \int_s^t -\tfrac{1}{2m}\left|\hat{S}^1_z(r,\zeta_r)\right|_c^2 + m\omega^2\Big([\zeta_r]_1^4 - 3[\zeta_r]_1^2[\zeta_r]_2^2 + (3/8)[\zeta_r]_2^4 \Big)\,dr \right.\\
&\qquad \left. + \sum_{l=0}^{4}\sum_{j=0}^{l} b^{\phi}_{4,l,j}[\zeta_t]_1^j[\zeta_t]_2^{l-j} \right\}. \qquad (2.66)
\end{aligned}$$

Note that the right-hand side of (2.65) consists of an expectation of a polynomial in ζ_t and an integral of a polynomial in $\zeta_\cdot$, and further, that the dynamics of ζ are linear in the state variable. Thus, we may anticipate that $\hat{S}^1(s,\cdot)$ may also be polynomial. Applying this anticipated form on the right-hand side of (2.66) suggests that the polynomial form will be inherited in each $\hat{S}^k$. This will form the basis of our computational scheme.

The computation of the expectations that generate the $\hat{S}^k$ for $k \geq 1$ will be obtained through the moments of the underlying diffusion process. Thus, the first step is solution of (2.49). We let the state transition matrices for deterministic linear systems $\dot{y}_r = -A_{>0}(r)y_r$ and $\dot{y}^{(2)}_r = -\bar{A}_{>0}(r)y^{(2)}_r$ be denoted by $\Phi(r,s)$ and $\Phi^{(2)}(r,s)$, respectively. More specifically, with initial (or terminal) conditions, $y_s = \bar{y}$ and $y^{(2)}_s = \bar{y}^{(2)}$, the solutions at time r are given by $y_r = \Phi(r,s)\bar{y}$ and $y^{(2)}_r = \Phi^{(2)}(r,s)\bar{y}^{(2)}$, respectively. The solutions of our SDEs are given by the following.

Lemma 7. *Linear SDE (2.49) has solution given by $\zeta_r = \mu_r + \Delta_r$, where*

$$\mu_r = \Phi(r,s)z + \int_s^r \Phi(r,\rho)\big(-b_{>0}(\rho)\big)\,d\rho, \quad \Delta_r = \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi(r,\rho)\,dB_\rho$$

for all $r \in [s,t]$. Linear SDE (2.56) has solution given by $X_r^{(2)} = \mu_r^{(2)} + \Delta_r^{(2)}$, where

$$\mu_r^{(2)} = \Phi^{(2)}(r,s)x^{(2)} + \int_s^r \Phi^{(2)}(r,\rho)\big(-\bar{b}_{>0}(\rho)\big)\,d\rho,$$
$$\Delta_r^{(2)} = \sqrt{\tfrac{\hbar}{2m}}\int_s^r \Phi^{(2)}(r,\rho)\bar{\mathscr{I}}\,dB_\rho$$

for all $r \in [s,t]$, where $x^{(2)} \doteq (x^T, y^T)^T$.

Proof. The case of (2.56) is standard,cf., [23]. We sketch the proof in the minor variant case of (2.49), where this uses the Itô-rule approach, but for the complex-valued diffusion case. For $0 \le s \le r \le t$, let $\alpha_r \doteq \Phi(s,r)\zeta_r = \Phi^{-1}(r,s)\zeta_r$. By Lemma 5,

$$\alpha_r = \int_s^r \Phi^{-1}(\rho,s)[-b_{>0}(\rho)]\,d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi^{-1}(\rho,s)\,dB_\rho,$$

which implies $\zeta_r = \int_s^r \Phi(r,\rho)[-b_{>0}(\rho)]\,d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r \Phi(r,\rho)\,dB_\rho$. □

Lemma 8. *For all $r \in [s,t]$, $X_r^{(2)}$ and ζ_r have normal distributions.*

Proof. The case of $X_r^{(2)}$ is standard, cf. [22], and then one notes $\zeta_r = \mathscr{V}_0(X_r^{(2)})$. □

Lemma 9. *For all $r \in [s,t]$, μ_r is the mean of ζ_r, and Δ_r is a zero-mean normal random variable with covariance given by $\mathbb{E}\big[\Delta_r\Delta_r^T\big] = \frac{i\hbar}{m}\int_s^r \Phi(r,\rho)\Phi^T(r,\rho)\,d\rho$, where further, $\mathbb{E}\big[(\zeta_r - \mu_r)(\zeta_r - \mu_r)^T\big] = \mathbb{E}\big[\Delta_r\Delta_r^T\big]$.*

Proof. That Δ_r has zero mean is immediate from its definition. Given Lemmas 7 and 8, it is sufficient to obtain the expression for $\mathbb{E}\big[\Delta_r\Delta_r^T\big]$. By Lemma 7,

$$\mathbb{E}\big[\Delta_r\Delta_r^T\big] = \tfrac{i\hbar}{m}\mathbb{E}\Big\{\big[\int_s^r \Phi(r,\rho)\,dB_\rho\big]\big[\int_s^r \Phi(r,\rho)\,dB_\rho\big]^T\Big\},$$

where the term inside the expectation is purely real, and consequently by standard results (cf., [22]), one obtains the asserted representation. □

As noted above, we will perform the computations mainly in the simpler, steady-state case of $\bar{k}_0 = 1$. In this case, we have

$$-A_{>0} = \omega \begin{pmatrix} -i & 0 \\ 2 & 0 \end{pmatrix}, \quad \text{and} \quad -b_{>0} = \tfrac{d}{2m} \begin{pmatrix} -2i \\ 1 \end{pmatrix}. \tag{2.67}$$

In the case $d = 0$, we have $-b_{>0} = 0$, while in the case $d = -2m\omega\hat{\delta}$, we have $-b_{>0} = \omega\hat{\delta}(2i, -1)^T$.

Theorem 4. *In the case $\bar{k}_0 = 1$, for all $r \in [s,t]$, ζ_r is a normal random variable with mean and covariance given by, with $\hat{d} \doteq d/(m\omega)$,*

$$\mu_r = \begin{pmatrix} \mu_r^1 \\ \mu_r^2 \end{pmatrix} \ \textit{and} \ \tilde{\Sigma}_r \doteq \begin{pmatrix} \tilde{\Sigma}_r^{1,1} & \tilde{\Sigma}_r^{1,2} \\ \tilde{\Sigma}_r^{2,1} & \tilde{\Sigma}_r^{2,2} \end{pmatrix}, \quad \textit{where}$$

$$\mu_r^1 = e^{-i\omega(r-s)} z_1 + \hat{d}(e^{-i\omega(r-s)} - 1),$$

$$\mu_r^2 = 2i[e^{-i\omega(r-s)} - 1] z_1 + z_2 + \hat{d}\big[2i((e^{-i\omega(r-s)} - 1) - 3\omega(r-s)/2\big],$$

$$\tilde{\Sigma}_r^{1,1} = \tfrac{\hbar}{m\omega}\tfrac{1}{2}\big(1 - e^{-2i\omega(r-s)}\big),$$

$$\tilde{\Sigma}_r^{1,2} = \tilde{\Sigma}_r^{2,1} = \tfrac{\hbar}{m\omega} i\big[2\big(e^{-i\omega(r-s)} - 1\big) - \big(e^{-2i\omega(r-s)} - 1\big)\big],$$

$$\tilde{\Sigma}_r^{2,2} = \tfrac{\hbar}{m\omega}\big[2\big(e^{-2i\omega(r-s)} - 1\big) - 8\big(e^{-i\omega(r-s)} - 1\big) - 3i\omega(r-s)\big].$$

Proof. The expression for μ_r is immediate from Lemma 7. To obtain the expression for the covariance, we evaluate the integral in Lemma 9. Letting $\tilde{\Sigma}_r \doteq \mathbb{E}\big[\Delta_r \Delta_r^T\big]$, component-wise, that integral is

$$\tilde{\Sigma}_r^{1,1} = \tfrac{i\hbar}{m} \int_s^r e^{-2i\omega(r-\rho)}\, d\rho, \qquad \tilde{\Sigma}_r^{1,2} = \tilde{\Sigma}_r^{2,1} = \tfrac{i\hbar}{m} \int_s^r 2i\big[e^{-2i\omega(r-\rho)} - e^{-i\omega(r-\rho)}\big]\, d\rho,$$

$$\tilde{\Sigma}_r^{2,2} = \frac{i\hbar}{m}\int_s^r -4\left[e^{-2i\omega(r-\rho)} - e^{-i\omega(r-\rho)}\right]^2 + 1\,d\rho.$$

Evaluating these, one obtains the asserted expression for the covariance. $\square$

Theorem 5. *For* $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$*,* $\hat{S}^1(s,z) = \sum_{l=0}^{3}\sum_{j=0}^{l}\hat{c}_{l,j}^1(s)z_1^j z_2^{l-j}$*, where the time-indexed coefficients,* $\hat{c}_{l,j}^1(\cdot)$ *are obtained by the evaluation of linear combinations of moments of up to third-order of the normal random variables* ζ_r *and closed-form time-integrals. For* $k > 1$ *and* $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$*, the* $\hat{S}^k$ *also take the similar forms,* $\hat{S}^k(s,z) = \sum_{l=0}^{k+2}\sum_{j=0}^{l}\hat{c}_{l,j}^k(s)z_1^j z_2^{l-j}$*. Given the coefficient functions* $\hat{c}_{l,j}^\kappa(s)$ *for* $\kappa < k$*, the time-indexed coefficients* $\hat{c}_{l,j}^k(s)$ *are obtained by the evaluation of linear combinations of moments of up to* $(k+2)^{th}$*-order of the normal random variables* ζ_r *and closed-form time-integrals.*

Proof. By Fubini's Theorem and Theorem 3,

$$\begin{aligned}\hat{S}^k(s,z) = \int_s^t -\frac{1}{2m}\sum_{\kappa=1}^{k-1}\mathbb{E}\Big\{\big[S_z^\kappa(r,\zeta_r)\big]^T S_z^{k-\kappa}(r,\zeta_r)\Big\} \\ + m\omega^2\sum_{j=0}^{k+2}c_{k+2,j}^V\mathbb{E}\big\{[\zeta_r]_1^j[\zeta_r]_2^{k+2-j}\big\}\,dr + \sum_{l=0}^{k+2}\sum_{j=0}^{l}b_{k+2,l,j}^\phi\mathbb{E}\big\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\big\}.\end{aligned} \tag{2.68}$$

In particular,

$$\hat{S}^1(s,z) = \int_s^t m\omega^2\Big[\mathbb{E}\big\{-[\zeta_r]_1^3\big\} + \tfrac{3}{2}\mathbb{E}\big\{[\zeta_r]_1[\zeta_r]_2^2\big\}\Big]\,dr + \sum_{l=0}^{3}\sum_{j=0}^{l}b_{3,l,j}^\phi\mathbb{E}\big\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\big\}. \tag{2.69}$$

We see that (2.69) immediately yields the assertions regarding $\hat{S}^1$. If for $\kappa < k$, the $\hat{S}^\kappa(s,z)$ are polynomials in z of order at most $\kappa+2$, then the products-of-derivatives, $[S_z^\kappa(r,\zeta_r)]^T S_z^{k-\kappa}(r,\zeta_r)$, in (2.68) are of order at most $k+2$ in ζ_r, and the asserted form follows. $\square$

2.7 The $\hat{S}^1$ term

In Section 2.5, steady-state and periodic solutions for the zeroth-order term in the expansion were computed. Here, we proceed an additional step, computing $\breve{S}^1 \doteq \hat{S}^0 + \frac{1}{\delta}\hat{S}^1$. We perform the actual computations for $\hat{S}^1$ only in the steady-state case of $\bar{k}_0 = 1$. For $(s,z) \in \overline{\mathscr{D}}_{\mathbb{C}}$, we may obtain $\hat{S}^1(s,z)$ from (2.69), using the expressions for the mean and variance of normal ζ_r given in Theorem 4. We see that we must evaluate integrals of the moments $\mathbb{E}\{[\zeta_r]_1^3\}$ and $\mathbb{E}\{[\zeta_r]_1[\zeta_r]_2^2\}$ as well as the general moments $\mathbb{E}\{[\zeta_t]_1^j[\zeta_t]_2^{l-j}\}$ for $j \in]0,l[$, $l \in]0,3[$. There are well-known expressions for all moments of normal random variables. In particular,

$$
\begin{aligned}
&\mathbb{E}\{[\zeta_r]_1^3\} = [\mu_r]_1^3 + 3[\mu_r]_1\tilde{\Sigma}_r^{1,1}, \\
&\mathbb{E}\{[\zeta_r]_1[\zeta_r]_2^2\} = [\mu_r]_1[\mu_r]_2^2 + [\mu_r]_1\tilde{\Sigma}_r^{2,2} + 2[\mu_r]_2\tilde{\Sigma}_r^{1,2}.
\end{aligned}
$$

This implies that for the integral term in (2.69), we must evaluate the integrals of moments given by $\int_s^t [\mu_r]_1^3\,dr$, $\int_s^t [\mu_r]_1\tilde{\Sigma}_r^{1,1}\,dr$, $\int_s^t [\mu_r]_1[\mu_r]_2^2\,dr$, $\int_s^t [\mu_r]_1\tilde{\Sigma}_r^{2,2}\,dr$, and $\int_s^t [\mu_r]_2\tilde{\Sigma}_r^{1,2}\,dr$. We note that, as our interest is in the solution of the original forward-time problem, it is sufficient to take $s = 0$. Further, as our interest will be in periodic-plus-drift solutions, we take $t = \tau \doteq 2\pi/\omega$. With assiduous effort, one eventually finds

$$
\begin{aligned}
\mathbb{E}\Big\{\int_0^\tau -\hat{V}^1(\zeta_r)\,dr\Big\} &= \int_0^\tau m\omega^2\Big[\mathbb{E}\{-[\zeta_r]_1^3\} + \tfrac{3}{2}\mathbb{E}\{[\zeta_r]_1[\zeta_r]_2^2\}\Big]\,dr \\
&= m\omega^2\Big\{\tfrac{3\pi\hat{d}}{\omega}[z_1^2 + iz_1z_2 - z_2^2] + c_1(\tau)(1,2i)z + c_2(\tau)\Big\},
\end{aligned}
\tag{2.70}
$$

where

$$
\begin{aligned}
&c_1(\tau) = (3\pi/\omega)\big[\hat{d}^2(1-3i\pi)/2 - \hbar/(m\omega)\big], \\
&c_2(\tau) = \tfrac{\pi\hat{d}\hbar}{m\omega^2}(18i\pi - 9/2) + \tfrac{3\pi\hat{d}^3}{2\omega}\big((1/3) - 3i\pi - 6\pi^2\big).
\end{aligned}
$$

From (2.70), we see that the expected value, $\mathbb{E}\int_0^\tau -\hat{V}^1(\zeta_r)\,dr$ has at most quadratic terms in z. (In contrast, for typical $t \neq \tau$, this integral is cubic in z.) Consequently, it may be of interest to take terminal cost, ϕ^1 to be quadratic rather than the more general hypothesized cubic form. Suppose we specifically take

$$
\phi^1(z) \doteq \tfrac{1}{2}z^T Q^1 z,
\tag{2.71}
$$

where Q^1 has components $Q^1_{j,k}$. Noting that we are seeking a solution of form $\breve{S}^1 = \hat{S}^0 + \frac{1}{\hat{\delta}}\hat{S}^1$, we find it helpful to now allow general $d \in \mathbb{C}$ with corresponding $\bar{\Lambda}^0$ given by (2.48). Also, note from Theorem 4 that

$$\mu_\tau = z - \left(0, \tfrac{3\pi d}{m\omega}\right)^T, \qquad \tilde{\Sigma}^{1,1}_\tau = \tilde{\Sigma}^{1,1}_\tau = 0, \qquad \tilde{\Sigma}^{2,2}_\tau = \tfrac{-6i\pi\hbar}{m\omega}. \tag{2.72}$$

Combining (2.69) and (2.70)–(2.72), we find

$$\hat{S}^1(\tau, z) = \tfrac{1}{2} z^T (Q^1 + Q^\Delta) z + b^T z + \rho^1(\tau), \tag{2.73}$$

where

$$Q^\Delta = 6\pi d \begin{pmatrix} 1 & i/2 \\ i/2 & -1 \end{pmatrix}, \qquad b = \left[\tilde{k}_1 \left(Q^1 + Q^\Delta\right) + \tilde{k}_2 Q^\Delta\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.74}$$

$$\rho^1(\tau) = \tilde{k}_1 \left[\tfrac{\tilde{k}_1}{2} + \tfrac{i\hbar}{d}\right] Q^1_{2,2} + \tfrac{\pi d\hbar}{m\omega}(18i\pi - 9/2) + \tfrac{3\pi d^3}{2m^2\omega^2}\left((1/3) - 3i\pi - 6\pi^2\right), \tag{2.75}$$

$$\tilde{k}_1 = \tfrac{-3\pi d}{m\omega}, \qquad \tilde{k}_2 = \tfrac{i\hbar}{d} + \tfrac{d}{2m\omega}(3\pi - i). \tag{2.76}$$

Recalling that $\hat{S}^0(\tau, z) = \frac{1}{2} z^T \bar{Q}^0 z + (\bar{\Lambda}^0)^T z + \rho^0(\tau)$, we find

$$\begin{aligned} \breve{S}^1(\tau, z) &= \hat{S}^0(\tau, z) + \tfrac{1}{\hat{\delta}}\hat{S}^1(\tau, z) \\ &= \tfrac{1}{2} z^T \left[\bar{Q}^0 + \tfrac{1}{\hat{\delta}}\left(Q^1 + Q^\Delta\right)\right] z + \left[\bar{\Lambda}^0 + \tfrac{1}{\hat{\delta}} b\right]^T z + \rho^0(\tau) + \tfrac{1}{\hat{\delta}}\rho^1(\tau) \end{aligned} \tag{2.77}$$

In the $\breve{S}^1$ solution given by (2.77), the Q^1 complex matrix coefficient, as well as complex d coefficient are free. Other potentially free parameters include the $\bar{k}_0$ parameter in $\hat{S}^0$ and terms that are not purely quadratic in the possible cubic ϕ^1.

2.8 Acknowledgment

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coordinates”, W.M. McEneaney and R. Zhao,, *Numerical Methods for Optimal Control Problems*, M. Falcone, R. Ferretti, L. Grune and W. McEneaney (Eds.), Springer INDAM Series, Vol. 29 (2018), 241–268. The dissertation author is a co-investigator and co-author of this article.

“A Diffusion-Based Solution Technique for Certain Schrödinger Equation Dynamical Systems”, W.M. McEneaney and R. Zhao, Proceedings of European Control Conference (2018).

Chapter 3

Staticization and Iterated Staticization

3.1 Introduction

Staticization maps a function into its values at stationary points (i.e., critical points). More specifically, the set-valued "stat" operator has as its range the set of such values, and if there is a unique such value, then that value is the output of the (single-valued) stat operator. This operator is obviously a generalization of the minimization and maximization operators for appropriate classes of differentiable functionals, and is also valid for functions with range other than the reals, including complex-valued functionals. The stat operator is at the heart of a new approach to solution of two-point boundary value problems (TPBVPs) in conservative dynamical systems [6, 7, 38, 40], as well as to solution of the Schrödinger equation [33, 35, 37]. A key component in this development is the theory that allows one to reorder stat operators under certain conditions, and that theory is the focus of the effort here. In order to motivate the theory, first let us indicate the application domain a bit further.

Recall that conservative dynamical systems propagate as stationary points of the action functional over the possible paths of the system. This stationary-action formulation has recently been found to be quite useful for generation of fundamental solutions to TPBVPs for conservative dynamical systems, cf. [6, 7, 38, 40]. To obtain a sense of this application domain, consider a finite-dimensional action-functional formulation of such a TPBVP. Let the path of the conservative system be denoted by ξ_r for $r \in [0,t]$ with $\xi_0 = \bar{x}$, in which case the action functional, with an appended terminal cost, may take the form

$$J(t,\bar{x},u) \doteq \int_0^t T(u_r) - V(\xi_r)\,dr + \phi(\xi_t), \tag{3.1}$$

where $\dot{\xi} = u$, $u \in \mathscr{U} \doteq L_2(0,t)$, $T(\cdot)$ denotes the kinetic energy associated to the momentum (specifically taken to be $T(v) \doteq \frac{1}{2}v^T \mathscr{M} v$ throughout, with $\mathscr{M}$ positive-definite and symmetric), and $V(\cdot)$ denotes a potential energy field. If, for example, one takes $\phi(x) \doteq -\bar{v}^T \mathscr{M} x$, a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\dot{\xi}_t = \bar{v}$; if one takes ϕ to be a min-plus delta function centered at z, then a stationary-action path satisfies the TPBVP with $\xi_0 = \bar{x}$ and $\xi_t = z$, cf. [7]. In the early work of Hamilton, it was formulated as the least-action principle [17], which states that a conservative dynamical system follows the trajectory that minimizes the action functional. However, this is typically only the case for relatively short-duration cases, cf. [14] and the references therein. In such short-duration cases, optimization methods and semiconvex duality are quite useful [6, 7, 40]. However, in order to extend to indefinitely long duration problems, it becomes necessary to apply concepts of stationarity [38].

It is worth noting that if one defines $\operatorname{stat}_{x\in\mathscr{X}} \phi(x)$ to be the critical value of ϕ (defined rigorously in Section 3.2.1), then a gravitational potential given as $V(x) = -\mu/|x|$ for $x \neq 0$ and constant $\mu > 0$, has the representation $V(x) = -(\frac{3}{2})^{3/2}\mu \operatorname{stat}_{\alpha>0}\{\alpha - \frac{\alpha^3|x|^2}{2}\}$, where we note that the argument of the stat operator is polynomial, [18, 40]. The Schrödinger equation in the context of a Coulomb potential may be similarly addressed. In that case, it is particularly helpful to consider an extension of the space variable to a vector space over the complex field, say $x \in \mathbb{C}^n$ rather than $x \in \mathbb{R}^n$. More specifically, for $x \in \mathbb{C}^n$, this representation takes a general form $V(x) = -(\frac{3}{2})^{3/2}\hat{\mu} \operatorname{stat}_{\alpha\in\mathscr{A}^R}\{\alpha - \frac{\alpha^3 x^T x}{2}\}$, where $\mathscr{A}^R \doteq \{\alpha = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C} \,|\, r \geq 0,\, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$ [33]. In the simple one-dimensional case, the resulting function on $\mathbb{C}$ has a branch cut along the negative imaginary axis, and this generalizes to higher-dimensional cases in the natural way.

