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RESEARCH PAPER

ON A METHOD OF SOLUTION OF SYSTEMS OF FRACTIONAL PSEUDO-DIFFERENTIAL EQUATIONS

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Abstract

This paper is devoted to the general theory of linear systems of fractional order pseudo-differential equations. Single fractional order differential and pseudo-differential equations are studied by many authors and several monographs and handbooks have been published devoted to its theory and applications. However, the state of systems of fractional order ordinary and partial or pseudo-differential equations is still far from completeness, even in the linear case. In this paper we develop a new method of solution of general systems of fractional order linear pseudo-differential equations and prove existence and uniqueness theorems in the special classes of distributions, as well as in the Sobolev spaces.

MSC 2010: Primary 35E15, 35R11; Secondary 35S10, 33E12

Key Words and Phrases: fractional system of differential equations; system of differential equations; fractional order differential equation; pseudo-differential operator; matrix symbol; solution operator; Mittag-Leffler function

1. Introduction

In the last few decades, fractional order differential equations have proved to be an essential tool in the modeling of dynamics of various complex stochastic processes arising in anomalous diffusion in physics [17, 33, 35, 52], finance [31, 45], hydrology [6], cell biology [32], and other fields of modern science and engineering. The complexity of stochastic processes

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includes phenomena such as the presence of weak or strong correlations, different sub- or super-diffusive modes and jump effects.

Various versions of fractional order differential and pseudo-differential equations are studied by many authors and several books have been published (see e.g. [21, 25, 40, 44, 47, 48]). However, the state of systems of fractional order ordinary and partial differential equations is still far from completeness, even in the linear case. At the same time systems of fractional order ordinary and partial differential equations have rich applications. For example, they are used in modeling of processes in biosystems [8, 15, 43], ecology [20, 42], epidemiology [19, 53], etc.

For some nonlinear systems of fractional order ordinary differential equations numerical and analytic approximate solution methods are developed; see e.g. [1, 2, 11, 36, 49, 51]. Many applied processes can be modeled by by-linear systems of fractional differential equations, including COVID-19 pandemic [3, 14, 28, 38, 41]. The advance of fractional order modeling is it adds parameters controlling effects like memory and correlations, leading to a better analysis and prediction.

In the linear case obtaining a representation for the solution is also possible. For example, in the paper [7], the authors prove existence and uniqueness of the system

$$D^{\alpha}[x(t) - x(0)] = Ax(t), \quad x(0) = x_0,$$

of time-fractional ordinary differential equations, where $x(t)$ is a vector-function, A is a nonsingular matrix, and $\alpha \in (0,1)$ is scalar, with the solution representation $x(t) = E_{\alpha}(t^{\alpha}A)x_0$. Here $E_{\alpha}(\mathcal{Z})$ is the matrix-valued Mittag-Leffler function of a matrix $\mathcal{Z}$. The paper [37] studies stability conditions for the system $D^{\alpha}u(t) = Au(t)$ of fractional order ordinary differential equations with a vector-order α, with components $\alpha_j \in (0,1),\ j = 1, \dots, m$.

More general cases of linear systems of the form $D^{\alpha}Lu(t) = Mu(t)$, where L and M are linear operators from a Banach space to another Banach space, were also considered. Gordievskikh and Fedorov [13] studied the Cauchy problem for degenerate operator L, that is $KerL \neq 0$. Regular case of the invertible operator L was studied in [4, 29]. Mamchuev [34] studied the boundary value problem for the fractional order system of the form

$$\sum_{i=1}^{m} A_i D_{x_i}^{\alpha_i} u(x) = Bu(x) + f(x),$$

with boundary conditions

$$D_{x_i}^{\alpha_i - 1} u(x)|_{x_i=0} = \phi(x_1, \dots x_{i-1}, x_{i+1}, \dots x_n), \ \ i = 1, \dots, m.$$

Here $A_i, i = 1, \dots m$, and B are $n \times n$-matrices and D^α is the Riemann-Liouville derivative. The existence and uniqueness theorem is obtained as well as a representation formula for the solution through the Green function.

An important aspect of systems with integer order derivatives is that one can reduce such a system to a first order system increasing the number of equations/unknowns. In general, this approach loses its meaning in the case of fractional order systems, though as shown in [10] in some cases the systems with distinct fractional orders can be reduced to a system with the same fractional order in each equation. However, in this case, on the one hand the orders of the original system assumed to be rational, and on the other hand the number of equations in the reduced system may increase significantly. For example, if the orders in the original system of 4 equations are $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$, and $\frac{1}{7}$, then the reduced system will contain 247 equations of order $\frac{1}{210}$. Therefore, developing the direct general techniques for solution and qualitative analysis of systems of fractional order differential equations with any positive real orders is important.

In what concerns systems of fractional order partial differential equations, many of them can be treated within the theory of fractional order operator-differential equations in Banach or topological-vector spaces [5, 27, 46, 47]. However, such systems are of single scalar order or distributed scalar order equations. They can not be of vector-order. Moreover, some important specific features of fractional order systems of partial differential equations , such as parabolicity or hyperbolicity properties, can not be captured by operator-differential equations. Kochubei [22, 23, 24] studied fractional (scalar) order generalizations of parabolic and hyperbolic systems and found the corresponding fundamental solutions. Vazquez and Mendes [50] and Pierantozzi [39] studied fractional (scalar) order systems of Dirac-like equations. Some other issues related to fractional order systems, such as stability problems, numerical solution, along with others, are considered in works [9, 12, 16, 30]. The orders of systems in these works are also scalar.

In this paper we will deal with the following general system of linear fractional vector-order pseudo-differential equations

$$\begin{cases} D^{\beta_1} u_1(t,x) = A_{1,1}(D)u_1(t,x) + \dots A_{1,m}(D)u_m(t,x) + h_1(t,x), \\ D^{\beta_2} u_2(t,x) = A_{2,1}(D)u_1(t,x) + \dots A_{2,m}(D)u_m(t,x) + h_2(t,x), \\ \dots \\ D^{\beta_m} u_m(t,x) = A_{m,1}(D)u_1(t,x) + \dots A_{m,m}(D)u_m(t,x) + h_m(t,x), \end{cases} \tag{1.1}$$

where D^{β_j}, $j = 1, \ldots, m$, is the fractional order derivative of order $0 < \beta_j \leq 1$ in the sense of Riemann-Liouville or Caputo, and $A_{j,k}(D)$ are pseudo-differential operators with (possibly singular) symbols depending only on dual variables (for simplicity) and described later. The obtained results can be extended for wider classes of pseudo-differential operators with symbols depending on time and spatial variables and non-symmetric as well, but this level of extension is not a goal of this paper. The initial conditions depend on the form of fractional derivatives. The results also can be extended to the case when the orders (some or all) $\beta_j \in (1, 2]$ adjusting properly the initial conditions.

The paper is organized as follows. Section 2 provides some preliminary facts on pseudo-differential operators with constant singular symbols, on the functional spaces where these pseudo-differential operators act continuously, and on fractional calculus used in this paper. In Section 3 we present main results. Here we prove the existence and uniqueness theorems in the general form for systems of time-fractional pseudo-differential equations. The representation formulas for solutions are also obtained in this section.

2. Preliminaries and auxiliaries

In this section we introduce some auxiliary notations and facts. We briefly recall definitions and related basic facts on general pseudo-differential operators without smoothness and growth restrictions to symbols as well as elliptic pseudo-differential operators and the spaces of distributions where these operators act. For details we refer the reader to the book [47].

