

UC Berkeley

UC Berkeley Electronic Theses and Dissertations

Title

Combinatorics and Computations in Tropical Mathematics

Permalink

<https://escholarship.org/uc/item/7cv95652>

Author

Lin, Bo

Publication Date

2017

Peer reviewed|Thesis/dissertation

Combinatorics and Computations in Tropical Mathematics

by

Bo Lin

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Mathematics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Bernd Sturmfels, Chair

Professor James Demmel

Professor Haiyan Huang

Summer 2017

Combinatorics and Computations in Tropical Mathematics

Copyright 2017
by
Bo Lin

Abstract

Combinatorics and Computations in Tropical Mathematics

by

Bo Lin

Doctor of Philosophy in Mathematics

University of California, Berkeley

Professor Bernd Sturmfels, Chair

In recent decades, tropical mathematics gradually evolved as a field of study in mathematics and it has more and more interactions with other fields and applications. In this thesis we solve combinatorial and computational problems concerning implicitization, linear systems and phylogenetic trees using tools and results from tropical mathematics.

In Chapter 1 we briefly introduce basic aspects of tropical mathematics, including tropical arithmetic, tropical polynomials and varieties, and the relationship between tropical geometry and phylogenetic trees.

In Chapter 2 we solve the implicitization problem for almost-toric hypersurfaces using results from tropical geometry. With results about the tropicalization of varieties, we present and prove a formula for the Newton polytopes of almost-toric hypersurfaces. This leads to a linear algebra approach for solving the implicitization problem. We implement our algorithm and show that it has better solving ability than some existing methods.

In Chapter 3 we compute linear systems on metric graphs. In algebraic geometry, linear systems on curves have been well-studied in the literature. In comparison, linear systems on metric graphs are their counterparts in tropical mathematics. We introduce the anchor divisors and anchor cells in order to compute the polyhedral cell complex structure of the linear system $|D|$ and the tropical semi-module $R(D)$. We develop algorithms, discuss their efficiency and present computational results for canonical linear systems on metric graphs.

In Chapter 4 we extend the previous chapter by introducing the tropical Hodge bundle, which is the counterpart of the Hodge bundle in tropical geometry. We study properties of tropical Hodge bundles and present computational results for some examples.

In Chapter 5 we study the geometry of metrics and convexity structures on the space of phylogenetic trees, which is here realized as the tropical linear space of all ultrametrics. The CAT(0)-metric of Billera-Holmes-Vogtman arises from the theory of orthant spaces. While its geodesics can be computed by the Owen-Provan algorithm, geodesic triangles are complicated. We show that the dimension of such a triangle can be arbitrarily high. Tropical convexity and the tropical metric exhibit properties that are desirable for geometric statistics, such as geodesics of small depth.

In Chapter 6 we study the Fermat-Weber points under the tropical metric. This is motivated by its application to the space of equidistant phylogenetic trees, here realized as the tropical linear space of all ultrametrics. While the Fréchet mean with the CAT(0)-metric of Billera-Holmes-Vogtman has been studied by many authors, the Fermat-Weber point under the tropical metric in tree spaces is not well understood. We investigate Fermat-Weber points under the tropical metric and we show that the set of tropical Fermat-Weber points for a given configuration is a classical convex polytope. We identify conditions under which this polytope is a point.

To My Family

Contents

Contents	ii
List of Figures	iv
List of Tables	vi
1 Introduction	1
1.1 Tropical semiring and semimodules	1
1.2 Tropical polynomials and varieties	2
1.3 Tropical geometry and phylogenetics	5
2 Almost-toric hypersurfaces	8
2.1 Formula for the Newton polygon	8
2.2 Proof, examples, and the Philippon-Sombra formula	10
2.3 Algorithm, implementation and case study	18
2.4 Summary	23
3 Computing linear systems on metric graphs	24
3.1 Definitions	24
3.2 The polyhedral cell complex $ D $	28
3.3 The tropical semimodule $R(D)$	31
3.4 Computations of $ D $ and $R(D)$	32
3.5 Canonical linear systems on some trivalent graphs	42
3.6 Future problems	44
4 Tropical Hodge bundles	45
4.1 Introduction of the tropical Hodge bundle	46
4.2 Moduli of tropical curves	47
4.3 Structure of the tropical Hodge bundle	51
4.4 Examples with small genus	56
4.5 The realizability problem	58
5 Convexity in tree spaces	61

5.1	Introduction	61
5.2	Orthant spaces	64
5.3	Geodesic triangles	68
5.4	Tropical convexity	74
5.5	Experiments with depth	79
5.6	Towards tropical statistics	82
6	Tropical Fermat-Weber points	87
6.1	Introduction	87
6.2	The tropical Fermat-Weber point	89
6.3	Uniqueness of the Fermat-Weber point	94
6.4	Fermat-Weber points within tree spaces	102
6.5	The m -ellipses under the tropical metric	103
6.6	Computing the Fermat-Weber points	105
	Bibliography	108

List of Figures

3.1	A metric graph Γ_1 with skeleton C_3 and metric 3, 3, 4.	25
3.2	A divisor $D = 2 \cdot A + 3 \cdot B + 1 \cdot C + (-1) \cdot E$ on Γ_1 in Figure 3.1.	26
3.3	the metric graph Γ	27
3.4	the divisor D	27
3.5	rational function f (blue) on the metric graph Γ (black)	27
3.6	the effective divisor $D + (f)$	27
3.7	Two cells in $ D $ only differ by m_e	29
3.8	The polyhedral complex $ K_{H_2} $	30
3.9	A divisor D such that $R(D)$ is not closed under the usual addition.	31
3.10	The graph of f (thick) on the edge e_j	34
3.11	The association map ϕ on one edge	39
3.12	The association map of $ K_{H_2} $. There are 8 anchor cells and 3 non-anchor cells.	40
3.13	A non-extremal divisor and the two subgraphs (thick) that can fire. The corresponding rational function takes value 1 on the thick lines and 0 on the normal lines and is linear with outgoing slope 1 on the dotted lines (from thick to normal lines).	42
3.14	The trivalent graph (020)	43
4.1	The face lattice of $\mathcal{H}_2^{trop}$. The numbers in green are the positive $h(v)$ and the numbers in black denote coefficients greater than 1 in the divisors.	48
4.2	The tropical curve $\Gamma_{\max}$ with $2g - 2$ vertices (bold) and $3g - 3$ edges.	53
4.3	The divisor D on $\Gamma_{\max}$	54
4.4	The polyhedral complex $ K_{G_\theta} $. Its f -vector is $(4, 3)$	56
4.5	The main skeleton of $ K_G $. An extra vertex is connected to the 10 vertices in red.	57
4.6	Each bat consists of 5 extra vertices (green). This bat appears when $M_{12} < M_{13}, M_{14}$	57
4.7	Numbers are the coefficients of divisors and the $\triangle$ symbols show the equal line segments.	58
4.8	Representatives of two chamber orbits of length 12.	58
4.9	Representatives of a chamber orbit of length 3.	58
4.10	Representatives of a chamber orbit of length 24.	58

5.1	The space of ultrametrics $\mathcal{U}_4$ is a two-dimensional fan with 15 maximal cones. Their adjacency forms a Petersen graph. Depicted on the right is a cycle of five cones.	62
5.2	The equidistant tree in $\mathcal{U}_4^{[1]}$ discussed in Example 5.7.	67
5.3	Three points in the open book with three pages. Their geodesic convex hull is shown on the right. The left diagram shows the subset formed by the pairwise geodesics.	70
5.4	The tropical triangle formed by three equidistant trees on five taxa.	76
5.5	Two convex hulls of three points a, b, c inside the open book with three pages. The geodesic triangle is shown on the left, while the tropical triangle is on the right.	83
6.1	The Fermat-Weber points of three points in Example 6.8 is a closed triangle (blue).	94
6.2	The 3-ellipse with $a = 8$. It is a hexagon.	104
6.3	The 3-ellipse with $a = 10$. It is a 13-gon.	105
6.4	The 3-ellipse with $a = 50$. It is a 18-gon.	106
6.5	The 3-ellipse with $a = 100$. It is a 18-gon.	107

List of Tables

2.1	Simple Examples	21
2.2	$n = 4, d = 4, k = 5$	22
3.1	A comparison of the two approaches from step (1).	38
3.2	Structure of $ K $ and R_K for different metrics on K_4	43
3.3	Structure of $ K $ and R_K for different metrics on (020)	44
3.4	Structure of $R_{M,k}$ and $ K $ for different metrics on $K_{3,3}$	44
5.1	The 23 ultrametrics (with tree topologies) at the nodes in Fig. 5.4.	77
5.2	The rows are labeled by the number m of taxa and the columns are labeled by the possible depths of a geodesic in tree space. The entries in each row sum to 100%. They are the frequencies of the depths among 1000 geodesics, randomly sampled using Algorithm 5.23.	81
5.3	Frequencies of the depths among 1000 tropical segments. The same input data as in Table 5.2 was used.	82
6.1	The maximal dimension of the set of Fermat-Weber points within the treespace: 60 samples of 4 or 5 or 6 phylogenetic trees with 4 leaves.	103

Acknowledgments

First of all, my deepest gratitude goes to my advisor, Prof. Bernd Sturmfels, for his constant guidance and encouragement throughout the doctorate program. It's him who led me into the beautiful world of tropical geometry, as well as other fields in mathematical research, for example computational algebraic geometry and algebraic statistics. In addition, from him I saw the substance of an outstanding scholar: enthusiasm in research and teaching, interests and insights in various branches of mathematics, and enthusiasm about discussions and collaborations. Hopefully I will be a scholar like him in the future.

I am grateful to my collaborators. I had a great time working with them and learned a lot from them. They are Martin Ulirsch on Chapter 4, Bernd Sturmfels, Xiaoxian Tang and Ruriko Yoshida on Chapter 5, and Ruriko Yoshida on Chapter 6. I also thank them for their generous permission on including these chapters in this dissertation.

During the writing of the chapters, I had helpful discussions with many people and received many useful comments from them. I would like to thank Carlos Amendola, Yang An, Matt Baker, Frank Ban, Madeline Brandt, Melody Chan, Angelica Cueto, Ruth Davidson, Maria Gillespie, Simon Hampe, Nathan Ilten, Steven Karp, Joe Kileel, Diane MacLagan, Madhusudan Manjunath, Ralph Morrison, Luke Oeding, Megan Owen, Kristian Ranestad, Qingchun Ren, Elina Robeva, Jose Rodriguez, Zvi Rosen, Anna Seigal, David Speyer, Ngoc Tran, Chu Wang, Lauren Williams, Yunlong Xiao, Josephine Yu and Xinyi Yuan. I also thank many anonymous referees for their critical comments on parts of this dissertation.

I acknowledge the supports by the National Institute for Mathematical Sciences in Daejeon, South Korea, a Peder Sather Grant, a Coleman Fellowship and the Max-Planck-Institute for Mathematics in the Sciences in Leipzig, Germany, when working on chapters of this dissertation. I really enjoy the time spent in South Korea, Norway and Germany.

Last but not least, I am grateful to my parents and other family members. I was so lucky to have their perpetual love and support. Since they live in China, studying abroad during the doctorate program means that I had very little time with them in recent years. Yet they never complained and helped me from many aspects. This dissertation is dedicated to them and I still owe them a lot. I would also thank my roommates Ze Xu, Pei Liu and Yichi Su, for maintaining a caring and harmonious atmosphere at home.

Chapter 1

Introduction

In this chapter we briefly introduce the tropical approach in mathematics. Along the way, we present an overview of the main results of this dissertation. Tropical mathematics is a rather young area, yet it has since matured and now it is a part of the field of algebraic geometry. It also expanded into other branches of the mathematical sciences, including number theory, computational biology, optimization, and beyond. We here introduce a few aspects of tropical mathematics, especially those tools and results that will appear in subsequent chapters in this dissertation. For a comprehensive introduction, see [78]. The word *tropical* was introduced by French mathematicians (for example Jean-Eric Pin [93]) in honor of their Brazilian colleague Imre Simon [97], who was one of the pioneers of the min-plus algebra. It simply stands for the French view of Brazil. We adapt the outline in [99].

1.1 Tropical semiring and semimodules

The basic object in tropical mathematics is the tropical semiring $(\overline{\mathbb{R}}, \oplus, \odot)$. The set $\overline{\mathbb{R}}$ is $\mathbb{R} \cup \{\infty\}$. For $a, b \in \mathbb{R}$ and $c \in \overline{\mathbb{R}}$, we define

$$a \oplus b = \min(a, b) \quad , \quad c \oplus \infty = c$$

and

$$a \odot b = a + b \quad , \quad c \odot \infty = \infty.$$

Then ∞ is the neutral element of the tropical addition $\oplus$ and 0 is the "one" element of the tropical multiplication $\odot$. Clearly the operations $\oplus, \odot$ are commutative, associative and they satisfy the distributive law. However, $(\overline{\mathbb{R}}, \oplus, \odot)$ is only a semiring instead of a ring, because we cannot define subtraction in it. For example, there does not exist $x \in \overline{\mathbb{R}}$ such that $x \oplus 0 = 3$, because $\min(x, 0)$ is at most 0.

Remark 1.1. *The above arithmetic is called the min-plus algebra. Symmetrically, there is the max-plus algebra, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ and $a \oplus b = \max(a, b)$. The two conventions are*

equivalent and they both appear in the literature. In this dissertation, we use the min-plus convention in Chapter 2, 5 and 6, and we use the max-plus convention in Chapter 3 and 4.

With the operations $\oplus$ and $\odot$, we can define tropical semimodules as follows:

Definition 1.2. Let $M \subset \overline{\mathbb{R}}^S$, where S is a set. The subset M is called a tropical semimodule (or tropically convex set) if

- for $x, y \in M$, we have that $x \oplus y \in M$;
- for $x \in M$ and $c \in \overline{\mathbb{R}}$, we have that $c \odot x \in M$.

In other words, M is closed under tropical addition and tropical scalar multiplication.

For a tropical semimodule, one natural question is whether it is finitely generated and if so, what is a minimal set of generators? In Chapter 3, the set $R(D)$ of rational functions for a divisor D on a metric graph has the structure of a tropical semimodule. We present algorithms and computational results concerning its minimal set of generators.

1.2 Tropical polynomials and varieties

In algebraic geometry, algebraic varieties are the central objects. As long as we can define the zeros of tropical polynomials, we can develop a parallel theory of tropical varieties. It not only has elegant results in its own right, but is also useful beyond tropical mathematics. Our Chapter 2 is an example, where we use results in tropical geometry to solve an implicitization problem for a type of projective varieties.

Let $x_1, x_2, \dots, x_n$ be variables that represent elements in the tropical semiring $(\overline{\mathbb{R}}, \oplus, \odot)$. A tropical monomial is any product of these variables, where repetition is allowed. Then each monomial is a linear function $\mathbb{R}^n \rightarrow \mathbb{R}$. For example,

$$x_2 \odot x_1 \odot x_3 \odot x_1 \odot x_2 = x_1^2 x_2^2 x_3 = 2x_1 + 2x_2 + x_3.$$

Then a tropical polynomial p is a finite linear combination of tropical monomials

$$p(x_1, x_2, \dots, x_n) = C_a \odot x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \oplus C_b \odot x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \oplus \dots$$

Here the coefficients $C_a, C_b, \dots$ are real numbers and the exponents a_i, b_i are integers. This means that p is a piecewise linear convex function $\mathbb{R}^n \rightarrow \mathbb{R}$:

$$p(x_1, x_2, \dots, x_n) = \min(C_a + \sum_{i=1}^n a_i x_i, C_b + \sum_{i=1}^n b_i x_i, \dots). \quad (1.1)$$

Definition 1.3. For a tropical polynomial p , a point $(x_1, x_2, \dots, x_n)$ is a zero of p if and only if the minimum in (1.1) is attained at least twice.

We can develop the theory of tropical varieties as the counterpart to varieties in algebraic geometry. Let K be a field with a possibly trivial valuation. We consider the ring $K[x_1^\pm, \dots, x_n^\pm]$ of Laurent polynomials over K with variables $x_1, x_2, \dots, x_n$. Then for a Laurent polynomial $f = \sum_{\mathbf{u} \in \mathbb{Z}^n} c_{\mathbf{u}} x^{\mathbf{u}}$ and a point $\mathbf{w} \in \mathbb{R}^n$, we have

$$\text{trop}(f)(\mathbf{w}) = \min_{\mathbf{u} \in \mathbb{Z}^n} (\text{val}(c_{\mathbf{u}}) + \sum_{i=1}^n u_i w_i).$$

For an ideal I in the Laurent polynomial ring, its variety $V(I)$ is contained in the n -dimensional torus $T^n = (K^*)^n$.

Definition 1.4. *The tropical hypersurface $V(\text{trop}(f))$ is the set*

$$\{\mathbf{w} \in \mathbb{R}^n \mid \text{the minimum in } \text{trop}(f)(\mathbf{w}) \text{ is attained at least twice}\}.$$

For a hypersurface $V(f)$, we define its tropicalization $\text{trop}(V(f))$ as $V(\text{trop}(f))$.

Definition 1.5. *For an ideal I in $K[x_1^\pm, \dots, x_n^\pm]$, we define the tropical variety $V(\text{trop}(I))$ as*

$$V(\text{trop}(I)) = \bigcap_{f \in I} V(\text{trop}(f)).$$

For a variety $X = V(I)$, we define its tropicalization $\text{trop}(X)$ as $V(\text{trop}(I))$.

Remark 1.6. *Unlike in the classical case, a finite set of polynomials that generate the ideal I may not be enough to cut out $V(\text{trop}(I))$.*

The following structure theorem characterizes tropical varieties.

Theorem 1.7. [78, Theorem 3.3.5] *Let X be an irreducible d -dimensional subvariety of the torus T^n . Then $\text{trop}(X)$ is the support of a balanced weighted polyhedral complex pure of dimension d .*

We now preview some of the main new results in this dissertation. In Chapter 2 we solve an implicitization problem with the help of tropical varieties. The almost-toric hypersurfaces $Z_{A,\mathbf{f}}$ are defined by parameterizations that map into projective space. In the parameterization of almost-toric hypersurfaces, each coordinate is a product of the following two factors: one is a monomial in variables $t_1, t_2, \dots, t_n$ (represented as a column of an $n \times (n+2)$ matrix A), the other is a univariate polynomial in variable x (represented as a coordinate of a vector $\mathbf{f}$). Our goal is to find the implicit defining polynomial of $Z_{A,\mathbf{f}}$. We use the fact that the Newton polytope of the defining polynomial of $Z_{A,\mathbf{f}}$ is a polygon. We first compute the tropicalization of $Z_{A,\mathbf{f}}$, which enables us to obtain a combinatorial formula of the Newton polygon of the implicit polynomial. Here the Plücker matrix P_A and the valuation matrix $V_{\mathbf{f}}$ are obtained directly from A and $\mathbf{f}$.

Theorem 2.1. *The edges of the Newton polygon of $Z_{A,\mathbf{f}}$ are the columns of the product of the Plücker matrix P_A and the valuation matrix $V_{\mathbf{f}}$. More precisely, given A and $\mathbf{f}$:*

- (a) *if $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 0$ then $Z_{A,\mathbf{f}}$ is not a hypersurface;*
- (b) *if $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 1$ then $Z_{A,\mathbf{f}}$ is a toric hypersurface;*
- (c) *if $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 2$ then $Z_{A,\mathbf{f}}$ is a hypersurface but not toric. The directed edges of the Newton polygon of $Z_{A,\mathbf{f}}$ are the nonzero column vectors of $P_A \cdot V_{\mathbf{f}}$.*

From the directed edges of the Newton polygon, we can find the lattice points in the interior and boundary of the Newton polygon of $Z_{A,\mathbf{f}}$. In other words, we know all possible terms that may appear in the implicit polynomial. In the end, we solve a linear system to determine the coefficients.

Tropical curves are the tropicalization of one-dimensional varieties. By Theorem 1.7, they are one-dimensional polyhedral complexes in $\mathbb{R}^n$, consisting of line segments and unbounded rays. If we disregard the unbounded rays and define metrics for the line segments, they become metric graphs. The linear systems on a metric graph Γ are well-studied in the literature, and parallel results of the theory for linear systems on algebraic curves have been obtained, for example, the tropical Riemann-Roch theorem [57, 82]. The article [64] studied the combinatorial structure of the linear system $|D|$ on a metric graph Γ . In particular, $|D|$ is a polyhedral cell complex. In Chapter 3 we focus on the computations of $|D|$ and $R(D)$, the corresponding set of rational functions. We present algorithms for computing the cells in $|D|$ and implementations for canonical linear systems, including our computational results for some examples. In order to find all cells in $|D|$, we introduce the anchor divisors and anchor cells. They are partially motivated from the following association map:

Theorem 3.47. *Let D be a vertex-supported effective divisor on a metric graph Γ .*

(1) *Suppose $\mathcal{C}$ is the set of all cells in $|D|$ and $\mathcal{A}$ is the set of all anchor cells in $|D|$. We define a function $\phi : \mathcal{C} \rightarrow \mathcal{A}$ as follows: let $C \in \mathcal{C}$ and $L \in C$ be a representative of C . Then we let $\phi(C)$ be the unique anchor cell represented by an anchor divisor obtained from C by bunching up all chips along each open edge. In other words, $\phi(C)$ is represented by an anchor divisor N such that*

- (i) *$N(v) = L(v)$ for all vertices v of Γ ;*
 - (ii) *suppose $f_N, f_L \in R(D)$ such that $N = D + (f_N)$ and $L = D + (f_L)$, then for every edge e of Γ and each endpoint v of e , f_N and f_L have the same outgoing slopes at v along e .*
- (2) *Let ϕ be the same map as in (1). Then for any anchor cell $A \in \mathcal{A}$, we have*

$$|\phi^{-1}(A)| = 2^{s(A)}.$$

Furthermore, for $0 \leq j \leq s(A)$, there are $\binom{s(A)}{j}$ cells of dimension $\dim A + j$ in $\phi^{-1}(A)$.

In Chapter 4 we introduce the tropical Hodge bundle, which is a cone complex whose base is the moduli space of tropical curves, and whose fibers are the corresponding canonical linear systems (as polyhedral complexes). We show that the dimension of tropical Hodge bundle is higher than that of the Hodge bundle in algebraic geometry (which is $4g - 4$):

Theorem 4.4. *Let $g \geq 2$.*

- (i) *The tropical Hodge bundle Λ_g^{trop} and the projective tropical Hodge bundle $\mathcal{H}_g^{\text{trop}}$ carry the structure of a generalized cone complex.*
- (ii) *The dimensions of Λ_g^{trop} and $\mathcal{H}_g^{\text{trop}}$ are given by*

$$\dim \Lambda_g^{\text{trop}} = 5g - 4 \quad \text{and} \quad \dim \mathcal{H}_g^{\text{trop}} = 5g - 5$$

respectively.

- (iii) *There is a proper subdivision of M_g^{trop} such that, for all $[\Gamma]$ in the relative interior of a cone in this subdivision, the canonical linear systems*

$$|K_\Gamma| = \pi_g^{-1}([\Gamma])$$

have the same combinatorial type.

In Section 4.4, we also present some computational results for the tropical Hodge bundle in the case of small genus.

1.3 Tropical geometry and phylogenetics

An important problem in computational biology is to construct a phylogenetic tree from distance data involving n leaves. Computational biologists call the leaves taxa. The taxa might be organisms or genes, each represented by a DNA sequence, and the tree structure records their evolutionary history [90]. Biologists are interested in pairwise distances between the taxa. In general, considering n taxa, the distance between taxon i and taxon j is a positive real number d_{ij} which has been determined by some biostatistical method [70]. So we have a real symmetric $n \times n$ -matrix $D = (d_{ij})$. We would like to find a tree with n leaves such that the pairwise distances between the leaves are also given by D .

Definition 1.8. *The matrix D is called a metric if $d_{ij} \geq 0$, $d_{ii} = 0$, and $d_{ij} \leq d_{ik} + d_{kj}$ for all indices $1 \leq i, j, k \leq n$.*

Remark 1.9. *Since the matrix D is symmetric and its diagonal entries are zero, we may view it as a vector in $\mathbb{R}^{\binom{n}{2}}$.*

Let Δ be the set of all such metrics in $\mathbb{R}^{\binom{n}{2}}$. Theorem 1.10 shows the connection between phylogenetic trees and tropical mathematics.

Theorem 1.10. [78, Theorem 4.3.5] *Up to sign, the tropicalization of the open Grassmannian $G^0(2, n)$ coincides with the space of phylogenetic trees. In symbols,*

$$\text{trop}(G^0(2, n)) = -\Delta.$$

Here $G^0(2, n)$ is the intersection of $G(2, n)$ and the torus $T^{\binom{n}{2}-1}$, and $G(2, n)$ is the Grassmannian variety of linear planes in K^n , where the field K has trivial valuation.

There are motivations from biology that lead us to consider the distance between phylogenetic trees, which means that we need a distance function to make the phylogenetic tree space into a metric space. The distance function suggested by Billera, Holmes and Vogtman in [21] is well-studied in the literature, and it is usually referred to as the BHV metric space. In Chapter 5, we focus on convexity in phylogenetic tree spaces. In the BHV metric space, there is a unique geodesic (shortest) path between any two points, and there is a polynomial time algorithm to compute the path. However, if we consider the geodesic triangle of three points, where we begin with a subset of the BHV metric space containing three points, and keep connecting points that are already in the subset by geodesic paths and including all points on these geodesic paths to our subset, we can get arbitrarily high dimensional subsets in the ambient space.

Theorem 5.14. *Let d be a positive integer. There exists an orthant space of dimension $2d$ and three points in that space such that their geodesic triangle contains a d -dimensional simplex.*

In comparison, we look at another distance function - the tropical metric (5.15) on the phylogenetic tree space. Under the tropical metric, the geodesic paths are not unique in general, while they might be more favorable in statistical applications, by the issue of stickiness in geometric statistics [67, 83]. In Section 5.5 we present some computational experiments with the two metric spaces.

In addition, in Section 5.6 we discuss possible statistical applications of the tropical metric. In Chapter 6, we focus on the Fermat-Weber points under the tropical metric. Unlike in the Euclidean case, the Fermat-Weber point of a finite sample of points under the tropical metric is not unique in general. In fact, the set of the Fermat-Weber points forms a (classical) convex polytope.

Proposition 6.7. *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The set of their Fermat-Weber points is a classical convex polytope in $\mathbb{R}^{n-1} \simeq \mathbb{R}^n/\mathbb{R}\mathbf{1}$.*

Nonetheless, a polytope can still be a point. Then we analyze when a sample of points has a unique Fermat-Weber point under the tropical metric. In general the uniqueness of Fermat-Weber points is rare.

Theorem 6.12. *Fix positive integers m and n . Consider the space $\mathbb{R}^{m(n-1)}$ of m points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The configurations representing an essential set of points with*

a unique Fermat-Weber point are contained in a finite union of proper linear subspaces in $\mathbb{R}^{m(n-1)}$.

Chapter 2

Almost-toric hypersurfaces

This chapter has been published under the same title [72]. In this chapter we solve the implicitization problem for almost-toric hypersurfaces, using results from tropical geometry [78]. An almost-toric hypersurface is parameterized by monomials multiplied by polynomials in one extra variable. We determine the Newton polytope of such a hypersurface, and apply this to give an algorithm for computing the implicit polynomial equation of an almost-toric hypersurface.

2.1 Formula for the Newton polygon

Toric varieties are parameterized by monomials. They form an important and rich class of examples in algebraic geometry and often provide a testing ground for theorems [46]. Here we work with toric varieties that need not be normal. We fix an algebraically closed field K , and we set $K^* = K \setminus \{0\}$. Our first ingredient is an arbitrary projective toric variety of codimension 2 in the projective space $\mathbb{P}^{n+1}$ over K . This is defined as follows.

