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PARABOLIC CYLINDER FUNCTIONS W(a>, ±X): EXPANSIONS FOR ALL ARGUMENTS

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PARABOLIC CYLINDER FUNCTIONS W(a>,±X): EXPANSIONS FOR ALL ARGUMENTS[†]

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ABSTRACT

Asymptotic expansions of the parabolic cylinder functions in terms of Airy functions have been derived, which contain no derivatives of Airy functions. Apart from its simplicity it may reduce computer times by ≈ 40%.

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I. INTRODUCTION

The two parabolic cylinder functions W(a,x), W(a,-x), with $x \geq 0$ are the two symmetric solutions of the differential equation

$$d_{xx}y + (1/4\ x^2 - a)y = 0. \tag{1}$$

Expansions of W(a,±x) which might be used for numerical calculations are

1) the ordinary power series as given in [1]. These however can be utilized because of the inherent round-off errors for all their leading terms for only up to approximately a,x<5, assuming a 48 bit mantisse-computer is used (e.g. CDC 6600). Otherwise the validity is even smaller. For illustration see fig. 1.

2) the two expansions of Darwin [2] hold only for $x^2 \ll 4a$ and $x^2 \gg 4a$ respectively. In fig. 1 the area is shown, where these expansions approximate the parabolic cylinder functions better than a relative 10^{-6} deviation.

3) the expansions [3] hold for $x \gg 1$ only. The 10^{-6} accuracy area is given again in fig. 1.

4) the asymptotic expansion in terms of Airy-functions [4]. Unfortunately its first correction term includes the derivative of Airy functions, which are elaborate to calculate. If this term would be neglected, the resulting 10^{-6} accuracy area would be so small that it is outside the frame of fig. 1.

In this paper we present a modified version of formula (12.18) and (12.20) in [4], which are easier and faster for numerical computation and which together with the expansions 1) and 3) cover the whole (x,a>0)- plane, thus allowing to compute numerically the parabolic cylinder functions W(a,±x) for all x, $a \geq 0$ to a relative accuracy better then 10^{-6}.

II. ASYMPTOTIC EXPANSION

The asymptotic expansions of Olver [4] for the parabolic cylinder functions are

$$W(a,x) \sim \sqrt{\pi}(4a)^{-1/4} e^{-1/2\pi a} \ell(a) \left(\frac{t}{\xi^2-1}\right)^{1/4} \left[\mathrm{Bi}(-T) \sum_{s=0}^{\infty} (-)^s \frac{A_s(\xi)}{(2a)^{2s}} + \frac{\mathrm{Bi}'(-T)}{(2a)^{4/3}} \sum_{s=0}^{\infty} (-)^s \frac{B_s(\xi)}{(2a)^{2s}}\right] \tag{2}$$

and

$$W(a,-x) \sim 2 \sqrt{\pi}\ (4a)^{-1/4} e^{1/2\pi a} \ell(a) \left(\frac{t}{\xi^2-1}\right)^{1/4} \left[\mathrm{Ai}(-T) \sum_{s=0}^{\infty} (-)^s \frac{A_s(\xi)}{(2a)^{2s}} + \frac{\mathrm{Ai}'(-T)}{(2a)^{4/3}} \sum_{s=0}^{\infty} (-s) \frac{B_s(\xi)}{(2a)^{2s}}\right], \tag{3}$$

where Ai and Bi are the Airy functions, and $\ell(a)$ is a power series in a^{-2},

$$\ell(a) = 1 - \frac{1}{1152} \cdot \frac{1}{(2a)^2} - \frac{16123}{39813120} \cdot \frac{1}{(2a)^4} + \ldots\ , \tag{4}$$

and

$$\xi = \frac{x}{2\sqrt{a}}\ , \tag{5}$$

$$T^{3/2} = t^{3/2} = (2a) \cdot 3/4[\xi \sqrt{\xi^2-1} - \ell n\ (\xi + \sqrt{\xi^2-1})]\ , \tag{6}$$

$$A_s(\xi) = \sum_{m=0}^{2s} b_m \left(\frac{2a}{t^{3/2}}\right)^m C_{2s-m}(\xi)\ , \tag{7}$$

$$\left(\frac{2a}{t^{3/2}}\right)^{1/3} B_s(\xi) = - \sum_{m=0}^{2s+1} a_m \left(\frac{2a}{t^{3/2}}\right)^m C_{2s-m+1}(\xi) \quad , \tag{8}$$

where $a_0 = 1$ and

$$a_m = \frac{(2m+1)(2m+3)\ldots(6m-1)}{m!\ (144)^m} \quad , \; b_m = - \frac{6m+1}{6m-1} am, \tag{9}$$

$$C_s = \frac{u_s(\xi)}{(\xi^2-1)^{3/2 \cdot s}} \quad . \tag{10}$$

$u_s(\xi)$ are polynomials, the first four of which are given by

$$u_0(\xi) = 1; \; u_1(\xi) = \frac{\xi^3-6\xi}{24} \; , \quad u_2(\xi) = \frac{-9\xi^4 + 249\xi^2 + 145}{1152} \tag{11}$$

$$u_3(\xi) = (-4042\xi^9 + 18189\xi^7 - 28287\xi^5 - 151995\xi^3 - 259290\xi)/1414720.$$