Although stationarity-based representations for gravitational and Coulomb potentials are inside the integral in (3.1), they may be moved outside through the introduction of α-valued processes, cf. [18, 40]. In particular, not only does one seek the stationary path for action J, but the action functional itself can be given as a stationary value of an integral of a polynomial, leading to an iterated-stat problem formulation for such TPBVPs. This may be exploited in the solution of TPBVPs in such systems, cf. [18, 38, 40], which will be discussed further in Section 3.5.

It has also been demonstrated that this stationary-action approach may be applied to TPBVPs for infinite-dimensional conservative systems described by classes of lossless wave equations, see for example [6, 7]. There, stat is used in the construction of fundamental solution groups for these wave equations by appealing to stationarity of action on longer horizons.

Lastly, it has recently been demonstrated that stationarity may be employed to obtain a Feynman-Kac type of representation for solutions of the Schrödinger initial value problem (IVP) for certain classes of initial conditions and potentials [37]. As with the conservative-system cases above, these representations are valid for indefinitely long duration problems, whereas with only the minimization operation, such representations are valid only on time intervals such that the action remains convex, which is always a bounded duration and potentially zero.

In all of these examples, one obtains the stationary value of an action functional where the action functional itself takes the form of a stationary value of a functional that is quadratic in the momentum (the $u.$ in put in (3.1)) and cubic in the newly introduced potential energy parameterization variable (a time-dependent form of the α parameter above). That is, the overall stationary value is obtained from iterated staticization operations, where the outer stat is over a variable in which the functional is quadratic. Thus, if one can invert the order of the of the stat operations, then the inner stat operation results in a functional that is obtained as a solution of a differential Riccati equation (DRE). (It should be noted that this DRE must typically be propagated past escape times, where this propagation may be efficiently performed through the use of what has been termed "stat duality", cf. [34].) Hence, after inversion of the order of the iterated stat operations, the problem may be reduced to a single stat operation such that the argument takes the form of a linear functional operating on a set of DRE solutions. Consequently, an issue of fundamental importance regards conditions under which one may invert the order of stat operations in an iterated staticization.

In Section 3.2, the stat operator will be rigorously defined, and a general problem class along with some corresponding notation will be indicated. Then, in Section 3.3, a somewhat general condition will be indicated, and it will shown that one may invert the order of staticization operations under that condition. This will be demonstrated by obtaining an equivalence between iterated staticization and full staticization over both variables together. Section 3.4 will present several classes of problems for which the general condition of Section 3.3 holds. Finally, in Section 3.5, a stationary-action application in astrodynamics will be discussed.

3.2 Problem and Stationarity Definitions

Before the issue to be studied can be properly expressed, it is necessary to define stationarity and the stat operator.

3.2.1 Stationarity definitions

As noted above, the motivation for this effort is the computation and propagation of stationary points of payoff functionals, which is unusual in comparison to the standard classes of problems in optimization (although one should note for example, [8]). In analogy with the language for minimization and maximization, we will refer to the search for stationary points as “staticization”, with these points being statica, in analogy with minima/maxima, and a single such point being a staticum in analogy with minimum/maximum. One might note that here that the term staticization is being derived from a Latin root, staticus (presumably originating from the Greek, statikós), in analogy with the Latin root, maximus, of “maximization”. We note that Ekeland [8] employed the term “extremization” for what is largely the same notion that is being referred to here as staticization, but with a very different focus. We make the following definitions. Let $\mathscr{F}$ denote either the real or complex field. Suppose $\mathscr{U}$ is a normed vector space (over $\mathscr{F}$)

with $\mathscr{A} \subseteq \mathscr{U}$, and suppose $G : \mathscr{A} \to \mathscr{F}$. We say $\bar{u} \in \operatorname{argstat}_{u\in\mathscr{A}} G(u) \doteq \operatorname{argstat}\{G(u)\,|\,u \in \mathscr{A}\}$ if $\bar{u} \in \mathscr{A}$ and either

$$\limsup_{u\to\bar{u}, u\in\mathscr{A}\setminus\{\bar{u}\}} \frac{|G(u)-G(\bar{u})|}{|u-\bar{u}|} = 0, \tag{3.2}$$

or there exists $\delta > 0$ such that $\mathscr{A} \cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). If $\operatorname{argstat}\{G(u)\,|\,u \in \mathscr{A}\} \neq \emptyset$, we define the possibly set-valued stat^s operation by

$$\operatorname*{stat}^s_{u\in\mathscr{A}} G(u) \doteq \operatorname{stat}^s\{G(u)\,|\,u \in \mathscr{A}\} \doteq \left\{G(\bar{u}) \,\middle|\, \bar{u} \in \operatorname{argstat}\{G(u)\,|\,u \in \mathscr{A}\}\right\}. \tag{3.3}$$

If $\operatorname{argstat}\{G(u)\,|\,u \in \mathscr{A}\} = \emptyset$, then $\operatorname{stat}^s_{u\in\mathscr{A}} G(u)$ is undefined. Where applicable, we are also interested in a single-valued stat operation (note the absence of superscript s). In particular, if there exists $a \in \mathscr{F}$ such that $\operatorname{stat}^s_{u\in\mathscr{A}} G(u) = \{a\}$, then $\operatorname{stat}_{u\in\mathscr{A}} G(u) \doteq a$; otherwise, $\operatorname{stat}_{u\in\mathscr{A}} G(u)$ is undefined. At times, we may abuse notation by writing $\bar{u} = \operatorname{argstat}\{G(u)\,|\,u \in \mathscr{A}\}$ in the event that the argstat is the single point $\{\bar{u}\}$.

In the case where $\mathscr{U}$ is a Banach space and $\mathscr{A} \subseteq \mathscr{U}$ is an open set, $G : \mathscr{A} \to \mathscr{F}$ is Fréchet differentiable at $\bar{u} \in \mathscr{A}$ with continuous, linear $DG(\bar{u}) \in \mathscr{L}(\mathscr{U};\mathscr{F})$ if

$$\lim_{w\to 0,\, \bar{u}+w\in\mathscr{A}\setminus\{\bar{u}\}} \frac{|G(\bar{u}+w)-G(\bar{u})-[DG(\bar{u})]w|}{|w|} = 0. \tag{3.4}$$

The following is immediate from the above definitions.

Lemma 10. *Suppose $\mathscr{U}$ is a Banach space, with open set $\mathscr{A} \subseteq \mathscr{U}$, and that G is Fréchet differentiable at $\bar{u} \in \mathscr{A}$. Then, $\bar{u} \in \operatorname{argstat}\{G(y)\,|\,y \in \mathscr{A}\}$ if and only if $DG(\bar{u}) = 0$.*

3.2.2 Problem definition

Throughout, let $\mathscr{U}, \mathscr{V}$ be Banach spaces with norm on $\mathscr{U}$ denoted by $|\cdot|_\mathscr{U}$, and similarly for $\mathscr{V}$. When $\mathscr{U}$ is also Hilbert, let the inner product be denoted by $\langle\cdot,\cdot\rangle_\mathscr{U}$, and similarly for $\mathscr{V}$. Let the resulting norms and inner products on $\mathscr{U}\times\mathscr{V}$ be denoted by $|\cdot|_{\mathscr{U}\times\mathscr{V}}$ and $\langle\cdot,\cdot\rangle_{\mathscr{U}\times\mathscr{V}}$. Let $\mathscr{A} \subseteq \mathscr{U}$ and $\mathscr{B} \subseteq \mathscr{V}$ be open. Throughout, we assume

$$G \in C^2(\mathscr{A} \times \mathscr{B}; \mathscr{F}). \tag{A.1}$$

Then, for each $u \in \mathscr{A}$, let $g^{1,u} \in C^2(\mathscr{B};\mathscr{F})$ be given by $g^{1,u}(v) \doteq G(u,v)$ for all $v \in \mathscr{B}$. Similarly, for each $v \in \mathscr{B}$, let $g^{2,v} \in C^2(\mathscr{A};\mathscr{F})$ be given by $g^{2,v}(u) \doteq G(u,v)$ for all $u \in \mathscr{A}$. Further, let

$$\mathscr{A}_G \doteq \big\{ u \in \mathscr{A} \,\big|\, \operatorname*{stat}_{v\in\mathscr{B}} g^{1,u}(v) \text{ exists} \big\} \;\text{ and }\; \mathscr{B}_G \doteq \big\{ v \in \mathscr{B} \,\big|\, \operatorname*{stat}_{u\in\mathscr{A}} g^{2,v}(u) \text{ exists} \big\}. \tag{3.5}$$

Given $u \in \mathscr{A}_G$, let $\mathscr{M}^1(u) \doteq \operatorname{argstat}_{v\in\mathscr{B}} g^{1,u}(v)$, and given $v \in \mathscr{B}_G$, let $\mathscr{M}^2(v) \doteq \operatorname{argstat}_{u\in\mathscr{A}} g^{2,v}(u)$. Next, define $\bar{G}^1 : \mathscr{A}_G \to \mathscr{F}$ and $\bar{G}^2 : \mathscr{B}_G \to \mathscr{F}$ by

$$\bar{G}^1(u) \doteq \operatorname*{stat}_{v\in\mathscr{B}} g^{1,u}(v) \quad \forall u \in \mathscr{A}_G \;\text{ and }\; \bar{G}^2(v) \doteq \operatorname*{stat}_{u\in\mathscr{A}} g^{2,v}(u) \quad \forall v \in \mathscr{B}_G.$$

Finally, let

$$\hat{\mathscr{A}}_G \doteq \operatorname*{argstat}_{u\in\mathscr{A}_G} \bar{G}^1(u) \;\text{ and }\; \hat{\mathscr{B}}_G \doteq \operatorname*{argstat}_{v\in\mathscr{B}_G} \bar{G}^2(v).$$

We will discuss conditions under which

$$\operatorname*{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) = \operatorname*{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v) = \operatorname*{stat}_{v\in\mathscr{B}_G} \bar{G}^2(v). \tag{3.6}$$

We will generally be concerned only with the left-hand equality in (3.6); obviously the right-hand equality would be obtained analogously. We refer to the left-hand object in (3.6) as iterated stat operations, while the center object will be referred to as a full stat operation. Although in some results, the existence of both the iterated and full stat operations are obtained, many of the results will assume the existence of one or both of these objects. We list the two potential assumptions below. In each result to follow, we will indicate when one or both of these is utilized. The full-stat assumption is as follows.

Assume $\operatorname{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$ exists (and let $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$). $(A.2f)$

Note that under Assumption $(A.2f)$,

$$\bar{u} \in \mathscr{A}_G,\ \bar{v} \in \mathscr{B}_G,\ \bar{v} \in \mathscr{M}^1(\bar{u}),\ \text{and}\ \bar{u} \in \mathscr{M}^2(\bar{v}). \tag{3.7}$$

The iterated-stat assumption is as follows.

$$\text{Assume } \operatorname{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) \text{ exists (and let } \bar{u} \in \hat{\mathscr{A}}_G). \tag{$A.2i$}$$

Note that under Assumption $(A.2i)$,

$$\exists\, \bar{v} \in \mathscr{M}^1(\bar{u}),\ \text{ and } \operatorname*{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) = g^{1,\bar{u}}(\bar{v}) = G(\bar{u},\bar{v}). \tag{3.8}$$

We will first obtain (3.6) under some general assumptions. After that, we will demonstrate that these assumptions are satisfied under certain other sets of assumptions, where the latter sets describe more commonly noted classes of functions (specifically, quadratic, semi-quadratic and Morse functions). Again, we mainly address only the left-hand equality of (3.6); the right-hand equality is handled similarly.

3.3 The General Case

Given $\mathscr{C} \subseteq \mathscr{V}$ and $\hat{v} \in \mathscr{V}$, we let $d(\hat{v},\mathscr{C}) \doteq \inf_{v\in\mathscr{C}} |v-\hat{v}|$, and use this distance notation more generally throughout. In addition to $(A.1)$, we assume the following throughout this section.

There exist $\delta = \delta(\bar{u},\bar{v}) > 0$ and $K = K(\bar{u},\bar{v}) < \infty$ such that $d(\bar{v},\mathscr{M}^1(u)) \leq K\,|\bar{u}-u|$ for all $u \in \mathscr{A}_G \cap B_\delta(\bar{u})$. $(A.3)$

We note that $(A.3)$ is trivially satisfied in the case that there exists $\delta > 0$ such that $B_\delta(\bar{u}) \cap \mathscr{A}_G = \emptyset$. It may be helpful to also note that $(A.3)$ is satisfied under the possibly more heuristically appealing, following assumption.

For every $\tilde{u} \in \mathscr{A}_G$ and every $\tilde{v} \in \mathscr{M}^1(\tilde{u})$, there exist $\delta = \delta(\tilde{u},\tilde{v}) > 0$ and $K = K(\tilde{u},\tilde{v}) < \infty$ such that $d(\tilde{v},\mathscr{M}^1(u)) \leq K\,|\tilde{u}-u|$ for all $u \in \mathscr{A}_G \cap B_\delta(\tilde{u})$. $(A.3')$

Lemma 11. *Assume* $(A.2f)$. *Then,* $\bar{u} \in \hat{\mathscr{A}}_G$ *and* $G(\bar{u},\bar{v}) \in \operatorname{stat}^s_{u\in\mathscr{A}_G} \bar{G}^1(u)$.

Proof. Let $(\bar{u},\bar{v})$ be as in $(A.2f)$. Let $R \doteq d\big((\bar{u},\bar{v}),(\mathscr{A}\times\mathscr{B})^c\big)$. By Assumption $(A.3)$, there exist $\delta \in (0,R/2)$ and $K < \infty$ such that for all $u \in \mathscr{A}_G \cap B_\delta(\bar{u})$ and all $\varepsilon \in (0,1)$, there exists $v \in \mathscr{M}^1(u)$ such that

$$|v-\bar{v}| \le (K+\varepsilon)|u-\bar{u}| \le (K+\varepsilon)\delta. \tag{3.9}$$

Let $\tilde{u} \in \mathscr{A}_G \cap B_{\delta/(K+1)}(\bar{u})$. By (3.7),

$$|\operatorname*{stat}_{v\in\mathscr{B}} g^{1,\tilde{u}}(v) - \operatorname*{stat}_{v\in\mathscr{B}} g^{1,\bar{u}}(v)| = |\operatorname*{stat}_{v\in\mathscr{B}} g^{1,\tilde{u}}(v) - G(\bar{u},\bar{v})|,$$

and by (3.9), there exists $\tilde{v} \in B_\delta(\bar{v})$ such that this is

$$= |G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})|. \tag{3.10}$$

Let $f \in C^\infty\big((-3/2,3/2);\mathscr{A}\times\mathscr{B}\big)$ be given by $f(\lambda) = \big(\bar{u}+\lambda(\tilde{u}-\bar{u}),\bar{v}+\lambda(\tilde{v}-\bar{v})\big)$ for all $\lambda \in (-3/2,3/2)$. Define $W^0(\lambda) = [G\circ f](\lambda)$ for all $\lambda \in (-3/2,3/2)$, and note that by Assumption $(A.1)$ and standard results, $W^0 \in C^2\big((-3/2,3/2);\mathscr{F}\big)$. Similarly, let $W^1(\lambda) = [(G_u,G_v)\circ f](\lambda) = \big(G_u(f(\lambda)),G_v(f(\lambda))\big)$. By Assumption $(A.1)$ and standard results, $W^1 \in C^1\big((-3/2,3/2);\mathscr{U}\times\mathscr{V}\big)$. Then, by the Mean Value Theorem (cf. [2, Th. 12.6]), there exists $\lambda_0 \in (0,1)$ such that

$$|G(\tilde{u},\tilde{v}) - G(\bar{u},\bar{v})| = |W^0(1)-W^0(0)| \le \Big|\frac{dG}{d(u,v)}(f(\lambda_0))\Big|\Big|\frac{df}{d\lambda}(\lambda_0)\Big|$$
$$= \big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\big|(\tilde{u}-\bar{u},\tilde{v}-\bar{v})\big|,$$

where $(u_0,v_0) \doteq f(\lambda_0)$, and which by (3.9),

$$\le \big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big|\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|. \tag{3.11}$$

Similarly, there exists $\lambda_1 \in (0,\lambda_0)$ such that

$$\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big) - \big(G_u(\bar{u},\bar{v}),G_v(\bar{u},\bar{v})\big)\big| = |W^1(\lambda_0)-W^1(0)|$$

$$\leq \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\Big|\frac{df}{d\lambda}(\lambda_1)\Big|\,|\lambda_1| \leq \Big|\frac{d^2G}{d(u,v)^2}(u_1,v_1)\Big|\,\big|(u_1-\bar{u},v_1-\bar{v})\big|,$$

where $(u_1,v_1) \doteq f(\lambda_1)$, and this is

$$\leq \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|.$$

Recalling $(\bar{u},\bar{v}) \in \operatorname{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$, this implies

$$\big|\big(G_u(u_0,v_0),G_v(u_0,v_0)\big)\big| \leq \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\sqrt{1+(K+1)^2}|\tilde{u}-\bar{u}|. \tag{3.12}$$

Combining (3.11) and (3.12) yields

$$|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})| \leq \Big|\frac{d^2G}{d(u,v)^2}(f(\lambda_1))\Big|\,[1+(K+1)^2]\,|\tilde{u}-\bar{u}|^2.$$

Let $K_1 \doteq \big|\frac{d^2G}{d(u,v)^2}(\bar{u},\bar{v})\big|$. By $(A.1)$, there exists $\hat{\delta} \in (0,\delta/(K+1))$ such that for all $(u,v) \in B_{\hat{\delta}}(\bar{u},\bar{v})$, $\big|\frac{d^2G}{d(u,v)^2}(u,v)\big| \leq K_1+1$. Hence, there exists $\bar{C} < \infty$ such that

$$|G(\tilde{u},\tilde{v})-G(\bar{u},\bar{v})| \leq \bar{C}|\tilde{u}-\bar{u}|^2 \quad \forall\, \tilde{u} \in \mathscr{A}_G \cap B_{\hat{\delta}/(K_1+1)}(\bar{u}). \tag{3.13}$$

Combining (3.10) and (3.13) one has $|\operatorname{stat}_{v\in\mathscr{B}} g^{1,\tilde{u}}(v) - \operatorname{stat}_{v\in\mathscr{B}} g^{1,\bar{u}}(v)| \leq \bar{C}|\tilde{u}-\bar{u}|^2$, which upon recalling that $\tilde{u} \in \mathscr{A}_G \cap B_{\hat{\delta}/(K+1)}(\bar{u})$ was arbitrary, yields the assertions. □

Theorem 6. *Assume $(A.2f)$. Then*

$$\operatorname*{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v).$$

Proof. The assertions follow directly from the assumption, $(A.2f)$ and Lemma 11. □

3.4 Some Specific Cases

We examine several classes of functionals that fit within the general class above.

3.4.1 The Quadratic Case

Throughout this section, we take $\mathscr{A} = \mathscr{U}$ and $\mathscr{B} = \mathscr{V}$, where $\mathscr{U}, \mathscr{V}$ are Hilbert. Let

$$
\begin{aligned}
G(u,v) &= \tfrac{c}{2} + \langle w,u\rangle_{\mathscr{U}} + \langle y,v\rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_1 u,u\rangle_{\mathscr{U}} + \langle \bar{B}_2 v,u\rangle_{\mathscr{U}} + \tfrac{1}{2}\langle \bar{B}_3 v,v\rangle_{\mathscr{V}} \\
&= \tfrac{c}{2} + \langle w,u\rangle_{\mathscr{U}} + \langle y,v\rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_1 u,u\rangle_{\mathscr{U}} + \langle \bar{B}_2' u,v\rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_3 v,v\rangle_{\mathscr{V}},
\end{aligned} \tag{3.14}
$$

for all $u \in \mathscr{U}$ and $v \in \mathscr{V}$, where $\bar{B}_1 \in \mathscr{L}(\mathscr{U};\mathscr{U})$, $\bar{B}_2 \in \mathscr{L}(\mathscr{V};\mathscr{U})$, $\bar{B}_3 \in \mathscr{L}(\mathscr{V};\mathscr{V})$, $w \in \mathscr{U}$, $y \in \mathscr{V}$ and $c \in \mathscr{F}$, where $\mathscr{L}(\cdot,\cdot)$ generically denotes a space of bounded linear operators, and $\bar{B}_1, \bar{B}_3$ are self-adjoint and closed. We present results under both the cases of $(A.2f)$ and $(A.2i)$.

When the full staticization is known to exist

We suppose $(A.2f)$. This subcase is fully covered in [34], and hence we will mainly only indicate an additional approach. More specifically, in [34], it is directly shown that under Assumption $(A.2f)$, in the case of (3.14), we have the following.

Theorem 7. *Assume $(A.2f)$. Then, $\operatorname{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u)$ exists, and*

$$
\operatorname*{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v). \tag{3.15}
$$

Here however, we show that Assumption $(A.3)$ is satisfied for G given by (3.14), and hence that assertions (3.15) of Theorem 7 follow as a special case of Theorem 6. We begin by noting the following, which follows directly from (3.14) and Lemma 10.

Lemma 12. *Let $\hat{u} \in \mathscr{A}$. Then $\hat{v} \in \mathscr{M}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3\hat{v} + y = 0$.*

We now indicate the key result of this section.

