Title
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Publication Date
1986-08-01

LBL-21632
Preprint

Lawrence Berkeley Laboratory

UNIVERSITY OF CALIFORNIA

Submitted to Annals of Physics

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August 1986

NUCLEAR SCIENCE DIVISION

Prepared for the U.S. Department of Energy under Contract DE-AC03-76SF00098

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LBL-21632

Quantum Transport Theory for Abelian Plasmas [1]

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August 7, 1986

Abstract: The gauge invariant relativistic quantum equations of motion for the fermion and photon Wigner operators are derived from QED. In the mean field (Hartree) approximation, we extract the generalized quantum Vlasov and mass-shell constraint equations for fermions. In addition, a complete spinor decomposition is performed. A systematic method for computing quantum corrections to all orders in $\hbar$ is developed. First order quantum (spin) corrections are computed explicitly. Finally, the relations between gauge dependent and independent definitions of the photon Wigner function and their corresponding transport equations are discussed.

[1] Work supported by the Director, Office of High Energy and Nuclear Physics of the Department of Energy under Contract DE-AC03-76SF00098. H.-Th. Elze and D. Vasak gratefully acknowledge support by DAAD-NATO Postdoctoral Fellowships.

1. Introduction

Interest in transport equations for gauge field theories has been stimulated by recent QCD calculations indicating the possible existence of a new phase of nuclear matter, the quark-gluon plasma (QGP) [1]. It is generally expected that this phase may be formed in violent collisions between heavy ions. Most signatures of QGP have been calculated using thermodynamic or hydrodynamic model assuming that local equilibrium is reached in such collisions. However, due to the short time scales involved, nonequilibrium and even quantum effects may play an important role. The impact of those effects on the proposed signatures of the QGP can be assessed only with the help of non-equilibrium quantum transport theory.

In dealing with fermions interacting via gauge bosons the transport theory should also be invariant under local gauge transformations. Recently, such a gauge covariant formulation of quantum transport theory for QCD has been proposed [2,3] (subsequently referred to as EGV). In order to gain further insight into the structure of that theory without the additional complication caused by specifically non-abelian effects, it is instructive to study the abelian (QED) limit in more detail. The primary objective of this paper is to complement EGV in this respect. In addition, gauge invariant abelian transport theory is interesting in its own right since up to now most formulations, e.g. [4,5], were implicitly based on gauge dependent definitions of the fermion Wigner function. Only the special cases of scalar QED in the semiclassical approximation [6] and the non-relativistic limit [7] were considered in a gauge invariant way.

Classical kinetic theory characterizes an ensemble of point-like particles by their one-particle phase-space distribution function $f(x,p)$, where x and p are the particle co-ordinate and kinetic momentum, respectively. The time development of the function $f(x,p)$ is governed by the Vlasov-Boltzmann transport equation [8]. Quantum corrections can be calculated rigorously by considering the quantum transport equation for the appropriate Wigner function [9], which is the quantum mechanical analogue of $f(x,p)$. Its equation of motion can be derived from Heisenberg's equations of motion for the associated field operators [10,11]. In the case of gauge theories though, the Wigner function must be defined in such a way as to insure gauge invariance [12].

In this paper we consider the Lorentz covariant, gauge invariant quantum transport theory of fermions and vector gauge bosons. After introducing our notation in section 2, we discuss the appropriate form of the gauge invariant fermion Wigner operator in section 3. The form of that operator is uniquely determined by requiring that the momentum variable correspond to the kinetic momentum. We derive the exact quantum equation for the Wigner operator in section 4. For arbitrary external or self-consistent (Hartree) fields, that equation reduces to a linear quantum operator equation which contains in the classical limit the familiar relativistic Vlasov equation modified by specific spin dependent terms. This is shown by isolating the hermitian and anti-hermitian parts of the quadratic form of that operator equation. In this way we derive not only the generalized quantum Vlasov equation but also a generalized mass-shell constraint equation. Quantum corrections to classical transport theory can be calculated systematically from those equations by an expansion in powers of the triangle operator, $\triangle = \partial_x \cdot \partial_p$, introduced in EGV. We find that unlike in classical transport theory, the quantum transport equation is incomplete without the generalized mass-shell constraint equation because of the uncertainty principle. Finally, we derive an expression for the quantum transport equation that follows when the operator nature of the electromagnetic field is taken into account.

Beyond the scope of EGV we perform in section 5 a complete decomposition of the Wigner function in spinor space. This spinor decomposition complements EGV in an important way because in that work only a decomposition in color space was considered. We show that the large set of coupled equations for the moments of the Wigner function reduces to a set of coupled equations for the scalar and axial vector components. The resulting equations and solutions are shown to have an especially transparent physical interpretation in the classical limit. Finally, in section 6 we propose an extension of the phase-space formalism to vector gauge bosons and derive transport equations for gauge dependent and gauge independent versions of the photon Wigner operator. We also discuss the additional complication in that case associated with the separation of coherent and chaotic field components.

2. Definitions

Our conventions, notation, and the basic relations employed in the following sections are listed below. In general, we follow the conventions of ref. [13]. Natural units ($\hbar = c = 1$) are used except for emphasis in cases where displaying $\hbar$ helps to clarify the presentation.

The Lagrangian for the abelian gauge theory of fermions (electrons) with mass m and charge e,

$$\mathcal{L}(x) = \bar{\psi}(x)\,(i\gamma\cdot\partial - m)\,\psi(x) - e\,\bar{\psi}(x)\,\gamma\cdot A(x)\,\psi(x) - \tfrac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x)\;,$$

leads to Dirac's equations for the fermion field operator and its adjoint,

$$[i\gamma^\mu D_\mu(x) - m]\;\psi(x) = 0 = \bar{\psi}(x)[i\gamma^\mu D^\dagger_\mu(x) + m]\;, \tag{2.1}$$

where the "gauge invariant derivatives"

$$D_\mu(x) \equiv \partial_\mu + ieA_\mu(x),\; D^\dagger_\mu(x) \equiv \partial^\dagger_\mu - ieA_\mu(x)$$

involve the ordinary derivatives, $\partial_\mu \equiv \frac{\overrightarrow{\partial}}{\partial x^\mu}$ and $\partial^\dagger_\mu \equiv \frac{\overleftarrow{\partial}}{\partial x^\mu}$, acting to the right and to the left, respectively. The gauge field tensor,

$$F_{\mu\nu}(x) \equiv \partial_\mu\, A_\nu(x) - \partial_\nu\, A_\mu(x)\;, \tag{2.2}$$

satisfies the homogeneous Maxwell's equation

$$\partial^\sigma\, F^{\mu\nu}(x) + \partial^\mu\, F^{\nu\sigma}(x) + \partial^\nu\, F^{\sigma\mu}(x) = 0\;, \tag{2.3}$$

and the inhomogeneous Maxwell's equation

$$\partial_\mu F^{\mu\nu}(x) = e\,\bar{\psi}(x)\,\gamma^\nu\,\psi(x) \equiv j^\nu(x)\,. \tag{2.4}$$

In quantum field theory the above equations are understood to be operator equations in the Heisenberg picture.

The form of the coupling between the fermions and the real vector field is motivated by the requirement of local gauge invariance. This means that under a local gauge transformation,

$$\psi(x) \to \psi'(x) = \psi(x)\,e^{ie\Lambda(x)}\,, \tag{2.5}$$

with an arbitrary real function $\Lambda(x)$, the equations of motion (2.1), (2.3) and (2.4) should retain their form. This can be achieved only if simultaneously with (2.5) the vector field transforms inhomogeneously according to

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) - \partial_\mu\,\Lambda(x)\,.$$

For later reference we also display here the quadratic form of Dirac's equation and its adjoint

$$\begin{aligned} (D_\mu D^\mu + \tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} + m^2)\,\psi &= 0\,, \\ \bar{\psi}\,(D_\mu^\dagger D^{\dagger\mu} + \tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu} + m^2) &= 0\,, \end{aligned} \tag{2.6}$$

where $\sigma^{\mu\nu} \equiv \frac{1}{2}i[\gamma^\mu,\gamma^\nu]$ and $g^{\mu\nu} = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\}$.

3. The Wigner Operator

The quantum mechanical analogue of the classical phase-space distribution function $f(t,\vec{x},\vec{p})$ is called the Wigner function $W(x,p)$ and is obtained by applying Weyl's correspondence principle [9]-[12]. In relativistic field theory that function corresponds to the ensemble average of the Wigner operator

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\,\bar{\psi}_\beta(x)\,e^{\frac{1}{2}y\cdot\partial^\dagger}\,e^{-\frac{1}{2}y\cdot\partial}\,\psi_\alpha(x)\,. \tag{3.1}$$

Here the derivatives $\partial_\mu^\dagger$ and ∂_μ play the role of generators of translations acting to the left and to the right, respectively. The spinors are Heisenberg operators. The attached indices α and β are spinor indices, such that $\hat{W}$ is a (4×4)-matrix. The Wigner function is then

$$W(x,p) \equiv \langle :\hat{W}(x,p): \rangle\,.$$

The brackets indicate ensemble averaging and the colons normal ordering with respect to the vacuum state. The physical interpretation of the Wigner function is made clear if we note that in terms of the four momentum operator, as represented here by $\hat{p}_\mu = \frac{1}{2}i(\partial_\mu - \partial_\mu^\dagger)$, the integration over y gives formally

$$W_{\alpha\beta}(x,p) = \langle :\bar{\psi}_\beta(x)\delta^4(p-\hat{p})\psi_\alpha(x): \rangle\,. \tag{3.2}$$

Thus $trW(x,p)$ measures the Lorentz scalar density of electrons at space-time point x_μ with four momentum p_μ. In terms of the Wigner function other quantities of interest such as the fermion vector current can be written as

$$\langle j^\mu(x)\rangle \equiv e\,\langle :\bar{\psi}(x)\gamma^\mu\psi(x): \rangle = e\,tr\int d^4p\,\gamma^\mu\langle :\hat{W}(x,p): \rangle = e\,tr\int d^4p\,\gamma^\mu W(x,p)\;. \tag{3.3}$$

The trace refers to spinor indices. This relation is obvious from (3.2). As another example, the ensemble average of the non-interacting part of the (unsymmetrized) fermion stress tensor [13] is given by

$$\langle T_{\mu\nu}(x)\rangle \equiv -\tfrac{1}{2}i\,\langle :\bar{\psi}(x)\left[\gamma_\mu\partial_\nu^\dagger - \gamma_\mu\partial_\nu\right]\psi(x): \rangle = tr\int d^4p\,p_\nu\gamma_\mu W(x,p)\;. \tag{3.4}$$

In order to describe the dynamics of an ensemble of interacting particles, we have to specify the equation of motion for the Wigner function $W(x,p)$. However, as we shall see below its derivation does not refer to the particular ensemble under consideration. Hence we may in the following address the Wigner operator $\hat{W}(x,p)$, as defined in eq. (3.1), directly. Note that unlike the classical distribution function, $f(t,\vec{x},\vec{p})$, the relativistic Wigner function depends on eight variables, (x_μ,p_μ).

In dealing with gauge theories such as QED, the definition of $\hat{W}(x,p)$ must be modified if we want to insure local gauge invariance. The problem is that ordinary translations produced by the operator $\exp(i\hat{p}\cdot y)$ are not well defined in a gauge theories. To see this, note that under a gauge transformation (2.5),

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\,\bar{\psi}_\beta(x+\tfrac{1}{2}y)\psi_\alpha(x-\tfrac{1}{2}y) \;\rightarrow$$

$$\int\frac{d^4y}{(2\pi)^4}e^{-i\left[p\cdot y - eA(x-\frac{1}{2}y)+eA(x+\frac{1}{2}y)\right]}\,\bar{\psi}_\beta(x+\tfrac{1}{2}y)\psi_\alpha(x-\tfrac{1}{2}y)\;.$$

The shift in the arguments of the fermion field operators caused by the action of the translation in eq. (3.1) has been written out explicitly. With the definition (3.1) physical quantities such as the momentum distribution of electrons would therefore not be gauge invariant.

