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MATRIX ANALYSIS OF INTERREGIONAL POPULATION GROWTH AND DISTRIBUTION

by Andrei Rogers*

1. INTRODUCTION

Among available operational methods for analyzing and forecasting population growth and change, one may broadly distinguish between those models which are concerned only with total births, deaths and migration from those which focus on the behavior of cohorts disaggregated by age and sex. The former class of models typically are referred to as components-of-change models; the latter have been defined as cohort-survival models. Component models begin by analyzing the behavior of each of the major contributors to population change: i.e., births, deaths and migration, and then combine this information with population data for a base period to establish an estimate of a future population. Cohort-survival models focus on the age distribution of a population on a given date and, by subjecting it to the appropriate age-specific rates of fertility, mortality, and mobility, arrive at the age distribution of survivors and descendants of the original population at successive intervals of time. In the interests of simplicity, the age-specific rates frequently are assumed to remain constant over the projection period, and often male and female cohorts are not identified for separate analysis.

During the past few years, several analysts have taken advantage of the conceptual elegance and computational simplicity that matrix representation of population change provides.[1] Together, their efforts have opened up a fertile area for significant research. Expressing the process of population growth as a matrix "operator," separates it from the population to which it is applied and thereby allows a clearer focus on the intrinsic characteristics of a particular growth structure, its impact on another population, and its long-run distributional consequences.

This paper summarizes the principal results of a few recent efforts to introduce the interregional dimension into the demographer's matrix model of population growth.[2]

2. MATRIX MODELS OF POPULATION GROWTH AND DISTRIBUTION

The Interregional Components-of-Change Model. Consider an interregional population distribution at some point in time, t. Denote this distribution by a column vector, $w^{(t)}$, whose ith element denotes the total population in region i

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[1] See: Keyfitz (9), (10), (11); and Lopez (14). References to citations will be found in the Bibliography which follows the text.

[2] See: Rogers (19), (22), (23), and (25).

at time t. Construct a matrix, $\boldsymbol{P}$, which when applied to $\boldsymbol{w}^{(t)}$ will move an appropriate number of people from region to region during the interval $(t, t+1)$ and produce a new interregional population distribution, $\boldsymbol{w}^{(t+1)}$. More specifically, define a matrix operator which satisfies the following equation:

$$\boldsymbol{P}\boldsymbol{w}^{(t)} = \boldsymbol{w}^{*(t+1)} . \qquad [1]$$

The asterisk over the derived interregional distribution, $\boldsymbol{w}^{(t+1)}$, serves to remind us that effects of births and deaths remain to be considered.

The matrix operator $\boldsymbol{P}$ (which "migrates" a population) has the following form:

$$\underset{m\times m}{\boldsymbol{P}} = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{m1} \\ p_{12} & p_{22} & \cdots & \cdot \\ \vdots & \vdots & \ddots & \vdots \\ p_{1m} & p_{2m} & \cdots & p_{mm} \end{pmatrix}$$

where p_{ij} is the proportion of people in region i at time t who are in region j at time $t+1$.

Consider now the matrix representation of the fertility and mortality process in an interregional system. The ordinary method for expressing the effects of births and deaths in components-of-change models is by the application of crude birth and death rates to the total number of people in each region of an interregional system. As in the case of migration, such a process may be expressed by premultiplying an interregional population distribution by a suitably constructed fertility or mortality operator. Such matrices have zeros everywhere except in the principal diagonal:

$$\underset{m\times m}{\boldsymbol{B}} = \begin{pmatrix} b_1 & 0 & \cdots & 0 \\ 0 & b_2 & \cdots & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & b_m \end{pmatrix} \qquad \underset{m\times m}{\boldsymbol{D}} = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & \cdot \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & d_m \end{pmatrix}$$

where $b_i =$ the crude birth rate in the ith region;

$d_i =$ the crude death rate in the ith region.

The effects of interregional mobility, fertility, and mortality may be combined, to find their joint contribution to an interregional growth process: addition of matrices $\boldsymbol{P}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ defines a "growth" operator, $\boldsymbol{G}$, which describes the regime of growth that exists in a given interregional system. Thus:

$$(\boldsymbol{P} + \boldsymbol{B} - \boldsymbol{D})\boldsymbol{w}^{(t)} = \boldsymbol{w}^{(t+1)}$$

equivalently: [2]

$$\boldsymbol{G}\boldsymbol{w}^{(t)} = \boldsymbol{w}^{(t+1)}$$

The Single-Region Cohort-Survival Model. Mathematical demographers have demonstrated that the process of surviving an age distribution forward through time may be expressed by means of matrix multiplication. The fundamental model may be summarized:

$$Sw^{(t)} = w^{(t+1)} \qquad [3]$$

where:

$$\underset{n\times n}{S} = \begin{pmatrix} 0 & 0 & \cdots & b_1 & b_2 & \cdots & b_u & 0 & \cdots & 0 \\ {}_2d_3 & 0 & 0 & \cdots & & & & & \cdots & 0 \\ 0 & {}_3d_4 & 0 & \cdots & & & & & \cdots & 0 \\ 0 & 0 & {}_4d_5 & \cdots & & & & & \cdots & 0 \\ \vdots & & \vdots & & & & & & & \vdots \\ 0 & 0 & 0 & \cdots & & & & & {}_{n-1}d_n & 0 \end{pmatrix} \qquad w^{(t)} = \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \vdots \\ w_n^{(t)} \end{pmatrix}$$

and:

$w_r^{(t)}$ = the population in the rth age group at time t;

b_r = the birth rate (number of births per person) of the rth child-bearing age group;

${}_rd_{r+1}$ = the proportion of people in the rth age group who survive to the r + 1st age group after a unit time interval.