Fix an $n \times (n+2)$ integer matrix A whose columns span the lattice $\mathbb{Z}^n$:

$$A = \begin{bmatrix} \mathbf{a}_0 & \mathbf{a}_1 & \cdots & \mathbf{a}_{n+1} \end{bmatrix}.$$

The column vectors $\mathbf{a}_i$ correspond to Laurent monomials $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{1,i}} t_2^{a_{2,i}} \cdots t_n^{a_{n,i}}$ in the variables $\mathbf{t} = (t_1, t_2, \dots, t_n)$. These $n+2$ monomials specify a monomial map

$$\Phi_A : (K^*)^n \rightarrow (K^*)^{n+2}, \mathbf{t} \rightarrow (\mathbf{t}^{\mathbf{a}_0}, \mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_{n+1}}).$$

Throughout this chapter we assume that all columns of A sum to the same positive integer d . Under this hypothesis, we obtain an induced map $\Phi_A : (K^*)^n \rightarrow \mathbb{P}^{n+1}$. The toric variety X_A is the closure in $\mathbb{P}^{n+1}$ of the image of Φ_A . The degree of X_A is the normalized volume of the polytope $\text{conv}(A)$, and the equations defining X_A form the *toric ideal*, a well-studied object in combinatorial commutative algebra and its applications [100].

A natural extension of toric theory is the study of *complexity one T -varieties* [68]. They are varieties with an action of a torus whose general orbits have codimension one. They can

be viewed as a family of (possibly reducible) toric varieties over a curve. Our aim here is to explore such T-varieties from the point of view of symbolic computation. For simplicity we assume that T-varieties are rational, i.e. the underlying curve is rational, and we focus on projective hypersurfaces.

Our second ingredient is a vector of univariate polynomials in a new variable x :

$$\mathbf{f} = (f_0(x), f_1(x), \dots, f_{n+1}(x)) \in K[x]^{n+2}. \quad (2.1)$$

This vector specifies a parametric curve $Y_{\mathbf{f}} \subset \mathbb{P}^{n+1}$, namely the closure of the set of points $(f_0(x) : f_1(x) : \dots : f_{n+1}(x))$. Let $Z_{A,\mathbf{f}}$ denote the Hadamard product in $\mathbb{P}^{n+1}$ of the toric variety X_A and the curve $Y_{\mathbf{f}}$. By definition, this is the Zariski closure of the set

$$\{(\mathbf{t}^{\mathbf{a}_0} f_0(x) : \mathbf{t}^{\mathbf{a}_1} f_1(x) : \dots : \mathbf{t}^{\mathbf{a}_{n+1}} f_{n+1}(x)) \in \mathbb{P}^{n+1} \mid \mathbf{t} \in (K^*)^n, x \in K\} \subset \mathbb{P}^{n+1}.$$

Under some mild hypotheses (see Theorem 2.1(a)), the variety $Z_{A,\mathbf{f}}$ has codimension 1, and we call it the *almost-toric hypersurface* associated with $(A, \mathbf{f})$. To find the defining equation of $Z_{A,\mathbf{f}}$ is an implicitization problem. Existing approaches include Gröbner Bases, resultants and syzygies [42]. We shall present a fast method for implicitizing $Z_{A,\mathbf{f}}$ and compare our method with existing ones in Section 2.3. The output of our algorithm is the irreducible polynomial in $K[u_0, u_1, \dots, u_{n+1}]$ that vanishes on $Z_{A,\mathbf{f}}$. The torus action given by A ensures that its Newton polytope $\text{Newt}(Z_{A,\mathbf{f}})$ lies in a plane in $\mathbb{R}^{n+2}$, so it is a polygon. Our first main result is the following combinatorial formula for this Newton polygon.

The ingredients in our formula are two matrices, which we now define. The *Plücker matrix* associated with A is the $(n+2) \times (n+2)$ -matrix $P_A = (p_{ij})$ with entries

$$p_{ij} = \begin{cases} \frac{1}{\delta}(-1)^{i+j} \det(A_{[i,j]}), & i < j; \\ -p_{ji}, & i > j; \\ 0, & i = j, \end{cases}$$

where δ is the greatest common divisor of all subdeterminants $\det(A_{[i,j]})$. Here $A_{[i,j]}$ is the $n \times n$ submatrix of A obtained by deleting the columns $\mathbf{a}_i$ and $\mathbf{a}_j$. The Plücker matrix P_A is skew-symmetric and has rank 2, and its row space is the kernel of A . The latter property implies that all rows and all columns of P_A sum to zero.

The *valuation matrix* associated with $\mathbf{f}$ is an integer matrix $V_{\mathbf{f}}$ with $n+2$ rows that is defined as follows. Since K is algebraically closed, each of our polynomials $f_i(x)$ factors into linear factors in $K[x]$. Let $g_1(x), g_2(x), \dots, g_m(x)$ be the list of all distinct linear factors of $f_0(x)f_1(x) \cdots f_{n+1}(x)$. We write $\text{ord}_{g_j} f_i$ for the order of vanishing of $f_i(x)$ at the unique root of $g_j(x)$. We organize these numbers into the vectors

$$\mathbf{u}_j = (\text{ord}_{g_j} f_0, \text{ord}_{g_j} f_1, \dots, \text{ord}_{g_j} f_{n+1}) \in \mathbb{N}^{n+2} \quad \text{for } 1 \leq j \leq m.$$

We now aggregate these vectors according to the lines they span. Let $S_{\mathbf{f}} = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$. If two vectors in $S_{\mathbf{f}}$ are linearly dependent, then we delete them and add their sum to

the set. We repeat this procedure. After finitely many steps, we end up with a new set $S'_f = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_\ell\}$ whose vectors span distinct lines. The *valuation matrix* is

$$V_f = \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T & \dots & \mathbf{v}_\ell^T & (-\sum_{j=1}^\ell \mathbf{v}_j)^T \end{bmatrix}.$$

The last vector represents the valuation at ∞ . It ensures that the rows of V_f sum to zero. The following theorem allows us to derive the Newton polygon from A and f .

Theorem 2.1. *The edges of the Newton polygon of $Z_{A,f}$ are the columns of the product of the Plücker matrix P_A and the valuation matrix V_f . More precisely, given A and f ,*

- (a) *if $\text{rank}(P_A \cdot V_f) = 0$ then $Z_{A,f}$ is not a hypersurface;*
- (b) *if $\text{rank}(P_A \cdot V_f) = 1$ then $Z_{A,f}$ is a toric hypersurface;*
- (c) *if $\text{rank}(P_A \cdot V_f) = 2$ then $Z_{A,f}$ is a hypersurface but not toric. The directed edges of the Newton polygon of $Z_{A,f}$ are the nonzero column vectors of $P_A \cdot V_f$.*

The rest of this chapter is organized as follows. In Section 2.2 we present the proof of Theorem 2.1, and we illustrate this result with several small examples. Earlier work of Philippon and Sombra [92, Proposition 4.1] yields an expression for the degree of the almost-toric hypersurface $Z_{A,f}$ as a certain sum of integrals over $\text{conv}(A)$.

Section 2.3 is concerned with computational issues. Our primary aim is to give a fast algorithm for computing the implicit equation of the almost-toric hypersurface $Z_{A,f}$. We develop such an algorithm and implement it in `Maple` 17. A case study of hard implicitization problems demonstrates that our method performs very well.

The mathematics under the hood of Theorem 2.1 is *tropical algebraic geometry* [78]. The proof relies on a technique known as tropical implicitization [102, 103, 104]. Thus this chapter offers a concrete demonstration that tropical implicitization can serve as an efficient and easy-to-use tool in computer algebra.

2.2 Proof, examples, and the Philippon-Sombra formula

In this section we prove Theorem 2.1. In our proof we use the technique of tropicalization. We then present examples to illustrate our main theorem. We also consider an easier task: finding the degree of the almost-toric hypersurface. We compare our result to existing results, including the Philippon-Sombra Formula in [92, Proposition 1.2] and another formula in [104].

First we prove some of the claims made in Section 2.1.

Lemma 2.2. *Let $Z_{A,f}$ be an almost-toric hypersurface. Then $\text{Newt}(Z_{A,f})$ is at most 2-dimensional in $\mathbb{R}^{n+2}$.*

Proof. Replacing the variables $u_0, \dots, u_{n+1}$ by the parameterization $u_i = \mathbf{t}^{\mathbf{a}_i} f_i(x)$ in the implicit equation $p(u_0, \dots, u_{n+1})$ of $Z_{A,\mathbf{f}}$, we get another polynomial in variables $t_1, t_2, \dots, t_n, x$. The latter polynomial is the zero polynomial. After this substitution, each term in p becomes the product of a monomial in variables $t_1, t_2, \dots, t_n$ and a polynomial in x . Since p is the generator of the principal ideal corresponding to the almost-toric hypersurface, all such monomials in the variables $t_1, t_2, \dots, t_n$ are the same; otherwise the almost-toric hypersurface would vanish on a polynomial that contains some terms of p , a contradiction. Suppose after substitution the monomial is $\prod_{i=1}^n t_i^{\alpha_i}$. If $\mathbf{v} = (v_0, \dots, v_{n+1})$ is a vertex of $\text{Newt}(Z_{A,\mathbf{f}})$, then p contains a term $\prod_{i=0}^{n+1} u_i^{v_i}$. Therefore $\mathbf{v} \cdot A^T = (\alpha_1, \dots, \alpha_n)$. So vertices of $\text{Newt}(Z_{A,\mathbf{f}})$ satisfy n independent linear equations and we conclude that $\text{Newt}(Z_{A,\mathbf{f}})$ is at most 2-dimensional. $\square$

Remark 2.3. *Even if $Z_{A,\mathbf{f}}$ is a hypersurface, its Newton polygon could be a degenerate polygon that has only two vertices.*

Example 2.4. *Let $Z_{A,\mathbf{f}}$ be the surface in $\mathbb{P}^3$ given by the following parametrization over $\mathbb{C}$:*

$$(s^2x(x-1) : st(x-1)(x-2) : stx(x-3) : t^2(x-2)(x-3)).$$

Then the defining equation of $Z_{A,\mathbf{f}}$ is $u_0u_3 - u_1u_2 = 0$, and $\text{Newt}(Z_{A,\mathbf{f}})$ is the line segment in $\mathbb{R}^4$ with endpoints $(1, 0, 0, 1)$ and $(0, 1, 1, 0)$.

Remark 2.5. *In the definition of the valuation matrix, we need to combine linearly dependent vectors in S , because they correspond to vectors with the same slope and thus parts of the same edge in $\text{Newt}(Z_{A,\mathbf{f}})$.*

Lemma 2.6. *Let P_A be the Plücker matrix defined in Section 2.1. Then P_A is skew-symmetric. The rank of P_A is 2 and the entries in each row and column of P_A sum to 0.*

Proof. The first claim follows directly from the definition of P_A . For the second claim, note that the inner product of the i -th row of P_A and the j -th row of A is the determinant of the following $(n+1) \times (n+1)$ matrix up to sign: append the j -th row of A to matrix A and then delete the i -th column of the new matrix. Therefore $\text{row}(P_A) \subseteq \text{row}(A)^\perp$, which means that the rank of P_A is at most 2. Since A has full rank, with the definition of P_A , there is a nonzero non-diagonal entry p_{ij} of P_A . By the first claim, p_{ji} is nonzero too. So we get a nonzero 2×2 minor in P_A . For the last claim, since P_A is skew-symmetric, it is enough to prove the claim for rows. It turns out that, up to sign, the sum of entries in the i -th row is the determinant of the matrix formed by $A_{[i]}$ and the vector $\mathbf{1} = (1, 1, \dots, 1) \in K^{n+1}$, where $A_{[i]}$ is the matrix obtained from A by deleting the i -th column. Since each column of A has sum d , the vector $\mathbf{1}$ lies in the row space of $A_{[i]}$, so the matrix is singular and its determinant is zero. $\square$

Next, we explore $Z_{A,\mathbf{f}}$. We consider its tropicalization (we follow the definition from [78, Definition 3.2.1]). Let $X_A, Y_{\mathbf{f}}, Z_{A,\mathbf{f}}$ be defined as in Section 2.1. Note that $Z_{A,\mathbf{f}}$ is the Hadamard product $\overline{X_A \star Y_{\mathbf{f}}}$ of the varieties X_A and $Y_{\mathbf{f}}$. We have the following result relating the tropicalizations of $X_A, Y_{\mathbf{f}}, Z_{A,\mathbf{f}}$:

Proposition 2.7. [47, Corollary 3.3.6] *We have that*

$$\text{trop}(Z_{A,\mathbf{f}}) = \text{trop}(X_A) + \text{trop}(Y_{\mathbf{f}}),$$

where the sum is Minkowski sum.

So in order to find $\text{trop}(Z_{A,\mathbf{f}})$, it is enough to find $\text{trop}(X_A)$ and $\text{trop}(Y_{\mathbf{f}})$.

Lemma 2.8. *If $\text{row}(A)$ is the row space of A with real coefficients, then*

$$\text{trop}(X_A) = \text{row}(A).$$

Proof. By Proposition 2.7, it is enough to prove the case when $n = 1$ and then use induction. For $n = 1$ we need to show that if $a_0, a_1, \dots, a_{n+1}$ are integers and

$$X_A = \text{cl}(\{(t^{a_0} : t^{a_1} : \dots : t^{a_{n+1}}) \in (\mathbb{P}^*)^{n+1} | t \in K^*\}),$$

then

$$\text{trop}(X_A) = \{r \cdot \mathbf{a} | r \in \mathbb{R}\},$$

where $\mathbf{a} = (a_0, a_1, \dots, a_{n+1})$. In this case the ideal $I(X_A)$ is generated by binomials as follows (cf. [100, Corollary 4.3]):

$$I(X_A) = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} | \mathbf{a} \cdot \mathbf{u} = \mathbf{a} \cdot \mathbf{v} \rangle.$$

This implies that all points in $\text{trop}(X_A)$ are scalar multiples of $\mathbf{a}$. $\square$

Lemma 2.9. *Let $Y_{\mathbf{f}}$ be defined as in (2.3). Then*

$$\text{trop}(Y_{\mathbf{f}}) = \bigcup_{z \in S_{\mathbf{f}}} \{\lambda(\text{ord}_z f_0, \text{ord}_z f_1, \dots, \text{ord}_z f_{n+1}) \in \mathbb{R}^{n+2} | \lambda \geq 0\}. \quad (2.2)$$

In addition, $\text{trop}(Y_{\mathbf{f}})$ is an 1-dimensional balanced polyhedral fan in $\mathbb{R}^{n+2}$ and the rays are spanned by the vectors $\mathbf{v}_z$, where $\mathbf{v}_z = (\text{ord}_z f_0, \text{ord}_z f_1, \dots, \text{ord}_z f_{n+1})$.

Proof. Let $K' = K\{\{t\}\}$ be the field of Puiseux series [78, Example 2.1.3] in the variable t with coefficients in K . Then K' has a nontrivial valuation and is algebraically closed. Let $Y'_{\mathbf{f}}$ be the variety parameterized as in (2.1) but x varies in K' instead. Note that $Y_{\mathbf{f}}$ and $Y'_{\mathbf{f}}$ have the same ideal, so $\text{trop}(Y_{\mathbf{f}}) = \text{trop}(Y'_{\mathbf{f}})$.

By [78, Theorem 3.2.3], $\text{trop}(Y'_{\mathbf{f}}) = \overline{\text{val}(Y'_{\mathbf{f}})}$, where

$$\text{val}(Y'_{\mathbf{f}}) = \{(\text{val}(f_0(x)), \dots, \text{val}(f_{n+1}(x))) | x \in K'\}.$$

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, the right hand side of (2.2) is the closure of

$$B = \bigcup_{z \in S} \{\lambda(\text{ord}_z f_0, \text{ord}_z f_1, \dots, \text{ord}_z f_{n+1}) \in \mathbb{R}^{n+2} | \lambda \in \mathbb{Q}^+\}.$$

It then suffices to show $B = \text{val}(Y'_{\mathbf{f}})$. We first show that $B \subseteq \text{val}(Y'_{\mathbf{f}})$. Fixing $z \in S$ and $\lambda > 0$, we get a vector

$$\mathbf{u} = \lambda(\text{ord}_z f_0, \text{ord}_z f_1, \dots, \text{ord}_z f_{n+1}) \in B.$$

- If $z \neq \infty$, then $z = g_j$ for some $1 \leq j \leq m$. Since each g_j is linear, we may assume that $g_j(x) = x - r_j$ where $r_j \in K$. For each i , we have $f_i(x) = (x - r_j)^{\text{ord}_{g_j} f_i} h_i(x)$, where $h_i(r_j) \neq 0$. Then $\text{val}(h_i(r_j)) = 0$. Now if we take $x = r_j + t^\lambda \in K'$, then $f_i(x) = t^{\lambda \text{ord}_{g_j} f_i} h_i(r_j + t^\lambda)$. Note that $h_i(x)$ is a polynomial in $K[x]$, so t^λ divides $h_i(r_j + t^\lambda) - h_i(r_j)$ in K' , thus $\text{val}(h_i(r_j + t^\lambda) - h_i(r_j)) > 0$. By the property of valuation, $\text{val}(h_i(r_j + t^\lambda)) = 0$. Thus $\text{val}(f_i(x)) = \lambda \text{ord}_{g_j} f_i$, which means $\mathbf{u} \in \text{val}(Y'_f)$.
- If $z = \infty$, then $\text{ord}_z f_i = -\deg(f_i)$. We take $x = t^{-\lambda} \in K'$. Among all terms in $f_i(x)$, the term with smallest valuations is the leading term, because $\lambda > 0$. By definition,

$$\text{val}(f_i(x)) = (-\lambda) \deg(f_i) = \lambda \text{ord}_\infty f_i.$$

So $\mathbf{u} \in \text{val}(Y'_f)$, too.

We next show that $\text{val}(Y'_f) \subseteq B$. Suppose $\mathbf{u} \in \text{val}(Y'_f)$. Then there exists $x_0 \in K'$ such that $\mathbf{u} = (\text{val}(f_0(x_0)), \dots, \text{val}(f_{n+1}(x_0)))$. We may assume that $\mathbf{u} \neq 0$. We need to deal with two cases.

- All terms of x_0 are nonnegative powers of t . Since $f_i(x) = \prod_{j=1}^m g_j(x)^{\text{ord}_{g_j} f_i}$, we have

$$\text{val}(f_i(x_0)) = \sum_{j=1}^m \text{ord}_{g_j} f_i \cdot \text{val}(g_j(x_0)).$$

Note that all g_j are linear functions, then for $1 \leq j < j' \leq m$ we have $\text{val}(g_j(x_0) - g_{j'}(x_0)) = 0$. Hence for $1 \leq j \leq m$ there is at most one nonzero $\text{val}(g_j(x_0))$, while they are not all zero because $\mathbf{u} \neq 0$. Suppose $\text{val}(g_j(x_0)) = \lambda > 0$. Then $\mathbf{u} = \lambda(\text{ord}_z f_0, \dots, \text{ord}_z f_{n+1})$ where $z = g_j \in S$ and $\mathbf{u} \in B$.

- At least one term of x_0 is a negative power of t . Suppose in x_0 the term with least degree is $ct^{-\frac{p}{q}}$, where $\frac{p}{q} \in \mathbb{Q}_{\geq 0}$. Then $\mathbf{u} = \frac{p}{q}(\text{ord}_\infty f_0, \text{ord}_\infty f_1, \dots, \text{ord}_\infty f_{n+1})$ is in B , too.

Finally, by [78, Proposition 3.4.13], $\text{trop}(Y'_f)$ is 1-dimensional and balanced. $\square$

By the definition of V_f , the tropicalization $\text{trop}(Y_f)$ is exactly the union of the rays generated by column vectors in V_f . This implies the following corollary:

Corollary 2.10. *The tropicalization of $Z_{A,f}$ is*

$$\{\mathbf{u} + \lambda \cdot \mathbf{v}^T \mid \mathbf{u} \in \text{row}(A), \mathbf{v} \text{ is a column vector of } V_f, \lambda \geq 0\}.$$

We now find $\text{Newt}(Z_{A,f})$.

Proposition 2.11. *The edges of $\text{Newt}(Z_{A,f})$ are parallel to the nonzero column vectors in $P_A \cdot V_f$.*

Proof. By [78, Proposition 3.1.10], $\text{trop}(Z_{A,\mathbf{f}})$ is the support of an $(n+1)$ -dimensional polyhedral fan, which is the $(n+1)$ -skeleton of the normal fan of $\text{Newt}(Z_{A,\mathbf{f}})$. Since $\text{Newt}(Z_{A,\mathbf{f}})$ is a polygon, every cone in the $(n+1)$ -skeleton of its normal fan is a cone spanned by $\text{row}(A)$ (which is exactly the orthogonal complement of the plane that contains the polygon) and a ray inside this plane that is orthogonal to the corresponding edge of the polygon. By Corollary 2.10, every one of these directed edges belongs to $\ker(A)$ and is orthogonal to the corresponding column vectors of $V_{\mathbf{f}}$. Let $\mathbf{v}$ be a column vector of $V_{\mathbf{f}}$. Since P_A is skew-symmetric, we have

$$\mathbf{v}^T \cdot P_A \cdot \mathbf{v} = 0.$$

Hence $P_A \cdot \mathbf{v}$ is orthogonal to $\mathbf{v}^T$, which means that there exists a scalar c such that $c(P_A \cdot \mathbf{v})$ represents an edge of $\text{Newt}(Z_{A,\mathbf{f}})$. $\square$

It remains to show that the length of the column vectors in $P_A \cdot V_{\mathbf{f}}$ coincide with the edges of $\text{Newt}(Z_{A,\mathbf{f}})$. To analyze these lengths we recall the notion of multiplicity of a polyhedron which is maximal in a weighted polyhedral complex. We adopt the definition in [78, Definition 3.4.3]. We have the following result:

Lemma 2.12. [78, Lemma 3.4.6] *The lattice length of any edge (defined as the number of lattice points on the edge minus 1) of $\text{Newt}(Z_{A,\mathbf{f}})$ is the multiplicity of the corresponding $(n+1)$ -dimensional polyhedron in the normal fan of $\text{Newt}(Z_{A,\mathbf{f}})$.*

Note that if an edge of $\text{Newt}(Z_{A,\mathbf{f}})$ is expressed by a vector, then its lattice length is the content of that vector. Therefore it is enough to determine $\text{mult}(\sigma)$ for each σ in the $(n+1)$ -skeleton of the normal fan of $\text{Newt}(Z_{A,\mathbf{f}})$. We cannot achieve this directly from the definition of multiplicity, because it involves the implicit polynomial of $Z_{A,\mathbf{f}}$, which is unknown to us. However, we can find out those multiplicities with the projection of polyhedral complexes [78, p. 134]. The following proposition will lead to our main theorem.

Proposition 2.13. *For every maximal cell σ' in the $(n+1)$ -skeleton of the normal fan of $\text{Newt}(Z_{A,\mathbf{f}})$, $\text{mult}(\sigma')$ is the content of the corresponding column vector of $P_A \cdot V_{\mathbf{f}}$.*

Proof. Let Σ' be $\text{trop}(Z_{A,\mathbf{f}})$, which is a pure weighted polyhedral complex in $\mathbb{R}^{n+2}$. Suppose there are m roots $z_1, \dots, z_m \in K$ of $\prod_{i=0}^{n+1} f_i(z)$. We define another pure weighted polyhedral complex $\Sigma \subseteq \mathbb{R}^{n+m+2}$ as follows:

$$\Sigma = \{(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{n+m+2} \mid \mathbf{u} \in \text{row}(A), \mathbf{v} = \lambda \cdot e_i \text{ for some } 1 \leq i \leq m+1 \text{ and } \lambda \geq 0\},$$

where e_i is the vector with i -th component 1 and others 0, for $1 \leq i \leq m$, and $e_{m+1} = -\mathbf{1}$. Note that

$$\Sigma = \text{trop}(Z),$$

where Z is the variety parameterized by

$$\{((\mathbf{t}^{\mathbf{a}_0} : \mathbf{t}^{\mathbf{a}_1} : \dots : \mathbf{t}^{\mathbf{a}_{n+1}}), (x - z_1, x - z_2, \dots, x - z_m)) \in \mathbb{P}^{n+1} \times \mathbb{C}^m \mid \mathbf{t} \in (K^*)^n, x \in K\}.$$

Note that we can get Σ' from Σ via a projection ϕ , where ϕ keeps $u \in \text{row}(A)$ fixed and sends each e_i to the transpose of the i -th column vector $\mathbf{v}_i$ in $V_{\mathbf{f}}$. So ϕ maps a maximal cell $\sigma \in \Sigma$ to a maximal cell $\sigma' \in \Sigma'$. This correspondence of maximal cells is a bijection. Suppose σ' in the $(n+1)$ -skeleton of the normal fan of $\text{Newt}(Z_{A,\mathbf{f}})$ corresponds to the vector $\mathbf{v}_i$. Then by [78, (3.6.2)], we have

$$\text{mult}(\sigma') = \text{mult}(\sigma) \cdot [N_{\sigma'} : \phi(N_{\sigma})],$$

where $N_{\sigma'} \subseteq \mathbb{R}^{n+2}$ is the lattice generated by all integer points in the span of row vectors of A and $\mathbf{v}_i^T$, and $N_{\sigma} \subseteq \mathbb{R}^{n+m+2}$ is the lattice generated by all integer points in the space $\text{row}(A) \oplus \mathbb{R}\mathbf{e}_i$. Let $N \subseteq \mathbb{R}^{n+2}$ be the lattice generated by row vectors of A and $\mathbf{v}_i^T$. Then $N \subseteq N_{\sigma'}, N_{\sigma}$ and $[N_{\sigma'} : \phi(N_{\sigma})] = \frac{[N_{\sigma'} : N]}{[N_{\sigma} : N]}$. The lattice index $[N_{\sigma'} : N]$ is the greatest common divisor of all maximal minors of the matrix formed by all generating vectors of $N_{\sigma'}$. Here this matrix is obtained from A by adding one row vector $\mathbf{v}_i^T$ and its maximal minors are the product of δ and one entry in the i -th column of $P_A \cdot V_{\mathbf{f}}$, so $[N_{\sigma'} : N]$ is the product of δ and the content of the i -th column vector of $P_A \cdot V_{\mathbf{f}}$. Similarly, $[N_{\sigma} : N]$ is the greatest common divisor of all maximal minors of A , which is δ .

Finally the initial ideal $\text{in}_{\sigma}(I(Z))$ with respect to σ of the ideal of Z is the direct sum of two prime ideals: the first one is toric, the second one is generated by linear polynomials. Then the quotient ring of $K[u_0, \dots, u_{n+1}]$ modulo the toric ideal is a domain. Note that the quotient ring of $K[u_0, \dots, u_{n+1}, v_1, \dots, v_m]$ modulo $\text{in}_{\sigma}(I(Z))$ is the quotient ring of $K[u_0, \dots, u_{n+1}]$ modulo the toric ideal adjoining some variable(s) in $\{v_1, \dots, v_m\}$, which is also a domain. Hence $\text{in}_{\sigma}(I(Z))$ is also prime. Then by the definition of multiplicity, $\text{mult}(\sigma) = 1$. This finishes the proof. $\square$

Proof of Theorem 2.1. If $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 0$ then all column vectors of $V_{\mathbf{f}}$ belong to $\text{row}(A)$, which means $Z_{A,\mathbf{f}}$ contains the toric variety X_A and has codimension 2, so it is not a hypersurface. If $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 1$, then essentially $\mathbf{f}$ provides one parameter not appearing in X_A , which means that $Z_{A,\mathbf{f}}$ has codimension 1 and is a toric hypersurface. If $\text{rank}(P_A \cdot V_{\mathbf{f}}) = 2$, then $\text{Newt}(Z_{A,\mathbf{f}})$ is a nondegenerate polygon and with Proposition 2.11, Lemma 2.12 and Proposition 2.13 we have proved Theorem 2.1. $\square$

We illustrate Theorem 2.1 with the following example.

