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Marked bumpless pipedreams and compatible pairs

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Abstract. We construct a bijection between marked bumpless pipedreams with reverse compatible pairs, which are in bijection with not-necessarily-reduced pipedreams. This directly unifies various formulas for Grothendieck polynomials in the literature. Our bijection is a generalization of a variant of the bijection of Gao and Huang in the unmarked, reduced case.

Keywords. Bumpless pipedreams, Grothendieck polynomials

Mathematics Subject Classifications. 05E05

1. Introduction

The polynomial ring $\mathbb{Z}[\beta][x_1, x_2, \cdots]$ admits operators

$$\partial_i := \frac{1 - s_i}{x_i - x_{i+1}} \text{ and } \pi_i := \partial_i(1 + \beta x_{i+1}),$$

where s_i acts by switching x_i and x_{i+1}, x_i acts as multiplication by x_i, and 1 denotes the identity operator. The β-*Grothendieck polynomials* $\mathfrak{G}^{(\beta)}_w$ for $w \in S_\infty := \bigcup_{i=1}^\infty S_n$ are the unique family of polynomials that satisfy

$$\begin{aligned} \mathfrak{G}^{(\beta)}_{w_0^n} &= x_1^{n-1}x_2^{n-2}\cdots x_{n-1} && \text{where } w_0^n := n\; n-1\; \cdots\; 1 \\ \pi_i \mathfrak{G}^{(\beta)}_w &= \mathfrak{G}^{(\beta)}_{ws_i} && \text{if } \ell(ws_i) = \ell(w) - 1. \end{aligned}$$

The β-Grothendieck polynomials indexed by $w \in S_n$ are a Schubert basis of connective K-theory of the flag variety [Hud14]. Specializing β yields two important families of polynomials, both of which were introduced in this form by Lascoux and Schützenberger [LS82]. The

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Grothendieck polynomials $\mathfrak{G}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=-1}$ represent structure sheaves of Schubert varieties in the flag variety. The Schubert polynomials $\mathfrak{S}_w = \mathfrak{G}_w^{(\beta)}|_{\beta=0}$ of [LS82] represent the cohomology classes of Schubert varieties in the flag variety [BGG73, Dem74, LS82]. Equivalently $\mathfrak{S}_w$ is the coefficient of β^0 in $\mathfrak{G}_w^{(\beta)}$ or equivalently the lowest x-degree component of $\mathfrak{G}_w^{(\beta)}$.

The β-Grothendieck polynomials satisfy the positivity $\mathfrak{G}_w^{(\beta)} \in \mathbb{Z}_{\geqslant 0}[\beta; x_1, x_2, \dots]$ for all $w \in S_\infty$. Due to this positivity, any combinatorial formula for $\mathfrak{G}_w^{(\beta)}$ based on a bijection between a set and the monomials of $\mathfrak{G}_w^{(\beta)}$, restricts to a combinatorial formula for Schubert polynomials.

There are many combinatorial formulas for Grothendieck polynomials. Fomin and Kirillov [FKsd, Prop. 3.3] gave a compatible pair formula for Grothendieck polynomials, extending the formula of Billey, Jockusch, and Stanley [BJS93] for Schubert polynomials. Knutson and Miller [KM05] interpreted Grothendieck polynomials as K-polynomials of matrix Schubert varieties (up to change of variables) and showed via Gröbner degeneration they can be computed with (not-necessarily-reduced) pipedreams, generalizing the reduced pipedream formula of Billey and Bergeron [BB93] for Schubert polynomials. Knutson and Miller [KM05] also showed that the Schubert polynomials are multidegrees of matrix Schubert varieties. Billey and Bergeron [BB93] gave a bijection between reduced compatible pairs and reduced pipedreams and their map generalizes directly to the not-necessarily-reduced case. On the other hand, using marked bumpless pipedreams, Weigandt [Wei21] reinterpreted Lascoux's formulas [Las08] for Grothendieck polynomials based on alternating sign matrices (ASMs), which extends the bumpless pipedream formula of [LLS21] for Schubert polynomials.

Gao and Huang [GH23] gave a bijection between reduced pipedreams and reduced bumpless pipedreams. The goal of this paper is to extend the bijection of Gao and Huang, furnishing a bijection between pipedreams and marked bumpless pipedreams. This provides a direct connection between the various kinds of combinatorial formulas for Grothendieck polynomials in the literature. In a later paper we shall prove that our bijection restricts to that of [GH23] in the reduced case. The Gao–Huang bijection processes tiles in a bumpless pipedream from top to bottom using a generalization of the column moves of [LLS21] which operate on adjacent columns. In particular, the generalization involves scanning over blank tiles. In the reduced case our map processes tiles from bottom to top, has no blank scanning, and operates on adjacent rows by row moves, the transpose of [LLS21] column moves.

In fact, long before the discovery of bumpless pipedreams [LLS21], it was known that the 2-enumeration of alternating sign matrices of size n is $2^{n(n-1)/2}$ [MRR83, EKLP92, Kup96], which is easily seen to be the number of not-necessarily reduced pipedreams of size n. Therefore, based on Weigandt's observation that connects alternating sign matrices and bumpless pipedreams [Wei21], our bijection can be viewed as a bijection between 2-enumerated ASMs and not-necessarily reduced pipedreams. This connection between ASMs and pipedreams was recently exploited by Striker and Huang [HS24] to establish a bijection between certain totally symmetric self-complementary plane partitions and certain ASMs, using the Gao–Huang bijection.

The not-necessarily-reduced setting introduces many complications which make it considerably more subtle than the reduced case.

1.1. Pipedreams

Let $[n] = \{1, 2, \ldots, n\}$. A (not-necessarily-reduced) pipedream is a tiling of $\{(i, j) \in [n] \times [n] : i + j \leqslant n + 1\}$, such that all the entries (i, j) with $i + j = n + 1$ have tiles , and all other entries are either or . We use the matrix-style notation $D_{i,j}$ for the tile in the i-th row and j-th column. Every such diagram can be viewed as n pipes entering from the top and exit from the left. We denote the set of pipedreams in the $n \times n$ grid with $\mathrm{PD}(n)$. Each $D \in \mathrm{PD}(n)$ has an associated permutation $w = w(D) \in S_n$ where the pipe entering at the left border at the i-th row and exits the top border at the $w(i)$-th column, with the proviso that if two pipes have already crossed, the subsequent crossings between this pair of pipes are ignored (that is, treated like .). For $w \in S_n$, let $\mathrm{PD}(w)$ denote the set of pipedreams with associated permutation w. For $w \in S_n$, the Grothendieck polynomial $\mathfrak{G}_w$ can be computed via pipedreams [FKsd]:

$$\mathfrak{G}_w^{(\beta)} = \sum_{D \in \mathrm{PD}(w)} \beta^{|\mathrm{cross}(D)| - \ell(w)} \prod_{(i,j) \in \mathrm{cross}(D)} x_i \tag{1.1}$$

where $\mathrm{cross}(D)$ denote the set of coordinates of the in D.

1.2. Compatible pairs

A *biletter* is an ordered pair of integers (i, a) with $1 \leqslant i \leqslant a < n$. A *reverse compatible pair* is a sequence of biletters

$$B = ((i_1, a_1), (i_2, a_2), \ldots, (i_\ell, a_\ell)) \tag{1.2}$$

which are strictly decreasing in the order given by $(i_1, a_1) > (i_2, a_2)$ if $i_1 > i_2$ or if $i_1 = i_2$ and $a_1 < a_2$.

Let $\mathrm{RCP}(n)$ denote the set of reverse compatible pairs.

The weight of B is defined by $\mathrm{wt}(B) = (m_1, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$ where m_i is the number of biletters in B of the form (i, a) for some a.

The Demazure or 0-Hecke product $*$ on permutations is the unique monoid structure such that for any simple reflection s_i and permutation w,

$$s_i * w = \begin{cases} s_i w & \text{if } \ell(s_i w) > \ell(w) \\ w & \text{otherwise.} \end{cases}$$

Every $B \in \mathrm{RCP}(n)$ has an associated permutation $w(B) = s_{a_\ell} * s_{a_{\ell-1}} * \cdots * s_{a_1}$ with notation as in (1.2). Note that the subscripts of a are *decreasing*.

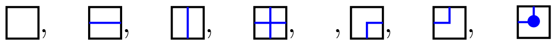
Denote by $\mathrm{RCP}(w)$ the set of $B \in \mathrm{RCP}(n)$ with associated permutation w. Let $|B|$ be the length of the sequence B.

Remark 1.1. There is a bijection $\mathrm{RCP}(n) \to \mathrm{PD}(n)$ that sends the compatible pair $B = ((i_1, a_1), \ldots, (i_\ell, a_\ell))$ to the pipedream with crossings at positions $(i_j, a_j - i_j + 1)$ for $1 \leqslant j \leqslant \ell$. This was proved in [BB93] for the reduced case but the proof works in the nonreduced setting. This bijection preserves the weight and the associated permutation. Therefore

$$\mathfrak{G}_w^{(\beta)} = \sum_{B \in \mathrm{RCP}(w)} \beta^{|B| - \ell(w)} x^{\mathrm{wt}(B)}. \tag{1.3}$$

1.3. Marked Bumpless pipedreams

We work with a set of tiles



named blank, horizontal, vertical, plus, R, J, and marked J.

We will say that a tile *connects to the right* (resp. left, up, down) if it contains a line segment going from its center to the right (resp. left, up, down).

A *marked bumpless pipedream* is an $[n] \times [n]$ matrix with entries in the above set of tiles such that every row has a pipe entering from the right, every column has a pipe leaving to the south, no pipe enters from the top, no pipe leaves to the left, and the "pipes are connected". Let $\mathrm{MBPD}(n)$ be the set of $[n] \times [n]$ marked bumpless pipedreams. Each $D \in \mathrm{MBPD}(n)$ has an *associated permutation* $w = w(D) \in S_n$ where the pipe entering the right border at the i-th row, exits the bottom border in the $w(i)$-th column, with the proviso that if two pipes have already crossed, then subsequent crossings between this pair of pipes are ignored, that is, treated like bump tiles . For $w \in S_n$ let $\mathrm{MBPD}_w = \{D \in \mathrm{MBPD}(n) \mid w(D) = w\}$, the bumpless pipedreams with associated permutation w. For every $w \in S_n$ the *Rothe bumpless pipedream* D_w is the unique element of MBPD_w whose i-th pipe turns only at position $(i, w(i))$ for all i.

Say that a tile is *heavy* if it is either a or a and *light* otherwise. The *weight* of $D \in \mathrm{MBPD}(n)$ is the sequence $\mathrm{wt}(D) = (m_1, m_2, \ldots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$ where m_i is the number of heavy tiles in the i-th row of D. The only element of $\mathrm{MBPD}(n)$ with no heavy tile is D_{id}.

Our main theorem is a constructive proof of the following.

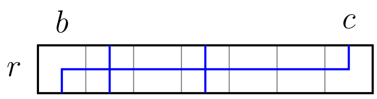
Theorem 1.2. *There are mutually-inverse bijections* $\Phi : \mathrm{MBPD}(n) \to \mathrm{RCP}(n)$ *and* $\Psi : \mathrm{RCP}(n) \to \mathrm{MBPD}(n)$ *which preserve weights and associated permutations. Furthermore, these bijections restrict to mutually-inverse bijections between reduced, unmarked bumpless pipedreams and reduced reverse compatible sequences.*

2. Terminology on tiles

2.1. Some tile notation

Given $D \in \mathrm{MBPD}(n)$ and intervals $I, J \subset [n]$ we denote by $D_{I,J}$ the submatrix of entries $D_{i,j}$ for $i \in I$ and $j \in J$. We write $D_{r,J}$ when $I = \{r\}$ is a singleton row index.

We say $D_{r,[b,c]}$ *is a pipe segment* if the interior part $D_{r,[b+1,c-1]}$ consists solely of tiles and . A *kink* is a pipe segment $D_{r,[b,c]}$ such that $D_{r,b} =$ and $D_{r,c} =$.



By definition, a non-blank tile $D_{r,[b,b]}$ is always a pipe segment. We also make the following observation which says gluing two pipe segments produces a pipe segment.

Lemma 2.1. *Suppose $D_{r,[a,b]}$ and $D_{r,[b,c]}$ are pipe segments, then so is $D_{r,[a,c]}$.*

Proof. For $a < j < c$, $D_{r,j}$ is clearly or if $j \neq b$. Since $D_{r,[b,c]}$ is a pipe segment, $D_{r,b}$ can be , or . Since $D_{r,[a,b]}$ is a pipe segment, we know $D_{r,b} \neq$, so it is or . □

A *light sequence* in $D \in \mathrm{MBPD}(n)$ is a set of tiles of the form $D_{r,[b,c]}$ which consists solely of light tiles. The RJ subsequence of a light sequence is the sequence of tiles obtained by only keeping the 's and 's. They come in four flavors.

1. Paired: Some number of copies of a pair given by an followed by a .
2. Type J: A followed by a paired sequence.
3. Type R: A paired sequence followed by an .
4. Type JR: A followed by a paired sequence followed by a .

Remark 2.2. Consider a light sequence S in a row of some $D \in \mathrm{MBPD}(n)$.

1. If S immediately follows a , , , or then S is paired or type R.
2. If S immediately follows a , , or then S is type J or type JR.
3. If S immediately precedes a , , or then S is paired or of type J.
4. If S immediately precedes a , , , or then S must be type R or type JR.

The following remark is a useful observation on MBPDs.

Remark 2.3. For any $D \in \mathrm{MBPD}(n)$ and heavy tile in D, there is a in its row to the right and a in its column below.

2.2. Drooping

For a row index r and column indices $1 \leqslant b < d \leqslant n$, the $(r, [b, d])$-droop and $(r, [b, d])$-undroop are operations that change an MBPD into another, only changing tiles in the "(un)droop rectangle" $[r, r + 1] \times [b, d]$.

