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DIFFERENTIAL COMPLEXES, HELMHOLTZ DECOMPOSITIONS, AND DECOUPLING OF MIXED METHODS

LONG CHEN AND XUEHAI HUANG*

ABSTRACT. A framework to systematically construct differential complex and Helmholtz decompositions is developed. The Helmholtz decomposition is used to decouple the mixed formulation of high order elliptic equations into combination of Poisson-type and Stokes-type equations. By finding the underlying complex, this decomposition is applied in the discretization level to design fast solvers for solving the linear algebraic system. It can be also applied in the continuous level first and then discretize the decoupled formulation, which leads to a natural superconvergence between the Galerkin projection and the decoupled approximation. Examples include but not limit to: biharmonic equation, triharmonic equation, fourth order curl equation, HHJ mixed method for plate problem, and Reissner-Mindlin plate model etc. As a by-product, Helmholtz decompositions for many dual spaces are obtained.

1. INTRODUCTION

Differential complexes and corresponding Helmholtz decomposition play the fundamental role in the design and analysis of mixed finite element methods. Among many others, the de Rham complex for Hodge Laplacian and the elasticity complex for linear elasticity are two successful examples [6, 7].

A direct and useful result of a differential complex is the Helmholtz decomposition. With the decomposition, the kernel spaces of differential operators involved in the complex are characterized clearly. The explicit expression of the kernel space can be used to develop fast solvers, see, for example, [43, 5, 44, 25, 26]. Helmholtz decomposition is also a key tool to construct the *a posteriori* error estimator of nonconforming and mixed finite element methods [1, 21, 22, 24, 46].

In this paper we shall develop a framework to systematically construct differential complex and Helmholtz decompositions. Based on a commutative diagram involving the complex, we shall present a standard mixed formulation and use the corresponding Helmholtz decomposition to decouple the mixed formulation into combination of Poisson-type and Stokes-type equations. As a by-product, we obtain Helmholtz decompositions for many dual spaces, such as $\boldsymbol{H}^{-1}(\operatorname{div},\Omega)$, $\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^n)$, $\boldsymbol{H}^{-1}(\operatorname{div}\operatorname{\mathbf{div}},\Omega;\mathbb{S})$, $\boldsymbol{H}^{-1}(\operatorname{curl},\Omega)$, $\boldsymbol{H}^{-2}(\operatorname{rot}\operatorname{\mathbf{rot}},\Omega;\mathbb{S})$ and so on.

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More precisely, our method is based on the following commutative diagram

$$
\begin{array}{ccccc}
X & \xrightarrow{J_X} & X' & & \\
 & & \cup & & \\
P & \xrightarrow{d^-} & \Sigma & \xrightarrow{d} & V' \\
 & & \uparrow{\scriptstyle \Pi_\Sigma} & & \uparrow{\scriptstyle J_V} \\
 & & \widetilde{\Sigma} & \xleftarrow{\Pi_V} & V
\end{array}
\quad , \tag{1.1}
$$

where the isomorphisms J_X and J_V are inverse of the Riesz representations, and the rest linear operators are all continuous but not necessarily isomorphic. A stable Helmholtz decomposition can be derived from (1.1)

$$
\Sigma = d^- P \oplus \Pi_\Sigma \Pi_V V. \tag{1.2}
$$

An abstract two-term mixed formulation based on the commutative diagram (1.1) is: given $g \in \Sigma'$ and $f \in V'$, find $(\sigma, u) \in \Sigma \times V$ such that

$$
(\sigma, \tau)_{X'} + \langle d\tau, u\rangle = \langle g, \tau\rangle \quad \forall\, \tau \in \Sigma, \tag{1.3}
$$

$$
\langle d\sigma, v\rangle = \langle f, v\rangle \quad \forall\, v \in V. \tag{1.4}
$$

To discretize the inner product $(\sigma, \tau)_{X'}$, we introduce $\phi = J_X^{-1}\sigma \in X$ and obtain an equivalent but unfolded three-term formulation: find $(\phi, u, \sigma) \in X \times V \times \Sigma$ such that

$$
(\phi, \psi)_X - \langle \sigma, d'v + \psi\rangle = -\langle f, v\rangle \quad \forall\, (\psi, v) \in X \times V, \tag{1.5}
$$

$$
\langle d'u + \phi, \tau\rangle = \langle g, \tau\rangle \qquad \forall\, \tau \in \Sigma. \tag{1.6}
$$

Applying the Helmholtz decomposition (1.2) to the unfolded formulation (1.5)-(1.6), we obtain a decoupled formulation: find $w, u \in V$, $\phi \in X$, and $p \in P/\ker d^-$ such that

$$
(w, v)_V = \langle f, v\rangle \qquad \forall\, v \in V, \tag{1.7}
$$

$$
(\phi, \psi)_X - \langle d^- p, \psi\rangle = \langle \Pi_\Sigma \Pi_V w, \psi\rangle \qquad \forall\, \psi \in X, \tag{1.8}
$$

$$
\langle d^- q, \phi\rangle = \langle g, d^- q\rangle \qquad \forall\, q \in P/\ker d^-, \tag{1.9}
$$

$$
(u, \chi)_V = \langle g - \phi, \Pi_\Sigma \Pi_V \chi\rangle \quad \forall\, \chi \in V. \tag{1.10}
$$

The middle system (1.8)-(1.9) of (ϕ, p) is now a Stokes-type system.

By finding the underlying complex, we can apply this decomposition in the discretization level to design fast solvers for solving the linear algebraic system. For example, we can solve

- the biharmonic equation discretized by conforming or non-conforming elements by two Poisson solvers and one Stokes solver;
- the HHJ method for plate problem by two Poisson solvers and one linear elasticity solver;
- the fourth order curl equation by solving two Maxwell equations and one Stokes equation;
- the triharmonic equation by solving two biharmonic equations and one tensorial Stokes equation.

We can also apply the decomposition in the continuous level first and then discretize the decoupled formulation. Compared to the original formulation, it is much easier to construct conforming finite element spaces for the decoupled formulation.

We shall show a natural superconvergence between the Galerkin projection and the decoupled approximation.

It should be mentioned that such decoupling is known for Reissner-Mindlin plate model [19] and recently is discovered for HHJ formulation of Kirchhoff plate model [49]. Similar results can be also found for biharmonic equations [45, 47] and the fourth order curl equation [62]. Our framework unifies those results and will lead to many more especial differential complexes for high order elliptic equations. Along this way, we can decouple the higher order partial differential equation into lower order ones, which makes the discretization easier. In addition, the structure revealed in our work will also play an important role in the *a posterior* error analysis which will be explored somewhere else.

We are confident that the abstract framework of Helmholtz decomposition, mixed method and its decomposition based on the commutative diagram developed in this paper will play a vital role in designing robust and convergent discrete method, the fast solver, and the optimal adaptive algorithm for partial differential equations.

The rest of this paper is organized as follows. In Section 2, we establish the abstract Helmholtz decomposition based on the commutative diagram and give several examples. The abstract mixed formulation and its decomposition based on the Helmholtz decomposition are present in Section 3. In Section 4, the discrete mixed method based on the commutative diagram and its decomposition are advised and analyzed. In Section 5, we discretize the decoupled formulation directly illustrated by two examples. Throughout this paper, we use "$\lesssim \cdots$" to mean that "$\leq C\cdots$", where C is a generic positive constant independent of meshsize h, which may take different values at different appearances. And $a \eqsim b$ means $a \lesssim b$ and $b \lesssim a$.

2. Abstract Helmholtz Decompositions

2.1. **Framework.** We start from a short exact sequence

$$\widetilde{W} \xrightarrow{\widetilde{d}_2} \widetilde{V} \xrightarrow{\widetilde{d}_1} \widetilde{U}, \tag{2.1}$$

and a bounded linear operator $d_1 : U \to V$. Here capital letters represent Banach spaces and $\widetilde{d}_i\,(i = 1, 2)$ are bounded linear operators. The sequence (2.1) is exact meaning that

$$\ker(\tilde{d}_1) = \operatorname{img}(\tilde{d}_2).$$

Let $I_V : V \to \widetilde{V}$ be a bounded linear operator, and $J_U : U \to \widetilde{U}$ be an isomorphism satisfying the assumption:

$$\widetilde{d}_1 I_V d_1 u = J_U u \quad \text{for all } u \in U. \tag{2.2}$$

That is we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{W} & \xrightarrow{\widetilde{d}_2} & \widetilde{V} & \xrightarrow{\widetilde{d}_1} & \widetilde{U} \\ & & \uparrow{\scriptstyle I_V} & & \uparrow{\scriptstyle J_U} \\ & & V & \xleftarrow{d_1} & U \end{array} . \tag{2.3}$$

Then we have an abstract Helmholtz decomposition as follows.

Theorem 2.1. *Suppose we have a short exact sequence (2.1). Assume the commutative diagram (2.3) holds with all the linear operators being bounded and $J_U : U \to \widetilde{U}$ being an isomorphism. Then we have a stable Helmholtz decomposition*

$$\widetilde{V} = \widetilde{d}_2 \widetilde{W} \oplus I_V d_1 U.$$

More precisely for any $\widetilde{v} \in \widetilde{V}$, there exist $\widetilde{w} \in \widetilde{W}/\ker \widetilde{d}_2$ and $u \in U$ such that

$$\widetilde{v} = \widetilde{d}_2 \widetilde{w} + I_V d_1 u, \tag{2.4}$$

$$\|\widetilde{w}\|_{\widetilde{W}} + \|u\|_U \lesssim \|\widetilde{v}\|_{\widetilde{V}}. \tag{2.5}$$

Proof. Let $u = J_U^{-1} \widetilde{d}_1 \widetilde{v}$, then it follows from (2.2)

$$\widetilde{d}_1(\widetilde{v} - I_V d_1 u) = \widetilde{d}_1 \widetilde{v} - J_U u = \widetilde{d}_1 \widetilde{v} - \widetilde{d}_1 \widetilde{v} = 0.$$

Due to the exactness, there exists $\widetilde{w} \in \widetilde{W}/\ker \widetilde{d}_2$ such that

$$\widetilde{v} - I_V d_1 u = \widetilde{d}_2 \widetilde{w},$$

which implies the decomposition (2.4).

By the definition of u,

$$\|u\|_U = \|J_U^{-1} \widetilde{d}_1 \widetilde{v}\|_U \lesssim \|\widetilde{d}_1 \widetilde{v}\|_{\widetilde{U}} \lesssim \|\widetilde{v}\|_{\widetilde{V}}. \tag{2.6}$$

Since $\widetilde{d}_2 \widetilde{W} = \ker \widetilde{d}_1$ is a closed subspace of $\widetilde{V}$, we get by open mapping theorem that $\widetilde{d}_2$ is an isomorphism from $\widetilde{W}/\ker \widetilde{d}_2$ onto $\widetilde{d}_2 \widetilde{W}$, which means

$$\|\widetilde{w}\|_{\widetilde{W}} \lesssim \|\widetilde{d}_2 \widetilde{w}\|_{\widetilde{V}} \quad \forall\, \widetilde{w} \in \widetilde{W}/\ker \widetilde{d}_2.$$

Hence it holds from (2.6)

$$\begin{aligned} \|\widetilde{w}\|_{\widetilde{W}} &\lesssim \|\widetilde{d}_2 \widetilde{w}\|_{\widetilde{V}} = \|\widetilde{v} - I_V d_1 u\|_{\widetilde{V}} \le \|\widetilde{v}\|_{\widetilde{V}} + \|I_V d_1 u\|_{\widetilde{V}} \\ &\lesssim \|\widetilde{v}\|_{\widetilde{V}} + \|d_1 u\|_V \lesssim \|\widetilde{v}\|_{\widetilde{V}} + \|u\|_U \lesssim \|\widetilde{v}\|_{\widetilde{V}}. \end{aligned}$$

We acquire (2.5) from (2.6) and the last inequality.

At last, we show that the Helmholtz decomposition is a direct sum. For any $\widetilde{v} \in \widetilde{d}_2 \widetilde{W} \cap I_V d_1 U$, there exist $\widetilde{w} \in \widetilde{W}$ and $u \in U$ such that $\widetilde{v} = \widetilde{d}_2 \widetilde{w} = I_V d_1 u$. By (2.2) and the exactness of (2.1),

$$J_U u = \widetilde{d}_1 I_V d_1 u = \widetilde{d}_1 \widetilde{d}_2 \widetilde{w} = 0.$$

Since J_U is an isomorphism, we have $u = 0$, which indicates $\widetilde{v} = 0$. □

Let W be a Banach space and $d_2 : V \to W$ be a bounded linear operator. If the following exact sequence

$$U \xrightarrow{d_1} V \xrightarrow{d_2} W \tag{2.7}$$

holds, we have $d_1 U = \ker(d_2)$ and obtain another form of decomposition

$$\widetilde{V} = \widetilde{d}_2 \widetilde{W} \oplus I_V \ker(d_2).$$

In Theorem 2.1, the decomposition is a direct sum but not necessarily orthogonal. Indeed in the proof we do not use the inner product structure. We now explore the orthogonality for Hilbert complexes. In what follows, we always denote by $\langle \cdot, \cdot \rangle$ the duality pairing and $(\cdot, \cdot)$ the L^2 inner product.

We can apply Theorem 2.1 to the dual complex of (2.7)

$$U' \xleftarrow{d_1'} V' \xleftarrow{d_2'} W', \tag{2.8}$$

where X' is the dual space of a linear space X and $T' : Y' \to X'$ is the dual of a linear operator $T : X \to Y$ defined as

$$\langle T'g, x\rangle := \langle g, Tx\rangle.$$

Assume the range d_2V of d_2 is closed in W, then the dual complex (2.8) is also exact if the complex (2.7) is exact (cf. [57, Remark 2.15]). When X is a Hilbert space with an inner product $(\cdot,\cdot)_X$ and X' is the continuous dual of X, by Riesz representation theorem, we have an isomorphism $J_X : X \to X'$: for any $w \in X$, define $J_X w \in X'$ as

$$\langle J_X w, v\rangle = (w, v)_X \quad \forall\ v \in X. \tag{2.9}$$

The induced inner product and norm for any $w', v' \in X'$ are given by

$$\begin{aligned}(w', v')_{X'} &:= (J_X^{-1}w', J_X^{-1}v')_X = \langle J_X^{-1}w', v'\rangle = \langle w', J_X^{-1}v'\rangle,\\ \|w'\|_{X'} &:= \|J_X^{-1}w'\|_X.\end{aligned} \tag{2.10}$$

With $\widetilde{X}\,(X = U, V, W)$ and $\widetilde{d}_i\,(i = 1, 2)$ replaced by $X'\,(X = U, V, W)$ and $d_i'\,(i = 1, 2)$, the assumption (2.2) becomes

$$d_1' J_V d_1 u = J_U u \quad \text{for all } u \in U, \tag{2.11}$$

and the commutative diagram is

$$\begin{array}{ccccc} W' & \xrightarrow{d_2'} & V' & \xrightarrow{d_1'} & U' \\ & & \uparrow{\scriptstyle J_V} & & \uparrow{\scriptstyle J_U} \\ & & V & \xleftarrow{d_1} & U \end{array} \quad . \tag{2.12}$$

Corollary 2.2. *Let U, V, W be Hilbert spaces. Assume the Hilbert complex (2.7) is exact with d_2V being closed in W and bounded linear operators $d_i(i = 1, 2)$, and the commutative diagram (2.12) holds with (inverse) Riesz representations J_V and J_U, then we have the $(\cdot,\cdot)_{V'}$-orthogonal Helmholtz decomposition*

$$V' = d_2'W' \oplus^{\perp} J_V d_1 U.$$

That is for any $v' \in V'$, there exist $w' \in W'/\ker d_2'$ and $u \in U$ such that

$$v' = d_2'w' + J_V d_1 u, \tag{2.13}$$

$$\|v'\|^2_{V'} = \|d_2'w'\|^2_{V'} + \|J_V d_1 u\|^2_{V'}. \tag{2.14}$$

Proof. We need only to verify the orthogonality which can be done as follows

$$(d_2'w', J_V d_1 u)_{V'} = \langle d_2'w', d_1 u\rangle = \langle w', d_2 d_1 u\rangle = 0.$$

□

Remark 2.3. A direct proof of Corollary 2.2 is given as follows. Noting that $\ker(d_1')$ is closed, we have the orthogonal decomposition

$$V' = \ker(d_1') \oplus^{\perp} \ker(d_1')^{\perp}.$$

Since d_1U is closed in V, $\ker(d_1')^o = d_1U$ where $\ker(d_1')^o$ the annihilator of $\ker(d_1')$. It is obvious that $\ker(d_1')^{\perp} = J_V \ker(d_1')^o$. Hence

$$V' = \ker(d_1') \oplus^{\perp} J_V d_1 U.$$

By the exactness of the dual complex (2.8), we get $\ker(d_1') = d_2' W'$, which ends the proof. ☐

Consider a special case when V is a dense subspace of a larger space Y endowed with the inner product $(\cdot,\cdot)_Y$. In most places in this paper, Y is the L^2 space for scalar or vector functions with $(\cdot,\cdot)_Y = (\cdot,\cdot)$ being the L^2-inner product. We can equip V with the graph norm

$$(w,v)_V := (w,v)_Y + (d_2 w, d_2 v)_W.$$

Or we can start from Y and define V as the subspace of Y with $\|\cdot\|_V < \infty$. By identifying Y' with Y, we have the rigged Hilbert space [17, 33]

$$V \subset Y \subset V'.$$

We obtain from the definition of J_V that for any $u \in U$ and $v \in V$

$$\langle J_V d_1 u, v\rangle = (d_1 u, v)_V = (d_1 u, v)_Y + (d_2 d_1 u, d_2 v)_W = (d_1 u, v)_Y.$$

Thus J_V is just identity operator on d_1U. If we choose $J_U = d_1' d_1$, in order to satisfy the assumption (2.11), we need to verify

$$d_1' d_1 : U \to U' \tag{2.15}$$

is an isomorphism which is equivalent to

$$\|u\|_U \lesssim \|d_1 u\|_V \quad \forall\, u \in U. \tag{2.16}$$

Such Poincaré type inequality holds for examples considered in this paper.

We shall present examples in the sequel. Let $\Omega \subset \mathbb{R}^n, n = 2, 3$, be a bounded Lipschitz domain. Denote by $\mathbb{T}$ the space of all $n \times n$ tensors and $\mathbb{S}$ the space of all symmetric $n\times n$ tensors. We use standard notation for Sobolev spaces and boldface letters for vector valued spaces. When we want to emphasize the spatial dimension, we include $\mathbb{R}^n$ into the notation of spaces.