Consider the contribution of net migration to population change in the above system. Denote the age-distributed migration component, during the interval $(t, t+1)$, by the vector $n^{(t)}$. Then:

$$Sw^{(t)} + n^{(t)} = w^{(t+1)} \qquad [4]$$

Define a matrix, M, which when applied to $w^{(t)}$ will generate the net migration vector $n^{(t)}$. That is, find a matrix operator which satisfies the following equation:

$$Mw^{(t)} = n^{(t)} \qquad [5]$$

A matrix operator which will premultiply $w^{(t)}$ into $n^{(t)}$ is of the form:

$$\underset{n\times n}{M} = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ m_1 & 0 & 0 & \cdots & & 0 \\ 0 & m_2 & 0 & \cdots & & 0 \\ 0 & 0 & m_3 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & m_{n-1} & 0 \end{pmatrix}$$

where m_i = the net migration rate for the ith age group.

The effects of survivorship and migration may be combined simply by adding the matrices S and M to define a "growth" matrix, G. This transforms equation [4] into:

$$Gw^{(t)} = w^{(t+1)} \qquad [6]$$

The Interregional Cohort-Survival Model. The above development of a matrix cohort-survival model has been limited to a consideration of a single region's population. Extension of this model to an interregional system of m regions is relatively straight-forward. Each population vector, $\boldsymbol{w}$, and survivorship matrix, $\boldsymbol{S}$, is specified as to region by a subscript:

$$\boldsymbol{S}_i \boldsymbol{w}_i^{(t)} = \boldsymbol{w}_i^{*(t+1)} \qquad (i = 1, \cdots, m) \qquad [7]$$

At this point an asterisk is introduced to remind us that the migration component of change, unique to interregional population systems, still needs to be incorporated. Since effects of out-migration may be included in the survivorship proportions, ${}_r d_{r+1}$, only the contribution of in-migration remains to be expressed. Finally, it should be noted that, in order to maintain the convenience of a single matrix operator, one may adopt the algebra of partitioned matrices and apply equation [7] to all m regions in one step:

$$\left(\begin{array}{c|c|c|c|c} \boldsymbol{S}_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \boldsymbol{S}_2 & 0 & & \vdots \\ \hline 0 & 0 & \boldsymbol{S}_3 & & \vdots \\ \hline \vdots & & & \ddots & 0 \\ \hline 0 & \cdots & \cdots & 0 & \boldsymbol{S}_m \end{array}\right) \cdot \left(\begin{array}{c} \boldsymbol{w}_1^{(t)} \\ \hline \boldsymbol{w}_2^{(t)} \\ \hline \vdots \\ \hline \boldsymbol{w}_m^{(t)} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{w}_1^{*(t+1)} \\ \hline \boldsymbol{w}_2^{*(t+1)} \\ \hline \vdots \\ \hline \boldsymbol{w}_m^{*(t+1)} \end{array}\right)$$

Consider migration as a component of population change in an interregional system. Associated with every region, i, is a population vector, $\boldsymbol{w}_i$. During each unit interval of time, a certain fraction of the ith region's population, in a given age group, r, migrates to region j and enters the $r + 1$st age group there. Thus for each age group we may construct an interregional transition matrix, ${}_r\boldsymbol{P}_{r+1}$, which describes the proportion of people, in the rth age group in region i, who during the specified time period move into the $r + 1$st age group in region j:

$$\underset{m \times m}{{}_r\boldsymbol{P}_{r+1}} = \begin{pmatrix} {}_{r,1}p_{r+1,1} & {}_{r,1}p_{r+1,2} & \cdots & {}_{r,1}p_{r+1,m} \\ {}_{r,2}p_{r+1,1} & {}_{r,2}p_{r+1,2} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ {}_{r,m}p_{r+1,1} & \cdots & \cdots & {}_{r,m}p_{r+1,m} \end{pmatrix} \qquad (r = 1, \cdots, n)$$

With an estimated set of n transition matrices, one may construct a series of matrix operators, $\boldsymbol{M}_{ij}$, which when applied to the age distribution at i will "migrate" the requisite number of people from region i to region j and survive them into the next age cohort:

$$\boldsymbol{M}_{ij} \boldsymbol{w}_i^{(t)} = \boldsymbol{k}_{ij} \qquad [9]$$

where:

$$\underset{n\times n}{\boldsymbol{M}_{ij}} = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ {}_{1,i}p_{2,j} & 0 & \cdots & \cdots & 0 \\ 0 & {}_{2,i}p_{3,j} & 0 & & \cdot \\ \cdot & & \ddots & & \cdot \\ \cdot & & & \ddots & \cdot \\ 0 & \cdots & \cdots & {}_{n-1,i}p_{n,j} & 0 \end{pmatrix} \qquad \boldsymbol{k}_{ij} = \begin{pmatrix} 0 \\ {}_2k_{ij} \\ \cdot \\ \cdot \\ \cdot \\ {}_nk_{ij} \end{pmatrix}$$

and:

${}_rk_{ij}$ = the total number of in-migrants, in the rth age group, into region j from region i.

Summing the in-migrants over all origins, i, one finds the total in-migration into region j:

$$ {}_rk_{\cdot j} = \sum_i {}_rk_{ij} \qquad (r = 1, 2, \cdots, n) \qquad [10]$$

As in the case of the fertility-mortality process, one may introduce partitioned matrices to express equation [10]. For example, for $j=1$:

$$\left(0 \;\middle|\; \boldsymbol{M}_{21} \;\middle|\; \boldsymbol{M}_{31} \;\middle|\; \cdots \;\middle|\; \boldsymbol{M}_{m1} \right) \cdot \begin{pmatrix} \boldsymbol{w}_1^{(t)} \\ \hline \boldsymbol{w}_2^{(t)} \\ \hline \cdot \\ \cdot \\ \hline \boldsymbol{w}_m^{(t)} \end{pmatrix} = \boldsymbol{k}_{\cdot 1} \qquad [11]$$