We say that $D \in \mathrm{MBPD}(n)$ admits the $(r, [b, d])$-*droop* if

- The droop rectangle contains only light tiles except possibly a at $(r + 1, d)$.
- $D_{r,[b,d]}$ is a pipe segment.
- $D_{r+1,[b+1,d-1]}$ is a paired light sequence.
- $D_{r,b} \neq$ and $D_{r,d} \neq$.

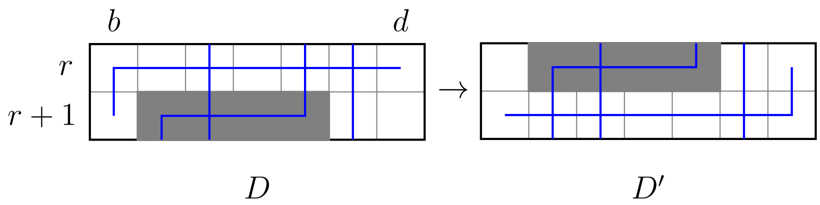
Remark 2.4. The pipe segment implies that $D_{r,b}$ is connected to the right and $D_{r,d}$ is connected to the left. The paired light sequence implies that $D_{r+1,b}$ is not connected to the right and $D_{r+1,d}$ is not connected to the left. We deduce that if D admits the $(r,[b,d])$-droop then

- $D_{r,b}$ is or (it connects to the right and down).
- $D_{r,d}$ is or (it connects to the left and not down).
- $D_{r+1,b}$ is or , (it connects up and not to the right).
- $D_{r+1,d}$ is or (it does not connect left nor up).

If D admits the $(r,[b,d])$-droop then we may produce $D' \in \mathrm{MBPD}(n)$ as follows; we say that $D \to D'$ is the $(r,[b,d])$-droop.

In the following pictures we only draw the parts of the upper right and lower left corner tiles $D_{r,d}$ and $D_{r+1,b}$ that change during the (un)droop; the other parts remain the same.

The pipe segment $D_{r,[b,d]}$, which connects down to $D_{r+1,b}$, "droops" to a pipe segment $D'_{r+1,[b,d]}$ which is connected upwards to $D'_{r,d}$. In columns x of D for $b < x < d$, the vertical pipes (x such that $b < x < d$ and $D_{r,x} = D_{r+1,x} =$) are unchanged. Each kink in row $r+1$ in D between columns b and d, is shifted up into row r in D'. A shifted kink is shaded gray in the following picture of a droop.

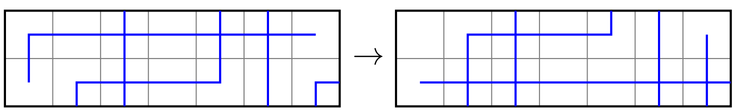


The naming of droops depends on the upper left and lower right corner tiles of the rectangle. The arrows indicates how the upper left and lower right tiles change during the droop. The upper left tile at (r,b) "loses an " while the lower right tile at $(r+1,d)$ "gains a ".

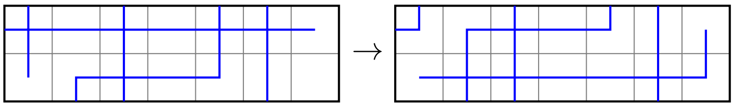
$D_{r,b} \backslash D_{r+1,d}$	$\to$	$\to$
$\to$	standard	fuse
$\to$	split	split-fuse

The above droop is standard. We imagine in D that the pipe in row r is a blue rope which is pinned in the center of the upper right and lower left boxes, and we are pulling the rope taut and holding it at the center of the upper left box. Then we let go. The rope falls to the floor (middle of row $r+1$). After it falls the rope forms a in the lower right.

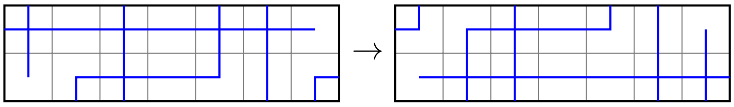
Here is a fuse droop: the in the lower right corner, when drooped upon, acquires a which is fused to it and becomes a .



For the split droop, we imagine that we cut the so it falls into the superposition of a and an . The droop loses the and produces a in the lower right corner.



Finally, a split-fuse droop, which combines the splitting and fusing.



By definition, $D \to D'$ is a $(r, [b, d])$-*undroop* if and only if $D' \to D$ is a $(r, [b, d])$-droop. Fusing (resp. splitting) undrooping is inverse to splitting (resp. fusing) drooping.

Explicitly, a $(r, [b, d])$-*undroop* $D \to D'$ is defined when

- The undroop rectangle contains only light tiles except possibly a at (r, b).
- $D_{r+1,[b,d]}$ is a pipe segment.
- $D_{r,[b+1,d-1]}$ is a paired light sequence.
- $D_{r+1,d} \neq$, $D_{r+1,b} \neq$.

Remark 2.5. If D admits a $(r, [b, d])$-undroop then

- $D_{r+1,d}$ is or (it connects to the left and up).
- $D_{r+1,b}$ is or (it connects to the right and not up).
- $D_{r,d}$ is or (it connects down and not to the left).
- $D_{r,b}$ is or (it does not connect to the right nor down).

It is straightforward to verify that the operations of drooping and undrooping, map MBPDs to MBPDs.

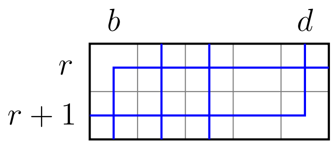
The following lemma is immediate from the definitions.

Lemma 2.6. *Let* $D \in \mathrm{MBPD}(n)$*.*

1. *If* D *admits the* $(r, [b, d])$*-droop producing* D'*, then* D' *admits the* $(r, [b, d])$*-undroop recovering* D*.*
2. *If* D' *admits the* $(r, [b, d])$*-undroop producing* D*, then* D *admits the* $(r, [b, d])$*-droop recovering* D'*.*

2.3. Double crosses

For $D \in \mathrm{MBPD}(n)$ the subdiagram $D_{[r,r+1],[b,d]}$ is a *double cross* if $D_{r,[b,d]}$ and $D_{r+1,[b,d]}$ are pipe segments, $D_{r,b} =$, and $D_{r+1,d} =$.



It is not hard to deduce that if $D_{[r,r+1],[b,d]}$ is a double cross, then $D_{r+1,b} = D_{r,d} =$. This is so named because the pipe through (r, b) and the pipe through $(r + 1, d)$ cross twice at (r, d) and $(r + 1, b)$.

Remark 2.7. There is at most one double cross $D_{[r,r+1],[b,d]}$ for each fixed pair (r, b) and also at most one such double cross for a fixed pair (r, d).

We describe how to find the "d" given (r, b).

Lemma 2.8. *Suppose $D_{[r,r+1],[b,d]}$ is a double cross. Then d is the smallest number such that $d > b$ and $D_{r+1,d} =$.*

Proof. Since $D_{r+1,[b,d]}$ is a pipe segment, there is no in $D_{r+1,[b,d)}$. □

3. F-moves

In this section and the next, we describe the basic combinatorial operations defined on MBPDs and their inverses that form the building blocks of our bijection.

3.1. f-targets and f^*-targets

Let $D \in \mathrm{MBPD}(n)$. Say that (r, c) is an f*-target* of D if the conditions (f1), (f2), and (f3) hold.

(f1) $D_{r,c}$ is the rightmost heavy tile in row r.

(f2) There is an index $c' > c$ such that $D_{r+1,c'} =$. Let c' be minimum with this property.

(f3) All tiles $D_{r+1,j}$ are light for $j < c'$.

We say that (r, c) is an f^**-target* of D if the conditions (f1), (f*2), and (f3) hold, where

(f*2) There is no nor in row $r + 1$ to the right of column c. In this case let c' be the maximum index such that $D(r, c') =$. This exists by Remark 2.3.

By abuse of language we will also say that $(D, (r, c))$ is an f-target (resp. f^*-target) to mean that $D \in \mathrm{MBPD}(n)$ and (r, c) is an f-target (resp. f^*-target) of D. We will write F to mean either f or f^*. So $(D, (r, c))$ is an F-target means it is either an f-target or f^*-target.

We define the *window* of an F-target $(D,(r,c))$ to be the two-row rectangle $[r,r+1]\times[b,c']$ where

$$b=\begin{cases} c & \text{if } D_{r,c}=\square \\ \text{maximum } b<c \text{ with } D_{r,b}= & \text{if } D_{r,c}=\boxdot \end{cases} \tag{3.1}$$

and c' is defined by (f2) in the case of an f-target and (f*2) in the case of an f^*-target.

The *maximum F-target* of $D\in\mathrm{MBPD}(n)\setminus\{D_{\mathrm{id}}\}$ is by definition its bottommost then rightmost heavy tile (say (r,c)), which is easily verified to be an F-target of D. If not explicitly specified otherwise, if we refer to the target of D we mean its maximum F-target. We say that an MBPD is *F-terminal* if its maximal F-target is an f^*-target and F-nonterminal otherwise.

For any f-target $(D,(r,c))$ we shall define an f-move $D\mapsto f_r(D)$ under which a heavy tile is removed from the r-th row and a heavy tile is put into the $(r+1)$-th row. In this case $w(D)=w(f_r(D))$. If $(D,(r,c))$ is an f^*-target we shall define an f^*-move $D\mapsto f_r^*(D)$ in which a heavy tile is removed from the r-th row. In this case $w(D)=w(f_r^*(D))*s_r$. Be aware that if we know $w(D)$, we cannot determine $w(f_r^*(D))$ which could be either $w(D)$ or $w(D)s_r$ (see Proposition 3.2). These moves are defined in §3.3 depending on a number of cases which are described below.

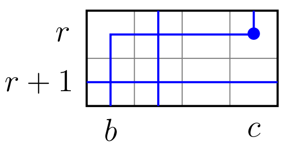
3.2. Two trichotomies of cases for F-moves

There are two trichotomies for an F-target $(D,(r,c))$, giving nine cases in all. They are called left and right because they describe the left and right sides of the window respectively.

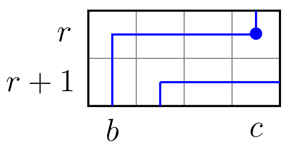
The left trichotomy for the F-target $(D,(r,c))$ asserts that exactly one of B, C, or C̸ holds.

(B) (Blank): This holds if $D_{r,c}=\square$.

- When not in Case B, $D_{r,c}=\boxdot$. Let b be as in (3.1).
  - (C) (Crossing): $D_{r+1,[b,c]}$ is a pipe segment. This is so named because in this case the pipe through (r,c) goes to the left and then crosses vertically with the above pipe segment at $D_{r+1,b}=\boxplus$.
  - (C̸) (Noncrossing): Neither Case B nor Case C holds.



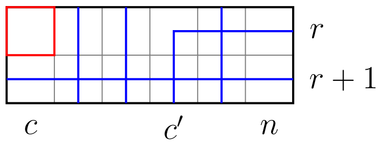
Case C



Case C̸

The right trichotomy says that for an F-target $(D,(r,c))$, exactly one T, D, or O holds.

(T) (Terminal) The tile (r,c) is an f^*-target of D.



Case T

- The complement of Case T : (r,c) is an f-target of D. There are two subcases.
  - (D) (Double cross): There is a double cross $D_{[r,r+1],[d,c']}$ for some d with $c < d < c'$.
  - (O) (Ordinary): Cases T and D do not hold.

The double cross requires a in row $r+1$ not allowed by Case T. Such a double cross is unique by Remark 2.7.

Given an F move, define the right droop column index ρ by

$$\rho = \begin{cases} c' & \text{for Cases T or O} \\ d & \text{for Case D.} \end{cases} \tag{3.2}$$

To be parallel with the upcoming definitions for E-moves, define the left droop column index λ by

$$\lambda = c \tag{3.3}$$

which is the column index of the F-target.

The following Lemma is used to prove the well-definedness of the F-moves to be defined later.

Lemma 3.1. *Suppose $(D,(r,c))$ is an F-target.*

1. *$D_{r+1,c}$ is either or .*
2. *$D_{r+1,[c,c']}$ is a pipe segment.*
3. *In Case T, $D_{r+1,c'} =$. Moreover there are no s in row $r+1$ to the right of column c.*
4. *$S = D_{r,[c+1,c'-1]}$ is a light sequence that is either paired or of type R.*
   - *(a) S is of type R if and only if $D_{r,c'} =$. In this case, suppose S has last at $D_{r,d}$ and we are not in Case T. Then $D_{[r,r+1],[d,c']}$ is a double cross (Case D holds).*
   - *(b) S is paired if and only if $D_{r,c'}$ is or . Then either Case T or Case O holds.*

Moreover the $(r,[c,\rho])$-droop applies (after removing the marking if $D_{r,c} =$).

Proof. By (f3) $D_{r+1,[c,c'-1]}$ is a light sequence. In particular $D_{r+1,c}$ is light. Since $D_{r,c}$ is heavy, $D_{r+1,c}$ does not connect upwards. Item (1) follows.

By Item (1) $D_{r+1,c}$ connects to the right. Following this pipe to the right in row $r+1$, suppose it turns before column c'. It turns at ┘ or ┘•. This is a contradiction for the f-target case by (f2) and (f3), and for the f^*-target case by (f*2). Item (2) follows. Moreover $D_{r+1,c'}$ must connect to the left. In the case that (r,c) is an f^*-target of D, $D_{r+1,c'}$ must also connect upwards since $D_{r,c'}$ = ┌. Therefore $D_{r+1,c}$ is ┌ or ─. The pipe leaving $(r+1,c)$ to the right, cannot turn upwards. Item (3) follows.