Recall the de Rham complexes in two dimensions

$$0 \longrightarrow H_0^1(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0(\text{div},\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \mathbb{R}, \tag{2.17}$$

$$\mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div},\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0, \tag{2.18}$$

$$0 \longrightarrow H_0^{s+2}(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0^{s+1}(\Omega;\mathbb{R}^2) \xrightarrow{\text{div}} H_0^s(\Omega) \longrightarrow \mathbb{R}, \tag{2.19}$$

$$\mathbb{R} \longrightarrow H^{s+2}(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}^{s+1}(\Omega;\mathbb{R}^2) \xrightarrow{\text{div}} H^s(\Omega) \longrightarrow 0, \tag{2.20}$$

and the de Rham complexes in three dimensions

$$0 \longrightarrow H_0^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}_0(\text{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow \mathbb{R}, \tag{2.21}$$

$$\mathbb{R} \longrightarrow H^1(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}(\text{curl}, \Omega) \xrightarrow{\text{curl}} \boldsymbol{H}(\text{div}, \Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0, \tag{2.22}$$

$$0 \longrightarrow H_0^{s+3}(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}_0^{s+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} \boldsymbol{H}_0^{s+1}(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} H_0^s(\Omega) \longrightarrow \mathbb{R}, \tag{2.23}$$

$$\mathbb{R} \longrightarrow H^{s+3}(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}^{s+2}(\Omega; \mathbb{R}^3) \xrightarrow{\text{curl}} \boldsymbol{H}^{s+1}(\Omega; \mathbb{R}^3) \xrightarrow{\text{div}} H^s(\Omega) \longrightarrow 0, \tag{2.24}$$

with $s \in \mathbb{R}$. When Ω is simply connected with connected boundary, the L^2 de Rham complexes (2.17)-(2.18) and (2.21)-(2.22) are exact [37, 6, 7], the complexes (2.20) and (2.24) are exact if s is an integer, and the complexes (2.19) and (2.23) are exact if s is a nonnegative integer [57, 32]. When Ω is a bounded domain starlike with respect to a ball, the complexes (2.20) and (2.24) are exact for any $s \in \mathbb{R}$, and the complexes (2.19) and (2.23) are exact if s is nonnegative and $s - \frac{1}{2}$ is not an integer [32, p. 301]. Hereafter, we always assume the bounded Lipschitz domain Ω is always simply connected with connected boundary in this paper.

2.2. **Helmholtz decomposition of L^2 functions.** Splitting $\Delta = \text{div}\, I\, \text{grad}$, we construct the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{H}^1(\Omega) & \xrightarrow{\text{curl}} & \boldsymbol{L}^2(\Omega; \mathbb{R}^n) & \xrightarrow{\text{div}} & H^{-1}(\Omega) \\ & & \uparrow I & & \uparrow \Delta \\ & & \boldsymbol{L}^2(\Omega; \mathbb{R}^n) & \xleftarrow{\text{grad}} & H_0^1(\Omega) \end{array},$$

where $\boldsymbol{H}^1(\Omega)$ means $H^1(\Omega)$ for $n = 2$ and $\boldsymbol{H}^1(\Omega; \mathbb{R}^3)$ for $n = 3$. By the exact sequences (2.20) and (2.24) with $s = -1$, the complex

$$\boldsymbol{H}^1(\Omega) \xrightarrow{\text{curl}} \boldsymbol{L}^2(\Omega; \mathbb{R}^n) \xrightarrow{\text{div}} H^{-1}(\Omega) \longrightarrow 0 \tag{2.25}$$

is exact. It's trivial that all the linear operators in the commutative diagram are bounded and the operator $\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism.

Applying Theorem 2.1, we get the standard L^2-orthogonal Helmholtz decomposition (cf. [37, 23])

$$\boldsymbol{L}^2(\Omega; \mathbb{R}^n) = \nabla H_0^1(\Omega) \oplus^{\perp} \text{curl}\ \boldsymbol{H}^1(\Omega).$$

The orthogonality can be checked as in the proof of Corollary 2.2. In fact, we also have the following L^2-orthogonal Helmholtz decomposition (cf. [37, Theorem 3.2 and Corollary 3.4 in Chapter I])

$$\boldsymbol{L}^2(\Omega; \mathbb{R}^n) = \nabla H_0^1(\Omega) \oplus^{\perp} \text{curl}\ \boldsymbol{H}(\text{curl}, \Omega).$$

2.3. **Helmholtz decomposition of $\boldsymbol{H}^{-1}(\mathrm{div})$ space.** Define

$$\boldsymbol{H}^{-1}(\mathrm{div},\Omega) := \{\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^n) : \mathrm{div}\,\boldsymbol{\phi} \in H^{-1}(\Omega)\},$$

with norm

$$\|\boldsymbol{\phi}\|_{H^{-1}(\mathrm{div})}^2 := \|\boldsymbol{\phi}\|_{-1}^2 + \|\mathrm{div}\,\boldsymbol{\phi}\|_{-1}^2$$

Lemma 2.4. *Assume that the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is simply connected with connected boundary. The following complex*

$$\boldsymbol{L}^2(\Omega) \xrightarrow{\mathrm{curl}} \boldsymbol{H}^{-1}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} H^{-1}(\Omega) \longrightarrow 0$$

is exact, where $\boldsymbol{L}^2(\Omega)$ means $L^2(\Omega)$ for $n=2$ and $\boldsymbol{L}^2(\Omega;\mathbb{R}^3)$ for $n=3$.

Proof. It has been proved that $\ker(\mathrm{div}) = \mathrm{img}(\mathrm{curl})$ in [19, Proposition 2.3] for $n=2$ and [57, Corollary 2.31] for $n=3$. By the exact sequence (2.25), it holds $\mathrm{div}\,\boldsymbol{L}^2(\Omega;\mathbb{R}^n) = H^{-1}(\Omega)$, which together with $\boldsymbol{L}^2(\Omega;\mathbb{R}^n) \subset \boldsymbol{H}^{-1}(\mathrm{div},\Omega)$ indicates $\mathrm{div}\,\boldsymbol{H}^{-1}(\mathrm{div},\Omega) = H^{-1}(\Omega)$. ☐

With this exact sequence, we build up the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{L}^2(\Omega) & \xrightarrow{\mathrm{curl}} & \boldsymbol{H}^{-1}(\mathrm{div},\Omega) & \xrightarrow{\mathrm{div}} & H^{-1}(\Omega) \\ & & \uparrow I & & \uparrow \Delta \\ & & \boldsymbol{H}_0(\mathrm{curl},\Omega) & \xleftarrow{\mathrm{grad}} & H_0^1(\Omega) \end{array} \quad . \tag{2.26}$$

By Theorem 2.1, We obtain the Helmholtz decomposition (cf. [19, Proposition 2.3] for two dimensional version)

$$\boldsymbol{H}^{-1}(\mathrm{div},\Omega) = \nabla H_0^1(\Omega) \oplus^{\perp} \mathrm{curl}\ \boldsymbol{L}^2(\Omega). \tag{2.27}$$

As we mentioned early that $J_{H(\mathrm{curl})}$ is just identity operator on $\nabla H_0^1(\Omega)$, thus the decomposition (2.27) is orthogonal in the inner product $(\cdot,\cdot)_{J^{-1}_{H(\mathrm{curl})}}$.

Lemma 2.5. *Assume that the bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ is simply connected with connected boundary. We have*

$$(\boldsymbol{H}_0(\mathrm{curl},\Omega))' = \boldsymbol{H}^{-1}(\mathrm{div},\Omega). \tag{2.28}$$

Proof. The proof of $\boldsymbol{H}^{-1}(\mathrm{div},\Omega) \subset (\boldsymbol{H}_0(\mathrm{curl},\Omega))'$ can be found in [15, p. 338] for $n=2$. We can prove it for $n=3$ in a similar way by using the Helmholtz decomposition (2.27).

On the other hand, for each $\boldsymbol{\phi} \in (\boldsymbol{H}_0(\mathrm{curl},\Omega))'$, let $\boldsymbol{\varphi} \in \boldsymbol{H}_0(\mathrm{curl},\Omega)$ be the solution of

$$(\boldsymbol{\varphi},\boldsymbol{\psi}) + (\mathrm{curl}\,\boldsymbol{\varphi}, \mathrm{curl}\,\boldsymbol{\psi}) = \langle \boldsymbol{\phi},\boldsymbol{\psi}\rangle \quad \forall\ \boldsymbol{\psi} \in \boldsymbol{H}_0(\mathrm{curl},\Omega).$$

Then it holds

$$\boldsymbol{\phi} = \boldsymbol{\varphi} + \mathrm{curl}\,\mathrm{curl}\,\boldsymbol{\varphi}.$$

Obviously $\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^n)$, and

$$\mathrm{div}\,\boldsymbol{\phi} = \mathrm{div}\,\boldsymbol{\varphi} \in H^{-1}(\Omega).$$

Thus $\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\mathrm{div},\Omega)$. Therefore $(\boldsymbol{H}_0(\mathrm{curl},\Omega))' \subset \boldsymbol{H}^{-1}(\mathrm{div},\Omega)$. ☐

2.4. **Helmholtz decomposition of H^{-1} functionals.** By the exact sequences (2.19) and (2.23) with $s=0$, the complex

$$\boldsymbol{H}_0^2(\Omega) \xrightarrow{\text{curl}} \boldsymbol{H}_0^1(\Omega;\mathbb{R}^n) \xrightarrow{\text{div}} L_0^2(\Omega) \longrightarrow 0 \tag{2.29}$$

is exact, where $\boldsymbol{H}_0^2(\Omega)$ means $H_0^2(\Omega)$ for $n=2$ and $\boldsymbol{H}_0^2(\Omega;\mathbb{R}^3)$ for $n=3$. Based on this exact sequence, we set up the commutative diagram

$$\begin{array}{ccccc} L_0^2(\Omega) & \xrightarrow{\text{grad}} & \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^n) & \xrightarrow{\text{curl}} & \boldsymbol{H}^{-2}(\Omega) \\ & & \Delta \uparrow & & \uparrow \Delta^2 \\ & & \boldsymbol{H}_0^1(\Omega;\mathbb{R}^n) & \xleftarrow{\text{curl}} & \boldsymbol{H}_0^2(\Omega) \end{array}.$$

is the dual complex of the exact sequence (2.29). Applying Corollary 2.2, we then acquire the H^{-1}-orthogonal decomposition of $\boldsymbol{H}^{-1}$ (c.f. [55, Lemma 2.4])

$$\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^n) = \nabla L_0^2(\Omega) \oplus^{\perp} \Delta(\operatorname{curl} \boldsymbol{H}_0^2(\Omega)).$$

2.5. **Helmholtz decomposition of $\boldsymbol{H}^{-1}(\text{curl})$ space.** Define

$$\boldsymbol{H}'(\text{curl},\Omega) := \{\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{\phi} \in (\boldsymbol{H}_0(\text{curl},\Omega))'\},$$

with graph norm

$$\|\boldsymbol{\phi}\|^2_{H'(\text{curl})} := \|\boldsymbol{\phi}\|^2_{-1} + \|\operatorname{curl}\boldsymbol{\phi}\|^2_{(\boldsymbol{H}_0(\text{curl},\Omega))'}.$$

Due to (2.28), it holds

$$\boldsymbol{H}'(\text{curl},\Omega) = \{\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3) : \operatorname{curl}\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3)\} =: \boldsymbol{H}^{-1}(\text{curl},\Omega).$$

Following [27], we introduce the space

$$K_0^c := \{\boldsymbol{\phi} \in \boldsymbol{H}_0(\text{curl},\Omega) : \text{div}\boldsymbol{\phi} = 0\} = \boldsymbol{H}_0(\text{curl},\Omega) \cap \operatorname{curl}\boldsymbol{H}^1(\Omega;\mathbb{R}^3).$$

The operator $\operatorname{curl} : K_0^c \to \operatorname{curl}\boldsymbol{H}_0(\text{curl},\Omega)$ is an isomorphism (cf. [57, section 2.4]). Then $(\operatorname{curl}\cdot, \operatorname{curl}\cdot)$ defines an inner product on K_0^c and $(\operatorname{curl}\operatorname{curl})^{-1} : (K_0^c)' \to K_0^c$ is an isomorphism due to the Poincaré inequality (2.16) holding on K_0^c (cf. [57, section 2.4]). Given a $\boldsymbol{f} \in (K_0^c)'$, find $\boldsymbol{u} \in K_0^c$ such that $\operatorname{curl}\operatorname{curl}\boldsymbol{u} = \boldsymbol{f}$ in $(K_0^c)'$ is the Maxwell's equation with divergence free constraint.

Noting that $\operatorname{curl} K_0^c = \operatorname{curl}\boldsymbol{H}_0(\text{curl},\Omega)$, we get the following exact sequence from the de Rham complex (2.21)

$$0 \longrightarrow K_0^c \xrightarrow{\text{curl}} \boldsymbol{H}_0(\text{div},\Omega) \xrightarrow{\text{div}} L_0^2(\Omega) \longrightarrow 0.$$

Since $\boldsymbol{H}'(\text{curl},\Omega) = \boldsymbol{H}^{-1}(\text{curl},\Omega)$ and $K_0^c \subset \boldsymbol{H}_0(\text{curl},\Omega)$, we have

$$\operatorname{curl}\boldsymbol{H}^{-1}(\text{curl},\Omega) = \operatorname{curl}\boldsymbol{H}'(\text{curl},\Omega) \subset (\boldsymbol{H}_0(\text{curl},\Omega))' \subset (K_0^c)'.$$

Lemma 2.6. *Assume that the bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ is simply connected with connected boundary. The following complex*

$$0 \longrightarrow L_0^2(\Omega) \xrightarrow{\text{grad}} \boldsymbol{H}^{-1}(\text{curl},\Omega) \xrightarrow{\text{curl}} (K_0^c)' \longrightarrow 0$$

is exact.

Proof. For any $\boldsymbol{\phi} \in \ker(\mathrm{curl})$, it means that $\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3)$ and $\mathrm{curl}\,\boldsymbol{\phi} = \boldsymbol{0}$. Then by (2.24) with $s=-3$, there exists $v \in L_0^2(\Omega)$ satisfying $\boldsymbol{\phi} = \nabla v$. Hence $\ker(\mathrm{curl}) \subset \mathrm{img}(\mathrm{grad})$.

For any $\boldsymbol{\phi} \in (K_0^c)'$, by the fact that $(\mathrm{curl}\,\mathrm{curl})^{-1} : (K_0^c)' \to K_0^c$ is an isomorphism, let $\boldsymbol{\varphi} = \mathrm{curl}((\mathrm{curl}\,\mathrm{curl})^{-1}\boldsymbol{\phi})$. It is easy to check that $\boldsymbol{\varphi} \in \boldsymbol{H}^{-1}(\mathrm{curl},\Omega)$ and $\mathrm{curl}\,\boldsymbol{\varphi} = \boldsymbol{\phi}$. Therefore $\mathrm{img}(\mathrm{curl}) = (K_0^c)'$. □

Then we construct the commutative diagram

$$\begin{array}{ccccc} L_0^2(\Omega) & \xrightarrow{\mathrm{grad}} & \boldsymbol{H}^{-1}(\mathrm{curl},\Omega) & \xrightarrow{\mathrm{curl}} & (K_0^c)' \\ & & \big\uparrow I & & \big\uparrow \mathrm{curl}\,\mathrm{curl} \\ & & \boldsymbol{H}_0(\mathrm{div},\Omega) & \xleftarrow{\mathrm{curl}} & K_0^c \end{array} .$$

Using Theorem 2.1, it holds the stable Helmholtz decomposition

$$\boldsymbol{H}^{-1}(\mathrm{curl},\Omega) = \nabla L_0^2(\Omega) \oplus \mathrm{curl}\, K_0^c = \nabla L_0^2(\Omega) \oplus \mathrm{curl}\, \boldsymbol{H}_0(\mathrm{curl},\Omega). \tag{2.30}$$

By Helmholtz decomposition (2.30) and the similar argument used in the proof of Lemma 2.5, we can prove that

$$(\boldsymbol{H}_0(\mathrm{div},\Omega))' = \boldsymbol{H}^{-1}(\mathrm{curl},\Omega),$$

if $\Omega \subset \mathbb{R}^3$ is simply connected with connected boundary.