Analytic continuation shows that these formulaes hold for all $x \geq 0$. But there is a unique way to get the same expansions (2) and (3), just by dropping the Bi' terms completely and instead adding to T a power series in $(2a)^{-2}$, multiplied by $(2a)^{-4/3}$ and also adding a correction term to the A_s. As both Ai and Bi obey the differential equation

$$d_{xx}y = xy \; , \tag{12}$$

the derivative of any order is equivalent to a sum of y and y' each multiplied by a polynomial in x. Thus in our case making a Taylor expansion about T and collecting terms of the same order in a, that is a^0, a^{-2},,...; $a^{-4/3}$, $a^{-10/3}$... one gets in a straightforward manner the corrections mentioned above. The results up to the order $a^{-10/3}$ are

$$W(a,x) \sim \sqrt{\pi}\ (4a)^{-1/4}\ e^{-1/2\ \pi a}\left(1-\frac{1}{4608\cdot a^2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4}\ \mathrm{Bi}(-\tilde{T})\left(1-\frac{\tilde{A}_1}{4a^2}\right) \tag{13}$$

and

$$W(a,-x) \sim 2\sqrt{\pi}\ (4a)^{-1/4}\ e^{1/2\ \pi a}\left(1-\frac{1}{4608\cdot a^2}\right)\left(\frac{t}{\xi^2-1}\right)^{1/4}\ \mathrm{Ai}(-\tilde{T})\left(1-\frac{\tilde{A}_1}{4a^2}\right) \tag{14}$$

where

$$\xi = \frac{x}{2\sqrt{a}}\ , \tag{15}$$

$$t = (2a)^{2/3}\ \tau, \tag{16}$$

$$\tau = (3/4)^{2/3}\begin{cases} -[\arccos \xi - \xi\sqrt{1-\xi^2}\ ]^{2/3} & \text{for } \xi \le 1 \\ [\xi\sqrt{\xi^2-1} - \ln\ (\xi + \sqrt{\xi^2-1})]^{2/3} & \text{for } \xi > 1 \end{cases}, \tag{17}$$

$$\tilde{T} = t - (2a)^{-4/3}\left\{B_o + \frac{1}{4a^2}\ [-B_1 + B_o(\tilde{A}_1 + \tau\ \frac{B_o^2}{6}) + \frac{1}{1152}]\right\}\ , \tag{18}$$

and

$$B_o = -\ |\tau|^{-1/2}\left(\frac{\xi^3 - 6\ \xi}{24\sqrt{|\xi^2-1|^3}} + 5/48\ |\tau|^{-\ 3/2}\right)\ , \tag{19}$$

$$B_1 = -\ |\tau|^{-1/2}\left(\frac{-4042\xi^9 + 18189\xi^7 - 28287\xi^5 - 151995\xi^3 - 259290\xi}{414720\ \sqrt{|\xi^2-1|^9}}\right. \tag{20}$$

$$+\ 5/48\ |\tau|^{-3/2}\ \frac{-9\xi^4 + 249\xi^2 + 145}{1152|\xi^2-1|^3} + \frac{345}{4608}\ |\tau|^{-3}\ \frac{\xi^3 - 6\xi}{24\sqrt{|\xi^2-1|^3}}$$

$$\left. + \frac{85085}{663552}\ |\tau|^{-9/2}\right)\ ,$$

$$\tilde{A}_1 = \left(\frac{-9\xi^2 + 249\xi^2 + 145}{1152|\xi^2-1|^3} - 7/48\ |\tau|^{-3/2}\ \frac{\xi^3-6\xi}{24\sqrt{|\xi^2-1|}^{\,3}} - \frac{455}{4608}\ |\tau|^{-3}\right) \tag{21}$$

$$\times \begin{Bmatrix} 1 & \xi>1 \\ -1 & \xi<1 \end{Bmatrix} - \frac{\tau B_o^2}{2} \quad .$$

For $\xi \to 1$ these expressions become undefined, since both τ and ξ^2-1 tend to zero. But the coefficients can be analytically approximated in the interval $(\xi-1) \varepsilon\ [-0.003, 0.004]$ by

$$B_o \approx -0.0404974\ (1.484193 - 0.484193\cdot\xi) \quad , \tag{22}$$

and

$$A_1 \approx -0.008646 \quad , \tag{23}$$

while t is to second order

$$t \approx 0.2a^{2/3}\ (\xi-1)(9+\xi) \tag{24}$$

and $t/(\xi^2-1)$ may be written as

$$\frac{t}{\xi^2-1} \approx a^{2/3}(7/5 - 2/5\xi) \quad . \tag{25}$$

Furthermore B_1 can be replaced for all arguments by the much simpler expression

$$B_1 \approx -B_o\ \frac{0.43 + 0.2992\cdot\xi}{0.44 + 1.23\cdot\xi} \tag{26}$$

without any significant loss of accuracy.