Proposition 1. *Assumption $(A.3)$ is satisfied.*

Proof. We suppose $\mathscr{A}_G \neq \{\bar{u}\}$; otherwise the result is trivial. Let $\hat{u} \in \mathscr{A}_G \setminus \{\bar{u}\}$. By Lemma 12, $\hat{v} \in \mathscr{M}^1(\hat{u})$ if and only if $\bar{B}_2'\hat{u} + \bar{B}_3\hat{v} + y = 0$. However, by (3.7), $\bar{v} \in \mathscr{M}^1(\bar{u})$, and hence by

Lemma 12, $\bar{B}_2'\bar{u}+\bar{B}_3\bar{v}+y=0$. Combining these two inequalities, we see that $\hat{v}\in\mathscr{M}^1(\hat{u})$ if and only if $\bar{B}_2'(\hat{u}-\bar{u})+\bar{B}_3(\hat{v}-\bar{v})=0$. We take $\hat{v}\doteq\bar{v}-\bar{B}_3^{\#}\bar{B}_2'(\hat{u}-\bar{u})$, where the # superscript indicates the Moore-Penrose pseudo-inverse, where existence follows by the closedness of $\bar{B}_3$, cf. [3, 53]. Then, $\hat{v}\in\mathscr{M}^1(\hat{u})$ and $|\hat{v}-\bar{v}|\le|\bar{B}_3^{\#}||\bar{B}_2'||\hat{u}-\bar{u}|$, where the induced norms on the operators are employed, which yields the desired assertion. □

By Proposition 1, we may apply Theorem 6 to obtain the leftmost assertion of (3.15) in Theorem 7, if $\operatorname{stat}_{u\in\mathscr{A}_G}\bar{G}^1(u)$ exists. The existence of $\operatorname{stat}_{u\in\mathscr{A}_G}\bar{G}^1(u)$ given $(A.2f)$ is obtained in [34], and the proof is not repeated here.

When the iterated staticization is known to exist

We suppose $(A.2i)$. We will find that $\operatorname{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}}G(u,v)$ exists, and obtain the equivalence between full and iterated staticization. We begin with a lemma (which is similar to Lemma 10 of [34]).

Lemma 13. *Given any $\tilde{u}\in\mathscr{A}_G$, $\mathscr{M}^1(\tilde{u})$ is an affine subspace, and further, $G(\tilde{u},\cdot)$ is constant on $\mathscr{M}^1(\tilde{u})$.*

Proof. Let $\tilde{u}\in\mathscr{A}_G$. By Lemma 12, $v\in\mathscr{M}^1(\tilde{u})$ if and only if $\bar{B}_3v=-(\bar{B}_2'\tilde{u}+y)$, which yields the first assertion. Suppose $\tilde{v},\hat{v}\in\mathscr{M}^1(\tilde{u})$. Then, using (3.8),

$$\begin{aligned}G(\tilde{u},\tilde{v})-G(\tilde{u},\hat{v})&=\langle\bar{B}_2'\tilde{u}+y,\tilde{v}-\hat{v}\rangle_{\mathscr{V}}+\tfrac{1}{2}\langle\bar{B}_3\tilde{v},\tilde{v}\rangle_{\mathscr{V}}-\tfrac{1}{2}\langle\bar{B}_3\hat{v},\hat{v}\rangle_{\mathscr{V}},\\&=\langle-\tfrac{1}{2}(\bar{B}_3\tilde{v}+\bar{B}_3\hat{v}),\tilde{v}-\hat{v}\rangle_{\mathscr{V}}+\tfrac{1}{2}\langle\bar{B}_3\tilde{v},\tilde{v}\rangle_{\mathscr{V}}-\tfrac{1}{2}\langle\bar{B}_3\hat{v},\hat{v}\rangle_{\mathscr{V}}=0.\end{aligned}$$

□

Theorem 8. *Assume $(A.2i)$, and let $\bar{v}$ be as given in (3.8). Then,* $\operatorname{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}}G(u,v)$ *exists, and* $\operatorname{stat}_{(u,v)\in\mathscr{U}\times\mathscr{V}}G(u,v)=G(\bar{u},\bar{v})=\operatorname{stat}_{u\in\mathscr{A}_G}\bar{G}^1(u)$.

Proof. Assume $(A.2i)$, and let $\bar{v}$ be as given in (3.8). First, note that the assertion that $G(\bar{u},\bar{v}) = \mathrm{stat}_{u\in\mathscr{A}_G}\bar{G}^1(u)$ will follow from the other assertions and (3.8). By Lemma 12, $v \in \mathscr{M}^1(\bar{u})$ if and only if $\bar{B}_2'\bar{u} + \bar{B}_3 v + y = 0$. For $u \in \mathscr{A}_G$, let

$$\check{v}(u) \doteq \bar{v} - \bar{B}_3^{\#}\big[\bar{B}_2' u + y - (\bar{B}_2'\bar{u} + y)\big], \tag{3.16}$$

and note that

$$\check{v}(\bar{u}) = \bar{v}. \tag{3.17}$$

Let $\tilde{v} \doteq -\bar{B}_3^{\#}[\bar{B}_2'\bar{u} + y]$, and note that as $\bar{v}$ and $\tilde{v}$ are both in $\mathscr{M}^1(\bar{u})$, by Lemma 12,

$$0 = \bar{B}_3[\bar{v} - \tilde{v}] = \bar{B}_3\big[\bar{v} + \bar{B}_3^{\#}(\bar{B}_2'\bar{u} + y)\big]. \tag{3.18}$$

Then, using (3.16) and (3.18), we see that for $u \in \mathscr{A}_G$,

$$\begin{aligned}\bar{B}_3\check{v}(u) + \bar{B}_2' u + y &= \bar{B}_3\big[\bar{v} - \bar{B}_3^{\#}(\bar{B}_2' u - \bar{B}_2'\bar{u})\big] + \bar{B}_2' u + y \\ &= \bar{B}_3\big[-\bar{B}_3^{\#}(\bar{B}_2' u + y)\big] + \bar{B}_2' u + y,\end{aligned}$$

which by definition of the pseudo-inverse and the fact that $\bar{B}_2'\bar{u} + y \in \mathrm{Range}(\bar{B}_3)$ for $u \in \mathscr{A}_G$,

$$= 0.$$

Hence, $\check{v}(u) \in \mathscr{M}^1(u) \quad \forall u \in \mathscr{A}_G$, and consequently,

$$\bar{G}^1(u) = G(u, \check{v}(u)) \quad \forall u \in \mathscr{A}_G. \tag{3.19}$$

Then, by $(A.2i)$ and the choice of $\bar{u}$,

$$0 = \frac{d\bar{G}^1}{du}(\bar{u}),$$

which by (3.16), (3.19), $(A.1)$ and the chain rule,

$$= G_u(\bar{u},\breve{v}(\bar{u})) + G_v(\bar{u},\breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}),$$

which by (3.17) and our choice of $\bar{v}$,

$$= G_u(\bar{u},\bar{v}) + G_v(\bar{u},\bar{v})\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u},\bar{v}).$$

From this and the choice of $\bar{v}$, we see that

$$(\bar{u},\bar{v}) \in \underset{(u,v)\in\mathscr{A}\times\mathscr{B}}{\operatorname{argstat}} G(u,v) \;\text{ and }\; G(\bar{u},\bar{v}) \in \underset{(u,v)\in\mathscr{A}\times\mathscr{B}}{\operatorname{stat}^s} G(u,v). \tag{3.20}$$

Now suppose there exists $(\hat{u},\hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)\setminus\{(\bar{u},\bar{v})\}$. This implies

$$G_u(\hat{u},\hat{v}) = 0, \;\text{ and }\; G_v(\hat{u},\hat{v}) = 0, \tag{3.21}$$

and consequently,

$$\hat{v} \in \mathscr{M}^1(\hat{u}), \;\text{ and }\; \bar{G}^1(\hat{u}) = G(\hat{u},\hat{v}).$$

Let

$$\breve{v}'(u) \doteq \hat{v} - \bar{B}_3^{\#}\big[\bar{B}_2' u + y - (\bar{B}_2'\hat{u} + y)\big] \quad \forall u \in \mathscr{A}_G, \tag{3.22}$$

and note that

$$\breve{v}'(\hat{u}) = \hat{v}. \tag{3.23}$$

Let $\hat{\hat{v}} \doteq -\bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)$, and note that $\hat{v},\hat{\hat{v}} \in \mathscr{M}^1(\hat{u})$. Similar to the above, we see that

$$0 = \bar{B}_3(\hat{v} - \hat{\hat{v}}) = \bar{B}_3\big[\hat{v} + \bar{B}_3^{\#}(\bar{B}_2'\hat{u} + y)\big]. \tag{3.24}$$

Then, again similar to the above, using (3.24) and the definition of the pseudo-inverse and

$\bar{B}_2'\bar{u}+y \in \text{Range}(\bar{B}_3)$, we see that

$$\bar{B}_3\breve{v}'(u)+\bar{B}_2'u+y = \bar{B}_3\big[\hat{v}-\bar{B}_3^{\#}\big(\bar{B}_2'u+y-(\bar{B}_2'\hat{u}+y)\big)\big]+\bar{B}_2'u+y$$
$$= \bar{B}_3\big[\hat{v}-\bar{B}_3^{\#}(\bar{B}_2'u+y)\big]+\bar{B}_2'u+y = 0,$$

which implies that $\breve{v}'(u) \in \mathscr{M}^1(u)$ for all $u \in \mathscr{A}_G$. Hence,

$$\bar{G}^1(u) = G(u,\breve{v}'(u)) \quad \forall u \in \mathscr{A}_G. \tag{3.25}$$

By (3.22), (3.25), $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1}{du}(\hat{u}) = G_u(\hat{u},\breve{v}'(\hat{u}))+G_v(\hat{u},\breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}),$$

which by (3.21) and (3.23),

$$= G_u(\hat{u},\hat{v})+G_v(\hat{u},\hat{v})\frac{d\breve{v}'}{du}(\hat{u}) = 0,$$

which implies that $\hat{u} \in \hat{\mathscr{A}}_G$. Using this, (3.20) and $(A.2i)$, we see that $G(\hat{u},\hat{v}) = G(\bar{u},\bar{v})$. As $(\hat{u},\hat{v}) \in \text{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v) \setminus \{(\bar{u},\bar{v})\}$ was arbitrary, we have the desired result. □

3.4.2 The Semi-Quadratic Case

Throughout this section, we take $\mathscr{A} \subseteq \mathscr{U}$ and $\mathscr{B} = \mathscr{V}$, with $\mathscr{V}$ being Hilbert. Let

$$G(u,v) \doteq f_1(u)+\langle f_2(u),v\rangle_{\mathscr{V}}+\tfrac{1}{2}\langle\bar{B}_3(u)v,v\rangle_{\mathscr{V}}, \tag{3.26}$$

for all $u \in \mathscr{A}$ and $v \in \mathscr{V}$, where $f_1 \in C^2(\mathscr{A};\mathscr{F})$, $f_2 \in C^2(\mathscr{A};\mathscr{V})$ and $\bar{B}_3 \in C^2(\mathscr{A};\mathscr{L}(\mathscr{V},\mathscr{V}))$, and $\bar{B}_3(u)$ is self-adjoint and closed for all $u \in \mathscr{A}$. For each $u \in \mathscr{A}$, let $\bar{B}_3^{\#}(u) \doteq [\bar{B}_3(u)]^{\#}$ denote the Moore-Penrose pseudo-inverse of $\bar{B}_3(u)$. Assume that there exists a constant $D > 0$ such that $|\bar{B}_3^{\#}(u)| \le D$ for all $u \in \mathscr{A}_G$. Similar to Lemma 12, the next lemma follows directly from (3.26) and Lemma 10.

Lemma 14. *Let $\hat{u} \in \mathscr{A}$. Then $\hat{v} \in \mathscr{M}^1(\hat{u})$ if and only if $f_2(\hat{u}) + \bar{B}_3(\hat{u})\hat{v} = 0$.*

When the full staticization is known to exist

Lemma 15. *Assume $(A.2f)$. Then assumption $(A.3)$ is satisfied.*

Proof. The result is trivial for $\mathscr{A}_G = \{\bar{u}\}$. Suppose $\mathscr{A}_G \neq \{\bar{u}\}$. Choose any $\delta > 0$ such that $\mathscr{A}_G \cap (B_\delta(\bar{u}) \backslash \{\bar{u}\}) \neq \emptyset$. Let $\hat{u} \in [\mathscr{A}_G \cap B_\delta(\bar{u})] \setminus \{\bar{u}\}$. Let $\hat{v} = \bar{v} - \bar{B}_3^\#(\hat{u}) f_2(\hat{u}) - \bar{B}_3^\#(\hat{u}) \bar{B}_3(\hat{u}) \bar{v}$. Note that as $f_2(\hat{u}) \in \mathrm{Range}(\bar{B}_3(\hat{u}))$,

$$\begin{aligned}\bar{B}_3(\hat{u})\hat{v} + f_2(\hat{u}) &= \bar{B}_3(\hat{u})[\bar{v} - \bar{B}_3^\#(\hat{u}) f_2(\hat{u}) - \bar{B}_3^\#(\hat{u}) \bar{B}_3(\hat{u})\bar{v}] + f_2(\hat{u}) \\ &= \bar{B}_3(\hat{u})\bar{v} - f_2(\hat{u}) - \bar{B}_3(\hat{u})\bar{v} + f_2(\hat{u}) = 0.\end{aligned}$$

Therefore, $\hat{v} \in \mathscr{M}^1(\hat{u})$ by Lemma 14. We have

$$|\hat{v} - \bar{v}| = \left|\bar{B}_3^\#(\hat{u}) f_2(\hat{u}) + \bar{B}_3^\#(\hat{u}) \bar{B}_3(\hat{u})\bar{v}\right|,$$

and noting that by Lemma 14, $\bar{B}_3(\bar{u})\bar{v} + f_2(\bar{u}) = 0$, this is

$$\begin{aligned}&= \left|\bar{B}_3^\#(\hat{u})[f_2(\hat{u}) - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v} + \bar{B}_3(\hat{u})\bar{v}]\right| \\ &\leq |\bar{B}_3^\#(\bar{u})|\,|f_2(\hat{u}) - f_2(\bar{u}) + (\bar{B}_3(\hat{u}) - \bar{B}(\bar{u}))\bar{v}|,\end{aligned}$$

and letting $K_f = \max_{\lambda \in [0,1]} \left|\frac{df_2}{du}(\lambda\hat{u} + (1-\lambda)\bar{u})\right|$ and $K_B = \max_{\lambda \in [0,1]} \left|\frac{d\bar{B}_3}{du}(\lambda\hat{u} + (1-\lambda)\bar{u})\right|$, this is

$$\leq D\big[K_f|\hat{u} - \bar{u}| + K_B|\bar{v}||\hat{u} - \bar{u}|\big],$$

which yields $(A.3)$. □

Theorem 9. *Assume $(A.2f)$. Then $\mathrm{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u)$ exists, and*

$$\operatorname*{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v) \in \mathscr{A} \times \mathscr{B}} G(u, v).$$

Proof. This follows immediately from Lemma 15 and Theorem 6. □

When the iterated staticization is known to exist

The case where the iterated staticization is known to exist appears to require a substantial additional assumption.

Theorem 10. *Assume (A.2i). Also assume that* $f_2(u) \in \mathrm{Range}[\bar{B}_3(u)]$ *for all* $u \in \mathscr{A}_G$*. Then* $\mathrm{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$ *exists, and*

$$\operatorname*{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u) = G(\bar{u},\bar{v}) = \operatorname*{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v).$$

Proof. Note that by assumption and Lemma 14, $\mathscr{A}_G = \mathscr{A}$. Let $\bar{u} \in \hat{\mathscr{A}}_G$, and let $\bar{v}$ be as in (3.8), which implies

$$G_v(\bar{u},\bar{v}) = 0. \tag{3.27}$$

Suppose $G_u(\bar{u},\bar{v}) \neq 0$. Then there exists $\varepsilon > 0$, sequence $\{u_n\}$ with elements $u_n \in \mathscr{A}\setminus\{\bar{u}\}$ and $u_n \to \bar{u}$, and $\tilde{n} = \tilde{n}(\varepsilon) \in \mathbb{N}$ such that

$$|G(u_{\tilde{n}},\bar{v}) - G(\bar{u},\bar{v})| > \varepsilon|u_n - \bar{u}| \quad \forall n \geq \tilde{n}. \tag{3.28}$$

Let

$$v_n \doteq \bar{v} - \bar{B}_3^{\#}(u_n)[f_2(u_n) + \bar{B}_3(u_n)\bar{v}] \quad \forall n \in \mathbb{N}. \tag{3.29}$$

Then, using Lemma 14,

$$|v_n - \bar{v}| \leq |\bar{B}_3^{\#}(u_n)|\,\big|f_2(u_n) + \bar{B}_3(u_n)\bar{v} - f_2(\bar{u}) - \bar{B}_3(\bar{u})\bar{v}\big|,$$

which by assumption,

$$\leq D\big(|f_2(u_n) - f_2(\bar{u})| + |\bar{B}_3(u_n) - \bar{B}_3(\bar{u})||\bar{v}|\big). \tag{3.30}$$

Now, by the Mean value Theorem (cf. [2]), for each $n \in \mathbb{N}$, there exist $\lambda_n, \hat{\lambda}_n \in [0,1]$ such that

$$|f_2(u_n) - f_2(\bar{u})| \leq \left|\frac{df_2}{du}(\lambda_n u_n + (1-\lambda_n)\bar{u})\right| |u_n - \bar{u}|,$$
$$|\bar{B}_3(u_n) - \bar{B}_3(\bar{u})| \leq \left|\frac{d\bar{B}_3}{du}(\lambda_n u_n + (1-\lambda_n)\bar{u})\right| |u_n - \bar{u}|,$$

and hence by the smoothness of $f_2, \bar{B}_3$ and (3.30), there exist $K < \infty$ and $\hat{n} \in \mathbb{N}$ such that

$$|v_n - \bar{v}| \leq DK(1 + |\bar{v}|)|u_n - \bar{u}| \quad \forall n \geq \hat{n}. \tag{3.31}$$

Also, using (3.29),

$$\bar{B}_3(u_n)v_n + f_2(u_n) = \bar{B}_3(u_n)\left[\bar{v} - \bar{B}_3^{\#}(u_n)f_2(u_n) - \bar{B}_3^{\#}(u_n)\bar{B}_3(u_n)\bar{v}\right] + f_2(u_n),$$

which by assumption and the properties of the pseudo-inverse,

$$= \bar{B}_3(u_n)\bar{v} - f_2(u_n) - \bar{B}_3(u_n)\bar{v} + f_2(u_n) = 0, \tag{3.32}$$

and hence, $v_n \in \mathscr{M}_1(u_n)$ for all $n \in \mathbb{N}$. Hence, by $(A.2i)$, there exists $\bar{n} = \bar{n}(\varepsilon)$ such that for all $n \geq \bar{n}$,

$$|G(u_n, v_n) - G(\bar{u}, \bar{v})| = |\bar{G}^1(u_n) - \bar{G}^1(\bar{u})| < \frac{\varepsilon}{2}|u_n - \bar{u}|,$$

which implies

$$|G(u_n, v_n) - G(u_n, \bar{v}) + G(u_n, \bar{v}) - G(\bar{u}, \bar{v})| < \frac{\varepsilon}{2}|u_n - \bar{u}|,$$

and hence

$$|G(u_n, \bar{v}) - G(\bar{u}, \bar{v})| < \frac{\varepsilon}{2}|u_n - \bar{u}| + |G(u_n, v_n) - G(u_n, \bar{v})| \quad \forall n \geq \bar{n}. \tag{3.33}$$

Now by (3.26),

$$G(u_n, \bar{v}) - G(u_n, v_n) = \langle f_2(u_n), \bar{v} - v_n \rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\bar{v}, \bar{v} \rangle_{\mathscr{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n \rangle_{\mathscr{V}},$$

which by (3.29),

$$= \langle f_2(u_n), \bar{B}_3^{\#}(u_n)[f_2(u_n) + \bar{B}_3(u_n)\bar{v}]\rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\bar{v}, \bar{v}\rangle_{\mathscr{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n\rangle_{\mathscr{V}},$$

and by Lemma 14 and the self-adjointness of $\bar{B}_3$, this is,

$$\begin{aligned} &= \langle -\bar{B}_3(u_n)v_n, \bar{B}_3^{\#}(u_n)\bar{B}_3(u_n)(\bar{v}-v_n)]\rangle_{\mathscr{V}} + \tfrac{1}{2}\langle \bar{B}_3(u_n)\bar{v}, \bar{v}\rangle_{\mathscr{V}} - \tfrac{1}{2}\langle \bar{B}_3(u_n)v_n, v_n\rangle_{\mathscr{V}}, \\ &= \langle \bar{B}_3(u_n)(\bar{v}-v_n), (\bar{v}-v_n)\rangle_{\mathscr{V}}. \end{aligned} \tag{3.34}$$

Applying (3.31) in (3.34), we see that there exists $K_1 < \infty$ such that $|G(u_n,\bar{v}) - G(u_n, v_n)| \le K_1|u_n - \bar{u}|^2$ for all $n \ge \hat{n}$, and consequently, there exists $\bar{n}_1 = \bar{n}_1(\varepsilon) \in (\hat{n}, \infty)$ such that

$$|G(u_n,\bar{v}) - G(u_n,v_n)| < \frac{\varepsilon}{2}|u_n - \bar{u}| \quad \forall n \ge \bar{n}_1. \tag{3.35}$$

By (3.33) and (3.35),

$$|G(u_n,\bar{v}) - G(\bar{u},\bar{v})| < \varepsilon|u_n - \bar{u}| \quad \forall n \ge \bar{n} \wedge \bar{n}_1. \tag{3.36}$$

However, (3.36) contradicts (3.28), and consequently,

$$G_u(\bar{u},\bar{v}) = 0. \tag{3.37}$$

By (3.27) and (3.37),

$$(\bar{u},\bar{v}) \in \operatorname*{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v) \;\text{ and }\; G(\bar{u},\bar{v}) \in \operatorname*{stat^s}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v).$$

Now suppose there exists $(\hat{u},\hat{v}) \in \operatorname{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v) \setminus \{(\bar{u},\bar{v})\}$. This implies

$$G_u(\hat{u},\hat{v}) = 0, \qquad G_v(\hat{u},\hat{v}) = 0, \qquad \hat{v} \in \mathscr{M}^1(\hat{u}), \;\text{ and }\; \bar{G}^1(\hat{u}) = G(\hat{u},\hat{v}). \tag{3.38}$$

Similar to (3.29), let

$$\check{v}(u) \doteq \hat{v} - \bar{B}_3^{\#}(u) f_2(u) - \bar{B}_3^{\#}(u)\bar{B}_3(u)\hat{v} \quad \forall u \in \mathscr{A},$$

and note that $\check{v}(\hat{u}) = \hat{v}$. Also, similar to (3.32), we see that

$$\bar{B}_3(u)\check{v}(u) + f_2(u) = \bar{B}_3(u)\hat{v} - f_2(u) - \bar{B}_3(u)\hat{v} + f_2(u) = 0,$$

which implies that $\check{v}(u) \in \mathscr{M}^1(u)$ for all $u \in \mathscr{A}$. Hence, $\bar{G}^1(u) = G(u, \check{v}(u))$ for all $u \in \mathscr{A}$. Note that

$$\begin{aligned} |\bar{G}^1(u) - \bar{G}^1(\hat{u})| &= |G(u, \check{v}(u)) - G(\hat{u}, \hat{v})| \\ &\leq |G(u, \check{v}(u)) - G(u, \hat{v})| + |G(u, \hat{v}) - G(\hat{u}, \hat{v})|, \end{aligned}$$

and note that by (3.38), given $\varepsilon > 0$, there exists $\hat{\delta}_1 = \hat{\delta}_1(\varepsilon) > 0$ such that for all $|u - \hat{u}| < \hat{\delta}_1$,

$$\leq \frac{\varepsilon}{2}|u - \hat{u}| + |G(u, \check{v}(u)) - G(u, \hat{v})|. \tag{3.39}$$

Also, similar to the estimate (3.35), we find that there exists $\hat{\delta}_2 = \hat{\delta}_2(\varepsilon) > 0$ such that

$$|G(u, \check{v}(u)) - G(u, \hat{v})| < \frac{\varepsilon}{2}|u - \hat{u}| \quad \forall |u - \hat{u}| < \hat{\delta}_2.$$

Using this in (3.39), we see that

$$|\bar{G}^1(u) - \bar{G}^1(\hat{u})| < \varepsilon|u - \hat{u}| \quad \forall |u - \hat{u}| < \hat{\delta}_1 \wedge \hat{\delta}_2. \tag{3.40}$$

Hence, $\frac{d\bar{G}^1}{du}(\hat{u}) = 0$, which implies that $\hat{u} \in \hat{\mathscr{A}}_G$. Using this and $(A.2i)$, we see that $G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v})$. As $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v) \in \mathscr{A} \times \mathscr{B}} G(u, v) \setminus \{(\bar{u}, \bar{v})\}$ was arbitrary, we have the desired result.