Example 2.14. Let $Z_{A,\mathbf{f}}$ admit the following parameterization over $\mathbb{C}$:

$$(t_1^2(x^2 + 1) : t_1 t_2 x^3(x - 1) : t_1 t_3 x(x + 1) : t_2^2(x - 2)(x^2 + 1) : t_3^2(x - 1)^2(x + 1)).$$

In this example, $A = \begin{bmatrix} 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$, the degree of all monomials in the toric part is

$$d = 2. \text{ Then } P_A = \begin{bmatrix} 0 & -2 & 2 & 1 & -1 \\ 2 & 0 & -4 & 0 & 2 \\ -2 & 4 & 0 & -2 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -2 & 0 & 1 & 0 \end{bmatrix} \text{ and } \delta = 2. \text{ The linear factors of the univariate}$$

polynomials are $x, x-1, x+i, x-i, x+1, x-2$. But we can combine $x \pm i$ into $x^2 + 1$. So the vectors in S'_f are

$$(0, 3, 1, 0, 0), (0, 1, 0, 0, 2), (2, 0, 0, 2, 0), (0, 0, 1, 0, 1), (0, 0, 0, 1, 0), (-2, -4, -2, -3, -3).$$

Then the valuation matrix of $\text{Newt}(Z_{A,f})$ is

$$V_f = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & -2 \\ 3 & 1 & 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 2 & 0 & 1 & -3 \\ 0 & 2 & 0 & 1 & 0 & -3 \end{bmatrix}. \quad (2.3)$$

Using ideal elimination in *Macaulay 2* we can compute the implicit polynomial of $\text{Newt}(Z_{A,f})$ in variables u_0, u_1, u_2, u_3, u_4 :

$$\begin{aligned} & 16u_1^4u_2^{16}u_3^2 - 40u_0u_1^4u_2^{14}u_3^2u_4 + 8u_0^2u_1^2u_2^{14}u_3^3u_4 - 16u_0u_1^6u_2^{12}u_3u_4^2 + 20u_0^2u_1^4u_2^{12}u_3^2u_4^2 \\ & + 159u_0^3u_1^2u_2^{12}u_3^3u_4^2 + u_0^4u_2^{12}u_3^4u_4^2 + 54u_0^2u_1^6u_2^{10}u_3u_4^3 - 77u_0^3u_1^4u_2^{10}u_3^2u_4^3 + 379u_0^4u_1^2u_2^{10}u_3^3u_4^3 \\ & + 5u_0^2u_1^8u_2^8u_4^4 - 27u_0^3u_1^6u_2^8u_3u_4^4 - 29u_0^4u_1^4u_2^8u_3^2u_4^4 + 163u_0^5u_1^2u_2^8u_3^3u_4^4 - 12u_0^3u_1^8u_2^6u_4^5 \\ & - 35u_0^4u_1^6u_2^6u_3u_4^5 - 425u_0^5u_1^4u_2^6u_3^2u_4^5 + 4u_0^6u_1^2u_2^6u_3^3u_4^5 + 87u_0^5u_1^6u_2^4u_3u_4^6 + 717u_0^6u_1^4u_2^4u_3^2u_4^6 \\ & + 103u_0^6u_1^6u_2^2u_3u_4^7 - 115u_0^7u_1^4u_2^2u_3^2u_4^7 + 12u_0^7u_1^6u_3u_4^8 + 4u_0^8u_1^4u_2^2u_4^8. \end{aligned}$$

The vertices of the Newton polygon of this implicit polynomial are

$$(0, 4, 16, 2, 0), (2, 8, 8, 0, 4), (3, 8, 0, 6, 5), (7, 6, 0, 1, 8), (8, 4, 0, 2, 8), (4, 0, 12, 4, 2).$$

The directed edges are

$$\begin{aligned} & (2, 4, -8, -2, 4), (-4, 4, 4, -2, -2), (-4, -4, 12, 2, -6) \\ & (1, -2, 0, 1, 0), (4, -2, -6, 1, 3), (1, 0, -2, 0, 1), \end{aligned}$$

and the product $P_A \cdot V_f$ is

$$\begin{bmatrix} -4 & -4 & 2 & 1 & 1 & 4 \\ -4 & 4 & 4 & -2 & 0 & -2 \\ 12 & 4 & -8 & 0 & -2 & -6 \\ 2 & -2 & -2 & 1 & 0 & 1 \\ -6 & -2 & 4 & 0 & 1 & 3 \end{bmatrix}.$$

The column vectors of this matrix are the directed edges of $\text{Newt}(Z_{A,f})$.

Next we compare our result with existing work. The Philippon-Sombra formula [92, Proposition 4.1] computes the degree of $Z_{A,f}$ from A and f . Let B be a $2 \times (n+2)$ matrix such that its entries are integers and its row vectors span the kernel of A , and $\mathbf{b}_0, \dots, \mathbf{b}_{n+1}$ be

the column vectors of B . Given any column vector $\mathbf{v} = (v_0, \dots, v_{n+1})^T \in \mathbb{R}^{n+2}$, we define a polytope as the convex hull of the following $n+2$ vertices: $\mathbf{c}_i$ is the vector formed by the i -th column vector of A and v_{i-1} for $1 \leq i \leq n+2$. Then the vector $B \cdot \mathbf{v}$ admits a triangulation T of this polytope: for all $0 \leq i < j \leq n+1$, the n -dimensional simplex formed by vertices excluding $\mathbf{c}_i, \mathbf{c}_j$ belongs to this triangulation if and only if the vector $B \cdot \mathbf{v}$ belongs to the nonnegative span of $\mathbf{b}_i$ and $\mathbf{b}_j$. Now we define a sum

$$\partial_{\mathbf{v}}(A) = \frac{1}{\delta} \sum_{\sigma \in T} |A_{\sigma}| \sum_{i \in \sigma} v_i.$$

Here A_{σ} is the $n \times n$ minor of A that contains all columns corresponding to σ .

Proposition 2.15. *The degree of an almost-toric hypersurface $Z_{A,\mathbf{f}}$ is*

$$\sum_{\mathbf{v} \text{ is a column vector of } V_{\mathbf{f}}} \partial_{\mathbf{v}}(A).$$

This proposition is a direct corollary of [92, Proposition 4.1].

Remark 2.16. *For some $\mathbf{v}$, $\partial_{\mathbf{v}}(A)$ is negative, so the computation of degree using the Philippon-Sombra formula is not very simple. We attempted to obtain an alternative interpretation of the formula such that all summands are positive, but failed. However in Section 2.3 we present an algorithm to compute the implicit polynomial of $\text{Newt}(Z_{A,\mathbf{f}})$, and in step (3) we can compute the degree of this polynomial efficiently.*

Theorem 5.2 in [103] also provides an alternative way to compute the degree of $Z_{A,\mathbf{f}}$, using tropical geometry. Applying this theorem to our almost-toric hypersurface $Z_{A,\mathbf{f}}$ we have the following corollary.

Corollary 2.17. *For a generic column vector $\mathbf{w} \in \mathbb{R}^{n+2}$, the i -th coordinate of the vertex $\text{face}_{\mathbf{w}}(\text{Newt}(Z_{A,\mathbf{f}}))$ is the number of intersection points, each counted with its intersection multiplicity, of the tropical hypersurface $\text{trop}(Z_{A,\mathbf{f}})$ with the half line $\mathbf{w} + \mathbb{R}_{\geq 0}\mathbf{e}_i$.*

Here intersection multiplicity is defined in the remark after [103, Theorem 5.2] and $\mathbf{e}_i \in \mathbb{R}^{n+2}$ is the column vector with i -th component 1 and others 0 for $1 \leq i \leq n+2$. From Corollary 2.17 we get the following proposition.

Proposition 2.18. *Let $p(u_0, \dots, u_{n+1})$ be the implicit polynomial of $Z_{A,\mathbf{f}}$. Then for a generic vector $\mathbf{w} \in \mathbb{R}^{n+2}$, the initial monomial in $_{\mathbf{w}}p$ is*

$$\prod_{(i,j) \in U} u_i^{|\mathbf{e}_i P_A \mathbf{v}_j|},$$

where

$$U = \{(i,j) | 1 \leq i \leq n+2, \mathbf{v}_j \text{ is a column vector of } V_{\mathbf{f}}, \\ \mathbf{e}_i^T P_A \mathbf{w}, \mathbf{e}_i^T P_A \mathbf{v}_j, \mathbf{v}_j^T P_A \mathbf{w} \text{ have the same sign.}\}.$$

Hence the degree of $Z_{A,\mathbf{f}}$ is

$$\sum_{(i,j) \in U} |\mathbf{e}_i P_A \mathbf{v}_j|.$$

Proof. If $\mathbf{w}$ is generic, then $\text{trop}(Z_{A,\mathbf{f}})$ and $\mathbf{w} + \mathbb{R}_{\geq 0} \mathbf{e}_i$ have at most one intersection point. Let $\mathbf{v}_j$ be the j -th column vector of matrix $V_{\mathbf{f}}$ and $\text{col}(A^T) = \{\mathbf{u}^T | \mathbf{u} \in \text{row}(A)\}$. Suppose there is an intersection point of the maximal cone spanned by $\text{row}(A)$ and the half line $\mathbb{R}_{\geq 0} \mathbf{v}_j$ and $\mathbf{w} + \mathbb{R}_{\geq 0} \mathbf{e}_i$. Since $\mathbf{w}$ is generic, the 2-dimensional subspace spanned by $\{\mathbf{w}, \mathbf{e}_i\}$ has a unique common point with a translation of a codimension 2 subspace: $\mathbf{w} - \text{col}(A^T)$. Then there exists a unique pair of nonzero (because $\mathbf{w}$ is generic) real numbers λ_1, λ_2 and $\mathbf{u} \in \text{row}(A)$ such that $\mathbf{u}^T + \lambda_1 \mathbf{v}_j = \mathbf{w} + \lambda_2 \mathbf{e}_i$. Then the intersection point exists if and only if $\lambda_1, \lambda_2 > 0$. Note that $\lambda_1 > 0$ if and only if $\mathbf{v}_j$ and $\mathbf{w}$ are on the same side of the hyperplane spanned by $\text{col}(A^T) = \{\mathbf{u}^T | \mathbf{u} \in \text{row}(A)\}$ and $\mathbf{e}_i$ (if they don't span a hyperplane, then $\mathbf{e}_i \in \text{col}(A^T)$ and no intersection point exists since $\mathbf{w}$ is generic). So $\det([\mathbf{v}_j \ \mathbf{e}_i \ A^T])$ and $\det([\mathbf{w} \ \mathbf{e}_i \ A^T])$ have the same sign.

Note that for any column vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{n+2}$, $\det\left(\begin{bmatrix} \mathbf{b}^T \\ \mathbf{a}^T \\ A \end{bmatrix}\right) = \det([\mathbf{b} \ \mathbf{a} \ A^T]) = \delta \mathbf{a}^T P_A \mathbf{b}$,

where δ is introduced in the definition of Plücker matrix. Hence $\lambda_1 > 0$ if and only if $\mathbf{e}_i^T P_A \mathbf{w}$ and $\mathbf{e}_i^T P_A \mathbf{v}_j$ have the same sign. Similarly, $\lambda_2 > 0$ if and only if $\mathbf{w}$ and $\mathbf{e}_i$ are on the opposite sides of the hyperplane spanned by $\text{col}(A^T)$ and $\mathbf{v}_j$, which is equivalent to $\mathbf{e}_i^T P_A \mathbf{v}_j$ and $\mathbf{v}_j^T P_A \mathbf{w}$ having the same sign. So $(i, j) \in S$ if and only if there is an intersection point of the maximal cone spanned by $\text{row}(A)$ and the half line $\mathbb{R}_{\geq 0} \mathbf{v}_j$ with the half line $\mathbf{w} + \mathbb{R}_{\geq 0} \mathbf{e}_i$. Next it suffices to show that the intersection multiplicity of this point is $|\mathbf{e}_i P_A \mathbf{v}_j|$. By the definition of intersection multiplicity, for this point it is the lattice index of the lattice spanned by $\mathbf{e}_i, \mathbf{v}_j$ and the transpose of row vectors of A , so the intersection multiplicity is $|\mathbf{e}_i P_A \mathbf{v}_j|$. $\square$

Remark 2.19. Proposition 2.18 enables us to compute the degree of $Z_{A,\mathbf{f}}$ without knowing $\text{Newt}(Z_{A,\mathbf{f}})$.

2.3 Algorithm, implementation and case study

Algorithm to compute the implicit polynomial

For an almost-toric hypersurface, we would like to compute its implicit polynomial in $n + 2$ variables $u_0, u_1, \dots, u_{n+1}$ from A and $\mathbf{f}$. An existing approach uses ideal elimination with Gröbner bases, which is inefficient when n is large. Based on Theorem 2.1 we have the following alternative approach:

1. Compute P_A from A , factorize $f_0, f_1, \dots, f_{n+1}$ over K into irreducible factors to get $V_{\mathbf{f}}$.
2. Compute $P_A \cdot V_{\mathbf{f}}$ and verify that it has rank 2.

3. Find $\text{Newt}(Z_{A,\mathbf{f}})$ using Theorem 2.1.
4. Determine all possible monomials in variables $u_0, u_1, \dots, u_{n+1}$ that could appear in the implicit polynomial.
5. Use linear algebra to compute the coefficients of these monomials.

We now explain our implementation of this method using the software **Maple 17**.

Step (1) and (2)

Among the five steps, the first and second are trivial to implement (**Maple 17** has the command *Afactor* which factors a polynomial into irreducible factors over a given field).

Step (3)

Theorem 2.1 tells us that the set of directed edges are the column vectors of $P_A \cdot V_{\mathbf{f}}$. Then we need to arrange them in the correct order. We could project these vectors to a 2-dimensional space, by choosing two of the coordinates $1 \leq c_1 < c_2 \leq n+2$. There is still the problem of orientation: these directed edges admit two different arrangements. The correct orientation is determined by the sign of the entry $(P_A)_{c_1, c_2}$.

Now suppose all directed edges are arranged in correct order and are the column vectors of a matrix

$$[c_{i,j}]_{1 \leq i \leq n+2, 1 \leq j \leq m}.$$

If the vertex that corresponds to the first and m -th edges has coordinates $r_1, \dots, r_{n+2}$, then the other vertices have coordinates

$$(r_1 + \sum_{j=1}^k c_{1,j}, r_2 + \sum_{j=1}^k c_{2,j}, \dots, r_{n+2} + \sum_{j=1}^k c_{n+2,j}), k = 1, \dots, m-1.$$

We notice that for $1 \leq i \leq n+2$, the i -th coordinate of the vector of vertices corresponds to the exponents of t_i . Since the implicit polynomial is irreducible, the minimum of these exponents must be 0:

$$\min_{1 \leq k \leq m} \{r_i + \sum_{j=1}^k c_{i,j}\} = 0, 1 \leq i \leq n+2.$$

Hence

$$r_i = - \min_{1 \leq k \leq m} \left\{ \sum_{j=1}^k c_{i,j} \right\}, 1 \leq i \leq n+2.$$

So $\text{Newt}(Z_{A,\mathbf{f}})$ is uniquely determined by $P_A \cdot V_{\mathbf{f}}$ and we can compute its vertices using the formula above.

Step (4)

Given the vertices of $\text{Newt}(Z_{A,\mathbf{f}})$, we need to find all lattice points of this polygon. Since all of them lie in the translation of a 2-dimensional subspace, projection onto 2 coordinates would work. We find all lattice points within a convex polygon (which may be degenerate) as follows:

- compute the upper and lower bound for both coordinates of such a lattice point and thus find a rectangle of the form $[l_1, u_1] \times [l_2, u_2]$ containing the polygon;
- list the defining inequalities of the polygon and for each lattice point in the rectangle, check whether it satisfies all defining inequalities of the polygon.

Then we recover the corresponding lattice points in $\text{Newt}(Z_{A,\mathbf{f}})$. These lattice points correspond to all possible monomials in the implicit polynomial of Z : components of each vector are the exponents of variables $u_0, u_1, \dots, u_{n+1}$.

Step (5)

Given all monomials that can occur in the implicit polynomial $p(u_0, \dots, u_{n+1})$, it is enough to find their coefficients. Consider the coefficients as undetermined unknowns. After the substitution $u_i = \mathbf{t}^{\mathbf{a}_i} f_i(x)$ we get another polynomial $q(t_1, \dots, t_n, x)$. Each term in q is the product of a coefficient, a monomial in the variables $t_1, \dots, t_n$ and a polynomial in the variable x . We claim that all these monomials in variables $t_1, \dots, t_n$ are the same. Assume the opposite situation. Then we can pick all terms in q with a particular monomial in variables $t_1, \dots, t_n$ and get the corresponding monomials in p , which form another polynomial $p'(u_0, \dots, u_{n+1})$ with less terms than p . Since p is the implicit polynomial, the polynomial q must be identically zero. Then p' vanishes everywhere too, so p' belongs to the principal ideal of our hypersurface, a contradiction! So after the substitution we can cancel the unique monomial in the variables $t_1, \dots, t_n$ and get a univariate polynomial in x with undetermined coefficients. Since this polynomial is identically zero, the undetermined coefficients satisfy a system of homogeneous linear equations.

Next we use interpolation. Suppose there are k possible monomials in the implicit polynomial, then we replace x by integers ranging from $-r$ to r , where $r = \lfloor \frac{k}{2} \rfloor$. Each interpolation gives a linear equation with k coefficients. Then we use the *solve* command in **Maple 17** to solve these coefficients. Since this is a homogeneous linear system, the solution space should be 1-dimensional. This leads us to add another equation, for example $a_1 = 1$ where a_1 is one of the coefficients, to guarantee the uniqueness of solution. After getting the solution, if all coefficients are rational, we normalize them so that their content is 1.

Example 2.20. Let $Z_{A,\mathbf{f}}$ be the almost-toric surface in $\mathbb{P}^3$ parameterized by

$$(s^3x^2(x-1) : s^2t(x^2+1) : st^2x(x+1)^2 : t^3(x-1)(x-2)).$$

Then $A = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $V_f = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 2 & 0 & 0 & -2 \\ 1 & 0 & 0 & 2 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1 & -2 \end{bmatrix}$. The vertices of $\text{Newt}(Z_{A,f})$ are

$$(6, 0, 0, 6), (4, 0, 6, 2), (2, 2, 8, 0), (1, 4, 7, 0), (0, 7, 4, 1), (2, 6, 0, 4).$$

We find all monomials in the implicit polynomial to be

$$\begin{aligned} & u_0^6 u_3^6, u_0^5 u_1 u_2 u_3^5, u_0^5 u_2^3 u_3^4, u_0^4 u_1^3 u_3^5, u_0^4 u_1^2 u_2^2 u_3^4, u_0^4 u_1 u_2^4 u_3^3, u_0^4 u_2^6 u_3^2, u_0^3 u_1^4 u_2 u_3^4, u_0^3 u_1^3 u_2^3 u_3^3, \\ & u_0^3 u_1^2 u_2^5 u_3^2, u_0^3 u_1 u_2^7 u_3, u_0^2 u_1^6 u_3^4, u_0^2 u_1^5 u_2^2 u_3^3, u_0^2 u_1^4 u_2^4 u_3^2, u_0^2 u_1^3 u_2^6 u_3, u_0^2 u_1^2 u_2^8, u_0 u_1^6 u_2^3 u_3^2, \\ & u_0 u_1^5 u_2^5 u_3, u_0 u_1^4 u_2^7, u_1^7 u_2^4 u_3. \end{aligned}$$

We use interpolation to solve for the coefficients of these monomials and obtain the implicit polynomial

$$\begin{aligned} & 8u_0^6 u_3^6 + 52u_0^5 u_1 u_2 u_3^5 - 28u_0^5 u_2^3 u_3^4 - 48u_0^4 u_1^3 u_3^5 + 58u_0^4 u_1^2 u_2^2 u_3^4 - 82u_0^4 u_1 u_2^4 u_3^3 \\ & + 25u_0^4 u_2^6 u_3^2 - 1312u_0^3 u_1^4 u_2 u_3^4 - 175u_0^3 u_1^3 u_2^3 u_3^3 + 476u_0^3 u_1^2 u_2^5 u_3^2 - 50u_0^3 u_1 u_2^7 u_3 \\ & + 72u_0^2 u_1^6 u_3^4 - 5760u_0^2 u_1^5 u_2^2 u_3^3 + 5056u_0^2 u_1^4 u_2^4 u_3^2 - 1194u_0^2 u_1^3 u_2^6 u_3 + 25u_0^2 u_1^2 u_2^8 \\ & - 4176u_0 u_1^6 u_2^3 u_3^2 - 3256u_0 u_1^5 u_2^5 u_3 - 72u_0 u_1^4 u_2^7 - 576u_1^7 u_2^4 u_3. \end{aligned}$$

Comparison with Method of Gröbner bases

To test the efficiency of our algorithm, we try some simple examples using our implementation in Maple 17 and using ideal elimination in Macaulay 2. The result is in Table 2.1:

sample	degree	# of terms	our time cost	ideal elimination's time cost
Example 2.14	22	24	0.094s	1.875s
Example 2.20	10	16	0.047s	0.078s.

Table 2.1: Simple Examples

We then try some examples that both Macaulay 2 and Sagemath cannot solve in a reasonable time. We generate some samples as the input using the following method: let n be the dimension of the torus, d the degree of the homogeneous monomials and k a positive integer. Then we choose $n + 2$ degree d monomials randomly from all possible $\binom{n+d-1}{d}$ choices. For the univariate polynomials, we choose $n + 2$ polynomials of the form $(x - 2)^*(x - 1)^* \cdots (x + 1)^*(x + 2)^*$, where each $*$ is a random integer between 0 and k . Table 2.2 shows the time needed to find the implicit polynomial of almost-toric hypersurfaces given by randomly generated inputs. It turns out that our implementation improves the efficiency of finding the implicit polynomial of almost-toric hypersurfaces.

The codes of these tests are available at <https://sites.google.com/site/bolinmath/research/codes/ath>.

sample	degree	# of terms	time cost	sample	degree	# of terms	time cost
1	213	109	12.484s	6	179	97	8.110s
2	109	80	1.594s	7	40	32	0.156s
3	172	129	10.421s	8	27	14	0.140s
4	474	275	156.969s	9	79	71	1.766s
5	291	137	20.375s	10	281	148	20.719s

 Table 2.2: $n = 4, d = 4, k = 5$

Comparison with Method of Syzygies

We consider another example ([26, Example 3.2]), where a hypersurface in $\mathbb{P}^3$ is parameterized by

$$a = su^2, b = t^2(s + u), c = st(s + u), d = tu(s + u). \quad (2.4)$$

In [26] methods called moving planes and moving quadrics based on syzygies are used to solve implicitization problems similar to ours. For this example the parameterization has base points, including $(0 : 1 : 0)$. So one needs several assumptions and one assumption does not hold and the method of moving quadrics fails (cf. [26, Example 2.6]). Let's find the implicit polynomial for (2.4) using our algorithm.

In (2.4) the polynomials in s, t, u are homogeneous, while for almost-toric hypersurface, our counterparts are not homogeneous because of the coefficient part. But we can modify the parametrization to make it consistent with our definition. Let $x = \frac{u}{s}$. Then (2.4) becomes

$$a = s^3x^2, b = st^2(x + 1), c = s^2t(x + 1), d = s^2tx(x + 1). \quad (2.5)$$

Then we apply our algorithm. The matrix A is

$$\begin{bmatrix} 3 & 1 & 1 & 2 \\ 0 & 2 & 2 & 1 \end{bmatrix}.$$

Then $\delta = 3$ and

$$P_A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -2 \\ -1 & -1 & 2 & 0 \end{bmatrix}.$$

As for the coefficient part, $\mathbf{f} = (x^2, x + 1, x + 1, x(x + 1))$, there are only two irreducible factors x and $x + 1$. Their corresponding vectors are $(2, 0, 0, 1)$ and $(0, 1, 1, 1)$. They are linearly independent, so

$$V_{\mathbf{f}} = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & -2 \end{bmatrix}.$$

Next

$$P_A \cdot V_f = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \\ -2 & 1 & 1 \end{bmatrix}.$$

By Theorem 2.1 these three column vectors correspond to the directed edges of $\text{Newt}(Z_{A,f})$. As for the orientation, we project onto the 2-dimensional subspace by the third and fourth coordinates: $(0, -2), (-1, 1), (1, 1)$. Since $(P_A)_{3,4} = -2 < 0$, the orientation is clockwise, i.e. the edges are arranged in the following order:

$$(-1, -1, 1, 1), (1, 1, 0, -2), (0, 0, -1, 1).$$

Then by step (3) of our algorithm we can determined the three vertices of $\text{Newt}(Z_{A,f})$:

$$(1, 1, 0, 1), (0, 0, 1, 2), (1, 1, 1, 0).$$

Once again we consider its projection in the 2-dimensional subspace. The triangle formed $(0, 1), (1, 2), (1, 0)$ has no lattice point in its interior and one point $(1, 1)$ in the interior of its boundary, which has preimage $(\frac{1}{2}, \frac{1}{2}, 1, 1)$ and is not a lattice point in $\text{Newt}(Z_{A,f})$. Then by step (4), all monomials appearing in the implicit polynomial are

$$abd, cd^2, abc.$$

Note that the part of univariate polynomial in them are

$$x^3(x+1)^2, x^2(x+1)^3, x^2(x+1)^2,$$

and

$$-x^3(x+1)^2 + x^2(x+1)^3 = x^2(x+1)^2,$$

then the implicit polynomial is

$$abc + abd - cd^2 = 0.$$

2.4 Summary

Our solution of the implicitization problem of almost-toric hypersurface demonstrates an application of tropical geometry to algebraic geometry. The discussion in Section 2.3 shows the significant efficiency improvement of computing the implicit polynomial. We do not know whether our method could be applied to more general algebraic varieties. One example is the case when there are multiple factors of univariate polynomials in the product of the parameters.

Chapter 3

Computing linear systems on metric graphs

This chapter is posted on the [arXiv \[73\]](#), as a single-authored paper with the same title. In this chapter we focus on linear systems on metric graphs. In algebraic geometry, linear systems of divisors on curves are well studied (cf. [62, §3]). In tropical geometry, the analogue of algebraic curves are tropical curves. They can be viewed as metric graphs. Many scholars studied the linear systems of divisors on metric graphs, for example [10, 12, 64]. The article [13] proved a graph-theoretic analogue of the Riemann-Roch Theorem, and it was generalized to tropical curves (which may contain unbounded rays) independently by [57] and by [82]. The theory of linear systems on metric graphs is applied to algebraic geometry, for example in [14].

Haase, Musiker and Yu [64] studied the cell complex structure of $|D|$ and the tropical semimodule structure of $R(D)$, where D is a divisor on a metric graph. This chapter is an extension of [64]. We focus on the computation of the cell complex $|D|$ and the tropical semimodule $R(D)$, namely given a metric graph Γ and a divisor D on it, how to find the cells in $|D|$, the f -vector of $|D|$ and the extremal generators of $R(D)$.

The outline of this chapter is as follows: in Section 3.1 we recall the definitions of the objects that we study; in Section 3.2 and Section 3.3 we introduce the anchor divisors and anchor cells, and discuss the properties of $|D|$ and $R(D)$ respectively; in Section 3.4 we present our main theorem (Theorem 3.47), introduce our algorithms and discuss their efficiency; and in Section 3.5 we present computational results on some examples of canonical linear systems. In addition, in Chapter 4 we define the tropical Hodge bundle, a counterpart of the classical Hodge bundle in tropical geometry. Its fibers over the moduli space are canonical linear systems on metric graphs.

3.1 Definitions

In this section we recall the definition of linear systems on metric graphs.

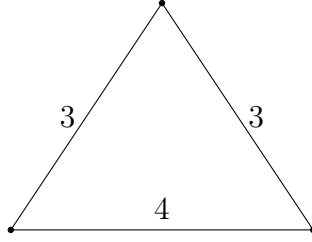


Figure 3.1: A metric graph Γ_1 with skeleton C_3 and metric 3, 3, 4.

3.1.1 Metric graphs

Definition 3.1. A metric graph Γ is a connected undirected finite graph whose edges have positive lengths. A metric graph Γ is determined by its graph-theoretic type and the lengths of its edges. The graph-theoretic type of Γ is called the skeleton of Γ , and the lengths of the edges of Γ are called the metric of Γ and denoted by M . For an edge e of Γ we denote by M_e the length of e in M .

Figure 3.1 shows a metric graph Γ_1 .

Remark 3.2. We do not require a metric graph to be simple, because metric graphs with loops or parallel edges play an important role in Berkeovich theory [18]. See also [25, Figure 4].

Remark 3.3. Tropical curves are closely related to metric graphs. As 1-dimensional polyhedral complexes, they may have different models $\Gamma = (V, E)$: different models have different set of vertices.

Definition 3.4. The genus of a metric graph $\Gamma = (V, E)$ is defined as $|E| - |V| + 1$.