Thus, for $j = 1, 2, \cdots, m$ we have:

$$\left(\begin{array}{c|c|c|c|c} 0 & \boldsymbol{M}_{21} & \boldsymbol{M}_{31} & \cdots & \boldsymbol{M}_{m1} \\ \hline \boldsymbol{M}_{12} & 0 & \boldsymbol{M}_{32} & \cdots & \boldsymbol{M}_{m2} \\ \hline \boldsymbol{M}_{13} & \boldsymbol{M}_{23} & 0 & \cdots & \boldsymbol{M}_{m3} \\ \hline \vdots & & & \ddots & \vdots \\ \hline \boldsymbol{M}_{1m} & \cdots & \cdots & \cdots & 0 \end{array} \right) \cdot \begin{pmatrix} \boldsymbol{w}_1^{(t)} \\ \hline \boldsymbol{w}_2^{(t)} \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \boldsymbol{w}_m^{(t)} \end{pmatrix} = \begin{pmatrix} \boldsymbol{k}_{\cdot 1} \\ \hline \boldsymbol{k}_{\cdot 2} \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \boldsymbol{k}_{\cdot m} \end{pmatrix} \qquad [12]$$

The combined effects of fertility, mortality, and geographical mobility on an interregional population system, may be expressed by adding the "super" matrix operators of equations [8] and [12]. This defines an overall "growth" matrix operator, $\boldsymbol{G}$, which when applied to an interregional population distribution will carry it forward through time. That is:

$$G = \left(\begin{array}{c|c|c|c|c} S_1 & M_{21} & M_{31} & \cdots & M_{m1} \\ \hline M_{12} & S_2 & M_{32} & \cdots & M_{m2} \\ \hline M_{13} & M_{23} & S_3 & \cdots & M_{m3} \\ \hline \vdots & & & \ddots & \vdots \\ \hline M_{1m} & \cdots & \cdots & \cdots & S_m \end{array}\right) \qquad w^{(t)} = \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ w_m^{(t)} \end{pmatrix}$$

and:

$$Gw^{(t)} = w^{(t+1)} \qquad [13]$$

Each submatrix, S_i, in G, now accounts for the effects of fertility and mortality in region i and for the migration out of region i. The matrices M_{ij} account for the migration of people into region j from all origins i ($i \neq j$).

Two-Region Example. The application of the above three matrix models may be demonstrated by a simple example involving two regions: California, and The Rest of the United States.[3] First, calibrating equation [2] on the basis of data for the 1955–1960 time period, we define the interregional components-of-change growth process for California and the Rest of the United States (in thousands) as follows:

$$\begin{array}{ccc} G & w^{1955} & w^{1960} \\ \begin{pmatrix} 1.0215 & .0127 \\ .0627 & 1.0667 \end{pmatrix} & \begin{pmatrix} 12988 \\ 152082 \end{pmatrix} = & \begin{pmatrix} 15199 \\ 163040 \end{pmatrix} \end{array}$$

Second, disaggregating into five-year age groups and focusing solely on California, we may estimate equation [6] for the same time interval (see Table 1). Finally, simultaneously considering California and The Rest of the United States, we find the process of equation [13] described in Table 2.[4] Here the eighteen five-year ago groups have been collapsed into nine ten-year cohorts for ease of exposition. Thus the interval has been expanded into the ten-year period 1950–1960.[5]

3. STABLE INTERREGIONAL POPULATION THEORY

In the preceding paragraphs, we have outlined three matrix models of population growth and distribution. In all three, a matrix operator, G, was defined which if applied to a population vector, $w^{(t)}$, "grew" it to the population vector of the subsequent time period, $w^{(t+1)}$. Consider now the population projection

[3] Without loss of generality, assume that the total population of this two-region system is "closed," i.e., undisturbed by emigration and immigration. This assumption produces 1960 population totals that are slightly lower than those reported by the 1960 U.S. Census.

[4] Because migration data for the 1950–1960 interval are not available, the elements of the matrix operator were estimated so as to exactly reproduce the 1960 population vector. Thus California's 1960 population in Table 2 differs slightly from that derived in Table 1.

[5] There is limited experience with the use of these models for forecasting purposes; see Rogers (18), (20), and (21) for some examples.

[13]

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TABLE 1: The Single-Region Cohort-Survival Process For California, 1955–1960*

$$
\underset{G_{Cal}}{\begin{pmatrix}
0 & 0 & 0 & .2884 & .7762 & .4969 & .2762 & .1380 & .0321 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1.0622 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0759 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.1175 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.2126 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.1570 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.1131 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0802 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0561 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0368 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0149 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9944 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9691 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .9381 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .8776 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .7856 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .6417 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .4129 & 0
\end{pmatrix}}
\underset{w^{1955}_{Cal}}{\begin{pmatrix} 1399 \\ 1317 \\ 1003 \\ 804 \\ 725 \\ 909 \\ 1057 \\ 992 \\ 938 \\ 820 \\ 710 \\ 621 \\ 525 \\ 458 \\ 332 \\ 210 \\ 107 \\ 61 \end{pmatrix}}
=
\underset{w^{1960}_{Cal}}{\begin{pmatrix} 1705 \\ 1486 \\ 1417 \\ 1121 \\ 975 \\ 839 \\ 1012 \\ 1142 \\ 1048 \\ 973 \\ 832 \\ 706 \\ 602 \\ 493 \\ 402 \\ 261 \\ 135 \\ 44 \end{pmatrix}}
$$

* Population data in thousands.