The unwanted gauge dependence of $\hat{W}(x,p)$ can be removed [12] by inserting a suitable phase factor of the form

$$U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y) = e^{ief(A;x,y)}$$

in the definition (3.1):

$$\hat{W}_{\alpha\beta}(x,p) \equiv \int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\,\bar{\psi}_\beta(x+\tfrac{1}{2}y)\,U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\,\psi_\alpha(x-\tfrac{1}{2}y)\;. \tag{3.5}$$

In order to cancel the gauge dependent phases generated by the spinors and to retain the property (3.3), the phase $f(A;x,y)$ has to satisfy

$$f(A;x,y=0)=0\;,$$

$$f(A-\partial\Lambda;x,y)=f(A;x,y)+\Lambda(x+\tfrac{1}{2}y)-\Lambda(x-\tfrac{1}{2}y)\ .$$

Due to the inhomogeneous transformation property of the gauge fields, it is readily seen that the *ansatz*

$$f(A;x,y)\equiv -y^\mu\int_0^1 ds\ A_\mu(x-\tfrac{1}{2}y+sy)\ ,$$

involving a line integral over the gauge field A^μ along a straight path between the points $x-\frac{1}{2}y$ and $x+\frac{1}{2}y$, satisfies the above requirements. When A_μ is regarded as a field operator, the unitary phase factor must be written in terms of a path ordered exponential

$$P\,U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)=P\,e^{-ie\,f(A;x,y)}=P\,e^{-ie\,y^\mu\int_0^1 ds\,A_\mu(x-\frac{1}{2}y+sy)}\ , \qquad (3.6)$$

where P denotes an ordering with respect to the parameter s in the same way as the familiar Dyson time ordering deals with the time variable $t=x^0$ [13]. The Wigner operator (3.5) with this unitary phase factor is indeed gauge invariant. The fact that $A^\mu(x)$ is an operator rather than an ordinary function implies that we cannot move the phase factor (3.6) outside the "sandwiching" fermion operators. Only if the gauge field operators are approximated by c-number fields we can drop the difficulty of operator ordering alltogether.

For further manipulations it is convenient to define the gauge invariant Wigner kernel

$$\Psi_{\alpha\beta}(x,y)\equiv\bar\psi_\beta(x+\tfrac{1}{2}y)\ P\ U(A,x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\ \psi_\alpha(x-\tfrac{1}{2}y)\ ,$$

which is related to the Wigner operator by

$$\hat W_{\alpha\beta}(x,p)=\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\ \Psi_{\alpha\beta}(x,y)\ .$$

It is important to note that the short-hand notation for f,

$$f(A)=-\int_{x-\frac{1}{2}y}^{x+\frac{1}{2}y}dz^\mu\ A_\mu(z)\ ,$$

should be avoided because it suggests path independence of the integral. However, $f(A)$ is known to depend on the path in general [14]. It is independent of the path only in the trivial case that the vector field is pure gauge, i.e., $A^\mu(x)=\partial^\mu\Lambda(x)$. The question then arises whether the gauge invariant definition (3.5) is unique in view of our choice of the straight line in the path integral in (3.6). It is indeed possible to choose other paths which still leave $\hat W(x,p)$ gauge invariant. We only need to assign an arbitrary path, $z^\mu(s;x_1,x_2)$, to every pair of points, (x_1,x_2), such that $z^\mu(0;x_1,x_2)=x_1$ and $z^\mu(1;x_1,x_2)=x_2$, and define the phase factor as [2]

$$PU(A;x_1,x_2)=\lim_{n\to\infty}(1-ie\ dz_n\cdot A(z_n))\cdots(1-ie\ dz_1\cdot A(z_1))\ , \qquad (3.7)$$

where $z_i=z(s=i/n;x_1,x_2)$ and $dz_i=z_i-z_{i-1}$ with $i=1,\cdots,n$. However, we find in section 4 that the specific choice of the path enters explicitly in the equation of motion for $\hat W$.

This path ambiguity can be removed only by requiring the proper physical interpretation of the momentum variable in $W(x,p)$. It was recognized long ago [15] that the straight line path is special in connection with identifying p as the *kinetic* momentum. Recall that in classical electrodynamics [16] the kinetic momentum is given by $\pi^\mu = p^\mu - eA^\mu$, where p^μ is the *canonical* momentum conjugate to the coordinate x^μ. In quantum mechanics p^μ can be represented by the operator $\hat{p}^\mu$ as defined above (3.2). Thus, the operator representing the kinetic momentum is given by $\hat{\pi}_\mu = \hat{p}_\mu - eA_\mu(x) = \frac{1}{2}i(D - D^\dagger)$ in terms of the *gauge invariant* derivatives. Note that up to spin corrections $\hat{\pi}^2 = m^2$ follows from the quadratic Dirac equation (2.6). We are thus led to define the gauge invariant Wigner operator by substituting the gauge invariant derivative D^μ and its adjoint in place of ∂^μ and its adjoint in the original gauge dependent definition (3.1). Applying this minimal substitution rule turns the familiar translation operators into gauge invariant ones:

$$\hat{W}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, \bar{\psi}(x) e^{\frac{1}{2}y\cdot D^\dagger(x)} \otimes e^{-\frac{1}{2}y\cdot D(x)} \psi(x)\ . \tag{3.8}$$

The symbol $\otimes$ denotes a tensor product in spinor space. Integrating over y gives formally

$$\hat{W}(x,p) = \bar{\psi}(x) \otimes\ \delta^4(p - \hat{\pi}(x))\ \psi(x)\ , \tag{3.9}$$

which shows explicitly that this Wigner operator measures the density of particles at x with kinetic momentum p. Remarkably, the definition (3.8) of the Wigner operator based on these physical arguements is completely equivalent to the definition (3.5) in terms of the phase factor (3.6). The path is determined uniquely to be a straight line with the help of the generalized Baker-Campbell-Haussdorff formula,

$$e^{-y\cdot D(x)}\psi(x) = U(A; x, x-y)\psi(x-y)\ \ , \tag{3.10}$$

that holds only if U is given by (3.6). This relation was proved in EGV (see also ref. [12]). We thus see that the path ambiguity is removed by requiring the proper physical interpretation of $W(x,p)$.

We note in passing that under hermitian conjugation the gauge invariant Wigner operator behaves like an ordinary γ-matrix:

$$\hat{W}^\dagger(x,p) = \gamma^0\, \hat{W}(x,p)\, \gamma^0\ . \tag{3.11}$$

This property insures that physical currents, such as $tr(\gamma_\mu W)$, are real.

4. Dynamics of the Wigner Operator

4.1 Exact Linear Quantum Equation

The equation of motion for the Wigner operator is determined by eq. (2.1) for the fermion Heisenberg field operators. To derive that equation we observe first that using the short-hand notation $x_1 \equiv x + \frac{1}{2}y$ and $x_2 \equiv x - \frac{1}{2}y$

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\,\Psi_{\alpha\beta}(x,y) = \quad & \left[\frac{\partial}{\partial x_1^\mu}\,\bar{\psi}_\beta(x_1)\right]\,PU(A;x_1,x_2)\,\psi_\alpha(x_2) \\ & + \bar{\psi}_\beta(x_1)\,PU(A;x_1,x_2)\,\left[\frac{\partial}{\partial x_2^\mu}\,\psi_\alpha(x_2)\right] \\ & - ie\,\bar{\psi}_\beta(x_1)\,P\left[y^\nu\int_0^1 ds\;\partial_\mu A_\nu(x-\tfrac{1}{2}y+sy)\;U(A;x_1,x_2)\right]\,\psi_\alpha(x_2)\,.\end{aligned} \tag{4.1}$$

The chain rule then allows us to re-write the path integral in the third term as follows:

$$\begin{aligned} & y^\nu\int_0^1 ds\;\frac{\partial}{\partial x^\mu}A_\nu(x-\tfrac{1}{2}y+sy) = y^\nu\int_0^1 ds\;\left[\partial_\nu A_\mu + F_{\mu\nu}\right] \\ & = \int_0^1 ds\;\frac{d}{ds}\,A_\mu(x-\tfrac{1}{2}y+sy) + y^\nu\int_0^1 ds\;F_{\mu\nu}(x-\tfrac{1}{2}y+sy) \\ & = A_\mu(x+\tfrac{1}{2}y) - A_\mu(x-\tfrac{1}{2}y) + y^\nu\int_0^1 ds\;F_{\mu\nu}(x-\tfrac{1}{2}y+sy)\,.\end{aligned} \tag{4.2}$$

Because the gauge fields commute at equal times, the above relation applies within the path ordered product. Therefore, we find that

$$\begin{aligned} i\frac{\partial}{\partial x^\mu}\,\Psi_{\alpha\beta}(x,y) = \quad & \bar{\psi}_\beta(x_1)\,iD_\mu^\dagger(x_1)\,PU(A;x_1,x_2)\,\psi_\alpha(x_2) \\ & + \bar{\psi}_\beta(x_1)\,PU(A;x_1,x_2)\,iD_\mu(x_2)\,\psi_\alpha(x_2) \\ & + e\,\bar{\psi}_\beta(x_1)\,P\left[y^\nu\int_0^1 ds\;F_{\mu\nu}(x-\tfrac{1}{2}y+sy)\;U(A;x_1,x_2)\right]\,\psi_\alpha(x_2)\,.\end{aligned} \tag{4.3}$$

In a similar way we find that

$$\begin{aligned} i\frac{\partial}{\partial y^\mu}\Psi_{\alpha\beta}(x,y) \;=\; & \tfrac{1}{2}\,\bar{\psi}_\beta(x_1)\,iD_\mu^\dagger(x_1)\,PU(A;x_1,x_2)\,\psi_\alpha(x_2) \\ & - \tfrac{1}{2}\,\bar{\psi}_\beta(x_1)\,PU(A;x_1,x_2)\,iD_\mu(x_2)\,\psi_\alpha(x_2) \\ & - e\,\bar{\psi}_\beta(x_1)\,P\left[y^\nu\int_0^1 ds\,(\tfrac{1}{2}-s)\,F_{\mu\nu}(x-\tfrac{1}{2}y+sy)\;U(A;x_1,x_2)\right]\,\psi_\alpha(x_2)\,.\end{aligned} \tag{4.4}$$

Taking next the Fourier transform of $i\gamma\cdot(\frac{1}{2}\partial_x - \partial_y)\Psi(x,y)$ and using the Dirac equation (2.1) leads finally to the exact linear quantum equation of motion

$$\begin{aligned} & \left[m - \gamma\cdot(p+\tfrac{1}{2}i\partial)\right]\;\hat{W}(x,p) = ie\,\frac{\partial}{\partial p_\mu}\int\frac{d^4y}{(2\pi)^4}e^{-ip\cdot y}\;\bar{\psi}(x+\tfrac{1}{2}y) \\ & \otimes P\left[\int_0^1 ds\,(1-s)\,F_{\mu\nu}(x-\tfrac{1}{2}y+sy)\;U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\right]\;\gamma^\nu\,\psi(x-\tfrac{1}{2}y)\end{aligned} \tag{4.5}$$

where $i\partial/\partial p_\mu$ arose by pulling y^μ out of the integral. This equation for the gauge invariant fermion Wigner operator $\hat{W}(x,p)$ is completely equivalent to Heisenberg's equation (2.1) for the spinor field operator. This equation is clearly invariant under local gauge transformations. On the other hand, we see that the choice of the path enters this equation because derivatives of the phase factor depend on it. Had we chosen another path $z^\mu(s;x_1,x_2)$, the factor $y_\alpha g^{\alpha\mu}(1-s)$ would, for example, be replaced by

$$\frac{dz_\alpha}{ds}\left(\frac{1}{2}\frac{\partial z^\mu}{\partial x_\alpha}-\frac{\partial z^\mu}{\partial y_\alpha}\right) .$$

Because of the conjugation property (3.11) of $\hat{W}(x,p)$ the equation which is equivalent to the adjoint Dirac equation can be obtained from eq. (4.5) by hermitian conjugation. Hence the Wigner operator contains the same amount of information as the field operators. In this sense we might call it "complete".