$\square$

3.4.3 The Uniformly Locally Morse Case

Throughout this section, we will assume that G is uniformly locally Morse in v in the following sense. We assume that for all $(\hat{u},\hat{v}) \in \mathscr{A} \times \mathscr{B}$ such that $G_v(\hat{u},\hat{v}) = 0$, there exist $\tilde{\varepsilon} = \tilde{\varepsilon}(\hat{u},\hat{v}) > 0$ and $\tilde{K} = \tilde{K}(\hat{u},\hat{v}) < \infty$ such that $G_{vv}(u,v)$ is invertible with $\left|[G_{vv}(u,v)]^{-1}\right| \le \tilde{K}$ for all $(u,v) \in B_{\tilde{\varepsilon}}(\hat{u},\hat{v})$. We also assume that $G_{uv}(u,v)$ is bounded on bounded sets. Under these assumptions and $(A.1)$, we will find that Assumption $(A.3')$ holds (and consequently, $(A.3)$). Hence, one may apply Theorem 6. We present results under both the cases of $(A.2f)$ and $(A.2i)$.

When the full staticization is known to exist

We suppose $(A.2f)$. We will find that $\operatorname{stat}_{u\in\mathscr{A}_G} \bar{G}^1(u)$ exists, and obtain the equivalence between full and iterated staticization.

Lemma 16. *Assume* $(A.2f)$. *There exist* $\varepsilon, \delta > 0$ *and* $\check{v} \in C^1(B_\varepsilon(\bar{u}); \mathscr{B} \cap B_\delta(\bar{v}))$ *such that* $B_\varepsilon(\bar{u}) \subseteq \mathscr{A}_G$, $\check{v}(\bar{u}) = \bar{v}$, $G_v(u,\check{v}(u)) = 0$ *and* $\frac{d\check{v}}{du}(u) = -\left[G_{vv}(u,v)\big|_{(u,\check{v}(u))}\right]^{-1} G_{uv}(u,v)\big|_{(u,\check{v}(u))}$ *for all* $u \in B_\varepsilon(\bar{u})$.

Proof. The first two assertions are simply the implicit mapping theorem, cf. [28]. The final assertion then follows from an application of the chain rule, that is, noting that $G_v(u,\check{v}(u)) = 0$ on $B_\varepsilon(\bar{u})$,

$$0 = \frac{dG_v(u,\check{v}(u))}{du} = G_{uv}(u,v)\big|_{(u,\check{v}(u))} + G_{vv}(u,v)\big|_{(u,\check{v}(u))} \frac{d\check{v}}{du}(u) \quad \forall u \in B_\varepsilon(\bar{u}).$$

□

By Lemma 16 and the definition of $\mathscr{A}_G$,

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathscr{B}} g^{1,u}(v) = G(u,\check{v}(u)) \quad \forall u \in B_\varepsilon(\bar{u}) \subseteq \mathscr{A}_G. \tag{3.41}$$

Then, by (3.41), the chain rule, $(A.1)$ and Lemma 16,

$$\bar{G}^1(\cdot) \in C^1(B_\varepsilon(\bar{u}); \mathscr{F}). \tag{3.42}$$

Lemma 17. *Assume $(A.2f)$. Then, there exists $K < \infty$ and $\delta \in (0, \varepsilon)$ such that $|\check{v}(u) - \check{v}(\bar{u})| = |\check{v}(u) - \bar{v}| \leq K|u - \bar{u}|$ for all $u \in B_\delta(\bar{u}) \subseteq \mathscr{A}_G$.*

Proof. By Lemma 16, $\frac{d\check{v}}{du}(\cdot)$ is continuous on $B_\varepsilon(\bar{u}) \subseteq \mathscr{A}_G$. Further, by the final assertion of Lemma 16, the uniformly locally Morse assumption and the boundedness assumption of the lemma,

$$\left|\frac{d\check{v}}{du}(u)\right| = \left|\left[G_{vv}(u,v)\Big|_{(u,\check{v}(u))}\right]^{-1}\right|\left|G_{uv}(u,v)\Big|_{(u,\check{v}(u))}\right| \leq \tilde{K}\hat{K},$$

where $\hat{K}$ is a bound on $\left|G_{uv}(u,\check{v}(u))\Big|_{(u,\check{v}(u))}\right|$ over $B_\delta(\bar{u})$. Hence, by an application of the mean value theorem, we obtain the asserted bound. $\square$

Note that Lemma 17 implies that Assumption $(A.3)$ is satisfied, and hence one may apply Theorem 6, which implies that the equivalence of stat and iterated stat holds under the assumption of existence of the latter.

Lemma 18. *Assume $(A.2f)$. Then, $\operatorname{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u)$ exists.*

Proof. Note first that by (3.41), (3.42) and the chain rule,

$$\frac{d}{du}\bar{G}^1(u)\Big|_{u=\bar{u}} = \frac{d}{du}G(u,\check{v}(u))\Big|_{u=\bar{u}} = G_u(\bar{u},\check{v}(\bar{u})) + G_v(\bar{u},\check{v}(\bar{u}))\frac{d\check{v}}{du}(\bar{u}),$$

which by $(A.2f)$ and Lemma 16,

$$= 0.$$

Consequently,

$$\bar{u} \in \operatorname*{argstat}_{u \in \mathscr{A}_G} \bar{G}^1(u) \text{ and } \bar{G}^1(\bar{u}) \in \operatorname*{stat}^s_{u \in \mathscr{A}_G} \bar{G}^1(u). \tag{3.43}$$

Suppose $\hat{u} \neq \bar{u}$ is such that

$$\hat{u} \in \operatorname*{argstat}_{u \in \mathscr{A}_G} \bar{G}^1(u). \tag{3.44}$$

Then, by $(A.2f)$, there exists $\hat{v} \in \mathscr{M}^1(\hat{u})$. Recalling that G is uniformly locally Morse in v, and applying the implicit mapping theorem again, we find that there exists $\varepsilon' > 0$ and $\breve{v}' \in C^1(B_{\varepsilon'}(\hat{u}); \mathscr{B})$ such that $B_{\varepsilon'}(\hat{u}) \subseteq \mathscr{A}_G$ and

$$\breve{v}'(\hat{u}) = \hat{v} \text{ and } G_v(u, \breve{v}'(u)) = 0 \quad \forall u \in B_{\varepsilon'}(\hat{u}) \subseteq \mathscr{A}_G. \tag{3.45}$$

Then, by (3.44), another application of the chain rule and (3.45),

$$0 = \frac{d}{du} \bar{G}^1(u)\big|_{u=\hat{u}} = G_u(\hat{u}, \breve{v}'(\hat{u})) + G_v(\hat{u}, \breve{v}'(\hat{u})) \frac{d\breve{v}'}{du}(\hat{u}) = G_u(\hat{u}, \hat{v}). \tag{3.46}$$

By (3.45) and (3.46), $(\hat{u}, \hat{v}) \in \operatorname{argstat}_{(u,v) \in \mathscr{A} \times \mathscr{B}} G(u, v)$, and hence by $(A.2f)$,

$$G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v}). \tag{3.47}$$

Recalling from (3.7) that $\hat{v} \in \operatorname{argstat}_{v \in \mathscr{B}} g^{1,\hat{u}}(v)$, and using (3.47), we have

$$\bar{G}^1(\hat{u}) = g^{1,\hat{u}}(\hat{v}) = G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v}).$$

As $\hat{u} \in \operatorname{argstat}_{u \in \mathscr{A}_G} \bar{G}^1(u) \setminus \{\bar{u}\}$ was arbitrary, we have the desired result. $\square$

Theorem 11. *Assume* $(A.2f)$. *Then,* $\operatorname{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u)$ *exists, and*

$$\operatorname*{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{(u,v) \in \mathscr{U} \times \mathscr{V}} G(u, v).$$

Proof. The assertion of the existence of $\operatorname{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u)$ is simply Lemma 18. Then, noting that Lemma 17 implies that Assumption $(A.3)$ is satisfied, one may apply Theorem 6 to obtain the second assertion of the theorem. $\square$

When the iterated staticization is known to exist

We suppose $(A.2i)$. We will find that $\text{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$ exists, and obtain the equivalence between full and iterated staticization.

Lemma 19. *Assume* $(A.2i)$. *Then,* $\text{stat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$ *exists.*

Proof. By $(A.2i)$, (3.8), the uniform Morse property and the implicit mapping theorem, there exists $\delta > 0$ and $\breve{v} \in C^1(B_\delta(\bar{u});\mathscr{B})$ such that $B_\delta \subseteq \mathscr{A}_G$,

$$\breve{v}(\bar{u}) = \bar{v} \;\text{ and }\; G_v(u,\breve{v}(u)) = 0 \quad \forall u \in B_\delta(\bar{u}). \tag{3.48}$$

By the differentiability of $\breve{v}$, $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1}{du}(\bar{u}) = G_u(\bar{u},\breve{v}(\bar{u})) + G_v(\bar{u},\breve{v}(\bar{u}))\frac{d\breve{v}}{du}(\bar{u}) = G_u(\bar{u},\bar{v}) + G_v(\bar{u},\bar{v})\frac{d\breve{v}}{du}(\bar{u}).$$

Using $(A.2i)$ and (3.8), this implies $0 = G_u(\bar{u},\bar{v})$, and hence $(\bar{u},\bar{v}) \in \text{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v)$.

Now suppose there exists $(\hat{u},\hat{v}) \in \text{argstat}_{(u,v)\in\mathscr{A}\times\mathscr{B}} G(u,v) \setminus \{(\bar{u},\bar{v})\}$, which implies

$$G_u(\hat{u},\hat{v}) = 0 \;\text{ and }\; G_v(\hat{u},\hat{v}) = 0. \tag{3.49}$$

By (3.49), $(A.1)$, the uniform Morse property and the implicit mapping theorem, there exists $\delta' > 0$ and $\breve{v}' \in C^1(B_{\delta'}(\hat{u});\mathscr{B})$ such that $B_{\delta'}(\hat{u}) \subseteq \mathscr{A}_G$,

$$\breve{v}'(\hat{u}) = \hat{v} \;\text{ and }\; G_v(u,\breve{v}'(u)) = 0 \quad \forall u \in B_{\delta'}(\hat{u}). \tag{3.50}$$

Further, combining the definition of $\mathscr{A}_G$ and (3.50), we see that

$$\bar{G}^1(u) = \operatorname*{stat}_{v\in\mathscr{B}} g^{1,u}(v) = G(u,\breve{v}'(u)) \quad \forall u B_{\delta'}(\hat{u}). \tag{3.51}$$

Then, by (3.50), (3.51), $(A.1)$ and the chain rule,

$$\frac{d\bar{G}^1(\hat{u})}{du}(\hat{u}) = G_u(\hat{u}, \breve{v}'(\hat{u})) + G_u(\hat{u}, \breve{v}'(\hat{u}))\frac{d\breve{v}'}{du}(\hat{u}),$$

which by (3.49) and the definition of $\breve{v}'(u)$,

$$= 0.$$

That is, $\hat{u} \in \mathrm{argstat}_{u \in \mathscr{A}_G} \bar{G}^1(u)$, and using $(A.2i)$, this implies $\bar{G}^1(\hat{u}) = \bar{G}^1(\bar{u})$. Combining this with (3.50) and (3.51), we see that

$$G(\hat{u}, \hat{v}) = \bar{G}^1(\hat{u}) = \bar{G}^1(\bar{u}).$$

and then by the definition of $\bar{G}^1$ and (3.8), this is

$$= g^{1,\bar{u}}(\bar{v}) = G(\bar{u}, \bar{v}).$$

As $(\hat{u}, \hat{v}) \in \mathrm{argstat}_{(u,v) \in \mathscr{A} \times \mathscr{B}}$ was arbitrary,

$$G(\hat{u}, \hat{v}) = G(\bar{u}, \bar{v}) \text{ for all } (\hat{u}, \hat{v}) \in \operatorname*{argstat}_{(u,v) \in \mathscr{A} \times \mathscr{B}} G(u, v).$$

$\square$

By Lemma 19 and Theorem 11 we have the following.

Theorem 12. *Assume* $(A.2i)$. *Then,* $\mathrm{argstat}_{(u,v) \in \mathscr{A} \times \mathscr{B}} G(u, v)$ *exists, and*

$$\operatorname*{stat}_{(u,v) \in \mathscr{U} \times \mathscr{V}} G(u, v) = G(\bar{u}, \bar{v}) = \operatorname*{stat}_{u \in \mathscr{A}_G} \bar{G}^1(u).$$

3.5 Application to Astrodynamics

As noted in the introduction, there are two classes of problems in dynamical systems that have motivated the above development. The first class consists of TPBVPs in Astrodynamics, and we discuss that here. Specifically, one may obtain fundamental solutions to TPBVPs in astrodynamics through a stationary-action based approach [18, 19, 38, 40]. We briefly recall the case of the n-body problem. In this case, the action functional with an appended terminal cost (cf. [40]) takes the form indicated in (3.1), where now $x = \big((x^1)^T, (x^2)^T, \dots (x^n)^T\big)^T$, where each $x^j \in \mathbb{R}^3$ denotes a generic position of body j for $j \in \mathscr{N} \doteq \{1,2,\dots n\}$, and $\xi_\cdot$, $u_\cdot$ of (3.1) are similarly constructed. The kinetic-energy term is $T(u_r) \doteq \frac{1}{2}\sum_{j=1}^n m_j |u_r^j|^2$, where m_j is the mass of the j^{th} body.

Suppose $x^i \neq x^j$ for all $i \neq j$. Then, the additive inverse of the potential is given by

$$
\begin{aligned}
-V(x) = \sum_{(i,j)\in\mathscr{I}^\Delta} \frac{\Gamma m_i m_j}{|x^i - x^j|} &= \max_{\alpha\in\mathscr{M}_{(0,\infty)}} \sum_{(i,j)\in\mathscr{I}^\Delta} \left(\tfrac{3}{2}\right)^{3/2} \Gamma m_i m_j \left[\alpha_{i,j} - \frac{\alpha_{i,j}^3 |x^i - x^j|^2}{2}\right] \\
&\doteq \max_{\alpha\in\mathscr{M}_{(0,\infty)}} \left[-\tilde{V}(x,\alpha)\right] = -\tilde{V}(x,\bar{\alpha}),
\end{aligned} \tag{3.52}
$$

where Γ is the universal gravitational constant, $\mathscr{I}^\Delta \doteq \{(i,j) \in \mathscr{N}^2 \,|\, j > i\}$, $\mathscr{M}_{(0,\infty)}$ denotes the set of arrays indexed by $(i,j) \in \mathscr{I}^\Delta$ with elements in $(0,\infty)$, and $\bar{\alpha}_{i,j} = \bar{\alpha}_{i,j}(x) = \left[2/(3|x^i - x^j|^2)\right]^{1/2}$ for all $(i,j) \in \mathscr{I}^\Delta$; see [40]. Recalling the discussion in Section 3.1, we note that solutions of stationary-action problems with these kinetic and potential energy functions will yield solutions of TPBVPs for the n-body dynamics. Letting $\mathscr{U}_{0,t} \doteq L_2\big((0,t);\mathbb{R}^{3n}\big)$, one finds that the problem becomes that of finding the stationary-action value function given by

$$
W(t,x) = \operatorname*{stat}_{u\in\mathscr{B}} J^0(t,x,u), \tag{3.53}
$$

where

$$
J^0(t,x,u) \doteq \int_0^t T(u_r) - V(\xi_r)\, dr + \phi(\xi_t) = \int_0^t T(u_r) + \max_{\alpha\in\mathscr{M}_{(0,\infty)}} \left[-\tilde{V}(x,\alpha)\right] dr + \phi(\xi_t),
$$

$$\mathscr{B} \subseteq \{u \in \mathscr{U}_{0,t} | \quad \forall (i,j) \in \mathscr{I}^\Delta, \text{ for a.e. } r \in (0,t), |\xi_r^i - \xi_r^j| \neq 0\}. \tag{3.54}$$

Remark 5. *Throughout the discussion to follow, we assume that $W(t,x)$ given by (3.53) exists. In particular, we assume that $\mathscr{B}$ is open, and that there exists $\bar{u} \in \mathscr{B}$ such that* $\operatorname{argstat}_{u\in\mathscr{B}} J^0(t,x,u) = \{\bar{u}\}$. *In the case where the problem corresponds to a TPBVP, this amounts to an assumption that if there are multiple solutions to the TPBVP, then the solutions are isolated, cf. [18, 19, 40].*

Let $\tilde{\mathscr{A}}_{(0,t)} \doteq C\big((0,t);\mathscr{M}_{(0,\infty)}\big)$ and $\tilde{\mathscr{A}}^B_{(0,t)} \doteq C\big((0,t);\mathscr{M}_{\mathbb{R}}\big)$, where $\mathscr{M}_{\mathbb{R}}$ denotes the set of arrays indexed by $(i,j) \in \mathscr{I}^\Delta$ with elements in $\mathbb{R}$, and where we note that the former is a subset of the latter, which is a Banach space.

Lemma 20. *Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$ and $\mathscr{B} \subseteq \mathscr{U}_{0,t}$. Then,*

$$W(t,x) = \operatorname*{stat}_{u\in\mathscr{B}} \operatorname*{stat}_{\tilde{\alpha}\in\tilde{\mathscr{A}}_{(0,t)}} J(t,x,u,\tilde{\alpha}),$$

where

$$J(t,x,u,\tilde{\alpha}) \doteq \int_0^t T(u_r) - \tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr + \phi(\xi_t). \tag{3.55}$$

Further, if $\mathscr{A} \subset \tilde{\mathscr{A}}_{(0,t)}$ is open and such that $\bar{\tilde{\alpha}}^{i,j} \in \mathscr{A}$ where $\bar{\tilde{\alpha}}_r^{i,j} = \bar{\alpha}_{i,j}(\xi_r)$ for all $(i,j) \in \mathscr{I}^\Delta$ and a.e. $r \in (0,t)$, where $\xi_r = x + \int_0^r u_\rho\,d\rho$, then $W(t,x) = \operatorname{stat}_{u\in\mathscr{B}} \operatorname{stat}_{\tilde{\alpha}\in\mathscr{A}} J(t,x,u,\tilde{\alpha}) =$ $\operatorname{stat}_{u\in\mathscr{B}} J(t,x,u,\bar{\tilde{\alpha}})$.

Proof. Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$, $u \in \mathscr{B} \subseteq \mathscr{U}_{0,t}$ and $\mathscr{A} = \tilde{\mathscr{A}}_{(0,t)}$. By [40, Theorem 4.7], we find $J^0(t,x,u) = \max_{\tilde{\alpha}\in\mathscr{A}} J(t,x,u,\tilde{\alpha})$, where $J(t,x,u,\tilde{\alpha})$ is given by (3.55). Noting that $J(t,x,u,\cdot)$ is differentiable and strictly convex then yields $J^0(t,x,u) = \operatorname{stat}_{\tilde{\alpha}\in\mathscr{A}} J(t,x,u,\tilde{\alpha})$. Combining this with (3.53) yields the first assertion. The second assertion then follows by noting the argmax of (3.52). □

If one is able to reorder the stat operations, then the stat representation of Lemma 20 may be decomposed as

$$W(t,x) \doteq \underset{\tilde{\alpha}\in\mathscr{A}}{\operatorname{stat}} \tilde{W}(t,x,\tilde{\alpha}), \tag{3.56}$$

$$\tilde{W}(t,x,\tilde{\alpha}) \doteq \underset{u\in\mathscr{B}}{\operatorname{stat}} \left\{ \int_0^t T(u_r) - \tilde{V}(\xi_r,\tilde{\alpha}_r)\,dr + \phi(\xi_t) \right\}. \tag{3.57}$$

Further, suppose ϕ is a quadratic form, say

$$\phi(x) = \phi(x;z) \doteq \tfrac{1}{2}(x-z)^T P_0 (x-z) + \gamma_0, \tag{3.58}$$

where $z \in \mathbb{R}^{3n}$ and P_0 is symmetric, positive-definite. Then, the argument of stat in (3.57) will be quadratic in u, and we will have

$$\tilde{W}(t,x,\tilde{\alpha}) = \tfrac{1}{2}(x^T P_t^{\tilde{\alpha}} x + x^T Q_t^{\tilde{\alpha}} z + z^T Q_t^{\tilde{\alpha}} x + z^T R_t^{\tilde{\alpha}} z + \gamma_t^{\tilde{\alpha}}), \tag{3.59}$$

where $P_\cdot^{\tilde{\alpha}}, Q_\cdot^{\tilde{\alpha}}, R_\cdot^{\tilde{\alpha}}$ may be obtained from solution of $\tilde{\alpha}$-indexed DREs, and $\gamma_t^{\tilde{\alpha}}$ is obtained from an integral [34, 40]. It will now be demonstrated that in the case of quadratic ϕ, we may reorder the stat operators.