TABLE 2: The Multiregional Growth Process for California and the Rest of the United States, 1950–1960*

$$G=\left(\begin{array}{cc} \dfrac{S_{\mathrm{Cal}}}{M_{\mathrm{Cal,\,US}}} & \dfrac{M_{\mathrm{US,\,Cal}}}{S_{\mathrm{US}}} \end{array}\right) \qquad w=\left(\begin{array}{c} \dfrac{w_{\mathrm{Cal}}}{w_{\mathrm{US}}} \end{array}\right)_{1950} \qquad w=\left(\begin{array}{c} \dfrac{w_{\mathrm{Cal}}}{w_{\mathrm{US}}} \end{array}\right)_{1960}$$

																		1950 w	1960 w
0	.3375	1.1861	.4789	.0424	0	0	0	0	0	0	0	0	0	0	0	0	0	1947	3346
.8731	0	0	0	0	0	0	0	0	.0297	0	0	0	0	0	0	0	0	1283	2514
0	.9228	0	0	0	0	0	0	0	0	.0399	0	0	0	0	0	0	0	1698	2000
0	0	.7821	0	0	0	0	0	0	0	0	.0450	0	0	0	0	0	0	1750	2319
0	0	0	.7891	0	0	0	0	0	0	0	0	.0316	0	0	0	0	0	1438	2045
0	0	0	0	.6905	0	0	0	0	0	0	0	0	.0300	0	0	0	0	1125	1528
0	0	0	0	0	.7938	0	0	0	0	0	0	0	0	.0145	0	0	0	819	1102
0	0	0	0	0	0	.6288	0	0	0	0	0	0	0	0	.0137	0	0	404	655
0	0	0	0	0	0	0	.3540	0	0	0	0	0	0	0	0	.0126	0	122	208
0	0	0	0	0	0	0	0	0	0	.2257	.9746	,4252	.0364	0	0	0	0	27417	35667
.1682	0	0	0	0	0	0	0	0	.9903	0	0	0	0	0	0	0	0	20453	27478
0	.1503	0	0	0	0	0	0	0	0	.9523	0	0	0	0	0	0	0	22026	19670
0	0	.2536	0	0	0	0	0	0	0	0	.9843	0	0	0	0	0	0	21013	22111
0	0	0	.1565	0	0	0	0	0	0	0	0	.9594	0	0	0	0	0	17836	20434
0	0	0	0	.0988	0	0	0	0	0	0	0	0	.9176	0	0	0	0	14382	16508
0	0	0	0	0	.0670	0	0	0	0	0	0	0	0	.8499	0	0	0	10243	12298
0	0	0	0	0	0	.0583	0	0	0	0	0	0	0	0	.7171	0	0	5160	7393
0	0	0	0	0	0	0	.0560	0	0	0	0	0	0	0	0	.3921	0	1581	2046

* Population data in thousands.

that is generated by an unchanging regime of growth, G. Since:

$$Gw^{(t)} = w^{(t+1)}$$

and:

$$Gw^{(t+1)} = w^{(t+2)}$$

by substitution:

$$G^2 w^{(t)} = w^{(t+2)}$$

and, in general:

$$G^n w^{(t)} = w^{(t+n)} \qquad [14]$$

Asymptotic Behavior and the Intrinsic Rate of Growth. It has been demonstrated that the growth process of equation [14] gradually stabilized as the number of time periods, n, is increased.[6] Raising G to successively higher powers ultimately generates matrices which are scalar multiples of the preceding arrays. More specifically, at stability, say at and beyond $n = s$, we find that:

$$G^s = \lambda G^{s-1} \qquad [15]$$

and, since by [14]:

$$G^s w^{(1)} = Gw^{(s)}$$

by simple substitution, we have:

$$\lambda w^{(s)} = w^{(s+1)} \qquad [16]$$

Equation [16] establishes that a population vector, $w^{(t)}$, if subjected to an unchanging schedule of fertility, mortality and mobility, will ultimately increase at a constant "intrinsic" rate of growth, λ. Moreover, since λ is a scalar, it follows that, at stability, the corresponding elements of successive population vectors differ only by a constant factor, λ. Thus each entry of w receives a constant share of the total population. Denoting this allocation structure by the vector v, we have the following additional relationship at stability:

$$v^{(t+1)} = v^{(t)} \qquad [17]$$

The Theory of Non-Negative Matrices. The growth operator, G, in the three matrix models of population growth, is square and composed of non-negative elements. Therefore, Frobenius' theory holds for this class of matrices, and a considerable literature in mathematics and economics is directly applicable.[7] The relevant essentials for our immediate purposes may be summarized by the following theorem: (*Theorem 1*): If G is a non-negative matrix, then there exists a unique (except for a positive scalar factor) characteristic vector x, with all elements non-negative, associated with a non-negative characteristic root, λ, which is greater or equal to the absolute value of any other characteristic root of the matrix G.

Theorem 1 suggests the following formal formulation of the stable interregional population problem. Consider the relationship at stability expressed by

[6] See: Keyfitz (10), Rogers (23).

[7] See: Debreu and Herstein (3); Gantmacher (6); and Solow (28).

[16]. Since, by [13]:

$$Gw^{(s)} = w^{(s+1)} ,$$

it follows that:

$$Gw^{(s)} = \lambda w^{(s)} . \qquad [18]$$

The problem of deriving the set of λ and $\boldsymbol{w}$ for a known $\boldsymbol{G}$, in equation [18], is the classical matrix exercise of finding the characteristic roots and vectors of a non-negative square matrix. We proceed by solving the characteristic equation

$$|| G - \lambda I || = 0 \qquad [19]$$

This produces a set of n roots, λ_i $(i = 1, 2, \cdots, n)$. Theorem 1 ensures that there exists at least one positive root, λ, which is greater or equal to the absolute value of all others. It also establishes the existence of an associated positive characteristic vector, $\boldsymbol{w}$, whose components are all positive. Hence to derive the stable interregional vector $\boldsymbol{v}$, we need only to apply the relation:

$$v_i = \frac{w_i}{\sum_i w_i} . \qquad [20]$$

The Stable Interregional Population Distribution. Returning to our two-region example for California and The Rest of the United States, we may easily derive the stable interregional population distribution that is implied by the growth process defined in Section 2, above. First solving for the characteristic roots of the characteristic equation:

$$\left\| \begin{matrix} 1.0215 - \lambda & .0127 \\ .0627 & 1.0667 - \lambda \end{matrix} \right\| = 0$$

we have:

$$\lambda^2 - 2.0882\lambda + 1.0888 = 0$$

Thus: $\lambda_1 = 1.0802$ and $\lambda_2 = 1.0080$. The dominant root, $\lambda_1 = 1.0802$, is the "intrinsic" rate of growth of the interregional population at stability, and its associated characteristic vector may be found by solving equation [18] for this particular value of the root. Normalizing the elements of the resulting vector

TABLE 3: Past and Asymptotic Interregional Population Distributions: California and The Rest of the United States

Region	1955	1960	Stable distribution
California	.0787	.0876	.1772
5-year growth rate	1.2269	1.2101	λ=1.0802
Rest of the United States	.9213	.9124	.8228
5-year growth rate	1.0854	1.0758	λ=1.0802
United States 5-year growth rate	1.0954	1.0863	λ=1.0802

to unity (equation [20]), we find the stable interregional population $v = (.1772; .8228)$. Table 3 presents these results along with recent historical distributions for comparison.