For (4), by (f1) S is a light sequence. It is preceded by a heavy tile $D_{r,c}$, so S must be paired or of type R. S is followed by the tile $D_{r,c'}$, which is light by (f1) and which connects downwards, since $D_{r+1,c'}$ is either ┘ or ┼. So $D_{r,c'}$ must be │, ┌, or ┼.

If S is of paired type then $D_{r,c'}$ does not connect to the left, and is therefore │ or ┌. If S is of type R then $D_{r,c'}$ connects to the left and is therefore ┼.

Suppose S is of paired type. By Item (1) $D_{r+1,c}$ does not connect upwards and in our current case $D_{r,c'}$ does not connect to the left. It follows that D admits the $(r,[c,c'])$-undroop, proving (4b).

Otherwise S is of type R and $D_{r,c'}$ = ┼. Let S have last ┌ at position (r,d) where $c<d<c'$. Suppose also that we are in Case T so that $D_{r+1,c'}$ = ┘ by Item (3). Now $D_{r+1,d}$ connects upwards since $D_{r,d}$ = ┌ connects downwards and $D_{r+1,d}$ connects to the right by Item (2). Therefore $D_{r+1,d}$ = ┼. By the definition of d $D_{r,[d,c']}$ is a pipe segment. By Item (2) $D_{r+1,[d,c']}$ is a pipe segment. We have verified that $D_{[r,r+1]\times[d,c']}$ is a double cross.

We have $D_{r,d}$ = ┌ and $D_{r+1,d}$ = ┼. $D_{r+1,c}$ does not connect upwards since $D_{r,c}$ is heavy. $D_{r+1,[c,d]}$ is a pipe segment by Item (2) and $D_{r,[c,d]}$ is a paired light sequence since the ending ┌ of the type R light sequence S has been removed. Hence the $(r,[c,d])$-undroop of D is well-defined and (4a) holds. □

3.3. F-moves

Let $(D,(r,c))$ be an F-target. We define $F_r(D) \in \mathrm{MBPD}(n)$ to be the result of the following steps:

- If $D_{r,c}$ = ┘• remove the marking.
- In Cases B and C perform the $(r,[c,\rho])$-undroop; see (3.2).
- If the tile at $(r+1,c')$ is ┘, mark it.

The following proposition summarizes how F-moves affect the permutation associated to the MBPD.

Proposition 3.2. *Let $(D,(r,c))$ be an F-target. Then in all the non-terminal cases, $w(f_r(D)) = w(D)$. In the terminal cases, $w(f_r^*(D)) * s_r = w(D)$.*

Proof. In Case T, by the definition of f_r^*-target, the pipes p and q that enter from row r and $r+1$ necessarily cross within the window of the target. Therefore, s_r is a right descent of $w(D)$. In Case $\not{\mathsf{C}}$, only the marking changes, so the permutation does not change. When not in Cases $\not{\mathsf{C}}$

or T, the $(r,[c,\rho])$-undroop first possibly remove a crossing from a double cross, then possibly create a double cross, so the permutation is unchanged. Finally in Cases TB and TC, the unique crossing between pipes p and q within the window is uncrossed, and a new double cross which does not affect the permutation is created in Case TC. If the pipes that enter from row r and $r+1$ in $f_r^*(D)$ no longer cross then $w(f_r^*(D)) = w(D)s_r$; otherwise $w(f_r^*(D)) = w(D)$. The claim then follows. $\square$

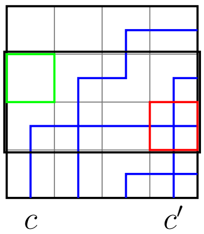
The F_r move removes a heavy tile in row r and creates a heavy tile in row $r+1$ unless it is in Case T. This behavior is captured by the following lemma.

Lemma 3.3. *Assume* $(D,(r,c))$ *is an* F*-target with window* $[r,r+1]\times[b,c']$*. Then all tiles in* $F_r(D)_{[r,r+1],[b,c']}$ *are light except possibly* $F_r(D)_{r+1,c'}$*. It is light if and only if* $(D,(r,c))$ *is in Case* T.

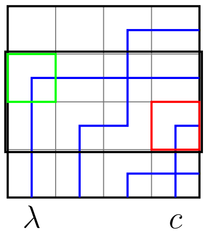
Proof. By (f1) and (f3), the only heavy tile in $D_{[r,r+1],[b,c']}$ is $D_{r,c}$. We first prove the lemma in Case $\cancel{\mathsf{C}}$. Since $D_{r,c} =$, the F_r move unmarks it. Then F_r marks $D_{r+1,c'}$ if and only if $D_{r+1,c'} =$, which is equivalent to we are not in case T.

Next we prove the lemma outside of Case $\cancel{\mathsf{C}}$. After the $(r,[c,\rho])$-undroop, the only tile that might be heavy in $[r,r+1]\times[b,c']$ is $(r+1,\rho)$. In Case T, $D_{r+1,\rho} = D_{r+1,c'} =$, so $F_r(D)_{r+1,\rho} =$ is not heavy. In Case D, $D_{r+1,\rho} =$, so $F_r(D)_{r+1,\rho} =$ is not heavy. Then $D_{r,c'} =$ is marked and become the only heavy tile in $[r,r+1]\times[b,c']$. In Case O, $D_{r+1,\rho} = D_{r+1,c'} =$, so $F_r(D)_{r+1,\rho} =$ is heavy. $\square$

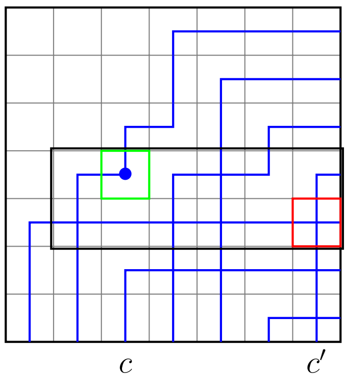
We now give illustrations for the F-moves in all nine cases. The labels for the results of these moves will be explained later in Section 4 when we define E-moves. The window of the move is indicated by a thick two-row rectangle. The box (r,λ) is colored green and the box $(r+1,\rho)$ is colored red. When (un)droops occur, these boxes will be the corners of the (un)droop window.



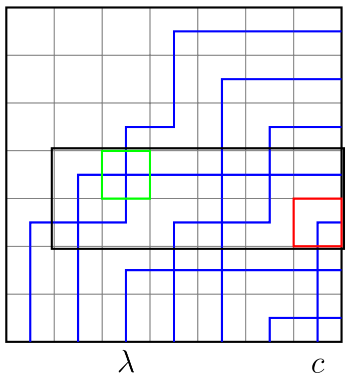
TB



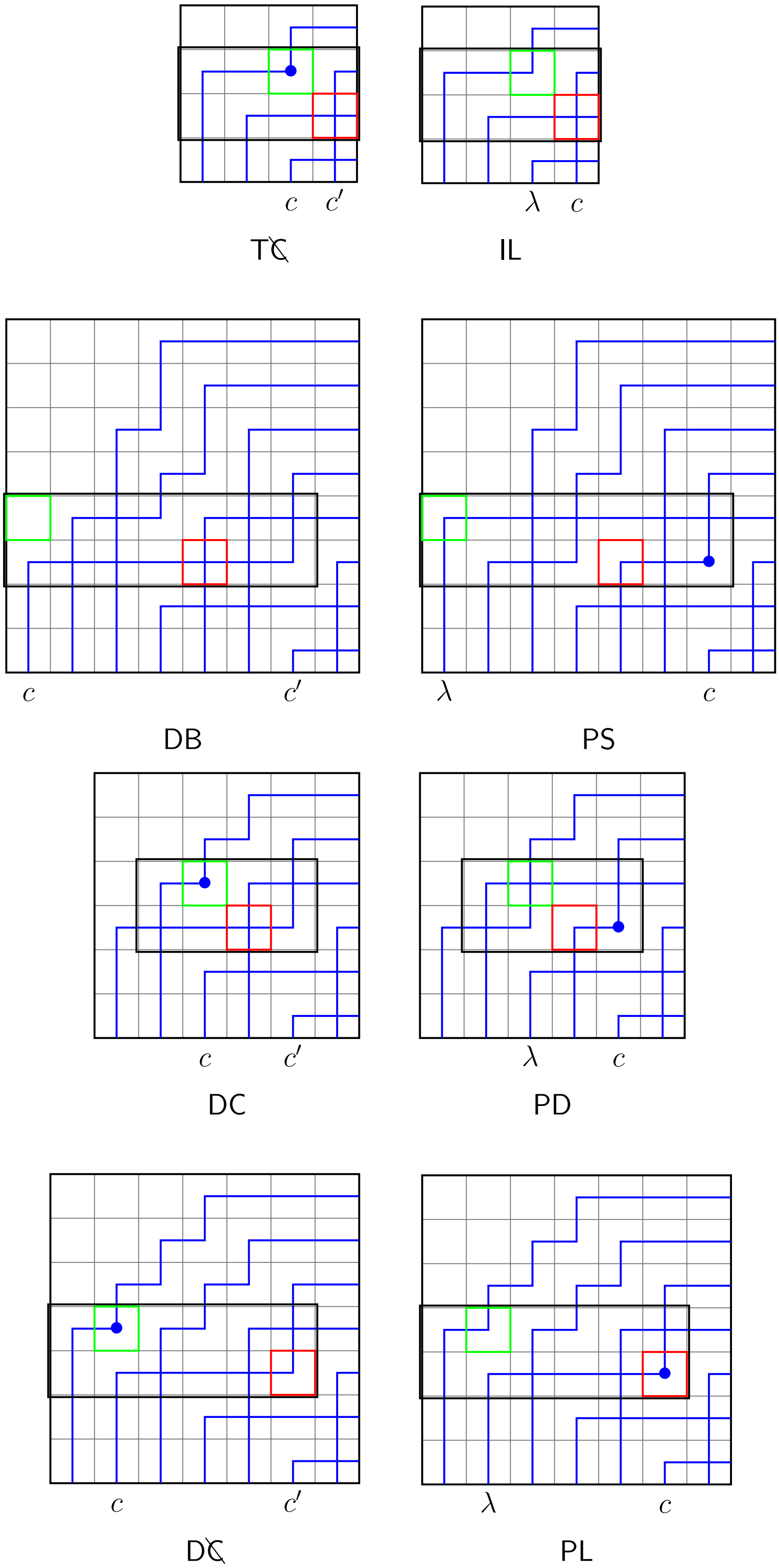
IS



TC



ID


c c'
T$\not{C}$
λ c
IL
c c'
DB
λ c
PS
c c'
DC
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PD
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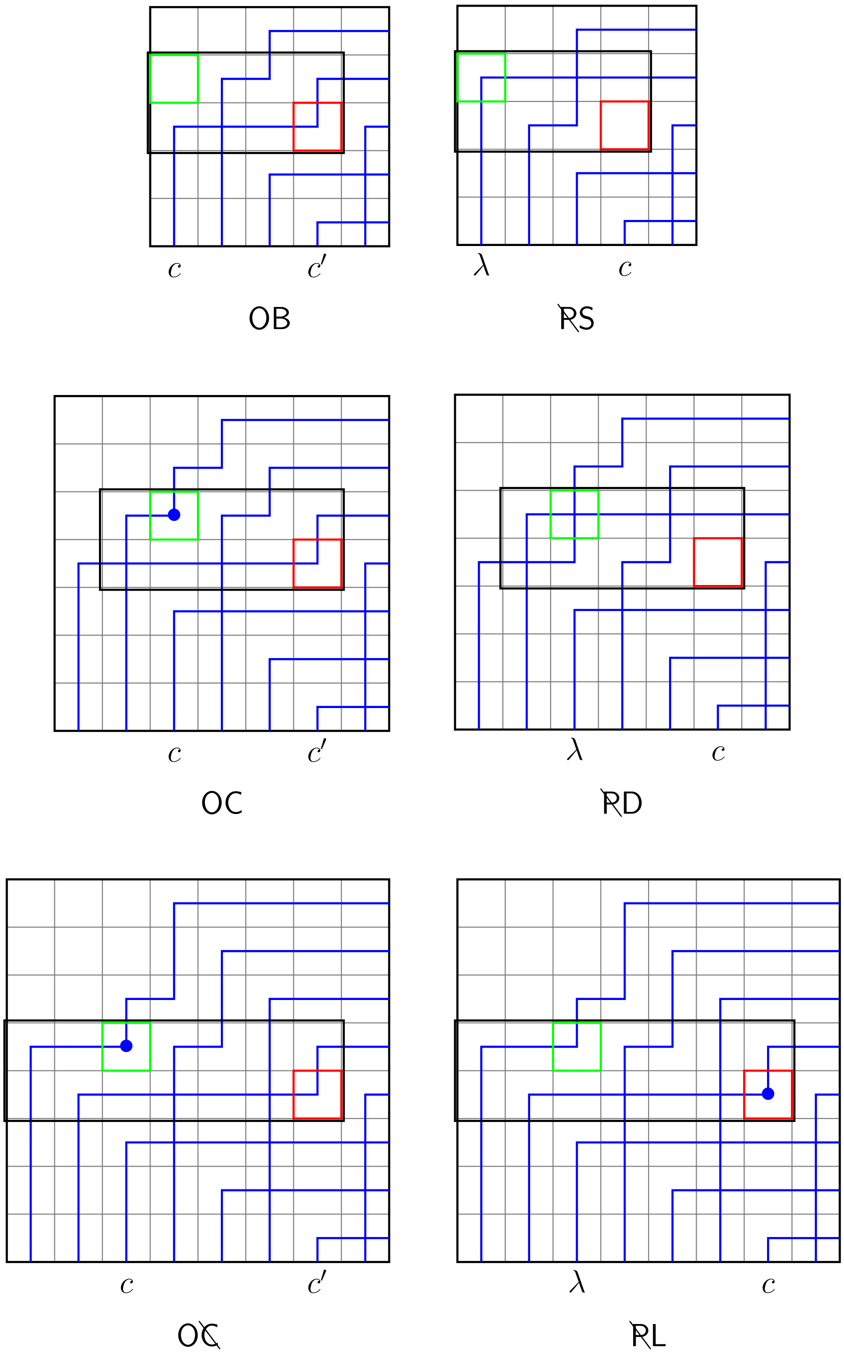
D$\not{C}$
λ c
PL



4. E-moves

4.1. E-targets

We say the tile $(r+1, c)$ is an *e-target* of $D \in \mathrm{MBPD}(n)$ if the following are satisfied.