Remark 6. *We remark that different forms of ϕ may be used such that payoffs (3.55) (which will be shown to be equivalent to (3.56)) correspond to different TPBVPs for the n-body problem; see Section 3.1 and [40]. The means by which this may be utilized for efficient generation of fundamental solutions is indicated in [18, 19, 40].*

Remark 7. *It can be shown that for sufficiently short time intervals, $J^0(t,x,\cdot)$ is convex and coercive, and one then has $W(t,x) = \min_{u\in\mathscr{B}} \max_{\tilde{\alpha}\in\mathscr{A}} J(t,x,u,\tilde{\alpha})$ for appropriate $\mathscr{A},\mathscr{B}$. In that case, one also finds that $W(t,x) = \max_{\tilde{\alpha}\in\mathscr{A}} \min_{u\in\mathscr{B}} J(t,x,u,\tilde{\alpha})$, and one proceeds similarly to the case here. That is, one again has (3.59), where the coefficients satisfy DREs. See [40] for the details. Here, we will employ the reordering of iterated stat operations to obtain $W(t,x)$ in a similar form, i.e., in the form (3.56).*

Lemma 21. *Let $x \in \mathbb{R}^{3n}$, $t \in (0,\infty)$ and $\tilde{\alpha} \in \tilde{\mathscr{A}}_{(0,t)}$. Suppose ϕ has the form (3.58). Then,*

$$J(t,x,u,\tilde{\alpha}) \doteq f_1(\tilde{\alpha}) + \langle f_2(\tilde{\alpha}), u\rangle_{\mathscr{U}_{0,t}} + \tfrac{1}{2}\langle \bar{B}_3(\tilde{\alpha})u, u\rangle_{\mathscr{U}_{0,t}} \quad \forall u \in \mathscr{U}_{0,t},$$

where $f_1(\tilde{\alpha}) \in \mathbb{R}$, $f_2(\tilde{\alpha}) \in \mathscr{U}_{0,t}$ and $\bar{B}_3(\tilde{\alpha}) \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$.

Proof. Using (3.52) and (3.55), we see that

$$J(t,x,u,\tilde{\alpha}) = \int_0^t \tfrac{1}{2}\sum_{j=1}^n m_j |u_r^j|^2 + \sum_{(i,j)\in\mathscr{I}^\Delta} \left(\tfrac{3}{2}\right)^{3/2} \Gamma m_i m_j \left[\tilde{\alpha}_r^{i,j} - \frac{(\tilde{\alpha}_r^{i,j})^3 |\xi_r^i - \xi_r^j|^2}{2}\right] dr + \phi(\xi_t). \tag{3.60}$$

Note that for the kinetic-energy term, we have the Riesz representation

$$\int_0^t \tfrac{1}{2}\sum_{j=1}^n m_j |u_r^j|^2 dr = \tfrac{1}{2}\langle Q_1 u, u\rangle_{\mathscr{U}_{0,t}}, \tag{3.61}$$

where the operator $Q_1 \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$ is given by $[Q_1 u]_r \doteq \bar{Q}_1 u_r$ for all $r \geq 0$, and $\bar{Q}_1$ is the $3n\times 3n$ block-diagonal matrix with blocks $m_1 I_3, m_2 I_3, \ldots m_n I_3$.

Let $\hat{\Gamma} \doteq \left(\tfrac{3}{2}\right)^{3/2}\Gamma$. Similarly, we have find that the potential term in J may be decomposed as

$$\begin{aligned}
&\hat{\Gamma}\sum_{(i,j)\in\mathscr{I}^\Delta} m_i m_j \int_0^t \left[\tilde{\alpha}_r^{i,j} - \left(\tilde{\alpha}_r^{i,j}\right)^3 \frac{|\xi_r^i - \xi_r^j|^2}{2}\right] dr \\
&= \hat{\Gamma}\sum_{(i,j)\in\mathscr{I}^\Delta} -m_i m_j \int_0^t \left[\left(\tilde{\alpha}_r^{i,j}\right)^3 \frac{|\int_0^r u_\rho^i d\rho|^2 + |\int_0^r u_\rho^j d\rho|^2 - 2\left(\int_0^r u_\rho^i d\rho\right)^T \int_0^r u_\tau^j d\tau}{2}\right] dr \\
&\quad + \hat{\Gamma}\sum_{(i,j)\in\mathscr{I}^\Delta} -m_i m_j \int_0^t \left[\left(\tilde{\alpha}_r^{i,j}\right)^3 \frac{2(x^i - x^j)^T \left(\int_0^r u_\rho^i d\rho\right) + 2(x^j - x^i)^T \left(\int_0^r u_\rho^j d\rho\right)}{2}\right] dr \\
&\quad + \hat{\Gamma}\sum_{(i,j)\in\mathscr{I}^\Delta} m_i m_j \int_0^t \left[\tilde{\alpha}_r^{i,j} - \left(\tilde{\alpha}_r^{i,j}\right)^3 \frac{|x^i|^2 + |x^j|^2 - 2(x^i)^T x^j}{2}\right] dr \qquad (3.62) \\
&\doteq \tfrac{1}{2}\langle Q_2(\tilde{\alpha})u, u\rangle_{\mathscr{U}_{0,t}} + \langle R_2(\tilde{\alpha}), u\rangle_{\mathscr{U}_{0,t}} + S_2(\tilde{\alpha}) \quad \forall u \in \mathscr{U}_{0,t},
\end{aligned}$$

where we will obtain explicit expressions for $Q_2(\tilde{\alpha}) \in L(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$, $R_2(\tilde{\alpha}) \in \mathscr{U}_{0,t}$ and $S_2(\tilde{\alpha}) \in \mathbb{R}$. Considering a single generic component inside the first summation on the right-hand side of (3.62), note that

$$\int_0^t \left(\tilde{\alpha}_r^{i,j}\right)^3 \Big(\int_0^r u_\rho^i d\rho\Big)^T \int_0^r u_\tau^j d\tau\, dr = \int_0^t \int_0^t \int_0^t \mathscr{I}_{(0,r)}(\rho)\mathscr{I}_{(0,r)}(\tau)\left(\tilde{\alpha}_r^{i,j}\right)^3 \left(u_\rho^i\right)^T u_\tau^j\, d\rho\, d\tau\, dr,$$

where generically, $\mathscr{I}_{\mathscr{C}}$ denotes the indicator function on set $\mathscr{C}$, and that this is

$$= \int_0^t \int_0^t \int_0^t \mathscr{I}_{(\rho,t)}(r)\mathscr{I}_{(\tau,t)}(r)\left(\tilde{\alpha}_r^{i,j}\right)^3 \left(u_\rho^i\right)^T u_\tau^j\, dr\, d\rho\, d\tau,$$

$$= \int_0^t \left(u_\rho^i\right)^T \Big\{ \int_0^t \Big[\int_{\rho\vee\tau}^t \left(\tilde{\alpha}_r^{i,j}\right)^3 dr \Big] u_\tau^j\, d\tau \Big\}\, d\rho$$

Combining all these generic terms and rearranging our choice of dummy variables, we find that for all $u \in \mathscr{U}_{0,t}$, $[Q_2(\tilde{\alpha})u]_r = \int_0^t [\bar{Q}_2(\tilde{\alpha})](r,\tau)u_\tau\, d\tau$, where $[\bar{Q}_2(\tilde{\alpha})](r,\tau)$ is given as follows. For $i,j \in]1,n[$ such that $i \neq j$, let

$$[\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,j} \doteq \hat{\Gamma} m_i m_j \int_{\tau\vee r}^t \left(\tilde{\alpha}_\sigma^{i,j}\right)^3 d\sigma,$$

and for $i \in]1,n[$, let

$$[\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,i} \doteq -\sum_{j\in]1,n[,\, j\neq i} [\hat{Q}_2(\tilde{\alpha})](r,\tau)]_{i,j}.$$

Then, $[\bar{Q}_2(\tilde{\alpha})](r,\tau) = [\hat{Q}_2(\tilde{\alpha})](r,\tau)] \otimes I_3$, where $\otimes$ denotes the Konecker product here.

Proceeding similarly, we find that $R_2(\tilde{\alpha}) \in \mathscr{U}_{0,t}$ has the Riesz representation

$$R_2(\tilde{\alpha}) = \left(([\hat{R}_2(\tilde{\alpha})(r)]_1)^T, ([\hat{R}_2(\tilde{\alpha})(r)]_2)^T, \ldots ([\hat{R}_2(\tilde{\alpha})(r)]_n)^T \right)^T$$

where for $i \in]1,n[$,

$$[\hat{R}_2(\tilde{\alpha})(r)]_i = -\hat{\Gamma} \sum_{j\neq i} m_i m_j \int_r^t \left(\tilde{\alpha}_\tau^{i,j}\right)^3 d\tau\, (x^i - x^j).$$

For the zeroth order in the expansion of the integral of the potential term, we have

$$S_2(\tilde{\alpha}) = \sum_{(i,j)\in\mathscr{I}^\Delta} \hat{\Gamma} m_i m_j \int_0^t \left[\tilde{\alpha}_r^{i,j} - \left(\tilde{\alpha}_r^{i,j}\right)^3\right] dr\, \frac{|x^i|^2 + |x^j|^2 - 2(x^i)^T x^j}{2}.$$

Now, we turn to the terminal cost. Recalling (3.58), we have

$$\begin{aligned}\phi(\xi_t) &= \tfrac{1}{2}\Big(\int_0^t u_\rho d\rho\Big)^T P_0\Big(\int_0^t u_\rho d\rho\Big) + \big(x-z\big)^T P_0\Big(\int_0^t u_\rho d\rho\Big) + \tfrac{1}{2}(x-z)^T P_0(x-z) + \gamma_0 \\ &\doteq \tfrac{1}{2}\langle Q_3 u, u\rangle_{\mathscr{U}_{0,t}} + \langle R_3, u\rangle_{\mathscr{U}_{0,t}} + S_3,\end{aligned}$$

where $Q_3 \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$, $R_3 \in \mathscr{U}_{0,t}$ and $S_3 \in \mathbb{R}$. In particular, we have $[Q_3 u]_r = P_0 \int_0^t u_\rho\, d\rho$ and $[R_3]_r = P_0(x-z)$ for all $r \in (0,t)$, and $S_3 = \frac{1}{2}(x-z)^T P_0(x-z) + \gamma_0$. Combining the terms, we have the asserted form for $J(t,x,u,\tilde{\alpha})$, where

$$f_1(\tilde{\alpha}) = S_2(\tilde{\alpha}) + S_3, \quad f_2(\tilde{\alpha}) = R_2(\tilde{\alpha}) + R_3, \text{ and } \bar{B}_3(\tilde{\alpha}) = Q_1 + Q_2(\tilde{\alpha}) + Q_3.$$

That $\bar{B}_3(\tilde{\alpha}) \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$ and $f_2(\tilde{\alpha}) \in \mathscr{U}_{0,t}$ is easily seen from the above expressions. □

Theorem 13. *Let $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$. Suppose $W(t,x)$ given by (3.53) exists. Let $\bar{\tilde{\alpha}}^{i,j} \in \tilde{\mathscr{A}}_{(0,t)}$ be as in Lemma 20, and $D > |\bar{B}_3^\#(\bar{\tilde{\alpha}})|$. Let $\mathscr{A} \doteq \{\tilde{\alpha} \in \tilde{\mathscr{A}}_{(0,t)} \,|\, |\bar{B}_3^\#(\tilde{\alpha})| < D\}$. Then,*

$$W(t,x) = \operatorname*{stat}_{u\in\mathscr{B}} \operatorname*{stat}_{\tilde{\alpha}\in\mathscr{A}} J(t,x,u,\tilde{\alpha}) = \operatorname*{stat}_{(u,\tilde{\alpha})\in\mathscr{B}\times\mathscr{A}} J(t,x,u,\tilde{\alpha}) = \operatorname*{stat}_{\tilde{\alpha}\in\mathscr{A}} \operatorname*{stat}_{u\in\mathscr{B}} J(t,x,u,\tilde{\alpha}).$$

Proof. Fix $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$. Note that by the conditions of Remark 5, $\mathscr{B}$ is open. By Lemma 21, $\bar{B}_3(\tilde{\alpha}) \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$ for all $\tilde{\alpha} \in \tilde{\mathscr{A}}_{(0,t)}$, where this implies that all such $\bar{B}_3(\tilde{\alpha})$ are closed operators, and hence $[\bar{B}_3^\#(\tilde{\alpha})] \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$ exists for all $\tilde{\alpha} \in \tilde{\mathscr{A}}_{(0,t)}$. Let $g : \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t}) \to \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$ be given by $g(B) \doteq B^\#$ for all $B \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t})$. Let D be as given and $\hat{D} \in (D,\infty)$. Let the open ball of radius D be denoted by $\mathscr{D}_D \doteq \{B \in \mathscr{L}(\mathscr{U}_{0,t};\mathscr{U}_{0,t}) \,|\, |B| < D\}$, and similarly for $\hat{D}$. Let $Q_D \doteq g^{-1}(\mathscr{D}_D)$ and $Q_{\hat{D}} \doteq g^{-1}(\mathscr{D}_{\hat{D}})$, and note that g is continuous on $Q_{\hat{D}}$ [20, 50]. Hence, Q_D is open, and as $\bar{B}_3(\cdot)$ is continuous, we find that $\mathscr{A} = (\bar{B}_3)^{-1}(Q_D)$ is open. The first asserted equality then follows from Lemma 20. Further, this implies that Assumption $(A.2i)$ is satisfied by the expression on the right-hand side of the first equality. Hence, if the conditions of Section 3.4.3 are met, then Theorem 12 will yield the second equality. In this case here, the Morse condition of Section 3.4.3 is that for all $(\tilde{\alpha}, u) \in \mathscr{A} \times \mathscr{B}$, $D^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) \in \mathscr{L}(\tilde{\mathscr{A}}^B_{(0,t)}; \tilde{\mathscr{A}}^B_{(0,t)})$ is invertible with locally bounded inverse. From Lemma 27, the differential $D^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma$ for $\gamma \in \mathscr{A}^B_{(0,t)}$ has representation with components given by

$$[\nabla^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma]^{i,j}_r = -3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \quad \forall (i,j) \in \mathscr{I}^\Delta, \text{ a.e. } r \in (0,t).$$

As $\tilde{\alpha}^{i,j}_r, |\xi^i_r - \xi^j_r| > 0$ for all $(i,j) \in \mathscr{I}^\Delta$ and $r \in (0,t)$, one finds that operator $D^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ is indeed invertible with locally bounded inverse for all $(\tilde{\alpha}, u) \in \mathscr{A} \times \mathscr{B}$. Lastly, noting the representation given in Lemma 28, one may easily show that $D^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ is bounded on bounded sets. Hence, the conditions of Section 3.4.3 are met, and one may apply Theorem 12 to obtain the second equality.

Note that the second equality also implies that the expression on the right-hand side of that equality satisfies Assumption $(A.2f)$. If the conditions of Theorem 9 are satisfied, we will have the final equality. It is sufficient to show that, as a function of $(\tilde{\alpha}, u) \in \mathscr{A} \times \mathscr{B}$, $J(t,x,u,\tilde{\alpha})$ satisfies the conditions of Section 3.4.2. That is, suppressing the dependence on (t,x), we must have

$$J(t,x,u,\tilde{\alpha}) = f_1(\tilde{\alpha}) + \langle f_2(\tilde{\alpha}), u \rangle_{\mathscr{U}_{0,t}} + \tfrac{1}{2} \langle \bar{B}_3(\tilde{\alpha}) u, u \rangle_{\mathscr{U}_{0,t}},$$

with $f_1, f_2, \bar{B}_3$ satisfying the conditions indicated there. From Lemma 21, we see that $f_1, f_2, \bar{B}_3$ are C^2, and $\bar{B}^{\#}_3(\tilde{\alpha})$ exists and is uniformly bounded over $\mathscr{A}$. The result follows from Theorem 9. □

Remark 8. *It should be noted that the assertions of Theorem 13 allow the staticization problem of (3.53) to be reduced to staticization over the set of DRE solutions and integrals,* $\mathscr{P} \doteq$

$\{(P_t^{\tilde{\alpha}}, Q_t^{\tilde{\alpha}}, R_t^{\tilde{\alpha}}, \gamma_t^{\tilde{\alpha}}) \,|\, \tilde{\alpha} \in \mathscr{A}\}$*, as noted in (3.59). In cases where the terminal cost,* ϕ*, (indexed by z) has been constructed so that the staticization problems correspond to TPBVPs, the set* $\mathscr{P}$ *provides a fundamental solution object for a set of TPBVPs. One may see [18, 19, 40] for more detailed discussions regarding the calculations.*

3.6 Schrödinger IVPs

We indicate that application to Schrödinger IVPs. The general outline is similar to that of the previous subsection, but where the dynamics are now stochastic and complex-valued. In order to simplify matters, in this case we consider only the problem of a single particle in a central Coulomb field. The Schrödinger IVP is

$$0 = i\hbar\psi_t(s,y) + \tfrac{\hbar^2}{2m}\Delta\psi(s,y) - \psi(s,y)V(y), \;(s,y) \in \mathscr{D},$$
$$\psi(0,y) = \psi_0(y), \quad y \in \mathbb{R}^n,$$

where $m \in (0,\infty)$ denotes particle mass, $\hbar$ denotes the Planck constant, initial condition ψ_0 takes values in $\mathbb{C}$, V denotes the Coulomb potential function, Δ denotes the Laplacian with respect to the space (second) variable, $\mathscr{D} \doteq (0,t) \times \mathbb{R}^n$, and subscript t will denote the derivative with respect to the time variable (the first argument of ψ here) regardless of the symbol being used for time in the argument list. We consider what is sometimes referred to as the Maslov dequantization of the solution of the Schrödinger equation (cf. [29]), which is $S : \bar{\mathscr{D}} \to \mathbb{C}$ given by $\psi(s,y) = \exp\{\frac{i}{\hbar}S(s,y)\}$. We also extend the space from $\mathbb{R}^n$ to $\mathbb{C}^n$, and reverse the time variable. The resulting transformed problem is given by [33, 35, 37]

$$0 = S_t(s,x) + \tfrac{i\hbar}{2m}\Delta S(s,x) + H(x, S_x(s,x)), (s,x) \in \mathscr{D}_{\mathbb{C}} \doteq (0,t) \times \mathbb{C}^n, \tag{3.63}$$
$$S(t,x) = \phi(x), \quad x \in \mathbb{C}^n, \tag{3.64}$$

where $H : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is the Hamiltonian given by

$$H(x,p) \doteq -\left[\tfrac{1}{2m}|p|_c^2 + V(x)\right] = \underset{u^0\in\mathbb{C}^n}{\text{stat}} \left\{(u^0)^T p + \tfrac{m}{2}|u^0|_c^2 - V(x)\right\},$$

where for $x \in \mathbb{C}^n$, $|x|_c^2 \doteq \sum_{j=1}^n x_j^2$. (We remark that notation $|\cdot|_c^2$ is not intended to indicate a squared norm; the range is complex.) We fix $t \in (0,\infty)$, and allow s to vary in $(0,t]$.

Under certain conditions, the solution of this dequantized form of the Schrödinger IVP has a representation in the form of the value function of staticization controlled diffusion equation [37]. In particular, we suppose the solution satisfies $|S_{xx}| \le C(1+|x|^{2q})$ for some $q \in \mathbb{N}$. We let $(\Omega, \mathscr{F}, P)$ be a probability triple, where Ω denotes a sample space, $\mathscr{F}$ denotes a σ-algebra on Ω, and P denotes a probability measure on $(\Omega,\mathscr{F})$. Let $\{\mathscr{F}_s \,|\, s\in[0,t]\}$ denote a filtration on $(\Omega,\mathscr{F},P)$, and let $B_\cdot$ denote an $\mathscr{F}_\cdot$-adapted Brownian motion taking values in $\mathbb{R}^n$. For $s \in [0,t]$, let

$$\mathscr{U}_s \doteq \{u\colon [s,t]\times\Omega \to \mathbb{C}^n \,|\, u \text{ is } \mathscr{F}_\cdot\text{-adapted, right-continuous and such that}$$
$$\mathbb{E}\textstyle\int_s^t |u_r|^m\,dr < \infty\ \forall m\in\mathbb{N}\}.$$

We supply $\mathscr{U}_s$ with the norm $\|u\|_{\mathscr{U}_s} \doteq \max_{m\in\{1,\dots\bar{M}\}} \left[\mathbb{E}\int_s^t |u_r|^m\,dr\right]^{1/m}$, where $\bar{M} \ge 8q$. We will be interested in diffusion processes given by

$$\begin{aligned}\xi_r = \xi_r^{(s,x)} &= x + \int_s^r u_\rho\,d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}}\int_s^r dB_\rho \\ &\doteq x + \int_s^r u_\rho\,d\rho + \sqrt{\tfrac{\hbar}{m}}\tfrac{1+i}{\sqrt{2}} B_r^\Delta,\end{aligned}$$

where $x\in\mathbb{C}^n$, $s\in[0,t]$, $u\in\mathscr{U}_s$, and $B_r^\Delta \doteq B_r - B_s$ for $r\in[s,t]$. For $s\in(0,t)$ and $\hbar\in(0,1]$, we define payoff $J(s,\cdot,\cdot)\colon \mathbb{C}^n\times\mathscr{U}_s \to \mathbb{C}$ and stationary value, $\bar{S}\colon \mathscr{D}_\mathbb{C}\to\mathbb{C}$ by

$$J(s,x,u) \doteq \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 - V(\xi_r)\,dr + \phi(\xi_t)\right\},$$
$$S(s,x) \doteq \underset{u\in\mathscr{U}_s}{\text{stat}}\, J(s,x,u) \quad \forall (s,x)\in\mathscr{D}_\mathbb{C}.$$

The Coulomb potential generated by a point charge in the central field takes the form of $\bar{V}(x) = -\hat{\mu}/|x|, \quad \forall x \in \mathbb{R}^n$, where $\hat{\mu}$ is a constant. This may be extended to the complex domain as (abusing notation)

$$V(x) \doteq -\hat{\mu}/\sqrt{|x|_c^2} = -\bar{c} \operatorname*{stat}_{\bar{\alpha}\in\mathscr{A}^R} \{\alpha - \tfrac{\bar{\alpha}^3|x|_c^2}{2}\} \quad \forall x \in \mathbb{C}^n$$

where $\bar{c} \doteq (\frac{3}{2})^{3/2}\hat{\mu}$ and $\mathscr{A}^R \doteq \{\bar{\alpha} = r[\cos(\theta) + i\sin(\theta)] \in \mathbb{C} \,|\, r \geq 0,\, \theta \in (\frac{-\pi}{2}, \frac{\pi}{2}]\}$.