2.6. **Helmholtz decomposition of symmetric tensors: HHJ complex.** Let

$$\boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \mathrm{div}\,\mathbf{div}\boldsymbol{\tau} \in H^{-1}(\Omega)\},$$
$$\boldsymbol{H}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) := \{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \mathrm{div}\,\mathbf{div}\boldsymbol{\tau} \in L^2(\Omega)\}.$$

We introduce the symmetric curl operator $\nabla^s \times \boldsymbol{\phi} := \left(\mathbf{curl}\boldsymbol{\phi} + (\mathbf{curl}\boldsymbol{\phi})^T\right)/2$. And recall the Hellan-Herrmann-Johnson (HHJ) exact sequence (cf. [26, Lemma 2.2])

$$\boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\nabla^s\times} \boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) \xrightarrow{\mathrm{div}\,\mathbf{div}} H^{-1}(\Omega). \tag{2.31}$$

Given a scalar function v, we can embed it into the symmetric tensor space as $\boldsymbol{\pi}(v) = v\boldsymbol{I}_{n\times n}$. Since $\Delta v = \mathrm{div}\,\mathbf{div}\boldsymbol{\pi}(v)$, we have the commutative diagram in two dimensions (cf. [49])

$$\begin{array}{ccccc} \boldsymbol{H}^1(\Omega;\mathbb{R}^2) & \xrightarrow{\nabla^s\times} & \boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) & \xrightarrow{\mathrm{div}\,\mathbf{div}} & H^{-1}(\Omega) \\ & & \big\uparrow I & & \big\uparrow \Delta \\ & & \boldsymbol{H}_0^1(\Omega;\mathbb{S}) & \xleftarrow{\boldsymbol{\pi}} & H_0^1(\Omega) \end{array} .$$

According to Theorem 2.1, we thus recover the Hemholtz decomposition in [49, Theorem 3.1]

$$\boldsymbol{H}^{-1}(\mathrm{div}\,\mathbf{div},\Omega;\mathbb{S}) = \nabla^s \times \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \oplus \boldsymbol{\pi} H_0^1(\Omega). \tag{2.32}$$

Then we build up the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{H}^1(\Omega;\mathbb{R}^2) & \xrightarrow{\nabla^s\times} & \boldsymbol{L}^2(\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}\mathbf{div}} & H^{-2}(\Omega) \\ & & \uparrow I & & \uparrow \Delta^2 \\ & & \boldsymbol{L}^2(\Omega;\mathbb{S}) & \xleftarrow{\boldsymbol{\nabla}^2} & H_0^2(\Omega) \end{array}.$$

Due to this commutative diagram and the exact sequence (2.31), the following complex is exact:

$$\boldsymbol{H}^1(\Omega;\mathbb{R}^2) \xrightarrow{\nabla^s\times} \boldsymbol{L}^2(\Omega;\mathbb{S}) \xrightarrow{\operatorname{div}\mathbf{div}} H^{-2}(\Omega) \longrightarrow 0. \tag{2.33}$$

By Theorem 2.1, we have the L^2-orthogonal Helmholtz decomposition obtained in [46, Lemma 3.1]

$$\boldsymbol{L}^2(\Omega;\mathbb{S}) = \nabla^s\times\boldsymbol{H}^1(\Omega;\mathbb{R}^2)\oplus^{\perp}\boldsymbol{\nabla}^2 H_0^2(\Omega).$$

Similarly, smoothness of the symmetric tensor can be further increased to

$$\begin{array}{ccccc} \boldsymbol{H}^1(\Omega;\mathbb{R}^2) & \xrightarrow{\nabla^s\times} & \boldsymbol{H}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}\mathbf{div}} & L^2(\Omega) \\ & & \uparrow I & & \uparrow I \\ & & H_0^1(\Omega;\mathbb{S}) & \xleftarrow{\boldsymbol{\pi}\Delta^{-1}} & L^2(\Omega) \end{array},$$

which leads to the Helmholtz decomposition

$$\boldsymbol{H}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}) = \nabla^s\times\boldsymbol{H}^1(\Omega;\mathbb{R}^2)\oplus\boldsymbol{\pi}\Delta^{-1}L^2(\Omega).$$

2.7. **Helmholtz decomposition of symmetric tensors: linear elasticity.** Let $\mathbb{C}$ be the elasticity tensor and $\mathbb{A}=\mathbb{C}^{-1}$ be the compliance tensor. Recall that the symmetric gradient $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\nabla u + (\nabla u)^T)/2$. Construct the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{H}^1(\Omega;\mathbb{R}^2) & \xrightarrow{\boldsymbol{\varepsilon}} & \boldsymbol{L}^2(\Omega;\mathbb{S}) & \xrightarrow{\operatorname{rot}\mathbf{rot}} & H^{-2}(\Omega) \\ & & \uparrow \mathbb{C}I & & \uparrow \operatorname{rot}\mathbf{rot}(\mathbb{C}\mathbf{curl}\operatorname{curl}) \\ & & \boldsymbol{L}^2(\Omega;\mathbb{S}) & \xleftarrow{\mathbf{curl}\operatorname{curl}} & H_0^2(\Omega) \end{array}.$$

The Hilbert complex on the top is exact since it is just the rotation of the exact sequence (2.33). Then by Theorem 2.1 we have a Helmholtz-type decomposition

$$\boldsymbol{L}^2(\Omega,\mathbb{S}) = \boldsymbol{\varepsilon}\boldsymbol{H}^1(\Omega;\mathbb{R}^2)\oplus^{\perp}\mathbb{C}\mathbf{curl}\operatorname{curl}H_0^2(\Omega), \tag{2.34}$$

and the decomposition (2.34) is orthogonal in the weighted L^2-inner product $(\cdot,\cdot)_{\mathbb{C}^{-1}} = (\cdot,\cdot)_{\mathbb{A}}$.

We can relax the smoothness of symmetric tensors by considering

$$\boldsymbol{H}^{-2}(\operatorname{rot}\mathbf{rot},\Omega;\mathbb{S}) := \{\boldsymbol{\tau}\in\boldsymbol{H}^{-1}(\Omega;\mathbb{S}) : \operatorname{rot}\mathbf{rot}\boldsymbol{\tau}\in H^{-2}(\Omega)\}$$

with graph norm

$$\|\boldsymbol{\tau}\|^2_{\boldsymbol{H}^{-2}(\operatorname{rot}\mathbf{rot})} := \|\boldsymbol{\tau}\|^2_{-1} + \|\operatorname{rot}\mathbf{rot}\boldsymbol{\tau}\|^2_{-2}.$$

Lemma 2.7. *Assume that the bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ is simply connected. Then the complex*

$$\boldsymbol{L}^2(\Omega;\mathbb{R}^2) \xrightarrow{\boldsymbol{\varepsilon}} \boldsymbol{H}^{-2}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) \xrightarrow{\mathrm{rot}\,\mathbf{rot}} H^{-2}(\Omega) \longrightarrow 0 \tag{2.35}$$

is exact.

Proof. It's trivial that (2.35) is a complex. Next we show the exactness. For any $\boldsymbol{\tau} \in \ker(\mathrm{rot}\,\mathbf{rot})$, by the rotated version of the exact sequence (2.20) with $s = -3$, there exists $v \in H^{-1}(\Omega)$ satisfying $\mathbf{rot}\boldsymbol{\tau} = \nabla v$. Since $\nabla v = \mathbf{rot}\begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}$, we have

$$\mathbf{rot}\left(\boldsymbol{\tau} - \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}\right) = \mathbf{0}.$$

Thus $\boldsymbol{\tau} = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix} + \boldsymbol{\nabla}\boldsymbol{\phi}$ with $\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^2)$, which means $\boldsymbol{\tau} = \boldsymbol{\varepsilon}(\boldsymbol{\phi})$. Hence $\ker(\mathrm{rot}\,\mathbf{rot}) \subset \mathrm{img}(\boldsymbol{\varepsilon})$.

For any $v \in H^{-2}(\Omega)$, by the rotated version of the exact sequence (2.20) with $s = -2$, there exists $\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2)$ satisfying $v = \mathrm{rot}\,\boldsymbol{\phi}$. Again, $\boldsymbol{\phi} = \mathbf{rot}\boldsymbol{\tau}$ with $\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega,\mathbb{M})$. Then it holds $v = \mathrm{rot}\,\mathbf{rot}\boldsymbol{\tau}$. Note that $\mathrm{rot}\,\mathbf{rot}\boldsymbol{\tau} = \mathrm{rot}\,\mathbf{rot}\,\mathrm{sym}\boldsymbol{\tau}$ where $\mathrm{sym}\boldsymbol{\tau} = (\boldsymbol{\tau} + \boldsymbol{\tau}^T)/2$. Hence $v = \mathrm{rot}\,\mathbf{rot}\,\mathrm{sym}\boldsymbol{\tau}$, which implies $\mathrm{img}(\mathrm{rot}\,\mathbf{rot}) = H^{-2}(\Omega)$. ☐

With complex (2.35) and the fact that $\Delta^2 = \mathrm{rot}\,\mathbf{rot}\boldsymbol{I}\mathbf{curl}\,\mathrm{curl}$, we have the commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{L}^2(\Omega;\mathbb{R}^2) & \xrightarrow{\boldsymbol{\varepsilon}} & \boldsymbol{H}^{-2}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) & \xrightarrow{\mathrm{rot}\,\mathbf{rot}} & H^{-2}(\Omega) \\
 & & \uparrow{\scriptstyle \boldsymbol{I}} & & \uparrow{\scriptstyle \Delta^2} \\
 & & \boldsymbol{H}_0(\mathrm{div},\Omega;\mathbb{S}) & \xleftarrow{\mathbf{curl}\,\mathrm{curl}} & H_0^2(\Omega)
\end{array}$$

which leads to a Helmholtz decomposition

$$\boldsymbol{H}^{-2}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) = \boldsymbol{\varepsilon}\boldsymbol{L}^2(\Omega;\mathbb{R}^2) \oplus \mathbf{curl}\,\mathrm{curl}\,H_0^2(\Omega). \tag{2.36}$$

By Helmholtz decomposition (2.36) and the similar argument used in the proof of Lemma 2.5, we can prove that

$$(\boldsymbol{H}_0(\mathrm{div},\Omega;\mathbb{S}))' = \boldsymbol{H}^{-2}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}),$$

if $\Omega \subset \mathbb{R}^2$ is simply connected.

More differential complex and Helmholtz decompositions can be obtained and some of them will be discussed along with mixed formulations of elliptic systems.

3. Abstract Mixed Formulation and Its Decomposition

In this section we present an abstract mixed formulation and use a Helmholtz decomposition to decouple the saddle point system into several elliptic problems.

3.1. **Framework.** Assume we have the exact sequence

$$P \xrightarrow{\;d^-\;} \Sigma \xrightarrow{\;d\;} V', \tag{3.1}$$

and the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{J_X} & X' & & \\ & & \cup & & \\ P & \xrightarrow{d^-} & \Sigma & \xrightarrow{d} & V' \\ & & \uparrow{\scriptstyle \Pi_\Sigma} & & \uparrow{\scriptstyle J_V} \\ & & \widetilde{\Sigma} & \xleftarrow{\Pi_V} & V \end{array}\quad, \tag{3.2}$$

where the isomorphisms J_X and J_V are given by (2.9), i.e., the inverse of the Riesz representation operator, and the rest linear operators are all continuous but not necessarily isomorphic. By Theorem 2.1, we have a stable Helmholtz decomposition

$$\Sigma = d^- P \oplus \Pi_\Sigma \Pi_V V. \tag{3.3}$$

We emphasize that we do not need to know neither the short exact sequence at the bottom nor the space $\widetilde{\Sigma}$ in a very precise form (i.e. $\widetilde{\Sigma}$ can be reasonably enlarged to include the image space $\Pi_V V$).

3.1.1. *Two-term formulation.* An abstract mixed formulation based on the commutative diagram (3.2) is: given $g \in \Sigma'$ and $f \in V'$, find $(\sigma, u) \in \Sigma \times V$ such that

$$(\sigma, \tau)_{X'} + \langle d\tau, u\rangle = \langle g, \tau\rangle \quad \forall\, \tau \in \Sigma, \tag{3.4}$$

$$\langle d\sigma, v\rangle = \langle f, v\rangle \quad \forall\, v \in V. \tag{3.5}$$

The operator system for the mixed formulation (3.4)-(3.5) is

$$\begin{aligned} I' A_{X'} I\sigma + d'u &= g \quad \text{ in } \Sigma', \\ d\sigma &= f \quad \text{ in } V'. \end{aligned}$$

Here I is the injection operator from Σ into X', and $A_{X'} : X' \to X$ means the operator induced by the bilinear form $(\cdot,\cdot)_{X'}$, i.e.

$$\langle A_{X'}\tau, \varsigma\rangle = (\tau, \varsigma)_{X'} \quad \forall\, \tau, \varsigma \in X'.$$

Remark 3.1. The bilinear form $(\cdot,\cdot)_{X'}$ is not necessary to be an inner product unless we intend to involve X and J_X in the mixed formulation. We only assume $(\cdot,\cdot)_{X'}$ is positive semidefinite and symmetric. □

To show the well-posedness of the mixed formulation (3.4)-(3.5), we assume the following norm equivalence

$$\|\tau\|_\Sigma^2 \eqsim \|\tau\|_{X'}^2 + \|d\tau\|_{V'}^2 \quad \forall\, \tau \in \Sigma. \tag{3.6}$$

This norm equivalence is usually trivial, and it holds apparently for all the examples in this paper.

Theorem 3.2. *Assume the exact sequence* (3.1)*, the commutative diagram* (3.2) *and the norm equivalence* (3.6) *hold, then the mixed formulation* (3.4)*-*(3.5) *is uniquely solvable. Moreover, we have the stability result*

$$\|\sigma\|_\Sigma + \|u\|_V \lesssim \|g\|_{\Sigma'} + \|f\|_{V'}.$$

Proof. It is trivial that the bilinear forms in the mixed formulation (3.4)-(3.5) are continuous due to (3.6). Using (3.6) again, it is also obvious that

$$\|\tau\|_\Sigma \lesssim \|\tau\|_{X'} + \|d\tau\|_{V'} = \|\tau\|_{X'} \quad \forall\, \tau \in \ker d.$$

By Babuška-Brezzi theory (cf. [9, 18, 13]), it suffices to prove the inf-sup condition

$$\|v\|_V \lesssim \sup_{\tau\in\Sigma} \frac{\langle d\tau, v\rangle}{\|\tau\|_\Sigma} \quad \forall\, v \in V. \tag{3.7}$$

For each $v \in V$, let $\tau = \Pi_\Sigma \Pi_V v$. It is apparent that

$$\|\tau\|_\Sigma = \|\Pi_\Sigma \Pi_V v\|_\Sigma \lesssim \|v\|_V.$$

Then making use of the commutative diagram (3.2) and (2.9), it follows

$$\langle d\tau, v\rangle = \langle d\Pi_\Sigma \Pi_V v, v\rangle = \langle J_V v, v\rangle = \|v\|_V^2.$$

Hence we have

$$\|v\|_V \|\tau\|_\Sigma \lesssim \|v\|_V^2 = \langle d\tau, v\rangle,$$

which means the inf-sup condition (3.7). □

3.1.2. *Unfolded three-term formulation.* We derive an equivalent three-term formulation of the mixed formulation (3.4)-(3.5) when X' is a Sobolev space of negative order. To this end, we assume the bilinear form $(\cdot,\cdot)_{X'}$ is the corresponding inner product of X'.

Let $\phi = J_X^{-1}\sigma \in X$. By (2.10), we can rewrite (3.4) as

$$\langle \tau, \phi\rangle + \langle d\tau, u\rangle = \langle g, \tau\rangle \quad \forall\, \tau \in \Sigma.$$

Noting that $\sigma = J_X\phi$, it follows from (2.9)

$$\langle \sigma, \psi\rangle = \langle J_X\phi, \psi\rangle = (\phi, \psi)_X \quad \forall\, \psi \in X.$$

Therefore the mixed formulation (3.4)-(3.5) is equivalent to an unfolded three-term formulation: find $(\phi, u, \sigma) \in X \times V \times \Sigma$ such that

$$(\phi, \psi)_X - \langle \sigma, d'v + \psi\rangle = -\langle f, v\rangle + \langle g_X, \psi\rangle \quad \forall\, (\psi, v) \in X \times V, \tag{3.8}$$

$$\langle d'u + \phi, \tau\rangle = \langle g, \tau\rangle \qquad \forall\, \tau \in \Sigma, \tag{3.9}$$

with $g_X = 0$. It is interesting to note that the variable σ can be formally interpreted as the Lagrange multiplier to impose the constraint $\phi = -d'u$ in Σ' if $g = 0$.

According to Theorem 3.2, we immediately obtain the well-posedness of the mixed formulation (3.8)-(3.9). The well-posedness of the mixed formulation (3.8)-(3.9) for general $g_X \in X'$ is given as follows.

Theorem 3.3. *Assume the exact sequence* (3.1)*, the commutative diagram* (3.2) *and the norm equivalence* (3.6) *hold, then the unfolded mixed formulation* (3.8)-(3.9) *is uniquely solvable. Moreover, we have the stability result*

$$\|\phi\|_X + \|\sigma\|_\Sigma + \|u\|_V \lesssim \|g_X\|_{X'} + \|g\|_{\Sigma'} + \|f\|_{V'}.$$

Proof. By (3.6), all the bilinear forms in the mixed formulation (3.8)-(3.9) are obviously continuous. Let $(\psi, v) \in X \times V$ satisfy

$$\langle \tau, \psi\rangle + \langle d\tau, v\rangle = 0 \quad \forall\, \tau \in \Sigma.$$

Then $\psi = -d'v$. Due to (2.9) and the commutative diagram (3.2), we get

$$\begin{aligned}\|v\|_V^2 =& \langle J_V v, v\rangle = \langle d\Pi_\Sigma\Pi_V v, v\rangle = \langle \Pi_\Sigma\Pi_V v, d'v\rangle \\ \le& \|\Pi_\Sigma\Pi_V v\|_\Sigma \|d'v\|_{\Sigma'} \lesssim \|v\|_V \|d'v\|_{\Sigma'}.\end{aligned}$$

Noting the fact that $X \subset \Sigma'$, we have

$$\|v\|_V \lesssim \|d'v\|_X = \|\psi\|_X,$$

which implies the coercivity on the kernel.

On the other hand, for any $\tau \in \Sigma$ it follows from (3.6) that

$$\|\tau\|_\Sigma \lesssim \|\tau\|_{X'} + \|d\tau\|_{V'} = \sup_{\psi\in X}\frac{\langle \tau,\psi\rangle}{\|\psi\|_X} + \sup_{v\in V}\frac{\langle d\tau, v\rangle}{\|v\|_V} \lesssim \sup_{\psi\in X, v\in V}\frac{\langle \tau,\psi\rangle + \langle d\tau, v\rangle}{\|\psi\|_X + \|v\|_V},$$

which is just the inf-sup condition. Therefore the required result is guaranteed by Babuška-Brezzi theory. □

3.1.3. *Decoupled formulation.* We decompose the mixed formulation (3.4)-(3.5) using the Helmholtz decomposition (3.3). Applying the Helmholtz decomposition (3.3) to both the trial and test functions

$$\sigma = d^- p + \Pi_\Sigma \Pi_V w, \quad \tau = d^- q + \Pi_\Sigma\Pi_V \chi,$$

where $p, q \in P/\ker d^-$, and $w, \chi \in V$. Then substituting them into the mixed formulation (3.4)-(3.5), we have

$$(d^- p + \Pi_\Sigma\Pi_V w, \Pi_\Sigma\Pi_V\chi)_{X'} + \langle d\Pi_\Sigma\Pi_V\chi, u\rangle = \langle g, \Pi_\Sigma\Pi_V\chi\rangle, \tag{3.10}$$

$$(d^- p + \Pi_\Sigma\Pi_V w, d^- q)_{X'} = \langle g, d^- q\rangle, \tag{3.11}$$

$$\langle d\Pi_\Sigma\Pi_V w, v\rangle = \langle f, v\rangle, \tag{3.12}$$

for any $\chi \in V$, $q \in P/\ker d^-$ and $v \in V$. We obtain from the commutative diagram (3.2) and (2.9) again

$$\langle d\Pi_\Sigma\Pi_V w, v\rangle = \langle J_V w, v\rangle = (w, v)_V, \quad \langle d\Pi_\Sigma\Pi_V\chi, u\rangle = (\chi, u)_V.$$

Therefore, the mixed formulation (3.10)-(3.12) is equivalent to (in backwards): find $w \in V$, $p \in P/\ker d^-$, and $u \in V$ such that

$$(w, v)_V = \langle f, v\rangle \qquad \forall\, v \in V, \tag{3.13}$$

$$(d^- p, d^- q)_{X'} = \langle g, d^- q\rangle - (\Pi_\Sigma\Pi_V w, d^- q)_{X'} \qquad \forall\, q \in P/\ker d^-, \tag{3.14}$$

$$(u, \chi)_V = \langle g, \Pi_\Sigma\Pi_V\chi\rangle - (\sigma, \Pi_\Sigma\Pi_V\chi)_{X'} \qquad \forall\, \chi \in V, \tag{3.15}$$

where $\sigma = d^- p + \Pi_\Sigma\Pi_V w$.