(e1) It is the leftmost heavy tile on row $r+1$.

(e2) There is an index $c' < c$ such that $D_{r,c'} = \boxed{\ulcorner}$ and $D_{r+1,[c',c]}$ is not a pipe segment. Let c' be maximum with this property.

(e3) On the right of (r, c') in row r there are no heavy tiles.

We say the tile $(r + 1, c)$ is an e^*-*target* of $D \in \mathrm{MBPD}(n)$ if the conditions (e*1), (e2), and (e3) are satisfied, where the condition (e1) for an e-target has been replaced by the condition

(e*1) There are no heavy tiles in row $r + 1$, and c is the largest such that $D_{r,c}$ or $D_{r+1,c}$ is $\boxed{\ulcorner}$.

Again we abuse language by saying that an e-*target* (resp. e^*-target) is a pair $(D, (r + 1, c))$ where $D \in \mathrm{MBPD}(n)$ and $(r + 1, c)$ is an e-target (resp. e^*-target) of D. Similar to the F-case, an E-target is either an e-target or e^*-target.

We define the *window* of an E-target $(D, (r + 1, c))$ to be the two-row rectangle $[r, r + 1] \times [c', c]$ with c' as in (e2).

If $(D, (r + 1, c))$ is an e-target, we shall define an e-move $D \mapsto e_r(D)$ under which a heavy tile is removed from the $(r + 1)$-th row and a heavy tile is put into the r-th row. In this case $w(D) = w(e_r(D))$. If $(D, (r+1, c))$ is an e^*-target we shall define an e^*-move $D \mapsto e^*_r(D)$ in which a heavy tile is inserted into the r-th row. In this case $w(e^*_r(D)) = w(D) * s_r$. The moves will only affect tiles in the windows.

After some preliminary work these moves are defined in §4.3.

4.2. Two trichotomies of cases for E-moves

Let $(D, (r + 1, c))$ be an E-target. We have 9 cases with three choices for the first symbol and three for the second symbol.

Here is the right trichotomy for E-moves. Exactly one of them holds for an E-target.

- I (Initial): $(r + 1, c)$ is an e^*-target.
- P (Plus): $(r + 1, c)$ is an e-target and $D_{r,c} = \boxplus$.
- $\not{\mathsf{P}}$ (No plus): $(r + 1, c)$ is an e-target and $D_{r,c} \neq \boxplus$.

Define the *right droop column* ρ of the E-target $(D, (r + 1, c))$ by

$$\rho = \begin{cases} c & \text{in Cases } \mathsf{I} \text{ and } \not{\mathsf{P}} \\ \max\left\{ d' \mid d' < c \text{ and } D_{r+1,d'} = \boxed{\ulcorner} \right\} & \text{in Case } \mathsf{P}. \end{cases} \tag{4.1}$$

The left trichotomy for E-moves are the following three cases, exactly one of which holds for any E-target $(D, (r + 1, c))$ with window $[r, r + 1] \times [c', c]$.

- L (Left turn): $D_{r,[c',c]}$ is not a pipe segment. That is, the pipe leaving the $\boxed{\ulcorner}$ at (r, c') to the right, must turn upwards (make a left turn) before arriving at (r, c).
- D (Double cross): $D_{r,[c',c]}$ is a pipe segment and the tile (r, c') is the top-left corner of a double cross.

- S (Straight): $D_{r,[c',c]}$ is a pipe segment and the tile (r, c') is not the top-left corner of a double cross.

Define the *left droop column* λ of the E-target $(D, (r+1, c))$ by

$$\lambda = \begin{cases} c' & \text{in Case S} \\ \min\left\{ d \mid d > c' \text{ such that } D_{r+1,d} = \right\} & \text{in Case D} \\ \min\left\{ d \mid d > c' \text{ such that } D_{r,d} = \right\} & \text{in Case L.} \end{cases} \tag{4.2}$$

We start by characterizing what the tile $(r+1, c)$ can be if it is an E-target.

Lemma 4.1. *Let D have E-target $(r+1, c)$. In Case* L*, $D_{r+1,c}$ is or . Otherwise, $D_{r+1,c}$ is , or .*

Proof. We first show that $D_{r+1,c}$ is one of , , , or . Then we show that if $D_{r+1,c}$ is or then $(D, (r+1, c))$ is not in case L, and if $D_{r+1,c} =$ then $(D, (r+1, c))$ is in case L.

In Case I, we know $D_{r,c}$ or $D_{r+1,c}$ is the only in $D_{[r,r+1],[c,n]}$. In the former situation, we know the rightmost in row $r+1$ is on the left of $D_{r+1,c}$, so $D_{r+1,c} =$. In the latter situation, $D_{r+1,c} =$. If we are not in Case I, $D_{r+1,c}$ is heavy, so it is or .

Suppose $D_{r+1,c}$ is or , so $D_{r,c}$ is not connected to the bottom. By (e3), $D_{r,c}$ is light, so it connects to the left. Let $d < c$ be the largest such that $D_{r,d} =$. Then $D_{r+1,[d,c]}$ is not a pipe segment. We must have $d = c'$ and $D_{r,[c',c]}$ is a pipe segment, so we cannot be in Case L.

Suppose $D_{r+1,c} =$. We are in Case I since it is light. By (e*1), $D_{r,c} =$. Then $D_{r,[c',c]}$ cannot be a pipe segment, so we are in Case L. □

Lemma 4.2. *Let D have E-target $(r+1, c)$.*

1. *If Case* L *holds then $S = D_{r,[c'+1,c-1]}$ has type J or JR.*
2. *Suppose Case* L *does not hold. Let $S' = D_{r+1,[c'+1,c-1]}$. Case* D *holds if and only if S' is of type J or JR. In this case, $D_{[r,r+1],[c',\lambda]}$ is a double cross.*

Proof. Suppose Case L holds. Since $D_{r,c'} =$ we deduce Item (1) by Remark 2.2 (2).

Suppose Case L does not hold, so $D_{r,[c',c]}$ is a pipe segment. If S' has type J or JR, we can find smallest d such that $c' < d < c$ and $D_{r+1,d} =$. Then $D_{r+1,[c',d]}$ is a pipe segment. Since $D_{r,[c',c]}$ is a pipe segment, so is $D_{r,[c',d]}$. We have $D_{[r,r+1],[c',d]}$ is a double cross, so we are in Case D and $\lambda = d$. Conversely, if the double cross exists, then $D_{r+1,[c',\lambda]}$ is a pipe segment with $D_{r+1,\lambda} =$. Thus, S' has type J or JR. □

Lemma 4.3. *If not in Case* L*, D admits the $(r, [\lambda, \rho])$-droop.*

Proof. We check the conditions of drooping.

- All tiles in $D_{[r,r+1],[c',c]}$ are not heavy except possibly $(r+1, c)$. Since $c' \leqslant \lambda < \rho \leqslant c$, all tiles in $D_{[r,r+1],[\lambda,\rho]}$ are not heavy except possibly $(r+1, \rho)$.
- Since we are not in Case L, $D_{r,[c',c]}$ is a pipe segment and thus $D_{r,[\lambda,\rho]}$ is also.

- Let $S' = D_{r+1,[\lambda+1,\rho-1]}$. To show S' is paired, we first show S' cannot have type J or JR.
  - In case S, we know $\lambda = c'$. By Lemma 4.2, $D_{r+1,[c'+1,c-1]}$ does not have type J or JR. Thus, S' does not have type J or JR.
  - In case D, we know $D_{r+1,\lambda} =$ ┘. Thus, S' does not have type J or JR.

  Finally, we just need to check S' cannot have type R, which is equivalent to showing that $D_{r+1,\rho}$ does not connect to the left. Suppose it does. Then $D_{r+1,\rho}$ cannot be ┌, so we are not in Case P and $\rho = c$. By Lemma 4.1, since $D_{r+1,c}$ connects to the left, it is $\boxplus$ or $\boxdot$, so it connects to the top. Since we are not in Case P, $D_{r,c}$ must be ┌ or │. Thus, $D_{r,[c',c]}$ cannot be a pipe segment, which is a contradiction.

- We check $D_{r,\lambda} \neq \boxminus$. It is ┌ in Case S and ┘ in Case L. In case D, we know $D_{r+1,\lambda} =$ ┘, so $D_{r,\lambda}$ cannot be $\boxminus$.

  Then we check $D_{r,\rho} \neq \boxplus$. In Case P, $D_{r+1,\rho} =$ ┌ so $D_{r,\rho} \neq \boxplus$. Otherwise, $\rho = c$ and we know $D_{r,c} \neq \boxplus$. ☐

4.3. The E-moves

We shall now define e-moves, which are the inverses of f-moves. Given an E-target $(D,(r+1,c))$, define $E_r(D) \in \mathrm{MBPD}(n)$ to be the result of the following steps:

- If $D_{r+1,c} = \boxdot$, remove the marking.
- In Cases S and D, perform the $(r,[\lambda,\rho])$-droop where λ and ρ are defined respectively by (4.2) and (4.1).
- If (r,λ) is ┘, mark it.

The E_r move creates a heavy tile in row r and removes a heavy tile in row $r+1$ unless it is in Case I. This behavior is captured by the following lemma.

Lemma 4.4. *Assume $D_{r+1,c}$ is a E-target with window $[r,r+1]\times[c',c]$. Then all tiles in $E_r(D)_{[r,r+1],[c',c]}$ are light except $E_r(D)_{r,\lambda}$, where λ is the left droop column. Moreover, $E_r(D)_{r+1,c} = \boxplus$ if we are in Case* I *and* ┘ *otherwise.*

Proof. By (e1), (e*1) and (e3), Tiles in $D_{[r,r+1],[c',c]}$ are all light except $D_{r+1,c}$, which is light if and only if we are in Case I. We first prove the lemma in Case L. By Lemma 4.1, $D_{r+1,c} = \boxplus$ in Case I and $\boxdot$ otherwise. Thus, $E_r(D)_{r+1,c} = \boxplus$ in Case I and ┘ otherwise. In either case, it is light. Since $D_{r,\lambda} =$ ┘, if will be marked and become the only heavy tile in $E_r(D)_{[r,r+1],[c',c]}$

Next we prove the lemma outside of Case L. We first understand $E_r(D)_{r,\lambda}$ by the three possibilities of $D_{r+1,c}$ in Lemma 4.1.

- If $D_{r,c} =$ ┌, we know we are in Case I since it is light and Case ᖇ since it is not connected to top. Then $\rho = c$. After the droop, we know $E_r(D)_{r,c} = \boxplus$.
- If $D_{r,c} = \square$, we know we are not in Case ᖇ since it is light and not connected to top. Then $\rho = c$, and if would become ┘ after the droop.
- If $D_{r,c} = \boxdot$, then E_r would unmark it. It will stay ┘ after the droop.

After the $(r,[\lambda,\rho])$-droop, (r,λ) is the only tile that might be heavy in $[r,r+1]\times[\lambda,\rho]$, and therefore in $[r,r+1]\times[c',c]$ We know (r,λ) is $\square$ or $\boxed{\lrcorner}$ after the droop. In the latter case, E_r would mark it, so $E_r(D)_{r,\lambda}$ must be heavy. $\square$

5. Tables of cases

We give a diagrammatic summary of the F- and E-moves, as described in Section 3 and 4, for the readers' convenience. Small examples of windows of the nine cases for f and f^* are pictured in Figure 5.1. The green square has coordinate (r,λ) and the red square has coordinate $(r+1,\rho)$. In F-move cases other than $\not\mathsf{C}$ and E-move cases other than L, the green and red squares are the corners of the (un)droop window. The parts of tiles not pictured remain the same. Small examples of the nine cases for e and e^* are pictured in Figure 5.2.

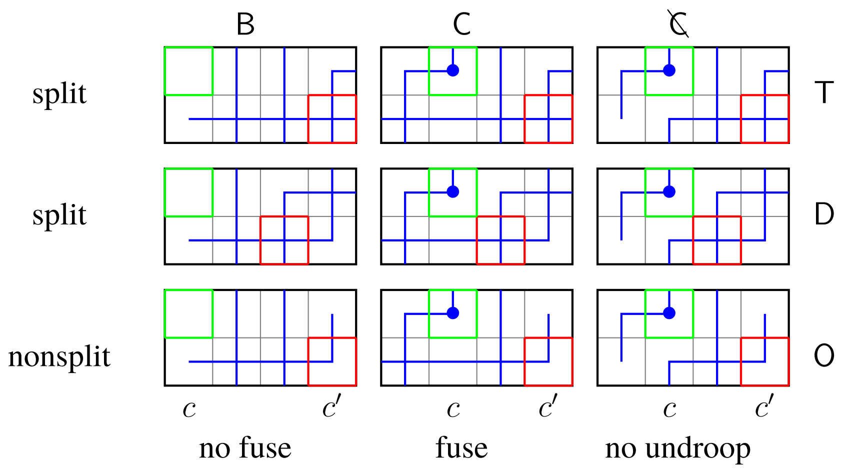


Figure 5.1: Cases for f and f^*.