Lemma 22. *Let* $\mathscr{A} \doteq L^2(\Omega; L^2([s,t]; \mathscr{A}^R))$. *Then*

$$\operatorname*{stat}_{\alpha\in\mathscr{A}} \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\,dr + \phi(\xi_t)\right\} = \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c}\operatorname*{stat}_{\bar{\alpha}\in\mathscr{A}^R}[\bar{\alpha} - \frac{\bar{\alpha}^3|\xi_2|_c^2}{2}]\,dr + \phi(\xi_t)\right\}.$$

The problem of solving for $S(s,x)$ then becomes that of finding the stationary-action value function given by

$$S(s,x) \doteq \operatorname*{stat}_{u\in\mathscr{U}_s} \operatorname*{stat}_{\alpha\in\mathscr{A}} \mathbb{E}\Big\{\int_s^t \tfrac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\,dr + \phi(\xi_t)\Big\} \quad \forall (s,x) \in \mathscr{D}_{\mathbb{C}}.$$

which may be decomposed as

$$\begin{aligned}
S(t,x) &\doteq \operatorname*{stat}_{\alpha\in\mathscr{A}} \tilde{S}(t,x,\tilde{\alpha}),\\
\tilde{S}(t,x,\alpha) &\doteq \operatorname*{stat}_{u\in\mathscr{U}_s} \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 - \tilde{V}(\xi_r,\alpha_r)\,dr + \phi(\xi_t)\right\}
\end{aligned}$$

where $\tilde{V}(\xi_r,\alpha_r) \doteq -\bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]$. Again, the problem becomes that of interchanging the order of the staticization operators, where we note that the functional is semi-quadratic in u and uniformly Morse in $\tilde{\alpha}$. Once that is achieved, one has

$$\begin{aligned}
S(t,x) &\doteq \operatorname*{stat}_{\alpha\in\mathscr{A}} \tilde{S}(t,x,\tilde{\alpha}),\\
\tilde{S}(t,x,\alpha) &\doteq \operatorname*{stat}_{u\in\mathscr{U}_s} \mathbb{E}\left\{\int_s^t \tfrac{m}{2}|u_r|_c^2 - \tilde{V}(\xi_r,\alpha_r)\,dr + \phi(\xi_t)\right\}
\end{aligned}$$

where $\tilde{V}(\xi_r, \alpha_r) \doteq -\bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]$. As the $\tilde{S}$ value function is that of a linear-quadratic problem for each $\tilde{\alpha}$, it may be solved through solution of a set of associated DREs. We specifically require the following.

Theorem 14. *The functional given by* $\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\,dr + \phi(\xi_t)\right\}$ *is twice Fréchet differentiable in* α.

For x and u taking values in $\mathbb{R}^3$, we have the following results.

Theorem 15. $\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|^2 dr + \int_s^t \frac{\bar{c}}{\sqrt{|\xi_r|_c^2}} dr + \phi(\xi_t)\right\}$ *is differentiable with respect to u everywhere.*

This theorem follows from the following lemmas.

Lemma 23. *There exists a probability measure Q such that Q and P are mutually absolutely continuous, and* $d\xi_s = \sqrt{\frac{\hbar}{m}}\frac{1+i}{\sqrt{2}}d\hat{B}_s$, *where* $\hat{B}_\cdot$ *is a* $Q-$*Brownian motion.*

Proof. This follows from the Girsanov Theorem. □

Theorem 16. *Let X be an open subset of* $\mathbb{C}$, *and* Ω *be a measure space. Let* $f : X \times \Omega$ *satisfy*

(1) $f(x,\omega)$ *is Lebesgue integrable in* ω *for each* $x \in X$.

(2) For a.e. ω, $f_x(x,\omega)$ *exists for a.e. x.*

(3) There exists $\theta : \Omega \to \mathbb{C}$ *s.t.* $|f_x(x,\omega)| \leq \theta(\omega)\ \forall x \in X$, *a.e.* ω.

Then $\frac{d}{dx}\int_\Omega f(x,\omega)d\omega = \int_\Omega f_x(x,\omega)d\omega$.

Lemma 24. *The function*

$$f(\bar{x}) \doteq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\frac{1}{\sqrt{(2\pi)^3}|\sigma|}\exp\left(-\frac{x^2+y^2+z^2}{2\sigma^2}\right)}{\sqrt{(x-\bar{x})^2+y^2+z^2}}dxdydz$$

is differentiable with respect to $\bar{x}$.

Proof. After changing to a spherical coordinates centered at $\bar{x}$, the result follows from an application of the Leibniz rule for Lebesgue integrals. $\square$

Lemma 25. *For each $r \in [s,t]$, $f(r,u) \doteq \mathbb{E}\left\{\frac{1}{\sqrt{|\xi_r|_c^2}}\right\}$ is differentiable with respect to u.*

Proof. Without loss of generality, we consider $x = (x,0,0) \in \mathbb{R}^3$. After a change of measure as in lemma 23, we have $f(r,u) = \mathbb{E}^Q\{1/\sqrt{|x+\sqrt{\frac{\hbar}{m}}\frac{1+i}{\sqrt{2}}\hat{B}_r|_c^2}\}$. Let $\bar{x} \doteq -x$. By lemma 24, the chain rule, and noting that $x + \int_s^r u_\rho d\rho$ is an affine functional of u, we have $f(r,u)$ is differentiable in u. $\square$

Note that so far we've only proved that $\mathbb{E}\left\{\int_s^t \frac{m}{2}|u_r|_c^2 + \bar{c}[\alpha_r - \frac{\alpha_r^3|\xi_2|_c^2}{2}]\,dr + \phi(\xi_t)\right\}$ is C^1 in u. Further work is needed in order to apply the results that follow from $(A.1)$, where the functional is C^2 in both variables.

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3.7 Appendix: Calculation of Derivatives

We begin by indicating some notation, and recalling standard results, cf. [2]. Let $f : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)} \to \mathbb{R}$ satisfy $f(u,\cdot) \in C^2(\tilde{\mathscr{A}}^B_{(0,t)};\mathbb{R})$, $f(\cdot,\tilde{\alpha}) \in C^2(\mathscr{U}_{0,t};\mathbb{R})$ for all $u \in \mathscr{U}_{0,t}$, $\tilde{\alpha} \in \tilde{\mathscr{A}}^B_{(0,t)}$. Let $D_u f : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)} \to \mathscr{L}(\mathscr{U}_{0,t};\mathbb{R})$ and $D_{\tilde{\alpha}} f : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)} \to \mathscr{L}(\tilde{\mathscr{A}}^B_{(0,t)};\mathbb{R})$ denote the Fréchet derivatives with respect to u and $\tilde{\alpha}$, respectively. Note that we have $[D_u f(u,\tilde{\alpha})]\delta_u \in \mathbb{R}$, $[D_\alpha f(u,\tilde{\alpha})]\delta_{\tilde{\alpha}} \in \mathbb{R}$, $\forall \delta_u \in \mathscr{U}_{0,t}$, $\delta_{\tilde{\alpha}} \in \tilde{\mathscr{A}}^B_{(0,t)}$. By the Riesz representation theorem, for each $\hat{u} \in \mathscr{U}_{0,t}$ and $\hat{\tilde{\alpha}} \in \tilde{\mathscr{A}}^B_{(0,t)}$, there exists unique $\nabla_u f(\hat{u},\hat{\tilde{\alpha}}) \in \mathscr{U}_{0,t}$ such that $D_u f(\hat{u},\hat{\tilde{\alpha}})\delta_u = \langle \delta_u, \nabla_u f(\hat{u},\hat{\tilde{\alpha}})\rangle_{\mathscr{U}_{0,t}} \quad \forall \delta_u \in \mathscr{U}_{0,t}$.

For $L \in L_2((0,t);\mathscr{M}_{\mathbb{R}})$ and $\gamma \in \tilde{\mathscr{A}}^B_{(0,t)}$, define the continuous, bilinear functional $\langle L,\gamma\rangle_2 = \langle \gamma, L\rangle_2 \doteq \sum_{(i,j)\in\mathscr{I}^\Delta} \int_0^t L_r^{i,j}\gamma_r^{i,j}\,dr$. Note that $\nabla_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}}) : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)} \to \tilde{\mathscr{A}}^B_{(0,t)}$, is a representation of $D_\alpha f(\hat{u},\hat{\tilde{\alpha}})\delta_{\tilde{\alpha}}$ everywhere in $\mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)}$ if $\langle \nabla_{\tilde{\alpha}} f(\hat{u},\hat{\tilde{\alpha}}), \delta_{\tilde{\alpha}}\rangle_2 = D_\alpha f(\hat{u},\hat{\tilde{\alpha}})\delta_{\tilde{\alpha}}$ for all $\delta_{\tilde{\alpha}} \in \tilde{\mathscr{A}}^B_{(0,t)}$, $(\hat{u},\hat{\tilde{\alpha}}) \in \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)}$.

Let $D^2_{\tilde{\alpha}} f : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}^B_{(0,t)} \to \mathscr{L}(\tilde{\mathscr{A}}^B_{(0,t)}, \mathscr{L}(\tilde{\mathscr{A}}^B_{(0,t)}, \mathbb{R}))$ denote the second Fréchet derivative with respect to $\tilde{\alpha}$. Note that for each $\delta_{\tilde{\alpha}} \in \tilde{\mathscr{A}}^B_{(0,t)}$ and pair $(\hat{u}, \hat{\tilde{\alpha}})$, we have $D^2_{\tilde{\alpha}} f(\hat{u}, \hat{\tilde{\alpha}})\delta_{\tilde{\alpha}} \in \mathscr{L}(\tilde{\mathscr{A}}^B_{(0,t)}; \mathbb{R})$. Further, $D^2_{\tilde{\alpha}} f(\hat{u}, \hat{\tilde{\alpha}})$ is the second Fréchet derivative with respect to $\tilde{\alpha}$ at $(\hat{u}, \hat{\tilde{\alpha}})$ if $D^2_{\tilde{\alpha}} f(\hat{u}, \hat{\tilde{\alpha}}) = D_{\tilde{\alpha}}[D_{\tilde{\alpha}} f](\hat{u}, \hat{\tilde{\alpha}})$. Analogous definitions hold for second derivatives with respect to u.

We now proceed to obtain certain derivatives and Riesz representations employed in the proof of Theorem 13. Let $J : (0,t) \times \mathbb{R}^{3n} \times \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}_{(0,t)}$ be given by (3.55) with quadratic terminal cost (3.58).

Lemma 26. *For any $t \in (0,\infty)$, $x \in \mathbb{R}^{3n}$ and $u \in \mathscr{U}_{0,t}$, $J(t,x,u,\cdot)$ is Fréchet differentiable over $\tilde{\mathscr{A}}^B_{(0,t)}$, and the Fréchet derivative has Riesz representation $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$, where $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ acting on $\gamma \in \tilde{\mathscr{A}}^B_{(0,t)}$ is given by $\langle \nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}), \gamma \rangle_2$, and*

$$[\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]^{i,j}_r = \hat{\Gamma} m_i m_j \Big[1 - \frac{3\big(\tilde{\alpha}^{i,j}_r\big)^2 |\xi^i_r - \xi^j_r|^2}{2} \Big] \quad \forall (i,j) \in \mathscr{I}^\Delta, r \in (0,t). \tag{3.65}$$

Proof. Let $\gamma \in \tilde{\mathscr{A}}^B_{(0,t)}$, and let L denote object indicated by the right-hand side of (3.65). With a small amount of algebra, one finds

$$\begin{aligned}
&|J(t,x,u,\tilde{\alpha}+\gamma) - J(t,x,u,\tilde{\alpha}) - \langle L, \gamma \rangle_2| \\
&= \left| \hat{\Gamma} \sum_{(i,j)\in\mathscr{I}^\Delta} \int_0^t \frac{-m_i m_j}{2} \big[3\tilde{\alpha}^{i,j}_r \big(\gamma^{i,j}_r\big)^2 + \big(\gamma^{i,j}_r\big)^3 \big] |\xi^i_r - \xi^j_r|^2 \, dr \right| \\
&\le \hat{\Gamma} \sum_{(i,j)\in\mathscr{I}^\Delta} \frac{m_i m_j}{2} \int_0^t \big(1 + 3\tilde{\alpha}^{i,j}_r\big) |\xi^i_r - \xi^j_r|^2 \, dr \sup_{r\in(0,t)} \big[\big|\gamma^{i,j}_r\big|^2 + \big|\gamma^{i,j}_r\big|^3 \big],
\end{aligned}$$

which for appropriate choice of $K_0(t,x,u,\tilde{\alpha}) < \infty$ and $|\gamma| \le 1$,

$$\le K_0(t,x,u,\tilde{\alpha}) |\gamma|^2,$$

which implies that the Fréchet derivative $D_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ exists, and has the indicated Riesz representation. □

Lemma 27. *For any $t \in (0,\infty)$, $x \in \mathbb{R}^{3n}$ and $u \in \mathscr{U}_{0,t}$, the second order Fréchet derivative $D^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$ exists for all $\tilde{\alpha} \in \mathscr{A}_{(0,t)}$, and the differential has representation $\nabla^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma$, which for all $\gamma \in \tilde{\mathscr{A}}^B_{(0,t)}$, is given by*

$$[\nabla^2_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})\gamma]^{i,j}_r = -3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \quad \forall (i,j) \in \mathscr{I}^\Delta, \text{ a.e. } r \in (0,t).$$

Proof. Recalling the above discussion, we obtain the second-derivative representation by examining the Fréchet derivative of $\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$. Let t,x,u be as specified, and take $\tilde{\alpha} \in \mathscr{A}_{(0,t)}$. Let $\gamma \in \tilde{\mathscr{A}}^B_{(0,t)}$, and let $[T\gamma]^{i,j}_r \doteq -3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r$ for all $i,j \in]1,n[$ and $r \in (0,t)$, where $\xi^i_r = x^i + \int_0^r u_\rho \, d\rho$. Note that

$$\begin{aligned}
&|\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}+\gamma) - \nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}) - [T\gamma]| \\
&= \left[\sum_{(i,j)\in\mathscr{I}^\Delta} \int_0^t \left| [\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha}+\gamma)]^{i,j}_r - [\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]^{i,j}_r + 3\hat{\Gamma} m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \right|^2 dr \right]^{1/2},
\end{aligned}$$

which by (3.65),

$$\begin{aligned}
&= \left[\hat{\Gamma} \sum_{(i,j)\in\mathscr{I}^\Delta} \int_0^t \left| \frac{-3}{2} m_i m_j \left[2\tilde{\alpha}^{i,j}_r \gamma^{i,j}_r + (\gamma^{i,j}_r)^2 \right] \left| \xi^i_r - \xi^j_r \right|^2 + 3 m_i m_j \tilde{\alpha}^{i,j}_r |\xi^i_r - \xi^j_r|^2 \gamma^{i,j}_r \right|^2 dr \right]^{1/2} \\
&= \left[\hat{\Gamma} \sum_{(i,j)\in\mathscr{I}^\Delta} \frac{9}{4} m_i^2 m_j^2 \int_0^t \left| (\gamma^{i,j}_r)^2 \left| \xi^i_r - \xi^j_r \right|^2 \right|^2 dr \right]^{1/2} \\
&\le \hat{\Gamma} \sum_{(i,j)\in\mathscr{I}^\Delta} \frac{9}{4} m_i^2 m_j^2 \Big(\int_0^t \left| \xi^i_r - \xi^j_r \right|^4 dr \Big)^{1/2} \sup_{r\in(0,t)} \left| \gamma^{i,j}_r \right|^2 \le K_1 |\gamma|^2,
\end{aligned}$$

for appropriate choice of $K_1 = K_1(t,x,u) < \infty$, and this yields the result. □

The following is obtained in a similar manner to Lemma 26, and the proof is not included.

Lemma 28. *For any $t \in (0,\infty)$ and $x \in \mathbb{R}^{3n}$, $J(t,x,\cdot,\cdot) : \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}_{(0,t)} \to \mathbb{R}$ has a mixed second partial Fréchet derivative, and this derivative, evaluated at $(u,\tilde{\alpha}) \in \mathscr{U}_{0,t} \times \tilde{\mathscr{A}}_{(0,t)}$, $D^2_{u,\tilde{\alpha}} J(t,x,u,\tilde{\alpha})$, has a representation comprised of the Riesz representations of the derivatives of $[\nabla_{\tilde{\alpha}} J(t,x,u,\tilde{\alpha})]^{i,j}_\cdot$*

with respect to u for $(i,j)\in\mathscr{I}^\Delta$. *More specifically, for* $\delta_u\in\mathscr{U}_{0,t}$ *and* $\delta_{\tilde\alpha}\in\tilde{\mathscr{A}}_{(0,t)}$,

$$\left[D^2_{u,\tilde\alpha}J(t,x,u,\tilde\alpha)\delta_{\tilde\alpha}\right]\delta_u=\left\langle\nabla^2_{u,\tilde\alpha}J(t,x,u,\tilde\alpha)\delta_{\tilde\alpha},\delta_u\right\rangle_{\mathscr{U}_{0,t}}=\sum_{k\in\mathscr{N}}\int_0^t\left[\nabla^2_{u,\tilde\alpha}J(t,x,u,\tilde\alpha)\delta_{\tilde\alpha}\right]^k_\rho[\delta_u]^k_\rho\,d\rho,$$

where

$$\left[\nabla^2_{u,\tilde\alpha}J(t,x,u,\tilde\alpha)\delta_{\tilde\alpha}\right]^k_\rho=\sum_{(i,j)\in\mathscr{I}^\Delta}\int_0^t\left[[\nabla_{\tilde\alpha,u}J(t,x,u,\tilde\alpha)]^{i,j}_r\right]^k_\rho[\delta_{\tilde\alpha}]^{i,j}_r\,dr\quad\forall k\in\mathscr{N},\,\rho\in(0,t),$$

$$\left[[\nabla_{\tilde\alpha,u}J(t,x,u,\tilde\alpha)]^{i,j}_r\right]^k_\rho\doteq\begin{cases}-3\hat\Gamma m_im_j(\tilde\alpha^{i,j}_r)^2(\xi^i_r-\xi^j_r)\mathscr{I}_{(0,r)}(\rho) & \textit{if } k=i,\\ 3\hat\Gamma m_im_j(\tilde\alpha^{i,j}_r)^2(\xi^i_r-\xi^j_r)\mathscr{I}_{(0,r)}(\rho) & \textit{if } k=j,\\ 0 & \textit{otherwise}\end{cases}$$

for all $r,\rho\in(0,t)$, $k\in\mathscr{N}$ *and* $(i,j)\in\mathscr{I}^\Delta$, *and we recall that* $\mathscr{I}_{(0,r)}(\cdot)$ *denotes the indicator function on set* $(0,r)$.

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R. Zhao, *Proceedings of the 12th Asian Control Conference*, 2019. The dissertation author is a co-investigator and co-author of this article.

Chapter 4

Strong Solutions for a Class of Degenerate SDEs

4.1 Introduction

Diffusion processes have long been a useful tool in the study of Hamilton-Jacobi partial differential equations (HJ PDEs). Recently, the use of stationary-action controlled diffusion process representation has been proved useful in solving the Schrödinger initial value problems (IVPs), cf. [32, 35, 43]. In this paper, we will prove the existence of strong solutions for a class of degenerate stochastic differential equations (SDEs) that arises in the staticization based diffusion representation for the solution of Schrödinger IVP associated with the Coulomb potential. In existing work on the existence of strong solutions of SDEs, the drift term may have singularities, but the diffusion coefficient of diffusion processs is assumed to be non-degenerate (cf. [26, 55]). The SDEs we consider in this paper have $2m \times m$ degenerate diffusion coefficients and may have discontinuities in the drift term on a manifold of codimension 1 and singularities along a manifold of codimension 2. We first demonstrate the existence of weak solutions of the SDE, which is obtained through passing to the limit of a sequence of diffusion processes with C^1 drifts. After that, the existence of a unique strong solution follows from pathwise uniqueness property of the process. The class of SDEs we will consider are of the form

$$d\eta_t = F(\eta_t, \zeta_t)dt + dB_t, \tag{4.1}$$

$$d\zeta_t = G(\eta_t, \zeta_t)dt, \quad t \in [0,T], \tag{4.2}$$

$$\eta_0 = y^0 \in \mathbb{R}^m, \; \zeta_0 = z^0 \in \mathbb{R}^m, \; l \doteq 2m. \tag{4.3}$$

where F, G are continuous functions outside a bounded region in $\mathbb{R}^l$ but have discontinuities inside the region. More specifically, for $\delta \geq 0$, let $\mathscr{G}_\delta \subset \mathbb{R}^l$ be such that $\mathscr{G}_{\delta_1} \subseteq \mathscr{G}_{\delta_2}$ for all $0 \leq \delta_1 \leq \delta_2$, and $\lim_{\delta \downarrow 0} \mathscr{G}_\delta = \mathscr{G}_0$. Let $h_0, h_1 \in C(\mathbb{R}^m; \mathbb{R})$, and

$$\mathscr{H}_0 \doteq \{(y,z) \in \mathbb{R}^l | h_0(y) - h_1(z) = 0\}. \tag{4.4}$$

We assume the following.

$$\begin{aligned} &F, G \in C^1([\mathscr{G}_0 \cup \mathscr{H}_0]^c). \\ &\text{For each } \delta > 0,\ F \text{ and } G \text{ are bounded on } \mathscr{G}_\delta^c. \\ &\text{For each } \delta > 0,\ \nabla_{(y,z)} F \text{ and } \nabla_{(y,z)} G \text{ are bounded on } [\mathscr{G}_\delta \cup \mathscr{H}_0]^c. \end{aligned} \tag{A.1}$$

If for sample point $\omega \in \Omega$, each component, say $[\eta_\cdot]_j(\omega)$ for $1 \leq j \leq m$, of sample path $\eta_\cdot(\omega)$ has infinite total variation on $[a,b] \subseteq [0,T]$, then $h_0(\eta_\cdot(\omega))$ has infinite total variation on $[a,b]$. $(A.2)$

Let $\bar{\mathscr{L}}$ denote the space of nonsingular $m \times m$ matrices, and let $I_{m\times m} \in \bar{\mathscr{L}}$ denote the identity matrix.