Notice that the ensemble average of the stress tensor for *interacting* fermions [13], which we obtain from eq. (3.4) by replacing the usual derivatives ∂ and $\partial^\dagger$ by their gauge invariant analogues D and $D^\dagger$, reads

$$\langle T_{\mu\nu}(x)\rangle \equiv -\tfrac{1}{2}i\,\langle :\bar{\psi}(x)\left[\gamma_\mu D^\dagger_\nu-\gamma_\mu D_\nu\right]\psi(x):\rangle = tr\int d^4p\; p_\nu\gamma_\mu W(x,p) ,$$

i.e., it retains its form. This is in consequence of eq. (4.4) and the fact that the last term there corresponds to a total derivative with respect to the kinetic momentum p^μ and therefore does not contribute to the integral.

The equation of motion for the Wigner *function*, $W(x,p)$, is obtained by taking the ensemble average of eq. (4.5). The averaged equation involves in general a two-body term due to the operator property of the field tensor $F^{\mu\nu}$ which, for a fully interacting system, is a linear functional of the fermion current operator. Since the two-body function satisfies an equation of motion depending on three-body functions etc., this generates the so-called BBGKY-hierarchy [10]. In principle, this is an infinite hierarchy of equations similar to the hierarchy of Dyson equations [13], and we shall encounter all the problems familiar in field theory like the appearance of infinities which require their regularization, the necessity of renormalization, etc. (for the nucleon-pion system this is discussed in ref. [17]).

4.2 Homogeneous External Fields

The physical content of the equation of motion (4.5) can be seen most clearly by considering the special case of slowly varying external (c-number) fields, which can be expanded as

$$F^{\mu\nu}_{ext}(x+x') = F^{\mu\nu}_{ext}(x) + x'\cdot\partial\, F^{\mu\nu}_{ext}(x) + \cdots . \tag{4.6}$$

We call a c-number field *homogeneous* if the derivative term in (4.6) can be neglected. We remark that for homogeneous fields, we can choose a gauge in which $A^\nu(x) = \frac{1}{2}x_\mu F^{\mu\nu}_{ext}$. In that gauge, the associated Wigner function can be shown to coincide with the *naive*, gauge dependent definition (3.1) except that the momentum p in the exponential of the Fourier transformation is replaced by $p+eA(x)$. Obviously this form of the Wigner function, which has been employed for example in ref. [18] for a constant magnetic field, is gauge dependent!

For a homogeneous external field, $F_{\mu\nu}$ can be pulled out of the y integral in (4.5), and the gauge invariant equation of motion reduces to

$$(\gamma\cdot K - m)\, W^E(x,p) = 0\ . \tag{4.7}$$

The superscript E on W indicates that the gauge field is a constant external c-number field. In this case the BBGKY-hierarchy truncates at the one body level. The operator K is defined by

$$K^\mu \equiv p^\mu + \tfrac{1}{2}i(\partial_x^\mu - eF_{ext}^{\mu\nu}\,\partial_\nu^p) = p^\mu + \tfrac{1}{2}i\nabla^\mu \tag{4.8}$$

with

$$\nabla^\mu \equiv \partial_x^\mu - eF_{ext}^{\mu\nu}\,\partial_\nu^p\ ,$$

where ∂_x (∂^p) is the derivative with respect to x (p). This is a coupled first-order equation for sixteen components of $W_{\alpha\beta}^E(x,p)$ which, as a matrix, is an element of the Clifford algebra.

To illustrate its relation to classical transport theory we convert it into a "quadratic" equation. By quadratic we mean that it involves combinations of p^μ and ∂_x^μ which we have seen above to generate second derivatives of the field operators. To this end we simply multiply eq. (4.7) by $(\gamma\cdot K + m)$ and obtain

$$\left[\, p^2 - m^2 - \tfrac{1}{4}(\partial_x - eF_{ext}\cdot\partial^p)^2 + i(p\cdot\partial_x - ep\cdot F_{ext}\cdot\partial^p) - \tfrac{1}{2}e\sigma_{\mu\nu}F_{ext}^{\mu\nu}\right]\, W^E(x,p) = 0\ .$$

The operator acting on the Wigner function is complex. As was already mentioned in EGV we can isolate the hermitian or anti-hermitian part of the this equation by adding or subtracting its adjoint. This leads, by virtue of eq. (3.11), to the following *two* equations:

$$(p\cdot\partial_x - ep\cdot F_{ext}\cdot\partial^p\ )\ W^E(x,p) = -\tfrac{1}{4}ieF_{ext}^{\mu\nu}\,[\sigma_{\mu\nu},\, W^E(x,p)]\ , \tag{4.9}$$

$$\left[p^2 - m^2 - \tfrac{1}{4}(\partial_x - eF_{ext}\cdot\partial^p)^2\right]\ W^E(x,p) = \tfrac{1}{4}eF_{ext}^{\mu\nu}\,\{\sigma_{\mu\nu},\, W^E(x,p)\}\ . \tag{4.10}$$

The left-hand side of the first equation is recognized to be the relativistic Vlasov equation known from classical kinetic theory. It describes the flow of particles influenced by the external Lorentz force. The *convective derivative* $p\cdot\partial_x$ on its left-hand side gives the total time rate of change of a quantity moving with the momentum p, and $eF_{ext}^{\mu\nu}p_\nu$ is the covariant Lorentz force. Its right-hand side displays corrections due to spin. The second equation generalizes the classical mass-shell condition and may be regarded as a constraint equation. In classical physics, the condition $p^2 = m^2$ is assumed implicitly. However, the uncertainty principle modifies the dispersion relation by an amount $\Delta m^2 = \frac{1}{4}\hbar^2(\partial_x - eF_{ext}\partial_p)^2$ plus spin corrections, where we have reinstated $\hbar$ to display the quantum origin of the term and where ∂_x and ∂_p can be replaced by the inverse characteristic length and momentum scale of variations of $W(x,p)$. Thus strongly varying Wigner functions necessarily lead to off mass-shell effects. Note that even for $F = 0$, off mass-shell effects become important due to the uncertainty principle if the plasma distribution varies rapidly on the scale of the Compton wavelength.

4.3 Arbitrary c-Number Fields

For the case of arbitrary external or self-consistent mean fields, eq. (4.7) can be easily generalized since the BBGKY-hierarchy still truncates at the one body level. In this case we approximate the quantum gauge field by the self-consistent mean field

$$\bar{F}^{\mu\nu}(x) \equiv \langle F^{\mu\nu}(x)\rangle \ . \tag{4.11}$$

This c-number field is calculable from the ensemble average of Maxwell's equations, (2.4), via

$$\partial_\mu \bar{F}^{\mu\nu}(x) = \langle j^\mu(x)\rangle = e\, tr \int d^4p\, \gamma^\mu W^H(x,p) \ . \tag{4.12}$$

This is referred to as the Hartree approximation (hence the superscript H). Of course, an arbitrary external current could be added to the r.h.s. of (4.12). Alternatively, choosing the gauge such that $\langle \partial\cdot A\rangle = 0$, we might employ the mean electromagnetic potential $\bar{A}^\mu \equiv \langle A^\mu\rangle$ and solve the Poisson equation

$$\Box \bar{A}^\mu(x) = \langle j^\mu(x)\rangle = e\, tr \int d^4p\, \gamma^\mu W^H(x,p)$$

with $\Box = \partial_\mu\partial^\mu$. The general solution of this equation can be expressed in terms of Green's functions which obey the equation

$$\Box D^{\mu\nu}(x,y) = g^{\mu\nu}\, \delta^4(x-y) \ ,$$

and are determined solely by the boundary conditions. If we choose the retarded Green's function $D^{\mu\nu}_{ret}(x,y) \sim \theta(x^0 - y^0)$, which vanishes in the backward light cone in consequence of causality, the potential is given by

$$\bar{A}^\mu(x) = \bar{A}^\mu_{in}(x) + \int d^4y\, D^{\mu\nu}_{ret}(x,y)\langle j_\nu(y)\rangle \ . \tag{4.13}$$

The incoming part of the mean field, $\bar{A}^\mu_{in} \equiv \langle A^\mu_{in}\rangle$, which is a solution of the homogeneous Poisson equation, vanishes in isotropic media where no direction is specified, e.g. in the vacuum or in equilibrated ensembles. It might, however, be important in practical applications, for example in situations where the plasma is heated by a laser beam. It should be noticed, incidentally, that the above expression contains radiation arising from the dynamics of the fermion current (coherent "bremsstrahlung"), cf. for example ref. [13]. The mean field strength tensor is then given by

$$\bar{F}^{\mu\nu}(x) = \bar{F}^{\mu\nu}_{in}(x) + \int d^4y D^{[\sigma,\mu]\nu}_{ret}(x,y)\langle j_\nu(y)\rangle$$

with

$$D^{[\sigma,\mu]\nu}_{ret}(x,y) \equiv \partial^\sigma D^{\mu\nu}_{ret}(x,y) - \partial^\mu D^{\sigma\nu}_{ret}(x,y) \ .$$

To generalize eq. (4.7) for arbitrarily varying c-number fields, we follow the method introduced in EGV. First, we note that we can write

$$\bar{F}^{\mu\nu}(x - \tfrac{1}{2}y + sy) = e^{(s-\frac{1}{2})\, y\cdot\partial_x}\, \bar{F}^{\mu\nu}(x) \ . \tag{4.14}$$

Next, we observe that formally

$$\int d^4y\, e^{-ip\cdot y} f(y)\, g(y) = f(i\partial_p) \int d^4y\, e^{-ip\cdot y} g(y) \ .$$

Therefore, we can pull $\bar{F}^{\mu\nu}$ out of the d^4y integral in eq. (4.5) replacing y by $i\partial^p$ and using eq. (4.14). The integral over s is then elementary giving rise to the following linear quantum equation of motion:

$$\left[m - \gamma\cdot(p + \tfrac{1}{2}i\hbar\partial_x)\right] W^H(x,p) = -\tfrac{1}{2}\hbar e \left[ij_0(\tfrac{1}{2}\hbar\triangle) + j_1(\tfrac{1}{2}\hbar\triangle)\right] \bar{F}^{\mu\nu}(x)\partial^p_\nu \gamma_\mu W^H(x,p) \ , \tag{4.15}$$

where $j_l(z)$ are conventional spherical Bessel funtions and the *triangle operator*, $\triangle$, denotes the mixed derivative

$$\triangle \equiv \partial^p\cdot\partial_x \ . \tag{4.16}$$

We emphasize that the derivative with respect to x in $\triangle$ acts only on the field strength tensor and not on the Wigner function. Of course, the derivative with respect to momentum acts here only on the Wigner function. We have explicitly reinstated $\hbar$ into eq. (4.15) to show the quantum character of that equation. Notice that the triangle operator has the dimension of inverse action. Therefore, a power series expansion of the Bessel functions coincides with an expansion in terms of the ratio of $\hbar$ to a characteristic angular momentum, L, of the plasma. That characteristic angular momentum measures the product of the spatial scale, ΔR_F, over which the field tensor, $\bar{F}^{\mu\nu}(x)$, varies appreciably and the momentum scale, ΔP_W, over which the Wigner function varies appreciably. Therefore, a necessary condition for the validity of a power series expansion of the Bessel functions is that

$$\hbar \ll \Delta R_F \Delta P_W \ . \tag{4.17}$$

Note that even for general mean fields the linear equation of motion (4.15) for the Wigner function may be cast into a form similar to (4.7), i.e.,

$$(\gamma\cdot K - m)\, W^H(x,p) = 0 \ , \tag{4.18}$$

in terms of the operator

$$K^\mu \equiv \Pi^\mu + \tfrac{1}{2}i\hbar\nabla^\mu \ , \tag{4.19}$$

with

$$\nabla^\mu \equiv \partial^\mu_x - ej_0(\tfrac{1}{2}\hbar\triangle)\, \bar{F}^{\mu\nu}\, \partial^p_\nu \ , \tag{4.20}$$

$$\Pi^\mu \equiv p^\mu - \tfrac{1}{2}\hbar e j_1(\tfrac{1}{2}\hbar\triangle)\, \bar{F}^{\mu\nu}\, \partial^p_\nu \ . \tag{4.21}$$