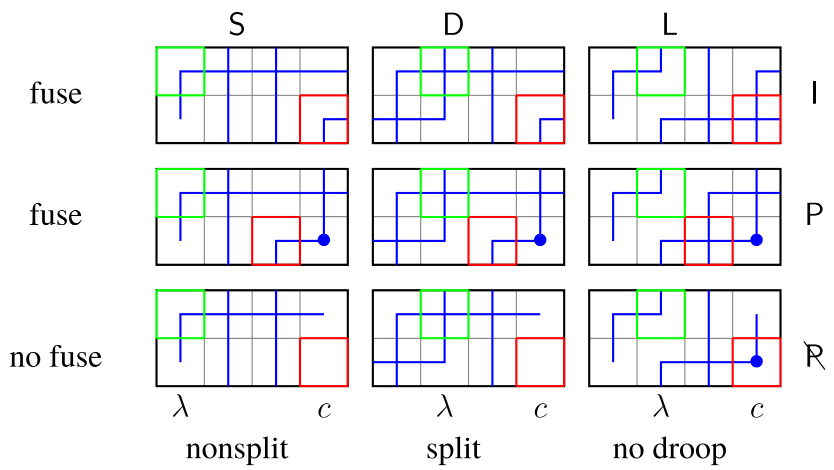


Figure 5.2: Cases for e and e^*.

6. F-moves and E-moves are inverses

In this section, we verify that F-moves and E-moves are inverses. More specifically, we establish Proposition 6.1 and Proposition 6.3.

Proposition 6.1. *Let D have F-target (r, c) with window $[r, r+1] \times [b, c']$.*

- *Then $F_r(D)$ has E-target $(r+1, c')$ with the same window.*
- *The case of $(D,(r, c))$ corresponds to the case of $(F_r(D),(r+1, c'))$ according to Figure 5.1 and Figure 5.2. For instance, if $(D,(r, c))$ is in case* BT*, then $(F_r(D),(r+1, c'))$ is in case* SI.
- *We have $E_r(F_r(D)) = D$.*

Proof. Let $\lambda = c$ and ρ be the left and right droop columns of $(D,(r, c))$. We start by proving a useful claim.

Claim 6.2. *If $(D,(r, c))$ is not in Case $\not\mathsf{C}$ then $F_r(D)_{r,[b,c']}$ is a pipe segment.*

Proof. We first check $F_r(D)_{r,[b,c]}$ is a pipe segment. If $(D,(r, c))$ is in case C, then $D_{r,[b,c]}$ is a pipe segment, and $F_r(D)_{r,[b,c]}$ is also. Otherwise we have $b = c$ and $F_r(D)_{r,[b,c]}$ is trivially a pipe segment.

Since F_r performs an $(r,[c, \rho])$-undroop, $F_r(D)_{r,[c,\rho]}$ is a pipe segment. By Lemma 2.1, it remains to check $F_r(D)_{r,[\rho,c']}$ is a pipe segment. If $(D,(r, c))$ is not in case D, $\rho = c'$ and $F_r(D)_{r,[\rho,c']}$ is trivially a pipe segment. Otherwise $D_{[r,r+1],[\rho,c']}$ is a double crossing. Thus $D_{r,[\rho,c']}$ is a pipe segment, and $F_r(D)_{r,[\rho,c']}$ is also. □

Now we check $Y = (F_r(D),(r+1, c'))$ is an E-target.

(e1) We suppose $(D,(r, c))$ is not in Case T and show Y satisfies (e1). By (f3) and Lemma 3.3, all tiles in $F_r(D)_{r+1,[1,c')}$ are light. By Lemma 3.3, $F_r(D)_{r+1,c'}$ is heavy, so it satisfies (e1).

(e*1) We assume $(D,(r, c))$ is in case T and show Y satisfies (e*1). By (f3) and Lemma 3.3, all tiles in $F_r(D)_{r+1,[1,c')}$ are light. By Lemma 3.1 (3) $D_{r+1,c'} = \boxplus$. By (f*2) there is neither nor in $D_{r+1,(c,n]}$. It follows that there are no heavy tile in $D_{r+1,[c',n]}$, or equivalently in $F_r(D)_{r+1,[c',n]}$. Thus, there are no heavy tiles in row $r+1$ of $F_r(D)$. We know there are no in $D_{[r,r+1],(c',n]}$; we already checked this for row $r+1$, and for row r it holds by the choice of c' in (f*2), because somewhere to the right of every there must be an . Therefore there is neither nor in $F_r(D)_{[r,r+1],(c',n]}$. It remains to check $F_r(D)_{r,c'}$ or $F_r(D)_{r+1,c'}$ is . By (f*2), $D_{r,c'} =$. If $(D,(r, c))$ is in Case $\not\mathsf{C}$, then F_r does no undroop and $F_r(D)_{r,c'} = D_{r,c'} =$. Otherwise, after the $(r,[c, \rho])$-undroop, $(r+1, c')$ changes from $\boxplus$ to .

(e2) We check $F_r(D)_{r,b}$ is the rightmost tile satisfying the condition of (e2) for Y. First, we check $F_r(D)_{r,b} =$: In case B, $b = c$ and (r, b) becomes after the undroop; Otherwise, $D_{r,b} =$ and stays unchanged by F_r.

Next, we check $F_r(D)_{r+1,[b,c']}$ is not a pipe segment. In case Ȼ, $D_{r+1,[b,c']}$ is not a pipe segment since $D_{r+1,[b,c]}$ is not, and there is no undroop. In case B or C, $F_r(D)_{r+1,\rho}$, being the bottom right corner after an undroop, must be or , both of which cannot be part of a pipe segment.

Finally, take $b < h < c'$ with $F_r(D)_{r,h} =$. We need to verify $F_r(D)_{r+1,[h,c']}$ is a pipe segment. If we are not in Case Ȼ, by Claim 6.2, there is no in $F_r(D)_{r,(b,c']}$, so such h cannot exist. Otherwise, $D_{r,[b,c]}$ is a pipe segment, so $c < h < c'$. Then since $D_{r+1,[c,c']}$ is a pipe segment by Lemma 3.1 (2), so is $D_{r+1,[h,c']}$. In Case Ȼ there is no undroop so the pipe segment still exists after F_r.

(e3) By (f1) for $(D,(r,c))$, there are no heavy tiles in $D_{r,(c',n]}$, or equivalently $F_r(D)_{r,(c',n]}$. By Lemma 3.3, there are no heavy tiles in $D_{r,[b,c']}$.

Thus, Y is an E-target with window $[r, r+1] \times [b, c']$. Next we go through all the cases of $(D,(r,c))$, assuming they hold and proving that the corresponding case of Y holds. For example, in the proof D $\to$ P, we assume $(D,(r,c))$ is in Case D and prove that Y is in Case P.

T $\to$ I: Already checked above. In fact it was shown above that T $\leftrightarrow$ I.

D $\to$ P: We already know I does not hold. By Lemma 3.1 (4) $D_{r,c'} =$. Whether or not F_r performs an $(r,[c,\rho])$-undroop, $F_r(D)_{r,c'}$ is still since $\rho < c'$.

O $\to$ Ꞧ: Again we know that I does not hold. By Lemma 3.1 (4), $D_{r,c'} \neq$. Whether or not F_r performs an $(r,[c,c'])$-undroop, $F_r(D)_{r,c'} \neq$.

B $\to$ S: In this case $c = b$ and F_r does an $(r,[c,\rho])$-undroop, so $F_r(D)_{r+1,c} \neq$. Thus, $F_r(D)_{r,c}$ cannot be the top-left corner of a double cross. By Claim 6.2, $F_r(D)_{r,[c,c']}$ is a pipe segment, so Y is in case S.

C $\to$ D: By Claim 6.2, $F_r(D)_{r,[b,c']}$ is a pipe segment. It is enough to show that $F_r(D)_{[r,r+1],[b,c]}$ is a double cross. First, $F_r(D)_{r,[b,c]}$ is a pipe segment since $F_r(D)_{r,[b,c']}$ is. By Case C, $D_{r+1,[b,c]}$ is a pipe segment. The F_r performs an $(r,[c,\rho])$-undroop, so $F_r(D)_{r+1,[b,c]}$ is a pipe segment. We have shown $F_r(D)_{r,b} =$. Finally, since $F_r(D)_{r+1,[b,c]}$ is a pipe segment, $F_r(D)_{r+1,c}$ can be , or . Since $F_r(D)$ is obtained by an $(r,[c,\rho])$-undroop, $F_r(D)_{r+1,c} =$.

Ȼ $\to$ L: We know $D_{r,c} =$, so $F_r(D)_{r,c} =$. Since $b < c < c'$, $F_r(D)_{r,[b,c']}$ cannot be a pipe segment.

Finally, we verify $E_r(F_r(D)) = D$. Let λ', ρ' be the left and right droop columns of Y. We may suppose that $(D,(r,c))$ is not in Case Ȼ and show that $\lambda' = c$ and $\rho' = \rho$. We prove $\lambda' = c$:

- Suppose the F-target $(D,(r,c))$ is in Case B. We have shown that the E-target Y is in Case S. We have $c = b$ from Case B and $b = \lambda'$ from Case S.

- Suppose the F-target $(D,(r,c))$ is in Case C. We know $D_{r+1,[b,c']}$ is a pipe segment. After applying the $(r,[c,\rho])$-undroop, $F_r(D)_{r+1,c} =$ and $F_r(D)_{r+1,[b,c]}$ is a pipe segment. Thus, $\lambda' = c$.

We prove $\rho' = \rho$:

- Suppose the F-target $(D, (r, c))$ is in Case D. We know $D_{r+1,[\rho,c']}$ is a pipe segment and $D_{r+1,\rho} =$. After applying the $(r, [c, \rho])$-undroop, $F_r(D)_{r+1,[\rho,c']}$ remains a pipe segment and $F_r(D)_{r+1,c} =$. Thus, $\rho' = \rho$.

- Suppose the F-target $(D, (r, c))$ is not in Case D. Then $\rho' = c' = \rho$. $\square$

Proposition 6.3. *Let D have E-target $(r + 1, c)$ with window $[r, r + 1] \times [c', c]$. Let λ, ρ be its left and right droop columns.*

- *Then $E_r(D)$ has F-target (r, λ) with the same window.*

- *The case of $(D, (r + 1, c))$ corresponds to the case of $(E_r(D), (r + 1, c'))$ according to Figure 5.1 and Figure 5.2.*

- *We have $F_r(E_r(D)) = D$.*

Proof. We start by proving a useful claim.

Claim 6.4. *$E_r(D)_{r+1,[\lambda,c]}$ is a pipe segment.*

Proof. We first consider Case L. It is enough to show $D_{r+1,[\lambda,c]}$ is a pipe segment. Let $d < c$ be the largest such that $D_{r+1,d} =$. By Lemma 4.1, $D_{r+1,c}$ connects to the left, so $D_{r+1,[d,c]}$ is a pipe segment. It is enough to show $d \leqslant \lambda$. Suppose $d > \lambda$. (e2) implies $d > c'$. By (e3) $D_{r,d}$ is light and not connected below, so it is connected to the left. Going down this pipe to the left, it must turn downwards at a before reaching the at (r, λ). This contradicts the choice of c' in (e2). Therefore $d \leqslant \lambda$.

Now suppose we are not in Case L. Then $E_r(D)$ is obtained from D via a $(r, [\lambda, \rho])$-droop and possibly marking or unmarking certain tiles. Thus $E_r(D)_{r+1,[\lambda,\rho]}$ is a pipe segment. By Lemma 2.1, it remains to show $E_r(D)_{r+1,[\rho,c]}$ is a pipe segment. If not in Case P, we are done since $c = \rho$. If in Case P, we know $D_{r+1,[\rho,c]}$ is a pipe segment, and hence so is $E_r(D)_{r+1,[\rho,c]}$. $\square$

Now we check $Y = (E_r(D), (r, \lambda))$ is an F-target.

(f1) By (e3) for $(D, (r + 1, c))$, there are no heavy tiles in $D_{r,(c,n]}$, or equivalently $E_r(D)_{r,(c,n]}$. By Lemma 4.4, (r, λ) is the only heavy tile in $E_r(D)_{r,[c',c]}$. Thus, there are no heavy tiles in $E_r(D)_{r,(\lambda,n]}$.

(f2) Assume $(D, (r+1, c))$ is not in case I. We show Y satisfies (f2) by checking $(r+1, c)$ is the leftmost in $E_r(D)_{r+1,[\lambda,n]}$. By Claim 6.4, we know there are no in $E_r(D)_{r+1,[\lambda,c]}$. Then by Lemma 4.4, $E_r(D)_{r+1,c} =$.

(f*2) Assume $(D, (r + 1, c))$ is in case I. We show Y satisfies (f*2) and verify $E_r(D)_{r,c}$ is the rightmost in row r of $E_r(D)$. By Claim 6.4, there are no in $E_r(D)_{r+1,[\lambda,c)}$. It remains to check $E_r(D)_{r,c}$ is the only in in $E_r(D)_{[r,r+1],[c,n]}$. This will imply there are no in $E_r(D)_{r+1,[c,n]}$

By (e*1) of $(r+1,c)$ in D, we know (r,c) or $(r+1,c)$ is the only ┌ in $D_{[r,r+1],[c,n]}$. If $D_{r,c} =$ ┌, we know $D_{r,[c',c]}$ is not a pipe segment, so we are in Case L. Then $E_r(D)_{r,c}$ remains the only ┌ in $E_r(D)_{[r,r+1],[c,n]}$. If $D_{r+1,c} =$ ┌, we know $D_{r,c} =$ ⊟. After the $(r,[\lambda,c])$-droop, $D_{r,c}$ becomes the only ┌ in $E_r(D)_{[r,r+1],[c,n]}$.

(f3) By (e1) or (e*1) for $(D,(r+1,c))$, there are no heavy tiles in $D_{r+1,[1,c')}$, or equivalently $E_r(D)_{r,[1,c')}$. By Lemma 4.4, there are no heavy tiles in $E_r(D)_{r+1,[c',c]}$.