The Stable Single-Region Age Distribution. The intrinsic rate of growth and the stable age distribution for the growth process of Table 1 may be derived by applying the method outlined above. However, for large matrices, more efficient algorithms based on "powering" are available.[8] Applying such a method to the growth matrix in Table 1, yields an intrinsic rate of growth $\lambda = 1.2393$ and the stable age vector presented in Table 4.

The Stable Interregional Age Structure. Repeating the "powering" method of solution on the growth operator in Table 2, we find the intrinsic decennial rate of growth of 1.2238 and an age-specific allocation of people to regions that we may define as the stable interregional age structure. Table 5 presents the detailed results.

TABLE 4: Past and Asymptotic Age Distributions: California

Age group	1950 Distribution	1960 Distribution	Stable age distribution
0– 4	.1037	.1111	.1364
5– 9	.0801	.1018	.1169
10–14	.0617	.0902	.1015
15–19	.0595	.0697	.0915
20–24	.0730	.0626	.0895
25–29	.0874	.0646	.0836
30–34	.0838	.0709	.0751
35–39	.0815	.0767	.0654
40–44	.0722	.0682	.0558
45–49	.0637	.0618	.0467
50–54	.0570	.0523	.0382
55–59	.0493	.0450	.0307
60–64	.0426	.0375	.0240
65–69	.0348	.0326	.0181
70–74	.0236	.0253	.0129
75–79	.0146	.0164	.0081
80–84	.0075	.0084	.0042
85+	.0040	.0049	.0014
5-year growth rate	1.1402	1.2101	$\lambda = 1.2393$

4. SPECIAL TOPICS

Conceptualizing an interregional demographic process in matrix terms generates a number of interesting sub-topics, which emerge as a direct consequence of the matrix formulation. As is so frequently true in theory-building, what

[8] See: Frazer *et al* (5), pp. 134–8.

TABLE 5: Past and Asymptotic Interregional and Intraregional Age Structures: California and The Rest of the United States

Age Group	1950		1960		Stability	
	Interregional Age Stucture	Intraregional Age Structure	Interregional Age Structure	Intraregional Age Structure	Interregional Age Structure	Intraregional Age Structure
California:						
0-9	.0129	.1839	.0187	.2129	.0808	.2775
10-19	.0085	.1212	.0140	.1600	.0612	.2102
20-29	.0113	.1604	.0112	.1273	.0504	.1731
30-39	.0116	.1653	.0129	.1475	.0362	.1244
40-49	.0095	.1358	.0114	.1301	.0259	.0889
50-59	.0075	.1063	.0085	.0972	.0166	.0570
60-69	.0054	.0774	.0061	.0701	.0115	.0396
70-79	.0027	.0382	.0036	.0417	.0064	.0220
80+	.0008	.0115	.0012	.0132	.0021	.0073
Subtotal	.0702	1.0000	.0876	1.0000	.2911	1.0000
Regional decennial growth rate	—	1.5326	—	1.4847	—	1.2238
Rest of the U.S.:						
0-9	.1819	.1957	.1989	.2180	.1471	.2075
10-19	.1357	.1460	.1532	.1680	.1301	.1836
20-29	.1462	.1572	.1097	.1202	.1088	.1534
30-39	.1394	.1500	.1233	.1351	.0979	.1382
40-49	.1184	.1273	.1140	.1249	.0814	.1148
50-59	.0954	.1026	.0921	.1009	.0631	.0891
60-69	.0680	.0731	.0686	.0752	.0448	.0631
70-79	.0343	.0368	.0412	.0452	.0268	.0378
80+	.0105	.0113	.0114	.0125	.0089	.0125
Subtotal	.9298	1.0000	.9124	1.0000	.7089	1.0000
Regional decennial growth rate	—	1.1230	—	1.1677	—	1.2238
Total	1.0000	—	1.0000	—	1.0000	—
National decennial growth rate	1.1445	—	1.1900	—	$\lambda=1.2238$	—

70–79	.0343	.0368	.0412	.0452	.0268	.0378
80+	.0105	.0113	.0114	.0125	.0089	.0125
Subtotal	.9298	1.0000	.9124	1.0000	.7089	1.0000
Regional decennial growth rate	—	1.1230	—	1.1677	—	1.2238
Total	1.0000	—	1.0000	—	1.0000	—
National decennial growth rate	1.1445	—	1.1900	—	$\lambda=1.2238$	—

at first appears to be a purely pragmatic conceptualization, ultimately becomes the vehicle for insights not suggested by conventional methods. Of particular relevance for regional analysis are the following problems:

(1) Given a matrix growth operator $\boldsymbol{G}$, calibrated on the basis of an n year interval, estimate $\boldsymbol{G}^{1/n}$, i.e., the growth operator for an "average" one-year interval.[9]

(2) Given time series data on an interregional population vector, $\boldsymbol{w}$ (i.e., $\boldsymbol{w}^{(t)}, \boldsymbol{w}^{(t+1)}, \cdots, \boldsymbol{w}^{(t+n)}$), estimate a matrix growth operator $\boldsymbol{G}$ which, when applied to $\boldsymbol{w}^{(t)}$, will reproduce as closely as possible the observed time series.[10]

(3) Given an initial interregional population vector, $\boldsymbol{w}$, and a desired goal distribution, $\boldsymbol{g}$, determine if the goal is feasible and, if it is, find an intervention vector, $\boldsymbol{f}$, which will redirect the system from its original asymptotic distribution toward the desired distribution.[11]

Preliminary explorations in these problem areas have provided encouraging results and are summarized below.