Let $\mathscr{I} \doteq [0,1]$, and let $p \in C^1(\mathscr{I}^0; \mathbb{R}^m) \cap C(\mathscr{I}; \mathbb{R}^m)$. Let $\bar{e} \in \mathbb{R}^m \setminus \{0\}$. Let $J \in C^2(\mathbb{R}^m \setminus \{0\}; \bar{\mathscr{L}})$ be given by $J(z) = (1/|z|)\Gamma(z)$ where $\Gamma(z)$ is orthonormal for all $z \in \mathbb{R}^m \setminus \{0\}$, and such that $J(z)z = \bar{e}$ for all $z \in \mathbb{R}^m \setminus \{0\}$, $[J(\bar{e})]^{-1} = I_{m\times m}$, and $\frac{dJ}{dz}$ is bounded on $\mathbb{R}^m \setminus B_\delta(0)$ for all $\delta > 0$. Finally, suppose $\mathscr{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, y \in \tilde{\mathscr{G}}_0(z)\}$ where for all $z \in \mathbb{R}^m$, $\tilde{\mathscr{G}}_0(z) \doteq \{y \in \mathbb{R}^m \,|\, \exists \lambda \in \mathscr{I} \text{ s.t. } y = [J(z)]^{-1} p(\lambda)\}$. $(A.3)$

An additional assumption will appear in Section 4.3, after some additional definitions, and it will be the final assumption.

Remark 9. *The above structure for $\mathscr{G}_0$, which may at first seem unusual, was chosen for the case where the singular set is defined in terms of η_t relative to ζ_t. A motivational example where these assumptions are satisfied is given in Section 4.2. In that case, $m = 3$, $\mathscr{H}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0\}$ and $\mathscr{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y|^2 - |z|^2 = 0,$ and $y^T z = 0\}$. In that case, one may take $\bar{e}$ to be $(1,0,0)^T$ and $p(\cdot)$ to be a parameterization of the unit circle in the plane perpendicular to $\bar{e}$.*

Remark 10. *The assumptions may be weakened to allow for a finite number of both discontinuity and singularity manifolds, with no fundamental change in the proofs. For clarity of exposition, we do not include the details.*

4.2 Motivation from the Schrödinger Initial Value Problem

We briefly discuss how the SDEs of (4.1)- (4.3) are related to the Schrödinger IVP. Recall that the Schrödinger IVP has the form

$$0 = i\hbar\psi_t(s,y) + \frac{\hbar^2}{2m}\Delta\psi(s,y) - \psi(s,y)V(y),\ (s,y) \in \mathscr{D}^{\mathbb{R}}. \tag{4.5}$$

$$\psi(0,y) = \psi_0(y),\ y \in \mathbb{R}^m, \tag{4.6}$$

where i is the imaginary unit, $\hbar$ is the Planck's constant, $m > 0$ denotes mass, $\mathscr{D}^{\mathbb{R}} \doteq (0,t) \times (\mathbb{R}^m \backslash \{0\})$, ψ, ψ_0 take values in $\mathbb{C}$, the subscript t denotes derivative with respect to the time (first) variable regardless of what letter is used for the variable, and Δ denotes the Laplacian of ψ in space (second) variables. We also let $\bar{\mathscr{D}}^{\mathbb{R}} \doteq (0,t] \times (\mathbb{R}^m \backslash \{0\})$. Employing the Maslov dequantization $\psi(s,y) \doteq \exp\{\frac{i}{\hbar}S(s,y)\}$ and reversing the time variable,we find that the equations

(4.5), (4.6) become the equivalent HJ PDE backward dynamic programming problem:

$$0 = S_t(r,x) + \tfrac{i\hbar}{2m}\Delta S(r,x) + \operatorname*{stat}_{v\in\mathbb{C}^n}[S_x(r,x)^T v + \tfrac{m}{2}|v|_c^2 - V(x)] \quad (r,x)\in\mathscr{D}^{\mathbb{R}} \tag{4.7}$$

$$S(t,x) = \phi(x), \quad x\in\mathbb{R}^m, \tag{4.8}$$

where the stat operator denotes the search for stationary values. Namely, Let $\mathscr{F}$ denote either the real or complex field, and suppose $\mathscr{U}$ is a normed vector space (over $\mathscr{F}$) with $\mathscr{A}\subseteq\mathscr{U}$, $\bar{G}:\mathscr{A}\to\mathscr{F}$. We say $\bar{u}\in\operatorname{argstat}_{u\in\mathscr{A}}\bar{G}(u)\doteq\operatorname{argstat}\{\bar{G}(u)\,|\,u\in\mathscr{A}\}$ if $\bar{u}\in\mathscr{A}$ and either

$$\limsup_{u\to\bar{u},u\in\mathscr{A}\setminus\{\bar{u}\}} \frac{|\bar{G}(u)-\bar{G}(\bar{u})|}{|u-\bar{u}|} = 0, \tag{4.9}$$

or there exists $\delta>0$ such that $\mathscr{A}\cap B_\delta(\bar{u}) = \{\bar{u}\}$ (where $B_\delta(\bar{u})$ denotes the ball of radius δ around $\bar{u}$). Since $S(s,y)$ takes value in $\mathbb{C}$, to allow complex-valued state processes, we find it convenient to extend the domain of the problem from $\mathbb{R}^m$ to $\mathbb{C}^m$. Consider the case when $m=3$. The Coulomb potential generated by a charge at the origin has the form $-V(y) = C/|y|$, for $y\in\mathbb{R}^3\setminus\{0\}$, where C is a constant. Let

$$\mathscr{C}_1 \doteq \{x\in\mathbb{C}^3 | \exists r\in(0,\infty), \omega\in(-\pi,\pi],\ s.t.\ |x|_c^2 \doteq r^2e^{2iw}\}$$

$$\mathscr{A}^q \doteq \{\alpha\in\mathbb{C} | \exists r_\alpha\in(0,\infty), \theta_\alpha\in(-\pi,\pi],\ s.t.\ \alpha = r_\alpha e^{i\theta_\alpha}\},$$

where $|x|_c^2 \doteq \sum_{j=1}^m x_j^2$ for $x\in\mathbb{C}^m$. We have (cf. [39]) for $x\in\mathscr{C}_1$,

$$-V(y) = \hat{C}\operatorname*{stat}_{\alpha\in\mathscr{A}^q}\big[\alpha - \frac{\alpha^3|x|_c^2}{2}\big], \tag{4.10}$$

where $\hat{C}\doteq(\frac{3}{2})^{(3/2)}C$. Extending the domain to $\mathbb{C}^m$, the equations (4.7), (4.8) become

$$0 = S_t(r,x) + \tfrac{i\hbar}{2m}\Delta S(r,x) + \operatorname*{stat}_{v\in\mathbb{C}^n}[S_x(r,x)^T v + \tfrac{m}{2}|v|_c^2 - V(x)] \quad (r,x)\in\mathscr{D}_1 \tag{4.11}$$

$$S(t,x) = \phi(x), \quad x \in \mathscr{C}_1, \tag{4.12}$$

where $\mathscr{D}_1 \doteq \{(0,t) \times \mathbb{C}^3\}$. One may check that the function $S^0 : [0,\infty) \times \mathbb{C}^3 \to \mathbb{C}$ given by

$$S^0(t,x) = \frac{-c_1^2}{2m} t + i c_1 \sqrt{x^T x},$$

is a solution of (4.11), (4.12), where $\sqrt{x} \doteq \exp\{\frac{1}{2}\log(x)\}$, $\log(x) \doteq \mathrm{Log}(r) + i\theta$ for $r \in (0,\infty)$, $\theta \in (-\pi,\pi]$. We remark that S^0 is derived from the solution of the Schrödinger IVP for the lowest energy shell ([10]).

The dynamics of the diffusion process generating the solution as the associated stationary value function are given by [39] $d\xi_r = (-1/\bar{m}) S_x^0(r,\xi_r)\,dr + \sqrt{\hbar/\bar{m}}\frac{1+i}{\sqrt{2}}\,dB_r$ with $\xi_0 = x$. One may separate the three-dimensional complex state, ξ_r, into its real and imaginary parts as $\xi_r = \hat{\eta}_r + i\hat{\zeta}_r$. Similarly, letting $S^0(r,x) = R^0(r,\hat{y},\hat{z}) + iT^0(r,\hat{y},\hat{z})$ with $x = \hat{y} + i\hat{z}$, and employing the Cauchy-Riemann equations, the SDE system becomes

$$\begin{aligned} d\hat{\eta}_r &= (-1/\bar{m}) R_y^0(r,\hat{\eta}_r,\hat{\zeta}_r)\,dr + \sqrt{\tfrac{\hbar}{2\bar{m}}}\,dB_r, \quad \hat{\eta}_0 = \hat{y}, \\ d\hat{\zeta}_r &= (1/\bar{m}) R_z^0(r,\hat{\eta}_r,\hat{\zeta}_r)\,dr + \sqrt{\tfrac{\hbar}{2\bar{m}}}\,dB_r, \quad \hat{\zeta}_0 = \hat{z}. \end{aligned}$$

Performing the change of coordinates $\eta_r = (1/\sqrt{2})[\hat{\eta}_r + \hat{\zeta}_r]$, $\zeta_r = (1/\sqrt{2})[-\hat{\eta}_r + \hat{\zeta}_r]$ yields

$$\begin{aligned} d\eta_r &= (1/\sqrt{2}\bar{m})[-R_y^0 + R_z^0](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\,dr + \sqrt{\tfrac{\hbar}{\bar{m}}}\,dB_r, \\ d\zeta_r &= (1/\sqrt{2}\bar{m})[R_y^0 + R_z^0](r, \tfrac{\eta_r - \zeta_r}{2}, \tfrac{\eta_r + \zeta_r}{2})\,dr, \end{aligned}$$

with $\eta_0 = y^0 \doteq (1/\sqrt{2})[\hat{y} + \hat{z}]$ and $\zeta_0 = z^0 \doteq (1/\sqrt{2})[-\hat{y} + \hat{z}]$. Using the specific form of S^0 in this example, this reduces to

$$d\eta_r = F(\eta_r,\zeta_r)\,dr + \sigma\,dB_r \tag{4.13}$$

$$\begin{aligned}
&\doteq \frac{c_1}{\bar{m}\sqrt{\tilde{R}_r}}[\sin(\tilde{\theta}_r)\eta_r - \cos(\tilde{\theta}_r)\zeta_r]\,dr + \sqrt{\tfrac{\hbar}{\bar{m}}}\,dB_r,\\
d\zeta_r &= G(\eta_r,\zeta_r)\,dr \qquad (4.14)\\
&\doteq \frac{-c_1}{\bar{m}\sqrt{\tilde{R}_r}}[\cos(\tilde{\theta}_r)\eta_r + \sin(\tilde{\theta}_r)\zeta_r]\,dr,
\end{aligned}$$

where $\tilde{R}_r \doteq \bar{R}(\eta_r,\zeta_r) \doteq \left[(-2\eta_r^T\zeta_r)^2 + (|\eta_r|^2 - |\zeta_r|^2)^2\right]^{1/2}$, $\cos(2\tilde{\theta}_r) = \frac{-2\eta_r^T\zeta_r}{\tilde{R}_r}$ and $\sin(2\tilde{\theta}_r) = \frac{|\eta_r|^2-|\zeta_r|^2}{\tilde{R}_r}$ with $\tilde{\theta}_r \in (-\pi/2, \pi/2]$.

In this case, $\mathscr{H}_0$ corresponds to the branch cut induced by $\sqrt{x^T x}$, which is at $|\hat{y}|^2 - |\hat{z}|^2 < 0$, $\hat{y}^T\hat{z} = 0$, or equivalently, at $y^T z > 0$, $|y|^2 - |z|^2 = 0$. That is, $\mathscr{H}_0 = \{(y,z) \in \mathbb{R}^l \,|\, |y| = |z|,\ y^T z > 0\}$. In particular, one may take $h_0(y) = |y|$ and $h_1(z) = |z|$. From this, one may easily verify Assumption $(A.2)$. Also, we see that the singularities occur on

$$\mathscr{G}_0 = \{(y,z) \in \mathbb{R}^l \,|\, \bar{R}(y,z) = 0\} = \{(y,z) \in \mathbb{R}^l \,|\, y^T z = 0 \text{ and } |y| = |z|\}.$$

If $\mathscr{G}_\delta$ is defined to be the set of points in $\mathbb{R}^l$ whose distance from $\mathscr{G}_0$ is at most δ, one easily finds that Assumption $(A.1)$ is satisfied. Lastly, to see that Assumption $(A.3)$ is satisfied, note that one may take $\tilde{\mathscr{G}}_0(z) \doteq \{y \in \mathbb{R}^m \,|\, y^T z = 0 \text{ and } |y| = |z|\}$. Note that if $z = (1,0,0)^T$, then $\tilde{\mathscr{G}}_0(z)$ is the unit circle in the (z_2,z_3)–plane. Hence, one may take $p(\lambda) \doteq (0, \cos(2\pi\lambda), \sin(2\pi\lambda))$ and $\bar{e} = (1,0,0)^T$. Then, for $z \in \mathbb{R}^m \setminus \{0\}$, one may then let

$$\Gamma(z) \doteq \begin{bmatrix} u^T \\ v^T \\ w^T \end{bmatrix} \quad \text{where } u \doteq \frac{z}{|z|}, \quad e_k \text{ denotes the } k^{th} \text{ standard basis vector in } \mathbb{R}^3,$$

$$\hat{v} \doteq \sum_{k=1}^{2} u \times e_k, \quad v = \frac{\hat{v}}{|\hat{v}|} \text{ and } w \doteq \frac{u \times v}{|u \times v|}.$$

One may then easily verify Assumption $(A.3)$.

We remark that another example, again associated to a classical energy shell, is given by

$$S^1(t,x) \doteq \frac{-c_{1,1}^2}{2m} t + ic_{1,1}\sqrt{x^T x} - i\hbar \log(x_1),$$

where $c_{1,1} \doteq \frac{mC}{2\hbar}$. In this case there are additional discontinuity and singularity manifolds. In particular, in addition to $\mathscr{H}_0$ and $\mathscr{G}_0$ from the S^0, we also have $\mathscr{H}_{0,1} \doteq \{(y,z) \in \mathbb{R}^l \,|\, z_1 = 0 \text{ and } y_1 < 0\}$ and $\mathscr{G}_{0,1} \doteq \{(y,z) \in \mathbb{R}^l \,|\, y_1 = z_1 = 0\}$.

4.3 The $\delta > 0$ Prelimit

We smooth the dynamics as follows. For $\delta > 0$, let $g^\delta, \tilde{g}^{\delta/4,\delta} \in C^\infty(\mathbb{R})$ be given by

$$g^\delta(\rho) \doteq \begin{cases} 1 - \exp\left\{\frac{1}{\delta^2} + \frac{1}{\rho^2 - \delta^2}\right\} & \text{if } |\rho| \in [0,\delta], \\ 1 & \text{if } |\rho| > \delta, \end{cases} \tag{4.15}$$

$$\tilde{g}^{\delta/4,\delta}(\rho) \doteq \begin{cases} 0 & \text{if } |\rho| \in [0,\delta/4], \\ g^{3\delta/4}(|\rho| - \delta/4) & \text{if } |\rho| > \delta/4. \end{cases} \tag{4.16}$$

Defining $\hat{R}(y,z) \doteq d\big((y,z),\mathscr{G}_0\big)$, we let

$$F^\delta(y,z) \doteq g^\delta(\hat{R}(y,z))F(y,z), \text{ and } G^\delta(y,z) \doteq \tilde{g}^{\delta/4,\delta}(|z|)g^\delta(\hat{R}(y,z))G(y,z) \tag{4.17}$$

for all $(y,z) \in \mathbb{R}^l$. Note that

$$F^\delta = F \text{ and } G^\delta = G \text{ on } \mathscr{G}_\delta^c \tag{4.18}$$

Our final assumption is that for each $\delta > 0$,

$$
\begin{aligned}
&F^{\delta}, G^{\delta} \in C^1(\mathscr{H}_0^c), \quad F^{\delta} \text{ and } G^{\delta} \text{ are bounded on } \mathbb{R}^l, \\
&\text{and } \nabla_{(y,z)} F^{\delta} \text{ and } \nabla_{(y,z)} G^{\delta} \text{ are bounded on } \mathscr{H}_0^c.
\end{aligned} \qquad (A.4)
$$

Note that $(A.4)$ holds for the examples given in Section 4.2, and that it will hold more generally when the dynamics are bounded by the the multiplicative inverse of appropriate polynomial forms.

Suppose $(y^0, z^0) \notin \mathscr{G}_0$. Consider the system with modified dynamics given in integral form as

$$
\eta_t^{\delta} = y^0 + \int_0^t F^{\delta}(\eta_r^{\delta}, \zeta_r^{\delta})\, dr + B_t, \qquad (4.19)
$$

$$
\zeta_t^{\delta} = z^0 + \int_0^t G^{\delta}(\eta_r^{\delta}, \zeta_r^{\delta}),\, dr \qquad (4.20)
$$

for $t \in [0,T]$. We demonstrate existence and uniqueness of a strong solution via application of the Girsanov transform approach to first obtain existence of a weak solution, followed by a demonstration of pathwise uniqueness to then obtain the strong-solution assertion.

Lemma 29. *Suppose η^{δ} is a brownian motion on probability space $(\Omega, \bar{\mathscr{F}}, \hat{P})$ where Ω, $\bar{\mathscr{F}}$ and $\hat{P}$ denote a sample space, σ-algebra and probability measure, and with filtration denoted by $\mathscr{F}_{\cdot}$. Let ζ^{δ} be continuous and of bounded variation on $[0,T]$. Then, for a.e. $\omega \in \Omega$, $\mu(\{t \in [0,T] \,|\, (\eta_t^{\delta}, \zeta_t^{\delta}) \in \mathscr{H}_0\}) = 0$, where μ denotes Lebesgue measure.*

Proof. Let $\eta^{\delta}, \zeta^{\delta}$ be as given. Let $\eta^{\delta}(\omega)$ denote a sample path of the brownian motion. By Assumption $(A.2)$, for $\omega \in \Omega^0$ with $\hat{P}(\Omega^0) = 1$, the total [linear] variation of $h_0(\eta^{\delta}(\omega))$ on any interval $[t_1, t_2] \in [0,T]$ with $t_2 > t_1$ is $\bar{T}(\eta^{\delta}(\omega); t_1, t_2) = \infty$. Let $\mathscr{A}(\omega) \doteq \{t \in (0,T) \,|\, h_0(\eta_t^{\delta}) - h_1(\zeta_t^{\delta}) = 0\}$, which implies that $\mathscr{A}^c(\omega) = \{t \in (0,T) \,|\, h_0(\eta_t^{\delta}) - h_1(\zeta_t^{\delta}) \in [(-\infty, 0) \cup (0, \infty)]\}$ is open. Then, $\mathscr{A}^c(\omega)$ is a countable collection of open intervals, say (c_j, d_j) for $j \in \mathscr{J}$. One then easily finds that $[0,T] \setminus \mathscr{A}^c(\omega)$ is a countable collection of closed intervals. Suppose there exists one such closed interval, say $[d_k, c_l]$ such that $d_k < c_l$. Then, $\bar{T}(h_0(\eta^\delta(\omega); d_k, c_l)) = \bar{T}(h_1(\zeta^\delta(\omega)); d_k, c_l) < \infty$, which is a contradiction. Hence, $d_k = c_l$ for all closed intervals in $[0,T] \setminus \mathscr{A}^c(\omega)$. $\square$

Lemma 30. *For a.e. $\omega \in \Omega$, There exists absolutely continuous, unique $\zeta^\delta_\cdot(\omega)$ satisfying (4.20).*

Proof. The proof follows the standard successive approximations approach. We indicate the main steps. By Assumption $(A.4)$, there exists $L < \infty$ such that $|\nabla_{(y,z)} G^\delta(y,z)| \leq L$ for all $(y,z) \in \mathscr{H}_0^c$. Let $0 = t_0 < t_1 < \ldots t_J = T$ where $t_{j+1} - t_j \in (0, 1/(2L))$ for all j. Fix $\omega \in \Omega^0$, where Ω^0 is defined in the proof of Lemma 29. Suppose we have a unique, absolutely continuous solution, $\zeta^\delta_\cdot(\omega)$, up to t_j (where j may be zero), and let $z^j = \zeta^\delta_{t_j}(\omega)$. We extend the solution to $[t_j, t_{j+1}]$. Let $\tilde{\zeta}^{\delta,0}$ be absolutely continuous (and hence of bounded variation) on $[t_j, t_{j+1}]$, with $\tilde{\zeta}^{\delta,0}(t_j) = z^j$. For $k \geq 0$, let

$$\tilde{\zeta}^{\delta,k+1}_t \doteq z^j + \int_{t_j}^t G^\delta(\eta^\delta_r(\omega), \tilde{\zeta}^{\delta,k}_r)\, dr \quad \forall t \in [t_j, t_{j+1}].$$