Thus, Y is an F-target. Its window is $[r,r+1]\times[b,c]$ for some b. Next we go through all the cases of $(D,(r+1,c))$, assuming they hold and proving that the corresponding case of Y holds. For example, in the proof P $\to$ D, we assume $(D,(r+1,c))$ is in Case P and prove that Y is in Case D. In the last three proofs, we also check the window of Y is the same as $(D,(r+1,c))$ by showing $b = c'$.

I $\to$ T: Already checked above. In fact it was shown above that I $\leftrightarrow$ T.

P $\to$ D: We already know T does not hold. By the definition of Case P, $D_{r,c} =$ ┼. Whether or not E_r performs an $(r,[\lambda,\rho])$-droop, $E_r(D)_{r,c}$ is still ┼ since $\rho < c$. By Lemma 3.1 (4), Y is in Case D.

Ɍ $\to$ O: Again we know that T does not hold. By the definition of Case Ɍ, $D_{r,c} \neq$ ┼. Whether or not E_r performs an $(r,[\lambda,\rho])$-droop, $E_r(D)_{r,c}$ is still not a ┼. By Lemma 3.1 (4), Y is in Case O.

S $\to$ B: In this case, we have $\lambda = c'$ by definition. Since $D_{r,\lambda} =$ ┌, $E_r(r,\lambda) =$ □ after the $(r,[\lambda,\rho])$-droop. Thus, Y is in Case B. In this case, b is defined as $\lambda = c'$.

D $\to$ C: We know $D_{r,[c',c]}$ is a pipe segment and $D_{r+1,\lambda} =$ ┘, so $D_{r,\lambda} =$ ┼. After making the $(r,[\lambda,\rho])$-droop and marking (r,λ), we know $E_r(D)_{r,\lambda} =$ ⊡. Since $D_{r,c'} =$ ┌, we know $E_r(D)_{r,c'} =$ ┌ and $E_r(D)_{r,[c',\lambda]}$ is a pipe segment. Thus, $b = c'$.

By Claim 6.4, $E_r(D)_{r+1,[\lambda,c]}$ is a pipe segment. Then since $D_{r+1,[c',\lambda]}$ is a pipe segment, so is $E_r(D)_{r+1,[c',\lambda]}$. By Lemma 2.1, $E_r(D)_{r+1,[c',c]}$ is a pipe segment, so Y is in Case C.

L $\to$ Ȼ: We know $D_{r,\lambda} =$ ┘ and is marked by E_r, so $E_r(D)_{r,\lambda} =$ ⊡. By how we defined λ, $D_{r,[c',\lambda]}$ is a pipe segment. $E_r(D)_{r,[c',\lambda]}$ remains a pipe segment and $E_r(D)_{r,c'} =$ ┌, so $b = c'$.

By condition (e2), $D_{r+1,[c',c]}$ is not a pipe segment. Thus, $E_r(D)_{r+1,[c',c]}$ is not a pipe segment and Y is in case Ȼ.

Finally, we check $F_r(E_r(D)) = D$. We may assume $(D,(r+1,c))$ is not in case L. Let λ' and ρ' be the left and right droop columns of Y.

We first check that $\lambda' = \lambda$.

- Suppose $(D,(r+1,c))$ is in case S. Then $\lambda = c'$. Y is in case B so $\lambda' = c'$ is the column of the □.

- Suppose $(D, (r+1, c))$ is in case D. Then $\lambda > c'$ is minimum such that $D_{r+1,\lambda}$ is . $E_r(D)_{r,\lambda} =$ so that $\lambda' = \lambda$ is the column of the .

We check that $\rho' = \rho$.

- Suppose $(D, (r+1, c))$ is in Case I or R. Then $\rho = c$. For Y we are in Case T or O respectively, and in both cases we have $\rho' = c$, the right end column index of the window.

- Suppose $(D, (r+1, c)$ is in Case P. We just need to check $[r, r+1] \times [\rho, c]$ is a double crossing in $E_r(D)$. Since $(D, (r+1, c)$ is not in Case L, $D_{r,[c',c]}$ is a pipe segment, so $E_r(D)_{r,[\rho,c]}$ remains a pipe segment. By Claim 6.4, $E_r(D)_{r+1,[\rho,c]}$ is a pipe segment. By Lemma 4.4, $E_r(D)_{r+1,c} =$. Finally, after the $(r, [\lambda, \rho])$-droop, $E_r(D)_{r,\rho}$ can be or . Since $E_r(D)_{r,[\rho,c]}$ is a pipe segment, $E_r(D)_{r,\rho} =$. □

7. Bijection between marked bumpless pipedreams and compatible pairs

7.1. $\Phi : \mathrm{MBPD}(n) \to \mathrm{RCP}(n)$ via F-moves.

The following two simple steps may be iterated to reduce any $B \in \mathrm{RCP}(n)$ to the empty biword. Suppose B is nonempty; let (i, a) its first biletter.

(RCP1) If $i = a$ remove the first biletter (i, i) from B. Write $X_i(B)$ for the resulting biword.

(RCP2) If $i < a$, replace the first biletter of B by $(i+1, a)$. Call $\uparrow B$ the result of this operation.

We define the left inverse operations.

(RRCP1) Let $I_i(B)$ be the result of prepending (i, i) to B.

(RRCP2) If (i, a) is the first biletter of B and $i > 1$, replace (i, a) by $(i-1, a)$. Call the result of this operation $\downarrow B$.

The bijection $\Phi : \mathrm{MBPD}(n) \to \mathrm{RCP}(n)$ is defined by specifying the operations on marked bumpless pipedreams which correspond to the above operations on biwords.

(0) If $D = D_{\mathrm{id}}$ then $\Phi(D) = ()$ is the empty sequence of biletters.

Otherwise D has a heavy tile. Let D have maximum F-target (i, j). By inductive hypothesis $\Phi(D)$ has the form $((i, a), (i_2, a_2), \dots)$ for some $a \geqslant i$.

(1) Suppose D is F-terminal. Then define $\Phi(D) = I_i \Phi(f_i^*(D))$.

(2) Suppose D is F-nonterminal. Then $f_i(D)$ has a single heavy tile in row $i+1$ and none in later rows, so its maximum F-target is in row $i+1$. By induction we have defined $\Phi(f_i(D)) = ((i+1, a), (i_2, a_2), (i_3, a_3), \dots)$ with $i+1 \leqslant a$. Define $\Phi(D) = \downarrow \Phi(f_i(D)) = ((i, a), (i_2, a_2), \dots)$.

$\mathrm{MBPD}(n)$ is the disjoint union of $\{D_{\mathrm{id}}\}$, the singleton containing the unique MBPD with no heavy tiles, the set $\mathrm{MBPD}(n)_{\mathrm{T}}$ of MBPDs whose maximal F-target is terminal, and the set $\mathrm{MBPD}(n)_{\overline{\mathrm{T}}}$ of MBPDs whose maximal F-target is nonterminal. We assume the maximal F-target is in row i. The terminal case is given by the following commutative diagram.

$$\begin{tikzcd} \mathrm{MBPD}(n)_{\mathrm{T}} \arrow[r, "\Phi"] \arrow[d, "f_i^*"'] & \mathrm{RCP}(n) \\ \mathrm{MBPD}(n) \arrow[r, "\Phi"'] & \mathrm{RCP}(n) \arrow[u, "I_i"'] \end{tikzcd}$$

The nonterminal case is given by the diagram

$$\begin{tikzcd} \mathrm{MBPD}(n)_{\overline{\mathrm{T}}} \arrow[r, "\Phi"] \arrow[d, "f_i"'] & \mathrm{RCP}(n) \\ \mathrm{MBPD}(n) \arrow[r, "\Phi"'] & \mathrm{RCP}(n) \arrow[u, "\downarrow"'] \end{tikzcd}$$

We group several f and f^* moves into a single operation called “row pop”. Then the computation of $\Phi(D)$ can be broken into a sequence of row pops. Starting with D with maximum target in row i, we apply f_i, then f_{i+1}, and so on, as long as the result is not terminal. For some $a \geqslant i$ the result $f_{a-1}\cdots f_{i+1}f_i(D)$ is terminal. Then we apply f_a^*. Let $\nabla_r D = f_a^* f_{a-1}\cdots f_{i+1}f_i(D)$. Then $\Phi(D) = (i,a)\cdot\Phi(\nabla_r D)$ which means prepending the biletter (i,a) to the biword $\Phi(\nabla_r D)$. We introduce the notation $\mathrm{rpop}(D) = (i,a)$ for this biletter. We call the map $D \mapsto (\mathrm{rpop}(D), \nabla_r D)$ *row pop*. Iterating row pop produces the biletters of $\Phi(D)$ from left to right.

Let $\mathbb{B}$ be the set of biletters. We have a commutative diagram

$$\begin{tikzcd} \mathrm{MBPD}(n) \setminus \{D_{\mathrm{id}}\} \arrow[r, dashed, "\Phi"] \arrow[d, "\mathrm{rpop} \times \nabla_r"'] & \mathrm{RCP}(n) \setminus \varnothing \\ \mathbb{B} \times \mathrm{MBPD}(n) \arrow[r, "\mathrm{id}_{\mathbb{B}} \times \Phi"'] & \mathbb{B} \times \mathrm{RCP}(n) \arrow[u, "\text{prepend}"'] \end{tikzcd}$$

Remark 7.1. To define the bijection $\Phi : \mathrm{MBPD}(n) \to \mathrm{RCP}(n)$ we only need to define moves using the maximum F-target (see §3.1). However to prove that the image biwords satisfy the compatibility condition, we find it convenient to use more general moves, which are defined in terms of F-targets which need not be maximum.

7.2. Well-definedness of Φ

We first prepare a lemma that addresses a commutativity property of f-moves.

Lemma 7.2. *If D has an F-target in both row i and $i+1$, then $F_iF_{i+1}(D) = F_{i+1}F_i(D)$.*

Proof. We note that the two moves only look at and make changes within their own windows, which are necessarily disjoint by the definition of F-target. □

Proposition 7.3. *The map* $\Phi : \mathrm{MBPD}(n) \to \mathrm{RCP}(n)$ *is well-defined and weight-preserving.*

Proof. Let $D \in \mathrm{MBPD}(n)$. The proof proceeds by induction on the number of heavy tiles in D and then on the number of operators f_k and f_k^* that must be applied to compute $\Phi(D)$.

We may assume D has at least one heavy tile. Let D have F-target at (i, j).

For every move f_r in the definition of Φ, the new heavy tile created by applying f_r is always southeast of the target. Thus the new target has fewer 's in the southeast. It follows that the terminal case must eventually be reached: D admits a unique sequence of operators of the form $f_a^* f_{a-1} \cdots f_{i+1} f_i$ for some $a \geqslant i$. Moreover all moves do not disturb heavy tiles in previous rows or columns.

Suppose D has only one heavy tile. Then after applying F, $F(D)$ has no heavy tiles: $F(D) = D_{\mathrm{id}}$. There is a unique diagram which f_a^* sends to D_{id}, namely, the Rothe BPD D_{s_a} for the simple reflection s_a. There is also a unique diagram that f_{a-1} sends to D_{s_a}. Working backwards there is one possibility for D: a single blank at (i, i), with 's at (j, j) for $i < j \leqslant a$. We get $\Phi(D) = ((i, a))$.

We now assume D has at least two heavy tiles. $\Phi(D)$ must start with (i, a) for some $a \geqslant i$. Suppose first D only has one heavy tile in row i. Let $i' < i$ be maximum such that D has a heavy tile in row i'. This tile is unchanged during the computation of $\nabla_r D = f_a^* f_{a-1} \cdots f_{i+1} f_i D$. Therefore $\nabla_r D$ has a target in row i'. Hence the second biletter in $\Phi(D)$ has the form (i', a') for some $a' \geqslant i'$. Since $i' < i$ the compatibility condition is satisfied between the first and second biletters. By the induction hypothesis $\Phi(\nabla_r D) \in \mathrm{RCP}(n)$. Therefore $\Phi(D) \in \mathrm{RCP}(n)$.

We may assume D has at least two heavy tiles in row i. Let $j' < j$ be maximum such that $D_{i,j'}$ is heavy. During the computation of $\nabla_r D$ the (i, j') tile remains unchanged. It follows that the second biletter of $\Phi(D)$ has the form (i, a') for some $a' \geqslant i$. We must show $a < a'$. This is equivalent to showing that D admits the operator

$$\varphi_{i,a} := f_a f_{a-1} \cdots f_i f_a^* f_{a-1} \cdots f_i \qquad \text{for } a \geqslant i. \tag{7.1}$$

We first argue that $F_i(D)$ admits f_i; that is, $F_i(D)$ with the maximal F-target (i, j') is F-nonterminal. Since $D_{i,j'}$ is or , it must be the case that $D_{i+1,j'}$ is or . If $D_{i+1,[j',n]}$ is a pipe segment, then $(D, (i, j))$ must be in case TB or TC. In these cases, $f_i^*(D)_{i+1,j} =$. It follows that (i, j') is the maximal f-target in $f_i^*(D)$, and hence F-nonterminal. Otherwise there exists $k > j'$ smallest such that $D_{i+1,k} =$. If $k > j$ then $(D, (i, j))$ is in Case D or O. In these cases $f_i(D)_{i+1,j} =$, which implies $(f_i(D), (i, j'))$ is nonterminal. If $j' < k < j$ then this tile is not changed by F_i on D, so $F_i(D)$ is F-nonterminal.

The argument above shows that if D is F-terminal (the case that $a = i$), then D admits $\varphi_{i,a} = \varphi_{i,i} = f_i f_i^*$.

Now suppose D is F-nonterminal. To prove that D admits $\varphi_{i,a}$ there are two cases depending on whether $f_i(D)$ is terminal or not. Since $f_i f_i D$ has two heavy tiles in row $i + 1$ and none in later rows, we may assume that $\varphi_{i+1,a}$ applies to $f_i f_i D$.