3.1.2 Divisors

The definition of divisors is exactly the same as that of divisors on algebraic curves.

Definition 3.5. A divisor D on Γ is a formal finite $\mathbb{Z}$ -linear combination $D = \sum_{x \in \Gamma} D(x) \cdot x$ of points x in the edges of Γ . The divisor is effective if $D(x) \geq 0$ for all $x \in \Gamma$. The degree of a divisor D is $\sum_{x \in \Gamma} D(x)$. The support of a divisor D on Γ is the set $\{x \in \Gamma \mid D(x) \neq 0\}$, denoted as $\text{supp}(D)$.

Example 3.6. Figure 3.2 shows an effective divisor D on the metric graph Γ_1 in Figure 3.1.

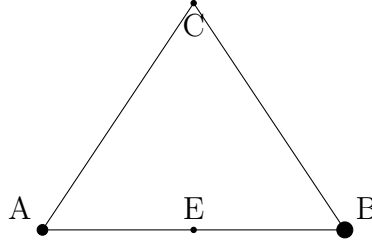


Figure 3.2: A divisor $D = 2 \cdot A + 3 \cdot B + 1 \cdot C + (-1) \cdot E$ on Γ_1 in Figure 3.1.

3.1.3 Rational functions

Definition 3.7. A (tropical) rational function f on Γ is a continuous function $f : \Gamma \rightarrow \mathbb{R}$ that is piecewise-linear on each edge of Γ with finitely many pieces and integer slopes. The order $\text{ord}_x(f)$ of f at a point $x \in \Gamma$ is the sum of the outgoing slopes at x along all directions.

Remark 3.8. Note that if x is an interior point of a linear piece of f , then there are two directions at x , and x has two opposite outgoing slopes, so $\text{ord}_x(f) = 0$.

Definition 3.9. The principal divisor associated to f is

$$(f) = \sum_{x \in \Gamma} \text{ord}_x(f) \cdot x. \quad (3.1)$$

By Remark 3.8, the support of (f) is finite, so (f) is well-defined.

Lemma 3.10. For any rational function f on a metric graph Γ , the degree of (f) is 0.

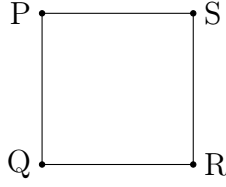
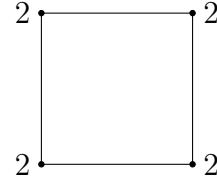
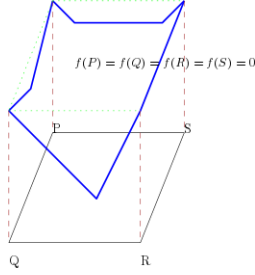
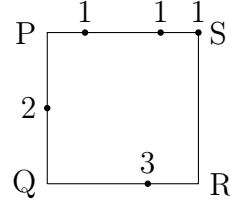
Proof. The metric graph Γ consists of finitely many linear pieces. For each linear piece $\overline{PQ}$, suppose the outgoing slope at point P towards Q is k . Then the outgoing slope at point Q towards P is $-k$. The edge $\overline{PQ}$ contributes a k and a $-k$ in the coefficients of the right hand side of (3.1). Thus $\deg((f)) = 0$. $\square$

Remark 3.11. This definition is independent of the orientation of an edge of the metric graph, because the outgoing slopes at any point $x \in \Gamma$ only depend on the rational function f .

3.1.4 Linear systems

Definition 3.12. Two divisors D and D' are linearly equivalent if $D - D' = (f)$ for some rational function f , denoted as $D \sim D'$. For any divisor D on Γ , let $R(D)$ be the set of all rational functions f on Γ such that the divisor $D + (f)$ is effective, and $|D| = \{D + (f) \mid f \in R(D)\}$, the linear system of D .

Example 3.13. Let Γ be a metric graph with graph-theoretic type C_4 and equal edge lengths. Below are examples of D and $f \in R(D)$.


 Figure 3.3: the metric graph Γ

 Figure 3.4: the divisor D

 Figure 3.5: rational function f (blue) on the metric graph Γ (black)

 Figure 3.6: the effective divisor $D + (f)$

When we fix the skeleton of Γ , for any metric M and divisor D on Γ , we denote by $R(D)_M$ the set $R(D)$ and by $|D|_M$ the linear system $|D|$.

Remark 3.14. Given a metric graph $\Gamma = (V, E, M)$ and a divisor D on Γ . The divisor D is said to be vertex-supported if $\text{supp}(D) \subseteq V$. Note that $\text{supp}(D)$ is always a finite set. If D is not vertex-supported, then we can refine Γ to get a new metric graph whose set of vertices is $\text{supp}(D)$. Also we may assume that $|D|$ is not empty, so if D is not effective, we can consider an effective divisor $D' \in |D|$. It is obvious that $|D'| = |D|$. We shall assume from now that D is vertex-supported.

3.1.5 Riemann-Roch theorem

For a divisor on a metric graph, we can still define its *rank*, and there is an analogue of the famous Riemann-Roch theorem in algebraic geometry.

Definition 3.15. Let D be a divisor on a metric graph Γ . The rank of D , denoted as $r(D)$, is the largest nonnegative integer r such that for any divisor E on Γ with $\deg(E) = r$, the linear system $|D - E|$ is nonempty. If such an r does not exist, then $r(D) = 0$.

Lemma 3.16. For any divisor D on a metric graph Γ , $r(D) \leq \max(0, \deg(D))$.

Proof. If $r(D) = 0$, the claim is trivial. Otherwise let $r(D) = r \geq 1$. By definition there exists a divisor E such that $\deg(E) = r$ and $|D - E| \neq \emptyset$. By Lemma 3.10, adding a principal divisor does not change the degree of a divisor. Since the linear system $|D - E|$ is nonempty, there is an effective divisor with the same degree as $D - E$. Thus $\deg(D - E) \geq 0$, and $r = \deg(E) \leq \deg(D) \leq \max(0, \deg(D))$. $\square$

In order to state the metric graph version of the Riemann-Roch theorem, we need to define the *canonical divisors*.

Definition 3.17. *For any metric graph Γ , the canonical divisor K of Γ is the divisor on Γ with $K(x) = d(x) - 2$ when x is a vertex of Γ and $K(x) = 0$ otherwise, where $d(x)$ is the valence of x in Γ .*

Theorem 3.18. [13] *Let Γ be a metric graph and D be a divisor on Γ . Then we have*

$$r(D) - r(K - D) = \deg(D) - g + 1, \quad (3.2)$$

where K is the canonical divisor of Γ .

3.2 The polyhedral cell complex $|D|$

In this section we present the cell complex structure of $|D|$.

3.2.1 The cell complex structure

Note that if c is a constant rational function on Γ , then for any rational function f on Γ , the divisor $D + (f + c)$ is equal to $D + (f)$. Let $\mathbf{1}$ be the set of constant functions on Γ . The set $R(D)/\mathbf{1}$ can be identified with the linear system $|D|$ by the map $\bar{f} \mapsto D + (f)$. We adapt the formulation of $|D|$ as a cell complex in [64], which originates from [57] and [82].

Definition 3.19. *We identify each open edge $e \in E$ with the interval $(0, M_e)$ (this implicitly gives an orientation of e , while it is independent of the cell complex). Each cell of $|D|$ is characterized by the following data:*

- a nonnegative integer d_v for each $v \in V$;
- an ordered partition $d_e = \sum_{i=1}^{r_e} d_e^i$ of positive integers for some of the edges $e \in E$;
- an integer m_e for each $e \in E$.

A divisor L belongs to this cell if and only if the following statements hold:

- $L(v) = d_v$ for each $v \in V$;
- for those $e \in E$ with the partition above, the divisor L on e is expressed as $\sum_{i=1}^{r_e} d_e^i x_i$, where $0 < x_1 < x_2 < \dots < x_{r_e} < M_e$; for other edges e , $L(x) = 0$ for all points x in the interior of e ;
- for any $f \in R(D)$ such that $L = D + (f)$, the outgoing slope of f at the point 0 is m_e for each $e \in E$.

Remark 3.20. The slopes m_e are also required because two distinct rational functions in $R(D)$ may lead to different effective divisors in $|D|$ with exactly the same values of the d_v and the ordered partitions of d_e .

Example 3.21. Let $\Gamma = (V, E)$ be a loop e with one vertex v and the length of e equal to 3. Let $D = 3 \cdot v$. Then the two effective divisors in Figure 3.7 belong to $|D|$. For both, all d_v are 0 and all d_e are 3 with one part in the partition, while the slopes m_e are different: one is -1 and the other is -2 .



Figure 3.7: Two cells in $|D|$ only differ by m_e

A cell is represented by any divisor $D + (f)$ in it, because once we have the rational function $f \in R(D)$, we can compute all data in Definition 3.19. There is a natural question: how can we tell the dimension of a cell from a representative? The following proposition is a good answer.

Proposition 3.22. [64, Proposition 13] Let D be a vertex-supported effective divisor on a metric graph Γ and V be the set of vertices in Γ . Let C be a cell in $|D|$ and L a representative of C . Let $I_L = \{x \in \Gamma - V \mid L(x) > 0\}$. Then $\dim C$ is one less than the number of connected components in the graph $\Gamma - I_L$.

Corollary 3.23. Let D be a vertex-supported effective divisor on a metric graph Γ and $d = \deg(D)$. The dimension of the cell complex $|D|$ is at most d . In addition, if Γ is 2-connected, then the dimension of the cell complex $|D|$ is at most $d - 1$.

Proof. Let C be a cell in $|D|$ and L a representative of C . We have $\deg(L) = \deg(D) = d$. So the support of L contains at most d points. By definition, $I_L \subseteq \text{supp}(L)$, so I_L contains at most d points as well. Now we consider the connected components in the graph $\Gamma - I_L$. First Γ is connected. And each time we delete a point in I_L , the number of connected components can increase by at most 1, because the deleted point is always interior to an edge of the current graph. Therefore there are at most $d + 1$ connected components in the graph $\Gamma - I_L$. Hence, by Proposition 3.22, we conclude that $\dim C \leq d$. If Γ is 2-connected, then when we delete one point in I_L from Γ , the remaining graph is still connected, so there are at most d connected components in the graph $\Gamma - I_L$. Hence $\dim C \leq d - 1$. $\square$

Remark 3.24. We cannot generalize Corollary 3.23 to metric graphs with higher connectivity. Because with appropriate metrics, there exists a degree- d divisor in $|D|$ whose support

consists of d distinct points on the same edge of Γ , then by Proposition 3.22, this divisor belongs to a $d - 1$ -dimensional cell. In other words, if we only delete up to one point on each edge, higher connectivity reduces the number of connected components in the resulting graph, while we can delete many points on the same edge and then higher connectivity makes no difference in this case.

In order to find the cells in $|D|$, we need to know whether there are finitely many of them. Fortunately we have the following theorem.

Theorem 3.25. [64, Theorem 14] *The cell complex $|D|$ has finitely many vertices.*

Since each cell is uniquely determined by its vertices, we have the following corollary.

Corollary 3.26. *The cell complex $|D|$ has finitely many cells.*

Example 3.27. Let H_2 be a hyperelliptic tropical curve with vertices P, Q as in Figure 3.8 : there is a loop at P , a loop at Q and an edge connecting P, Q . The canonical divisor on H_2 is $K_{H_2} = P + Q$. Then $|K_{H_2}|$ is a 2-dimensional polyhedral cell complex whose f -vector is $(5, 5, 1)$. Let P_1, Q_1 be the antipodal points of P, Q on the two loops, respectively. Then the 5 vertices of $|K_{H_2}|$ are:

$$P + Q, 2 \cdot P, 2 \cdot Q, 2 \cdot P_1, 2 \cdot Q_1.$$

Note that $|K_{H_2}|$ is not pure dimensional, but it is contractible.

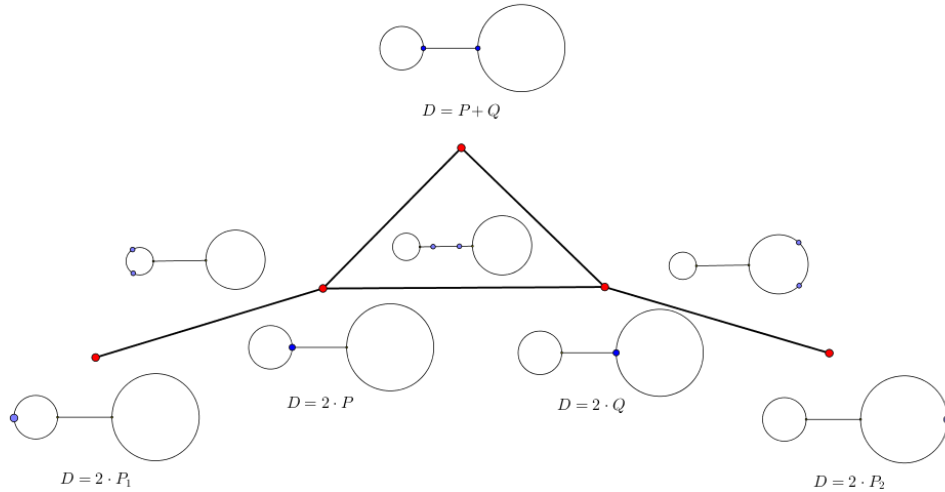


Figure 3.8: The polyhedral complex $|K_{H_2}|$.

3.3 The tropical semimodule $R(D)$

In this section we explore the algebraic properties of $R(D)$. We can define the sum of rational functions in $R(D)$ in a natural way via $(f + g)(x) = f(x) + g(x)$ for $x \in \Gamma$. Then the new function $f + g$ is still a tropical rational function on Γ . However, it may not belong to $R(D)$.

Example 3.28. Let Γ be a line segment PQ and $D = P + Q$. This function f belongs to $R(D)$, but $f + f = 2f$ does not. Because $(D + (2f))(P) = 1 + 2 \cdot (-1) = -1 < 0$.

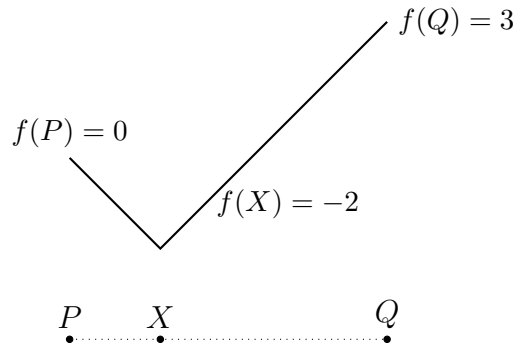


Figure 3.9: A divisor D such that $R(D)$ is not closed under the usual addition.

To make $R(D)$ closed under addition, we may consider the following operation.

Definition 3.29. If f and g are rational functions defined on Γ , we let $f \oplus g$ be the rational function on Γ with $(f \oplus g)(x) = \max(f(x), g(x))$ and $f \odot g$ be the rational function on Γ with $(f \odot g)(x) = f(x) + g(x)$. In other words, $\oplus$ and $\odot$ are the tropical operations in the set of rational functions on Γ . Here we choose the max-plus algebra.

Lemma 3.30. [64, Lemma 4] Let D be any divisor on a metric graph Γ . The space $R(D)$ is a tropical semimodule, i.e. it is closed under tropical addition and tropical scalar multiplication. We are thus in the setting of Definition 1.2, with $S = \Gamma$ and $M = R(D)$.

Definition 3.31. A function $f \in R(D)$ is called extremal if, for any $g_1, g_2 \in R(D)$, the decomposition $f = g_1 \oplus g_2$ implies that $f = g_1$ or $f = g_2$. Note that any generating set of $R(D)$ must contain all extremal functions up to tropical scalar multiplications.

It turns out that extremal functions are enough to generate $R(D)$.

Proposition 3.32. [64, Lemma 4] The tropical semimodule $R(D)$ is generated by the set of its extremal functions (up to tropical scalar multiplications).

Thus it is important to find the extremal functions of $R(D)$. We discuss the approach in Section 3.4.2.

3.4 Computations of $|D|$ and $R(D)$

There is a natural computational question: given a metric graph Γ and a divisor D on Γ , how to find the linear system $|D|$ and the tropical semimodule $R(D)$? In this section, we present general methods to compute the cells of $|D|$ and the extremal generators of $R(D)$. Even though we know that there are finitely many cells in $|D|$, the total number could be huge, as for a rational function f in $R(D)$, there might be many linear pieces on a single edge of the metric graph. To overcome this obstacle, we introduce the *anchor cells* that serve as landmarks to find all cells of $|D|$. We discuss the pros and cons of two linear programming approaches for computing the anchor cells of $|D|$. We also implement our algorithms and address issues about symbolic and numerical computations in our implementations. Based on the cells of $|D|$, we introduce an algorithm to compute the extremal generators of $R(D)$. Finally we present our computational results for some canonical linear systems.

3.4.1 Computing the cells of $|D|$

Given Γ and D , in order to find the cells of $|D|$, one approach is that we first find the vertices of $|D|$, and then find other cells of $|D|$. As for the vertices, it is straightforward to implement the method in the proof of [64, Theorem 6]. However, there might be many cells in $|D|$ with rather complicated structure, because there might be many parts in the partition on one edge in the data in Definition 3.19. To grasp the cells in $|D|$, we introduce an important type of divisors and cells in $|D|$ that serve as landmarks.

3.4.1.1 Anchor divisors and cells

Definition 3.33. *A divisor L on Γ is an anchor divisor if for each edge of Γ there is at most one interior point x with $L(x) > 0$. A cell C in $|D|$ is an anchor cell if all representatives L of C are anchor divisors.*

Remark 3.34. *The number of anchor divisors in $|D|$ could be infinite. If an anchor cell has dimension 1, then it contains infinitely many divisors and each such divisor is an anchor divisor, as in Example 3.35. Nonetheless, they all represent the same anchor cell because they share the same data in Definition 3.19.*

Example 3.35. *Let Γ and D be the same as in Example 3.28. Then for any $X \in \Gamma$, the anchor divisor $2 \cdot X$ belongs to $|D|$. But they represent the same 1-dimensional cell when X is an interior point. Figure 3.9 shows the rational function $f \in R(D)$ such that $D + (f) = 2 \cdot X$.*

One property of anchor cells is that the set of anchor cells contains the vertices of $|D|$.

Lemma 3.36. *Each vertex of $|D|$ is an anchor cell.*

Proof. Suppose that $L \in |D|$ is not an anchor divisor. There exists an edge e of Γ and two distinct interior points P, Q of e such that $L(P), L(Q) > 0$, then $P, Q \in I_L$. After deleting

P and Q the graph is no longer connected. So the number of connected components in the graph $\Gamma - I_L$ is at least 2. Hence, by Proposition 3.22, L is the representative of a cell with dimension at least 1. Therefore, if a divisor represents a vertex of $|D|$, it must be an anchor divisor. So each vertex is an anchor cell. $\square$

Lemma 3.37. *Let D be a vertex-supported effective divisor on a metric graph Γ . If $f \in R(D)$ and $D + (f)$ is an anchor divisor, then f has at most two linear pieces on each edge of Γ .*

Proof. For any edge e of Γ , suppose x is an interior point of e and it is the intersection of two linear pieces of f with different slopes. Then, by definition, we have $(f)(x) \neq 0$. Since D is vertex-supported, $D(x) = 0$. Then $(D + (f))(x) \neq 0$. Since $f \in R(D)$, the divisor $D + (f)$ is effective, so $(D + (f))(x) > 0$. However $D + (f)$ is an anchor divisor, meaning that there is at most one such point x , so f has at most two linear pieces on e .

Therefore if the outgoing slopes at the two endpoints of e sum to 0, then f is linear on e ; otherwise f has two linear pieces on e and there is one interior point x of e such that $(D + (f))(x) > 0$. $\square$

Corollary 3.38. *Let D be a vertex-supported divisor on a metric graph Γ . If C is an anchor cell and it is represented by a divisor $D + (f)$, then C is uniquely determined by the outgoing slopes of f at all vertices of Γ .*

Proof. It suffices to show that given all those slopes, the data of C are uniquely determined. Firstly, the m_e are determined given those slopes. Secondly we show that the d_e are also determined. If C is an anchor cell and $D + (f)$ is a representative of C , then $D + (f)$ is an anchor divisor. By Lemma 3.37, f has at most two linear pieces on each edge e of Γ . If f is linear on e , then the integer d_e is zero for $D + (f)$; otherwise f has two linear pieces on e . Suppose they have a common point v . Then $(D + (f))(v)$ is minus the sum of the two outgoing slopes at the endpoints of e , so d_e is still determined by those slopes. Finally for each vertex v of Γ , $d_v = (D + (f))(v)$. The coefficient $D(v)$ is known, and each $(f)(v)$, which is the sum of the outgoing slopes at v , is also determined, so d_v is determined too. $\square$

Lemma 3.39. [64, Lemma 7] *Let D be an effective divisor on a metric graph Γ and $f \in R(D)$. Then the slopes of all linear pieces of f are between $-\deg(D)$ and $\deg(D)$.*

With the anchor cells, we have the following approach for finding the cells of $|D|$:

- (1) computing the anchor cells of $|D|$;
- (2) given the anchor cells of $|D|$, computing the other cells of $|D|$.

3.4.1.2 Step (1) - linear programming algorithms

By Corollary 3.38 and Lemma 3.39, we can find all anchor cells by considering the $2|E|$ -tuple of outgoing slopes of all $f \in R(D)$ with at most two linear piece(s) on each edge of Γ . In particular we have a proof of the finiteness of anchor cells.

Next we implement this approach using linear programming algorithms. Suppose $\Gamma = (V, E)$, where $V = \{v_1, \dots, v_n\}$ is the set of vertices and $E = \{e_1, \dots, e_m\}$ is the set of edges. For $1 \leq j \leq m$, the edge e_j has endpoints $v_{j(1)}$ and $v_{j(2)}$, where $1 \leq j(1), j(2) \leq n$, and the length of e_j is M_j . For $1 \leq i \leq n$ let $d_i = D(v_i)$ and $d = \deg(D) = \sum_{i=1}^n d_i$.

Lemma 3.40. *Let f be a rational function defined on $\Gamma = (V, E)$ such that there are at most two linear pieces on each edge of Γ . Denote by $a_i = f(v_i)$ for $1 \leq i \leq n$ and by $s_{j,1}, s_{j,2}$ the outgoing slope at $v_{j(1)}$ and $v_{j(2)}$ of e_j for $1 \leq j \leq m$. Then $f \in R(D)$ if and only if*

- for each vertex v_i we have the inequalities

$$d_i + \sum_{j(1)=i} s_{j,1} + \sum_{j(2)=i} s_{j,2} \geq 0; \quad (3.3)$$

- for $1 \leq j \leq m$, either

$$s_{j,1} + s_{j,2} = 0, a_{j(1)} - a_{j(2)} + s_{j,1} \cdot M_j = 0 \quad (3.4)$$

or

$$s_{j,1} + s_{j,2} < 0, s_{j,2} \cdot M_j < a_{j(1)} - a_{j(2)} < -s_{j,1} \cdot M_j. \quad (3.5)$$

Example 3.41. Figure 3.10 shows the function f on the edge e_j . In this example, $M_j = 3$ and the lengths of $\overline{v_{j(1)}p}$ and $\overline{v_{j(2)}p}$ are 1 and 2 respectively. The outgoing slopes are $s_{j,1} = -3$ and $s_{j,2} = -1$. The values of f are $f(v_{j(1)}) = 4$, $f(v_{j(2)}) = 3$ and $f(p) = 1$.

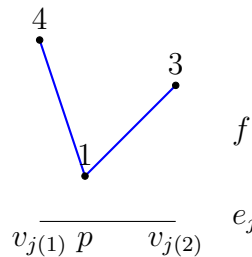


Figure 3.10: The graph of f (thick) on the edge e_j

Proof. Suppose $f \in R(D)$. Note that for each $1 \leq i \leq n$ we have $(f)(v_i) = \sum_{j(1)=i} s_{j,1} + \sum_{j(2)=i} s_{j,2}$, then

$$d_i + \sum_{j(1)=i} s_{j,1} + \sum_{j(2)=i} s_{j,2} = (D + (f))(v_i) \geq 0.$$

For each $1 \leq j \leq m$, f has either one or two linear piece(s) on the edge e_j . If there is one linear piece, then $s_{j,1} + s_{j,2} = 0$, and by definition of slope we have $a_{j(2)} - a_{j(1)} = s_{j,1} \cdot M_j$; if there are two linear pieces, then there exists an interior point p of e_j such that f is linear on both line

segments $v_{j(1)}p$ and $v_{j(2)}p$, and $s_{j,1} + s_{j,2} \neq 0$. Note that $(D + (f))(p) = (f)(p) = -s_{j,1} - s_{j,2}$, so $s_{j,1} + s_{j,2} < 0$. Let x be the length of the segment $v_{j(1)}p$, then the length of the segment $v_{j(2)}p$ is $M_j - x$. And by definition of slope we have

$$a_{j(1)} - f(p) + s_{j,1} \cdot x = 0, \quad a_{j(2)} - f(p) + s_{j,2} \cdot (M_j - x) = 0. \quad (3.6)$$

Since $f(p)$ can be any real number, we eliminate it and get

$$(s_{j,1} + s_{j,2})x = (a_{j(2)} - a_{j(1)} + s_{j,2}M_j).$$

Then

$$x = \frac{a_{j(2)} - a_{j(1)} + s_{j,2} \cdot M_j}{s_{j,1} + s_{j,2}}.$$

Since p is an interior point of e_j , we have $0 < x < M_j$. Then $a_{j(2)} - a_{j(1)} + s_{j,2} \cdot M_j < 0$ and $a_{j(1)} - a_{j(2)} + s_{j,1} \cdot M_j < 0$.

Conversely, suppose for each $1 \leq i \leq n$ (3.3) holds and for each $1 \leq j \leq m$, either (3.4) or (3.5) holds. For each $1 \leq j \leq m$, if $s_{j,1} + s_{j,2} = 0$ then by (3.4) f is well-defined on e_j and $D + (f)(q) \geq 0$ for all points q in the interior of e_j ; otherwise (3.5) holds, by the chain inequalities f is well-defined on e_j with two linear pieces. Since $s_{j,1} + s_{j,2} < 0$, we have $D + (f)(q) \geq 0$ for all points q in the interior of e_j . Finally for each $1 \leq i \leq n$, (3.3) holds. Then $(D + (f))(v_i) \geq 0$. So f is a well-defined rational function on Γ with at most two linear pieces on each edge of Γ and $f \in R(D)$. $\square$

Note that if $D + (f)$ is an anchor divisor in $|D|$, then it also has degree d and its support has at most one intersection point with each open e_i . Thus we obtain a partition of d into $n + m$ nonnegative integers: $d = \sum_{i=1}^n d'_i + \sum_{j=1}^m c_j$, where $d'_i = (D + (f))(v_i)$ and $c_j = \sum_{x \in e_j^\circ} (D + (f))(x)$. We call them *configurations* of anchor divisors. Note that there are $\binom{d+n+m-1}{d}$ configurations in total.

The following lemma follows from the definition of principal divisors (f) .

Lemma 3.42. *Suppose $f \in R(D)$ with the same notations as in Lemma 3.40. Let $d = \sum_{i=1}^n d'_i + \sum_{j=1}^m c_j$ be the configuration of the divisor $D + (f)$. For $1 \leq j \leq m$, we have*

$$c_j = -s_{j,1} - s_{j,2}. \quad (3.7)$$

And for $1 \leq i \leq n$ we have

$$d'_i = d_i + \sum_{j(1)=i} s_{j,1} + \sum_{j(2)=i} s_{j,2} \quad (3.8)$$

Now for each configuration $d = \sum_{i=1}^n d'_i + \sum_{j=1}^m c_j$, we consider the following system of linear constraints in (3.4), (3.5), (3.7), (3.8). Here d_i, d'_i, M_j, c_j are parameters and $a_i, s_{j,1}, s_{j,2}$ are variables. Then among the solutions of this system, each $2m$ -tuple of integers $s_{j,1}, s_{j,2}$ gives an anchor cell of $|D|$.

Remark 3.43. For each j , whether to apply (3.4) or (3.5) depends on the value of c_j .

Lemma 3.40 and Corollary 3.42 gives rise to Algorithm 1. The input is the metric graph $\Gamma = (V, E)$, the edge lengths M_j and the divisor D . The output L is a list of the anchor cells.