2.1. **Generalized function spaces $\Psi_{G,p}(\mathbb{R}^n)$, $\Psi_{-G,q}(\mathbb{R}^n)$.** Let $p > 1$, $q > 1$, $p^{-1} + q^{-1} = 1$ be two conjugate numbers. The generalized functions space $\Psi_{-G,q}(\mathbb{R}^n)$, which we are going to introduce is distinct from the classical spaces of generalized functions.

Let $G \subset \mathbb{R}^n$ be an open domain and a system $\mathcal{G} \equiv \{g_k\}_{k=0}^{\infty}$ of open sets be a locally finite covering of G, i.e., $G = \bigcup_{k=0}^{\infty} g_k$, $g_k \subset\subset G$. This means that any compact set $K \subset G$ has a nonempty intersection with a finite number of sets g_k. Denote by $\{\phi_k\}_{k=0}^{\infty}$ a smooth partition of unity for G. We set $G_N = \cup_{k=1}^{N} g_k$ and $\kappa_N(\xi) = \sum_{k=1}^{N} \phi_k(\xi)$. It is clear that $G_N \subset G_{N+1}$, $N = 1, 2, \ldots$, and $G_N \to G$ for $N \to \infty$. The support of a given f we denote by supp$f$. Further, by $F[f](\xi)$ (or $\hat{f}(\xi)$ for a given function $f(x)$ we denote its Fourier transform, and by $F^{-1}f$ the inverse Fourier transform:

$$F[f](\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{ix\xi} dx, \quad \xi \in \mathbb{R}^n,$$

and

$$F^{-1}[\hat{f}](\xi) = f(x) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \hat{f}(\xi) e^{-ix\xi} d\xi, \quad x \in \mathbb{R}^n.$$

Let $N \in \mathbb{N}$. Denote by $\Psi_{N,p}$ the set of functions $f \in L_p(\mathbb{R}^n)$ satisfying the conditions:

(1) $\mathrm{supp}\, F[f] \subset G_N$;
(2) $\mathrm{supp}\, F[f] \cap \mathrm{supp}\, \phi_j = \emptyset$ for $j > N$;
(3) $p_N(f) = \|F^{-1}\kappa_N F f\|_p < \infty$.

Lemma 2.1. *For $N = 1, 2, \dots,$ the relations*

(1) $\Psi_{N,p} \hookrightarrow \Psi_{N+1,p}$,
(2) $\Psi_{N,p} \hookrightarrow L_p(\mathbb{R}^n)$

are valid, where $\hookrightarrow$ denote the operation of continuous embedding.

It follows from Lemma 2.1 that $\Psi_{N,p}$ form an increasing sequence of Banach spaces. Its limit with the inductive topology we denote by $\Psi_{G,p}$. Thus,

$$\Psi_{G,p}(\mathbb{R}^n) = \mathrm{ind} \lim_{N \to \infty} \Psi_{N,p}. \tag{2.1}$$

The inductive limit topology of $\Psi_{G,p}(\mathbb{R}^n)$ is equivalent to the following convergence. A sequence of functions $f_m \in \Psi_{G,p}(\mathbb{R}^n)$ is said to converge to an element $f_0 \in \Psi_{G,p}(\mathbb{R}^n)$ iff:

(1) there exists a compact set $K \subset G$ such that $\mathrm{supp}\, \hat{f}_m \subset K$ for all $m \in \mathbb{N}$;
(2) $\|f_m - f_0\|_p = (\int_{R^n} |f_m - f_0|^p dx)^{\frac{1}{p}} \to 0$ for $m \to \infty$.

Remark 2.1. According to the Paley-Wiener-Schwartz theorem, elements of $\Psi_{G,p}(\mathbb{R}^n)$ are entire functions of exponential type which, restricted to $\mathbb{R}^n$, are in the space $L_p(\mathbb{R}^n)$.

The space topologically dual to $\Psi_{G,p}(\mathbb{R}^n)$, which is the projective limit of the sequence of spaces conjugate to $\Psi_{N,p}$, is denoted by $\Psi'_{-G,q}(\mathbb{R}^n)$, that is

$$\Psi'_{-G,q}(\mathbb{R}^n) = \mathrm{pr} \lim_{N \to \infty} \Psi^*_{N,p}. \tag{2.2}$$

In other words, $\Psi'_{-G,q}(\mathbb{R}^n)$ is the space of all linear bounded functionals defined on the space $\Psi_{G,p}(\mathbb{R}^n)$ endowed with the weak topology. Namely, a sequence of generalized functions $f_N \in \Psi'_{-G,q}(\mathbb{R}^n)$ converges to an element $f_0 \in \Psi'_{-G,q}(\mathbb{R}^n)$ in the weak sense, if for all $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ the sequence of numbers $\langle f_N, \varphi \rangle$ converges to $\langle f_0, \varphi \rangle$ as $N \to \infty$. We recall that the notation $\langle f, \varphi \rangle$ means the value of $f \in \Psi'_{-G,q}(\mathbb{R}^n)$ on an element $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$. For relations of the spaces $\Psi_{G,p}(\mathbb{R}^n)$ and its dual $\Psi'_{-G,q}(\mathbb{R}^n)$ to other spaces including Sobolev and Schwartz distributions see [47].

Further, we denote by $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ the m-times topological direct product

$$\mathbf{\Psi}_{G,p}(\mathbb{R}^n) = \Psi_{G,p}(\mathbb{R}^n) \otimes \cdots \otimes \Psi_{G,p}(\mathbb{R}^n),$$

of spaces $\Psi_{G,p}(\mathbb{R}^n)$. Elements of $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ are vector-functions $\Phi(x) = \big(\varphi_1(x), \ldots, \varphi_m(x)\big)$, where $\varphi_j(x) \in \Psi_{G,p}(\mathbb{R}^n), j = 1, \ldots, m$. The space, topologically dual to $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$, is the direct sum $\Psi'_{-G,p}(\mathbb{R}^n) \oplus \cdots \oplus \Psi'_{-G,p}(\mathbb{R}^n)$, which we denote by $\mathbf{\Psi}'_{-G,p}(\mathbb{R}^n)$. Elements of $\mathbf{\Psi}'_{-G,p}(\mathbb{R}^n)$ are m-tuples of generalized functions $F(x) = \big(f_1(x), \ldots, f_m(x)\big)$, and the value of $F \in \mathbf{\Psi}'_{-G,p}(\mathbb{R}^n)$ on $\Phi \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ is defined by

$$F(\Phi) = \langle F(x), \Phi(x) \rangle = \Big(\langle f_1(x), \varphi_1(x) \rangle, \ldots, \langle f_m(x), \varphi_m(x) \rangle\Big).$$

Finally for a topological vector space X we denote by $C^{(k)}[[a,b];X]$ the space of vector-functions $g(t),\ t \in [a,b]$, with values in X and k times differentiable in the sense of the topology of X. Similarly, one can define the space $C^{\infty}[[a,b];X]$.

2.2. **Pseudo-differential operators with constant symbols.** Now we introduce and consider some properties of pseudo-differential operators with constant (that is not depending on the variable x) symbols defined and continuous in a domain $G \subset \mathbb{R}^n$. Outside of G or on its boundary the symbol $a(\xi)$ may have singularities of arbitrary type. It is clear that the corresponding class of pseudo-differential operators are not in the frame of classic pseudo-differential operators with infinitely differentiable symbols, studied first in works by Kon-Nirenberg [26] and Hörmander [18]. For the systematic presentation of the theory of pseudo-differential operators being considered in this paper we refer the reader to [47].