To zeroth order in $\triangle$ we recover eq. (4.8). To second order these operators are given by

$$\begin{aligned} \nabla^\mu &= \partial^\mu_x - e\bar{F}^{\mu\nu}\partial^p_\nu + \tfrac{1}{24}e\hbar^2\triangle^2\bar{F}^{\mu\nu}\partial^p_\nu + O(\hbar^4) \ , \\ \Pi^\mu &= p^\mu - \tfrac{1}{12}e\hbar^2\,\triangle\bar{F}^{\mu\nu}\partial^p_\nu + O(\hbar^4) \ . \end{aligned} \tag{4.22}$$

The generalized constraint and transport equations can be extracted as in the homogeneous field example by first multiplying (4.18) by $(\gamma \cdot K + m)$. Noting that $\gamma^\mu\gamma^\nu = g^{\mu\nu} - i\sigma^{\mu\nu}$, the following quadratic quantum equation of motion follows:

$$(K^2 - m^2 - \tfrac{1}{2}i\sigma^{\mu\nu}[K_\mu, K_\nu])\, W^H(x,p) = 0 \ . \tag{4.23}$$

Adding and subtracting the adjoint of this equation then leads to

$$(\Pi^2 - m^2 - \tfrac{1}{4}\hbar^2\nabla^2)\, W^H(x,p) = -\tfrac{1}{4}\hbar I^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} + \tfrac{1}{4}i\hbar R^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] \ , \tag{4.24}$$

$$\Pi^\mu \nabla_\mu\, W^H(x,p) = \tfrac{1}{4}iI^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] + \tfrac{1}{4}R^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} \ , \tag{4.25}$$

where the l.h.s. of the quantum transport equation (4.25) acquires the remarkably simple form on account of the relation

$$[\nabla_\mu, \Pi^\mu] = 0 \ , \tag{4.26}$$

and where the real operators R and I on the r.h.s. are given by

$$\begin{aligned} R^{\mu\nu} + iI^{\mu\nu} &\equiv \hbar^{-1}[K^\mu, K^\nu] \\ &= -\tfrac{1}{2}\hbar\triangle j_0(\tfrac{1}{2}\hbar\triangle)\, e\bar{F}^{\mu\nu} - i[j_0(\tfrac{1}{2}\hbar\triangle) - \tfrac{1}{2}\hbar\triangle j_1(\tfrac{1}{2}\hbar\triangle)]\, e\bar{F}^{\mu\nu} \\ &= -ie\bar{F}^{\mu\nu} - \tfrac{1}{2}\hbar e\triangle\bar{F}^{\mu\nu} + O(\hbar^2 e\triangle^2\bar{F}^{\mu\nu}) \ . \end{aligned} \tag{4.27}$$

The generalized quantum constraint and quantum Vlasov equations, (4.24) and (4.25), specify the dynamics of abelian plasmas in the "collisionless" regime. Quantum corrections to any order in $\hbar$ may be computed systematically by expanding the Bessel functions in powers of the triangle operator. Together with eq. (4.12) they form a coupled set of equations to determine simultaneously the fermion Wigner function and the field $\bar{F}^{\mu\nu}(x)$ in a self-consistent way. They form a closed set of equations because the expectation value of the current is directly related to the Wigner function in the Hartree approximation. The "collision" terms neglected in this approximation follow from the operator equation (4.5) only when correlations such as $\langle F^{\mu\nu}\hat{W}\rangle - \bar{F}^{\mu\nu}W$ are not neglected, i.e., only if the operator character of the gauge field is explicitly taken into account. Using the methods outlined in ref. [10] for non-gauge theories, it should be possible to extract such collisions terms. However, for gauge theories with long range forces, such as QED, the infrared divergences arising in perturbation theory need special care. As in classical plasmas, Debye screening has to be taken into account, and we expect collision terms of the Balescu-Lenard form [8]. However, the explicit construction of such terms is beyond the scope of this paper.

To second order in $\hbar$, the quantum Vlasov equation including spin corrections is thus given by

$$\begin{aligned} &(p\cdot\partial_x - ep_\mu\bar{F}^{\mu\nu}\partial^p_\nu)\, W^H(x,p) + \tfrac{1}{4}ie\bar{F}^{\mu\nu}\left[\sigma_{\mu\nu}, W^H(x,p)\right] \\ &= -\tfrac{1}{12}\hbar^2 e\triangle\bar{F}_{\mu\nu}[\partial^\nu_x - e\bar{F}^{\nu\lambda}\partial^p_\lambda]\partial^\mu_p\, W^H(x,p) - \tfrac{1}{8}\hbar e\triangle\bar{F}^{\mu\nu}\left\{\sigma_{\mu\nu}, W^H(x,p)\right\} \ . \end{aligned} \tag{4.28}$$

It coincides with eq. (4.9) in the special case $\partial^\sigma \bar{F}^{\mu\nu} = 0$. In EGV eq. (4.28) was also derived directly via the triangle operator expansion method. Equation (4.28) reduces to Remler's result [6] for scalar QED when the spin terms are neglected.

It is important to note that the above quantum Vlasov equation is not complete without the quantum constraint equation (4.24). As soon as quantum corrections become important in the transport equation, quantum corrections to the constraint equation must also be considered. This important point was not discussed in ref. [6]. To second order in $\hbar$, the constraint equation in the Hartree approximation is given by

$$\begin{aligned}(p^2 - m^2)\, W^H(x,p) \;&=\; \tfrac{1}{4}\hbar e \bar{F}^{\mu\nu}\{\sigma_{\mu\nu}, W^H(x,p)\} - \tfrac{1}{8} i\hbar^2 e \triangle \bar{F}^{\mu\nu}[\sigma_{\mu\nu}, W^H(x,p)] \\ &\quad + \hbar^2(\tfrac{1}{6} e p \cdot \triangle \bar{F} \cdot \partial_p + \tfrac{1}{12} e(\partial_x^\mu \bar{F}_{\mu\nu})\partial_\nu^p + \tfrac{1}{4}(\partial_x - e\bar{F}\cdot\partial_p)^2) W^H(x,p) \ . \end{aligned} \tag{4.29}$$

From eqs. (4.28) and (4.29) we see that quantum transport theory reduces to classical transport theory only if several conditions are satisfied simultaneously. In addition to the condition (4.17) necessary for the validity of the triangle operator expansion, the field strength must be small compared to the typical energy scale of particles in the plasma, i.e.,

$$\hbar e \|\bar{F}^{\mu\nu}\| \ll m^2 + |\vec{p}|^2 \ . \tag{4.30}$$

Thus the field strength has to be relatively weak and slowly varying for classical theory to hold. In addition, spatial and momentum gradients of the Wigner function need to be sufficiently small such that none of the order $\hbar^2$ terms on the r.h.s. of (4.29) are large compared to the typical energy scale on the l.h.s.

In any specific application, where the solution, W_{cl}, of the classical ($\hbar = 0$) transport equation is known, the magnitude of quantum corrections can be estimated by substituting W_{cl} for W^H on the right hand side of the above equations. More specifically, writing $W^H = W_{cl} + \delta W$, the quantum correction, δW, obeys (4.28) and (4.29) with W^H replaced by δW on the l.h.s and W^H replaced by W_{cl} on the r.h.s. of those equations.

4.4 General Quantum Fields

For general quantum gauge fields, it is still possible to employ a generalized triangle operator expansion. In order to take the operator ordering into account we have to generalize the derivative ∂_x in the triangle operator $\triangle$ to include also a commutator term. We define a new derivative $\mathcal{D}(x)$ such that its action on the field operator $F^{\mu\nu}(x)$ is given by

$$\mathcal{D}(x)\, F^{\mu\nu}(x) \equiv \partial_x F^{\mu\nu}(x) + ie\,[A(x), F^{\mu\nu}(x)] \ . \tag{4.31}$$

This form bears resemblance to gauge covariant derivatives familiar from non-abelian gauge theories. However, whereas for non-abelian fields the commutator is also of algebraic origin and does not vanish in the limit of c-number fields, it vanishes in the abelian limit when the quantum nature of the vector field is neglected.

Now, in consequence of definition (4.31) the "translation"

$$e^{-z\cdot D(x)} F^{\mu\nu}(x) = P[U(A;x,x-z)\, F^{\mu\nu}(x-z)\, U(A;x-z,x)]\ ,$$

when employed on the right-hand side of eq. (4.5) with $z = (\frac{1}{2} - s)\, y$, automatically moves the gauge field operator to the right place in the path-ordered product (for a proof and further details see EGV). With this trick the path integral may be formally evaluated even in the fully general case of coupled quantum fields. We obtain again, in analogy to eq. (4.15), a sum of two spherical Bessel functions, since the integral under consideration is similar to that encountered above in the Hartree approximation. However, the triangle operator in their argument is now defined with the derivative ∂_x replaced by $D(x)$, the latter acting again on the quantum field $F^{\mu\nu}(x)$ only. Moreover, the Bessel functions have to be kept between the fermion operators. In the Hartree limit the arising equation consequently reduces to eq. (4.15).

In the previous section we have extracted the transport equation from the linear equation of motion (4.7) by squaring the operator acting on $W(x,p)$ and isolating the anti-hermitian part. This method is straightforward for the case of c-number fields but is much less transparent for the general quantum field case. In order to collect all quantum mechanical corrections to the Vlasov equation we shall therefore derive the quantum transport equation directly rather than employ the exact linear equation (4.5). This can be acomplished by calculating the action of the convective derivative $p \cdot \partial_x$ on the Wigner operator. By similar reasoning as was applied when deriving the linear equation (4.5) this automatically generates the anti-hermitian combination of the quadratic form of Dirac's equation and its adjoint for the fermion field operators, cf. eqs. (2.6), plus additional terms involving the field strength tensor operator $F^{\mu\nu}$. The quantum transport equation in natural units is then found to be

$$\begin{aligned}
p\cdot\partial_x\, \hat{W}_{\alpha\beta}(x,p) = \\
&\tfrac{1}{4} ie \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \Big\{ \big[\bar{\psi}(x+\tfrac{1}{2}y)\,\sigma^{\mu\nu}\big]_\beta F_{\mu\nu}(x+\tfrac{1}{2}y)\, PU(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\,\psi_\alpha(x-\tfrac{1}{2}y) \\
&\quad - \bar{\psi}_\beta(x+\tfrac{1}{2}y)\, PU(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\, F_{\mu\nu}(x-\tfrac{1}{2}y)\, \big[\sigma^{\mu\nu}\psi(x-\tfrac{1}{2}y)\big]_\alpha \Big\} \\
&+ ie\frac{\partial}{\partial p^\mu} \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, \bar{\psi}_\beta(x+\tfrac{1}{2}y) \\
&\quad \times \Big\{ P\Big[\int_0^1 ds\,(s-\tfrac{1}{2})\, F^{\mu\nu}(x-\tfrac{1}{2}y+sy)\, U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\Big]\ D_\nu \\
&\quad + D_\nu^\dagger P\Big[\int_0^1 ds\,(s-\tfrac{1}{2})\, F^{\mu\nu}(x-\tfrac{1}{2}y+sy)\, U(A;x+\tfrac{1}{2}y,x-\tfrac{1}{2}y)\Big] \Big\}\ \psi_\alpha(x-\tfrac{1}{2}y) \\
&- e\,p_\nu \frac{\partial}{\partial p^\mu} \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}
\end{aligned}$$

$$\times \bar{\psi}_\beta(x+\tfrac{1}{2}y)\, P \left[\int_0^1 ds\, F^{\mu\nu}(x-\tfrac{1}{2}y+sy)\, U(A; x+\tfrac{1}{2}y, x-\tfrac{1}{2}y) \right] \psi_\alpha(x-\tfrac{1}{2}y) \; . \tag{4.32}$$

To connect this equation with the abelian limit of the general quantum transport equation derived EGV, the derivatives D_ν and $D_\nu^\dagger$ must still be extracted from the middle term on the right hand side. However, the resulting equation is not significantly simpler than for the more general non-abelian case as a result of the necessity of keeping track of the operator ordering. The main simplicity arises only in the limit of c-number fields considered in the last section. We display (4.32) mainly to record the starting point from which future derivations of collision terms could begin using methods as discussed in ref. [10]. We note again that the general quantum constraint equation has to be considered at the same time. It can be derived along similar lines by computing $(p^2 - m^2)\, \hat{W}$.