Algorithm 1 Computing Anchor Cells

```

1: procedure ANCHORCELLS( $\Gamma = (V, E), M, D$ )
2:    $L \leftarrow \emptyset$ 
3:    $m \leftarrow |E|$ 
4:    $n \leftarrow |V|$ 
5:    $d \leftarrow \deg(D)$ 
6:    $s \leftarrow \{c = (c_1, \dots, c_m, d'_1, d'_2, \dots, d'_n) \mid c_j, d'_i \in \mathbb{N}, \sum_{j=1}^m c_j + \sum_{i=1}^n d'_i = d.\}$ 
7:   for  $c \in s$  do
8:      $S \leftarrow$  the system of linear constraints in (3.4), (3.5), (3.7), (3.8)
9:     if  $S$  does not have a solution with all  $s_{j,1}, s_{j,2}$  being integers then (*)
10:      next  $c$ 
11:   end if
12:    $a \leftarrow$  array of  $m$  entries
13:    $b \leftarrow$  array of  $m$  entries
14:   for  $j \leftarrow 1, m$  do
15:      $a[j] \leftarrow [\text{minimize}(s_{j,1}, S), \text{maximize}(s_{j,1}, S)] \cap \mathbb{Z}$ 
16:      $b[j] \leftarrow [\text{minimize}(s_{j,2}, S), \text{maximize}(s_{j,2}, S)] \cap \mathbb{Z}$ 
17:   end for
18:    $T \leftarrow$  the Cartesian product of  $a[1], \dots, a[m], b[1], \dots, b[m]$ 
19:   for  $v \in T$  do
20:     if  $S$  union  $\{s_{j,1} = v_j, s_{j,2} = v_{j+m} \mid j = 1, \dots, m\}$  is feasible then
21:        $L \leftarrow L \cup \{[v, c]\}$ 
22:     end if
23:   end for
24: end for
25: return  $L$ 
26: end procedure

```

Remark 3.44. In our implementation, we use the *LPSolve* command in *Maple 2015* for the step (*) in Algorithm 1. The codes are available at <https://sites.google.com/site/bolinmath/research/codes/linearsystem>.

We can also process more general input. Suppose the skeleton of Γ is fixed, but the metric may vary. In this case the input is the metric $(M_j)_{1 \leq j \leq m}$, and the desired output is the set of anchor cells in $|D|$. Apparently we can apply the above approach once the M_j are given,

but since Algorithm 1 needs to run the linear-programming subroutine $\binom{d+n+m-1}{d}$ times, it is not very efficient if we would like to compute for many different metrics.

Instead, we can also view the M_j 's as variables. However $s_{j,1} \cdot M_j$ appears in (3.5). In order to make the system of constraints linear, we have to let $s_{j,1}, s_{j,2}$ become parameters. So we take the approach in the proof of Theorem 3.25. We still have the equations and inequalities (3.4), (3.5), (3.7), (3.8), but $s_{j,1}, s_{j,2}$ are parameters instead. Now the parameters are $d_i, s_{j,1}, s_{j,2}$ and the variables are M_j, d'_i, a_i, c_j . Then we also need

$$c_j \geq 0 \quad \forall 1 \leq j \leq m \text{ and } d'_i \geq 0 \quad \forall 1 \leq i \leq n \quad (3.9)$$

for $D + (f)$ being effective and

$$M_j > 0 \quad \forall 1 \leq j \leq m. \quad (3.10)$$

for the edge lengths in M are all positive.

By Lemma 3.39, we have $|s_{j,1}|, |s_{j,2}| \leq d$. So there are finitely many possible values of them. Now we have an empty list first and for each choice of $(s_{j,k})$, we check the feasibility of the system of linear constraints formed by (3.4), (3.5), (3.7), (3.8), (3.9), (3.10). If it is feasible, then we find one anchor cell and save it to our list. The output is the set of anchor cells represented by divisors L such that there exists some metric $M = (M_j)_{1 \leq j \leq m}$ with $L \in |D|_M$.

The advantage of this approach is that we can compute this list of all possible anchor cells in $|D|$ beforehand. Then given a specific metric, we just plug in the values of M_j and check the feasibility of each anchor cell in the list.

Algorithm 2 Computing Anchor Cells

```

1: procedure ANCHORCELLS( $\Gamma = (V, E), M, D$ )
2:    $L \leftarrow \emptyset$ 
3:    $m \leftarrow |E|$ 
4:    $n \leftarrow |V|$ 
5:    $d \leftarrow \deg(D)$ 
6:    $P \leftarrow \{s = (s_{1,1}, s_{1,2}, \dots, s_{m,1}, s_{m,2}) \mid s_{j,1}, s_{j,2} \in [-d, d]\} = [-d, d]^{2m}$ 
7:   for  $s \in P$  do
8:      $S \leftarrow$  the system of linear constraints in (3.4), (3.5), (3.7), (3.8), (3.9), (3.10)
9:     if  $S$  does not have a solution then
10:       next  $c$ 
11:     else  $L \leftarrow L \cup \{s\}$ 
12:     end if
13:   end for
14:   return  $L$ 
15: end procedure
    
```

Table 3.1 is a comparison of Algorithm 1 and Algorithm 2.

	Slopes m_e, m'_e	d_v, d_e	values $f(v)$	number of cases	the problem in each case
1	fixed	fixed	variables	$(2 \deg(D) + 1)^{2 E }$	linear programming
2	integer variables	fixed	variables	$\binom{ V + E +\deg(D)-1}{\deg(D)-1}$	mixed integer programming

Table 3.1: A comparison of the two approaches from step (1).

3.4.1.3 Symbolic versus numerical

So far our approaches are purely symbolic. However, for the implementation of both Algorithm 1 and 2, there is a step that we need to check the feasibility of a system of linear equations and inequalities. Note that the inequalities (3.5) are strict, which is an obstacle in simplifying the system, as many existing softwares only allow nonstrict inequalities. Then we had to relax the inequalities to

$$s_{j,1} + s_{j,2} < 0, s_{j,2} \cdot M_j + \epsilon \leq a_{j(1)} - a_{j(2)} \leq -s_{j,1} \cdot M_j - \epsilon, \quad (3.11)$$

where $\epsilon > 0$ is very small. Then in Algorithm 1, when we check whether the slopes $s_{j,1}, s_{j,2}$ of a solution are integers, we also need to deal with the error coming from ϵ . In our implementation, we usually set $\epsilon = 0.1$ for metric graphs with small size. But we do not have a general strategy to deal with this symbolic/numerical issue.

3.4.1.4 Step (2) - The association map

The motivation for anchor cells is Theorem 3.47.

Definition 3.45. Let A be an anchor cell of $|D|$ and L be a divisor in A . For each (open) edge e_i of Γ , we let $c_i = \sum_{x \in e_i} L(x)$. Then c_i is the unique positive coefficient of L on e_i if the second part of the data of A in Definition 3.19 contains a trivial partition; otherwise $c_i = 0$. We define a function

$$s(A) = \sum_{i=1}^{|E|} \max(c_i - 1, 0). \quad (3.12)$$

Example 3.46. Let A_1, A_2 be the two 0-dimensional anchor cells in Figure 3.7. There is a unique edge in the metric graph and for both A_1 and A_2 , it corresponds to the trivial partition $3 = 3$ in the data. Then

$$s(A_1) = s(A_2) = 3 - 1 = 2.$$

In comparison, for the five 0-dimensional anchor cells in Figure 3.8, the cells corresponding to $2P_1$ and $2Q_2$ have an edge with the trivial partition $2 = 2$, and other three cells do not have any. Then (we denote the cells by their corresponding divisors)

$$s(2P_1) = s(2Q_2) = 2 - 1 = 1, s(2P) = s(2Q) = s(P + Q) = 0.$$

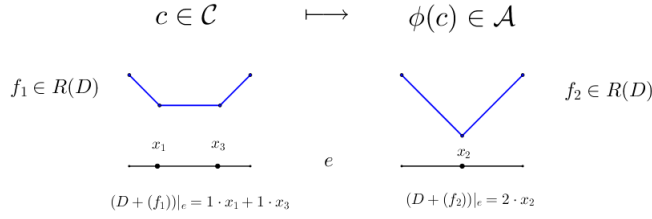


Figure 3.11: The association map ϕ on one edge

Theorem 3.47. (*Association to anchor cells*) Let D be a vertex-supported effective divisor on a metric graph Γ . (1) Suppose $\mathcal{C}$ is the set of all cells in $|D|$ and $\mathcal{A}$ is the set of all anchor cells in $|D|$. We define a function $\phi : \mathcal{C} \rightarrow \mathcal{A}$ as follows: let $C \in \mathcal{C}$ and $L \in C$ be a representative of C . Then we let $\phi(C)$ be the unique anchor cell represented by an anchor divisor obtained from C by bunching up all chips along each open edge. In other words, $\phi(C)$ is represented by an anchor divisor N such that

- (i) $N(v) = L(v)$ for all vertices v of Γ ;
 - (ii) suppose $f_N, f_L \in R(D)$ such that $N = D + (f_N)$ and $L = D + (f_L)$, then for every edge e of Γ and each endpoint v of e , f_N and f_L have the same outgoing slopes at v along e .
- (2) Let ϕ be the same map as in (1). Then for any anchor cell $A \in \mathcal{A}$, we have

$$|\phi^{-1}(A)| = 2^{s(A)}.$$

Furthermore, for $0 \leq j \leq s(A)$, there are $\binom{s(A)}{j}$ cells of dimension $\dim A + j$ in $\phi^{-1}(A)$.

Example 3.48. Figure 3.11 shows the association map ϕ of a cell $C \in \mathcal{C}$ on one edge. On this edge, the data of C contains a nontrivial partition $2 = 1 + 1$, and the data of $\phi(C)$ contains a trivial partition $2 = 2$.

Example 3.49. We consider Example 3.27 again. Figure 3.11 shows the association map ϕ of $|K_{H_2}|$.

Proof. (1) Given D and L , we construct N as follows. Let $f \in R(D)$ such that $L = D + (f)$. Then we construct another rational function $g \in R(D)$. First we let $g(v) = f(v)$ for vertices v of Γ ; next we let $g(x) = f(x)$ for all points $x \in e$, where e is an edge of Γ such that $\sum_{y \in e^\circ} L(y) = 0$; for other edges e of Γ , we regard the open edge e as the interval $(0, l_e)$. Suppose f has r linear pieces $[p_i, p_{i+1}]$ on e , with slope s_i , where $0 = p_0 < p_1 < \dots < p_r = l_e$. Then for $1 \leq i \leq r - 1$, we have $L(p_i) = s_{i+1} - s_i$. Since the adjacent linear pieces have different slopes, $s_{i+1} - s_i \neq 0$. While L is effective, so $s_{i+1} - s_i > 0$. Now we let g have two linear pieces on e , the one containing p_0 with slope s_1 , the one containing p_r with slope s_r . Note that if $r = 2$ then g coincides with f on e . If $r \geq 3$, since $s_1 < s_2 < \dots < s_r$, the graph

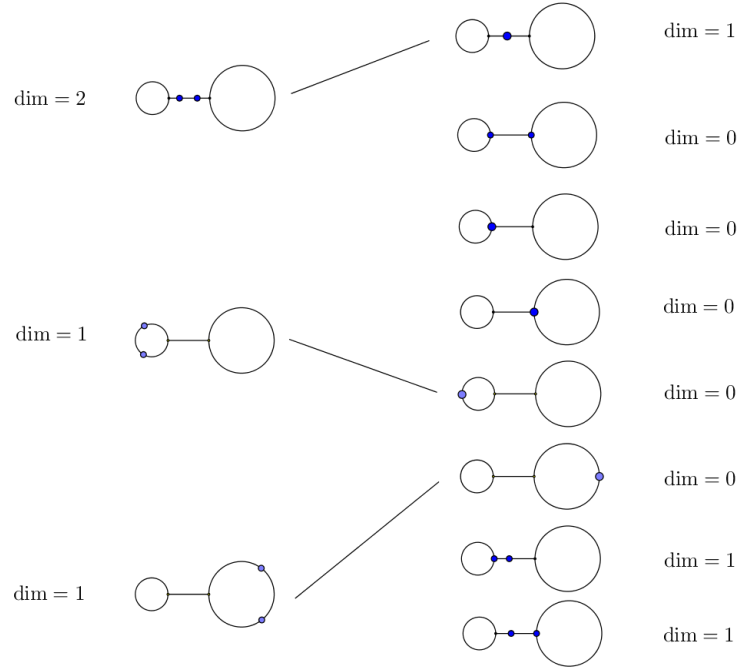


Figure 3.12: The association map of $|K_{H_2}|$. There are 8 anchor cells and 3 non-anchor cells.

of the two pieces of g will intersect within the interval, so g is well-defined on e . In addition let p be the intersection point of these two pieces, then $(D + (g))(p) = s_r - s_1 > 0$. We let $N = D + (g)$. By definition of g , N is effective and anchor, and N satisfies both (i) and (ii).

(2) Note that for $C \in \mathcal{C}$, the two cells C and $\phi(C)$ share the same first and third parts of their data, and given any edge e of Γ , either both do not have a partition, or they have the partitions of the same positive integer. The only difference is that the partition of $\phi(C)$ is always trivial, while the partition of C could be arbitrary. Given a positive integer t , there are 2^{t-1} ordered partitions of t objects in a row, because each partition corresponds to a $(t-1)$ -tuple of 0, 1, indicating whether or not to break the $t-1$ pairs of adjacent objects. In addition, for $1 \leq i \leq t$, there are $\binom{t-1}{i-1}$ ordered partitions with i parts.

Now if on edge e_i , the data of A contains a trivial partition $c_i = c_i$, then the partition on e_i of a preimage of A has 2^{c_i-1} choices. Once all partitions are determined, so is the preimage. Then $|\phi^{-1}(A)| = 2^{s(A)}$. Furthermore, note that if the number of parts in the partition on one edge increases by 1, the number of connected components in Proposition 3.22 also increases by 1, which means the dimension of the new cell is one more. Then for $0 \leq j \leq s(A)$, the number of cells with dimension $\dim A + j$ in $\phi^{-1}(A)$ equals to the number of the r -tuples of partitions of $c_1, \dots, c_r$ with $r + j$ parts in total. This number is $\binom{s(A)}{j}$ by an easy argument of generating functions. $\square$

The following corollary provides a combinatorial formula to compute the f -vector of $|D|$ given all of its anchor cells.

Corollary 3.50. *Let D be a vertex-supported effective divisor on a metric graph Γ . If $A_1, A_2, \dots, A_m$ are all anchor cells in $|D|$, then for each $d \in \mathbb{N}$ the number of d -dimensional cells in the cell complex $|D|$ is the coefficient of x^d in the generating function*

$$\sum_{i=1}^m x^{\dim A_i} (1+x)^{s(A_i)}.$$

Proof. By Theorem 3.47(1), each cell in $|D|$ is associated to a unique anchor cell of $|D|$, i.e. $\bigcup_{i=1}^m \phi^{-1}(A_i) = \mathcal{C}$. By Theorem 3.47(2), the generating function for $\phi^{-1}(A_i)$ is $x^{\dim A_i} (1+x)^{s(A_i)}$. So the total generating function is just their sum. $\square$

Example 3.51. *Let Γ have the skeleton of the complete bipartite graph $K_{3,3}$ and let K be the canonical divisor on Γ . If L is the left divisor in Figure 3.13, then L belongs to $|K|$ and it has two chips on two edges respectively. Then $2^{2-1} \cdot 2^{2-1} = 4$ cells of $|K|$ are associated to L . By Proposition 3.22, their dimensions are 0, 1, 1, 2 respectively.*

Remark 3.52. *By [64, Corollary 31], the polyhedral cell complex $|D|$ is always contractible. So we can think of the map ϕ in Theorem 3.47 as a deformation retract.*

3.4.2 Computing the extremal generators of $R(D)$

We would like to find a minimal set of generators of the tropical semimodule $R(D)$. We use the notion of *chip-firing*, which was introduced in [22].

For an effective divisor D on Γ we regard it as a distribution of $\deg(D)$ chips on Γ : there are $D(x)$ chips at each point $x \in \Gamma$. Suppose that D is effective and $f \in R(D)$. For each linear piece $\overline{PQ}$ of Γ , if f has slope $s \in \mathbb{Z}_-$ from P to Q , then we say that P fires s chips to Q when adding (f) to D .

3.4.2.1 Chip-firing

For a metric graph Γ , a *subgraph* is a compact subset with a finite number of components. Fix an effective divisor D on Γ . We say that a subgraph Γ' of Γ can *fire* for D if, for each boundary point x of $\Gamma' \cap \overline{\Gamma - \Gamma'}$, the number of edges pointing out of Γ' is no greater than $D(x)$.

Lemma 3.53. *[64, Theorem 14(a)] If D is a vertex-supported divisor on a metric graph Γ and $f \in R(D)$ is extremal, then $D + (f)$ is a vertex of the cell complex $|D|$.*

By Lemma 3.53, in order to find the extremal generators of $R(D)$, it suffices to search among the vertices of $|D|$. The next lemma is an important criterion for the extremal generators of $R(D)$.

Lemma 3.54. [64, Lemma 5] *Let D be any divisor on a metric graph Γ . Then $f \in R(D)$ is extremal if and only if there do not exist two proper subgraphs Γ_1 and Γ_2 of Γ such that they cover Γ and both can fire on $D + (f)$.*

Remark 3.55. *For a non-extremal $f \in R(D)$, the proper subgraphs Γ_1 and Γ_2 may not be obvious. See the example in Figure 3.13, where Γ is the metric graph with skeleton $K_{3,3}$ and all-equal metric, $D = K$ is the canonical divisor on Γ .*

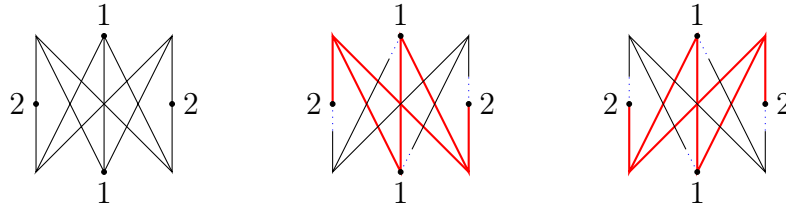


Figure 3.13: A non-extremal divisor and the two subgraphs (thick) that can fire. The corresponding rational function takes value 1 on the thick lines and 0 on the normal lines and is linear with outgoing slope 1 on the dotted lines (from thick to normal lines).

3.4.2.2 The algorithm

Our discussion suggests the following approach for computing all extremal generators of $R(D)$:

- (A) Find all anchor divisors in $R(D)$.
- (B) Filter them by Proposition 3.22 to get all vertices of $|D|$.
- (C) Filter the vertices by Lemma 3.54 to get all extremal generators of $R(D)$.

We introduce a method for Step (C). Given an effective divisor $L = D + (f)$ on Γ , we apply Lemma 3.54 to check whether f is extremal. Note that if a proper subgraph can fire for L , then its boundary is contained in $\text{supp}(L)$. So we partition Γ into connected subgraphs whose boundaries are contained in $\text{supp}(L)$ and we call them components. Since $\text{supp}(L)$ is finite, so is the number of such components. We conclude that any subgraph that can fire for L is a union of these components. For each such union, we ignore it if it is non-proper or cannot fire for L , then we have a finite list of all proper subgraphs of Γ that can fire for L . Finally we check whether there is a pair in the list that covers Γ .

3.5 Canonical linear systems on some trivalent graphs

In this section we apply our methods to some nontrivial examples. A metric graph is *trivalent* if the degree of every vertex is 3. Trivalent graphs appear in the Berkovich skeleton

of many curves [14, Example 5.29]. This fact motivates us to compute the examples of trivalent metric graphs and the canonical divisor K .

For the computations we performed below, our hardware is a laptop with Intel Core i5-6200U processor (2.3 GHz) and 8GB RAM. The software is Maple, version 2015. All computations were single-threaded.

3.5.1 The complete graph K_4

The complete graph K_4 has 4 vertices and its genus is 3. Table 3.2 shows the structure of $R(K)_M$ and $|K|_M$ on K_4 given a particular metric M . For $1 \leq i < j \leq 4$ let l_{ij} be the edge length between vertex i and j . A metric M is given by $(l_{12}, l_{13}, l_{14}, l_{23}, l_{24}, l_{34})$.

Metric	Anchor Cells	Extremal Generators	f -vector	Computational Time (s)
(1, 1, 1, 1, 1, 1)	30	7	(14, 28, 15)	3.1
(1, 1, 2, 2, 1, 1)	42	11	(26, 52, 31, 4)	3.4
(2, 2, 2, 2, 2, 3)	36	9	(20, 40, 23, 2)	3.0
(2, 2, 2, 2, 2, 1)	40	11	(24, 44, 21)	3.8
(4, 9, 7, 8, 6, 10)	50	15	(34, 60, 27)	4.1

Table 3.2: Structure of $|K|$ and R_K for different metrics on K_4

Conjecture 3.56. *If Γ has skeleton K_4 and $D = K$, then the number of anchor cells in $|D|$ minus the number of vertices in $|D|$ is always 16.*

3.5.2 The example of (020)

Following the notation of [25], we denote by (020) the following trivalent graph with 4 vertices and 6 edges (Figure 3.14). The metric is the vector $(a_1, a_2, b, c, d_1, d_2)$.

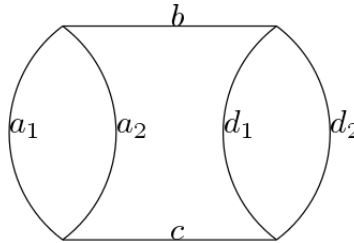


Figure 3.14: The trivalent graph (020)

Table 3.3 shows the structure of $R_{M,K}$ and $|K|_M$ on (020) for some metric M .

Metric	Anchor Cells	Extremal Generators	f -vector	Computational Time (s)
(1, 1, 1, 1, 1, 1)	20	42	(31, 61, 36, 5)	3.1
(1, 1, 1, 2, 1, 1)	12	44	(25, 47, 24, 1)	3.5
(1, 3, 2, 2, 1, 3)	20	42	(31, 61, 36, 5)	3.0

Table 3.3: Structure of $|K|$ and R_K for different metrics on (020)

3.5.3 The complete bipartite graph $K_{3,3}$

Table 3.4 shows the structure of $R(K)_M$ and $|K|_M$ on $K_{3,3}$ given a particular metric M . Here M is denoted by a 3×3 matrix whose (i, j) -th entry is the length of the edge connecting the i -th vertex of A and the j -th vertex of B , where A and B are the disjoint two parts of vertices in $K_{3,3}$.

Metric	Anchor Cells	Extremal Generators	f -vector	Computational Time (s)
All-equal	370	33	(130, 483, 630, 348, 81, 9)	171.9
$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$	460	63	(196, 615, 666, 276, 33, 3)	189.8
$\begin{bmatrix} 3 & 91 & 96 \\ 94 & 4 & 92 \\ 93 & 95 & 5 \end{bmatrix}$	730	84	(337, 936, 873, 273)	241.9

Table 3.4: Structure of $R_{M,k}$ and $|K|$ for different metrics on $K_{3,3}$

Remark 3.57. By [64, Corollary 31], the cell complex $|D|$ is contractible as a topological space. Thus the Euler characteristic of $|D|$ is always 1, which is consistent with all the f -vectors we computed above.

3.6 Future problems

There are still some directions to be pursued in the future. For example,

- given all cells of $|D|$, find the face lattice. In other words, for each cell, find all cells on its boundary.
- Characterize the set of convex polytopes that could be the closure of cells in some $|D|$.
- Find the linear independence relations among the extremal generators of $R(D)$.

Chapter 4

Tropical Hodge bundles

This chapter is a joint work with Martin Ulirsch. It is posted on the [arXiv](#) [75] with the title "Towards a tropical Hodge bundle", and it will be published in the book "Combinatorial Algebraic Geometry". It is a continuation of Chapter 3. The moduli space M_g^{trop} of tropical curves of genus g is a generalized cone complex that parametrizes metric vertex-weighted graphs of genus g . For each such graph Γ , the associated canonical linear system $|K_\Gamma|$ has the structure of a polyhedral complex. In this chapter we propose a tropical analogue of the Hodge bundle on M_g^{trop} and study its basic combinatorial properties. Our construction is illustrated with explicit computations and examples.

Let $g \geq 2$ and denote by $\mathcal{M}_g$ the moduli space of smooth algebraic curves of genus g . The *Hodge bundle* Λ_g is a vector bundle on $\mathcal{M}_g$ whose fiber over a point $[C]$ in $\mathcal{M}_g$ is the vector space $H^0(C, \omega_C)$ of holomorphic differentials on C . One can think of the total space of Λ_g as parameterizing pairs (C, ω) consisting of a smooth algebraic curve and a differential ω on C . Since for every curve C the canonical linear system $|K_C|$ can be identified with the projectivization $\mathbb{P}(H^0(C, \omega_C))$, the total space of the projectivization $\mathcal{H}_g = \mathbb{P}(\Lambda_g)$ of Λ_g parametrizes pairs (C, D) consisting of a smooth algebraic curve C and a canonical divisor D on C . This is referred to as the *projective Hodge bundle*. Let $\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g$ be the universal curve on $\mathcal{M}_g$. We may define Λ_g formally as the pushforward $\pi_* \omega_g$ of the relative dualizing sheaf ω_g on $\mathcal{C}_g$ over $\mathcal{M}_g$.

The Hodge bundle is of fundamental importance when describing the geometry of $\mathcal{M}_g$. For example, its Chern classes, the so-called *λ -classes*, form an important collection of elements in the tautological ring on $\mathcal{M}_g$ (see [108] for an introductory survey). The Hodge bundle admits a natural stratification by prescribing certain pole and zero orders $(m_1, \dots, m_n)$ such that $m_1 + \dots + m_n = 2g - 2$ and the study of natural compactifications of these components has recently seen an surge from both the perspective of algebraic geometry as well as from Teichmüller theory (see e.g. [9]).

In this chapter, we introduce the *tropical Hodge bundle*, which is an analogue of the Hodge bundle in tropical geometry. We note that the fibers of tropical Hodge bundles are exactly linear systems on metric graphs. In tropical geometry, the natural analogue of $\mathcal{M}_g$ is the moduli space M_g^{trop} that parametrizes isomorphism classes $[\Gamma]$ of stable tropical curves Γ of

genus g . In Section 4.2 we recall the construction of this moduli space. In particular, we show how this moduli space naturally admits the structure of a generalized cone complex whose cones are in a natural order-reversing one-to-one correspondence with the boundary strata of the Deligne-Mumford compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ (see [1] as well as Section 4.2 for details). In Section 4.3 we prove Theorem 4.4 by describing the polyhedral structures of both Λ_g^{trop} and $\mathcal{H}_g^{trop}$ simultaneously. Section 4.4 contains a selection of explicit (sometimes partial) calculations of the polyhedral structure of $\mathcal{H}_g^{trop}$ in some small genus cases. In Section 4.5 we describe a natural tropicalization procedure for the projective algebraic Hodge bundle via non-Archimedean analytic geometry and exhibit a natural *realizability problem*.

4.1 Introduction of the tropical Hodge bundle

Definition 4.1. A tropical curve is a finite metric graph Γ (with a fixed minimal model $G = (V(G), E(G))$, see Remark 3.3) together with a genus function $h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$.

Definition 4.2. The genus of a tropical curve Γ with a fixed minimal model G is defined as

$$g(\Gamma) = g(G) + \sum_{v \in V(G)} h(v).$$

In the above sum one should think of the vertex-weight terms as the contributions of $h(v)$ infinitesimally small loops at every vertex v .

As explained in [4, Section 5.2], the canonical divisor on Γ is defined as

$$K_\Gamma = K_G = \sum_{v \in V(G)} (2h(v) + |v| - 2)(v) ,$$

where $|v|$ denotes the valence of the vertex v . Note that this definition is different from Definition 3.17, because of the genus function h . Observe that $\deg(K_\Gamma) = 2g - 2$. The $h(v)$ -term in the sum should hereby be thought of as contributing $h(v)$ infinitely small loops at the vertex v . In fact, given a semistable curve C whose dual graph is G , the canonical divisor is the multidegree of the dualizing sheaf on C (see [3, Remark 3.1]).