For a function $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ the operator $A(D)$ corresponding to the symbol $A(\xi)$ is defined by the formula

$$A(D)\varphi(x) = \frac{1}{(2\pi)^n} \int_G A(\xi) F[\varphi](\xi) e^{ix\xi} d\xi \quad x \in \mathbb{R}^n. \tag{2.3}$$

We note that the assumption $\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ is crucial in the definition of $A(D)$ in (2.3). Generally speaking, $A(D)$ has no sense even for functions in the space $C_0^{\infty}(\mathbb{R}^n)$. In fact, let ξ_0 be a non-integrable singular point of $A(\xi)$ and denote by $O(\xi_0)$ some neighborhood of ξ_0. Let us take a function $\varphi \in C_0^{\infty}(\mathbb{R}^n)$ with $F[\varphi](\xi) > 0$ for $\xi \in O(\xi_0)$ and $F[\varphi](\xi_0) = 1$. Then it is easy to verify that $A(D)\varphi(x) = \infty$. On the other hand, for

$\varphi \in \Psi_{G,p}(\mathbb{R}^n)$ the integral in Eq. (2.3) is convergent due to the compactness of supp $F[\varphi] \subset G$. We define the operator $A(-D)$ acting in the space $\Psi'_{-G,q}(\mathbb{R}^n)$ by the duality formula

$$\langle A(-D)f, \varphi\rangle \ = \ \langle f, A(D)\varphi\rangle, \ f \in \Psi'_{-G,q}(\mathbb{R}^n), \ \varphi \in \Psi_{G,p}(\mathbb{R}^n). \tag{2.4}$$

Theorem 2.1. *The space $\Psi_{G,p}(\mathbb{R}^n)$ ($\Psi'_{-G,q}(\mathbb{R}^n)$) is invariant with respect to the action of an arbitrary pseudo-differential operator $A(D)$ (A(-D)), whose symbol is continuous in G. Moreover, if $A(\xi)k_N(\xi)$ is a multiplier in L_p for every $N \in \mathbb{N}$, then the operators*

$$A(D) : \Psi_{G,p}(\mathbb{R}^n) \to \Psi_{G,p}(\mathbb{R}^n),$$

and

$$A(-D) : \Psi'_{-G,q}(\mathbb{R}^n) \to \Psi'_{-G,q}(\mathbb{R}^n),$$

act continuously.

Remark 2.2. In the case $p = 2$ an arbitrary pseudo-differential operator whose symbol is continuous in G acts continuously without the additional condition for $A(\xi)k_N(\xi)$ to be a multiplier in L_2 for every $N \in \mathbb{N}$.

Finally, the following theorem establishes conditions for continuous closability of the pseudo-differential operator $A(D)$ acting in the space $\Psi_{G,p}(\mathbb{R}^n)$ to Sobolev spaces $W_p^s(\mathbb{R}^n)$ for $s \in \mathbb{R}$ and $p > 1$.

Theorem 2.2. ([47]) *Let $1 < p < \infty$, $-\infty < s, \ell < +\infty$ and $\mu(\mathbb{R}^n \setminus G)= 0$. For a pseudo-differential operator*

$$A(D) : \Psi_{G,p}(\mathbb{R}^n) \to \Psi_{G,p}(\mathbb{R}^n),$$

there exists a closed extension

$$\hat{A}(D) : W_p^s(\mathbb{R}^n) \to W_p^\ell(\mathbb{R}^n),$$

if and only if the symbol $A(\xi)$ satisfies the estimate

$$|A(\xi)| \le C(1 + |\xi|)^{s-\ell}, \quad C > 0, \ \xi \in R^n. \tag{2.5}$$

Theorems 2.1 and 2.2 can be extended to matrix pseudo-differential operators, elements of which satisfy theses theorems. Let

$$\mathbb{A}(D) = \begin{bmatrix} a_{1,1}(D) & \dots & a_{1,m}(D) \\ \dots & \dots & \dots \\ a_{m,1}(D) & \dots & a_{m,m}(D) \end{bmatrix}$$

be the matrix pseudo-differential operator with the matrix-symbol

$$\mathcal{A}(\xi) = \begin{bmatrix} a_{1,1}(\xi) & \dots & a_{1,m}(\xi) \\ \dots & \dots & \dots \\ a_{m,1}(\xi) & \dots & a_{m,m}(\xi) \end{bmatrix}, \quad \xi \in G \subset \mathbb{R}^n. \tag{2.6}$$

Namely, the following theorems are valid.

THEOREM 2.3. *The space $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ ($\mathbf{\Psi}'_{-G,q}(\mathbb{R}^n)$) is invariant with respect to the action of an arbitrary pseudo-differential operator $\mathbb{A}(D)$ ($\mathbb{A}(-D)$), whose symbol $\mathcal{A}(\xi)$ is continuous in G. Moreover, if $a_{j,k}(\xi)k_N(\xi)$, $j,k = 1,\dots,m$, are multipliers in L_p for every $N \in \mathbb{N}$, then the operators*

$$\mathbb{A}(D) : \mathbf{\Psi}_{G,p}(\mathbb{R}^n) \to \mathbf{\Psi}_{G,p}(\mathbb{R}^n),$$

and

$$\mathbb{A}(-D) : \mathbf{\Psi}'_{-G,q}(\mathbb{R}^n) \to \mathbf{\Psi}'_{-G,q}(\mathbb{R}^n),$$

act continuously.

THEOREM 2.4. *Let $1 < p < \infty$, $-\infty < s, \ell < +\infty$ and $\mu(\mathbb{R}^n \setminus G) = 0$. For a pseudo-differential operator*

$$\mathbb{A}(D) : \mathbf{\Psi}_{G,p}(\mathbb{R}^n) \to \mathbf{\Psi}_{G,p}(\mathbb{R}^n),$$

there exists a closed extension

$$\hat{\mathbb{A}}(D) : \mathbf{W}^s_p(\mathbb{R}^n) \to \mathbf{W}^\ell_p(\mathbb{R}^n),$$

if and only if each entry $a_{jk}(\xi)$ of the symbol $\mathcal{A}(\xi)$ satisfies the estimate

$$|a_{jk}(\xi)| \le C(1+|\xi|)^{s-\ell}, \quad C > 0, j,k = 1,\dots,m,\ \xi \in R^n. \tag{2.7}$$

Proofs of these statements directly follow from Theorems 2.1 and 2.2.

2.3. **Fractional integrals and derivatives.** Let a function $f(t)$ be defined and measurable on an interval (a,b), $a < b \le \infty$. The fractional integral of order $\beta > 0$ of the function f is defined by

$${}_aJ^\beta_t f(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} f(\tau) d\tau, \ \ t \in (a,b),$$

where $\Gamma(\beta)$ is Euler's gamma function, that is

$$\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt.$$

If $\beta = 0$, then we agree that ${}_aJ^0_t = I$, the identity operator. For arbitrary $\beta \ge 0$ and $\alpha \ge 0$ the following semigroup property holds:

$${}_aJ^\beta_t\, {}_aJ^\alpha_t ={}_a J^\alpha_t\, {}_aJ^\beta_t ={}_a J^{\beta+\alpha}_t. \tag{2.8}$$

Let m be a natural number and $m-1 \le \beta < m$. Then the fractional derivative of order β of a function f in the sense of Riemann–Liouville is defined as

$$ {}_aD_+^\beta f(t) = \frac{1}{\Gamma(m-\beta)} \frac{d^m}{dt^m} \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^{\beta+1-m}}, \tag{2.9} $$

provided the expression on the right exists. One can write ${}_aD_+^\beta$ in the operator form

$$ {}_aD_+^\beta = \frac{d^m}{dt^m} \, {}_aJ_t^{m-\beta}. \tag{2.10} $$

This operator is the left-inverse to the fractional integration operator ${}_aJ_t^\beta$. Indeed, due to relation (2.8), one has

$$ {}_aD_+^\beta \, {}_aJ_t^\beta = \frac{d^m}{dt^m} \, {}_aJ_t^{m-\beta} \, {}_aJ_t^\beta = \frac{d^m}{dt^m} \, {}_aJ_t^m = I. $$

To explore a domain of ${}_aD_+^\beta$ for any order β, consider first the case $0 < \beta < 1$. It follows from definition (2.9) that if $0 < \beta < 1$, then

$$ {}_aD_+^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_a^t \frac{f(\tau)d\tau}{(t-\tau)^\beta}. \tag{2.11} $$

The operator form of ${}_aD_+^\beta$ in this case is ${}_aD_+^\beta = \frac{d}{dt} \, {}_aJ_t^{1-\beta}$. Let $C^\lambda[a,b]$ denote the class of Hölder continuous functions of order $\lambda > 0$ on an interval $[a,b]$. The following statement says that if f is Hölder continuous of order $\lambda \in (0,1)$, then its fractional derivative of order $\beta < \lambda$ exists.