5. Spinor Decomposition

5.1 Arbitrary c-Number Fields

The equations of motion derived in the previous sections apply to the Wigner function $W(x,p)$ viewed as a four by four matrix in spinor indices. In this section we decompose the spinor structure of that equation and derive a set of coupled equations for the components of W. We consider here only the case of arbitrary c-number fields in the Hartree approximation.

The Wigner function matrix can always be expanded in terms of the sixteen independent generators of the Clifford algebra. We choose the conventional basis [13]

$$\Gamma_i = 1,\ \gamma^\mu,\ i\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3,\ \gamma^\mu\gamma^5,\ \sigma^{\mu\nu}\ .$$

Under hermitian conjugation $\Gamma_i^\dagger = \gamma^0\Gamma_i\gamma^0$. In this basis the expansion of the Wigner function (see also ref. [4]) is given by

$$W = \mathcal{F} + i\gamma^5\mathcal{P} + \gamma^\mu\mathcal{V}_\mu + \gamma^\mu\gamma^5\mathcal{A}_\mu + \tfrac{1}{2}\sigma^{\mu\nu}\mathcal{S}_{\mu\nu}\ , \tag{5.1}$$

where the superscript H has been suppressed. The sixteen components of W,

$$\mathcal{F}(x,p) \equiv \tfrac{1}{4}\, tr\, W(x,p)\ , \tag{5.2}$$

$$\mathcal{P}(x,p) \equiv -\tfrac{1}{4} i\, tr\, \gamma^5 W(x,p)\ , \tag{5.3}$$

$$\mathcal{V}_\mu(x,p) \equiv \tfrac{1}{4}\, tr\, \gamma_\mu W(x,p)\ , \tag{5.4}$$

$$\mathcal{A}_\mu(x,p) \equiv \tfrac{1}{4}\, tr\, \gamma_5\gamma_\mu W(x,p)\ , \tag{5.5}$$

$$\mathcal{S}_{\mu\nu}(x,p) \equiv \tfrac{1}{4}\, tr\, \sigma_{\mu\nu} W(x,p) = -\mathcal{S}_{\nu\mu}(x,p)\ , \tag{5.6}$$

are *real* due to property (3.11) of the Wigner function, and behave under Lorentz transformation like a scalar, pseudoscalar, vector, axial vector and an antisymmetric tensor, respectively. Next, using trace properties of the Γ-matrices listed in ref. [13], we decompose the first-order eq. (4.18) in this basis. The result is the following set of coupled equations:

$$K^\mu\mathcal{V}_\mu - m\mathcal{F} = 0\ , \tag{5.7}$$

$$iK^\mu\mathcal{A}_\mu + m\mathcal{P} = 0\ , \tag{5.8}$$

$$K_\mu\mathcal{F} - iK^\nu\mathcal{S}_{\mu\nu} - m\mathcal{V}_\mu = 0\ , \tag{5.9}$$

$$iK_\mu\mathcal{P} + \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho}K^\nu\mathcal{S}^{\sigma\rho} - m\mathcal{A}_\mu = 0\ , \tag{5.10}$$

$$i(K_\mu\mathcal{V}_\nu - K_\nu\mathcal{V}_\mu) - \epsilon_{\mu\nu\sigma\rho}K^\sigma\mathcal{A}^\rho + m\mathcal{S}_{\mu\nu} = 0\ , \tag{5.11}$$

where K^μ is given by (4.19) in terms of the operators ∇^μ and Π^μ defined in (4.20) and (4.21), respectively. Since the moments of the Wigner function are real but the operator K^μ is complex, the real and imaginary parts of the above set of equations have to be satisfied separately. The real parts of the above equations are given by

$$\Pi^\mu \mathcal{V}_\mu = m\mathcal{F} \ , \tag{5.12}$$

$$\hbar \nabla^\mu \mathcal{A}_\mu = 2m\mathcal{P} \ , \tag{5.13}$$

$$\Pi_\mu \mathcal{F} + \hbar \tfrac{1}{2} \nabla^\nu \mathcal{S}_{\mu\nu} = m\mathcal{V}_\mu \ , \tag{5.14}$$

$$-\hbar \nabla_\mu \mathcal{P} + \epsilon_{\mu\nu\sigma\rho} \Pi^\nu \mathcal{S}^{\sigma\rho} = 2m\mathcal{A}_\mu \ , \tag{5.15}$$

$$\tfrac{1}{2}\hbar \left(\nabla_\mu \mathcal{V}_\nu - \nabla_\nu \mathcal{V}_\mu\right) + \epsilon_{\mu\nu\sigma\rho} \Pi^\sigma \mathcal{A}^\rho = m\mathcal{S}_{\mu\nu} \ , \tag{5.16}$$

whereas the imaginary parts lead to

$$\hbar \nabla^\mu \mathcal{V}_\mu = 0 \ , \tag{5.17}$$

$$\Pi^\mu \mathcal{A}_\mu = 0 \ , \tag{5.18}$$

$$\tfrac{1}{2}\hbar \nabla_\mu \mathcal{F} = \Pi^\nu \mathcal{S}_{\mu\nu} \ , \tag{5.19}$$

$$\Pi_\mu \mathcal{P} = -\tfrac{1}{4}\hbar \epsilon_{\mu\nu\sigma\rho} \nabla^\nu \mathcal{S}^{\sigma\rho} \ , \tag{5.20}$$

$$\Pi_\mu \mathcal{V}_\nu - \Pi_\nu \mathcal{V}_\mu = \tfrac{1}{2}\hbar \, \epsilon_{\mu\nu\sigma\rho} \nabla^\sigma \mathcal{A}^\rho \ . \tag{5.21}$$

This generalized set of equations reduces to those derived in ref. [4] for the special case of a constant external magnetic field. However, our equations apply for the more general case of arbitrarily varying c-number fields.

The linear equations can also be combined to form quadratic equations for the different components. Substituting (5.9) into (5.7) gives, for example,

$$(K^2 - m^2)\mathcal{F} = \tfrac{1}{2} i [K^\mu, K^\nu] \mathcal{S}_{\mu\nu} \ . \tag{5.22}$$

Similarly, the quadratic equation satisfied by $\mathcal{A}^\mu$ is obtained by substituting (5.8) and (5.11) into (5.10) with the result

$$(K^2 - m^2)\mathcal{A}_\mu + [K^\mu, K^\nu]\mathcal{A}_\nu = \tfrac{1}{2} i \epsilon_{\mu\nu\sigma\rho} [K^\nu, K^\sigma] \mathcal{V}^\rho \ . \tag{5.23}$$

The real and imaginary parts of (5.22) and (5.23) together with equations (5.8), (5.9), and (5.11) are of course equivalent to the 32 equations (5.7)-(5.11) for the 16 components (5.2)-(5.6). As we show below, this particular regrouping of the equations is useful because it leads to a simple physical interpretation in the classical limit.

To first order in $\hbar$, $\Pi^\mu = p^\mu$ and $\nabla^\mu = \partial_x^\mu - eF^{\mu\nu}\partial_\nu^p$ and

$$[K^\mu, K^\nu] = -i\hbar e F^{\mu\nu} \ , \tag{5.24}$$

$$K^2 = p^2 + i\hbar p \cdot \nabla \ . \tag{5.25}$$

To this order, the imaginary parts of (5.22) and (5.23) correspond to the following transport equations for $\mathcal{F}$ and $\mathcal{A}$:

$$p\cdot\nabla\mathcal{F} = 0 \ , \tag{5.26}$$

$$p\cdot\nabla\mathcal{A}^\mu = eF^{\mu\nu}\mathcal{A}_\nu \ . \tag{5.27}$$

The real parts form the constraint equations

$$(m^2 - p^2)\,\mathcal{F} = -\tfrac{1}{2}\hbar eF^{\mu\nu}\mathcal{S}_{\mu\nu} \ , \tag{5.28}$$

$$(m^2 - p^2)\,\mathcal{A}^\mu = -\hbar e\tilde{F}^{\mu\nu}\mathcal{V}_\nu \ , \tag{5.29}$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\sigma\rho}F^{\sigma\rho}$ is the dual field strength tensor. $\mathcal{A}_\mu$ also satisfies the subsidiary condition (5.18), which to first order in $\hbar$ reads

$$p^\mu\mathcal{A}_\mu = 0 \ . \tag{5.30}$$

The pseudo-scalar, vector and tensor densities are given by (5.13), (5.14), and (5.16) respectively:

$$\mathcal{P} = \frac{\hbar}{2m}\nabla_\mu\mathcal{A}^\mu \ , \tag{5.31}$$

$$\mathcal{V}_\mu = \frac{1}{m}(\Pi_\mu\mathcal{F} + \tfrac{1}{2}\hbar\nabla_\nu\mathcal{S}^{\mu\nu}) \ , \tag{5.32}$$

$$\mathcal{S}_{\mu\nu} = \frac{1}{m}\epsilon_{\mu\nu\sigma\rho}\Pi^\sigma\mathcal{A}^\rho + \frac{\hbar}{2m}(\nabla_\mu\mathcal{V}_\nu - \nabla_\nu\mathcal{V}_\mu) \ . \tag{5.33}$$

These relations obviously hold only for $m \neq 0$. Solving for $\mathcal{V}_\mu$ and $\mathcal{S}_{\mu\nu}$ in terms of $\mathcal{F}$ and $\mathcal{A}_\mu$ to first order in $\hbar$ gives finally

$$\mathcal{V}_\mu = \frac{p_\mu}{m}\mathcal{F} + \frac{\hbar}{2m^2}\epsilon_{\mu\nu\sigma\rho}\,\nabla^\nu p^\sigma\mathcal{A}^\rho \ , \tag{5.34}$$

$$\mathcal{S}_{\mu\nu} = \frac{1}{m}\epsilon_{\mu\nu\sigma\rho}p^\sigma\mathcal{A}^\rho + \frac{\hbar}{2m^2}(\nabla_\mu p_\nu - \nabla_\nu p_\mu)\mathcal{F} \ . \tag{5.35}$$

The relation (5.34) corresponds to Gordon's decomposition of the vector current into a convection and a spin part [13] (the connection between the pseudovector current and spin will become transparent below). The relation (5.35) displays the first order quantum correction to the classical connection [16,19] between the spin tensor and the axial vector current. Note that ∇^ν in (5.34) acts also on the p^σ, and that

$$[\nabla^\alpha, p^\beta] = -eF^{\alpha\beta} + O(\hbar^2) \ . \tag{5.36}$$

We have thus shown that to first order in $\hbar$ all sixteen components of the Wigner matrix can be specified in terms of the four independent quantities $\mathcal{F}$ and $\mathcal{A}_\mu$, where one component of $\mathcal{A}_\mu$ is eliminated by (5.30). In fact, *to any finite order* in $\hbar$, $\mathcal{P}$,

$\mathcal{V}_\mu$, and $\mathcal{S}_{\mu\nu}$ can be computed in terms of $\mathcal{F}$ and $\mathcal{A}_\mu$. Equation (5.13) fixes $\mathcal{P}$ to all orders in $\hbar$. Furthermore, (5.14) and (5.16) have the generic structure

$$\begin{aligned} \mathcal{V} &= O_1\mathcal{F} + \hbar O_2 \mathcal{S} \ , \\ \mathcal{S} &= O_3\mathcal{A} + \hbar O_4 \mathcal{V} \ , \end{aligned} \tag{5.37}$$

where O_i are known operators expandable in a power series in $\hbar^2$ using (4.20) and (4.21):

$$O_i = O_{i,0} + \hbar^2 O_{i,2} + \cdots \ . \tag{5.38}$$

The solution of (5.37) to any order in $\hbar$ can thus be obtained by iteration:

$$\begin{aligned} \mathcal{V} &= O_{1,0}\mathcal{F} + \hbar O_{2,0}O_{3,0}\mathcal{A} + \cdots \ , \\ \mathcal{S} &= O_{3,0}\mathcal{A} + \hbar O_{4,0}O_{1,0}\mathcal{F} + \cdots \ . \end{aligned} \tag{5.39}$$

The explicit form of the lowest order operators can be read off from eqs. (5.34) and (5.35). Therefore, the quadratic equations (5.22) and (5.23) together with the subsidiary condition (5.30) form a closed system of equations for $\mathcal{F}$ and $\mathcal{A}_\mu$ to any finite order in $\hbar$. We have thus reduced the problem of the spinor decomposition of the Wigner function matrix to computing four independent phase-space densities, $\mathcal{F}$ and $\mathcal{A}_\mu$.