Remark 3.4. When the decomposition (3.3) is orthogonal with respect to $(\cdot,\cdot)_{X'}$ and $g = 0$, the second equation (3.14) will disappear. □

Applying the Helmholtz decomposition (3.3) to the unfolded formulation, the uncoupled mixed formulation (3.8)-(3.9) is equivalent to find $w, u \in V$, $\phi \in X$, and $p \in P/\ker d^-$ such that

$$(w, v)_V = \langle f, v\rangle \qquad \forall\, v \in V, \tag{3.16}$$

$$(\phi, \psi)_X - \langle d^- p, \psi\rangle = \langle \Pi_\Sigma\Pi_V w, \psi\rangle \qquad \forall\, \psi \in X, \tag{3.17}$$

$$\langle d^- q, \phi\rangle = \langle g, d^- q\rangle \qquad \forall\, q \in P/\ker d^-, \tag{3.18}$$

$$(u, \chi)_V = \langle g - \phi, \Pi_\Sigma\Pi_V\chi\rangle \qquad \forall\, \chi \in V. \tag{3.19}$$

The middle system (3.17)-(3.18) of (ϕ, p) is now a Stokes-type system.

We summarize the former derivation as follows.

Theorem 3.5. *Assume the exact sequence (3.1), the commutative diagram (3.2) and the norm equivalence (3.6) hold, then the mixed formulation (3.4)-(3.5) can be decoupled as three elliptic equations (3.13)-(3.15) or four equations (3.16)-(3.19).*

In the rest of this section, we shall apply our abstract framework to several concrete examples.

3.2. **HHJ mixed formulation.** Based on the commutative diagram

$$\begin{array}{ccccc} & & \boldsymbol{L}^2(\Omega;\mathbb{S}) & & \\ & & \cup & & \\ \boldsymbol{H}^1(\Omega;\mathbb{R}^2) & \xrightarrow{\nabla^s\times} & \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}) & \xrightarrow{\operatorname{div}\mathbf{div}} & H^{-1}(\Omega) \\ & & \boldsymbol{I}\uparrow & & \uparrow\Delta \\ & & \boldsymbol{H}_0^1(\Omega;\mathbb{S}) & \xleftarrow{\boldsymbol{\pi}} & H_0^1(\Omega) \end{array},$$

the mixed formulation is to find $(\boldsymbol{\sigma}, u) \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}) \times H_0^1(\Omega)$ such that

$$(\boldsymbol{\sigma},\boldsymbol{\tau}) + \langle \operatorname{div}\mathbf{div}\boldsymbol{\tau}, u\rangle = 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{H}^{-1}(\operatorname{div}\mathbf{div},\Omega;\mathbb{S}), \tag{3.20}$$

$$\langle \operatorname{div}\mathbf{div}\boldsymbol{\sigma}, v\rangle = \langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \tag{3.21}$$

where $f \in H^{-1}(\Omega)$. This is just the Hellan-Herrmann-Johnson (HHJ) mixed formulation [41, 42, 48]. According to Theorem 3.2, the HHJ mixed formulation is well-posed.

Applying Helmholtz decomposition (2.32), the mixed formulation (3.20)-(3.21) can be decoupled to find $w \in H_0^1(\Omega)$, $\boldsymbol{p} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RT}_0$ and $u \in H_0^1(\Omega)$ such that

$$(\nabla w, \nabla v) = -\langle f, v\rangle \quad \forall\, v \in H_0^1(\Omega), \tag{3.22}$$

$$(\nabla^s \times \boldsymbol{p}, \nabla^s \times \boldsymbol{q}) = -(\boldsymbol{\pi} w, \nabla^s \times \boldsymbol{q}) \quad \forall\, \boldsymbol{q} \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RT}_0, \tag{3.23}$$

$$(\nabla u, \nabla \chi) = (\boldsymbol{\sigma}, \boldsymbol{\pi}\chi) \quad \forall\, \chi \in H_0^1(\Omega), \tag{3.24}$$

where $\boldsymbol{\sigma} = \nabla^s \times \boldsymbol{p} + \boldsymbol{\pi} w$, and

$$\boldsymbol{RT}_0 := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{x} \right\}.$$

The second equation is also equivalent to the linear elasticity problem

$$(\boldsymbol{\varepsilon}(\boldsymbol{p}^\perp), \boldsymbol{\varepsilon}(\boldsymbol{q}^\perp)) = -(\boldsymbol{\pi} w, \boldsymbol{\varepsilon}(\boldsymbol{q}^\perp)) \quad \forall\, \boldsymbol{q}^\perp \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RM}$$

where rigid motion space

$$\boldsymbol{RM} := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{x}^\perp \right\}.$$

Such decomposition is firstly obtained in [49]. The equivalence between the mixed formulation (3.20)-(3.21) and formulation (3.22)-(3.24) can be extended to three dimensions [57].

3.3. **Ciarlet-Raviart mixed formulation of biharmonic equation.** Let

$$H^{-1}(\Delta,\Omega) := \{v \in L^2(\Omega) : \Delta v \in H^{-1}(\Omega)\},$$
$$H_0(\Delta,\Omega) := \{v \in L^2(\Omega) : \Delta v = 0\}.$$

Equip $H^{-1}(\Delta,\Omega)$ with norm (cf. [11, 2, 65])

$$\|v\|_{-1,\Delta}^2 := \|v\|_0^2 + \|\Delta v\|_{-1}^2.$$

Define $\boldsymbol{\pi}(v) = v\boldsymbol{I}_{2\times 2}$. The following commutative diagram

$$\begin{array}{ccccc} & & L^2(\Omega) & & \\ & & \cup & & \\ H_0(\Delta,\Omega) & \xrightarrow{I} & H^{-1}(\Delta,\Omega) & \xrightarrow{\Delta} & H^{-1}(\Omega) \\ & & {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle \Delta} \\ & & H_0^1(\Omega) & \xleftarrow{I} & H_0^1(\Omega) \end{array}$$

is apparent. And it's also trivial that the complex

$$H_0(\Delta,\Omega) \xrightarrow{I} H^{-1}(\Delta,\Omega) \xrightarrow{\Delta} H^{-1}(\Omega) \longrightarrow 0.$$

is exact.

According to Theorem 2.1, we thus acquire the Hemholtz decomposition (cf. [65, Lemma 3.1])

$$H^{-1}(\Delta,\Omega) = H_0(\Delta,\Omega) \oplus H_0^1(\Omega).$$

The mixed formulation is to find $(\sigma, u) \in H^{-1}(\Delta,\Omega) \times H_0^1(\Omega)$ such that (cf. [11, 2, 65])

$$(\sigma,\tau) + \langle \Delta\tau, u\rangle = 0 \qquad \forall\ \tau \in H^{-1}(\Delta,\Omega), \tag{3.25}$$
$$\langle \Delta\sigma, v\rangle = \langle f, v\rangle \quad \forall\ v \in H_0^1(\Omega). \tag{3.26}$$

Applying Theorems 3.2 and 3.5, this mixed formulation is well-posed, and can be formally decoupled to find $w \in H_0^1(\Omega)$, $p \in H_0(\Delta,\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned} (\nabla w, \nabla v) &= -\langle f, v\rangle && \forall\ v \in H_0^1(\Omega),\\ (p,q) &= -(w,q) && \forall\ q \in H_0(\Delta,\Omega),\\ (\nabla u, \nabla\chi) &= (p+w,\chi) && \forall\ \chi \in H_0^1(\Omega). \end{aligned}$$

The decoupled formulation is not easy to discretize since a finite element space of $H_0(\Delta,\Omega)$ seems difficult to construct. More details on $H^{-1}(\Delta,\Omega)$ and $H_0(\Delta,\Omega)$ can be found in [65].

3.4. **Biharmonic equation.** We consider a two dimensional and rotated version of (2.26). Apparently $\Delta = \operatorname{rot} I \operatorname{curl}$, thus we have the commutative diagram

$$\begin{array}{ccccc} & & \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2) \xrightarrow{\Delta} & \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2) & \\ & & & \cup & \\ L_0^2(\Omega) & \xrightarrow{\text{grad}} & & \boldsymbol{H}^{-1}(\text{rot},\Omega) \xrightarrow{\text{rot}} & H^{-1}(\Omega) \\ & & & {\scriptstyle I}\uparrow & \uparrow{\scriptstyle \Delta} \\ & & & \boldsymbol{H}_0(\text{div},\Omega) \xleftarrow{\text{curl}} & H_0^1(\Omega) \end{array}, \tag{3.27}$$

where recall that

$$\boldsymbol{H}^{-1}(\mathrm{rot},\Omega) := \{\boldsymbol{\phi} \in \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2) : \mathrm{rot}\,\boldsymbol{\phi} \in H^{-1}(\Omega)\}$$

with norm

$$\|\boldsymbol{\phi}\|^2_{H^{-1}(\mathrm{rot})} := \|\boldsymbol{\phi}\|^2_{-1} + \|\mathrm{rot}\,\boldsymbol{\phi}\|^2_{-1}.$$

According to Theorem 2.1, we have the Helmholtz decomposition

$$\boldsymbol{H}^{-1}(\mathrm{rot},\Omega) = \nabla L_0^2(\Omega) \oplus \mathrm{curl}\, H_0^1(\Omega). \tag{3.28}$$

The corresponding mixed formulation is to find $(\boldsymbol{\gamma}, u) \in \boldsymbol{H}^{-1}(\mathrm{rot},\Omega) \times H_0^1(\Omega)$ such that

$$(\boldsymbol{\gamma},\boldsymbol{\beta})_{-1} - \langle \mathrm{rot}\,\boldsymbol{\beta}, u\rangle = 0 \qquad \forall\ \boldsymbol{\beta} \in \boldsymbol{H}^{-1}(\mathrm{rot},\Omega), \tag{3.29}$$

$$\langle \mathrm{rot}\,\boldsymbol{\gamma}, v\rangle = \langle f, v\rangle \quad \forall\ v \in H_0^1(\Omega), \tag{3.30}$$

where $f \in H^{-1}(\Omega)$ and $(\boldsymbol{\gamma},\boldsymbol{\beta})_{-1} := -\langle \boldsymbol{\Delta}^{-1}\boldsymbol{\gamma}, \boldsymbol{\beta}\rangle = -\langle \boldsymbol{\gamma}, \boldsymbol{\Delta}^{-1}\boldsymbol{\beta}\rangle$.

By introducing variable $\boldsymbol{\phi} = -\boldsymbol{\Delta}^{-1}\boldsymbol{\gamma} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, the unfolded formulation is: find $(\boldsymbol{\gamma}, u, \boldsymbol{\phi}) \in \boldsymbol{H}^{-1}(\mathrm{rot},\Omega) \times H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$ such that

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + \langle \boldsymbol{\gamma}, \mathrm{curl}\, v - \boldsymbol{\psi}\rangle = \langle f, v\rangle \quad \forall\ (v,\boldsymbol{\psi}) \in H_0^1(\Omega) \times \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2),$$

$$\langle \boldsymbol{\beta}, \mathrm{curl}\, u - \boldsymbol{\phi}\rangle = 0 \qquad \forall\ \boldsymbol{\beta} \in \boldsymbol{H}^{-1}(\mathrm{rot},\Omega).$$

which is just the rotation of problem (2.4) in [20].

The second equation implies $\boldsymbol{\phi} = \mathrm{curl}\, u$, together with $\boldsymbol{\gamma} = -\boldsymbol{\Delta}\boldsymbol{\phi}$ and $\mathrm{rot}\,\boldsymbol{\gamma} = f$, hence we conclude u satisfies the biharmonic equation

$$\Delta^2 u = \mathrm{rot}\,\Delta\,\mathrm{curl}\, u = \mathrm{rot}\,\boldsymbol{\gamma} = f, \tag{3.31}$$

with homogenous Dirichlet boundary condition.

The decoupled and unfolded formulation is: find $w \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, $p \in L_0^2(\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$(\mathrm{curl}\, w, \mathrm{curl}\, v) = \langle f, v\rangle \qquad \forall\ v \in H_0^1(\Omega), \tag{3.32}$$

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\mathrm{div}\,\boldsymbol{\psi}, p) = (\mathrm{curl}\, w, \boldsymbol{\psi}) \qquad \forall\ \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2), \tag{3.33}$$

$$(\mathrm{div}\,\boldsymbol{\phi}, q) = 0 \qquad \forall\ q \in L_0^2(\Omega), \tag{3.34}$$

$$(\mathrm{curl}\, u, \mathrm{curl}\,\chi) = (\boldsymbol{\phi}, \mathrm{curl}\,\chi) \qquad \forall\ \chi \in H_0^1(\Omega). \tag{3.35}$$

Therefore we have shown that the biharmonic equation (3.31) is equivalent to two Poisson equations and one Stokes equation [45, 47].

Such decoupling of the biharmonic equation in two dimensions can be generalized in various ways. First, if we equip space $\boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$ with full norm $\|\cdot\|_1$, we derive the following system find $w \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, $p \in L_0^2(\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$(\mathrm{curl}\, w, \mathrm{curl}\, v) = \langle f, v\rangle \qquad \forall\ v \in H_0^1(\Omega),$$

$$(\boldsymbol{\phi}, \boldsymbol{\psi}) + (\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\mathrm{div}\,\boldsymbol{\psi}, p) = (\mathrm{curl}\, w, \boldsymbol{\psi}) \qquad \forall\ \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2),$$

$$(\mathrm{div}\,\boldsymbol{\phi}, q) = 0 \qquad \forall\ q \in L_0^2(\Omega),$$

$$(\mathrm{curl}\, u, \mathrm{curl}\,\chi) = (\boldsymbol{\phi}, \mathrm{curl}\,\chi) \qquad \forall\ \chi \in H_0^1(\Omega).$$

This system is equivalent to the following fourth order elliptic problem with homogenous Dirichlet boundary condition

$$\Delta^2 u - \Delta u = f.$$

Another direction is that the biharmonic equation (3.31) in three dimensions is equivalent to find $w \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$, $\boldsymbol{p} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)/\nabla H^1(\Omega)$ and $u \in H_0^1(\Omega)$ such that

$$\begin{aligned}(\nabla w, \nabla v) &= \langle f, v\rangle & \forall\, v \in H_0^1(\Omega),\\ (\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\operatorname{curl}\boldsymbol{\psi}, \boldsymbol{p}) &= (\nabla w, \boldsymbol{\psi}) & \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3),\\ (\operatorname{curl}\boldsymbol{\phi}, \boldsymbol{q}) &= 0 & \forall\, \boldsymbol{q} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)/\nabla H^1(\Omega),\\ (\nabla u, \nabla \chi) &= (\boldsymbol{\phi}, \nabla\chi) & \forall\, \chi \in H_0^1(\Omega).\end{aligned}$$

One more generalization to three dimensions will be discussed below.

3.5. **Fourth order** curl **equation.** By the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3) & \xrightarrow{\Delta} & \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^3) & & \\ & & \cup & & \\ L_0^2(\Omega) & \xrightarrow{\text{grad}} & \boldsymbol{H}^{-1}(\operatorname{curl},\Omega) & \xrightarrow{\text{curl}} & (K_0^c)' \\ & & I\uparrow & & \uparrow \operatorname{curl}\operatorname{curl} \\ & & \boldsymbol{H}_0(\operatorname{div},\Omega) & \xleftarrow{\text{curl}} & K_0^c \end{array},$$

the corresponding mixed formulation is to find $(\boldsymbol{\gamma}, \boldsymbol{u}) \in \boldsymbol{H}^{-1}(\operatorname{curl},\Omega) \times K_0^c$ such that

$$(\boldsymbol{\gamma}, \boldsymbol{\beta})_{-1} - \langle \operatorname{curl}\boldsymbol{\beta}, \boldsymbol{u}\rangle = 0 \qquad \forall\, \boldsymbol{\beta} \in \boldsymbol{H}^{-1}(\operatorname{curl},\Omega), \tag{3.36}$$

$$\langle \operatorname{curl}\boldsymbol{\gamma}, \boldsymbol{v}\rangle = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in K_0^c, \tag{3.37}$$

where $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega;\mathbb{R}^3)$ satisfying $\operatorname{div}\boldsymbol{f} = 0$.

By introducing variable $\boldsymbol{\phi}$ and applying the Helmholtz decomposition (2.30)

$$\boldsymbol{\phi} = -\boldsymbol{\Delta}^{-1}\boldsymbol{\gamma} = -\boldsymbol{\Delta}^{-1}(\operatorname{curl}\boldsymbol{w} + \nabla p) \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3),$$

the decoupled and unfolded system is: find $\boldsymbol{w} \in K_0^c$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$, $p \in L_0^2(\Omega)$ and $\boldsymbol{u} \in K_0^c$ such that

$$(\operatorname{curl}\boldsymbol{w}, \operatorname{curl}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in K_0^c, \tag{3.38}$$

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\operatorname{div}\boldsymbol{\psi}, p) = (\operatorname{curl}\boldsymbol{w}, \boldsymbol{\psi}) \qquad \forall\, \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3), \tag{3.39}$$

$$(\operatorname{div}\boldsymbol{\phi}, q) = 0 \qquad \forall\, q \in L_0^2(\Omega), \tag{3.40}$$

$$(\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\boldsymbol{\chi}) = (\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\chi}) \qquad \forall\, \boldsymbol{\chi} \in K_0^c. \tag{3.41}$$

According to (3.40)-(3.41), we have $\operatorname{curl}\boldsymbol{u} = \boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$. Note that

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) = (\operatorname{curl}\boldsymbol{\phi}, \operatorname{curl}\boldsymbol{\psi}) + (\operatorname{div}\boldsymbol{\phi}, \operatorname{div}\boldsymbol{\psi}).$$

Thus we get from (3.39)

$$(\operatorname{curl}\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\operatorname{curl}\boldsymbol{v}) = (\operatorname{curl}\boldsymbol{w}, \operatorname{curl}\boldsymbol{v})$$

for any $\boldsymbol{v} \in K_0^c$ satisfying $\operatorname{curl}\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^3)$. Combined with (3.38), formulation (3.38)-(3.41) is equivalent to find $\boldsymbol{u} \in \boldsymbol{H}_0^2(\operatorname{curl},\Omega)$ such that

$$(\operatorname{curl}\operatorname{curl}\boldsymbol{u}, \operatorname{curl}\operatorname{curl}\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^2(\operatorname{curl},\Omega),$$

where

$$\begin{aligned}\boldsymbol{H}_0^2(\operatorname{curl},\Omega) := \{&\boldsymbol{v} \in \boldsymbol{L}^2(\Omega,\mathbb{R}^3) : \operatorname{curl}\boldsymbol{v}, \operatorname{curl}\operatorname{curl}\boldsymbol{v} \in \boldsymbol{L}^2(\Omega,\mathbb{R}^3),\\ &\operatorname{div}\boldsymbol{v} = 0, \text{ and } \boldsymbol{v}\times\boldsymbol{n} = (\nabla\times\boldsymbol{v})\times\boldsymbol{n} = 0\}.\end{aligned}$$

This equivalence has been given recently in [62].