Proposition 2.1. ([44]) *Let $f \in C^\lambda[a,b]$, $0 < \lambda \le 1$. Then for any $\beta < \lambda$ the fractional derivative ${}_aD_+^\beta f(t)$ exists and can be represented in the form*

$$ {}_aD_+^\beta f(t) = \frac{f(a)}{\Gamma(1-\beta)(t-a)^\beta} + \psi(t), \tag{2.12} $$

where $\psi \in C^{\lambda-\beta}[a,b]$.

Let m be a natural number and $m-1 \le \beta < m$. Then the fractional derivative of order β of a function f in the sense of Caputo is defined as

$$ {}_aD_*^\beta f(t) = \frac{1}{\Gamma(m-\beta)} \int_a^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\beta+1-m}}, \; t > a, \tag{2.13} $$

provided the integral on the right exists.

The operator form of the fractional derivative ${}_aD_*^\beta$ of order β, $m-1 \le \beta < m$, in the Caputo sense is

$$ {}_aD_*^\beta = {}_aJ_t^{m-\beta} \frac{d^m}{dt^m}, \tag{2.14} $$

which is well defined, for instance, in the class of m-times differentiable functions defined on an interval $[a,b),\ b > a$. It follows from definition (2.13) that if $0 < \beta < 1$, then

$$ {}_aD_*^{\beta} f(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'(\tau)d\tau}{(t-\tau)^{\beta}}. \tag{2.15}$$

The operator form of ${}_aD_+^{\beta}$ in this case is ${}_aD_+^{\beta} = {}_aJ_t^{1-\beta} \frac{d}{dt}$.

Let $a = 0$. In this case we write simply J^{β}, D_+^{β} and D_*^{β} respectively instead of ${}_0J_t^{\beta}$, ${}_0D_+^{\beta}$ and ${}_0D_*^{\beta}$. Suppose f is a function defined on the semi-axis $[0,\infty)$ and for which $D_+^{\beta} f(t)$ and $D_*^{\beta}(t)$ exist.

Proposition 2.2. *Let $\beta > 0$. Then the Laplace transform of $J^{\beta} f(t)$ is*

$$L[J^{\beta} f](s) = s^{-\beta} L[f](s), \ \ s > 0. \tag{2.16}$$

Proposition 2.3. *Let $m-1 \le \beta < m$, $m = 1, 2, \dots$. Then the Laplace transform of $D_+^{\beta} f(x)$ is*

$$L[D_+^{\beta} f](s) = s^{\beta} L[f](s) - \sum_{k=0}^{m-1} (D^k J^{m-\beta} f)(0) s^{m-1-k}. \tag{2.17}$$

Proposition 2.4. *Let $m-1 < \beta \le m$. The Laplace transform of the Caputo derivative of a function $f \in C^{m-1}[0,\infty)$ is*

$$L[D_*^{\beta} f](s) = s^{\beta} L[f](s) - \sum_{k=0}^{m-1} f^{(k)}(0) s^{\beta-1-k}, \ \ s > 0. \tag{2.18}$$

For $\beta \in (0,1]$ formulas (2.17) and (2.18) respectively take the forms:

$$L[D_+^{\beta} f](s) = s^{\beta} L[f](s) - (J^{1-\beta} f)(0), \tag{2.19}$$
$$L[D_*^{\beta} f](s) = s^{\beta} L[f](s) - f(0) s^{\beta-1}. \tag{2.20}$$

We will use these formulas in the vector form:

$$L[D_+^{\mathcal{B}} < f_1, \dots, f_m >](s) = < s^{\beta_1} L[f_1](s) - (J^{1-\beta_1} f_1)(0), \dots, s^{\beta_m} L[f_m](s) - (J^{1-\beta_m} f_m)(0) > \tag{2.21}$$

$$L[D_*^{\mathcal{B}} < f_1, \dots, f_m >](s) = < s^{\beta_1} L[f_1](s) - f_1(0) s^{\beta_1 - 1}, \dots, s^{\beta_m} L[f_m](s) - f_m(0) s^{\beta_m - 1} > . \tag{2.22}$$

In these formulas $\mathcal{B} = \langle \beta_1, \dots, \beta_m \rangle$ is a vector-order with $0 < \beta_j \le 1,\ j = 1, \dots, m$, and

$$L[\mathcal{D}^{\mathcal{B}} < f_1, \dots, f_m >](s) =< L[\mathcal{D}^{\beta_1} f_1](s), \dots, L[\mathcal{D}^{\beta_m} f_m](s)$$

for both operators $\mathcal{D} = D_+$ and $\mathcal{D} = D_*$.

3. **Main results**

Consider the following system of fractional order differential equations

$$\begin{cases} \mathcal{D}^{\beta_1} u_1(t,x) = A_{1,1}(D) u_1(t,x) + \dots A_{1,m}(D) u_m(t,x) + h_1(t,x), \\ \mathcal{D}^{\beta_2} u_2(t,x) = A_{2,1}(D) u_1(t,x) + \dots A_{2,m}(D) u_m(t,x) + h_2(t,x), \\ \dots \\ \mathcal{D}^{\beta_m} u_m(t,x) = A_{m,1}(D) u_1(t,x) + \dots A_{m,m}(D) u_m(t,x) + h_m(t,x), \end{cases} \tag{3.1}$$

where $0 < \beta_j \le 1,\ j = 1, \dots, m$, and the operator $\mathcal{D}$ on the left expresses either the Riemann-Liouville derivative D_+ or the Caputo derivative D_*. We will specify the initial conditions later depending on whether $\mathcal{D}$ is the Riemann-Liouville or the Caputo derivative.

With the vector-order $\mathcal{B} = \langle \beta_1, \dots, \beta_m \rangle$, introducing vector-functions $U(t,x) = \langle u_1(t,x), \dots, u_m(t,x) \rangle$, $H(t,x) = \langle h_1(t,x), \dots, h_m(t,x) \rangle$, we can represent system (3.1) in the vector form:

$$D^{\mathcal{B}} U(t,x) = \mathbb{A}(D) U(t,x) + H(t,x), \tag{3.2}$$

where $\mathbb{A}(D)$ is the matrix pseudo-differential operator with the matrix-symbol $\mathcal{A}(\xi),\ \xi \in G$, defined in (2.6), and

$$D^{\mathcal{B}} U(t,x) = \langle D^{\beta_1} u_1(t,x), \dots, D^{\beta_m} u_m(t,x) \rangle.$$

For simplicity we assume that the matrix-symbol is symmetric, $a_{k,j}(\xi) = a_{j,k}(\xi)$ for all $k, j = 1, \dots, m$, and $\xi \in G$, and diagonalizable. Namely, there exists an invertible $(m \times m)$-matrix-function $M(\xi)$, such that

$$\mathcal{A}(\xi) = M^{-1}(\xi) \Lambda(\xi) M(\xi), \quad \xi \in G, \tag{3.3}$$

with a diagonal matrix

$$\Lambda(\xi) = \begin{bmatrix} \lambda_1(\xi) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \lambda_m(\xi) \end{bmatrix}. \tag{3.4}$$

We denote entries of matrices $M(\xi)$ and $M^{-1}(\xi)$ by $\mu_{j,k}(\xi),\ j,k = 1, \dots, m$, and $\nu_{j,k}(\xi),\ j,k = 1, \dots, m$, respectively.