Temporal Decomposition of Population Growth Operators. A problem of occasional interest to demographers is the decomposition of a growth structure describing behavior over an n unit time interval into the corresponding "average" growth structure for a single time interval. That is, given a growth operator, $\boldsymbol{G}$, calibrated over a time period of n years, find a one-year growth matrix, $\boldsymbol{P}$, such that:

$$\boldsymbol{P}^n = \boldsymbol{G} \tag{21}$$

A convenient solution method is to find a collinearity transformation of $\boldsymbol{G}$ such that:

$$\boldsymbol{G} = \boldsymbol{N}\boldsymbol{D}\boldsymbol{N}^{-1} \tag{22}$$

where $\boldsymbol{D}$ is a diagonal matrix and $\boldsymbol{N}$ a suitably constructed non-singular square matrix of the same order as $\boldsymbol{G}$. Such a transformation enables one to examine a number of polynomial functions of $\boldsymbol{G}$. For example:

$$\boldsymbol{G}^2 = (\boldsymbol{N}\boldsymbol{D}\boldsymbol{N}^{-1})(\boldsymbol{N}\boldsymbol{D}\boldsymbol{N}^{-1}) = \boldsymbol{N}\boldsymbol{D}(\boldsymbol{N}^{-1}\boldsymbol{N})\boldsymbol{D}\boldsymbol{N}^{-1} = \boldsymbol{N}\boldsymbol{D}^2\boldsymbol{N}^{-1}$$

Using the associative property of matrix multiplication, the above argument may be extended to higher powers and thus to polynomial functions of $\boldsymbol{G}$:

$$f(\boldsymbol{G}) = \boldsymbol{N}f(\boldsymbol{D})\boldsymbol{N}^{-1} \tag{23}$$

where $f(\cdot)$ is a polynomial function.

Since the temporal decomposition problem, in effect, is to find $\boldsymbol{G}^{1/n}$, we may use [23] to establish the relationship:

$$\boldsymbol{G}^{1/n} = \boldsymbol{N}\boldsymbol{D}^{1/n}\boldsymbol{N}^{-1} \tag{24}$$

A well-known result in matrix algebra establishes that for a $\boldsymbol{G}$ with distinct characteristic roots, a solution for $\boldsymbol{D}$, in [22], is a diagonal matrix with non-zero entries equal to the characteristic roots of the determinantal equation:

$$\| \boldsymbol{G} - \lambda \boldsymbol{I} \| = 0\,. \tag{25}$$

[9] See: Rogers (24).
[10] See: Rogers and Miller (26).
[11] See: Rogers (17).

Moreover, it is also established that the columns of N are the characteristic vectors which correspond to the characteristic roots of equation [25].[12] A unique solution does not exist; however, any solution will satisfy [24]. One such solution may be constructed by using the cofactors of the first row of $\|G - \lambda_i I\|$ for the ith column of N.

The two-region growth process examined above may serve to clarify the discussion. Recall that the characteristic roots of the growth operator are $\lambda_1 =$ 1.0802, and $\lambda_2 = 1.0080$. Hence:

$$D = \begin{pmatrix} 1.0802 & 0 \\ 0 & 1.0080 \end{pmatrix}$$

The two cofactors of the first row of $\|G - \lambda_1 I\|$ are:

$$n_{11} = -.0135 \qquad n_{21} = -.0627$$

Analogously, for $\|G - \lambda_2 I\|$ we have:

$$n_{12} = .0587 \qquad n_{22} = -.0627\,.$$

Hence we may assemble the matrix:

$$N = \begin{pmatrix} -.0135 & .0587 \\ -.0627 & -.0627 \end{pmatrix}$$

and find its inverse:

$$N^{-1} = \begin{pmatrix} -13.8504 & -12.9668 \\ 13.8504 & -\ 2.9821 \end{pmatrix}$$

As a check on our arithmetic, compute NDN^{-1} and confirm that the result is indeed the matrix G.

With the collinear transform of G we now are in a position to find $G^{1/5}$. The first step is to derive $D^{1/5}$. Since D is a diagonal matrix, we need only to find $\sqrt{d_{ii}}$, which in our example yields:

$$D^{1/5} = \begin{pmatrix} 1.0155 & 0 \\ 0 & 1.0016 \end{pmatrix}$$

Recalling equation [24] we have:

$$\begin{aligned} P = G^{1/5} &= ND^{1/5}N^{-1} \\ &= \begin{pmatrix} -.0135 & .0587 \\ -.0627 & -.0627 \end{pmatrix} \begin{pmatrix} 1.0155 & 0 \\ 0 & 1.0016 \end{pmatrix} \begin{pmatrix} -13.8504 & -12.9668 \\ 13.8504 & -\ 2.9821 \end{pmatrix} \\ &= \begin{pmatrix} 1.0042 & .0024 \\ .0121 & 1.0129 \end{pmatrix} \end{aligned}$$

Again, as a check on our arithmetic, we compute P^5 and confirm that the result is the matrix G.

Estimating a Population Growth Operator from Distributional Time Series. A common constraint in interregional population analysis and forecasting is the absence of reliable data concerning the behavior of the fundamental components of population change, particularly in- and out-migration. Thus considerable importance attaches to possible ways of circumventing this problem. Toward this end, the recent literature concerned with techniques for estimating the transi-

[12] See: Frazer *et al* (5), pp. 66–70.

tion matrices of Markov chains becomes especially relevant.[13] These efforts suggest a method for estimation of the growth regime of an interregional system solely on the basis of a history of interregional population distributions.