Therefore we can solve the four curl problem by solving two Maxwell's equations and one Stokes equation. The Maxwell's equation with divergence-free constraint can be further decoupled into one vector Poisson equation and one scalar Poisson equation [27].

3.6. **A strain-based mixed formulation for linear elasticity.** Let

$$\boldsymbol{H}^{-1}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) := \{\boldsymbol{\tau}\in\boldsymbol{L}^2(\Omega;\mathbb{S}) : \mathrm{rot}\,\mathbf{rot}\,\boldsymbol{\tau}\in H^{-1}(\Omega)\}.$$

The rotated complex of the exact sequence (2.31)

$$\boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RM} \xrightarrow{\boldsymbol{\varepsilon}} \boldsymbol{H}^{-1}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) \xrightarrow{\mathrm{rot}\,\mathbf{rot}} H^{-1}(\Omega) \tag{3.42}$$

is surely exact. Based on the commutative diagram

$$\begin{array}{ccccc}
 & & \boldsymbol{L}^2(\Omega;\mathbb{S}) & & \\
 & & \cup & & \\
\boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RM} & \xrightarrow{\boldsymbol{\varepsilon}} & \boldsymbol{H}^{-1}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) & \xrightarrow{\mathrm{rot}\,\mathbf{rot}} & H^{-1}(\Omega) \\
 & & \boldsymbol{I}\uparrow & & \uparrow\Delta \\
 & & \boldsymbol{H}_0^1(\Omega;\mathbb{S}) & \xleftarrow{\boldsymbol{\pi}} & H_0^1(\Omega)
\end{array},$$

the mixed formulation is to find $(\boldsymbol{\sigma},p)\in\boldsymbol{H}^{-1}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S})\times H_0^1(\Omega)$ such that

$$(\mathbb{C}\boldsymbol{\sigma},\boldsymbol{\tau}) + \langle\mathrm{rot}\,\mathbf{rot}\,\boldsymbol{\tau},p\rangle = (\boldsymbol{f},\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\boldsymbol{H}^{-1}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}), \tag{3.43}$$

$$\langle\mathrm{rot}\,\mathbf{rot}\,\boldsymbol{\sigma},q\rangle = 0 \qquad \forall\,q\in H_0^1(\Omega), \tag{3.44}$$

where $\boldsymbol{f}\in\boldsymbol{L}^2(\Omega;\mathbb{T})$.

The mixed formulation (3.43)-(3.44) is formally the rotation of the HHJ mixed formulation (3.20)-(3.21). On the other side, by the exactness of the complex (3.42) and (3.44), we have $\boldsymbol{\sigma}=\boldsymbol{\varepsilon}(\boldsymbol{u})$ for some $\boldsymbol{u}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RM}$. By substituting it into (3.43), the mixed formulation (3.43)-(3.44) will be equivalent to the pure traction problem of linear elasticity

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v})) = (\boldsymbol{f},\boldsymbol{\varepsilon}(\boldsymbol{v})) \quad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1(\Omega;\mathbb{R}^2)/\boldsymbol{RM}.$$

If we treat (3.44) as a constraint condition (known as Saint Venant compatibility condition) directly, then the mixed formulation (3.43)-(3.44) can be rewritten as a primal formulation with only the strain tensor field $\boldsymbol{\sigma}$ as unknown, which was studied recently in [29, 30, 58].

3.7. **Triharmonic equation.** In view of the commutative diagram

$$\begin{array}{ccccc}
\boldsymbol{H}_0^1(\Omega;\mathbb{S}) & \xrightarrow{\Delta} & \boldsymbol{H}^{-1}(\Omega;\mathbb{S}) & & \\
 & & \cup & & \\
\boldsymbol{L}^2(\Omega;\mathbb{R}^2) & \xrightarrow{\boldsymbol{\varepsilon}} & \boldsymbol{H}^{-2}(\mathrm{rot}\,\mathbf{rot},\Omega;\mathbb{S}) & \xrightarrow{\mathrm{rot}\,\mathbf{rot}} & H^{-2}(\Omega) \\
 & & \boldsymbol{I}\uparrow & & \uparrow\Delta^2 \\
 & & \boldsymbol{H}_0(\mathrm{div},\Omega;\mathbb{S}) & \xleftarrow{\mathbf{curl}\,\mathrm{curl}} & H_0^2(\Omega)
\end{array},$$

the corresponding mixed formulation is to find $(\boldsymbol{\gamma}, u) \in \boldsymbol{H}^{-2}(\operatorname{rot}\mathbf{rot}, \Omega; \mathbb{S}) \times H_0^2(\Omega)$ such that

$$(\boldsymbol{\gamma}, \boldsymbol{\beta})_{-1} - \langle \operatorname{rot}\mathbf{rot}\boldsymbol{\beta}, u\rangle = 0 \qquad \forall\ \boldsymbol{\beta} \in \boldsymbol{H}^{-2}(\operatorname{rot}\mathbf{rot}, \Omega; \mathbb{S}), \tag{3.45}$$

$$\langle \operatorname{rot}\mathbf{rot}\boldsymbol{\gamma}, v\rangle = \langle f, v\rangle \quad \forall\ v \in H_0^2(\Omega), \tag{3.46}$$

where $f \in H^{-2}(\Omega)$.

By introducing variable and applying the Helmholtz decomposition (2.36)

$$\boldsymbol{\phi} = -\boldsymbol{\Delta}^{-1}\boldsymbol{\gamma} = -\boldsymbol{\Delta}^{-1}(\boldsymbol{\varepsilon}(p) + \mathbf{curl}\operatorname{curl} w) \in \boldsymbol{H}_0^1(\Omega; \mathbb{S}),$$

the decoupled system is: find $w \in H_0^2(\Omega)$, $\boldsymbol{\phi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{S})$, $\boldsymbol{p} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^2)/\boldsymbol{RM}$ and $u \in H_0^2(\Omega)$ such that

$$(\mathbf{curl}\operatorname{curl} w, \mathbf{curl}\operatorname{curl} v) = \langle f, v\rangle \qquad \forall\ v \in H_0^2(\Omega), \tag{3.47}$$

$$(\boldsymbol{\nabla}\boldsymbol{\phi}, \boldsymbol{\nabla}\boldsymbol{\psi}) + (\mathbf{div}\boldsymbol{\psi}, \boldsymbol{p}) = (\mathbf{curl}\operatorname{curl} w, \boldsymbol{\psi}) \qquad \forall\ \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{S}), \tag{3.48}$$

$$(\mathbf{div}\boldsymbol{\phi}, \boldsymbol{q}) = 0 \qquad \forall\ \boldsymbol{q} \in \boldsymbol{L}^2(\Omega; \mathbb{R}^2)/\boldsymbol{RM}, \tag{3.49}$$

$$(\mathbf{curl}\operatorname{curl} u, \mathbf{curl}\operatorname{curl} \chi) = (\boldsymbol{\phi}, \mathbf{curl}\operatorname{curl} \chi) \qquad \forall\ \chi \in H_0^2(\Omega). \tag{3.50}$$

Hence

$$\Delta^3 u = \operatorname{rot}\mathbf{rot}\Delta\mathbf{curl}\operatorname{curl} u = -\operatorname{rot}\mathbf{rot}\boldsymbol{\gamma} = -f, \tag{3.51}$$

which is the triharmonic equation with homogeneous Dirichlet boundary condition. Hence we achieve that the triharmonic equation (3.51) is equivalent to two biharmonic equations and one Stokes equation, c.f. (3.47)-(3.50).

Recursively applying the decomposition, we can decouple the m-th harmonic equation $\Delta^m u = f$ with homogenous Dirichlet boundary condition, i.e., $u \in H_0^m(\Omega)$ into a sequence of Poisson and Stokes equations.

3.8. **Reissner-Mindlin plate.** Let $\mathcal{L}_E := -\mathbf{div}(\mathbb{C}\boldsymbol{\varepsilon}) : \boldsymbol{H}_0^1(\Omega; \mathbb{R}^2) \to \boldsymbol{H}^{-1}(\Omega; \mathbb{R}^2)$, and define

$$\|\boldsymbol{\gamma}\|_{-1,\mathcal{L}_E}^2 := \langle \mathcal{L}_E^{-1}\boldsymbol{\gamma}, \boldsymbol{\gamma}\rangle \qquad \forall\ \boldsymbol{\gamma} \in \boldsymbol{H}^{-1}(\Omega; \mathbb{R}^2),$$
$$\|\boldsymbol{\psi}\|_{1,\mathcal{L}_E}^2 := \langle \mathcal{L}_E\boldsymbol{\psi}, \boldsymbol{\psi}\rangle = (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\psi}), \boldsymbol{\varepsilon}(\boldsymbol{\psi})) \quad \forall\ \boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega; \mathbb{R}^2).$$

Then it holds for any $\boldsymbol{\gamma} \in \boldsymbol{H}^{-1}(\Omega; \mathbb{R}^2)$

$$\|\boldsymbol{\gamma}\|_{-1,\mathcal{L}_E} = \|\mathcal{L}_E^{-1}\boldsymbol{\gamma}\|_{1,\mathcal{L}_E} = \sup_{\boldsymbol{\psi}\in\boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)} \frac{\langle \boldsymbol{\gamma}, \boldsymbol{\psi}\rangle}{\|\boldsymbol{\psi}\|_{1,\mathcal{L}_E}}. \tag{3.52}$$

We shall consider the interpolation of the following two differential complexes:

$$\boldsymbol{H}^1(\Omega)/\mathbb{R} \xrightarrow{\operatorname{curl}} \boldsymbol{L}^2(\Omega, \mathbb{R}^2) \xrightarrow{\operatorname{div}} H^{-1}(\Omega), \tag{3.53}$$

$$\boldsymbol{L}_0^2(\Omega) \xrightarrow{\operatorname{curl}} \boldsymbol{H}^{-1}(\operatorname{div}, \Omega) \xrightarrow{\operatorname{div}} H^{-1}(\Omega) \tag{3.54}$$

to construct the commutative diagram

$$\begin{array}{ccccc} \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2) \xrightarrow{\mathcal{L}_E} \boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2) \supset & (\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)) & & \\ & \cup & & \\ L_0^2(\Omega)\cap t\,(H^1(\Omega)/\mathbb{R}) \xrightarrow{\text{curl}} & \boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2) & \xrightarrow{\text{div}} & H^{-1}(\Omega) \\ & {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle\Delta} \\ & \boldsymbol{H}_0(\operatorname{rot},\Omega)+t^{-1}\boldsymbol{L}^2(\Omega;\mathbb{R}^2) & \xleftarrow{\text{grad}} & H_0^1(\Omega) \end{array} . \tag{3.55}$$

Here $0 < t \lesssim 1$ is the thickness of the plate.

The intersection space $\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ is algebraically equivalent to $\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ but equipped with the squared norm [4]

$$\|\boldsymbol{\gamma}\|_{-1,\mathcal{L}_E}^2 + t^2\|\boldsymbol{\gamma}\|_0^2,$$

the space $L_0^2(\Omega)\cap t\,(H^1(\Omega)/\mathbb{R})$ in the second line is equipped with

$$\|v\|_0^2 + t^2\|v\|_1^2,$$

the space $\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ in the second line is equipped with

$$\|\boldsymbol{\gamma}\|_{-1,\mathcal{L}_E}^2 + \|\operatorname{div}\boldsymbol{\gamma}\|_{-1}^2 + t^2\|\boldsymbol{\gamma}\|_0^2, \tag{3.56}$$

and the space $\boldsymbol{H}_0(\operatorname{rot},\Omega)+t^{-1}\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ at the bottom is equipped with

$$\inf_{\substack{\boldsymbol{\gamma}=\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2\\ \boldsymbol{\gamma}_1\in\boldsymbol{H}_0(\operatorname{rot},\Omega),\,\boldsymbol{\gamma}_2\in\boldsymbol{L}^2(\Omega;\mathbb{R}^2)}} \|\boldsymbol{\gamma}_1\|_{H(\operatorname{rot},\Omega)}^2 + t^{-2}\|\boldsymbol{\gamma}_2\|_0^2.$$

And by (2.28) and $(\boldsymbol{L}^2(\Omega;\mathbb{R}^2))' = \boldsymbol{L}^2(\Omega;\mathbb{R}^2)$, we have

$$\big(\boldsymbol{H}_0(\operatorname{rot},\Omega)+t^{-1}\boldsymbol{L}^2(\Omega;\mathbb{R}^2)\big)' = \boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2) \tag{3.57}$$

For a quick introduction on the intersection and summation of Hilbert spaces, in particular a proof of (3.57), we refer to [55, § 2.2].

The two-term mixed formulation is to find $(\boldsymbol{\gamma},w)\in\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)\times H_0^1(\Omega)$ such that

$$(\mathcal{L}_E^{-1}\boldsymbol{\gamma},\boldsymbol{\beta}) + t^2(\boldsymbol{\gamma},\boldsymbol{\beta}) - (\boldsymbol{\beta},\nabla w) = 0 \qquad \forall\,\boldsymbol{\beta}\in\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2), \tag{3.58}$$

$$(\boldsymbol{\gamma},\nabla v) = \langle f, v\rangle \quad \forall\, v\in H_0^1(\Omega), \tag{3.59}$$

where $f\in H^{-1}(\Omega)$. Again space $\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ is algebraically equivalent to $\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ but equipped with a non-standard norm (3.56).

Now let us verify the boundedness of operators in the commutative diagram (3.55). The boundedness of the curl and div operators are simply combination of those in (3.53)-(3.54). The boundedness of the embedding operator $I:\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)\to\boldsymbol{H}^{-1}(\Omega;\mathbb{R}^2)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ is trivial. For any $\boldsymbol{\gamma}_1\in\boldsymbol{H}_0(\operatorname{rot},\Omega)$ and $\boldsymbol{\gamma}_2\in\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$, it holds from (3.52)

$$\begin{aligned} &\|\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2\|_{-1,\mathcal{L}_E}^2 + \|\operatorname{div}(\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2)\|_{-1}^2 + t^2\|\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2\|_0^2\\ &\lesssim \|\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2\|_0^2\\ &\lesssim \|\boldsymbol{\gamma}_1\|_{H(\operatorname{rot},\Omega)}^2 + t^{-2}\|\boldsymbol{\gamma}_2\|_0^2, \end{aligned}$$

which means $I:\boldsymbol{H}_0(\operatorname{rot},\Omega)+t^{-1}\boldsymbol{L}^2(\Omega;\mathbb{R}^2)\to\boldsymbol{H}^{-1}(\operatorname{div},\Omega)\cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ is bounded. The boundedness can be also derived from (3.57) as I is just the inverse of the Riesz representation which is obviously continuous. For any $v \in H_0^1(\Omega)$, by choosing $\boldsymbol{\gamma}_1 = \nabla v$, we have

$$\inf_{\substack{\nabla v = \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2 \\ \boldsymbol{\gamma}_1 \in \boldsymbol{H}_0(\mathrm{rot},\Omega), \boldsymbol{\gamma}_2 \in \boldsymbol{L}^2(\Omega;\mathbb{R}^2)}} \|\boldsymbol{\gamma}_1\|^2_{H(\mathrm{rot},\Omega)} + t^{-2}\|\boldsymbol{\gamma}_2\|_0^2 \le \|\nabla v\|^2_{H(\mathrm{rot},\Omega)} = \|\nabla v\|_0^2,$$

thus grad : $H_0^1(\Omega) \to \boldsymbol{H}_0(\mathrm{rot},\Omega) + t^{-1}\boldsymbol{L}^2(\Omega;\mathbb{R}^2)$ is bounded. As $t \lesssim 1$, we conclude that all the continuity constants of these operators can be chosen to be uniformly bounded to the parameter t. Therefore we obtain the well-posedness of (3.58)-(3.59) and the stability constant is independent of t.