First we derive a representation formula for the solution of the initial value problem for system (3.2) in the homogeneous case. Since initial conditions depend on the form of the fractional derivative on the left hand side of equation (3.2), the corresponding representations of solutions differ. We demonstrate the derivation in the case of Caputo fractional derivative. The case of Rieman-Liouville fractional derivative can be treated similarly.

Consider the following Cauchy problem:

$$D_*^{\mathcal{B}} U(t,x) = \mathbb{A}(D)U(t,x), \quad t>0,\ x\in\mathbb{R}^n, \tag{3.5}$$

$$U(0,x) = \Phi(x), \quad x\in\mathbb{R}^n, \tag{3.6}$$

where the fractional derivatives on the left are in the sense of Caputo. Applying Fourier transform we obtain a system of fractional order ordinary differential equations with a parameter ξ :

$$D_*^{\mathcal{B}} F[U](t,\xi) = \mathcal{A}F[U](t,\xi), \quad t>0,\ \xi\in G,$$

with the initial conditions

$$F[U](0,\xi) = F[\Phi](\xi), \quad \xi\in G.$$

Now applying the Laplace transform in the vector form (2.22), one has

$$\langle s^{\beta_1} LF[u_1](s,\xi),\dots,s^{\beta_m} LF[u_m](s,\xi)\rangle = \langle s^{\beta_1-1}\varphi_1(\xi),\dots,s^{\beta_m-1}\varphi_m(\xi)\rangle + \mathcal{A}(\xi)LF[U](s,\xi), \quad s>0,\ \xi\in G.$$

Taking into account (3.3) the letter can be rewritten in the form

$$M^{-1}(Is^{\mathcal{B}} - \Lambda(\xi))M(\xi)LF[U](s,\xi) = Is^{\mathcal{B}-1}F[\Phi](\xi),$$

where $Is^{\mathcal{B}}$, $Is^{\mathcal{B}-1}$ are diagonal matrices with diagonal entries $s^{\beta_j},\ s^{\beta_j-1}$ $j = 1,\dots,m$, respectively. The solution to the obtained system is

$$LF[U](s,\xi) = M(\xi)\mathcal{N}(s,\xi)M^{-1}(\xi)F[\Phi](\xi), \tag{3.7}$$

where

$$\mathcal{N}(s,\xi) = \begin{bmatrix} \frac{s^{\beta_1-1}}{s^{\beta}-\lambda_1(\xi)} & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & \frac{s^{\beta_m-1}}{s^{\beta_m}-\lambda_m(\xi)} \end{bmatrix}. \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$F[U](t,\xi) = M(\xi)\mathcal{E}_{\mathcal{B}}(\Lambda(\xi)t^{\mathcal{B}})M^{-1}(\xi)F[\Phi](\xi), \quad t>0,\ \xi\in G. \tag{3.9}$$

Here $\mathcal{E}_{\mathcal{B}}(\Lambda(\xi)t^{\mathcal{B}})$ is the diagonal matrix of the form

$$\mathcal{E}_{\mathcal{B}}(\Lambda(\xi)t^{\mathcal{B}}) = \begin{bmatrix} E_{\beta_1}(\lambda_1(\xi)t^{\beta_1}) & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & E_{\beta_m}(\lambda_m(\xi)t^{\beta_m}) \end{bmatrix}, \tag{3.10}$$

where $E_{\beta_j}(z), j = 1,\dots,m$, are the Mittag-Leffler functions of indices $\beta_1,\dots,\beta_m$. Thus, the solution of problem (3.5)-(3.6) has the representation

$$U(t,x) = S(t,D)\Phi(x), \quad t>0,\ x\in\mathbb{R}^n, \tag{3.11}$$

where $S(t,D)$ is the solution matrix pseudo-differential operator with the matrix-symbol

$$\mathcal{S}(t,\xi) = M(\xi)\ \mathcal{E}_{\mathcal{B}}\Big(\Lambda(\xi)t^{\mathcal{B}}\Big)\ M^{-1}(\xi), \quad t>0,\ \xi \in G, \tag{3.12}$$

whose entries are

$$s_{j,k}(t,\xi) = \sum_{\ell=1}^{m} \mu_{j,\ell}(\xi)\nu_{\ell,k}(\xi)E_{\beta_\ell}(\lambda_\ell(\xi)t^{\beta_\ell}), \quad j,k=1,\dots,m,$$

The explicit component-wise form of the solution is

$$\begin{aligned} u_j(t,x) &= \sum_{k=1}^{m} s_{j,k}(t,D)\varphi_k(x) \\ &= \frac{1}{(2\pi)^n}\sum_{k=1}^{m}\sum_{\ell=1}^{m}\int_{\mathbb{R}^n} e^{i\xi x}\mu_{j,\ell}(\xi)E_{\beta_\ell}\left(\lambda_\ell(\xi)t^{\beta_\ell}\right)\nu_{\ell,k}(\xi)F[\varphi_k](\xi)d\xi. \end{aligned}$$

Theorem 3.1. *Let $\mathbb{A}$ be a pseudo-differential operator with the symbol $\mathcal{A}(\xi)$ continuous on G and satisfying the condition* (3.3). *Assume that $\Phi(x) \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$, $H(t,x) \in AC[\mathbb{R}_+;\mathbf{\Psi}_{G,p}(\mathbb{R}^n)]$, and $D_+^{1-\mathcal{B}}H(\tau,x) \in C[\mathbb{R}_+;\mathbf{\Psi}_{G,p}(\mathbb{R}^n)]$. Then for any $T>0$ Cauchy problem*

$$D_*^{\mathcal{B}}U(t,x) = \mathbb{A}(D)U(t,x) + H(t,x), \quad t>0,\ x\in\mathbb{R}^n, \tag{3.13}$$

$$U(0,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{3.14}$$

has a unique solution $U(t,x) \in C^\infty[(0,T];\mathbf{\Psi}_{G,p}(\mathbb{R}^n)] \cap C[[0,T];\mathbf{\Psi}_{G,p}(\mathbb{R}^n)]$, having the representation

$$U(t,x) = S(t,D)\Phi(x) + \int_0^t S(t-\tau,D)D_+^{1-\mathcal{B}}H(\tau,x)d\tau, \quad t>0,\ x\in\mathbb{R}^n, \tag{3.15}$$

where $S(t,D)$ is the pseudo-differential operator with the matrix-symbol $\mathcal{S}(t,\xi)$ defined in (3.12).