For expositional purposes, let us first generate a time series for our two-region example involving California and The Rest of the United States. Recall the annual growth matrix derived in the preceding section:

$$\boldsymbol{G} = \begin{pmatrix} 1.0042 & .0024 \\ .0121 & 1.0129 \end{pmatrix}$$

and begin the time series with the 1955 population vector:

$$\boldsymbol{w}^{1955} = \begin{pmatrix} 12988 \\ 152082 \end{pmatrix}$$

Applying the above $\boldsymbol{G}$ to $\boldsymbol{w}^{1955}$ generates a 1956 population vector. Continuing in this manner, we derive the distributional time series presented in Table 6.

TABLE 6: Distributional Time Series Generated by a Constant Growth Operator: California and The Rest of the United States, 1955–1960

Region	1955	1956	1957	1958	1959	1960
California	12988	13408	13834	14267	14707	15155
Rest of the United States	152082	154201	156352	158536	160754	163006
Total	165070	167609	170186	172803	175461	178161

The projected 1960 population figures are slightly lower than those reported by the U.S. Census. However, they are adequate for our expositional purposes and will be used to demonstrate the estimation techniques.

Consider now the growth process for our two-region example over two sequential periods:

$$\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} w_1^{(t)} \\ w_2^{(t)} \end{pmatrix} = \begin{pmatrix} w_1^{(t+1)} \\ w_2^{(t+1)} \end{pmatrix}$$

and

$$\begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} w_1^{(t+1)} \\ w_2^{(t+1)} \end{pmatrix} = \begin{pmatrix} w_1^{(t+2)} \\ w_2^{(t+2)} \end{pmatrix}$$

Assume that the w_i have been observed and therefore are known, but that the matrix growth operator, $\boldsymbol{G}$, is unknown and has to be estimated solely on the basis of the observed w_i. We have, then, a problem in the solution of a system of four equations in four unknowns. Thus, for example, given the $\boldsymbol{w}^{1955}$, $\boldsymbol{w}^{1956}$ and $\boldsymbol{w}^{1957}$ of Table 6, we may solve the system described below and "retrieve" the matrix $\boldsymbol{G}$:

$$12988\ g_{11} + 152082\ g_{21} = 13408$$
$$12988\ g_{12} + 152082\ g_{22} = 154201$$
$$13408\ g_{11} + 154201\ g_{21} = 13834$$

[13] See: Lee *et al* (12); Madansky (15); Miller (16); Telser (29); and Theil and Rey (30).

$$13408\ g_{12} + 154201\ g_{22} = 156352$$

hence:

$$g_{11} = 1.0042,\ g_{12} = .0121,\ g_{21} = .0024,\ \text{and}\ g_{22} = 1.0129\,.$$

Consider next the more general problem of estimating an operator $\boldsymbol{G}$ which, when applied to the vector $\boldsymbol{w}^{1955}$, reproduces, as accurately as possible, the entire time series in Table 6. We now have a system of ten equations in four unknowns, and therefore a criterion of "goodness-of-fit" needs to be introduced in order to arrive at a unique solution. Three have been suggested in the literature cited earlier: (1) the unrestricted least-squares estimate; (2) the minimum absolute deviations estimate; and (3) the restricted least-squares estimate. The first adopts the well-known estimation technique traditionally used in regression analysis; the latter two require a mathematical programming formulation. In this paper we will consider only the least-squares solution.[14]

Applying the least-squares procedure to our two-region example, we once again "retrieve" the growth matrix $\boldsymbol{G}$:

$$\boldsymbol{G} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} = \begin{pmatrix} 1.0042 & .0024 \\ .0121 & 1.0129 \end{pmatrix}$$

Unrestricted least-squares estimation, however, leaves unresolved a serious problem in the estimation of growth operators. This is the possible entry of negative elements into $\boldsymbol{G}$. These, of course, are by definition inadmissible estimates. To cope with this problem, Telser has suggested a method whereby negative elements are set equal to zero and the matrix is then adjusted to compensate for this change.[15] Such a procedure, however, is unsatisfactory for our purposes, since setting a given g_{ij} equal to zero would imply either: (1) zero migration from i to j, or (2) the extinction of the population in region i. Thus, for a more reasonable approach, one must adopt a method which incorporates a non-negativity constraint in the estimation process. Such a restriction may be introduced in a mathematical programming formulation of the estimation problem.[16]

Distributional Goals and System Intervention. In the absence of system intervention, the matrix growth models described in this paper converge to a stable distribution. This distribution, however, may be incompatible with long-range planning goals. Thus, intervention to effect a desired allocation of population to regions may be in order. An intervention, for example, may take the form of a policy that seeks to attract a certain minimum number of people, each year, from one set of regions to another, thereby redirecting the interregional system toward the desired equilibrium. Two considerations immediately arise:

(1) Not all distributional goals are feasible given a particular regime of growth $\boldsymbol{G}$; and

(2) If the goal is feasible, the right number of people must be transferred during each time period.

Let us define the number of people to be supplied or withdrawn from each region by a vector $\boldsymbol{f}$ which may have both positive and negative components.

14 The mathematical programming solutions are described in a forthcoming paper; see: Rogers and Miller (26).

15 See: Telser (29).

16 See: Lee *et al* (12); Rogers and Miller (26); and Theil and Rey (30).

A positive f_i indicates the number of people which must be attracted into a region during each unit interval of time; a negative f_i denotes the population that has to be periodically withdrawn from region i. The vector $\boldsymbol{w}$, as before, describes a population distribution, $\boldsymbol{G}$ is the matrix growth operator, $\boldsymbol{v}$ is the stable distribution, and $\boldsymbol{g}$ denotes the desired equilibrium interregional distribution.