We say a tropical curve Γ (or the graph G) is *stable*, if for every vertex $v \in V(G)$ we have

$$2h(v) - 2 + |v| > 0 , \tag{4.1}$$

where $|v|$ denotes the valence of G at v .

Let Γ be a tropical curve. We denote by K_Γ the canonical divisor on Γ and by $\text{Rat}(\Gamma)$ the group of piecewise integer linear functions on Γ . In this note we propose tropical analogues of the affine and the projective Hodge bundle, and study their basic combinatorial properties.

Definition 4.3. *As a set, the tropical Hodge bundle Λ_g^{trop} is given as*

$$\Lambda_g^{trop} = \{([\Gamma], f) \mid [\Gamma] \in M_g^{trop} \text{ and } f \in \text{Rat}(\Gamma) \text{ such that } K_\Gamma + (f) \geq 0\}$$

and the projective tropical Hodge bundle $\mathcal{H}_g^{trop}$ is given as

$$\mathcal{H}_g^{trop} = \{([\Gamma], D) \mid [\Gamma] \in M_g^{trop} \text{ and } D \in |K_\Gamma|\} .$$

The associations $([\Gamma], f) \mapsto [\Gamma]$ and $([\Gamma], D) \mapsto [\Gamma]$ define natural projection maps $\Lambda_g^{trop} \rightarrow M_g^{trop}$ and $\mathcal{H}_g^{trop} \rightarrow M_g^{trop}$, which, in a slight abuse of notation, we denote both by π_g .

In Chapter 3 we studied the structure of a polyhedral complex on the linear system $|D|$ associated to a divisor D on a tropical curve Γ . This was built on prior work in [57, 64, 82]. Our main result in this chapter is the following:

Theorem 4.4. *Let $g \geq 2$.*

- (i) *The tropical Hodge bundle Λ_g^{trop} and the projective tropical Hodge bundle $\mathcal{H}_g^{trop}$ carry the structure of a generalized cone complex.*
- (ii) *The dimensions of Λ_g^{trop} and $\mathcal{H}_g^{trop}$ are given by*

$$\dim \Lambda_g^{trop} = 5g - 4 \quad \text{and} \quad \dim \mathcal{H}_g^{trop} = 5g - 5$$

respectively.

- (iii) *There is a proper subdivision of M_g^{trop} such that, for all $[\Gamma]$ in the relative interior of a cone in this subdivision, the canonical linear systems*

$$|K_\Gamma| = \pi_g^{-1}([\Gamma])$$

have the same combinatorial type.

We are going to refer to this subdivision of M_g^{trop} as the *wall-and-chamber* decomposition of M_g^{trop} . In general, the generalized cone complexes Λ_g^{trop} and $\mathcal{H}_g^{trop}$ are not pure dimensional. So Theorem 4.4 (ii) really states that the dimension of a maximal-dimensional cone in Λ_g^{trop} (or $\mathcal{H}_g^{trop}$) has dimension $5g - 4$ (or $5g - 5$ respectively).

As a first example we refer the reader to Figure 4.1 below, which depicts the face lattice of the tropical Hodge bundle in the case $g = 2$.

4.2 Moduli of tropical curves

Definition 4.5. *As a set, the moduli space M_g^{trop} of stable tropical curves of genus g is given as*

$$M_g^{trop} = \{ \text{isomorphism classes } [\Gamma] \text{ of stable tropical curves of genus } g \} .$$

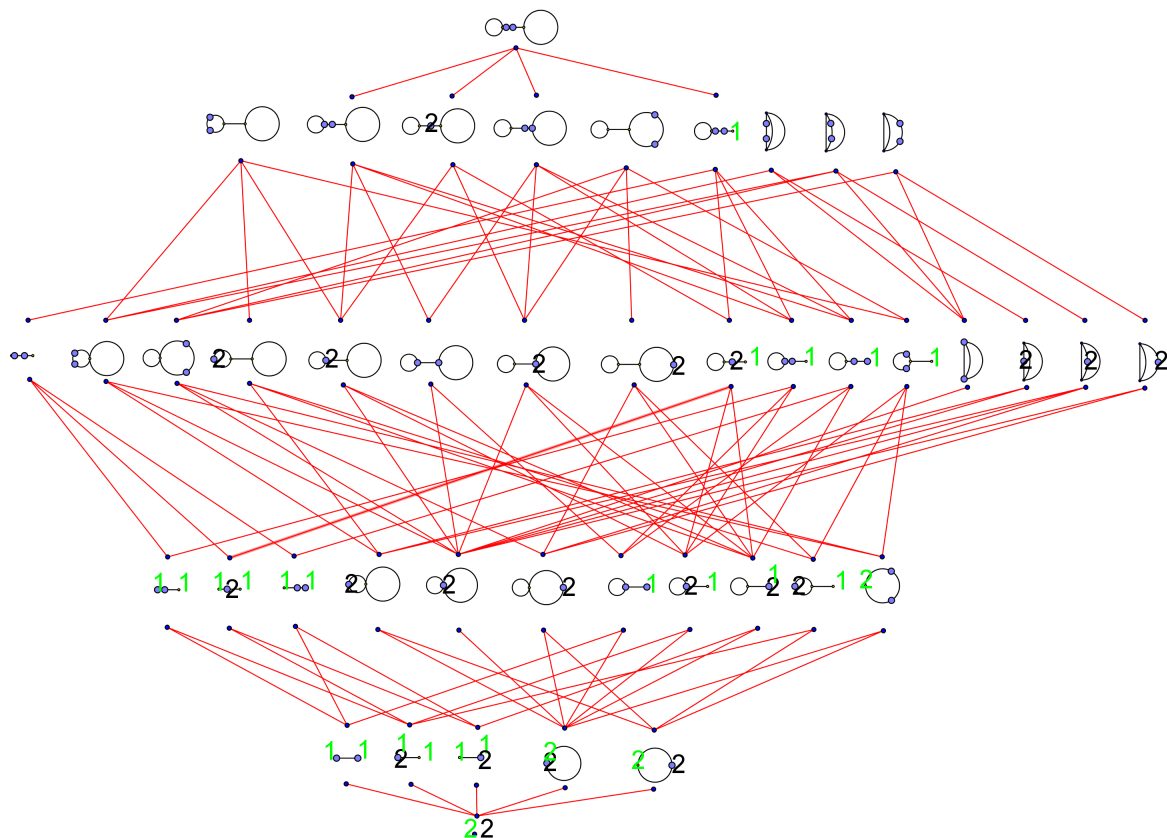


Figure 4.1: The face lattice of $\mathcal{H}_2^{trop}$. The numbers in green are the positive $h(v)$ and the numbers in black denote coefficients greater than 1 in the divisors.

Let us now recall from [1] the description of M_q^{trop} as a generalized extended cone complex.

Proposition 4.6 ([1] Section 4). *The moduli space M_g^{trop} carries the structure of a generalized rational polyhedral cone complex that is equi-dimensional of dimension $3g - 3$.*

First, recall that a morphism $\tau \rightarrow \sigma$ between rational polyhedral cones is said to be a *face morphism*, if it induces an isomorphism onto a face of σ . Note that we explicitly allow the class of face morphisms to include all isomorphisms. A *generalized (rational polyhedral) cone complex* is a topological space Σ that arises as a colimit of a finite diagram of face morphisms (see [1, Section 2] and [106, Section 3.5] for details).

In order to understand this structure on M_q^{trop} , we observe that it is given as a colimit

$$M_g^{trop} = \lim_{\rightarrow} \widetilde{M}_G \ ,$$

of rational polyhedral cones $\widetilde{M}_G$ taken over a category J_q . Let us go into some more detail:

1. The category J_q is defined as follows:

- its objects are stable vertex-weighted graphs (G, h) of genus g , and
- its morphisms are generated by *weighted edge contractions* $G \rightarrow G/e$ for an edge e of G as well as by the automorphisms of all (G, h) .

Here a *weighted edge contraction* $c : G \rightarrow G/e$ is an edge contraction such that for every vertex v in G/e we have

$$g(c^{-1}(v)) = h(v) .$$

2. Moreover, for every graph G we denote by

$$\widetilde{M}_G = \mathbb{R}_{\geq 0}^{E(G)}$$

the parameter space of all possible edge lengths on G .

3. The association $G \mapsto \widetilde{M}_G$ defines a contravariant functor $J_g \rightarrow \mathbf{RPC}_{\mathbb{Z}}$ from J_g to the category of rational polyhedral cones. It associates to a weighted edge contraction $G \rightarrow G/e$ the embedding of the corresponding face of $\widetilde{M}_G^{trop}$ and to an automorphism of G the automorphism of $\widetilde{M}_G$ that permutes the entries correspondingly.

We note hereby that we have a decomposition into locally closed subsets

$$M_g^{trop} = \bigsqcup_G \mathbb{R}_{>0}^{E(G)} / \text{Aut}(G) ,$$

where the disjoint union is taken over the objects in J_g , i.e. over all isomorphism classes of stable finite vertex-weighted graphs G of genus g .

Example 4.7 ([34] Theorem 2.12). *For a d -dimensional cone complex C , its f -vector is defined as $(f_0, f_1, \dots, f_d)$, where f_i is the number of i -dimensional cones in C . The 12-dimensional moduli space M_5^{trop} has 4555 cells; its f -vector is given by*

$$f(M_5^{trop}) = (1, 3, 11, 34, 100, 239, 492, 784, 1002, 926, 632, 260, 71) .$$

Remark 4.8. *Earlier approaches, such as [24], [27], [34], [37], [29] and [109], used to refer to the structure of a generalized cone complex as a stacky fan. Since there is a closely related, but not equivalent, notion of the same name in the theory of toric stacks we prefer to follow the terminology of generalized cone complexes introduced in [1].*

The spaces $S(D)$, $R(D)$, and $|D|$ are known to carry the structure of a polyhedral complex (see Definition 3.19). The following proposition is a more detailed version of [57, Lemma 1.9].

Proposition 4.9. *Given a divisor D on a tropical curve Γ , the space $S(D)$ has the structure of a polyhedral complex. Choose an orientation for each edge e of Γ , identifying it with the open interval $[0, l(e)]$. Then the cells of $S(D)$ can be described by the following (discrete) data:*

(i) a partition of $\{p_1, \dots, p_n\}$ into disjoint subsets P_e and P_v (indexed by $v \in V(G)$ and edges $e \in E(G)$) that tells us on which edge (or at which vertex) every p_i is located,

(ii) a total order on each P_e , and

(iii) the outgoing slope $m_e \in \mathbb{Z}$ of f at the starting point of e

such that for every vertex v the equality

$$\#P_v = D(v) + \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -(\#P_e + m_e)$$

holds. Furthermore, this polyhedral structure descends from $S(D)$ to $R(D) = S(D)/S_n$ (where S_n is the n -th symmetry group) and $|D| = R(D)/\mathbb{R}$.

Remark 4.10. This set $R(D)$ is consistent with the one defined in Section 3.1.4.

Proof. Set $d_v = \#P_v$ and $d_e = \#P_e$. We claim that the points in a cell of $S(D)$ can be parametrized by the following two types of continuous data:

- the value $f(v)$ at a vertex v , as well as
- the distance $d(p_i^e)$ of every $p_i^e \in P_e$ from $0 \in e = [0, l(e)]$.

The distances $d(p_i^e)$ immediately determine the p_i . In order to reconstruct f (if it exists) we write $\sum_{p \in P_e} p = \sum_j d_{e,j} x_j$ for points $0 < x_1 < \dots < x_r < l(e)$ on e , where the positive integers $d_{e,j}$ indicate the number of p_i^e that are all located at the same point x_j . The rational function f is then determined by taking the value $f(v)$ at the origin of every edge $e = [0, l(e)]$ and continuing it piecewise linearly with slope m_e until we hit x_1 , at which point we change the slope to $d_{e,1} + m_e$ until we hit x_2 , where we change the slope to $d_{e,2} + d_{e,1} + m_e$, and so on until we hit the vertex v' at the end of $e = [0, l(e)]$. So, by continuity, for every edge we obtain the linear condition

$$\begin{aligned} f(v') &= f(v) + m_e x_1 + \sum_{k=1}^r (m_e + \sum_{j=1}^k d_{e,j}) (x_{k+1} - x_k) = \\ &= f(v) + m_e l(e) + \sum_{i=1}^r d_{e,i} (l(e) - x_i) \end{aligned}$$

on the parameters of a cell in $S(D)$. This, together with the inequalities $0 < x_1 < \dots < x_r < l(e)$ determines the polyhedral structure of a cell in $S(D)$. Note that our parameters are still overdetermined in the sense that there may be no rational function f such that $D + (f) = p_1 + \dots + p_n \geq 0$ and which also fulfills all of the above inequalities; in this case we obtain an empty cell.

The conditions on the cells of $S(D)$ are all discrete and the points within one cell are all parametrized by the distances $d(p_i^e) \in (0, l(e))$ and the values $f(v)$ subject to these discrete

conditions. Therefore $S(D)$ is a polyhedral complex that does not depend on the choice of the orientation of Γ .

The action of the symmetric group S_n on every cell is affine linear and therefore the polyhedral structure descends to $R(D)$. Moreover, the additive group $\mathbb{R}$ acts on $R(D)$ by adding a constant to all $f(v)$ and therefore the polyhedral structure also descends to $|D|$. $\square$

4.3 Structure of the tropical Hodge bundle

Let Γ be a tropical curve with a fixed minimal model G . We recall Definition 4.3.

Definition 4.11. *Let $g \geq 2$. As a set, the tropical Hodge bundle Λ_g^{trop} is defined to be*

$$\Lambda_g^{trop} = \{([\Gamma], f) \mid [\Gamma] \in M_g^{trop} \text{ and } f \in \text{Rat}(\Gamma) \text{ such that } K_\Gamma + (f) \geq 0\}$$

and the projective tropical Hodge bundle $\mathcal{H}_g^{trop}$ as

$$\mathcal{H}_g^{trop} = \{([\Gamma], D) \mid [\Gamma] \in M_g^{trop} \text{ and } D \in |K_\Gamma|\}$$

The tropical Hodge bundles come with natural projection maps

$$\Lambda_g^{trop} \longrightarrow M_g^{trop} \quad \text{and} \quad \mathcal{H}_g^{trop} \longrightarrow M_g^{trop}$$

given by $([\Gamma], f) \mapsto [\Gamma]$ and $([\Gamma], D) \mapsto [\Gamma]$, which, in abuse of notation, we both denote by π_g .

In order to understand the structure of the tropical Hodge bundle Λ_g^{trop} we consider the pullback of Λ_g^{trop} and $\mathcal{H}_g^{trop}$ to $\widetilde{M}_G$, defined as

$$\widetilde{\Lambda}_G = \{([\Gamma], f) \mid [\Gamma] \in \widetilde{M}_G \text{ and } f \in \text{Rat}(\Gamma) \text{ such that } K_\Gamma + (f) \geq 0\}$$

and

$$\widetilde{\mathcal{H}}_G = \{([\Gamma], D) \mid [\Gamma] \in \widetilde{M}_G \text{ and } D \in |K_\Gamma|\}.$$

In analogy with the space $S(D)$, we also set

$$\begin{aligned} \widetilde{S}_G = \{([\Gamma], f, p_1, \dots, p_{2g-2}) \mid [\Gamma] \in \widetilde{M}_G \text{ and } f \in \text{Rat}(\Gamma) \\ \text{such that } K_\Gamma + (f) = p_1 + \dots + p_{2g-2} \geq 0\}. \end{aligned}$$

Proposition 4.12.

(i) *The action of S_{2g-2} on S_G that permutes the points $p_1, \dots, p_{2g-2}$ induces a natural bijection*

$$\widetilde{\Lambda}_G \simeq \widetilde{S}_G / S_{2g-2}.$$

- (ii) The action of the additive group $\mathbb{R} = (\mathbb{R}, +)$ on $\tilde{\Lambda}_G$, given by adding constant functions to f , induces a natural bijection

$$\tilde{\mathcal{H}}_G \simeq \tilde{\Lambda}_G / \mathbb{R} .$$

Proof. The projections $\tilde{S}_G \rightarrow \tilde{M}_G$ and $\tilde{\Lambda}_G \rightarrow \tilde{M}_G$ are both invariant under the action of S_{2g-2} and $\mathbb{R}$. Therefore our claims follow from the respective identities on the fibers. $\square$

Now we prove Theorem 4.4.

Proof of Theorem 4.4. Part (i): We are going to show that $\tilde{S}_G$ canonically carries the structure of a cone complex. Then, by Proposition 4.12 above, both $\tilde{\mathcal{H}}_G$ and $\tilde{\Lambda}_G$ carry the structure of a generalized cone complex.

Choose an orientation for each edge e of G , identifying it with the closed interval $[0, l(e)]$. As in Proposition 4.9 above, we can describe the cells of $\tilde{S}_G$ by the following discrete data:

- (i) a partition of $\{p_1, \dots, p_{2g-2}\}$ into disjoint subsets P_e and P_v (indexed by vertices $v \in V(G)$ and edges $e \in E(G)$) that tells us on which edge (or at which vertex) each p_i is located,
- (ii) a total order on each P_e , and
- (iii) the integer slope m_e of f at the starting point of e

such that for every vertex v the equality

$$d_v = 2h(v) - 2 + |v| + \sum_{\text{outward edges at } v} m_e + \sum_{\text{inward edges at } v} -(d_e + m_e)$$

holds, where $d_v = \#P_v$ and $d_e = \#P_e$. The continuous parameters describing all elements in our cell are given by

- (i) the values $f(v)$,
- (ii) the distances $d(p_i^e)$ of p_i^e from $0 \in [0, l(e)]$, and
- (iii) the lengths $l(e)$.

In order to find the conditions on those parameters, we again write $\sum_{p \in P_e} p = \sum d_{e,j} x_j$ for $x_1 < \dots < x_r$. Using this notation we have $0 < x_1 < \dots < x_r < l(e)$ as conditions on the parameters $d(p_i^e) = x_i$. And by the continuity of f , we have

$$\begin{aligned} m_e x_1 &= f(x_1) - f(v) \\ (m_e + d_{e,1})(x_2 - x_1) &= f(x_2) - f(x_1) \\ (m_e + d_{e,1} + d_{e,2})(x_3 - x_2) &= f(x_3) - f(x_2) \\ &\vdots \\ (m_e + \sum_{j=1}^r d_{e,j})(l(e) - x_r) &= f(v') - f(x_r) . \end{aligned}$$

We may use the Fourier-Motzkin elimination [112, Section 1.2] to eliminate the non-parameters $f(x_1), \dots, f(x_r)$. So we can combine the system of equations to

$$f(v') = f(v) + (m_e + d_e)l(e) - \sum_{j=1}^r d_{e,j}x_j . \quad (4.2)$$

Since these conditions are invariant under multiplying all parameters simultaneously by elements in $\mathbb{R}_{\geq 0}$, every non-empty cell in $\tilde{S}_G$ has the structure of a rational polyhedral cone.

Finally, the natural action of $\text{Aut}(G)$ on $\tilde{S}_G$, given by

$$\phi \cdot ([\Gamma], f, p_1, \dots, p_{2g-2}) = ([\phi(\Gamma)], f \circ \phi^{-1}, \phi(p_1), \dots, \phi(p_{2g-2}))$$

for $\phi \in \text{Aut}(G)$ is compatible with both the S_{2g-2} -operation and the $\mathbb{R}$ -operation. Moreover, given a weighted edge contraction $G' = G/e$ of G , the natural map $\tilde{S}_{G'} \hookrightarrow \tilde{S}_G$ identifies $\tilde{S}_{G'}$ with the subcomplex of $\tilde{S}_G$ given by the condition $l(e) = 0$ in the above coordinates.

Therefore we can conclude that both

$$\Lambda_g^{trop} = \lim_{\longrightarrow} \tilde{\Lambda}_G \quad \text{and} \quad \mathcal{H}_g^{trop} = \lim_{\longrightarrow} \tilde{\mathcal{H}}_G ,$$

where the limits are taken over the category J_g as in Section 4.2 above, carry the structure of a generalized cone complex.

Part (ii): We need to show that the dimension of a maximal-dimensional cone in $\mathcal{H}_g$ is $5g - 5$. By [24, Proposition 3.2.5 (i)], we have $\dim M_g^{trop} = 3g - 3$ and, by Corollary 3.23, the dimension of the fiber $|K_\Gamma|$ of a point $[\Gamma]$ is at most $\deg(K_\Gamma) = 2g - 2$. This shows that the dimension of $\mathcal{H}_g^{trop}$ is at most $(3g - 3) + (2g - 2) = 5g - 5$.

In addition we now exhibit a $(5g - 5)$ -dimensional cone in $\mathcal{H}_g^{trop}$ as follows: Consider the tropical curve $\Gamma_{\max}$ as indicated in Figure 4.2 and note that it has $2g - 2$ vertices and $3g - 3$ edges.

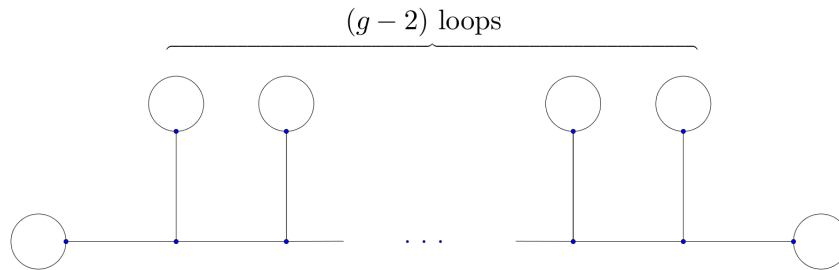


Figure 4.2: The tropical curve $\Gamma_{\max}$ with $2g - 2$ vertices (bold) and $3g - 3$ edges.

Lemma 4.13. *Let Γ be a tropical curve with minimal model $G = (V, E)$. Let D be a divisor on Γ such that the support of D is contained in V . Then the combinatorial structure of $|D|$ is independent of the length of any loop or bridge in G .*

A proof of Lemma 4.13 is provided below. Lemma 4.13 implies that the combinatorial structure of $|K_{\Gamma_{\max}}|$ is independent of the edge lengths, so we can choose a generic chamber. We obtain a divisor $D \in |K_{\Gamma_{\max}}|$ as indicated in Figure 4.3.

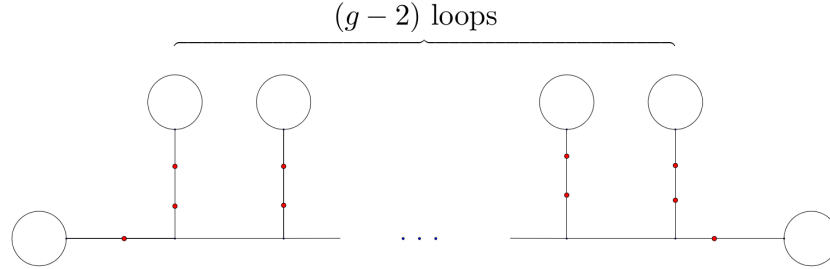


Figure 4.3: The divisor D on $\Gamma_{\max}$.

By [64, Proposition 13], the divisor D belongs to a $(2g - 2)$ -dimensional face in $|K_{\Gamma_{\max}}|$. Thus there is a $(5g - 5)$ -dimensional cone in $\mathcal{H}_g^{\text{trop}}$.

Part (iii): We use the coordinates described in part (i) above. For every edge e of G with $d_e = \#P_e = 0$ we have an equation $m_e l(e) = f(v) - f(v')$ which is parametrized by the $l(e)$. So suppose $d_e > 0$ and set $x = \frac{1}{d_e} \sum_{j=1}^r d_{e,j} x_j$. Then (4.2) can be rewritten as

$$f(v') = f(v) + (m_e + d_e)l(e) - d_e x .$$

Eliminating x from this equation, subject to the condition $0 < x < l(e)$ we obtain

$$\begin{aligned} (-d_e - m_e)l(e) &> f(v) - f(v') \\ m_e l(e) &< f(v') - f(v) \end{aligned}$$

and therefore we obtain that the images of cells in $\widetilde{M}_G$ are polyhedra. Moreover, the combinatorial type of $|K_{\Gamma}|$ is independent under scaling all edge lengths with a factor in $\mathbb{R}_{>0}$ and thus all these polyhedra determine a subdivision of $\widetilde{M}_G$ such that on each relatively open cell of this subdivision, the corresponding $|K_{\Gamma}|$ has the same set of cells, i.e., the combinatorial type of $\widetilde{S}_G$ is constant. $\square$

Proof of Lemma 4.13. Suppose $e_b \in E$ is a bridge in G . Let Γ_1 and Γ_2 be two tropical curves with minimal model G such that the length of e in Γ_1 and Γ_2 is l and $c \cdot l$ respectively (where $l, c > 0$), and for $e \in E - \{e_b\}$, the lengths of e in Γ_1 and Γ_2 are the same. It suffices to show that the sets of cells in $|D|_{\Gamma_1}$ and $|D|_{\Gamma_2}$ are exactly the same.

In Γ_1 , we view the bridge e_b as the open interval $(0, l)$. For any cell C_1 in $|D|_{\Gamma_1}$, its data consists of an integer m_{e_b} and a partition of nonnegative integers $d_{e_b} = \sum_{j=1}^r d_{e,j}$. Suppose a divisor $D + \text{Div}(f_1)$ is $\sum_{j=1}^r d_{e,j} \cdot x_j$ on the bridge e_b , where $0 < x_1 < \dots < x_{r(e)} < l$. Here the rational function f_1 is unique up to a translation. So we may assume that the value of f_1 is zero on the endpoint 0 of e_b . Then on e_b the function f_1 is defined inductively as follows:

- For $0 < x \leq x_1$ it is given by

$$f_1(x) = m_{e_b} \cdot x .$$

- Given $x_k < x < x_{k+1}$ for $1 \leq k \leq r(e)$ we have

$$f_1(x) = f_1(x_k) + \left(m_{e_b} + \sum_{j=1}^k d_{e_b,j} \right) \cdot (x - x_k)$$

Now on Γ_2 we also view the bridge e_b as the open interval $(0, c \cdot l)$, with the same orientation. We construct a rational function f_2 on Γ_2 . First, we define f_2 on the bridge e_b inductively as follows:

- For $0 < x \leq cx_1$ we define $f_2(x)$ by

$$f_2(x) = m_{e_b} \cdot x ,$$

- and, given $cx_k < x < cx_{k+1}$ for $1 \leq k \leq r(e)$, we set

$$f_2(x) = f_1(cx_k) + \left(m_{e_b} + \sum_{j=1}^k d_{e_b,j} \right) \cdot (x - cx_k)$$

Since e_b is a bridge in G , the graph $G - e_b$ consists of two connected components. We denote them by G_1 and G_2 , where G_1 contains the endpoint 0 of e_b and G_2 contains the endpoint cl of e_b . For convenience we let

$$f_1(l) = m_{e_b} \cdot x_1 + (m_{e_b} + d_{e_b,1}) \cdot x_2 + \cdots + \left(m_{e_b} + \sum_{j=1}^r d_{e_b,j} \right) \cdot (l - x_{r(e)})$$

and

$$f_2(cl) = c \cdot f_1(l).$$

Then we define f_2 on $G - e_b$ as follows:

$$f_2(x) = \begin{cases} f_1(x), & \text{if } x \in G_1; \\ f_1(x) + f_2(cl) - f_1(l) = f_1(x) + (c - 1)f_1(l), & \text{if } x \in G_2. \end{cases}$$

By definition, f_1 and f_2 admit the same data on G_i for $i = 1, 2$. In addition, on the bridge e_b , both functions admit the integer m_{e_b} and the same partition $\sum_{j=1}^r d_{e_b,j}$. So f_2 corresponds to a cell C_2 in $|D|_{\Gamma_2}$ that is exactly the same as C_1 . For the same reason we can get the cell C_1 from C_2 (just note that $l = (\frac{1}{c}) \cdot (cl)$). So Lemma 4.13 holds for bridges.