Noting Assumption $(A.4)$, we find that $\tilde{\zeta}^{\delta,k}$ is absolutely continuous for all $k \geq 0$. Letting $\mathscr{A}^k(\omega) \doteq \{t \in [t_j, t_{j+1}] \,|\, (\eta^\delta_t(\omega), \tilde{\zeta}^{\delta,k}_t) \in \mathscr{H}_0\}$, we see by Lemma 29 that $\mu(\mathscr{A}^k(\omega)) = 0$ for all k. Then, $\|\zeta^{\delta,k+1} - \zeta^{\delta,k}\|_{L_\infty(t_j,t_{j+1})} \leq \frac{1}{2}$. Application of the Banach Fixed Point Theorem then yields a unique, absolutely continuous extension of the solution, $\zeta^\delta_\cdot(\omega)$, to $[0, t_{j+1}]$. $\square$

Lemma 31. *Let $\delta > 0$. There exists a weak solution to (4.19)–(4.20).*

Proof. Let η^δ be a brownian motion as in Lemma 29, and let ζ^δ be the corresponding solution of (4.20) given by Lemma 30. Let $\nu^\delta_t(\omega) \doteq F^\delta(\eta^\delta_t, \zeta^\delta_t)$ for all $\omega \in \Omega^0$ (indicated in the proof of Lemma 29) and all $t \in [0,T]$. By Assumption $(A.4)$, there exists $D_1 < \infty$ such that $|\nu^\delta_t(\omega)| \leq D_1$ for all $\omega \in \Omega^0$ and $t \in [0,T]$. Let $B^\delta_t \doteq \eta^\delta_t - \int_0^t \nu^\delta_r\, dr$ for all $\omega \in \Omega^0$ and $t \in [0,T]$. We note that the Novikov condition is satisfied, and letting $P(\mathscr{C}) \doteq \mathbb{E}\big|_P \{\mathscr{I}_\mathscr{C} Z_T(\nu^\delta)\}$ for $\mathscr{C} \in \mathscr{F}_T$, with $Z_t(\nu^\delta) \doteq 1 + \sum_{j=1}^3 \int_0^t Z_r(\nu^\delta)[\nu^\delta_r]_j\, d[\nu^\delta_r]_j$, B^δ is a brownian motion on $(\Omega, \bar{\mathscr{F}}, P)$, with filtration

$\mathscr{F}_{\cdot}$. Then $(\eta^{\delta}, \zeta^{\delta})$ forms a solution to (4.19)–(4.20) with brownian motion B^{δ} and probability space $(\Omega, \bar{\mathscr{F}}, P)$. $\square$

Theorem 17. *Let $\delta > 0$. There exists a unique strong solution to (4.19)–(4.20).*

Proof. The strong solution will follow from a demonstration of pathwise uniqueness (cf. [22, Cor. 5.3.23]). Let $\gamma_t^{\delta} \doteq (\eta_t^{\delta^T}, \zeta_t^{\delta^T})^T$ for all $t \in [0,T]$, $H^{\delta} \doteq ([F^{\delta}]^T, [G^{\delta}]^T)^T$ and $\bar{\gamma}^0 \doteq ([y^0]^T, [z^0]^T)^T$, in which case,

$$\gamma_t^{\delta} = \bar{\gamma}^0 + \int_0^t H_r^{\delta}\, dr + \begin{bmatrix} \mathscr{I}_{m\times m} \\ 0 \end{bmatrix} B_t^{\delta} \quad \forall t \in [0,T]. \tag{4.21}$$

Letting γ^{δ} and $\tilde{\gamma}^{\delta}$ be two solution of (4.21), one sees from Assumption $(A.4)$ that there exists $\bar{L} < \infty$ such that

$$|\gamma_t^{\delta} - \tilde{\gamma}_t^{\delta}| \le \bar{L} \int_0^t |\gamma_r^{\delta} - \tilde{\gamma}_r^{\delta}|\, dr \quad \forall t \in [0,T].$$

Hence, by the Gronwall inequality, $\gamma^{\delta} = \tilde{\gamma}^{\delta}$, and we have pathwise uniqueness. $\square$

4.4 Taking $\delta \downarrow 0$

We obtain the limit result in the case where the dimension satisfies $m \ge 3$. This restriction is related to the form of $\tilde{\mathscr{G}}_0$, which takes the form of a curve in $\mathbb{R}^m$. It is expected that in the case where $\tilde{\mathscr{G}}_0$ is a point, the result would follow for $m \ge 2$.

Fix a probability space, say $(\Omega, \bar{\mathscr{F}}, \bar{P})$, and brownian motion, $B_{\cdot}$, with filtration $\mathscr{F}_{\cdot}$ generated by $B_{\cdot}$. Again, let $(y^0, z^0) \notin \mathscr{G}_0$, and note that there exists $\bar{\delta} > 0$ such that $(y^0, z^0) \notin \mathscr{G}_{\delta}$ for all $\delta \in [0, \bar{\delta}]$. Let $\delta_n \downarrow 0$ with $\delta_1 \in (0, \bar{\delta})$. Let the corresponding strong solutions of (4.19)–(4.20) be denoted by (η^n, ζ^n). Note that $G^{\delta_n}(y,z) = 0$ for all $z \in B_{\delta_n/4}(0)$, and hence

$$|\zeta_t^n| \ge \delta_n/4 \quad \forall t \in [0,T], \omega \in \Omega, n \in \mathbb{N}. \tag{4.22}$$

For $n \in \mathbb{N}$, let

$$\mathscr{A}_n \doteq \{ \omega \in \Omega | \ \nexists t \in [0,T] \text{ s.t. } (\eta_t^n, \zeta_t^n) \in \mathscr{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n/4}(0)] \}. \tag{4.23}$$

Recalling that $F^\delta = F$ on $\mathscr{G}_\delta^c$ and $G^\delta = G$ on $\mathscr{G}_\delta^c \cap B_{\delta/4}(0)^c$, we see that

$$(\eta^m, \zeta^m) = (\eta^n, \zeta^n) \quad \forall \omega \in \mathscr{A}_n \text{ and } m \geq n \geq 1. \tag{4.24}$$

Lastly, let

$$\tilde{\eta}_t^n = J(\zeta_t^n)\eta_t^n, \tag{4.25}$$

$$\tilde{\zeta}_t^n = J(\zeta_t^n)\zeta_t^n = \bar{e}, \tag{4.26}$$

for all $t \in [0,T]$.

Lemma 32. $(\eta_t^n, \zeta_t^n) \in \mathscr{G}_0$ *if and only if* $(\tilde{\eta}_t^n, \tilde{\zeta}_t^n) \in \mathscr{G}_0$ *if and only if* $\tilde{\eta}_t^n \in \tilde{\mathscr{G}}_0(\tilde{\zeta}_t^n)$ *if and only if there exists* $\lambda_t^n(\omega) \in \mathscr{I}$ *such that* $\tilde{\eta}_t^n = p(\lambda_t^n)$.

Proof. Recalling Assumption $(A.3)$ and (4.26), note that $(\tilde{\eta}_t^n, \tilde{\zeta}_t^n) \in \mathscr{G}_0$ if and only if $\tilde{\eta}_t^n \in \tilde{\mathscr{G}}_0(\tilde{\zeta}_t^n)$ if and only if there exists $\lambda_t^n(\omega) \in \mathscr{I}$ such that $\tilde{\eta}_t^n = [J(\tilde{\zeta}_t^n)]^{-1} p(\lambda_t^n) = [J(\bar{e})]^{-1} p(\lambda_t^n) = p(\lambda_t^n)$. Noting (4.25), we see that this is equivalent to $J(\zeta_t^n)\eta_t^n = p(\lambda_t^n)$, or $\eta_t^n = [J(\zeta_t^n)]^{-1} p(\lambda_t^n)$, which by definition, is true if and only if $\eta_t^n \in \tilde{\mathscr{G}}_0(\zeta_t^n)$. □

Lemma 33. *For each* $n \in \mathbb{N}$, *there exists a probability measure,* P_n, *mutually absolutely continuous with respect to* $\bar{P}$, *such that* η^n *is a brownian motion with respect to* P_n.

Proof. By the boundedness of F^{δ_n} and (4.19), one finds that the Novikov condition is satisfied, and hence the assertion follows from the Girsanov theorem, cf. [22]. □

Let

$$\hat{\mathscr{A}_n} \doteq \{\, \omega \in \Omega \,|\; \nexists t \in [0,T] \text{ s.t. either } \tilde{\eta}^n_t \in \tilde{\mathscr{G}_0}(\bar{e}) \text{ or } \tilde{\zeta}^n_t \in B_{\delta_n/4}(0) \,\}, \tag{4.27}$$

$$\tilde{\mathscr{A}_n} \doteq \{\, \omega \in \Omega \,|\; \nexists t \in [0,T] \text{ s.t. } \tilde{\eta}^n_t \in \tilde{\mathscr{G}_0}(\bar{e}) \,\}. \tag{4.28}$$

Using Lemma 32 and (4.22), we see that

$$\mathscr{A}_n = \hat{\mathscr{A}_n} = \tilde{\mathscr{A}_n}. \tag{4.29}$$

Lemma 34. *There exists a probability measure, $\breve{P}_n$, mutually absolutely continuous with respect to P_n, such that*

$$d\tilde{\eta}^n_t = J(\zeta^n_t)\, d\breve{\eta}^n_t,$$

where $\breve{\eta}^n_t$ is a brownian motion under $\breve{P}_n$.

Proof. Applying Itô's rule to $\tilde{\eta}^n$, and noting that $d\langle[\zeta^n]_k,[\zeta^n]_j\rangle_t \equiv 0$ for all $k,j \in]1,m[$, one sees that

$$d\tilde{\eta}^n_t = \bar{F}^n(\tilde{\eta}^n_t,\zeta^n_t)\,dt + J(\zeta^n_t)\,d\eta^n_t = J(\zeta^n_t)\Big[(J(\zeta^n_t))^{-1}\bar{F}^n(\tilde{\eta}^n_t,\zeta^n_t)\,dt + d\eta^n_t\Big], \tag{4.30}$$

where, component-wise,

$$\bar{F}^n_k(\tilde{\eta}^n_t,\zeta^n_t) = \sum_{j=1}^m \Big(\sum_{l=1}^m \big[\frac{\partial J}{\partial z_j}(\zeta^n_t)\big]_{k,l}[\tilde{\eta}^n_t]_l\Big)\big[G^{\delta_n}\big([J(\zeta^n_t)]^{-1}\tilde{\eta}^n_t,\zeta^n_t\big)\big]_j \tag{4.31}$$

for all $k \in]1,m[$. We examine $\bar{F}^n$. By Assumption $(A.4)$, there exists $M^1_n < \infty$ such that

$$|G^{\delta_n}\big([J(\zeta^n_t)]^{-1}\tilde{\eta}^n_t,\zeta^n_t\big)| \le M^1_n \quad \forall t \in [0,T],\, \omega \in \Omega. \tag{4.32}$$

Also, by (4.22) and Assumption $(A.3)$, there exists $M^2_n < \infty$ such that

$$|J(\zeta^n_t)|,\, \big|\frac{\partial J}{\partial z_j}(\zeta^n_t)\big| \le M^2_n \quad \forall j \in]1,m[,\, t \in [0,T],\, \omega \in \Omega. \tag{4.33}$$

Lastly, by (4.20), (4.32) and Assumption $(A.3)$ one sees that

$$|(J(\zeta_t^n))^{-1}| = |\zeta_t^n| \le |z^0| + M_n^1 T \doteq M_n^3 < \infty \quad \forall t \in [0,T],\, \omega \in \Omega. \tag{4.34}$$

By (4.31)–(4.34), we see that there exists $\bar{M}_n < \infty$ such that

$$|(J(\zeta_t^n))^{-1} \bar{F}^n(\tilde{\eta}_t^n, \zeta_t^n)| \le \bar{M}_n |\tilde{\eta}_t^n| \quad \forall t \in [0,T],\, \omega \in \Omega. \tag{4.35}$$

For integers $0 \le k \le K < \infty$, let $\Delta_K \doteq T/K$ and $t_k \doteq k\Delta_K$. By (4.35),

$$\begin{aligned}
&\mathbb{E}\Big\{\exp\big[\tfrac{1}{2}\int_{t_k}^{t_{k+1}} |(J(\zeta_t^n))^{-1} \bar{F}^n(\tilde{\eta}_t^n, \zeta_t^n)|^2\, dt\big]\Big\} \\
&\le \mathbb{E}\Big\{\exp\big[\tfrac{1}{2}\bar{M}_n^2 \Delta_K \big|\sup_{t\in[t_k,t_{k+1}]} \tilde{\eta}_t^n\big|^2 + \tfrac{1}{2}\bar{M}_n^2 \Delta_K \big|\inf_{t\in[t_k,t_{k+1}]} \tilde{\eta}_t^n\big|^2\big]\Big\} \\
&\le \mathbb{E}\Big\{\exp\big[\bar{M}_n^2 \Delta_K \big(\sup_{t\in[t_k,t_{k+1}]} |\tilde{\eta}_t^n|^2\big)\big]\Big\}^{\frac{1}{2}} \mathbb{E}\Big\{\exp\big[\bar{M}_n^2 \Delta_K \big(\inf_{t\in[t_k,t_{k+1}]} |\tilde{\eta}_t^n|^2\big)\big]\Big\}^{\frac{1}{2}}
\end{aligned}$$

for all $0 \le k \le K < \infty$ and $n \in \mathbb{N}$. However, recalling that $\tilde{\eta}^n$ is a brownian motion on measure P_n, by the reflection principle, this is finite for sufficiently large K. Hence, a weak Novikov condition is satisfied, cf. [22, Cor. 3.5.14], and we may apply a Girsanov transformation, yielding measure $\breve{P}_n$, mutually absolutely continuous with respect to P_n, given by $d\breve{P}_n \doteq \tilde{\mu}_T^n dP_n$, where

$$\tilde{\mu}_T^n \doteq \exp\Big[-\int_0^T (v_t^n)^T\, d\eta_t^n - \tfrac{1}{2}\int_0^T |v_t^n|^2\, dt\Big],$$

with $v_t^n \doteq (J(\zeta_t^n))^{-1} \bar{F}^n(\tilde{\eta}_t^n, \zeta_t^n)$, and such that under $\breve{P}_n$, the process $\breve{\eta}_t^n \doteq \int_0^t v_r^n\, dr + \eta_t^n$ is a brownian motion. Recalling (4.30), we have $d\tilde{\eta}_t^n = J(\zeta_t^n)\, d\breve{\eta}_t^n$. □

Lemma 35. *There exists a random-time process, α_t^n, and $0 \le \underline{\alpha} \le \bar{\alpha} < \infty$ such that $\alpha_{t+r}^n - \alpha_t^n \in [\underline{\alpha} r, \bar{\alpha} r]$ for all $0 \le t, r < \infty$, and such that $\tilde{\tilde{\eta}}_t^n \doteq \tilde{\eta}_{\alpha_t^n}^n$ is a $\breve{P}_n$ brownian motion, with filtration $\tilde{\tilde{\mathscr{F}}}_t \doteq \mathscr{F}_{\alpha_t}$.*

Proof. Note that $J(\zeta_t^n)J^T(\zeta_t^n) = |\zeta_t^n|^{-2}I_{m\times m}|$ for all $t \in [0,T]$, $\omega \in \Omega$. Define processes $c_t^n \doteq |\zeta_t^n|^{-2}$, $\beta_t^n \doteq \int_0^t c_r^n\, dr$ and $\alpha_t^n \doteq \inf\{r \in [0,\infty) \,|\, \beta_r^n > t\}$. The asserted bounds on $\alpha_\cdot^n$ then follow from Assumption $(A.3)$, (4.22) and (4.34). The second assertion is a direct application of well-known result [47, Th. 8.15]. $\square$

Lemma 36. *Let $m \geq 3$. $P_n(\mathscr{A}_n^c) = 0$ for all $n \in \mathbb{N}$.*

Proof. For $\bar{T} \in [0,\infty)$, let $\breve{\mathscr{A}}_n^{\bar{T}} \doteq \{\, \omega \in \Omega \,|\, \nexists t \in [0,T] \text{ s.t. } \tilde{\eta}^n_{\alpha_t^n} \in \tilde{\mathscr{G}}_0(\bar{e})\,\}$. Noting that $\tilde{\eta}^n_{\alpha_t^n} = \tilde{\tilde{\eta}}_t^n$ is a $\breve{P}_n$ brownian motion, by classical results (see Appendix 4.5) $\breve{P}\big((\breve{\mathscr{A}}_n^{\bar{T}})^c\big) = 0$ for all $\bar{T} \in [0,\infty)$. As this is true for all such $\bar{T}$, by (4.28), one finds that $\breve{P}\big((\tilde{\mathscr{A}_n})^c\big) = 0$. Noting Lemmas 33 and 34, this yields $\bar{P}\big((\tilde{\mathscr{A}_n})^c\big) = 0$. Combining this with (4.29), one has the asserted result. $\square$

Let $\bar{\mathscr{A}} \doteq \bigcap_{n\in\mathbb{N}} \mathscr{A}_n$. By Lemma 36 we immediately obtain the following.

Lemma 37. $\bar{P}(\bar{\mathscr{A}}) = 1$.

For any $n \in \mathbb{N}$, consider (4.1)–(4.2) with $(y_0, z_0) \notin \mathscr{G}_0$ up to the stopping time $\tau_{\delta_n} \doteq \inf\{t \geq 0; (\eta_t, \zeta_t) \in \mathscr{G}_{\delta_n} \cup [\mathbb{R}^m \times B_{\delta_n/4}(0)]\}$. Clearly, there exists a strong solution, $(\eta_{\tau_{\delta_n}}, \zeta_{\tau_{\delta_n}})$, up to the stopping time τ_{δ_n}. On the other hand, for all $\omega \in \bar{\mathscr{A}}^c$, there exists $\bar{n} = \bar{n}(\omega) \in \mathbb{N}$ such that $\omega \notin \mathscr{A}_{\bar{n}}$. For $\omega \in \bar{\mathscr{A}}^c$, let

$$(\bar{\eta}, \bar{\zeta})(\omega) \doteq (\eta^{\bar{n}}, \zeta^{\bar{n}})(\omega).$$

However, by (4.18), $(\eta^{\bar{n}}, \zeta^{\bar{n}})(\omega)$ satisfies (4.1)–(4.2) for all $\omega \in \mathscr{A}_{\bar{n}}$, and hence $(\bar{\eta}, \bar{\zeta})$ satisfies (4.1)–(4.2) on $\mathscr{A}_{\bar{n}}$ for all $n \geq \bar{n}$, which implies $(\bar{\eta}, \bar{\zeta})$ satisfies (4.1)–(4.2) on $\bar{\mathscr{A}}$. We have

$$(\eta_{\tau_{\delta_n}}, \zeta_{\tau_{\delta_n}}) \stackrel{d}{=} (\bar{\eta}_t, \bar{\zeta}_t).$$

Let $\tau_0 \doteq \inf\{t \geq 0; (\eta_t, \zeta_t) \in \mathscr{G}_0\}$. Noting Lemma 37, we have

$$\bar{P}(\tau_0 \leq T) \leq \lim_{\delta_n \to 0} \bar{P}(\tau_{\delta_n} \leq T) = \lim_{\delta_n \to 0} P_n(\tau_{\delta_n} \leq T) = 0.$$

Following this, we have the next result.

Theorem 18. *Let $m \geq 3$. There exists a strong solution to (4.1)–(4.2) for $t \in [0, T]$, $(y_0, z_0) \notin \mathscr{G}_0$.*

4.5 Appendix: Results from Potential Theory

In this section we briefly summarize some standard results from classical potential theory. We refer to [27] and [49] for more details.

4.5.1 Hitting Distribution

Let $B.$ be a Brownian motion in $\mathbb{R}^n$ and P^x be the unique probability distribution corresponding to the Brownian motion starting at $x \in \mathbb{R}^n$. We denote by $\mathscr{S}$ the collection of F_σ sets in $\mathbb{R}^n$. Note that every closed set is in $\mathscr{S}$. For $S \in \mathscr{S}$, let τ_S be the hitting time of S by the Brownian motion, that is $\tau_S \doteq \inf\{t > 0 : B_t \in S\}$, and set $\tau_S = \infty$ if $B_t \notin S$ for all $t > 0$. We make the following definition of polar sets. A set $S \in \mathscr{S}$ is said to be *polar* if $P^x(\tau_S < \infty) = 0$ for all $x \in \mathbb{R}^n$.

Let $S \in \mathscr{S}$. By Blumenthal's zero-one law for each $x \in \mathbb{R}^n$, $P^x(\tau_S = 0)$ equals zero or one. The point x is called *regular* if $P^x(\tau_S = 0) = 1$ and *irregular* if $P^x(\tau_S = 0) = 0$. The set of points which are regular for S is denoted by S^r. If S is polar, the set S^r is clearly empty.

4.5.2 Newtonian Potential, Equilibrium Measure, and Polar Sets

The *Newtonian potential kernel g* on $\mathbb{R}^n$, $n \geq 3$, is defined by

$$g(x) \doteq \frac{\Gamma(n/2 - 1)}{2\pi^{n/2}} |x|^{2-n}.$$

Let $g(x,y) \doteq g(y-x)$ for $x, y \in \mathbb{R}^n$. The *Newtonian potential* of a measure μ is defined by

$$g\mu(x) \doteq \int_{R^n} g(x,y)\mu(dy), \text{ for } x \in \mathbb{R}^n.$$

Let $S \in \mathscr{S}$ be bounded. There is at most one measure on $\mathbb{R}^n$ that is concentrated on S^r, the set of regular points for S, and has newtonian potential 1 on S^r. If such a measure exists, it is called the *equilibrium measure* of S and is denoted by μ_S. The *capacity* of S is defined by

$$C(S) \doteq \mu_S(\mathbb{R}^n),$$

and the Newtonian potential $g\mu_S$ is called the *equilibrium potential*.

The following lemma (cf. [49]) connects Newtonian capacity to polar sets of Brownian motion in $\mathbb{R}^n$, $n \geq 3$.

Lemma 38. *Let $S \in \mathscr{S}$ be bounded. Then S is polar if and only if $C(S) = 0$.*

Lemma 39. *Any curve in $\mathbb{R}^3$ of bounded variation has zero capacity and hence is polar for 3-dimensional Brownian motion.*

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