By introducing variable $\boldsymbol{\theta} = \mathcal{L}_E^{-1}\boldsymbol{\gamma} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, the mixed formulation (3.58)-(3.59) will be equivalent to the unfolded three-term formulation: find $(\boldsymbol{\gamma},\boldsymbol{\theta},w) \in \boldsymbol{H}^{-1}(\mathrm{div},\Omega) \cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2) \times \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2) \times H_0^1(\Omega)$ such that (cf. [19, 20])

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\theta}),\boldsymbol{\varepsilon}(\boldsymbol{\psi})) + (\boldsymbol{\gamma},\nabla v - \boldsymbol{\psi}) = \langle f, v\rangle \quad \forall\ (\boldsymbol{\psi},v) \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2) \times H_0^1(\Omega), \tag{3.60}$$

$$(\boldsymbol{\beta},\nabla w - \boldsymbol{\theta}) - t^2(\boldsymbol{\gamma},\boldsymbol{\beta}) = 0 \qquad \forall\ \boldsymbol{\beta} \in \boldsymbol{H}^{-1}(\mathrm{div},\Omega) \cap t\boldsymbol{L}^2(\Omega;\mathbb{R}^2). \tag{3.61}$$

Applying the Helmholtz decomposition $\boldsymbol{\gamma} = \mathrm{curl}\, p + \nabla r$ where $p \in L_0^2(\Omega) \cap t\,(H^1(\Omega)/\mathbb{R})$ and $r \in H_0^1(\Omega)$, the mixed formulation (3.60)-(3.61) can be further decoupled to find $r \in H_0^1(\Omega)$, $\boldsymbol{\theta} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, $p \in L_0^2(\Omega) \cap t\,(H^1(\Omega)/\mathbb{R})$ and $w \in H_0^1(\Omega)$ such that (cf. [19, (2.10)-(2.13)])

$$(\nabla r, \nabla v) = \langle f, v\rangle, \tag{3.62}$$

$$(\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{\theta}),\boldsymbol{\varepsilon}(\boldsymbol{\psi})) - (\mathrm{curl}\, p, \boldsymbol{\psi}) = (\nabla r, \boldsymbol{\psi}), \tag{3.63}$$

$$(\mathrm{curl}\, q, \boldsymbol{\theta}) + t^2(\mathrm{curl}\, p, \mathrm{curl}\, q) = 0, \tag{3.64}$$

$$(\nabla w, \nabla s) = (\boldsymbol{\theta}, \nabla s) + t^2\langle f, v\rangle, \tag{3.65}$$

for any $v \in H_0^1(\Omega)$, $\boldsymbol{\psi} \in \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$, $q \in L_0^2(\Omega) \cap t\,(H^1(\Omega)/\mathbb{R})$ and $s \in H_0^1(\Omega)$.

4. Discrete Mixed Method and Its Decomposition

In this section, we will develop discrete mixed methods based on a commutative diagram. Denote by $\mathcal{T}_h$ a partition of domain Ω. All discrete spaces in this paper are defined on $\mathcal{T}_h$. For a generic finite dimensional space V_h, we can always identify V_h' as V_h using the Riesz representation of L^2-inner product. In other words, in implementation, we do not see the dual space but just vector space $\mathbb{R}^{\dim V_h}$. This oversimplification is, somehow, misleading in the error analysis. We shall reveal the importance of a discrete V_h' norm in the error analysis.

4.1. **Setting.**

For a finite dimensional Hilbert space $V_h \subset L^2(\Omega)$, we use the Riesz representation of L^2 inner product to define an isomorphism $Q_h : V_h' \to V_h$ as

$$(Q_h v_h', w_h) = \langle v_h', w_h\rangle \quad \forall\ v_h' \in V_h',\ w_h \in V_h.$$

The $L^2(\Omega)$ is the so-called 'pivot' space. In the continuous level, a continuous functional $v' \in V'$ may not be a continuous linear functional in L^2-norm and thus the Riesz representation operator of L^2-inner product is unbounded. While in the discrete level, such mapping is always well defined by inverting a square mass matrix. The domain of Q_h is still algebraically equal to V_h, but the imagine is a realization of V_h'. To emphasize the difference, we shall denote $Q_h(V_h')$ by V_h^T and usually equip it with a norm of V_h'.

Suppose V_h is equipped with inner product $(\cdot,\cdot)_{V_h}$. Define $J_{V_h} : V_h \to V_h^T$ as

$$(J_{V_h} v_h, w_h) = (v_h, w_h)_{V_h} \quad \forall\, w_h \in V_h.$$

Then it holds

$$\|J_{V_h} v_h\|_{V_h'} = \sup_{w_h\in V_h} \frac{(J_{V_h} v_h, w_h)}{\|w_h\|_{V_h}} = \sup_{w_h\in V_h} \frac{(v_h, w_h)_{V_h}}{\|w_h\|_{V_h}} = \|v_h\|_{V_h} \quad \forall\, v_h \in V_h.$$

Hence J_{V_h} is an isomorphism if we equip the range space V_h^T with $\|\cdot\|_{V_h'}$ norm not $\|\cdot\|_{V_h}$ norm. The definition of J_{V_h} is slightly different with (2.9). Indeed it is a composition of

$$V_h \xrightarrow{\ J\ } V_h' \xrightarrow{\ Q_h\ } V_h^T = V_h.$$

Here J is the inverse of Riesz isomorphic from V_h onto V_h', i.e.

$$\langle J v_h, w_h\rangle = (v_h, w_h)_{V_h} \quad \forall\, v_h, w_h \in V_h.$$

Assume we have the discrete commutative diagram

$$\begin{array}{ccccc}
 & & X_h & \xrightarrow{J_{X_h}} & X_h^T \\
 & & & & \cup \\
P_h & \xrightarrow{d_h^-} & \Sigma_h & \xrightarrow{d_h} & V_h^T \\
 & & \uparrow{\scriptstyle \Pi_{\Sigma_h}} & & \uparrow{\scriptstyle J_{V_h}} \\
 & & \widetilde{\Sigma}_h & \xleftarrow{\Pi_{V_h}} & V_h
\end{array}, \tag{4.1}$$

that is the complex in the middle is exact, and

$$d_h \Pi_{\Sigma_h} \Pi_{V_h} v_h = J_{V_h} v_h \quad \text{for all } v_h \in V_h. \tag{4.2}$$

Note that (4.2) implies d_h is surjective. At the top $J_{X_h} : X_h \to X_h^T$ is defined similarly as J_{V_h}.

All the operators in the commutative diagram (4.1) are continuous. Applying Theorem 2.1, we acquire a stable Helmholtz decomposition

$$\Sigma_h = d_h^- P_h \oplus \Pi_{\Sigma_h} \Pi_{V_h} V_h. \tag{4.3}$$

That is for any $\tau_h \in \Sigma_h$, there exist $q_h \in P_h / \ker d_h^-$ and $v_h \in V_h$ such that

$$\tau_h = d_h^- q_h \oplus \Pi_{\Sigma_h} \Pi_{V_h} v_h,$$
$$\|\tau_h\|_{\Sigma_h} \eqsim \|d_h^- q_h\|_{\Sigma_h} + \|v_h\|_{V_h}.$$

4.2. **Two-term discretization.** The two-term mixed formulation is suitable when $(\cdot,\cdot)_{X_h'} = (\cdot,\cdot)_{X'}$ which will be assumed in this subsection. A simple example for $(\cdot,\cdot)_{X'}$ is the L^2 inner product or a weighted version, e.g., $(\cdot,\cdot)_{\mathbb{C}}$, which might be only positive semidefinite.

A discrete mixed method associated with mixed formulation (3.4)-(3.5) is to find $(\sigma_h, u_h) \in \Sigma_h \times V_h$ such that

$$(\sigma_h, \tau_h)_{X'} + (d_h \tau_h, u_h) = (g, \tau_h) \quad \forall\, \tau_h \in \Sigma_h, \tag{4.4}$$

$$(d_h \sigma_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h, \tag{4.5}$$

where $f, g \in L^2(\Omega)$.

We also assume the discrete norm equivalence

$$\|\tau_h\|_{\Sigma_h}^2 \eqsim \|\tau_h\|_{X'}^2 + \|d_h \tau_h\|_{V_h'}^2 \quad \forall\, \tau_h \in \Sigma_h. \tag{4.6}$$

Let (σ, u) be the solution to (3.4)-(3.5) and smooth enough. We assume the consistency of discrete mixed method (4.4)-(4.5): there exists $\sigma_I \in \Sigma_h$ an interpolation of σ, and $u_I \in V_h$ an interpolation of u satisfying

$$(\sigma, \tau_h)_{X'} + (d_h\tau_h, u_I) = (g, \tau_h) \quad \forall\ \tau_h \in \Sigma_h, \tag{4.7}$$

$$(d_h\sigma_I, v_h) = (f, v_h) \quad \forall\ v_h \in V_h. \tag{4.8}$$

Applying the Helmholtz decomposition (4.3), we have

$$\sigma_h = d_h^- p_h + \Pi_{\Sigma_h}\Pi_{V_h} w_h, \quad \tau_h = d_h^- q_h + \Pi_{\Sigma_h}\Pi_{V_h}\chi_h,$$

where $p_h, q_h \in P_h/\ker d_h^-$, and $w_h, \chi_h \in V_h$, then the discrete mixed method (4.4)-(4.5) can be decoupled as follows: find $w_h \in V_h$, $p_h \in P_h/\ker d_h^-$ and $u_h \in V_h$ such that

$$(w_h, v_h)_{V_h} = (f, v_h) \qquad \forall\ v_h \in V_h, \tag{4.9}$$

$$(d_h^- p_h, d_h^- q_h)_{X'} = (g, d_h^- q_h) - (\Pi_{\Sigma_h}\Pi_{V_h} w_h, d_h^- q_h)_{X'} \qquad \forall\ q_h \in P_h/\ker d_h^-, \tag{4.10}$$

$$(u_h, \chi_h)_{V_h} = (g, \Pi_{\Sigma_h}\Pi_{V_h}\chi_h) - (\sigma_h, \Pi_{\Sigma_h}\Pi_{V_h}\chi_h)_{X'} \quad \forall\ \chi_h \in V_h. \tag{4.11}$$

We now present error analysis based on the commutative diagram and the consistency equations.

Theorem 4.1. *Assume both the discrete commutative diagram* (4.1) *and the discrete norm equivalence* (4.6) *hold. Then the discrete mixed method* (4.4)-(4.5) *is uniquely solvable and can be decoupled into the discrete method* (4.9)-(4.11)*. Moreover, when the consistency* (4.7)-(4.8) *hold, we have*

$$\begin{aligned}\|\sigma - \sigma_h\|_{X'} + \|\sigma_I - \sigma_h\|_{\Sigma_h} + \|u_I - u_h\|_{V_h} \lesssim & \|\sigma - \sigma_I\|_{X'},\\ \|\sigma - \sigma_h\|_{\Sigma_h} \lesssim & \|\sigma - \sigma_I\|_{X'} + \|\sigma - \sigma_I\|_{\Sigma_h},\\ \|u - u_h\|_{V_h} \lesssim & \|\sigma - \sigma_I\|_{X'} + \|u - u_I\|_{V_h}.\end{aligned}$$

Proof. The following stability can be proved using the arguments in Theorem 3.2: for any $(\widetilde{\sigma}_h, \widetilde{u}_h) \in \Sigma_h \times V_h$,

$$\|\widetilde{\sigma}_h\|_{\Sigma_h} + \|\widetilde{u}_h\|_{V_h} \lesssim \sup_{(\tau_h, v_h)\in\Sigma_h\times V_h} \frac{(\widetilde{\sigma}_h, \tau_h)_{X'} + (d_h\tau_h, \widetilde{u}_h) + (d_h\widetilde{\sigma}_h, v_h)}{\|\tau_h\|_{\Sigma_h} + \|v_h\|_{V_h}}. \tag{4.12}$$

Subtracting (4.4)-(4.5) from (4.7)-(4.8), we derive the error equation

$$(\sigma - \sigma_h, \tau_h)_{X'} + (d_h\tau_h, u_I - u_h) + (d_h(\sigma_I - \sigma_h), v_h) = 0 \quad \forall\ \tau_h \in \Sigma_h, v_h \in V_h.$$

Rewriting it, we get

$$(\sigma_I - \sigma_h, \tau_h)_{X'} + (d_h\tau_h, u_I - u_h) + (d_h(\sigma_I - \sigma_h), v_h) = (\sigma_I - \sigma, \tau_h)_{X'},$$

for any $\tau_h \in \Sigma_h$ and $v_h \in V_h$.

Then taking $\widetilde{\sigma}_h = \sigma_I - \sigma_h$ and $\widetilde{u}_h = u_I - u_h$ in the inf-sup condition (4.12), we obtain from (4.6)

$$\|\sigma_I - \sigma_h\|_{\Sigma_h} + \|u_I - u_h\|_{V_h} \lesssim \sup_{(\tau_h, v_h)\in\Sigma_h\times V_h} \frac{(\sigma_I - \sigma, \tau_h)_{X'}}{\|\tau_h\|_{\Sigma_h} + \|v_h\|_{V_h}} \lesssim \|\sigma - \sigma_I\|_{X'},$$

which together with the triangle inequality ends the proof. □

Then to obtain optimal order of convergence, it suffices to construct (σ_I, u_I) with desirable approximation property. Note that, due to the existence of σ_I, the error estimate of σ is irrelevant of that of u which is in the spirit of Falk and Osborn [36].

4.3. **HHJ mixed finite element.** For each $K \in \mathcal{T}_h$, denote by $\boldsymbol{n}_K = (n_1, n_2)^T$ the unit outward normal to ∂K and write $\boldsymbol{t}_K := (t_1, t_2)^T = (-n_2, n_1)^T$, a unit vector tangent to ∂K. Without causing any confusion, we will abbreviate $\boldsymbol{n}_K$ and $\boldsymbol{t}_K$ as $\boldsymbol{n}$ and $\boldsymbol{t}$ respectively for simplicity. Let $\mathcal{E}_h$ be the union of all edges of the triangulation $\mathcal{T}_h$ and $\mathcal{E}_h^i$ the union of all interior edges of the triangulation $\mathcal{T}_h$. For a second order tensor-valued function $\boldsymbol{\tau}$, set

$$M_n(\boldsymbol{\tau}) := \boldsymbol{n}^T\boldsymbol{\tau}\boldsymbol{n}, \quad M_{nt}(\boldsymbol{\tau}) := \boldsymbol{t}^T\boldsymbol{\tau}\boldsymbol{n},$$

on each edge $e \in \mathcal{E}_h$. The corresponding finite element spaces are given by

$$\begin{aligned}
&\boldsymbol{\Sigma} := \left\{\boldsymbol{\tau} \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}|_K \in \boldsymbol{H}^1(K;\mathbb{S}) \ \ \forall\, K \in \mathcal{T}_h \text{ and } [M_n(\boldsymbol{\tau})]|_{\mathcal{E}_I} = 0\right\},\\
&V := \left\{v \in H_0^1(\Omega) : v|_K \in H^2(K) \ \ \forall\, K \in \mathcal{T}_h\right\},\\
&\boldsymbol{P}_h := \left\{\boldsymbol{q}_h \in \boldsymbol{H}^1(\Omega;\mathbb{R}^2) : \boldsymbol{q}_h|_K \in \boldsymbol{P}_k(K;\mathbb{R}^2) \ \ \forall\, K \in \mathcal{T}_h\right\},\\
&\boldsymbol{\Sigma}_h := \left\{\boldsymbol{\tau}_h \in \boldsymbol{L}^2(\Omega;\mathbb{S}) : \boldsymbol{\tau}_h|_K \in \boldsymbol{P}_{k-1}(K;\mathbb{S}) \ \ \forall\, K \in \mathcal{T}_h \text{ and } [M_n(\boldsymbol{\tau}_h)]|_{\mathcal{E}_I} = 0\right\},\\
&V_h := \left\{v \in H_0^1(\Omega) : v_h|_K \in P_k(K) \ \ \forall\, K \in \mathcal{T}_h\right\}
\end{aligned}$$

with integer $k \geq 1$.

Define discrete differential operator $(\operatorname{div}\mathbf{div})_h : \boldsymbol{\Sigma} \to V_h$ as

$$((\operatorname{div}\mathbf{div})_h\boldsymbol{\tau}, v_h) = -(\mathbf{div}_h\boldsymbol{\tau}, \boldsymbol{\nabla} v_h) + \sum_{K\in\mathcal{T}_h}\int_{\partial K} M_{nt}(\boldsymbol{\tau})\partial_t v_h \,\mathrm{d}s \quad \forall\, v_h \in V_h.$$

and $\Delta_h : H_0^1(\Omega) \to V_h$ as

$$(\Delta_h v, w_h) = -(\nabla v, \nabla w_h) \quad \forall\, w_h \in V_h.$$

For any element $K \in \mathcal{T}_h$, define interpolation $\boldsymbol{\Pi}_K : \boldsymbol{H}^1(K,\mathbb{S}) \to \boldsymbol{P}_{k-1}(K,\mathbb{S})$ in the following way (cf. [10, 36, 31, 13]): given $\boldsymbol{\tau} \in \boldsymbol{H}^1(K,\mathbb{S})$, for any element $K \in \mathcal{T}_h$ and any edge e of K,

$$\int_e M_n\left((\boldsymbol{\tau} - \boldsymbol{\Pi}_K\boldsymbol{\tau})|_K\right)\mu \,\mathrm{d}s = 0 \quad \forall\, \mu \in P_{k-1}(e),$$

$$\int_K (\boldsymbol{\tau} - \boldsymbol{\Pi}_K\boldsymbol{\tau}) : \boldsymbol{\varsigma} \,\mathrm{d}x = 0 \quad \forall\, \boldsymbol{\varsigma} \in \boldsymbol{P}_{k-2}(K,\mathbb{S}).$$

The associated global interpolation operator $\boldsymbol{\Pi}_h : \boldsymbol{\Sigma} \to \boldsymbol{\Sigma}_h$ is given by

$$(\boldsymbol{\Pi}_h)|_K := \boldsymbol{\Pi}_K \quad \text{ for all } K \in \mathcal{T}_h.$$

From the definition of $\boldsymbol{\Pi}_h$, it holds

$$(\operatorname{div}\mathbf{div})_h(\boldsymbol{\Pi}_h\boldsymbol{\tau}) = (\operatorname{div}\mathbf{div})_h\boldsymbol{\tau} \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}. \tag{4.13}$$

Define $\boldsymbol{\pi}_h v = \boldsymbol{\Pi}_h(v\boldsymbol{I}_{2\times 2})$, then it follows from (4.13), the definitions of $(\operatorname{div}\mathbf{div})_h$ and Δ_h

$$(\operatorname{div}\mathbf{div})_h(\boldsymbol{\pi}_h v) = (\operatorname{div}\mathbf{div})_h(v\boldsymbol{I}_{2\times 2}) = \Delta_h v \quad \forall\, v \in H_0^1(\Omega). \tag{4.14}$$

Define $I_K : H^2(K) \to P_k(K)$ in the following way (cf. [10, 36, 31, 60]): given $w \in H^2(K)$, any vertex a of K, and any edge e of K,

$$I_K w(a) = w(a),$$

$$\int_e (w - I_K w)v \,\mathrm{d}s = 0 \quad \forall\, v \in P_{k-2}(e),$$

$$\int_K (w - I_K w)v \,\mathrm{d}x = 0 \quad \forall\, v \in P_{k-3}(K).$$

The associated global interpolation operator $I_h : V \to V_h$ is given by

$$(I_h)|_K := I_K \quad \text{for all } K \in \mathcal{T}_h.$$

We use the traditional H^1 seminorm for V_h. The discrete H^{-1} norm is given by

$$\|v\|_{-1,h} := \sup_{w_h \in V_h} \frac{(v, w_h)}{|w_h|_1} \quad \forall\, v \in L^2(\Omega).$$

Then we equip $\boldsymbol{\Sigma}_h$ with the following mesh-dependent norm (cf. [49])

$$\|\boldsymbol{\tau}\|_{\Sigma_h}^2 := \|\boldsymbol{\tau}\|_0^2 + \|(\operatorname{div}\mathbf{div})_h \boldsymbol{\tau}\|_{-1,h}^2,$$

which implies (4.6).