Case $f_i(D)$ is terminal . In this case $a = i + 1$ and $\varphi_{i,i+1} = f_{i+1} f_i f_{i+1}^* f_i$. By induction $\varphi_{i+1,i+1}$ applies to $f_i f_i D$. By Lemma 7.2 we have

$$\begin{aligned} \varphi_{i+1,i+1} f_i f_i D &= f_{i+1} f_{i+1}^* f_i f_i D \\ &= f_{i+1} f_i f_{i+1}^* f_i D = \varphi_{i,i+1} D \end{aligned}$$

and $\varphi_{i,i+1}$ applies to D as required.

Case $f_i(D)$ is nonterminal . Here $a \geqslant i+2$. By induction $\varphi_{i+1,a}$ applies to $f_i f_i D$. By Lemma 7.2 we have

$$\begin{aligned}\varphi_{i+1,a} f_i f_i D &= f_a f_{a-1} \cdots f_{i+1} f_a^* f_{a-1} \cdots f_{i+1} f_i f_i D \\ &= f_a f_{a-1} \cdots f_{i+1} f_a^* f_{a-1} \cdots f_i f_{i+1} f_i D \\ &= f_a f_{a-1} \cdots f_{i+1} f_i f_a^* f_{a-1} \cdots f_{i+1} f_i D \\ &= \varphi_{i,a} D\end{aligned}$$

as required, using the obvious commutativity of f_i with operators that don't change rows i and $i+1$. $\square$

7.3. $\Psi : \mathrm{RCP}(n) \to \mathrm{MBPD}(n)$ via E-moves

Recall the operations I_i, X_i, $\uparrow$ and $\downarrow$ on $\mathrm{RCP}(n)$ defined in §7.1. Define the map $\Psi : \mathrm{RCP}(n) \to \mathrm{MBPD}(n)$ as follows. Ψ sends the empty biword to D_{id}. Let $B = ((i,a),(i_2,a_2),(i_3,a_3),\dots) \in \mathrm{RCP}(n)$ be nonempty. If $i = a$ let $\Psi(B) = e_i^*(\Psi(X_i(B))) = e_i^*(\Psi((i_2,a_2),(i_3,a_3),\dots))$. Otherwise let $\Psi(B) = e_i(\Psi(\uparrow B)) = e_i(\Psi((i+1,a),(i_2,a_2),(i_3,a_3),\dots))$. Similar to the situation for Φ, the computation of the bijection Ψ can be broken into coarser operations. Let $B = ((i,a),(i_2,a_2),\dots) \in \mathrm{RCP}(n)$. By induction we have computed $D' := \Psi((i_2,a_2),(i_3,a_3),\dots) \in \mathrm{MBPD}(n)$. Then $D := \Psi(B) = e_i \cdots e_{a-2} e_{a-1} e_a^*(D')$. The operation going from D' to D is called the *row push* of (i,a) into D'.

Let $B = ((i_1,a_1),(i_2,a_2),\dots,(a_\ell,i_\ell)) \in \mathrm{RCP}(n)$. Starting with $D_{\mathrm{id}} \in \mathrm{MBPD}(n)$, performing row push starting with (a_ℓ,i_ℓ) and iterating through (i_1,a_1), we obtain $\Psi(B) = D \in \mathrm{MBPD}(n)$.

7.4. Well-definedness of Ψ

Lemma 7.4. *Suppose $D \in \mathrm{MBPD}(n)$ has no heavy tiles in rows r.*

1. *If D has no heavy tiles in row $r+1$, then $e_r^*(D)$ is defined.*
2. *If D has a single heavy tile $D_{r+1,d}$ in row $r+1$, then $e_r(D)$ is defined.*

Proof. For (1), let d be maximum such that either $D_{r,d} =$ or $D_{r+1,d} =$. We need to verify that $(D,(r+1,d))$ is an e^*-target. The conditions (e*1) and (e3) are straightforward, so we check (e2).

Suppose $D_{r,d} =$. Let ρ be maximum such that $D_{r+1,\rho} =$; necessarily $\rho < d$. $D_{r,\rho}$ is light and does not connect downwards so it must connect to the left. Therefore there is a $d' < \rho$ such that $D_{r,d'} =$; take d' maximum.

Suppose $D_{r+1,d} =$. $D_{r,d}$ is light and does not connect downwards so it connects to the left. Therefore there is a $d' < d$ such that $D_{r,d'} =$; take d' maximum.

For (2), we verify that $(D,(r+1,d))$ is an e-target and we only need to check (e2). If $D_{r+1,d} =$, then $D_{r,d} =$ or . Following this pipe to the left, we will find some $d' < d$ maximal such that $D_{r,d'} =$. It follows that $(r+1,d)$ is an e-target. If $D_{r+1,d} =$, we follow this pipe to the left and find $d' < d$ such that $D_{r+1,d'} =$. Then $D_{r,d} =$ or . Following this pipe to the left we will find $d'' < d$ maximal such that $D_{r,d''} =$. □

Remark 7.5. In Lemma 7.4 the window of the e^*-move is $[r, r+1] \times [d', d]$. The assumption that there are no heavy tiles in row r, was used to rule out heavy tiles in the range $D_{r,(d',d)}$.

Lemma 7.6. *Suppose (r,c) is an f-target in $D \in \mathrm{MBPD}(n)$. Then*

(1) If there are no heavy tiles in row $r+1$, then row $r+1$ has an e^*-target at $(r+1,d)$ for some $d > c$. If (r,c) is also an e-target, then $D' := e_{r-1}e_r^*(D) = e_r^*e_{r-1}(D)$.

(2) If $(r+1,d)$ is the leftmost heavy tile in row $r+1$ with $d > c$, then it is an e-target. If (r,c) is also an e-target, then $D' := e_{r-1}e_r(D) = e_re_{r-1}(D)$.

For both (1) and (2), suppose D'_{r-1,c_1} and D'_{r,d_1} are the new heavy tiles in row $r-1$ and r as compared to D, then $c_1 < d_1$.

Proof. Since (r,c) is an f-target, by (f1), it is the rightmost heavy tile in row r; by (f2), there is a minimal $c' > c$ such that $D_{r+1,c'} =$; by (f3) all $D_{r+1,j}$ are light for $j < c'$.

For (1), we make the same choices as in Lemma 7.4 to construct d and d'. We notice that necessarily $c < d'$. The property (f1) for the F-target $(D,(r,c))$ implies that $(D,(r+1,d))$ is an e^*-target. If (r,c) is also an e-target, the window for the e-target $(D,(r,c))$ has (r,c) as the bottom right corner, whereas the top left corner of the window for the e-target $(D,(r+1,d))$ is (r,d') so these two windows are necessarily disjoint, implying the desired commutativity as well as the last statement, since the heavy targets are moved within the windows.

For (2), if $D_{r+1,d} =$, then $D_{r,d}$ is or . It cannot be since (r,c) is the rightmost heavy tile in row r. The rest of the argument for this case is similarly to (1) when $D_{r+1,d} =$. If $D_{r+1,d} =$, we follow this pipe to the left and find $D_{r+1,d'} =$ with $d' < d$ largest. The rest of the argument is similar to (1) when $D_{r,d} =$. □

Theorem 7.7. *The map $\Psi : \mathrm{RCP}(n) \to \mathrm{MBPD}(n)$ is well-defined and weight-preserving.*

Proof. Let $B \in \mathrm{RCP}(n)$. We proceed by induction on the length of B, and then on the number of operators e_k and e_k^* that must be applied to compute $\Psi(B)$.

Suppose B has length 1. If $B = ((a,a))$, $\Psi(B) = e_a^*(D_{\mathrm{id}}) = D_{s_a}$. If $B = ((i,a))$ for $1 \leqslant i < a$, then $\Psi(B) = e_ie_{i+1}\cdots e_{a-1}e_a^*$ is the unique bumpless pipedream for s_a with a single blank tile in row i.

We now assume that $B = ((i,a),(i_2,a_2),\cdots)$ has length at least 2. Let $B' := ((i_2,a_2),\cdots)$ be B without the first biletter.

Suppose first that $i > i_2$. By the induction hypothesis, $D' := \Psi(B')$ is well-defined and all its heavy tiles are at or above row i_2.

If $i = a$, we check that the conditions for applying e_a^* on D' must hold. This follows from Lemma 7.4(1), since there are no heavy tiles in row a and $a+1$.

If $i < a$, we check that the conditions for applying e_i on $\Psi(\uparrow B)$ must hold. By induction hypothesis, $\Psi(\uparrow B)$ is defined, and contains a single heavy tile $D_{i+1,d}$ in row $i+1$, and all other heavy tiles are in rows $\leqslant i_2$. In particular, there are no heavy tiles in row i. That $e_i(\Psi(\uparrow B))$ is defined follows from Lemma 7.4(2).

Now suppose $i = i_2$. Since $B \in \mathrm{RCP}$, we must have $i \leqslant a < a_2$. If $i = a$, by the induction hypothesis $\Psi(B')$ is defined, all heavy tiles are in rows $\leqslant i$, and $\Psi(B') = e_i\Psi(((i+1, a_2), \cdots))$. Since f_i and e_i are inverses, $\Psi(B')$ contains an f-target in row i. Since there are no heavy tiles in row $i+1$, by Lemma 7.6(1), row $i+1$ of $\Psi(B')$ has an e^*-target.

If $i < a$, by induction hypothesis and the definition of Ψ, $\Psi(\uparrow B)$ is defined and

$$\Psi(\uparrow B) = e_{i+1}e_{i+2}\cdots e^*_a e_i \cdots e_{a-1}e_a D^{a+1}$$

for some $D_{a+1} \in \mathrm{MBPD}(n)$, where D_{a+1} has a single heavy tile in row $a+1$ and all other heavy tiles are in rows $\leqslant i$. By the apparent commutativity of the operators indexed by non-adjacent numbers, we have

$$\Psi(\uparrow B) = e_{i+1}e_i e_{i+2}e_{i-1}\cdots e_{a-1}e_{a-2}e^*_a e_{a-1}e_a D^{a+1}.$$

(Colors are used for visual aid only.) Since f_a and e_a are inverses, $e_a D^{a+1}$ has an f-target in row a, say at (a, c_a). Since there are no heavy tiles below row a in $e_a D^{a+1}$, $e_a D^{a+1}$ has an e^*-target at $(a+1, d_{a+1})$ for some $d_{a+1} > c_a$. Since $e_{a-1}e_a D^{a+1}$ is defined, (a, c_a) is also an e-target in $e_a D^{a+1}$. Since there are no heavy tiles below row a in $e_a D^{a+1}$, by Lemma 7.6(1), we have

$$D^a := e^*_a e_{a-1}e_a D^{a+1} = e_{a-1}e^*_a e_a D^{a+1}.$$

Furthermore, if e_{a-1} moved the heavy tile $(e_a D^{a+1})_{a,c_a}$ to $D^a_{a-1,c_{a-1}}$ and e^*_a created a heavy tile D^a_{a,d_a}, then $c_{a-1} < d_a$.

Now notice $(D^a, (a-1, c_{a-1}))$ is an f-target and $(D^a, (a, d_a))$ is an e-target. If $i < a-1$ then $(D^a, (a-1, c_{a-1}))$ is also an e-target, we can then apply Lemma 7.6(2) and get

$$D^{a-1} := e_{a-1}e_{a-2}D^a = e_{a-2}e_{a-1}D^a.$$

which shows that D^{a-1} admits f_{a-2}. Furthermore, if e_{a-2} moved the heavy tile $D^a_{a-1,c_{a-1}}$ to $D^{a-1}_{a-2,c_{a-2}}$ and e_{a-1} moved D^a_{a,d_a} to $D^{a-1}_{a-1,d_{a-1}}$, then $c_{a-1} < d_{a-1}$. Keep applying Lemma 7.6(2) and continue the same reasoning, eventually we can conclude that $D^{i+1} = \Psi(\uparrow B)$ admits f_i, contains a single heavy tile in row $i+1$ with column index larger than the f-target in row i. Finally by Lemma 7.6(2) again we conclude that $\Psi(\uparrow B)$ admits e_i. $\square$

7.5. Proof of the main theorem

Equipped with the well-definedness of Φ and Ψ, we are ready to prove our main theorem.

Proof of Theorem 1.2. To show $\Psi \circ \Phi = \mathrm{id}$, we proceed by induction on the number of F-operators that can be applied on $D \in \mathrm{MBPD}(n)$. The base case is apparent. If $D \neq D_{\mathrm{id}}$ is F-terminal, then

$$\Psi(\Phi(D)) = \Psi(I_i\Phi(f^*_i(D))) = e^*_i(\Psi(\Phi(f^*_i(D)))) = e^*_i(f^*_i(D)) = D.$$

If $D \neq D_{\mathrm{id}}$ is F-nonterminal, then

$$\begin{aligned}\Psi(\Phi(D)) &= \Psi(\downarrow \Phi(f_i(D))) = e_i(\Psi(\uparrow\downarrow \Phi(f_i(D)))) \\ &= e_i(\Psi(\Phi(f_i(D)))) = e_i(f_i(D)) = D.\end{aligned}$$

Similarly, to show $\Phi \circ \Psi = \mathrm{id}$, we proceed by induction on the number of X and $\uparrow$ operators that can be applied on $B \in \mathrm{RCP}(n)$. If $B = ((i,i),(i_2,a_2),\dots) \in \mathrm{RCP}(n)$, then

$$\Phi(\Psi(B)) = \Phi(e_i^*(\Psi(X_i(B)))) = I_i(\Phi(f_i^* e_i^*(\Psi(X_i(B))))) = B.$$

If $B = ((i_1,a_1),(i_2,a_2),\dots) \in \mathrm{RCP}(n)$, with $i_1 < a_1$, then

$$\Phi(\Psi(B)) = \Phi(e_i(\Psi(\uparrow (B)))) = \downarrow (\Phi(f_i e_i(\Psi(\uparrow B)))) = B.$$

It then follows that Φ is indeed a bijection.