It has been shown earlier that if $\boldsymbol{w}$ is the initial distribution of people over regions then $\boldsymbol{Gw}$ is the distribution after one time period, $\boldsymbol{Gw}^2$ after two, and $\boldsymbol{G}^n\boldsymbol{w}$ after n time periods. We now introduce the effects of system intervention. The number of people transferred from region to region during the initial transition period changes the total in each region by $\boldsymbol{G}^{n-1}\boldsymbol{f}$ after n periods. The addition of the vector $\boldsymbol{f}$ after the following transition alters the total in each region by $\boldsymbol{G}^{n-2}\boldsymbol{f}$ after n periods, and so on. Thus, the total population after n periods is

$$\boldsymbol{G}^n\boldsymbol{w} + \sum_{k=0}^{n-1} \boldsymbol{G}^k\boldsymbol{f} \qquad [26]$$

Since the goal of the policy-making body is represented by $\boldsymbol{g}$, clearly what is desired is that equation [26] should, in the limit, converge to $\boldsymbol{g}$. For such a goal to be feasible, however, we must include the restriction that during this limiting process, the population of each region never becomes negative, i.e.,

$$\boldsymbol{G}^n\boldsymbol{w} + \sum_{k=0}^{n-1} \boldsymbol{G}^k\boldsymbol{f} \geqq 0 \quad \text{for all } k = 1, 2, \cdots, n-1 \qquad [27]$$

There are three possible classes of growth regimes:

(1) which maintains a constant population, i.e., the characteristic root $\lambda = 1$;

(2) which generates a gradually declining population, i.e., the characteristic root $\lambda < 1$; and, most commonly,

(3) a growth regime which develops a continually increasing population, i.e., the characteristic root $\lambda > 1$.

A recent paper develops in great detail the policy model defined by equation [26] for a constant population and for a gradually declining one.[17] The more complex problem of a growth regime with $\lambda > 1$ still remains to be solved.[18] For a growth regime with $\lambda = 1$, the relationship which links the goal distribution, $\boldsymbol{g}$, with the required intervention vector, $\boldsymbol{f}$, is:

$$\boldsymbol{f} = (\boldsymbol{I} - \boldsymbol{G})\boldsymbol{g} \qquad [28]$$

Table 7 presents the intervention that is required to achieve a particular goal distribution for our two-region example. Recall that in this instance we are assuming an interregional population that remains constant over time. Thus

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17 See: Rogers (17).

18 A possible line of approach is to find the intervention vector for the model with a constant population and then increase this vector after each time period by the intrinsic rate of growth, λ. Thus the population, after n periods, becomes

$$\boldsymbol{G}^n\boldsymbol{w} + \sum_{k=0}^{n-1} \boldsymbol{G}^k \lambda^{n-k} \boldsymbol{f}$$

and at stability both the interregional population and the intervention vector grow at the same rate, λ.

the stable distribution in the absence of intervention differs from that found in Table 3.

Consider the stable distribution that results in the absence of intervention. According to Table 7, this particular equilibrium indicates that 16.84% of the national population ultimately will reside in California. Assume that it is desired to limit this expected long-run share to only 12.00%. Is such a goal feasible? What transfer levels are required at each iteration to direct the system toward this allocation?

Table 7 shows that to redirect the interregional system away from a stable distribution of (.1684/.8316) toward one of (.1200/.8800) one must, during each five-year time interval, discourage 603,000 people from moving into California, i.e., roughly one-fourth of the volume of flow that occurred during the 1955-1960 period.

TABLE 7: Policy Model for California and The Rest of the United States: Constant Population*

Stable Distribution Without Intervention= (.1684; 8316)
Goal Distribution= (.1200; .8800)

$$\underset{G}{\begin{pmatrix} .9373 & .0127 \\ .0627 & .9873 \end{pmatrix}} \underset{w\ 1955}{\begin{pmatrix} 12988 \\ (.0787) \\ 152082 \\ (.9213) \end{pmatrix}} = \begin{pmatrix} 14105 \\ 150965 \end{pmatrix} + \underset{f}{\begin{pmatrix} -603 \\ 603 \end{pmatrix}} = \underset{w\ 1960}{\begin{pmatrix} 13502 \\ (.0818) \\ 151568 \\ (.9182) \end{pmatrix}} \begin{matrix} \textit{Cal.} \\ \textit{U.S.} \end{matrix}$$

$$\begin{pmatrix} .9373 & .0127 \\ .0627 & .9873 \end{pmatrix} \begin{pmatrix} 13502 \\ (.0818) \\ 151568 \\ (.9182) \end{pmatrix} = \begin{pmatrix} 14580 \\ 150490 \end{pmatrix} + \begin{pmatrix} -603 \\ 603 \end{pmatrix} = \begin{pmatrix} 13977 \\ (.0847) \\ 151093 \\ (.9153) \end{pmatrix} \begin{matrix} \textit{Cal.} \\ \textit{U.S.} \end{matrix}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\begin{pmatrix} .9373 & .0127 \\ .0627 & .9873 \end{pmatrix} \begin{pmatrix} 19808 \\ (.1200) \\ 145262 \\ (.8800) \end{pmatrix} = \begin{pmatrix} 20411 \\ 144659 \end{pmatrix} + \begin{pmatrix} -603 \\ 603 \end{pmatrix} = \begin{pmatrix} 19808 \\ (.1200) \\ 145262 \\ (.8800) \end{pmatrix} \begin{matrix} \textit{Cal.} \\ \textit{U.S.} \end{matrix}$$

* Population data in thousands.

5. CONCLUSION

This paper summarizes progress on the development of a matrix-based theory of interregional population growth and distribution. Perhaps the most important contribution of the matrix formulation is the separation of process from the population that is undergoing this process. The use of a projection operator to "grow" an interregional population system forward through time allows one to focus on the projection process itself, its application to another population, and its ultimate consequences. The insights yielded by such an analysis are not attainable with conventional techniques. Moreover, the matrix representation of population growth and change leads to a method for estimating a growth regime in the absence of fertility, mortality and mobility data. Because of the general paucity of adequate data on migration, this feature assumes considerable importance. Finally, in recent years many demographers have directed their attention to population problems in underdeveloped areas. Empirical investigations have noted the relatively unchanging patterns of fertility and stable age distributions that exist in such regions. From this one may infer that stable population theory may provide an adequate model of their demographic structure.

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