5.2 Consistency of First Order Equations

We show next that the transport and constraint equations for $\mathcal{F}$, (5.26) and (5.28), and $\mathcal{A}_\mu$, (5.27) and (5.29), together with the subsidiary condition (5.30) insure that to first order in $\hbar$ all 32 coupled equations are satisfied with $\mathcal{V}_\mu$ and $\mathcal{S}_{\mu\nu}$ given by (5.34) and (5.35).

Consider first eq. (5.12), where to this order $\Pi^\mu = p^\mu$. Substituting (5.34) for $\mathcal{V}^\mu$ gives

$$p^\mu \mathcal{V}_\mu = \frac{p^2}{m}\mathcal{F} + \frac{\hbar}{2m^2}\epsilon_{\mu\nu\sigma\rho}p^\mu \nabla^\nu p^\sigma \mathcal{A}^\rho \ . \tag{5.40}$$

Using (5.36), $\nabla^\nu p^\sigma$ can be replaced by $-eF^{\nu\sigma}$ since by symmetry $\epsilon_{\mu\nu\sigma\rho}p^\mu p^\sigma = 0$. From (5.35) we see that $\hbar\epsilon_{\mu\nu\sigma\rho}p^\mu\mathcal{A}^\rho = \hbar m \mathcal{S}_{\nu\sigma}$ to first order, and thus

$$p^\mu \mathcal{V}_\mu = \frac{1}{m}(p^2\mathcal{F} - \tfrac{1}{2}\hbar e F^{\nu\sigma}\mathcal{S}_{\nu\sigma}) = m\mathcal{F} \ , \tag{5.41}$$

where the last equality follows from the constraint equation (5.28). This proves that (5.12) is satisfied.

Equation (5.13) is just the definition of $\mathcal{P}$ in terms of $\mathcal{A}$ to all orders in $\hbar$. Equations (5.14) and (5.16) were used to calculate $\mathcal{V}$ and $\mathcal{S}$ in terms of $\mathcal{F}$ and $\mathcal{A}$ in the first place. That leaves only (5.15) among the first set of equations to be checked.

Since $\hbar\nabla_\mu \mathcal{P} = O(\hbar^2)$ by (5.13), we need only to calculate to first order

$$\epsilon_{\mu\nu\sigma\rho}\Pi^\nu \mathcal{S}^{\sigma\rho} = \frac{1}{m}\epsilon_{\mu\nu\sigma\rho}p^\nu \epsilon^{\sigma\rho\alpha\beta}p_\alpha \mathcal{A}_\beta \ . \tag{5.42}$$

Using the contraction property $\epsilon_{\mu\nu\sigma\rho}\epsilon^{\sigma\rho\alpha\beta} = -2(g^\alpha_\mu g^\beta_\nu - g^\beta_\mu g^\alpha_\nu)$, we see that the r.h.s. of (5.42) reduces to

$$-\frac{2}{m}[p_\mu(p_\nu \mathcal{A}^\nu) - p^2 \mathcal{A}_\mu] = 2m\mathcal{A}_\mu \ ,$$

where $p_\mu \mathcal{A}^\mu = 0$ as a result of the contraint equation (5.30).

Moving on to (5.17), we compute

$$\hbar\nabla^\mu \mathcal{V}_\mu = \hbar\nabla^\mu(p_\mu \mathcal{F}) = \hbar p_\mu \nabla^\mu \mathcal{F} = 0 \ ,$$

as follows from (5.26). Equation (5.18) is of course the subsidiary condition on $\mathcal{A}$. To verify (5.19) we compute

$$\begin{aligned} p^\nu \mathcal{S}_{\mu\nu} &= \frac{\hbar}{2m^2}(p^\nu\nabla_\mu p_\nu - p^\nu\nabla_\nu p_\mu)\mathcal{F} \\ &= \frac{\hbar}{2m^2}(\nabla_\mu p^2 - p_\mu(p\cdot\nabla))\mathcal{F} \\ &= \frac{\hbar}{2}\nabla_\mu \mathcal{F} \ , \end{aligned}$$

where we used (5.26), (5.28), and (5.36).

To check (5.20), note that

$$\begin{aligned} \hbar\nabla^\nu\epsilon_{\mu\nu\sigma\rho}\mathcal{S}^{\sigma\rho} &= \frac{\hbar}{m}\nabla^\nu\epsilon_{\mu\nu\sigma\rho}\epsilon^{\sigma\rho\alpha\beta}p_\alpha \mathcal{A}_\beta \\ &= -\frac{2\hbar}{m}\nabla^\nu(p_\mu \mathcal{A}_\nu - p_\nu \mathcal{A}_\mu) \\ &= -4p_\mu \mathcal{P} + \frac{2\hbar}{m}(p\cdot\nabla \mathcal{A}_\mu - F_{\mu\nu}\mathcal{A}^\nu) \ . \end{aligned}$$

In the last line we used the definition (5.13). The desired relation (5.20) follows to order $\hbar$ using the transport equation (5.27).

The final equation (5.21) can be verified by contracting both sides by $\epsilon^{\alpha\beta\mu\nu}$. The convection part of $\mathcal{V}$ does not contribute by symmetry. The spin part gives

$$\begin{aligned} \epsilon^{\alpha\beta\mu\nu}(p_\mu \mathcal{V}_\nu - p_\nu \mathcal{V}_\mu) &= \frac{\hbar}{m^2}\epsilon^{\alpha\beta\mu\nu}\epsilon_{\nu\sigma\gamma\rho}p_\mu\nabla^\sigma p^\gamma \mathcal{A}^\rho \\ &= -\frac{\hbar}{m^2}p_\mu\{\nabla^\alpha(p^\beta \mathcal{A}^\mu - p^\mu \mathcal{A}^\beta) - (\alpha\leftrightarrow\beta) - (\alpha\leftrightarrow\mu)\} \ . \end{aligned} \tag{5.43}$$

To simplify (5.43), we can employ the following first order relations obtained from eqs. (5.27), (5.29), (5.30), and (5.36):

$$
\begin{aligned}
p_\mu \nabla^\alpha p^\mu \mathcal{A}^\beta &= m^2 \nabla^\alpha \mathcal{A}^\beta - p_\mu e F^{\mu\alpha} \mathcal{A}^\beta \ , \\
p_\mu \nabla^\alpha p^\beta \mathcal{A}^\mu &= -[\nabla^\alpha, p_\mu] p^\beta \mathcal{A}^\mu = p^\beta e F^{\alpha\mu} \mathcal{A}_\mu \ , \\
p_\mu \nabla^\mu p^\alpha \mathcal{A}^\beta &= p^\alpha p \cdot \nabla \mathcal{A}^\beta - p_\mu e F^{\mu\alpha} \mathcal{A}^\beta \\
&= p^\alpha e F^{\beta\mu} \mathcal{A}_\mu - p_\mu e F^{\mu\alpha} \mathcal{A}^\beta \ .
\end{aligned}
$$

With these relations the r.h.s. of (5.43) reduces to $\hbar(\nabla^\alpha \mathcal{A}^\beta - \nabla^\beta \mathcal{A}^\alpha)$. Finally, contracting with $\epsilon_{\mu\nu\alpha\beta}$ proves that (5.21) is satisfied.

This completes the demonstration of the consistency of the coupled spinor equations to first order in $\hbar$. Although we have not constructed a general proof of consistency of the iterative solution (5.39) to all orders, we naturally expect this to hold.

5.3 Physical Interpretation in the Classical Limit

The physical interpretation of $\mathcal{F}$ and $\mathcal{A}_\mu$ becomes especially clear in the classical limit corresponding to the $\hbar = 0$ of the above equations. In that limit both densities vanish unless $p^2 = m^2$ according to the constraint equations (5.28) and (5.29). Furthermore, $\mathcal{P} = 0$ and

$$
\begin{aligned}
\gamma^\mu \mathcal{V}_\mu &= \gamma^\mu p_\mu \mathcal{F}/m \ , \\
\tfrac{1}{2}\sigma^{\mu\nu} \mathcal{S}_{\mu\nu} &= \tfrac{1}{2}\sigma^{\mu\nu} \epsilon_{\mu\nu\alpha\beta} p^\alpha \mathcal{A}^\beta / m \\
&= -i\gamma_5 \sigma_{\alpha\beta} p^\alpha \mathcal{A}^\beta / m \\
&= \gamma_5 \gamma_\alpha \gamma_\beta p^\alpha \mathcal{A}^\beta / m \ ,
\end{aligned}
$$

where the last line follows using the subsidiary condition (5.30). The complete Wigner matrix (5.1) in this limit is thus given by

$$
W = \frac{1}{m}(m + \gamma \cdot p)(\mathcal{F} - \gamma_5 \gamma \cdot \mathcal{A}) \ . \tag{5.44}
$$

To rewrite W in terms of conventional spin projection operators, $P(s) = \frac{1}{2}(1 \pm \gamma_5 \gamma \cdot s)$, we note that the unit axial vector field, $s^\mu(x,p) = \mathcal{A}^\mu / (\mathcal{A} \cdot \mathcal{A})^{\frac{1}{2}}$, satisfies on account of (5.30) the constraint

$$
p_\mu s^\mu(x,p) = 0 \ . \tag{5.45}
$$

Therefore, s^μ must be space-like with

$$
s_\mu(x,p) s^\mu(x,p) = -1 \ . \tag{5.46}
$$

Recalling conventional Dirac theory, s_μ is therefore a natural candidate for the spin phase-space density. As we show below, this interpretation is in fact correct. In terms of this field,

$$\mathcal{A}_\mu = -s_\mu(s \cdot \mathcal{A}) \ . \tag{5.47}$$

From the transport equation for $\mathcal{A}_\mu$, we see that s_μ obeys

$$p \cdot \nabla s^\mu(x,p) = eF^{\mu\nu}(x) s_\nu(x,p) \ , \tag{5.48}$$

since $p \cdot \nabla \mathcal{A}^2 = 0$.

Using (5.47) we can express the Wigner matrix (5.44) in terms of spin projection operators by rewriting

$$\mathcal{F} - \gamma_5 \gamma \cdot \mathcal{A} = \tfrac{1}{2}(1 + \gamma_5 \gamma \cdot s)\mathcal{N}_s + \tfrac{1}{2}(1 - \gamma_5 \gamma \cdot s)\mathcal{N}_{-s} \ . \tag{5.49}$$

The spin "up" and "down" phase-space densities are thus given by

$$\mathcal{N}_{\pm s} = \mathcal{F} \pm s \cdot \mathcal{A} \ . \tag{5.50}$$

The transport equations obeyed by these densities follow from (5.26), (5.27), and (5.48):

$$p \cdot \nabla \mathcal{N}_s = \mathcal{A}_\mu (p \cdot \nabla s^\mu - eF^{\mu\nu} s_\nu) = 0 \ . \tag{5.51}$$

Recall that in the classical limit $\nabla^\mu = \partial_x^\mu - eF^{\mu\nu}\partial_\nu^p$. Hence, we find that these particular spin components decouple and obey the usual relativistic Vlasov equation.

To understand why this decoupling occurs, we show next that eq. (5.48) is just the phase-space formulation of the classical Bargmann-Michel-Telegdi (BMT) equation [21] for spin. Equation (5.51) specifies contours in phase space $(x(\tau), p(\tau))$ along which the density of spin "up" and "down" particles does not change. The parameter τ can be thought of as a label specifying where along that contour a test particle is located. The equations of motion for that test particle can be deduced directly from the Vlasov equation. Because the number of test particles along $(x(\tau), p(\tau))$ does not change,

$$\frac{d}{d\tau}\mathcal{N}_s(x(\tau), p(\tau)) = (\dot{x} \cdot \partial_x + \dot{p} \cdot \partial_p)\mathcal{N}_s(x(\tau), p(\tau)) = 0 \ . \tag{5.52}$$

If we follow those contours which can be parametrized such that

$$m\frac{d}{d\tau}x^\mu(\tau) = p^\mu(\tau) \ , \tag{5.53}$$

then since both (5.52) and (5.51) must hold, we see that the test particle must obey the classical equations of motion,

$$m\frac{d}{d\tau}p^\mu(\tau) = eF^{\mu\nu}p_\nu(\tau) \ . \tag{5.54}$$

In this case τ can be interpreted as the proper time of that particle along its world line $x(\tau)$.