P r o o f. The representation (3.15) follows directly from (3.11) and from fractional Duhamel's principle [46, 47]. Denote the first and second terms on the right of (3.15) by $V(t,x)$ and $W(t,x)$, respectively:

$$V(t,x) = S(t,D)\Phi(x), \quad x\in\mathbb{R}^n, \tag{3.16}$$

$$W(t,x) = \int_0^t S(t-\tau,D)D_+^{1-\mathcal{B}}H(\tau,x)d\tau, \quad t\geq 0,\ x\in\mathbb{R}^n. \tag{3.17}$$

Then, in accordance with Theorem (2.3) $V(t,x) \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ for every fixed $t \geq 0$, continuous on $[0,T]$, and infinitely differentiable on $(0,T)$ in the topology of $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$ due to the construction of the solution operator $S(t,D)$. Further, there exists a sequence (see (2.1))

$$H_N(t,x) \in \mathbf{\Psi}_{N,p}(\mathbb{R}^n) \equiv \Psi_{G,p}(\mathbb{R}^n) \otimes \cdots \otimes \Psi_{G,p}(\mathbb{R}^n),$$

such that $H_N(t,x) \to H(t,x)$ as $N \to \infty$ in the topology of $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$. Moreover, $p_N(H_N) = \|H_N\|_p$. Let

$$W_N(t,x) = \int_0^t S(t-\tau,D) D_+^{1-\mathcal{B}} H_N(\tau,x) d\tau, \quad N = 1,2,\ldots.$$

Then we have $p_N(W_N) = \|F^{-1}\kappa_N F[W_N]\|_p \leq T\|W_N\|_p < \infty$ for all $N \geq 1$. It follows that $W_N \in \mathbf{\Psi}_{N,p}(\mathbb{R}^n)$ and $W_N(t,x) \to W(t,x)$, as $N \to \infty$, in the topology of $\mathbf{\Psi}_{N,p}(\mathbb{R}^n)$ for each fixed $t \in [0,T]$. The continuity of $W(t,x)$ on $[0,T]$ in the variable t and its infinite differentiability on $(0,T)$ follows from the construction of the solution operator $S(t,D)$ in standard way. □

THEOREM 3.2. *Let p and q, $1 < p,q < \infty$, be a conjugate pair and $\mathbb{A}$ be a pseudo-differential operator with the symbol $\mathcal{A}(\xi)$ continuous on G and satisfying the condition* (3.3). *Assume that $\Phi(x) \in \mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)$, $H(t,x) \in AC[\mathbb{R}_+;\mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)]$, and $D_+^{1-\mathcal{B}} H(\tau,x) \in C[\mathbb{R}_+;\mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)]$. Then for any $T > 0$ Cauchy problem*

$$D_*^{\mathcal{B}} V(t,x) = \mathbb{A}(-D)V(t,x) + H(t,x), \quad t > 0, \ x \in \mathbb{R}^n, \tag{3.18}$$

$$V(0,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{3.19}$$

has a unique solution $V(t,x) \in C^\infty[(0,T];\mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)] \cap C[[0,T];\mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)]$, having the representation

$$V(t,x) = S(t,-D)\Phi(x) + \int_0^t S(t-\tau,-D) D_+^{1-\mathcal{B}} H(\tau,x) d\tau, \quad t > 0, \ x \in \mathbb{R}^n, \tag{3.20}$$

where $S(t,-D)$ is the pseudo-differential operator with the matrix-symbol $\mathcal{S}(t,-\xi)$ defined in (3.12) * .

P r o o f. We note that elements $D_*^{\mathcal{B}} V(t,x)$ and $\mathbb{A}(-D)V(t,x)$ belong to the space $\mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)$ if $V(t,x) \in \mathbf{\Psi'}_{-G,q}(\mathbb{R}^n)$ for each fixed $t \geq 0$. This fact follows from the definition of the fractional derivative $D_*^{\mathcal{B}}$ and Theorem 2.3.

* with $-\xi$ instead of ξ

The solution $V(t,x)$ of the Cauchy problem (3.18) – (3.19), by definition, must satisfy the following conditions:

$$\langle D_*^{\mathcal{B}} V(t,x), F(x)\rangle = \langle V(t,x), \mathbb{A}(D)F(x)\rangle + \langle H(t,x), F(x)\rangle, \quad t>0, \tag{3.21}$$

$$\langle V(0,x), F(x)\rangle = \langle \Phi(x), F(x)\rangle, \tag{3.22}$$

for an arbitrary element $F(x)$ in the space $\mathbf{\Psi}_{G,p}(\mathbb{R}^n)$. We show that $U(t,x)$ defined in (3.20) satisfies both conditions in (3.21) and (3.22). Indeed, to show this fact let us first assume that $H(t,x)=0$ ** for all $t \geq 0$. Then (3.21) takes the form

$$\Big\langle \Big[D_*^{\mathcal{B}} S(t,-D) - \mathbb{A}(-D)\Big]\Phi(x), F(x)\Big\rangle = \Big\langle \Phi(x), \Big[D_*^{\mathcal{B}} S(t,D) - \mathbb{A}(D)\Big]F(x)\Big\rangle$$
$$= 0, \quad t>0.$$

The operator $S(t,D)$ is constructed so that $D_*^{\mathcal{B}} S(t,D) - \mathbb{A}(D) = 0$. Indeed, if $U(t,x)$ is a solution to equation (3.5), then it follows from representation (3.11) that $D_*^{\mathcal{B}} U(t,x) = D_*^{\mathcal{B}} S(t,D)\Phi(x) = \mathbb{A}(D)\Phi(x)$ for any fixed $\Phi \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$. This implies the equality $D_*^{\mathcal{B}} S(t,D) = \mathbb{A}(D)$. Thus, condition (3.21) is verified.

Further, it follows from (3.12) that the symbol $\mathcal{S}(t,\xi)$ at $t=0$ reduces to the identity matrix, since the matrix $\mathcal{E}_{\mathcal{B}}(0)$ is the identity matrix. Therefore, the operator corresponding to the matrix-symbol $\mathcal{S}(0,\xi)$ is the identity pseudo-differential operator. Hence, $V(0,x) = S(0,-D)\Phi(x) = \Phi(x)$. Thus, condition (3.22) is also verified.

In the general case, for non-zero $H(t,x)$, the representation (3.20) is an implication of the fractional Duhamel principle [46, 47]. □

Now consider the following initial-value problem

$$D_+^{\mathcal{B}} U(t,x) = \mathbb{A}(D)U(t,x), \quad t>0,\ x \in \mathbb{R}^n, \tag{3.23}$$

$$J^{1-\mathcal{B}} U(0,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{3.24}$$

where the fractional derivatives on the left hand side of equation (3.23) are in the sense of Riemann-Liouville. Performing similar calculations, in this case for the solution we obtain the representation

$$U(t,x) = S_+(t,D)\Phi(x), \quad t>0,\ x\in\mathbb{R}^n, \tag{3.25}$$

where $S_+(t,D)$ is the solution matrix pseudo-differential operator with the matrix-symbol

$$\mathcal{S}_+(t,\xi) = M(\xi)\ J^{1-\mathcal{B}} \mathcal{E}_{\mathcal{B}}\Big(\Lambda(\xi)t^{\mathcal{B}}\Big)\ M^{-1}(\xi), \quad t>0,\ \xi \in G, \tag{3.26}$$

** as an element of $\mathbf{\Psi}'_{-G,q}(\mathbb{R}^n)$

whose the entries are

$$s_{j,k}^{+}(t,\xi) = \sum_{\ell=1}^{m} \mu_{j,\ell}(\xi)\nu_{\ell,k} J^{1-\beta_\ell} E_{\beta_\ell}(\lambda_\ell(\xi)t^{\beta_\ell}), \quad j,k = 1,\dots,m,$$

The explicit component-wise form of the solution is

$$\begin{aligned} u_j(t,x) &= \sum_{k=1}^{m} s_{j,k}^{+}(t,D)\varphi_k(x) \\ &= \frac{1}{(2\pi)^n} \sum_{k=1}^{m}\sum_{\ell=1}^{m} \int_{\mathbb{R}^n} e^{i\xi x}\mu_{j,\ell}(\xi) J^{1-\beta_\ell} E_{\beta_\ell}\left(\lambda_\ell(\xi)t^{\beta_\ell}\right) \nu_{\ell,k}(\xi) F[\varphi_k](\xi) d\xi. \end{aligned}$$