Lemma 4.2. *We have commutative diagram for HHJ method*

$$\begin{array}{ccccc} & & L^2(\Omega) & & \\ & & \cup & & \\ \boldsymbol{P}_h & \xrightarrow{\nabla^s\times} & \boldsymbol{\Sigma}_h & \xrightarrow{(\operatorname{div}\mathbf{div})_h} & V_h^T \\ & & \boldsymbol{I}\uparrow & & \uparrow\Delta_h \\ & & \boldsymbol{\Sigma}_h & \xleftarrow{\boldsymbol{\pi}_h} & V_h \end{array} \tag{4.15}$$

Proof. The complex in the middle has been proved to be exact in [26, 46]. The commutation is just (4.14). All the operators $\boldsymbol{I}$, $(\operatorname{div}\mathbf{div})_h$ and Δ_h are continuous by definitions. The continuity of $\boldsymbol{\pi}_h$ follows from (4.14). □

Then HHJ mixed method based on the commutative diagram (4.15) is to find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + ((\operatorname{div}\mathbf{div})_h \boldsymbol{\tau}_h, u_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h, \tag{4.16}$$

$$((\operatorname{div}\mathbf{div})_h \boldsymbol{\sigma}_h, v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{4.17}$$

The well-posedness of HHJ mixed method (4.16)-(4.17) is covered by Theorem 4.1.

To carry out *a priori* error estimate, we choose $\boldsymbol{\sigma}_I = \boldsymbol{\Pi}_h \boldsymbol{\sigma}$ and $u_I = I_h u$ to verify the consistency (4.7)-(4.8). The consistency (4.7) holds as when the solution $(\boldsymbol{\sigma}, u)$ is smooth enough [10, p. 1058],

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}_h) = -(\nabla^2 u, \tau_h) = -((\operatorname{div}\mathbf{div})_h \boldsymbol{\tau}_h, I_h u).$$

Recall the following commuting diagram [26, Theorem 2.7]

$$\begin{array}{ccc} \boldsymbol{\Sigma} & \xrightarrow{\operatorname{div}\mathbf{div}} & H^{-1}(\Omega) \\ \boldsymbol{\Pi}_h \downarrow & & \downarrow Q_h \\ \boldsymbol{\Sigma}_h & \xrightarrow{(\operatorname{div}\mathbf{div})_h} & V_h^T \end{array},$$

which can be also verified from (4.13) directly. Therefore (4.8) holds with $\boldsymbol{\sigma}_I = \boldsymbol{\Pi}_h \boldsymbol{\sigma}$. Consequently optimal order of convergence follows from Theorem 4.1 and the interpolation error estimate [10, 36, 31, 60].

By the commutative diagram (4.15), we have the stable Helmholtz decomposition [49]

$$\boldsymbol{\Sigma}_h = \nabla^s \times \boldsymbol{P}_h \oplus \boldsymbol{\pi}_h V_h. \tag{4.18}$$

The HHJ mixed method (4.16)-(4.17) will be decoupled as: find $w_h \in V_h$, $\boldsymbol{p}_h \in \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot}$ and $u_h \in V_h$ such that

$$
\begin{aligned}
(\nabla w_h, \nabla v_h) &= -(f, v_h) && \forall\, v_h \in V_h,\\
(\nabla^s \times \boldsymbol{p}_h, \nabla^s \times \boldsymbol{q}_h) &= -(\boldsymbol{\pi}_h w_h, \nabla^s \times \boldsymbol{q}_h) && \forall\, \boldsymbol{q}_h \in \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot},\\
(\nabla u_h, \nabla \chi_h) &= (\boldsymbol{\sigma}_h, \boldsymbol{\pi}_h \chi_h) && \forall\, \chi_h \in V_h,
\end{aligned}
$$

where $\boldsymbol{\sigma}_h = \nabla^s \times \boldsymbol{p}_h + \boldsymbol{\pi}_h w_h$. Therefore we can solve the HHJ system by solving two Poisson equations and one linear elasticity problem which is established recently in [49].

4.4. **Unfolded three-term discretization.** When $(\cdot,\cdot)_{X'}$ is an inner product of negative order, e.g. $(\cdot,\cdot)_{-1}$, it is better to use the unfolded three-term mixed formulation to further discretize this inner product. In this subsection, we assume $X_h \subset X$ and $(\cdot,\cdot)_{X_h} = (\cdot,\cdot)_X$. Since $\Sigma_h \subset H^{-1}$ is less smooth, the differential operator d_h is usually understood in the weak sense. The space V_h is smooth enough such that d_h^T the L^2-adjoint of d_h, i.e., $(d_h^T v_h, \tau_h) := (v_h, d_h \tau_h)$ is a conforming discretization of a continuous operator $d' : V \to L^2$. We will thus shift the differential operator to space V_h.

Let $\phi_h = J_{X_h}^{-1}\sigma_h \in X_h$. Then the discrete mixed method (4.4)-(4.5) is equivalent to an unfolded discrete three-term formulation: find $(\phi_h, u_h, \sigma_h) \in X_h \times V_h \times \Sigma_h$ such that

$$
(\phi_h, \psi_h)_X - (\sigma_h, d_h^T v_h + \psi_h) = -(f, v_h) \quad \forall\, (\psi_h, v_h) \in X_h \times V_h, \tag{4.19}
$$

$$
(d_h^T u_h + \phi_h, \tau_h) = (g, \tau_h) \quad \forall\, \tau_h \in \Sigma_h. \tag{4.20}
$$

Let (ϕ, u, σ) be the solution to (3.8)-(3.9). To formulate the consistency, we will further assume $V_h \subset V$ and d_h^T is a conforming discretization of a continous operator $d' : V \to L^2$, and (ϕ, u, σ) satisfies the consistency equation:

$$
(\phi, \psi_h)_X - (\sigma, d' v_h + \psi_h) = -(f, v_h) \quad \forall\, (\psi_h, v_h) \in X_h \times V_h, \tag{4.21}
$$

$$
(d' u + \phi, \tau_h) = (g, \tau_h) \quad \forall\, \tau_h \in \Sigma_h. \tag{4.22}
$$

Applying the Helmholtz decomposition (4.3), the uncoupled discrete method (4.19)-(4.20) is equivalent to find $w_h, u_h \in V_h$, $\phi_h \in X_h$, and $p_h \in P_h/\ker d_h^-$ such that

$$
(w_h, v_h)_{V_h} = (f, v_h) \quad \forall\, v_h \in V_h, \tag{4.23}
$$

$$
(\phi_h, \psi_h)_{X_h} - (d_h^- p_h, \psi_h) = (\Pi_{\Sigma_h}\Pi_{V_h} w_h, \psi_h) \quad \forall\, \psi_h \in X_h, \tag{4.24}
$$

$$
(d_h^- q_h, \phi_h) = (g, d_h^- q_h) \quad \forall\, q_h \in P_h/\ker d_h^-, \tag{4.25}
$$

$$
(u_h, \chi_h)_{V_h} = (g - \phi_h, \Pi_{\Sigma_h}\Pi_{V_h}\chi_h) \quad \forall\, \chi_h \in V_h. \tag{4.26}
$$

We shall present an error analysis of the unfolded three-term formulation. Note that since we use different consistency equations, the obtained error estimate is also different with that in Theorem 4.1.

Theorem 4.3. *Assume both the discrete commutative diagram* (4.1) *and the discrete norm equivalence* (4.6) *hold, then the unfolded discrete method* (4.19)-(4.20) *is uniquely solvable and can be decoupled into the discrete method* (4.23)-(4.26).

Moreover, when the consistency (4.21)-(4.22) *hold, we have*

$$\begin{aligned}&\|\phi-\phi_h\|_X+\|\sigma-\sigma_h\|_{\Sigma_h}+\|u-u_h\|_{V_h}\\ \lesssim&\inf_{\psi_h\in X_h}\|\phi-\psi_h\|_X+\inf_{\tau_h\in\Sigma_h}\|\sigma-\tau_h\|_{\Sigma_h}+\inf_{v_h\in V_h}\|u-v_h\|_{V_h}.\end{aligned}$$

Proof. The following stability can be proved using the arguments in Theorem 3.3: for any $(\widetilde{\phi}_h,\widetilde{u}_h,\widetilde{\sigma}_h)\in X_h\times V_h\times\Sigma_h$,

$$\begin{aligned}&\|\widetilde{\phi}_h\|_X+\|\widetilde{u}_h\|_{V_h}+\|\widetilde{\sigma}_h\|_{\Sigma_h}\\ \lesssim&\sup_{(\widetilde{\psi}_h,\widetilde{v}_h,\widetilde{\tau}_h)\in X_h\times V_h\times\Sigma_h}\frac{(\widetilde{\phi}_h,\widetilde{\psi}_h)_X-(\widetilde{\sigma}_h,d_h^T\widetilde{v}_h+\widetilde{\psi}_h)-(d_h^T\widetilde{u}_h+\widetilde{\phi}_h,\widetilde{\tau}_h)}{\|\widetilde{\psi}_h\|_X+\|\widetilde{v}_h\|_{V_h}+\|\widetilde{\tau}_h\|_{\Sigma_h}}.\end{aligned}\tag{4.27}$$

Subtracting (4.19)-(4.20) from (4.21)-(4.22), we derive the error equation

$$(\psi_h-\phi_h,\widetilde{\psi}_h)_X-(\tau_h-\sigma_h,d_h^T\widetilde{v}_h+\widetilde{\psi}_h)=(\psi_h-\phi,\widetilde{\psi}_h)_X-(\tau_h-\sigma,d_h^T\widetilde{v}_h+\widetilde{\psi}_h),$$
$$(d_h^T(v_h-u_h)+\psi_h-\phi_h,\widetilde{\tau}_h)=(d_h^T(v_h-u)+\psi_h-\phi,\widetilde{\tau}_h),$$

for any $\psi_h,\widetilde{\psi}_h\in X_h$, $v_h,\widetilde{v}_h\in V_h$ and $\tau_h,\widetilde{\tau}_h\in\Sigma_h$. On the other hand, thanks to the discrete norm equivalence (4.6), we get

$$\sup_{\widetilde{\psi}_h\in X_h}\frac{(\psi_h-\phi,\widetilde{\psi}_h)_X-(\tau_h-\sigma,\widetilde{\psi}_h)}{\|\widetilde{\psi}_h\|_X}\lesssim\|\phi-\psi_h\|_X+\|\sigma-\tau_h\|_{\Sigma_h},$$

$$\sup_{\widetilde{v}_h\in V_h}\frac{-(\tau_h-\sigma,d_h^T\widetilde{v}_h)}{\|\widetilde{v}_h\|_{V_h}}\lesssim\sup_{\widetilde{v}_h\in V_h}\frac{\|\sigma-\tau_h\|_{\Sigma_h}\|d_h^T\widetilde{v}_h\|_{\Sigma_h'}}{\|\widetilde{v}_h\|_{V_h}}\lesssim\|\sigma-\tau_h\|_{\Sigma_h},$$

$$\begin{aligned}\sup_{\widetilde{\tau}_h\in\Sigma_h}\frac{(d_h^T(v_h-u)+\psi_h-\phi,\widetilde{\tau}_h)}{\|\widetilde{\tau}_h\|_{\Sigma_h}}&=\sup_{\widetilde{\tau}_h\in\Sigma_h}\frac{(v_h-u,d_h\widetilde{\tau}_h)+(\psi_h-\phi,\widetilde{\tau}_h)}{\|\widetilde{\tau}_h\|_{\Sigma_h}}\\&\lesssim\|u-v_h\|_{V_h}+\|\phi-\psi_h\|_X.\end{aligned}$$

Then we obtain from Theorem 3.3 and (4.27)

$$\|\psi_h-\phi_h\|_X+\|v_h-u_h\|_{V_h}+\|\tau_h-\sigma_h\|_{\Sigma_h}\lesssim\|\phi-\psi_h\|_X+\|u-v_h\|_{V_h}+\|\sigma-\tau_h\|_{\Sigma_h}.$$

Finally the required results will be derived by using the triangle inequality. □

4.5. **Primal discrete methods for biharmonic equation.** Suppose that we have the following discrete Stokes complex in two dimensions

$$0\longrightarrow V_h\xrightarrow{\operatorname{curl}_h}\boldsymbol{\Sigma}_h\xrightarrow{\operatorname{div}_h}P_h\longrightarrow 0,\tag{4.28}$$

which is exact. Here curl_h and div_h are elementwise counterparts of curl and div with respect to $\mathcal{T}_h$ since the discrete spaces may not be conforming to spaces in Stokes complex (**??**). We also assume the discrete Poincaré inequality holds

$$\|\boldsymbol{\tau}_h\|_0\lesssim|\boldsymbol{\tau}_h|_{1,h}\qquad\forall\ \boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h.\tag{4.29}$$

Denoted by $\operatorname{rot}_h^w=\operatorname{curl}_h^T:\boldsymbol{\Sigma}_h^T\to V_h^T$ and $\operatorname{grad}_h^w=\operatorname{div}_h^T:P_h^T\to\boldsymbol{\Sigma}_h^T$ the adjoints of curl_h and div_h with respect to the L^2-inner product, i.e.

$$(\operatorname{rot}_h^w\boldsymbol{\tau}_h,v_h)=(\boldsymbol{\tau}_h,\operatorname{curl}_h v_h),\quad(\operatorname{grad}_h^w q_h,\boldsymbol{\tau}_h)=(q_h,\operatorname{div}_h\boldsymbol{\tau}_h),$$

for any $\boldsymbol{\tau}_h\in\boldsymbol{\Sigma}_h$, $v_h\in V_h$ and $q_h\in P_h$.

There are many discrete Stokes complexes in the literatures. For example, the famous Argyris element [3] with the conforming Stokes complex developed in [35], the

H^2 conforming element corresponding to Scott-Vogelius element for Stokes equation in [59, 38], the Morley element [53, 50] with complex proved in [34], a quadrilateral Morley element [56] with complex established in [63], Bogner-Fox-Schmit element [14, 28] with complex proved in [8, 51], the modified Morley element [54] with complex given in [52], a nonconforming Stokes complex in [39], and the singular Zienkiewicz finite element [64, 28] with complex devised in [40].

We equip P_h with L^2 norm, V_h with discrete H^1 seminorm $|\cdot|_{1,h}$, and $\boldsymbol{\Sigma}_h$ with discrete $\boldsymbol{H}(\mathrm{div},\Omega)$ norm. Again the discrete norm equivalence (4.6) is trivial. Define $\Delta_h : V_h \to V_h$ as

$$(\Delta_h v_h, w_h) = -(\nabla_h v_h, \nabla_h w_h) \quad \forall\, w_h \in V_h.$$

Lemma 4.4. *Assume the discrete Stokes complex (4.28) is exact and the discrete Poincaré inequality (4.29) holds. We then have the discrete commutative diagram*

$$\begin{array}{ccccc} & & \boldsymbol{\Sigma}_h & \xrightarrow{\Delta_h} & \boldsymbol{\Sigma}_h^T \\ & & & & \cup \\ & & P_h^T & \xrightarrow{\mathrm{grad}_h^w} & \boldsymbol{\Sigma}_h^T & \xrightarrow{\mathrm{rot}_h^w} & V_h^T \\ & & & & I\uparrow & & \uparrow\Delta_h \\ & & & & \boldsymbol{\Sigma}_h & \xleftarrow{\mathrm{curl}_h} & V_h \end{array} \tag{4.30}$$

Here the spaces $\boldsymbol{\Sigma}_h, \boldsymbol{\Sigma}_h^T$ in the top are equipped with discrete H^1 and H^{-1} norms respectively, $\boldsymbol{\Sigma}_h^T$ in the middle is equipped with norm

$$\|\boldsymbol{\tau}_h\|_{H_h^{-1}(\mathrm{rot}_h)} := \sup_{\boldsymbol{\varsigma}_h \in \boldsymbol{\Sigma}_h} \frac{(\boldsymbol{\tau}_h, \boldsymbol{\varsigma}_h)}{\|\boldsymbol{\varsigma}_h\|_{H_h(\mathrm{div}_h)}},$$

and $\boldsymbol{\Sigma}_h$ in the bottom is equipped with discrete $H(\mathrm{div})$ norm. Moreover, the Helmholtz decomposition

$$\boldsymbol{\Sigma}_h^T = \mathrm{grad}_h^w P_h^T \oplus \mathrm{curl}_h V_h$$

is stable.

Proof. First we note that the middle complex is exact since it is just the adjoint complex of the exact sequence (4.28). By the definitions of rot_h^w and Δ_h, it holds $\Delta_h = \mathrm{rot}_h^w \mathrm{curl}_h$. Hence diagram (4.30) is commutative.