Furthermore, along the contours specified by (5.53) and (5.54) we can interpret $s^\mu(\tau) = s^\mu(x(\tau), p(\tau))$ as the unit spin vector specifying the direction of the intrinsic magnetic moment of our test particle. The equation of motion of that spin vector following from (5.48) is given by

$$m\frac{d}{d\tau}s^\mu(\tau) = p(\tau)\cdot\nabla s^\mu(\tau) = eF^{\mu\nu}(\tau)s_\nu(\tau) \ , \tag{5.55}$$

Equation (5.55) is recognized as the BMT-equation for spinning particles with gyromagnetic ratio $g = 2$. What we have shown, therefore, is that the familiar relativistic classical mechanics of spin-$\frac{1}{2}$ particles emerges naturally from the $\hbar \to 0$ limit of the gauge invariant transport theory developed here.

We now also see explicitly why the transport equations decouple for the different spin components. As the particles move along the classical trajectories determined from the self-consistent mean field, their spin vectors wobble around as described by the BMT-equation. The number of particles with spin parallel or antiparallel to the varying $s^\mu(x,p)$ does not change in the local co-moving frame of those particles. Hence, s^μ is the optimal choice of the spin quantization direction.

Having fully resolved the spin structure of the Wigner function in the classical limit, we turn finally to the decomposition in terms of positive and negative energy components. The dispersion relation, $p^2 = m^2$, obviously has two branches with $p^0 > 0$ and $p^0 < 0$. As usual a negative energy spin "down" electron has the interpretation of a positive energy positron with spin "up". Therefore, we can decompose the spin phase-space densities as

$$\begin{aligned} \mathcal{N}_s(x,p) &= \delta(p^2 - m^2)\ \left[\theta(p^0)\ f_s^{(+)}(x,p) + \theta(-p^0)\ f_{-s}^{(-)}(x,-p)\right] \\ &= \int d^4q\ \theta(q^0)\delta(q^2 - m^2)\ \left[\delta^4(p-q)\ f_s^{(+)}(x,q) + \delta^4(p+q)\ f_{-s}^{(-)}(x,q)\right] \ . \end{aligned}$$

The superscripts $+$ and $-$ refer to electrons with positive and negative energies, respectively. The "components" $f_s^{(\pm)}(x, \pm p)$ of $\mathcal{N}_s$ are defined in disjoint portions (on different sheets of the mass shell) of the eight-dimensional phase space. Hence the transport equation (5.51) holds separately for particles with a definite sign of energy,

$$p\cdot(\partial_x \mp ep\cdot F\cdot\partial_p)\ f_s^{(\pm)}(x,p) = 0 \ . \tag{5.56}$$

Note that the negative energy component obeys the Vlasov equation for positrons.

The electron and positron phase-space densities for each spin component can therefore be expressed as

$$\mathcal{N}_s^{\pm}(x,p) = \int d^4q\ \theta(q^0)\delta(q^2 - m^2)\ \delta^4(p \mp q)\ f_{\pm s}^{(\pm)}(x,q) \ . \tag{5.57}$$

In terms of these densities the scalar and axial vector densities are then given by

$$\begin{aligned} \mathcal{F} &= \mathcal{N}_s^+ + \mathcal{N}_{-s}^+ + \mathcal{N}_s^- + \mathcal{N}_{-s}^- \ , \\ \mathcal{A}_\mu &= -s_\mu(\mathcal{N}_s^+ - \mathcal{N}_{-s}^+ - \mathcal{N}_s^+ + \mathcal{N}_{-s}^+) \ . \end{aligned}$$

These relations make the physical interpretation of those densities clear. However, beyond the classical limit, this simple physical interpretation gets obscured by the fact that the dispersion relation for spin up and down particles changes to first order in $\hbar$, and $\mathcal{P}$, $\mathcal{V}_\mu$ and $\mathcal{S}_{\mu\nu}$ acquire spin corrections to order $\hbar$. We note that in the case of non-abelian theories, there is the additional complication that even in the classical limit the spin motion is coupled to the motion of the internal color vector [19,20].

6. Transport Equation for Photons

Up to now, we have only considered quantum transport theory for fermions. In ultra-relativistic plasmas we may need to consider the transport of photons as well. For dense enough plasmas, it may be important to take into account Compton scattering or even Delbruck scattering in order to study the approach to local thermal equilibrium. In such situations a need for a phase-space description of gauge bosons might arise. A more immediate motivation for studying photon transport theory is to gain insight into theory of gauge bosons in non-abelian theories discussed elsewhere [3]. In this section we discuss the basic starting point for the formulation of quantum transport theory for photons.

The extention of transport formalism to scalar bosons is well known [10]. However, for gauge vector bosons a Lorentz covariant, gauge invariant formulation has to our knowledge not been discussed up to now. The naive *ansatz* for the Wigner operator would be

$$\hat{X}^{\mu\nu}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, A^\nu(x+\tfrac{1}{2}y)\, A^\mu(x-\tfrac{1}{2}y)\ . \tag{6.1}$$

However, this construction is obviously not gauge invariant. Since all gauge independent quantities associated with the gauge fields may be expressed in terms of the field strengths $F^{\mu\nu}$, we propose instead the following definition for the gauge invariant phase-space distribution operator for photons:

$$\hat{Y}^{\mu\nu\sigma\rho}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, F^{\sigma\rho}(x+\tfrac{1}{2}y)\, F^{\mu\nu}(x-\tfrac{1}{2}y)\ . \tag{6.2}$$

By antisymmetry of the field operators, this operator is antisymmetric separately in the first and second pair of indices.

This gauge invariant operator is related to the energy momentum flux of vector gauge bosons rather than to the number density as is the case for fermions. Specifically, the ensemble average of stress tensor of the gauge field [13] is given by

$$\begin{aligned}
\langle T^\mu{}_\nu(x)\rangle &\equiv \langle :F^{\mu\sigma}(x)\, F_{\sigma\nu}(x) + \tfrac{1}{4}g^\mu{}_\nu F^{\sigma\rho}(x)\, F_{\sigma\rho}(x):\rangle \\
&= \int d^4p\, \langle :\hat{Y}^{\mu\sigma}{}_{\sigma\nu}(x,p) + \tfrac{1}{4}\, g^\mu{}_\nu\, \hat{Y}^{\sigma\rho}{}_{\sigma\rho}(x,p)\ :\rangle\ .
\end{aligned} \tag{6.3}$$

A novel aspect of the transport theory of photons that does not arise for fermions is that both definitions of the gauge boson Wigner operator may include a mean field part if $\langle F_{\mu\nu}\rangle \neq 0$. However, that part is already taken into account when we solve the mean field Maxwell's equations (4.12), and therefore, it should be subtracted to avoid redundancy. The part left over describes only the fluctuating or "chaotic" component of the field. We adopt the simplified definition of chaotic fields as those characterized by $\langle F^{\mu\nu}(x)\rangle = 0$ but with $\langle F^{\mu\nu}F_{\mu\nu}\rangle \neq 0$. The simplest example [10] of such a field is a photon gas in local thermal equilibrium with $\langle F^{\mu\nu}\rangle = 0$ but with

$$\langle T^{\mu\nu}(x,p)\rangle \propto \delta(p^2)p^\mu p^\nu (e^{p\cdot u(x)/T(x)} - 1)^{-1} \ , \tag{6.4}$$

in terms of the fluid four velocity and temperature fields, $u^\mu(x)$ and $T(x)$. Clearly, quantum fluctuations also lead to $\langle F^{\mu\nu}F_{\mu\nu}\rangle \neq 0$. However, we are particularly interested in fluctuations that survive in the classical limit and are associated with a chaotic gas of photons interacting with the surrounding plasma.

We therefore redefine the gauge invariant photon distribution as

$$\hat{Y}_{fl}^{\mu\nu\sigma\rho}(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y} \left[F^{\sigma\rho}(x+\tfrac{1}{2}y)\, F^{\mu\nu}(x-\tfrac{1}{2}y) - \langle F^{\sigma\rho}(x+\tfrac{1}{2}y)\rangle \langle F^{\mu\nu}(x-\tfrac{1}{2}y)\rangle\right] . \tag{6.5}$$

Similarly, we redefine the gauge dependent Wigner operator (6.1). The subscript fl emphasizes that only the fluctuation parts of the fields contribute.

It is important to recall that mean fields not only describe the self-consistent Coulomb and magnetic fields in the plasma but also coherent radiation fields, such as bremsstrahlung or synchrotron radiation. In general the mean field can be decomposed from (4.13) as

$$\langle A^\mu(x)\rangle = \bar{A}^\mu_{in}(x) + \bar{A}^\mu_{part}(x) + \bar{A}^\mu_{rad}(x) \ , \tag{6.6}$$

where the "near and intermediate" Coulomb and magnetic fields surrounding the particles are given by [13]

$$\bar{A}^\mu_{part}(x) = \int d^4y\, D^{\mu\nu}_{adv}(x,y)\langle j_\nu(y)\rangle \ , \tag{6.7}$$

and the radiation field is given by

$$\bar{A}^\mu_{rad}(x) = \int d^4y\, D^{\mu\nu}_{(-)}(x,y)\langle j_\nu(y)\rangle \ . \tag{6.8}$$

The advanced Green's function vanishes in the asymptotic future, while $D_{(-)} \equiv D_{ret} - D_{adv}$ satisfies homogeneous Maxwell's equations. The incoming field, $\bar{A}^\mu_{in}$, may specify for example an external laser or Coulomb field as was noted before.

The radiation field described by the mean field is characterized by a coherent state [22]. Of course, one could consider ensembles leading to partially coherent states (see, e.g., ref. [23]) as well. In such cases, the separation between coherent and chaotic field components is not so simple and more complex correlation functions are required to characterize the fields. We will assume below that such a complication does not arise and that there is a clean separation between coherent and chaotic components.

The gauge independent definition (6.2) of the photon Wigner operator can be expected to obey more complicated equations of motion than the gauge dependent one since, in general, it is more difficult to solve the coupled set of Maxwell's equations directly rather than detouring to the Poisson equation via the electromagnetic potential method. Nevertheless, since $F^{\mu\nu}$ is simply related to A^μ, we can easily relate the gauge invariant Wigner operators for photons to the gauge dependent one. As we have already encountered when discussing the fermion problem, the operator

$$k^\mu \equiv p^\mu + i\tfrac{1}{2}\partial_x^\mu ,$$

applied to the mixed Fourier transform

$$F(x,p) \equiv \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, B_1(x+\tfrac{1}{2}y)\, B_2(x-\tfrac{1}{2}y)$$

of two arbitrary functions (or operators) $B_1(x)$ and $B_2(x)$, acts as a derivative on the second function under the integral:

$$k^\mu\, F(x,p) = \int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\, B_1(x+\tfrac{1}{2}y)\, i\partial^\mu B_2(x-\tfrac{1}{2}y) \,.$$

Similarly the operator

$$k^* = p - i\tfrac{1}{2}\partial_x$$

acts as a derivative on the first function under the integral. Hence, once we know the Wigner operator $\hat{X}^{\mu\nu}(x,p)$ we might readily construct its gauge invariant alternative:

$$\hat{Y}_{\mu\nu\sigma\rho} = -k_\nu k^*_\sigma\, \hat{X}_{\mu\rho} + k_\nu k^*_\rho\, \hat{X}_{\mu\sigma} - k_\sigma k^*_\mu\, \hat{X}_{\nu\rho} + k_\mu k^*_\rho\, \hat{X}_{\nu\sigma} \,.$$

It is thus a matter of taste to prefer one or the other definition in the case of abelian vector bosons. It should be noticed that neither of them is hermitian, and they transform under hermitian conjugation according to

$$\hat{X}^\dagger_{\mu\nu}(x,p) = \hat{X}_{\nu\mu}(x,p) \,,$$

$$\hat{Y}^\dagger_{\mu\nu\sigma\rho}(x,p) = \hat{Y}_{\sigma\rho\mu\nu}(x,p) \,.$$

Next we derive the equations of motion for both types of Wigner operators. Rather than starting with Maxwell's equations in analogy to the way we proceeded in the fermion case in section 4, we compute directly the action of k^2 and k^{*2} on those operators. Notice that (reinstating $\hbar$)

$$k^2 = (p^2 - \hbar^2\tfrac{1}{4}\Box) + i\hbar p\cdot\partial_x \,. \tag{6.9}$$

Therefore, the hermitian part of this operator leads to the constraint equation and the anti-hermitian part leads to the transport equation as was the case for fermions using the operator $(\gamma\cdot K)^2 - m^2$. Notice that k^μ corresponds to the fermion operator K^μ in (4.19) with $\bar{F}^{\mu\nu} = 0$. This just reflects the fact that photons do not interact with the mean field. We stress that the full quantum transport theory for photons is again not complete without the constraint equation.