The resulting series (13,14) with (15-21) and (22-26) give the parabolic cylinderfunction $W(a,\pm x)$ for a wide range of a and x (see fig. 1) in terms of Airy functions only. To cover the whole a,x-plane they have to be complemented by the series P and X.

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FOOTNOTES AND REFERENCES

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2. National Physical Laboratory, Tables of Weber parabolic cylinder functions. Computed by Scientific Computing Service Ltd. (Her Majesty's Stationary Office, London, England, 1955), 84-85.

3. See ref. [1], p. 693, chapter 19.19.

4. F. W. J. Olver, Uniform asymptotic expansions for Weber parabolic cylinder functions of large order, J. Research NBS 63B, 2, 131-169 (1959).

FIGURE CAPTIONS

Fig. 1. The region where various expansions of the parabolic cylinder functions have at least a relative accuracy of 10^{-6} (as computed on a CDC 6600).

P: Power series solution

D_1: Darwins expansion for $x^2 \ll 4a$

D_2: Darwins expansion for $x^2 \gg 4a$

X : Expansion for $x \gg 1$ in [3], the 10^{-6} area is actually much larger than $x \gg a$, as assumed by [3].

A: Uniform asymptotic expansion to the order $a^{-10/3}$.

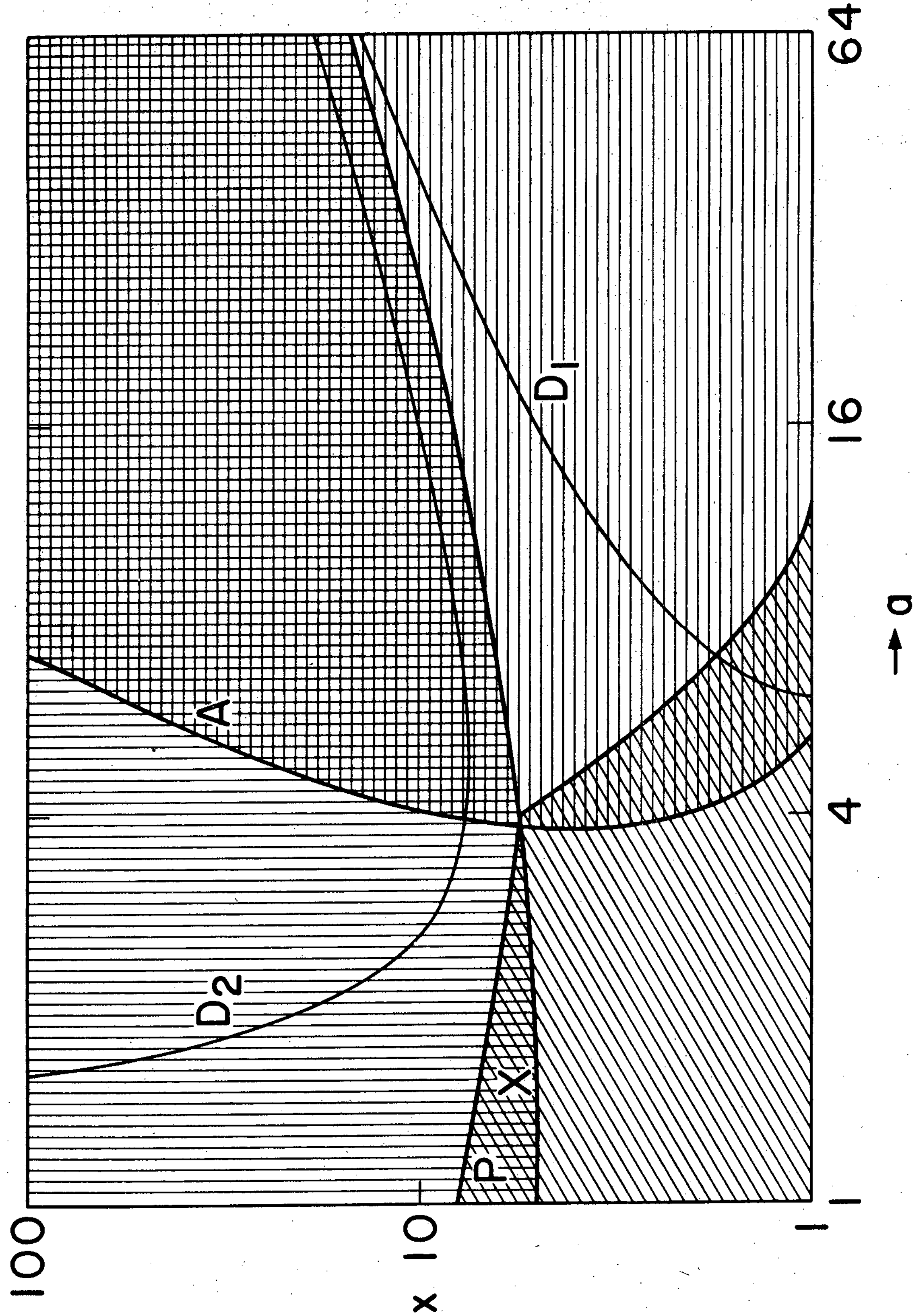

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Fig. 1

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