THEOREM 3.3. *Let $\mathbb{A}$ be a pseudo-differential operator with the symbol $\mathcal{A}(\xi)$ continuous on G and satisfying the condition* (3.3) *and $\Phi \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$. Then for any $T > 0$ Cauchy problem*

$$D_{+}^{\mathcal{B}} U(t,x) = \mathbb{A}(D)U(t,x) + H(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{3.27}$$

$$J^{1-\mathcal{B}} U(0,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{3.28}$$

has a unique solution $U(t,x) \in C^\infty[(0,T]; \mathbf{\Psi}_G(\mathbb{R}^n)]$, having the representation

$$U(t,x) = S_{+}(t,D)\Phi(x) + \int_0^t S_{+}(t-\tau,D)H(\tau,x)d\tau, \quad t > 0,\ x \in \mathbb{R}^n, \tag{3.29}$$

where $S_{+}(t,D)$ is the pseudo-differential operator with the matrix-symbol $\mathcal{S}_{+}(t,\xi)$ defined in (3.26).

THEOREM 3.4. *Let $\mathbb{A}$ be a pseudo-differential operator with the symbol $\mathcal{A}(\xi)$ continuous on G and satisfying the condition* (3.3) *and $\Phi \in \mathbf{\Psi}_{G,p}(\mathbb{R}^n)$. Then for any $T > 0$ Cauchy problem*

$$D_{+}^{\mathcal{B}} U(t,x) = \mathbb{A}(-D)U(t,x) + H(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \tag{3.30}$$

$$J^{1-\mathcal{B}} U(0,x) = \Phi(x), \quad x \in \mathbb{R}^n, \tag{3.31}$$

has a unique solution $U(t,x) \in C^\infty[(0,T]; \mathbf{\Psi}_G(\mathbb{R}^n)]$, having the representation

$$U(t,x) = S_{+}(t,-D)\Phi(x) + \int_0^t S_{+}(t-\tau,-D)H(\tau,x)d\tau, \quad t > 0,\ x \in \mathbb{R}^n, \tag{3.32}$$

where $S_{+}(t,-D)$ is the pseudo-differential operator with the matrix-symbol $\mathcal{S}_{+}(t,-\xi)$ defined in (3.26).

The proofs of Theorems 3.3 and 3.4 are omitted, since they are similar to the proofs of Theorems 3.1 and 3.2, respectively.

The properties of the solutions of problems (3.13)-(3.14) and (3.27)-(3.28) essentially depend on the asymptotic behavior of the functions

$$E_{\beta_\ell}\left(\lambda_\ell(\xi)t^{\beta_\ell}\right), \quad \ell = 1, \dots, m,$$

which form the symbols of solution operators; see (3.12) and (3.26). It is known that for $0 < \beta < 2$ the Mittag-Leffler function $E_\beta(z)$ has asymptotic behavior $\sim \exp(z^{1/\beta})$, $|z| \to \infty$, if $|\arg(z)| \le \beta\pi/2$; and $E_\beta(z) \sim 1/|z|, |z| \to \infty$, if $\beta\pi/2 \le |\arg(z) \le \pi$. Therefore, if a symbol $A(\xi)$ is complex-valued, then $E_\beta(A(\xi)t^\beta)$ may have an exponential growth as $|\xi| \to \infty$, even though $A(\xi)$ has a polynomial growth at infinity.

Now suppose that the pseudo-differential operator $\mathbb{A}(D)$ satisfies the following ellipticity condition: the symbol $\mathcal{A}(\xi)$, $\xi \in \mathbb{R}^n$, is symmetric, satisfies the condition (3.3) with a diagonal matrix $\Lambda(\xi)$, and there exists a number $R_0 > 0$ such that for the entries $\lambda_\ell(\xi)$, $\ell = 1, \dots, m$, of $\Lambda(\xi)$ the inequalities

$$-\Re(\lambda_\ell(\xi)) \le \eta|\xi|^{r_\ell}, \ \ell = 1, \dots, m, \tag{3.33}$$

where $\Re(z)$ is the real part of z, hold for all $\xi : \ |\xi| \ge R_0$; $\eta > 0$, $r_\ell \in \mathbb{R}$, $\ell = 1, \dots, m$, are constants. In this case we have

$$\left|E_{\beta_\ell}(\lambda_\ell(\xi)t^{\beta_\ell})\right| \le C_1(1 + |\lambda_\ell(\xi)|)^{-1} \le C_2(1 + |\xi|)^{-r_\ell}, \quad \xi \in \mathbb{R}^n, \tag{3.34}$$

with some C_1, C_2 positive constants. In the theorem below we use the notation $\mathbf{r} = (r_1, \dots, r_m)$.

Theorem 3.5. *Let the following conditions be verified:*

1. *the operator* $\mathbb{A}$ *is an elliptic pseudo-differential operator satisfying the condition* (3.34)*;*
2. *the symbol* $\mathcal{A}(\xi)$ *of the operator* $\mathbb{A}$ *is symmetric, continuous on* $\mathbb{R}^n$*, and satisfies the condition* (3.3)*;*
3. $\Phi \in \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)$*, where* $1 < p < \infty$ *and* $\mathbf{s} = (s_1, \dots, s_m)$*,* $s_j \in \mathbb{R}$*,* $j = 1, \dots, m$*;*
4. $H(t,x) \in AC[\mathbb{R}_+; \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)]$ *and* $D_+^{1-\mathcal{B}} H(\tau, x) \in C[\mathbb{R}_+; \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)]$.

Then for any $T > 0$ *Cauchy problem* (3.13)-(3.14) *has a unique solution* $U(t,x) \in C^\infty[(0,T]; \mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)] \cap C[[0,T]; \mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)]$*, having the representation*

$$U(t,x) = \hat{S}(t,D)\Phi(x) + \int_0^t \hat{S}(t-\tau, D) D_+^{1-\mathcal{B}} H(\tau, x) d\tau, \quad t > 0, \ x \in \mathbb{R}^n, \tag{3.35}$$

where $\hat{S}(t,D)$ is the closure of the pseudo-differential operator with the matrix-symbol $\mathcal{S}(t,\xi)$ defined in (3.12) *in the space $\mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)$.*

P r o o f. Let components of $\Phi(x)$ are $\varphi_k \in W_p^{s_k}(\mathbb{R}^n),\ k=1,...,m,$ and components $h_k(t,x),\ k=1,\dots,m$ of $H(t,x)$ for each fixed t, belong to $W_p^{s_k}$, respectively. We can choose any domain G whose complement $\mathbb{R}^n \setminus G$ has zero measure. In particular, one can take $G=\mathbb{R}^n$. Then the denseness $\overline{\Psi_{G,p}(\mathbb{R}^n)} = W_p^{s_k}(\mathbb{R}^n)$ (see [47]) holds for each $k=0,\dots,m-1$. Hence, for each φ_k and $h_k(t,\cdot)$ we have an approximating sequences of functions $\Phi_N = (\varphi_{1,N},\dots,\varphi_{m,N})$ $H_N(t,\cdot) = \big(h_{1,N}(t,\cdot),\dots,h_{m,N}(t,\cdot)\big)$ with $\varphi_{k,N},\ h_{k,N}(t,\cdot) \in \Psi_{G,p}(\mathbb{R}^n),\ N=0,1,2,\dots,$ such that $\varphi_{k,N} \to \varphi_k$ and $h_{k,N}(t,\cdot) \to h_k(t,\cdot)$ in the topology of $\Psi_{G,p}(\mathbb{R}^n)$. For fixed N, due to Theorem 3.1, there exists a unique solution of the Cauchy problem (3.13)-(3.14), where the initial data $\Phi(x)$ and $H(t,x)$ are replaced by $\Phi_N(x)$ and $H_N(t,x)$ respectively, and this solution is represented by the formula

$$U_N(t,x) = S(t,D)\Phi_N(x) + \int_0^t S(t-\tau,D) D_+^{1-\mathcal{B}} H_N(\tau,x)\,d\tau, \quad t>0,\ x\in\mathbb{R}^n. \tag{3.36}$$