Suppose e_l is a loop in G . In this case almost the same proof works, except that the condition $f_1(l) = 0$ must hold, and $G' = G - e_l$ is connected. We have $f_2(cl) = 0$; therefore we may define f_2 on e_l in the same way as on e_b , as well as $f_2(x) = f_1(x)$ for all $x \in G'$. Thus our claim also holds for loops. $\square$

4.4 Examples with small genus

In this section we present some computational results on the polyhedral structure of tropical Hodge bundles of small genus. In order to describe all cones in Λ_g^{trop} we first list all cones in M_g^{trop} . Then for each cone, we compute its subdivision by the structure of $|K_\Gamma|$. It turns out that already the two cases $g = 2$ and $g = 3$ show a surprisingly distinct behavior.

Proposition 4.14. *Let Γ be a tropical curve in M_2^{trop} . Then the combinatorial structure of $|K_\Gamma|$ is uniquely determined by the minimal model G of Γ . In other words, it is independent of the edge lengths in Γ .*

Proof. There are 7 faces in M_2^{trop} as in [34, Figure 4]. For 6 faces among them, all edges are loops or bridges, so the claim follows from Lemma 4.13. For the "theta graph" G_θ , by explicit computation we know that the canonical linear system $|K_{G_\theta}|$ is always a one-dimensional polyhedral complex with three segments, as in Figure 4.4. $\square$

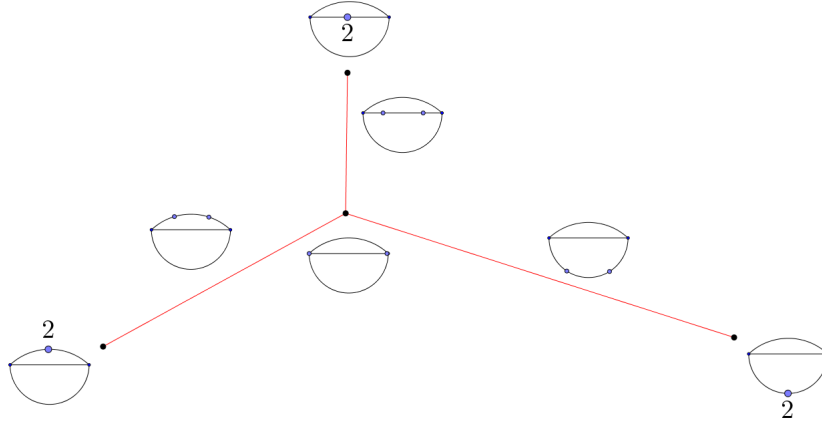


Figure 4.4: The polyhedral complex $|K_{G_\theta}|$. Its f -vector is $(4, 3)$.

The face lattice of Λ_2^{trop} is visualized in Figure 4.1.

Remark 4.15. *The f -vector of Λ_2^{trop} is $(1, 5, 11, 16, 9, 1)$, which is consistent with Theorem 4.4 (ii). The unique 5-dimensional face consists of the "dumbbell" graph and a triangular cell in $|K_\Gamma|$. In other words, any divisor in this cell is of the form $P + Q$, where P and Q are distinct points in the interior of the bridge in the dumbbell graph.*

When $g = 3$, the counterpart of Proposition 4.14 is no longer true. One counterexample consists of the 6-dimensional cone $C \simeq \mathbb{R}_{\geq 0}^6$ in M_3^{trop} parameterizing tropical curves whose minimal model G is a complete metric graph K_4 . The following proposition characterizes the open chambers of C regarding the structure of $|K_G|$. It is a result of explicit computations using the algorithm described in Section 3.2.

Proposition 4.16. (i) *There are 51 open chambers in C . For all metrics in the same chamber, the canonical linear system $|K_G|$ has the same set of cells. In that case, the polyhedral complex $|K_G|$ always has 34 vertices, 60 edges, and 27 two-dimensional faces (12 triangles and 15 quadrilaterals). However, there are 4 non-isomorphic combinatorial structures of $|K_G|$.*

(ii) *Let $M = (M_{12}, M_{13}, M_{14}, M_{23}, M_{24}, M_{34})$ be a metric. Consider the following four 3-subsets:*

$$\{M_{12}, M_{13}, M_{14}\}, \{M_{12}, M_{23}, M_{24}\}, \{M_{13}, M_{23}, M_{34}\}, \{M_{14}, M_{24}, M_{34}\}. \quad (4.3)$$

Then M belongs to an open chamber if and only if among the elements of each 3-subset in (4.3), the minimum is attained only once.

Remark 4.17 (The structure of $|K_G|$ for a generic metric). *If M belongs to an open chamber, the canonical linear system $|K_G|$ always has the 13 vertices in Figure 4.5. Among them, the*

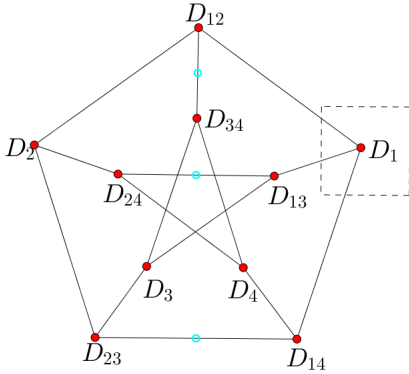


Figure 4.5: The main skeleton of $|K_G|$. An extra vertex is connected to the 10 vertices in red.

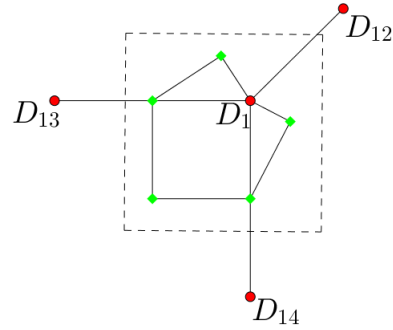


Figure 4.6: Each bat consists of 5 extra vertices (green). This bat appears when $M_{12} < M_{13}, M_{14}$.

10 labeled vertices are all connected to an extra vertex that is the divisor K_G . The remaining 20 vertices come from 4 copies of a sub-structure (which we call a bat) attached at D_1 , D_2 , D_3 , and D_4 . Note that some edges in Figure 4.5 are subdivided by other vertices in the bats.

The 4 distinct combinatorial types of $|K_G|$ come from different ways of attaching the bats. Since M belongs to an open chamber, the minimum of M_{12}, M_{13}, M_{14} appears only once. Suppose it is M_{12} , then the bat at D_1 is attached along the edges towards D_{13} and D_{14} , as in Figure 4.6. Figure 4.7 shows the divisors D_i and D_{ij} .

Remark 4.18 (The boundary of the 51 open chambers). *The action of the permutation group S_4 on the vertices of K_4 induces 4 orbits among the 51 open chambers, with lengths 24, 12, 12, and 3. Each orbit corresponds to a combinatorial type of $|K_G|$. Each open chamber is an open cone in C , defined by homogeneous linear inequalities involving M_{12} ,*

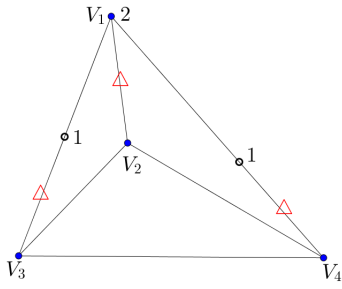
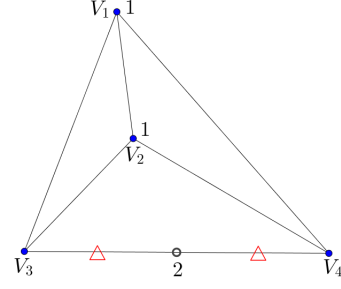

 The divisor D_1 .

 The divisor D_{34} .

Figure 4.7: Numbers are the coefficients of divisors and the $\triangle$ symbols show the equal line segments.

M_{13} , M_{14} , M_{23} , M_{24} , and M_{34} . The inequalities are displayed as the covers in a lattice. For example, M_{13} covering M_{12} means that the inequality $M_{13} > M_{12}$ holds.

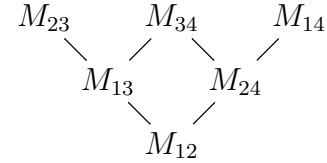
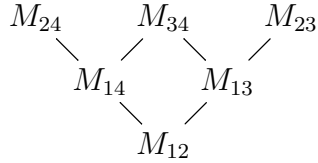


Figure 4.8: Representatives of two chamber orbits of length 12.

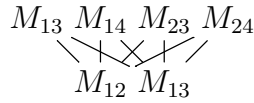


Figure 4.9: Representatives of a chamber orbit of length 3.

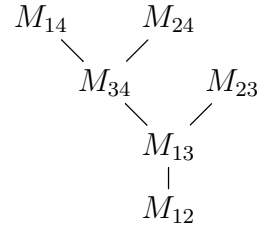


Figure 4.10: Representatives of a chamber orbit of length 24.

4.5 The realizability problem

Let k be an algebraically closed field carrying the trivial absolute value. In [1], expanding on earlier work (see e.g. [109]), the authors have constructed a natural (continuous) tropicalization map

$$\begin{aligned} \text{trop}_{\mathcal{M}_g} : \mathcal{M}_g^{an} &\longrightarrow M_g^{trop} \\ x &\longmapsto [\Gamma_x] \end{aligned}$$

from the non-Archimedean analytic moduli space $\mathcal{M}_g^{an}$ onto M_g^{trop} . Let us recall the construction of Γ_x : A point $x \in \mathcal{M}_g^{an}$ parametrizes an algebraic curve C over some non-Archimedean extension K of k . Possibly after a finite extension $K'|K$ we can extend C to a stable model $\mathcal{C} \rightarrow \text{Spec } R'$ over the valuation ring R' of K' . Denote by G_x the weighted dual graph of the special fiber $\mathcal{C}_s$ of $\mathcal{C}$, whose vertices correspond to the components of $\mathcal{C}_s$ and in which we have an edge e between two vertices v and v' for every node connecting the two corresponding components C_v and $C_{v'}$. The vertex weight function is given by

$$h(v) = g(\tilde{C}_v) ,$$

where $\tilde{C}_v$ denotes the normalization of C_v . Finally, around every node p_e in $\mathcal{C}_s$ we can find formal coordinates x and y of $\mathcal{C}$ such that $xy = t$ for some element t in the base. Then the edge length of e is given by $l(e) = \text{val}(t)$.

Denote by $\mathcal{H}_g^{an}$ the non-Archimedean analytification of the total space of the algebraic Hodge bundle $\mathcal{H}_g$.

Proposition 4.19. *There is a natural tropicalization map $\text{trop}_{\mathcal{H}_g} : \mathcal{H}_g^{an} \rightarrow \mathcal{H}_g^{trop}$ that makes the diagram*

$$\begin{array}{ccc} \mathcal{H}_g^{an} & \xrightarrow{\text{trop}_{\mathcal{H}_g}} & \mathcal{H}_g^{trop} \\ \downarrow & & \downarrow \\ \mathcal{M}_g^{an} & \xrightarrow{\text{trop}_{\mathcal{M}_g}} & M_g^{trop} \end{array}$$

commute.

Proof. An element $x \in \mathcal{H}_g^{an}$ parametrizes a tuple (C, K_C) consisting of a smooth projective curve C over a non-Archimedean extension K of k together with a canonical divisor on C . Then we may associate to (C, K_C) the point $([\Gamma_x], \tau_*(K_C))$, where

$$\tau_* : \text{Div}(C_{\bar{K}}) \longrightarrow \text{Div}(\Gamma)$$

denotes the specialization map constructed in [10, Section 2.3] that is given by pushing K_C forward to the non-Archimedean skeleton of $\mathcal{C}$. As shown in [10, Section 2.3] and the references therein, this is well-defined and the commutativity of the above diagram is an immediate consequence of the definition. $\square$

It is well-known that $\text{trop}_g : \mathcal{M}_g^{an} \rightarrow M_g^{trop}$ is surjective. By Theorem 4.4 (ii) we have that

$$\dim_{\mathbb{C}} \mathcal{H}_g = 4g - 4 < 5g - 5 = \dim \mathcal{H}_g^{trop}$$

and therefore the analogous statement for $\mathcal{H}_g$ appears to be false. This gives rise to the following problem.

Problem 4.20. *Find a characterization of the locus $\text{trop}_{\mathcal{H}_g}(\mathcal{H}_g^{an})$ in $\mathcal{H}_g^{trop}$, the so-called realizability locus in $\mathcal{H}_g^{trop}$.*

In other words, given a (stable) tropical curve Γ of genus g together with a divisor D that is equivalent to K_Γ , find algebraic and combinatorial conditions that ensure that there is an algebraic curve C over a non-Archimedean field extension K of k together with a canonical divisor $\tilde{D}$ on C such that

$$\text{trop}_{\mathcal{H}_g}([C], \tilde{D}) = ([\Gamma], D) .$$

Since $\text{trop}_{\mathcal{M}_g}$ is surjective, we know already that every tropical curve Γ can be lifted to a smooth algebraic curve C . In the case of Γ having integer edge lengths $l(e)$ we can give a constructive approach to this problem: Consider a special fiber $\mathcal{C}_s$ over k whose weighted dual graph is G , then apply logarithmically smooth deformation theory to find a smoothing of $\mathcal{C}_s$ to a stable family $\mathcal{C} \rightarrow \text{Spec } R$ with deformation parameters $l(e)$ at each node (see e.g. [63, Proposition 3.38]). If all $l(e) = 1$, we may alternatively also proceed as in [10, Appendix B].

Now let $\tilde{D}$ be a divisor on C that specializes to the given canonical divisor D on Γ . Since $\deg \tilde{D} = \deg D = 2g - 2$, Clifford's theorem (or alternatively Baker's Specialization Lemma [10, Corollary 2.11]) shows that the rank of $\tilde{D}$ is at most $g - 1$. If the rank of D is smaller than $g - 1$ it cannot be a canonical divisor. If, however, the divisor $\tilde{D}$ has rank $g - 1$, then, by Riemann-Roch, it is a canonical divisor. So the realizability problem reduces to finding a lift of the divisor D of rank $g - 1$.

The existence of such a divisor would follow, for example, from the smoothness of a suitable moduli space of limit linear series (see e.g. [52] and [88]). Unfortunately the machinery of limit linear series is not available in full generality (i.e. for nodal special fibers that are not of compact type). However, considerations undertaken from the point of view of compactifications of the moduli space of abelian differentials and its strata (see in particular [9]) treating the special case of limits of canonical linear systems seem to provide us with a very promising approach for future investigations into this question.

Chapter 5

Convexity in tree spaces

This chapter is a joint work with Bernd Sturmfels, Xiaoxian Tang and Ruriko Yoshida. It is posted on the [arXiv](#) [74] with the same title. This chapter is motivated from the study of phylogenetic trees in biology. We consider not only a well-studied metric space (the BHV space), but also another metric space motivated from tropical geometry. We compare the properties of both metric spaces involving convexity. In summary, the geodesic paths are unique in the BHV space, while we showed that the dimension of a geodesic triangle can be arbitrarily high; the geodesic paths are not unique in the alternative metric space, while this metric space is better in terms of the depth (the largest codimension of cones traversed).

5.1 Introduction

A finite metric space with m elements is represented by a nonnegative symmetric $m \times m$ -matrix $D = (d_{ij})$ with zero entries on the diagonal such that all triangle inequalities are satisfied:

$$d_{ik} \leq d_{ij} + d_{jk} \quad \text{for all } i, j, k \text{ in } [m] := \{1, 2, \dots, m\}.$$

The set of all such metrics is a full-dimensional closed polyhedral cone, known as the *metric cone*, in the space $\mathbb{R}^{\binom{m}{2}}$ of symmetric matrices with vanishing diagonal. For many applications one requires the following strengthening of the triangle inequalities:

$$d_{ik} \leq \max(d_{ij}, d_{jk}) \quad \text{for all } i, j, k \in [m]. \tag{5.1}$$

If (5.1) holds then the metric space D is called an *ultrametric*. An ultrametric (d_{ij}) can be considered as the weighted adjacency matrix of an *equidistant tree*, i.e. a metric tree where any two leaves (taxa) have the same distance to their common ancestor. The set of all ultrametrics contains the ray $\mathbb{R}_{\geq 0}\mathbf{1}$ spanned by the all-one metric $\mathbf{1}$, which is defined by $d_{ij} = 1$ for $1 \leq i, j \leq m$. The image of the set of ultrametrics in the quotient space $\mathbb{R}^{\binom{m}{2}}/\mathbb{R}\mathbf{1}$ is denoted $\mathcal{U}_m$ and called the *space of ultrametrics*. It is known in tropical geometry [8, 78] and in phylogenetics [60, 96] that $\mathcal{U}_m$ is the support of a pointed simplicial fan of dimension

$m - 2$. That fan has $2^m - m - 2$ rays, namely the *clade metrics* D_σ . A *clade* σ is a proper subset of $[m]$ with at least two elements, and D_σ is the ultrametric whose ij -th entry is 0 if $i, j \in \sigma$ and 1 otherwise. Each cone in that fan structure consists of all ultrametrics whose tree has a fixed topology. We encode each topology by a *nested set* [53], i.e. a set of clades $\{\sigma_1, \dots, \sigma_d\}$ such that

$$\sigma_i \subset \sigma_j \quad \text{or} \quad \sigma_j \subset \sigma_i \quad \text{or} \quad \sigma_i \cap \sigma_j = \emptyset \quad \text{for all } 1 \leq i < j \leq d. \quad (5.2)$$

Here d can be any integer between 1 and $m - 2$. The nested set represents the d -dimensional cone spanned by $\{D_{\sigma_1}, D_{\sigma_2}, \dots, D_{\sigma_d}\}$ inside $\mathcal{U}_m \subset \mathbb{R}^{\binom{m}{2}} / \mathbb{R}\mathbf{1}$. For an illustration of this fan structure, consider equidistant trees on $m = 4$ taxa. The space of these is a two-dimensional fan over the *Petersen graph*, shown on the left in Fig. 5.1.

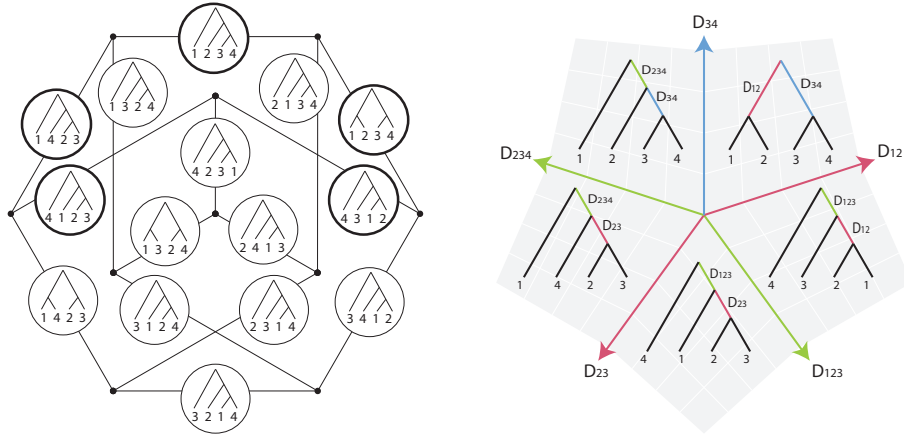


Figure 5.1: The space of ultrametrics $\mathcal{U}_4$ is a two-dimensional fan with 15 maximal cones. Their adjacency forms a Petersen graph. Depicted on the right is a cycle of five cones.

At this point it is essential to stress that we have *not* yet defined convexity or a metric on $\mathcal{U}_m$. So far, our tree space $\mathcal{U}_m$ is nothing but a subset of $\mathbb{R}^{\binom{m}{2}} / \mathbb{R}\mathbf{1}$. It is the support of the fan described above, but even that fan structure is not unique. There are other meaningful fan structures, classified by the *building sets* in [53]. An important one is the τ -space of [60, §2], known to combinatorialists as the order complex of the partition lattice [8].

The aim of this chapter is to compare different geometric structures on $\mathcal{U}_m$, and to explore statistical issues. The first geometric structure is the metric proposed by Billera, Holmes and Vogtman in [21]. In their setting, each metric cone is isometric to $\mathbb{R}_{\geq 0}^{m-2}$ for each binary tree topology. The *BHV metric* is the unique metric on $\mathcal{U}_m$ that restricts to the usual Euclidean distance on each such cone. For this to be well-defined, we must fix a simplicial fan structure on $\mathcal{U}_m$. This issue is subtle, as explained by Gavryushkin and Drummond in [60]. The BHV metric has the CAT(0)-property [21, §4.2]. This implies that between any two points there is a unique geodesic. Owen and Provan [89] proved that these geodesics can be computed in polynomial time.

Remark 5.1. *For experts in phylogenetics, we note that our tree space $\mathcal{U}_m$, with its BHV metric, is very similar to the BHV space of [21], but the two are not identical. A difference arises because we restrict to equidistant trees. Given two equidistant trees, their geodesic in the space of [21] can pass through trees that are not equidistant. It may even traverse some tree topologies not seen in the BHV geodesic on $\mathcal{U}_m$. From a mathematical perspective, this is only a minor technical point, but it will require some thought when applying our results and techniques to data in evolutionary biology.*

In Section 5.2 we give the precise definition for the BHV metric in our setting. This is done in the setting of orthant spaces $\mathcal{F}_\Omega$ associated with flag simplicial complexes Ω . In Section 5.3 we study geodesically closed subsets of an orthant space $\mathcal{F}_\Omega$, with primary focus on geodesic triangles. Problem 5.9 asks whether these are always closed. Our main result, Theorem 5.14, states that the dimension of a geodesic triangle can be arbitrarily large. The same is concluded for the tree space $\mathcal{U}_m$ in Corollary 5.15.

Tropical geometry [78] furnishes an alternative geometric structure on $\mathcal{U}_m$, via the graphic matroid of the complete graph [78, Example 4.2.14]. More generally, for any matroid M , the tropical linear space $\text{trop}(M)$ is tropically convex by [78, Proposition 5.2.8], and it is a metric space with the tropical distance to be defined in (5.15). Tropical geodesics are not unique, but tropically convex sets have desirable properties. In particular, triangles are always 2-dimensional.

Section 5.5 offers an experimental study of Euclidean geodesics and tropical segments. The latter are better than the former with regard to *depth*, i.e. the largest codimension of cones traversed. This is motivated by the issue of *stickiness* in geometric statistics [67, 83]. Section 5.6 advocates tropical geometry for statistical applications.

One of the great opportunities offered by modern genomics is that phylogenetics applied on a genomic scale (phylogenomics) is powerful for elucidating gene and genome evolution, relationships among species and populations, and processes of speciation and molecular evolution. However, a well-recognized hurdle is the sheer volume of genomic data that can now be generated relatively cheaply and quickly, but for which analytical tools are lagging. There is a major need to explore new approaches to undertake comparative genomic and phylogenomic studies more rapidly and robustly than existing tools allow. In statistics, many classical methods are based on linear algebra, such as linear regression and principal component analysis. Since it is not clear how to combine linear algebra with BHV metrics, we suggest to use the tropical metric for statistical methods in tree spaces. Starting from Nye's principal component analysis in [86], we propose two geometric tools for future data analyses: computation of tropical centroids, and nearest-point projection onto tropical linear spaces.

It can be advantageous to replace $\mathcal{U}_m$ by a compact subspace. We define *compact tree space* $\mathcal{U}_m^{[1]}$ to be the image in $\mathcal{U}_m$ of the set of ultrametrics $D = (d_{ij})$ that satisfy $\max_{ij} \{d_{ij}\} = 1$. This is a polyhedral complex consisting of one convex polytope for each nested set $\{\sigma_1, \sigma_2, \dots, \sigma_d\}$. This polytope consists of all $(\ell_1, \ell_2, \dots, \ell_d) \in [0, 1]^d$ such that $\ell_{i_1} + \ell_{i_2} + \dots + \ell_{i_d} \leq 1$ whenever $\sigma_{i_1} \subset \sigma_{i_2} \subset \dots \subset \sigma_{i_d}$. In phylogenetics, these are equidis-

tant trees of height $\frac{1}{2}$ with a fixed tree topology. For instance, $\mathcal{U}_4^{[1]}$ is a polyhedral surface, consisting of 12 triangles and 3 squares, glued along 10 edges.

5.2 Orthant spaces

In order to understand the geometry of tree spaces, we work in the more general setting of globally nonpositively curved (NPC) spaces. This was suggested by Miller, Owen and Provan in [84, §6]. We follow their set-up.

A simplicial complex Ω on the set $[n] = \{1, 2, \dots, n\}$ is a *flag complex* if all its minimal non-faces have two elements. Equivalently, a flag complex is determined by its edges: a subset σ of $[n]$ is in Ω if and only if $\{i, j\} \in \Omega$ for all $i, j \in \sigma$. Every simplicial complex Ω on $[n]$ determines a simplicial fan $\mathcal{F}_\Omega$ in $\mathbb{R}^n$. The cones in $\mathcal{F}_\Omega$ are the orthants $\mathcal{O}_\sigma = \text{pos}\{e_i : i \in \sigma\}$ where $\sigma \in \Omega$. Here $\{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$. We say that $\mathcal{F}_\Omega$ is an *orthant space* if the simplicial complex Ω is flag. The support of the fan $\mathcal{F}_\Omega$ is turned into a metric space by fixing the usual Euclidean distance on each orthant. For a path connecting two points in $\mathcal{F}_\Omega$, it consists of finitely many line segments where each line segment is contained in one orthant. We define the *length* of this path to be the sum of the lengths of these line segments. A path between two points v and w in $\mathcal{F}_\Omega$ is usually denoted by a map $\gamma : [0, 1] \rightarrow \mathcal{F}_\Omega$ such that $\gamma(0) = v$ and $\gamma(1) = w$. A path of minimal length between two points is a *geodesic*.

Proposition 5.2. *Let $\mathcal{F}_\Omega$ be an orthant space. For any two points v and w in $\mathcal{F}_\Omega$ there exists a unique geodesic between v and w . This geodesic is denoted $G(v, w)$.*

The uniqueness of geodesics is attributed to Gromov. We refer to [7, Theorem 2.12] and [84, Lemma 6.2] for expositions and applications of this important result. The main point is that orthant spaces, with their Euclidean metric as above, satisfy the CAT(0)-property, provided Ω is flag. This property means for any point u in the orthant space $\mathcal{F}_\Omega$, for any geodesic $\gamma : [0, 1] \rightarrow \mathcal{F}_\Omega$ and for any $t \in [0, 1]$,

$$G(u, \gamma(t))^2 \leq (1-t)G(u, \gamma(0))^2 + tG(u, \gamma(1))^2 - t(1-t)G(\gamma(0), \gamma(1))^2.$$

Geometrically, the CAT(0)-property means that chords in triangles are no longer than the corresponding chords in Euclidean triangles [7, Definition 2.11]. The metric spaces $\mathcal{F}_\Omega$ coming from flag complexes Ω are called *global NPC orthant spaces* in [84]. For simplicity we here use the term *orthant space* for $\mathcal{F}_\Omega$. Throughout this chapter, we use the term *Euclidean metric* for the metric on orthant spaces defined above, with the tacit understanding that this is really a piecewise Euclidean metric.

Example 5.3. *Let $n = 3$ and $\Omega = \{12, 23, 31\}$, i.e. the 3-cycle. The fan $\mathcal{F}_\Omega$ is the boundary of the nonnegative orthant in $\mathbb{R}^3$. This is not an orthant space because Ω is not flag. Some geodesics in $\mathcal{F}_\Omega$ are not unique: the points $v = (1, 0, 0)$ and $w = (0, 1, 1)$ have distance $\sqrt{5}$, and there are two geodesics: one passing through $(0, \frac{1}{2}, 0)$ and the other passing through*

$(0, 0, \frac{1}{2})$. By contrast, let $n = 4$ and $\Omega = \{12, 23, 34, 41\}$, i.e. the 4-cycle. Then $\mathcal{F}_\Omega$ is a 2-dimensional orthant space in $\mathbb{R}^4$. The Euclidean geodesics on that surface are unique.

Owen and Provan showed that we can compute the unique geodesics in polynomial time in terms of the number of leaves in [89]. In [89], the focus was on tree space of Billera-Holmes-Vogtman [21]. It was argued in [84, Corollary 6.19] that the result extends to arbitrary orthant spaces. Owen and Provan gave a polynomial-time algorithm whose input consists of two points v and w in $\mathcal{F}_\Omega$ and whose output is the geodesic $G(v, w)$. We shall now describe their method.