We proceed again induction on the number of F-operators that can be applied to D to show that Φ is permutation-preserving. If $D \neq D_{\mathrm{id}}$ is F-terminal, then

$$w(\Phi(D)) = w(I_i\Phi(f_i^*(D))) = w(\Phi(f_i^*(D))) * s_i = w(f_i^*(D)) * s_i = w(D)$$

where the last step is by Proposition 3.2. Finally, if $D \neq D_{\mathrm{id}}$ is F-nonterminal,

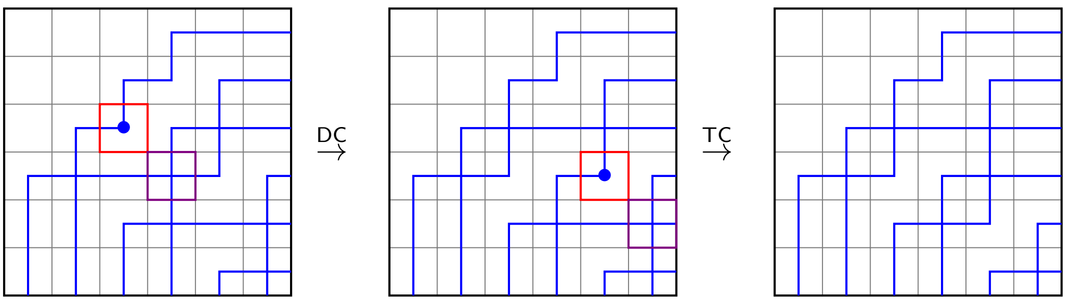
$$w(\Phi(D)) = w(\downarrow \Phi(f_i(D))) = w(\Phi(f_i(D))) = w(f_i(D)) = w(D).$$

That Ψ is permutation-preserving follows from it being the inverse of Φ.

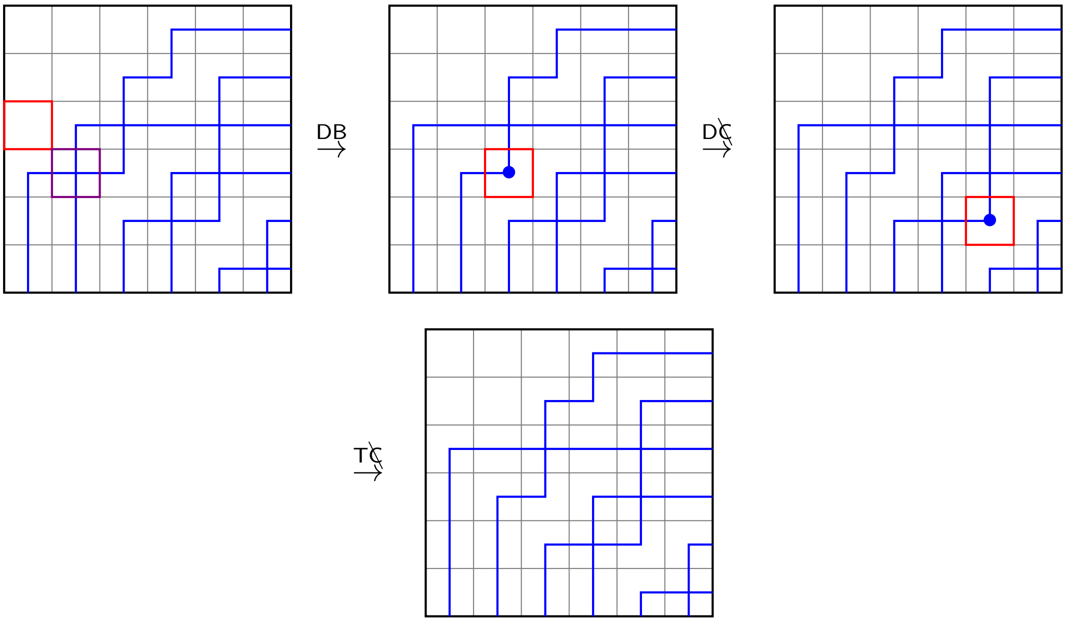
Finally, when restricted to the reduced case, the only relevant cases for Φ are TB and OB, and the corresponding relevant cases for Ψ are IS and $\cancel{\mathsf{R}}$S. □

8. An example

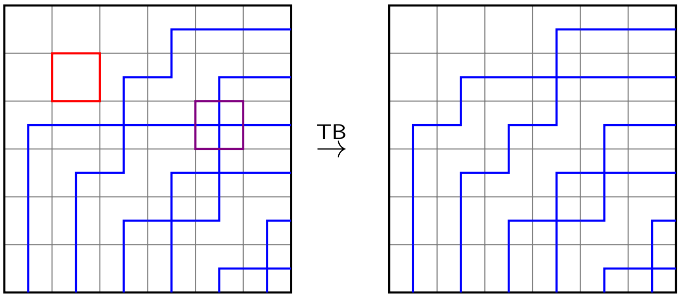
In the following diagrams the red square is the target and the violet square is the opposite corner of the undroop.



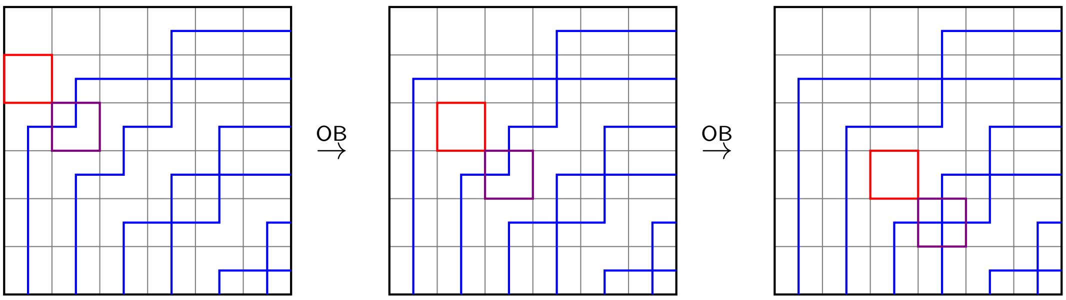
The first row pop produces the pair $(3,4)$.

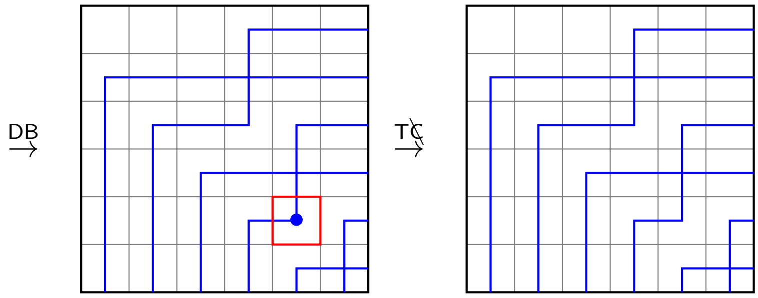


The second row pop yields $(3, 5)$.

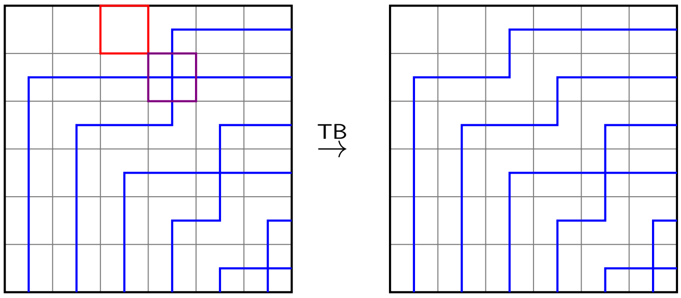


The third row pop produces $(2, 2)$.



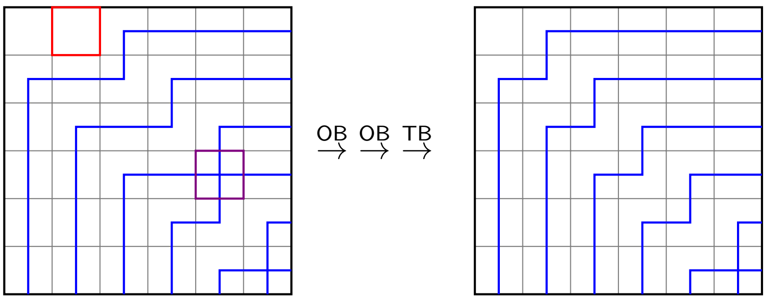


This row pop yields $(2, 5)$.



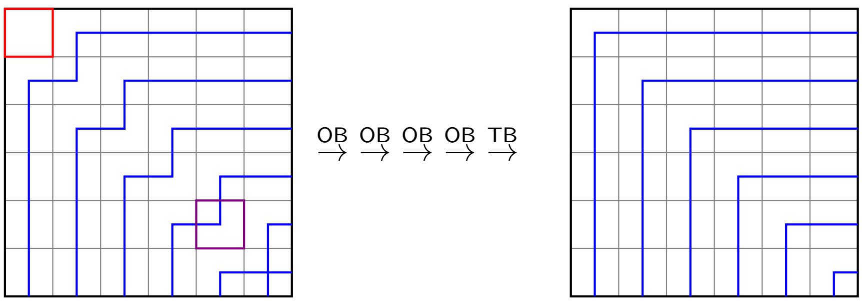
This row pop yields $(1, 1)$.

We do the entire next row pop, which consists of three f-moves.



This row pop yields $(1, 3)$.

We do the final row pop in one step.

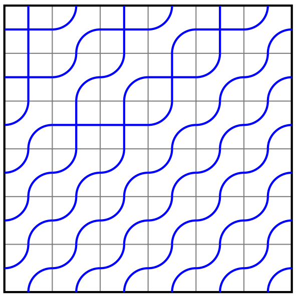


This row pop yields $(1, 5)$. The final compatible pair is

$$B = ((3, 4), (3, 5), (2, 2), (2, 5), (1, 1), (1, 3), (1, 5)).$$

By Remark 1.1 B corresponds to the PD having $\boxplus$'s at positions

$$((3,2),(3,3),(2,1),(2,4),(1,1),(1,3),(1,5)).$$



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References

[BB93] Nantel Bergeron and Sara Billey. RC-graphs and Schubert polynomials. *Experiment. Math.*, 2(4):257–269, 1993. URL: `http://projecteuclid.org/euclid.em/1048516036`, `doi:10.1080/10586458.1993.10504567`.

[BGG73] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand. Schubert cells, and the cohomology of the spaces G/P. *Uspehi Mat. Nauk*, 28(3(171)):3–26, 1973. `doi:10.1070/RM1973v028n03ABEH001557`.

[BJS93] Sara C. Billey, William Jockusch, and Richard P. Stanley. Some combinatorial properties of Schubert polynomials. *J. Algebraic Combin.*, 2(4):345–374, 1993. `doi:10.1023/A:1022419800503`.

[Dem74] Michel Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. URL: `http://www.numdam.org/item?id=ASENS_1974_4_7_1_53_0`.

[EKLP92] Noam Elkies, Greg Kuperberg, Michael Larsen, and James Propp. Alternating-sign matrices and domino tilings. I. *J. Algebraic Combin.*, 1(2):111–132, 1992. `doi:10.1023/A:1022420103267`.

[FKsd] Sergey Fomin and Anatol N. Kirillov. Grothendieck polynomials and the Yang-Baxter equation. In *Formal power series and algebraic combinatorics/Séries formelles et combinatoire algébrique*, pages 183–189. DIMACS, Piscataway, NJ, sd.

[GH23] Yibo Gao and Daoji Huang. The canonical bijection between pipe dreams and bumpless pipe dreams. *Int. Math. Res. Not. IMRN*, 2023(21):18629–18663, 2023. `doi:10.1093/imrn/rnad083`.

[HS24] Daoji Huang and Jessica Striker. A pipe dream perspective on totally symmetric self-complementary plane partitions. *Forum Math. Sigma*, 12:Paper No. e17, 19, 2024. doi:10.1017/fms.2023.131.

[Hud14] Thomas Hudson. A Thom-Porteous formula for connective K-theory using algebraic cobordism. *J. K-Theory*, 14(2):343–369, 2014. doi:10.1017/is014005031jkt266.

[KM05] Allen Knutson and Ezra Miller. Gröbner geometry of Schubert polynomials. *Ann. of Math.*, 161(3):1245–1318, 2005. doi:10.4007/annals.2005.161.1245.

[Kup96] Greg Kuperberg. Another proof of the alternating-sign matrix conjecture. *Internat. Math. Res. Notices*, 1996(3):139–150, 1996. doi:10.1155/S1073792896000128.

[Las08] Alain Lascoux. Chern and yang through ice. 2008. URL: https://www.math.uwaterloo.ca/~opecheni/lascoux-ice.pdf.

[LLS21] Thomas Lam, Seung Jin Lee, and Mark Shimozono. Back stable Schubert calculus. *Compos. Math.*, 157(5):883–962, 2021. doi:10.1112/S0010437X21007028.

[LS82] Alain Lascoux and Marcel-Paul Schützenberger. Structure de Hopf de l'anneau de cohomologie et de l'anneau de Grothendieck d'une variété de drapeaux. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(11):629–633, 1982.

[MRR83] W. H. Mills, David P. Robbins, and Howard Rumsey, Jr. Alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 34(3):340–359, 1983. doi:10.1016/0097-3165(83)90068-7.

[Wei21] Anna Weigandt. Bumpless pipe dreams and alternating sign matrices. *J. Combin. Theory Ser. A*, 182:Paper No. 105470, 52, 2021. doi:10.1016/j.jcta.2021.105470.