Since the components of the symbol $\mathcal{S}(t,\xi)$ of the solution operator $S(t,D)$ satisfy the estimate (3.34), it follows from Theorem 2.4 that there exists a unique continuous closure $\hat{S}(t,D)$ of the operator $S(tD)$, such that

$$\hat{S}(t,D) : \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n) \to \mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)$$

is continuous. Thus for the solution $U(t,x)$ we have representation (3.6). The fact that $U(t,x) \in C^\infty[(0,T];\mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)] \cap C[[0,T];\mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)]$ follows from the construction of the solution through the sequence (3.36), due to the density of $\Psi_{G,p}(\mathbb{R}^n)$ in $W_p^{s_k+r_k}(\mathbb{R}^n),\ k=1,\dots,m.$ □

Similarly one can prove the existence of a unique solution in the Sobolev spaces of the Cauchy problem (3.27)-(3.28). Below is the formulation of the corresponding theorem.

THEOREM 3.6. *Let the following conditions be verified:*

(1) *the operator $\mathbb{A}$ is an elliptic pseudo-differential operator satisfying the condition* (3.34)*;*
(2) *the symbol $\mathcal{A}(\xi)$ of the operator $\mathbb{A}$ is symmetric, continuous on $\mathbb{R}^n$, and satisfies the condition* (3.3)*;*
(3) *$\Phi \in \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)$, where $1<p<\infty$ and $\mathbf{s}=(s_1,\dots,s_m),\ s_j\in\mathbb{R},\ j=1,\dots,m;$*
(4) *$H(t,x) \in AC[\mathbb{R}_+;\mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)]$ and $D_+^{1-\mathcal{B}} H(\tau,x) \in C[\mathbb{R}_+;\mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)]$.*

Then for any $T > 0$ *Cauchy problem* (3.27)-(3.28) *has a unique solution* $U(t,x) \in C^{\infty}[(0,T];\mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)] \cap C[[0,T];\mathbf{W}_p^{\mathbf{s}+r}(\mathbb{R}^n)]$, *having the representation*

$$U(t,x) = \hat{S}_+(t,D)\Phi(x) + \int_0^t \hat{S}_+(t-\tau,D)D_+^{1-\mathcal{B}}H(\tau,x)d\tau, \quad t>0,\ x\in\mathbb{R}^n,$$

where $\hat{S}_+(t,D)$ *is the closure of the pseudo-differential operator with the matrix-symbol* $\mathcal{S}_+(t,\xi)$ *defined in* (3.26) *in the space* $\mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)$.

Remark 3.1. (1) The results of Theorems 3.1 - 3.6 coincide with the known results in 1-D case, see, e.g. [47].

(2) The results obtained in Theorems 3.1 - 3.6 can be extended to the case, when $0 < \beta_j \le 1,\ j = 1,\dots,m_0$ and $1 < \beta_j \le 2,\ j = m_0+1,\dots,m$, where $0 \le m_0 \le m$, with properly adjusted initial conditions.

(3) The results also can be extended to the case of fractional distributed order differential operators (DODE) on the left hand side of the considered systems.

Example. To illustrate the theorems proved above consider the following Cauchy problem

$$D_*^{\beta_1}u_1(t,x) = -D^2u_1(t,x) - Du_2(t,x), \quad t>0,\ -\infty<x<\infty, \tag{3.37}$$

$$D_*^{\beta_2}u_2(t,x) = -Du_1(t,x) - D^2u_2(t,x), \quad t>0,\ -\infty<x<\infty, \tag{3.38}$$

$$u_1(0,x) = \varphi_1(x), \quad u_2(0,x) = \varphi_2(x), \quad -\infty<x<\infty. \tag{3.39}$$

It is not hard to see that the symbol of the operator on the right hand side of (3.37)-(3.38) is symmetric and has the representation

$$\mathcal{A}(\xi) = \begin{bmatrix} -\xi^2 & -\xi \\ -\xi & -\xi^2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -\xi^2+\xi & 0 \\ 0 & -\xi^2-\xi \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{3.40}$$

As is seen from (3.40) that $\lambda_1(\xi) = -\xi^2+\xi$ and $\lambda_2(\xi) = -\xi^2-\xi$. The symbol of the solution operator $S(t,D)$ is the matrx $\mathcal{S}(t,\xi) = \{s_{j,k}(t,\xi)\},\ j,k = 1,2$, with entries

$$s_{1,1}(t,\xi) = s_{2,2}(t,\xi) = \frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) + \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2}), \tag{3.41}$$

$$s_{1,2}(t,\xi) = s_{2,1}(t,\xi) = \frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) - \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2}). \tag{3.42}$$

Therefore, the solution $U(t,x) = \langle u_1(t,x), u_2(t,x)\rangle$ to Cauchy problem (3.37)-(3.39) has the representation:

$$\begin{aligned}
u_1(t,x) &= \left[\frac{1}{2}E_{\beta_1}((-D^2+D)t^{\beta_1}) + \frac{1}{2}E_{\beta_2}((-D^2-D)t^{\beta_2})\right]\varphi_1(x)\\
&\quad + \left[\frac{1}{2}E_{\beta_1}((-D^2+D)t^{\beta_1}) - \frac{1}{2}E_{\beta_2}((-D^2-D)t^{\beta_2})\right]\varphi_2(x);\\
u_2(t,x) &= \left[\frac{1}{2}E_{\beta_1}((-D^2+D)t^{\beta_1}) - \frac{1}{2}E_{\beta_2}((-D^2-D)t^{\beta_2})\right]\varphi_1(x)\\
&\quad + \left[\frac{1}{2}E_{\beta_1}((-D^2+D)t^{\beta_1}) + \frac{1}{2}E_{\beta_2}((-D^2-D)t^{\beta_2})\right]\varphi_2(x).
\end{aligned}$$

In the explicit form this solution has the form

$$\begin{aligned}
u_1(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) + \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2})\right]F[\varphi_1](\xi)d\xi\\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) - \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2})\right]F[\varphi_2](\xi)d\xi;\\
u_2(t,x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) - \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2})\right]F[\varphi_1](\xi)d\xi\\
&\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{1}{2}E_{\beta_1}((-\xi^2+\xi)t^{\beta_1}) + \frac{1}{2}E_{\beta_2}((-\xi^2-\xi)t^{\beta_2})\right]F[\varphi_2](\xi)d\xi.
\end{aligned}$$

Moreover, obviously, $\lambda_k(\xi) \le 0,\ k = 1,2$, for all ξ satisfying the inequality $|\xi| \ge 1$. Applying Theorem 3.5 we have $U(t,x) \in C^\infty[(0,T]; \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)] \cap C[[0,T]; \mathbf{W}_p^{\mathbf{s}}(\mathbb{R}^n)]$, where $\mathbf{s} = (s_1, s_2)$, if $\varphi_k \in \mathbf{W}_p^{s_k}(\mathbb{R}^n),\ k = 1,2$.

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