The quantum constraint and transport equations for the gauge dependent Wigner operator in the Feynman gauge [13], where $\Box A^\mu = j^\mu$, is then given by

$$\left.\begin{array}{c} p^2 - \frac{1}{4}\Box \\ -ip\cdot\partial_x \end{array}\right\} \hat{X}^{\mu\nu}(x,p) = \tfrac{1}{2}(k^{*2} \pm k^2)\,\hat{X}^{\mu\nu}(x,p) = -\tfrac{1}{2}\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}$$

$$\times \left[\Box A^\nu(x+\tfrac{1}{2}y)\,A^\mu(x-\tfrac{1}{2}y) \pm A^\nu(x+\tfrac{1}{2}y)\,\Box\,A^\mu(x-\tfrac{1}{2}y)\right]$$

$$= -\tfrac{1}{2}\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\left[j^\nu(x+\tfrac{1}{2}y)\,A^\mu(x-\tfrac{1}{2}y) \pm A^\nu(x+\tfrac{1}{2}y)\,j^\mu(x-\tfrac{1}{2}y)\right] . \tag{6.10}$$

The equation of motion for the fluctuation part, $\hat{X}^{\mu\nu}_{fl}$ is given by (6.10) with the replacement

$$j^\nu A^\mu \to j^\nu A^\mu - \langle j^\nu\rangle\langle A^\mu\rangle \ ,$$

etc.

The equations for the gauge independent Wigner function are obtained similarly and are

$$\left.\begin{array}{c} p^2 - \frac{1}{4}\Box \\ -ip\cdot\partial_x \end{array}\right\} \hat{Y}^{\mu\nu\sigma\rho}(x,p) = \tfrac{1}{2}(k^{*2} \pm k^2)\,\hat{Y}^{\mu\nu\sigma\rho}(x,p)$$

$$= -\tfrac{1}{2}\int \tfrac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\left[\Box F^{\sigma\rho}(x+\tfrac{1}{2}y)\,F^{\mu\nu}(x-\tfrac{1}{2}y) \pm F^{\sigma\rho}(x+\tfrac{1}{2}y)\,\Box\,F^{\mu\nu}(x-\tfrac{1}{2}y)\right]$$

$$= -\tfrac{1}{2}\int \tfrac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\left\{[\partial^\sigma j^\rho - \partial^\rho j^\sigma](x+\tfrac{1}{2}y)\,F^{\mu\nu}(x-\tfrac{1}{2}y) \pm F^{\sigma\rho}(x+\tfrac{1}{2}y)\,[\partial^\mu j^\nu - \partial^\nu j^\mu](x-\tfrac{1}{2}y)\right\} . \tag{6.11}$$

The last line follows from the quadratic equation for the field tensor,

$$\Box\, F^{\mu\nu} = \partial^\mu j^\nu - \partial^\nu j^\mu \ , \tag{6.12}$$

which involves the Lorentz-covariant *curl* of the fermion current. Notice that this equation invariant against *curl*-free modifications of the fermion current.

Equation (6.11) can be derived also by isolating the hermitian and anti-hermitian parts of the equations that follow directly from Maxwell's equations. The homogeneous Maxwell's equations (2.3) for the field strength leads in particular to

$$k^\alpha\,\hat{Y}^{\mu\nu\sigma\rho}(x,p) + k^\mu\,\hat{Y}^{\nu\alpha\sigma\rho}(x,p) + k^\nu\,\hat{Y}^{\alpha\mu\sigma\rho}(x,p) = 0 \ . \tag{6.13}$$

The inhomogeneous equations lead to

$$k_\mu\,\hat{Y}^{\mu\nu\sigma\rho}(x,p) = i\int \frac{d^4y}{(2\pi)^4} e^{-ip\cdot y}\,F^{\sigma\rho}(x+\tfrac{1}{2}y)\,j^\nu(x-\tfrac{1}{2}y) \ . \tag{6.14}$$

Multiplying (6.13) with k_α and combining it with (6.14) and then adding or subtracting the adjoint equations gives directly (6.11).

The large number of coupled equations arising here is not surprising in view of the large number of indices in the gauge invariant definition. This means that the gauge invariant formulation is much more awkward to deal with. The main advantage of having a gauge invariant formulation is that approximations to the collision terms on the right hand side may be obtained in a manifestly gauge invariant way. In practical calculations, we will always have to specify a gauge, in which case the gauge dependent formulation will probably be more useful.

To compute the collision terms on the r.h.s. of these transport equations we would have to proceed with the program of extracting binary, tertiary, and higher order collision terms via the method discussed in detail in [10]. The above quantum transport equations provide the basic starting point for such a program. However, we shall not pursue this path further here. We conclude by noting that the constraint equation shows that the uncertainty principle brings even free photons off-shell by the amount $\frac{1}{4}\Box$ just as in the case of fermions. An obvious difference is the absence of a Vlasov term and associated off-shell corrections because the photons do not interact directly with each other. Of course, a Vlasov term does arise in the non-abelian case as shown in ref. [3]. The absence of direct interactions in the abelian case also leads to a very much simpler spin structure. Different helicity states only couple via the collision terms.

7. Summary and Conclusions

In this paper we addressed the phase-space formulation of abelian quantum gauge field theory for fermions and vector gauge bosons. First the translation of classical phase-space distribution function into quantum theory was discussed with particular emphasis on the question of gauge independence. Gauge independence was shown to emerge automatically if we demand that the momentum variable in the Wigner function corresponds to the kinetic rather than the canonical momentum.

Next an exact, complete and gauge independent linear equation of motion for the gauge invariant fermion Wigner operator was derived. By complete and linear we mean its equivalence to first order Dirac's equations of motion for the fermion Heisenberg field operator and its adjoint. The physical content of this equation and its relation to transport theory was discussed in detail. We employed the quadratic (second order) form of Dirac's equations to generalize the classical Vlasov equation to include quantum and spin corrections. An additional constraint equation, redundant in classical physics, emerges in quantum theory. That equation shows that the restriction of fermion kinetic momenta to mass shell may in general be violated by Heisenberg's uncertainty principle and spin interaction effects.

In the Hartree approximation we demonstrated how the linear and quadratic equations may be expanded systematically in derivatives of the physical electric and magnetic c-number fields. This expansion, which can be formulated with help of the triangle operator $\triangle = \partial_x \cdot \partial^p$, coincides with a semiclassical expansion in powers of $\hbar$. In the limit $\hbar \to 0$ the relativistic Vlasov equation is recovered.

Furthermore, we performed a decomposition of the Wigner function and its linear equation of motion in spinor space. We found that in the case of arbitrary c-number fields only four out of the sixteen components of the Wigner function are dynamically independent. We showed that the scalar and axial vector components are sufficient to define the Wigner matrix and have a particularly simple physical interpretation in the classical limit. First order quantum corrections, corresponding to spin interaction effects, were considered explicitly, and we proposed a systematic method to unravel the 32 coupled equations for the components of W to any finite order in $\hbar$.

Finally, two alternative (gauge dependent and gauge independent) ways to formulate the quantum transport theory for photons were proposed. A novel aspect of that theory is the necessity to separate coherent and chaotic field effects. The fluctuation part of the photon Wigner operator was found to obey a general constraint and transport equation. Only the collision terms complicate the problem because of the absence of Vlasov type terms.

Although in this paper we have studied quantum transport theory in the "collisionless" (Hartree) regime in most detail, the general quantum equations derived here should provide a natural starting point for further work on deriving generalized quantum collision terms. The structure of collision terms in gauge theories, with special attention paid to infrared singularities associated with long range forces, is a particularly interesting problem for future investigation.

Acknowledgement: We thank Walter Greiner for many fruitful discussions.

Bibliography

[1] H. Satz, ed.: Thermodynamics of Quarks and Hadrons, North-Holland, Amsterdam, 1981; T. W. Ludlam and H. E. Wegner, eds.: Quark Matter '83, Nucl.Phys. A418 (1984); K. Kajantie: Quark Matter '84, Springer Lecture Notes in Physics, Springer, Berlin, 1985; E. V. Shuryak, Phys.Rep. 115 (1984), 151; B. Müller: The Physics of the Quark-Gluon Plasma, Springer Lecture Notes in Physics, Springer, Berlin, 1985; J. Cleymans, R. V. Gavai, and E. Suhonen, Phys. Rep. 130 (1986), 217; Proc. of the 5^{th} International Conference on Ultra-Relativistic Nucleus-Nucleus Collisions, Asilomar, USA, 1986, to be published in Nucl. Phys.

[2] H.-Th. Elze, M. Gyulassy and D. Vasak, Nucl. Phys. B276 (1986), 706.

[3] H.-Th. Elze, M. Gyulassy and D. Vasak: Transport Equations for the QCD Gluon Wigner Operator, preprint LBL-21652, Phys. Lett. B, in press.

[4] R. Hakim, Riv. Nuovo Cim. 1 no. 6 (1978); R. Hakim and H. Sivak, Ann. Phys. (N.Y.) 139 (1982), 230; H. D. Sivak, Ann. Phys. 159 (1985) 351, Phys. Rev. C34 (1986), 653.

[5] M. Ploszajczak, M. J. Rhoades-Brown: Non-Equilibrium Truncation Schemes for Statistical Mechanics of Relativistic Matter, Stony Brook Preprint (1986) to be published.

[6] E. A. Remler, Phys. Rev. D 16 (1977), 3464.

[7] O. T. Serimaa, J. Javanainen and S. Varró, Phys. Rev. A 33 (1986), 2913.

[8] S. Ichimaru: Basic Principles of Plasma Physics, W.A. Benjamin Inc., London, 1973.

[9] E. Wigner, Phys. Rev. 40 (1932), 749.

[10] S. R. de Groot, W. A. van Leeuwen and Ch. G. van Weert: Relativistic Kinetic Theory, North-Holland, Amsterdam, 1980.

[11] P. Carruthers and F. Zachariasen, Rev. Mod. Phys. 55 (1983), 245.

[12] I. Bialynicki-Birula, Acta Phys. Austriaca, Suppl. XVIII (1977), 111.

[13] C. Itzykson and J.-B. Zuber: Quantum Field Theory, McGraw-Hill, New York, 1980.

[14] S. Mandelstam, Ann. Phys. (N.Y.) 19 (1962), 1.

[15] V. Weisskopf, Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. XIV (1936), no. 6.

[16] J. D. Jackson: Classical Electrodynamics, Wiley, New York, 1975.

[17] J. D. Alonso, Ann. Phys. (N.Y.) 160 (1985), 1.

[18] R. D. Tenreiro and R. Hakim, Phys. Rev. D 15 (1977), 1435.

[19] U. Heinz, Phys. Lett. 144B (1984) 228, Ann. Phys. (N.Y.) 161 (1985), 48.

[20] E. A. Remler: Classical Spin Equations of Motion in an External Color Field via the Wigner Representation, College of William and Mary Preprint, 1986.

[21] V. Bargmann, L. Michel and V. L. Telegdi, Phys. Rev. Lett. 2 (1959), 435.

[22] R.J. Glauber, Proc. Les Houches Summer School, 1964, eds. C. DeWitt *et al.*, Gordon and Beach, Pubs., New York, 1964, p.63.

[23] M. Gyulassy, S.K. Kauffmann, L.W. Wilson, Phys. Rev. C20 (1979), 2267.

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