Next we show that all the operators are continuous. The boundedness of $\mathrm{rot}_h^w : \boldsymbol{\Sigma}_h^T \to V_h^T$ and $\mathrm{grad}_h^w : P_h^T \to \boldsymbol{\Sigma}_h^T$ is trivial by the definitions of the dual norms. For any $v_h \in V_h$, we have

$$\|\mathrm{curl}_h v_h\|_{\Sigma_h} = \|\mathrm{curl}_h v_h\|_0 \lesssim |v_h|_{1,h}.$$

Therefore $\mathrm{curl}_h : V_h \to \boldsymbol{\Sigma}_h$ is continuous. □

The mixed finite element method based on the commutative diagram (4.30) is to find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h^T \times V_h$ such that

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{-1,h} - (\boldsymbol{\tau}_h, \mathrm{curl}_h u_h) = 0 \qquad \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^T, \tag{4.31}$$

$$(\boldsymbol{\sigma}_h, \mathrm{curl}_h v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{4.32}$$

Here the space $\boldsymbol{\Sigma}_h^T = \boldsymbol{\Sigma}_h$ but is equipped with $\|\cdot\|_{H_h^{-1}(\mathrm{rot}_h)}$ norm and

$$(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{-1,h} = (\boldsymbol{\nabla}_h \boldsymbol{\Delta}_h^{-1} \boldsymbol{\sigma}_h, \boldsymbol{\nabla}_h \boldsymbol{\Delta}_h^{-1} \boldsymbol{\tau}_h) = -(\boldsymbol{\Delta}_h^{-1} \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h).$$

The well-posedness of the mixed method (4.31)-(4.32) is covered by Theorem 4.1. By (4.31), $\boldsymbol{\sigma}_h = -\boldsymbol{\Delta}_h \operatorname{curl}_h u_h$. Then substituting it into (4.32), we have the conforming or nonconforming finite element method for the biharmonic

$$(\nabla_h^2 u_h, \nabla_h^2 v_h) = (f, v_h) \quad \forall\, v_h \in V_h. \tag{4.33}$$

On the other hand, let $\boldsymbol{\phi}_h = -\boldsymbol{\Delta}_h^{-1}\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$. Then the mixed finite element method (4.31)-(4.32) will be rewritten as: find $\boldsymbol{\phi}_h \in \boldsymbol{\Sigma}_h, \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h^T$, and $u_h \in V_h$ such that

$$\begin{aligned}
(\boldsymbol{\sigma}_h, \operatorname{curl}_h v_h) &= (f, v_h) && \forall\, v_h \in V_h.\\
(\nabla_h \boldsymbol{\phi}_h, \nabla_h \boldsymbol{\psi}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\psi}_h) &= 0 && \forall\, \boldsymbol{\psi}_h \in \boldsymbol{\Sigma}_h,\\
(\boldsymbol{\phi}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\tau}_h, \operatorname{curl}_h u_h) &= 0 && \forall\, \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h^T.
\end{aligned}$$

Here the space $\boldsymbol{\Sigma}_h$ in the middle and bottom are equipped with norms $|\cdot|_{1,h}$ and $\|\cdot\|_{H_h^{-1}(\operatorname{rot}_h)}$ accordingly.

When $V_h \in H_0^1(\Omega)$ and $\Sigma_h \in \boldsymbol{H}_0^1(\Omega, \mathbb{R}^2)$, i.e., for a conforming discretization of biharmonic equation, the *a priori* error estimate of this discrete method is covered by Theorem 4.3.

Applying the Helmholtz decomposition $\boldsymbol{\sigma}_h = \operatorname{curl}_h w_h - \operatorname{grad}_h^w p_h$, the mixed finite element method (4.31)-(4.32) is equivalent to find $w_h, u_h \in V_h$, $\boldsymbol{\phi}_h \in \boldsymbol{\Sigma}_h$ and $p_h \in P_h$ such that

$$(\operatorname{curl}_h w_h, \operatorname{curl}_h v_h) = (f, v_h) \qquad \forall\, v_h \in V_h. \tag{4.34}$$

$$(\nabla_h \boldsymbol{\phi}_h, \nabla_h \boldsymbol{\psi}_h) + (\operatorname{div}_h \boldsymbol{\psi}_h, p_h) = (\operatorname{curl}_h w_h, \boldsymbol{\psi}_h) \quad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{\Sigma}_h, \tag{4.35}$$

$$(\operatorname{div}_h \boldsymbol{\phi}_h, q_h) = 0 \qquad \forall\, q_h \in P_h, \tag{4.36}$$

$$(\operatorname{curl}_h u_h, \operatorname{curl}_h \chi_h) = (\boldsymbol{\phi}_h, \operatorname{curl}_h \chi_h) \quad \forall\, \chi_h \in V_h. \tag{4.37}$$

Therefore several conforming or nonconforming finite element method (4.33) for biharmonic equation can be decomposed into two discrete Poisson equations (4.34)-(4.37) and one discrete Stokes equations (4.35)-(4.36).

Note that the error analysis cannot cover the non-conforming method. For example, it is well known that discretization of the Poisson equation using Morley element will not converge [61, 54]. In this scenario, the decoupling can be used to design fast solvers for biharmonic equations [45, 47] since many fast solvers for Poisson and Stokes equations are available.

5. Discretization Based on Decoupled Formulation

In this section, we will consider discretization based on the decoupled formulation. That is we decouple first and then discretize while in Section 4 we discretize and then decouple. By decoupling the fourth order equation into second order equations, we can easily use conforming finite element spaces. Furthermore, we can easily derive the superconvergence to Galerkin projection.

5.1. **Decoupled discretization of HHJ formulation.** Let $f \in L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$ and $\boldsymbol{P}_h \subset \boldsymbol{H}^1(\Omega;\mathbb{R}^2)$. The discrete method based on formulation (3.22)-(3.24) is to find $w_h \in V_h$, $\boldsymbol{p}_h \in \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot}$ and $u_h \in V_h$ such that

$$(\nabla w_h, \nabla v_h) = -(f, v_h) \quad \forall\, v_h \in V_h, \tag{5.1}$$

$$(\nabla^s \times \boldsymbol{p}_h, \nabla^s \times \boldsymbol{q}_h) = -(\boldsymbol{\pi} w_h, \nabla^s \times \boldsymbol{q}_h) \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot}, \tag{5.2}$$

$$(\nabla u_h, \nabla \chi_h) = (\boldsymbol{\sigma}_h, \boldsymbol{\pi}\chi_h) \quad \forall\, \chi_h \in V_h, \tag{5.3}$$

where $\boldsymbol{\sigma}_h = \nabla^s \times \boldsymbol{p}_h + \boldsymbol{\pi} w_h$.

Define projection $\boldsymbol{P}_h^{cs} : \boldsymbol{H}^1(\Omega;\mathbb{R}^2) \to \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot}$ by

$$(\nabla^s \times \boldsymbol{P}_h^{cs}\boldsymbol{p}, \nabla^s \times \boldsymbol{q}_h) = (\nabla^s \times \boldsymbol{p}, \nabla^s \times \boldsymbol{q}_h).$$

Similarly, denote by $P_h^{\rm grad}$ the H^1 orthogonal projection onto V_h. Let

$$\boldsymbol{\sigma}_h^* := \nabla^s \times \boldsymbol{P}_h^{cs}\boldsymbol{p} + \boldsymbol{\pi} w_h.$$

Lemma 5.1. *Let $(w, \boldsymbol{p}, u)$ be the solution of HHJ mixed formulation (3.22)-(3.24) and $(w_h, \boldsymbol{p}_h, u_h)$ be the solution of (5.1)-(5.3). We then have the estimates*

$$|w - w_h|_1 \lesssim \inf_{v_h \in V_h} |w - v_h|_1,$$

$$\|\nabla^s \times (\boldsymbol{P}_h^{cs}\boldsymbol{p} - \boldsymbol{p}_h)\|_0 + \|\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}_h\|_0 \lesssim \|w - w_h\|_0,$$

$$|P_h^{\rm grad}u - u_h|_1 \lesssim \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 + \|w - w_h\|_0.$$

Proof. Subtracting (5.1)-(5.3) from (3.22)-(3.24), we get the error equations

$$(\nabla(w - w_h), \nabla v_h) = 0 \quad \forall\, v_h \in V_h,$$

$$(\nabla^s \times (\boldsymbol{P}_h^{cs}\boldsymbol{p} - \boldsymbol{p}_h), \nabla^s \times \boldsymbol{q}_h) = (\boldsymbol{\pi}(w_h - w), \nabla^s \times \boldsymbol{q}_h) \quad \forall\, \boldsymbol{q}_h \in \boldsymbol{P}_h/\boldsymbol{RM}^{\rm rot},$$

$$(\nabla(P_h^{\rm grad}u - u_h), \nabla \chi_h) = (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\pi}\chi_h) \quad \forall\, \chi_h \in V_h.$$

Then all the error estimates hold by standard argument. ☐

Furthermore, assume

$$\|w - w_h\|_0 \lesssim h^\delta |w - w_h|_1, \quad \|\boldsymbol{p} - \boldsymbol{p}_h\|_0 \lesssim h^\delta \|\nabla^s \times (\boldsymbol{p} - \boldsymbol{p}_h)\|_0, \tag{5.4}$$

where $\delta \in (1/2, 1]$ is the regularity constant depending on the shape of Ω. This assumption can be proved by duality argument (cf. [28, 16]).

Theorem 5.2. *Let $(w, \boldsymbol{p}, u)$ be the solution of HHJ mixed formulation (3.22)-(3.24) and $(w_h, \boldsymbol{p}_h, u_h)$ be the solution of (5.1)-(5.3). We then have the estimates*

$$\|\nabla^s \times (\boldsymbol{p} - \boldsymbol{p}_h)\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \lesssim \inf_{\boldsymbol{q}_h \in \boldsymbol{P}_h} \|\nabla^s \times (\boldsymbol{p} - \boldsymbol{q}_h)\|_0 + \inf_{v_h \in V_h} |w - v_h|_1,$$

$$|u - u_h|_1 \lesssim \inf_{v_h \in V_h} |u - v_h|_1 + \inf_{\boldsymbol{q}_h \in \boldsymbol{P}_h} \|\nabla^s \times (\boldsymbol{p} - \boldsymbol{q}_h)\|_0 + \inf_{v_h \in V_h} |w - v_h|_1.$$

Moreover if assumption (5.4) is true, we have the improved error estimates

$$\|\nabla^s \times (\boldsymbol{P}_h^{cs}\boldsymbol{p} - \boldsymbol{p}_h)\|_0 + \|\boldsymbol{\sigma}_h^* - \boldsymbol{\sigma}_h\|_0 \lesssim h^\delta \inf_{v_h \in V_h} |w - v_h|_1, \tag{5.5}$$

$$|P_h^{\rm grad}u - u_h|_1 \lesssim h^\delta \inf_{\boldsymbol{q}_h \in \boldsymbol{P}_h} \|\nabla^s \times (\boldsymbol{p} - \boldsymbol{q}_h)\|_0 + h^\delta \inf_{v_h \in V_h} |w - v_h|_1. \tag{5.6}$$

Proof. The first two error estimates can be derived from Lemma 5.1 and Poincaré inequality. We can acquire (5.5)-(5.6) from Lemma 5.1 and (5.4). ☐

Remark 5.3. The error estimates (5.5)-(5.6) are superconvergent if we use equal order finite element spaces for V_h and $\boldsymbol{P}_h$. ☐

5.2. **Decoupled discretization for biharmonic equation.** In Section 4.5, we show that the conforming or nonconforming finite element method (4.33) for biharmonic equation is decomposed as (4.34)-(4.37) based on the discrete Stokes complex (4.28). Now we discretize formulation (3.32)-(3.35) using more general finite element spaces without satisfying the discrete Stokes complex (4.28) [13, Subsection 10.4.5.1].

Let $f \in L^2(\Omega)$, $V_h \subset H_0^1(\Omega)$, $\boldsymbol{X}_h \subset \boldsymbol{H}_0^1(\Omega;\mathbb{R}^2)$ and $P_h \subset L^2(\Omega)$. The discrete method based on formulation (3.32)-(3.35) is to find $w_h, u_h \in V_h$, $\boldsymbol{\phi}_h \in \boldsymbol{X}_h$ and $p_h \in P_h$ such that

$$(\operatorname{curl} w_h, \operatorname{curl} v_h) = (f, v_h) \qquad \forall\, v_h \in V_h. \tag{5.7}$$

$$(\nabla \boldsymbol{\phi}_h, \nabla \boldsymbol{\psi}_h) + (\operatorname{div} \boldsymbol{\psi}_h, p_h) = (\operatorname{curl} w_h, \boldsymbol{\psi}_h) \qquad \forall\, \boldsymbol{\psi}_h \in \boldsymbol{X}_h, \tag{5.8}$$

$$(\operatorname{div} \boldsymbol{\phi}_h, q_h) = 0 \qquad \forall\, q_h \in P_h, \tag{5.9}$$

$$(\operatorname{curl} u_h, \operatorname{curl} \chi_h) = (\boldsymbol{\phi}_h, \operatorname{curl} \chi_h) \qquad \forall\, \chi_h \in V_h. \tag{5.10}$$

We assume $(\boldsymbol{X}_h, P_h)$ is a stable finite element pair for Stokes equation (cf. [13, 12]), i.e. it holds the inf-sup condtion

$$\|q_h\|_0 \lesssim \sup_{\boldsymbol{\psi}_h \in \boldsymbol{\Sigma}_h} \frac{(\operatorname{div} \boldsymbol{\psi}_h, q_h)}{|\boldsymbol{\psi}_h|_1} \quad \forall\, q_h \in P_h. \tag{5.11}$$

To analyze the discrete method (5.7)-(5.10), we rewrite it as a standard mixed finite element method

$$\begin{aligned} a(\boldsymbol{\phi}_h, u_h; \boldsymbol{\psi}_h, v_h) + b(\boldsymbol{\psi}_h, v_h; p_h, w_h) &= (f, v_h) \quad && \forall\, (\boldsymbol{\psi}_h, v_h) \in \boldsymbol{X}_h \times V_h, \\ b(\boldsymbol{\phi}_h, u_h; q_h, \chi_h) &= 0 && \forall\, (q_h, \chi_h) \in P_h \times V_h, \end{aligned}$$

where

$$\begin{aligned} a(\boldsymbol{\phi}_h, u_h; \boldsymbol{\psi}_h, v_h) &:= (\nabla \boldsymbol{\phi}_h, \nabla \boldsymbol{\psi}_h), \\ b(\boldsymbol{\phi}_h, u_h; q_h, \chi_h) &:= (\operatorname{div} \boldsymbol{\phi}_h, q_h) - (\boldsymbol{\phi}_h, \operatorname{curl} \chi_h) + (\operatorname{curl} u_h, \operatorname{curl} \chi_h). \end{aligned}$$

Lemma 5.4. *Assume the inf-sup condition (5.11), the following inf-sup condition holds*

$$\|q_h\|_0 + |\chi_h|_1 \lesssim \sup_{(\boldsymbol{\psi}_h, v_h) \in \boldsymbol{\Sigma}_h \times V_h} \frac{b(\boldsymbol{\psi}_h, v_h; q_h, \chi_h)}{|\boldsymbol{\psi}_h|_1 + |v_h|_1} \quad \forall\, (q_h, \chi_h) \in P_h \times V_h. \tag{5.12}$$

Proof. It is easy to see that

$$|\chi_h|_1 = \sup_{v_h \in V_h} \frac{(\operatorname{curl} v_h, \operatorname{curl} \chi_h)}{|v_h|_1} \le \sup_{(\boldsymbol{\psi}_h, v_h) \in \boldsymbol{\Sigma}_h \times V_h} \frac{b(\boldsymbol{\psi}_h, v_h; q_h, \chi_h)}{|\boldsymbol{\psi}_h|_1 + |v_h|_1}.$$

It follows from (5.11) and Poincaré inequality

$$\begin{aligned} \|q_h\|_0 &\lesssim \sup_{\boldsymbol{\psi}_h \in \boldsymbol{\Sigma}_h} \frac{(\operatorname{div} \boldsymbol{\psi}_h, q_h)}{|\boldsymbol{\psi}_h|_1} = \sup_{\boldsymbol{\psi}_h \in \boldsymbol{\Sigma}_h} \frac{b(\boldsymbol{\psi}_h, 0; q_h, \chi_h) + (\boldsymbol{\psi}_h, \operatorname{curl} \chi_h)}{|\boldsymbol{\psi}_h|_1} \\ &\lesssim |\chi_h|_1 + \sup_{(\boldsymbol{\psi}_h, v_h) \in \boldsymbol{\Sigma}_h \times V_h} \frac{b(\boldsymbol{\psi}_h, v_h; q_h, \chi_h)}{|\boldsymbol{\psi}_h|_1 + |v_h|_1}. \end{aligned}$$

Therefore the inf-sup condition (5.12) will be derived by combining the last two inequalities. $\square$

Theorem 5.5. *Let $(w, \boldsymbol{\phi}, p, u)$ be the solution of the mixed formulation (3.32)-(3.35), and $(w_h, \boldsymbol{\phi}_h, p_h, u_h) \in V_h \times \boldsymbol{X}_h \times P_h \times V_h$ be the solution of the discrete method (5.7)-(5.10). Assume both V_h and $\boldsymbol{X}_h$ are H^1 conforming, the inf-sup condition (5.11) holds, and the discrete spaces are consistent with respect to the mixed formulation (3.32)-(3.35), then*

$$\|w - w_h\|_1 + \|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \|p - p_h\|_0 + \|u - u_h\|_1 \tag{5.13}$$

$$\lesssim \inf_{\chi_h \in V_h} \|w - \chi_h\|_1 + \inf_{\boldsymbol{\psi}_h \in \boldsymbol{X}_h} \|\boldsymbol{\phi} - \boldsymbol{\psi}_h\|_1 + \inf_{q_h \in P_h} \|p - q_h\|_0 + \inf_{v_h \in V_h} \|u - v_h\|_1. \tag{5.14}$$

Moreover, if

$$\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_0 \lesssim h^\delta \left(\|\boldsymbol{\phi} - \boldsymbol{\phi}_h\|_1 + \inf_{q_h \in P_h} \|p - q_h\|_0 \right), \tag{5.15}$$

then

$$|P_h^{\rm grad} u - u_h|_1 \lesssim h^\delta \left(\inf_{\chi_h \in V_h} \|w - \chi_h\|_1 + \inf_{\boldsymbol{\psi}_h \in \boldsymbol{X}_h} \|\boldsymbol{\phi} - \boldsymbol{\psi}_h\|_1 + \inf_{q_h \in P_h} \|p - q_h\|_0 \right). \tag{5.16}$$

Proof. For any $(\boldsymbol{\psi}_h, v_h) \in \Sigma_h \times V_h$ satisfying

$$b(\boldsymbol{\psi}_h, v_h; q_h, \chi_h) = 0 \quad \forall\ (q_h, \chi_h) \in P_h \times V_h,$$

we have

$$(\boldsymbol{\psi}_h, \operatorname{curl} \chi_h) = (\operatorname{curl} v_h, \operatorname{curl} \chi_h) \quad \forall\ \chi_h \in V_h,$$

which implies

$$|v_h|_1 \leq \|\boldsymbol{\psi}_h\|_0 \lesssim |\boldsymbol{\psi}_h|_1.$$

Thus

$$|\boldsymbol{\psi}_h|_1^2 + |v_h|_1^2 \lesssim |\boldsymbol{\psi}_h|_1^2 = a(\boldsymbol{\psi}_h, v_h; \boldsymbol{\psi}_h, v_h).$$

Combining the inf-sup condition (5.12), we will obtain the error estimate (5.13) by standard mixed finite element method theory in [13]. And (5.16) can be derived using the similar argument adopted in Section 5.1. □

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