Let σ be a simplex in a flag complex Ω , with corresponding orthant $\mathcal{O}_\sigma$ in $\mathcal{F}_\Omega$. Consider a point $v = \sum_{i \in \sigma} v_i e_i$ in $\mathcal{O}_\sigma$ and any face τ of σ . We write $v_\tau = \sum_{i \in \tau} v_i e_i$ for the projection of v into $\mathcal{O}_\tau$. Its Euclidean length $\|v_\tau\| = (\sum_{i \in \tau} v_i^2)^{1/2}$ is called the *projection length*.

We now assume that Ω is pure $(d - 1)$ -dimensional, i.e. all maximal simplices in Ω have the same dimension $d - 1$. This means that all maximal orthants in $\mathcal{F}_\Omega$ have dimension d . Consider two general points v and w in the interiors of full-dimensional orthants $\mathcal{O}_\sigma$ and $\mathcal{O}_\tau$ of $\mathcal{F}_\Omega$ respectively. We also assume that $\sigma \cap \tau = \emptyset$. Combinatorially, the geodesic $G(v, w)$ is then encoded by a pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{A} = (A_1, \dots, A_q)$ is an ordered partition of $\sigma = \{\sigma_1, \dots, \sigma_d\}$ and $\mathcal{B} = (B_1, \dots, B_q)$ is an ordered partition of $\tau = \{\tau_1, \dots, \tau_d\}$. These two partitions have the same number q of parts, and they satisfy the following three properties:

(P1) for all pairs $i > j$, the set $A_i \cup B_j$ is a simplex in Ω .

(P2) $\|v_{A_1}\|/\|w_{B_1}\| \leq \|v_{A_2}\|/\|w_{B_2}\| \leq \dots \leq \|v_{A_q}\|/\|w_{B_q}\|$.

(P3) For $i = 1, \dots, q$, there do not exist nontrivial partitions $L_1 \cup L_2$ of A_i and $R_1 \cup R_2$ of B_i such that $R_1 \cup L_2 \in \Omega$ and $\|v_{L_1}\|/\|w_{R_1}\| < \|v_{L_2}\|/\|w_{R_2}\|$.

The following result is due to Owen and Provan [89]. They proved it for the case of BHV tree space. The general result is stated in [84, Corollary 6.19].

Theorem 5.4 (Owen-Provan). *Given points $v, w \in \mathcal{F}_\Omega$ satisfying the hypotheses above, there exists an ordered pair of partitions $(\mathcal{A}, \mathcal{B})$ satisfying (P1), (P2), (P3). The geodesic is a sequence of $q + 1$ line segments, $G(v, w) = [v, u^1] \cup [u^1, u^2] \cup \dots \cup [u^q, w]$. Its length equals*

$$d(v, w) = \sqrt{\sum_{j=1}^q (\|v_{A_j}\| + \|w_{B_j}\|)^2}. \quad (5.3)$$

In particular, $(\mathcal{A}, \mathcal{B})$ is the pair of ordered partitions that minimizes (5.3). The breakpoint u^i lives in the orthant of $\mathcal{F}_\Omega$ that is indexed by $\bigcup_{j=1}^{i-1} B_j \cup \bigcup_{j=i+1}^q A_j$. Setting $u^0 = v$, the coordinates of the breakpoints are computed recursively by the formulas

$$u_k^i = \frac{\|u_{A_i}^{i-1}\| \cdot w_k + \|w_{B_i}\| \cdot u_k^{i-1}}{\|u_{A_i}^{i-1}\| + \|w_{B_i}\|} \quad \text{for } k \in \bigcup_{j=1}^{i-1} B_j, \quad (5.4)$$

$$u_l^i = \frac{u_l^{i-1}}{\|u_{A_j}^{i-1}\|} \cdot \frac{\|w_{B_i}\| \cdot \|u_{A_j}^{i-1}\| - \|u_{A_i}^{i-1}\| \cdot \|w_{B_j}\|}{\|u_{A_i}^{i-1}\| + \|w_{B_i}\|} \quad \text{for } l \in A_j, \ i+1 \leq j \leq q. \quad (5.5)$$

Computing the geodesic between v and w in the orthant space $\mathcal{F}_\Omega$ means identifying the optimal pair $(\mathcal{A}, \mathcal{B})$ among all pairs. This is a combinatorial optimization problem. Owen and Provan [89] gave a polynomial-time algorithm for solving this.

We implemented this algorithm in **Maple**. The source code is available at

<https://sites.google.com/site/bolinmath/research/codes/gtp>.

Our code works for any flag simplicial complex Ω and for any points v and w in the orthant space $\mathcal{F}_\Omega$, regardless of whether they satisfy the hypotheses of Theorem 5.4. The underlying geometry is as follows:

Theorem 5.5. *The degenerate cases not covered by Theorem 5.4 are dealt with as follows:*
 (1) *If $\sigma \cap \tau \neq \emptyset$, then an ordered pair of partitions $(\mathcal{A}, \mathcal{B})$ is constructed for $(\sigma \setminus \tau, \tau \setminus \sigma)$. The breakpoint u^i lives in the orthant of $\mathcal{F}_\Omega$ indexed by $\bigcup_{j=1}^{i-1} B_j \cup \bigcup_{j=i+1}^q A_j \cup (\sigma \cap \tau)$. It satisfies*

$$u_k^i = \frac{\|u_{A_i}^{i-1}\| \cdot w_k + \|w_{B_i}\| \cdot u_k^{i-1}}{\|u_{A_i}^{i-1}\| + \|w_{B_i}\|} \quad \text{for } k \in \sigma \cap \tau. \quad (5.6)$$

(2) *If $|\sigma| < d$ or $|\tau| < d$, i.e. v or w is in a lower-dimensional orthant of $\mathcal{F}_\Omega$, then we allow one (but not both) of the parts A_i and B_i to be empty, and we replace (P2) by*

$$(P2') \quad \|v_{A_i}\| \cdot \|w_{B_{i+1}}\| \leq \|w_{B_i}\| \cdot \|v_{A_{i+1}}\| \quad \text{for } 1 \leq i \leq q-1.$$

We still get an ordered pair $(\mathcal{A}, \mathcal{B})$ satisfying (P1), (P2'), (P3), but it may not be unique. If $i \geq 0$ is the largest index with $A_i = \emptyset$ and $j \leq q+1$ is the smallest index with $B_j = \emptyset$, then $G(v, w)$ is a sequence of $j - i - 1$ segments. The breakpoints are given in (5.4), (5.5), (5.6).

(3) *Now, the length of the geodesic $G(v, w)$ is equal to*

$$d(v, w) = \sqrt{\sum_{k \in \sigma \cap \tau} (v_k - w_k)^2 + \sum_{j=1}^q (\|v_{A_j}\| + \|w_{B_j}\|)^2}.$$

Proof. This is an extension of the discussion for BHV tree space in [89, §4]. □

Our primary object of interest is the space of ultrametrics $\mathcal{U}_m$. We shall explain its metric structure as an orthant space. It is crucial to note that this structure is not unique. Indeed, any polyhedral fan Σ in $\mathbb{R}^d$ can be refined to a simplicial fan with n rays whose underlying simplicial complex Ω is flag. That fan still lives in $\mathbb{R}^d$, whereas the orthant space $\mathcal{F}_\Omega$ lives in $\mathbb{R}^n$. There is a canonical piecewise-linear isomorphism between $\mathcal{F}_\Omega$ and the support $|\Sigma|$, and the Euclidean metric on the former induces the metric on the latter. The resulting metric on $|\Sigma|$ depends on the choice of simplicial subdivision. A different Ω gives a different metric.

Example 5.6. Let $d = 2$ and $\Sigma = \mathbb{R}_{\geq 0}^2$ be the cone spanned by $v^1 = (1, 0)$ and $v^2 = (0, 1)$. In the usual metric on Σ , the distance between v^1 and v^2 is $\sqrt{2}$, and the geodesic passes through $(\frac{1}{2}, \frac{1}{2})$. We refine Σ by adding the ray spanned by $v^3 = (1, 1)$. The simplicial complex Ω has facets 13 and 23, and $\mathcal{F}_\Omega$ is the fan in $\mathbb{R}^3$ with maximal cones $\text{pos}(e_1, e_3)$ and $\text{pos}(e_2, e_3)$ where $\text{pos}(e_i, e_j) = \{a \cdot e_i + b \cdot e_j \mid a, b \geq 0\}$. The map from this orthant space onto Σ induces a different metric. In that new metric, the distance between v^1 and v^2 is 2. The geodesic is the cone path passing through $(0, 0)$.

Let $n = 2^m - m - 2$. The standard basis vectors in $\mathbb{R}^n$ are denoted e_σ , one for each clade σ on $[m]$. Let Ω be the flag simplicial complex whose simplices are the nested sets $\{\sigma_1, \dots, \sigma_d\}$ as defined in (5.2). The piecewise-linear isomorphism from the orthant space $\mathcal{F}_\Omega$ to $\mathcal{U}_m$ takes the basis vector e_σ to the clade metric D_σ , and this induces the BHV metric on the space $\mathcal{U}_m$ of ultrametrics. Each orthant in $\mathcal{F}_\Omega$ represents the set of all equidistant trees in $\mathcal{U}_m$ that have a fixed topology.

In Section 5.1, each ultrametric $D = (d_{ij})$ is an element of $\mathbb{R}^{\binom{m}{2}}/\mathbb{R}\mathbf{1}$. It has a unique representation

$$D = \ell_1 D_{\sigma_1} + \ell_2 D_{\sigma_2} + \dots + \ell_d D_{\sigma_d}, \quad (5.7)$$

where $\{\sigma_1, \dots, \sigma_d\} \in \Omega$ and $\ell_1, \dots, \ell_d \geq 0$. The coefficient ℓ_i is twice the length of the edge labeled σ_i in the tree given by D . It can be recovered from D by the formula

$$\ell_i = \min\{d_{rt} : r \in \sigma_i, t \notin \sigma_i\} - \max\{d_{rs} : r, s \in \sigma_i\}. \quad (5.8)$$

The maximal nested sets have cardinality $m - 2$, so this is the dimension of the orthant space $\mathcal{F}_\Omega$. The number of maximal nested sets is $(2m - 3)!! = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2m - 3)$. For instance, Fig. 5.1 shows that $\mathcal{U}_4$ consists of 15 two-dimensional cones and 10 rays.

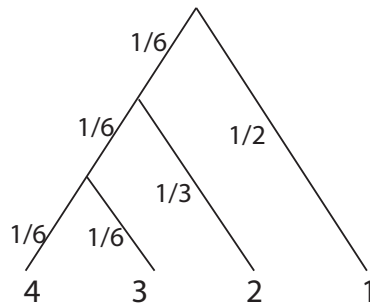


Figure 5.2: The equidistant tree in $\mathcal{U}_4^{[1]}$ discussed in Example 5.7.

Example 5.7. The equidistant tree in Fig. 5.2 corresponds to the ultrametric

$$D = (d_{12}, d_{13}, d_{14}, d_{23}, d_{24}, d_{34}) = \left(1, 1, 1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) \in \mathcal{U}_4^{[1]} \subset \mathcal{U}_4.$$

Consider the two clades $\sigma_1 = \{3, 4\}$ and $\sigma_2 = \{2, 3, 4\}$. If we cut the tree at those two internal edges, then we will get these two clades. Notice that $\{\sigma_1, \sigma_2\}$ is a nested set since $\sigma_1 \subset \sigma_2$. The two corresponding clade metrics are

$$D_{\sigma_1} = D_{34} = (1, 1, 1, 1, 1, 0) \quad \text{and} \quad D_{\sigma_2} = D_{234} = (1, 1, 1, 0, 0, 0).$$

The formula (5.8) gives $\ell_1 = \ell_2 = 1/3$. Since all vectors live in $\mathbb{R}^6/\mathbb{R}\mathbf{1}$, we have

$$\ell_1 D_{\sigma_1} + \ell_2 D_{\sigma_2} = \left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0\right) = D.$$

But, the orthant space that endows $\mathcal{U}_4$ with its metric lives in $\mathbb{R}^{10}$, and not in $\mathbb{R}^6/\mathbb{R}\mathbf{1}$.

We close this section by reiterating this extremely important remark. In Section 5.1 we introduced the set $\mathcal{U}_m$ of ultrametrics as a subset of a low-dimensional ambient space, having dimension $\binom{m}{2} - 1$. In Section 5.2 we elevated $\mathcal{U}_m$ to live in a high-dimensional ambient space, having dimension $2^m - m - 2$. It is only the latter realization, as an orthant space, that is used when we compute Euclidean distances. In other words, when we compute geodesics in the BHV metric on tree space, we use the coordinates $\ell_1, \dots, \ell_d$. These are local coordinates on the right-angled cones. The coordinates $d_{12}, \dots, d_{m-1,m}$ are never to be used for computing BHV geodesics.

5.3 Geodesic triangles

Biologists are interested in spaces of phylogenetic trees as a statistical tool for studying evolution. The geometric structures in [21, 77, 84] are motivated by applications such as [86]. This requires a notion of convexity.

We fix a flag simplicial complex Ω on $[n]$. The orthant space $\mathcal{F}_\Omega \subset \mathbb{R}^n$ with its intrinsic Euclidean metric has unique geodesics $G(v, w)$ as in Theorems 5.4 and 5.5. A subset T of $\mathcal{F}_\Omega$ is called *geodesically convex* if, for any two points $v, w \in T$, the unique geodesic $G(v, w)$ is contained in T .

Given a subset S of $\mathcal{F}_\Omega$, its *geodesic convex hull* $\text{gconv}(S)$ is the smallest geodesically convex set in $\mathcal{F}_\Omega$ that contains S . If $S = \{v^1, v^2, \dots, v^s\}$ is a finite set then we say that $\text{gconv}(S)$ is a *geodesic polytope*. If $s = 2$ then we recover the geodesic segment $\text{gconv}(v^1, v^2) = G(v^1, v^2)$. If $s = 3$ then we obtain a *geodesic triangle* $\text{gconv}(S) = \text{gconv}(v^1, v^2, v^3)$. The main point of this section is to demonstrate that geodesic triangles are rather complicated objects.

We begin with an iterative scheme for computing geodesic polytopes. Let S be any subset of $\mathcal{F}_\Omega$. Then we can form the union of all geodesics with endpoints in S :

$$g(S) = \bigcup_{v, w \in S} G(v, w).$$

For any integer $t \geq 1$, define $g^t(S)$ recursively as $g^t(S) = g(g^{t-1}(S))$, with $g^0(S) = S$.

Lemma 5.8. *Let S be a set of points in $\mathcal{F}_\Omega$. Then its geodesic convex hull equals*

$$\text{gconv}(S) = \bigcup_{t=0}^{\infty} g^t(S).$$

In words, the geodesic convex hull of S is the set of all points in $\mathcal{F}_\Omega$ that can be generated in finitely many steps from S by taking geodesic segments.

Proof. If T is a subset of $\text{gconv}(S)$ then $g(T) \subseteq \text{gconv}(S)$. By induction on t , we see that $g^t(S) \subseteq \text{gconv}(S)$ for all $t \in \mathbb{N}$. Therefore, $\bigcup_{t=0}^{\infty} g^t(S) \subseteq \text{gconv}(S)$. On the other hand, for any two points $v, w \in \bigcup_{t=0}^{\infty} g^t(S)$, there exist positive integers t_1 and t_2 such that $v \in g^{t_1}(S)$ and $w \in g^{t_2}(S)$. The geodesic path $G(v, w)$ is contained in $g^{\max(t_1, t_2)+1}(S)$, so $G(v, w)$ is in $\bigcup_{t=0}^{\infty} g^t(S)$. So, this set is geodesically convex, and we conclude that it equals $\text{gconv}(S)$. $\square$

Lemma 5.8 gives a numerical method for approximating geodesic polytopes by iterating the computation of geodesics. However, it is not clear whether this process converges. The analogue for negatively curved continuous spaces arises in [56, Lemma 2.1], along with a pointer to the following open problem stated in [17, Note 6.1.3.1]:

”An extraordinarily simple question is still open (to the best of our knowledge). What is the convex envelope of three points in a 3- or higher-dimensional Riemannian manifold? We look for the smallest possible set which contains these three points and which is convex. For example, it is unknown if this set is closed. The standing conjecture is that it is not closed, except in very special cases, the question starting typically in $\mathbb{CP}^2$. The only text we know of addressing this question is [23].”

It seems that the same question is also open in our setting:

Problem 5.9. *Are geodesic triangles in orthant spaces $\mathcal{F}_\Omega$ always closed?*

If T is a geodesically convex subset of $\mathcal{F}_\Omega$ then its restriction $T_\sigma = T \cap \mathcal{O}_\sigma$ to any orthant is a convex set in the usual sense. If Problem 5.9 has an affirmative answer then one might further conjecture that each geodesic polytope T is a polyhedral complex with cells T_σ . This holds in the examples we computed, but the matter is quite subtle. The segments of the pairwise geodesics need not be part of the complex $\{T_\sigma\}$, as the following example shows.

Example 5.10. *Consider a 2-dimensional orthant space that is locally an open book [67] with three pages. We pick three points a, b, c on these pages as shown in Fig. 5.3 and 5.5. The pairwise geodesics $G(a, b)$, $G(a, c)$ and $G(b, c)$ determine a set that is not a polyhedral complex unless one triangle is subdivided. That set, shown on the left in Fig. 5.3, is not geodesically convex. We must enlarge it to get the geodesic triangle $\text{gconv}(a, b, c)$, shown on the right in Fig. 5.3. It consists of three classical triangles, one in each page of the book. Note that the geodesic from a to c travels through the interiors of two classical triangles.*

We next present a sufficient condition for a set T to be geodesically convex. We regard each orthant $\mathcal{O}_\sigma \simeq \mathbb{R}_{\geq 0}^d$ as a poset by taking the component-wise partial order “ $\leq$ ”. We say

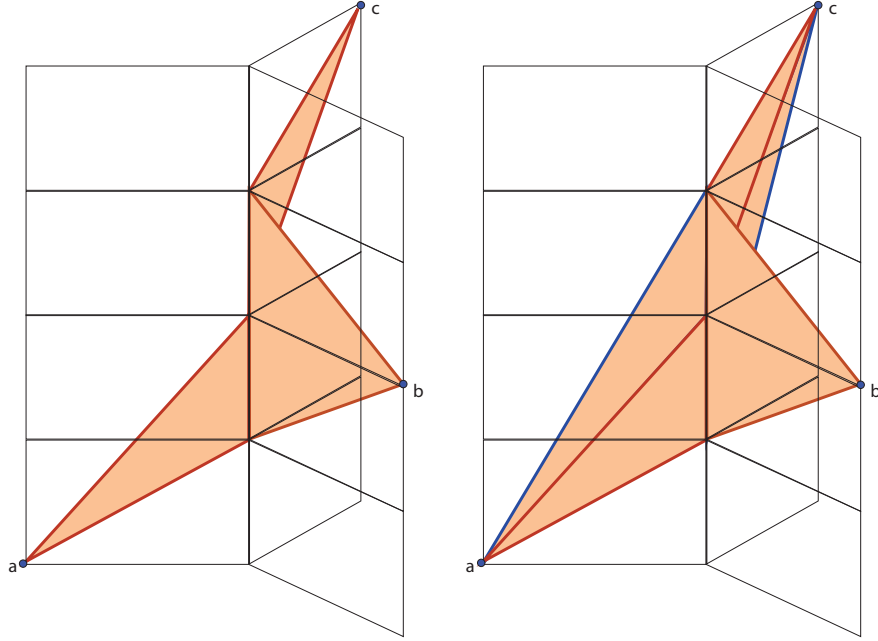


Figure 5.3: Three points in the open book with three pages. Their geodesic convex hull is shown on the right. The left diagram shows the subset formed by the pairwise geodesics.

a subset $T_\sigma \subset \mathcal{O}_\sigma$ is an order ideal in $\mathcal{O}_\sigma$, if for any $u \in T_\sigma$ and for any $v \in \mathcal{O}_\sigma$, if $v \leq u$, then $v \in T_\sigma$.

Theorem 5.11. *Let T be a subset of an orthant space $\mathcal{F}_\Omega$ such that, for each simplex $\sigma \in \Omega$, the restriction T_σ is both convex and an order ideal in $\mathcal{O}_\sigma$. Then T is geodesically convex.*

Proof. Let $v \in T_\sigma$, $w \in T_\tau$ and $G(v, w) = [v, u^1] \cup [u^1, u^2] \cup \dots \cup [u^q, w]$. Suppose $\mathcal{A} = (A_1, \dots, A_q)$ and $\mathcal{B} = (B_1, \dots, B_q)$ form the pair of ordered partitions corresponding to $G(v, w)$. In order to prove $G(v, w) \subset T$, it suffices to show $u^i \in T$ for all i , since the restriction of T to each orthant is convex. We first prove $u^1 \in T$ by constructing a point $u^* \in T_\sigma$ such that $u^1 \leq u^*$. We let

$$u^* = \lambda v_{\sigma \setminus A_1} + (1 - \lambda) w_{\sigma \cap \tau}, \quad \text{where } \lambda = \frac{\|w_{B_1}\|}{\|v_{A_1}\| + \|w_{B_1}\|}.$$

Since the restriction of T to each orthant is an order ideal, we know $v_{\sigma \setminus A_i} \leq v \in T_\sigma$. We also have $w_{\sigma \cap \tau} \leq w \in T_\tau \subset T$ and hence $w_{\sigma \cap \tau} \in T_\sigma$. Thus, $u^* \in T_\sigma$ since T_σ is convex. By formula (5.5), $u_k^* = u_k^1 + (1 - \lambda) \frac{\|w_{B_j}\|}{\|v_{A_j}\|} v_k$ if $k \in A_j$ for some $j = 2, \dots, q$. By formula (5.4), $u_k^* = u_k^1$ if $k \in \sigma \setminus (\bigcup_{j=2}^q A_j)$. Since $\lambda \leq 1$, we conclude $u^1 \leq u^*$. This implies $u^1 \in T_\sigma \subset T$. By a similar argument, $u^i \in T$ since u^i is the first breakpoint of $G(u^{i-1}, w)$ for every $i = 2, \dots, q$. $\square$

Corollary 5.12. *The compact tree space $\mathcal{U}_m^{[1]}$ is geodesically convex.*

Proof. If $T = \mathcal{U}_m^{[1]}$ then T_σ is a subpolytope of the cube $[0, 1]^d$, with coordinates $\ell_1, \dots, \ell_d$ as in the end of Section 5.1. The polytope T_σ is an order ideal because decreasing the edge lengths in a phylogenetic tree can only decrease the distances between pairs of leaves. $\square$

The sufficient condition in Theorem 5.11 is far from necessary. It is generally hard to verify that a set T is geodesically convex, even if $\{T_\sigma\}_{\sigma \in \Omega}$ is a polyhedral complex.

Example 5.13 (A Geodesic Triangle). *Let Ω be the 2-dimensional flag complex with facets 123, 234, 345 and 456. We consider eight points in the 3-dimensional orthant space $\mathcal{F}_\Omega \subset \mathbb{R}^6$:*

$$\begin{aligned} a &= (4, 6, 6, 0, 0, 0), \quad b = (0, 5, 8, 0, 0, 0), \quad c = (0, 0, 0, 1, 2, 3), \quad d = (0, 0, \frac{1}{7}, \frac{5}{7}, 0, 0), \\ e &= (0, 0, 0, \frac{8}{11}, \frac{1}{11}, 0), \quad f = (0, 0, 0, \frac{14}{25}, 0, 0), \quad h = (0, \frac{14}{19}, \frac{14}{19}, 0, 0, 0), \quad o = (0, 0, 0, 0, 0, 0). \end{aligned}$$

The geodesic triangle $T = \text{gconv}(a, b, c)$ is a 3-dimensional polyhedral complex. Its maximal cells are three tetrahedra $T_{123} = \text{conv}\{a, b, h, o\}$, $T_{345} = \text{conv}\{d, e, f, o\}$, $T_{456} = \text{conv}\{c, e, f, o\}$, and one bipyramid $T_{234} = \text{conv}\{b, d, f, h, o\}$. These four 3-polytopes are attached along the triangles $T_{23} = \text{conv}\{b, h, o\}$, $T_{34} = \text{conv}\{d, f, o\}$, and $T_{45} = \text{conv}\{e, f, o\}$.

Verifying this example amounts to a non-trivial computation. We first check that d, e, f, h, o are in the geodesic convex hull of a, b, c . This is done by computing geodesic segments as in Section 5.2. We find $G(a, c) = [a, o] \cup [o, c]$ and $G(b, c) = [b, d] \cup [d, e] \cup [e, c]$. Next, $G(a, e) = [a, x] \cup [x, y] \cup [y, e]$ where $x = (0, \frac{11}{13}, \frac{12}{13}, 0, 0, 0)$ and $y = (0, 0, \frac{6}{67}, \frac{44}{67}, 0, 0)$. If we now take $s = \frac{9}{22}b + \frac{13}{22}x$ in $\mathcal{O}_{23}$ then $G(s, c) = [s, f] \cup [f, c]$, and finally $G(a, f) = [a, h] \cup [h, f]$. Now, we need to show that the union of the four 3-polytopes is geodesically convex. For any two points v and w from distinct polytopes we must show that $G(v, w)$ remains in that union. Here, it does not suffice to take vertices. This is a quantifier elimination problem in piecewise-linear algebra, and we are proud to report that we completed this computation.

The following theorem is the main result in this section.

Theorem 5.14. *Let d be a positive integer. There exists an orthant space of dimension $2d$ and three points in that space such that their geodesic triangle contains a d -dimensional simplex.*

Proof. Fix the simplicial complex Ω on the vertex set $\{1, 2, \dots, 4d\}$ whose $2d + 1$ facets are $\{i+1, i+2, \dots, i+2d\}$ for $i = 0, 1, \dots, 2d$. This simplicial complex is flag because the minimal non-faces are the $\binom{2d+1}{2}$ pairs $\{i, j\}$ for $j - i \geq 2d$. The corresponding orthant space $\mathcal{F}_\Omega$ has dimension $2d$. We here denote the maximal orthants in $\mathcal{F}_\Omega$ by $\mathcal{O}_i = \text{pos}\{e_{i+1}, e_{i+2}, \dots, e_{i+2d}\}$.

For each positive integer i , we define an integer v_i as follows. We set $v_i = \frac{i}{2}$ if i is even and $v_i = \frac{7(i+1)}{2}$ if i is odd. We fix the points $a = \sum_{i=1}^{2d} v_i e_i$ and $b = \sum_{i=1}^{2d} e_i$ in the first orthant $\mathcal{O}_0$ and the point $c = \sum_{i=1}^{2d} e_{2d+i}$ in the last orthant $\mathcal{O}_{2d}$. Explicitly,

$$\begin{aligned} a &= (7, 1, 14, 2, 21, 3, 28, 4, \dots, 0, 0, 0, 0, 0, 0, 0, \dots), \\ b &= (1, 1, 1, 1, 1, 1, 1, 1, \dots, 0, 0, 0, 0, 0, 0, 0, \dots), \\ c &= (0, 0, 0, 0, 0, 0, 0, 0, \dots, 1, 1, 1, 1, 1, 1, 1, \dots). \end{aligned}$$

Consider the geodesic triangle $\text{gconv}(a, b, c)$ in the orthant space $\mathcal{F}_\Omega$. We shall construct a simplex P of dimension d that is contained in the convex set $\text{gconv}(a, b, c) \cap \mathcal{O}_0$.

We begin with the geodesic segment $G(a, c)$. The pair of ordered partitions is

$$\begin{aligned} & ((\{1, 2\}, \{3, 4\}, \dots, \{2d-1, 2d\}), \\ & (\{2d+1, 2d+2\}, \{2d+3, 2d+4\}, \dots, \{4d-1, 4d\})). \end{aligned} \quad (5.9)$$

We write the corresponding decompositions into classical line segments as follows:

$$G(a, c) = [a, u^1] \cup [u^1, u^2] \cup \dots \cup [u^{d-1}, u^d] \cup [u^d, c].$$

Each u^i , for $1 \leq i \leq d$, lies in the relative interior of an orthant of dimension $2d-2$:

$$u^i \in \text{pos}\{e_{2i-1}, \dots, e_{2d+2i-2}\} \cap \text{pos}\{e_{2i+1}, \dots, e_{2d+2i}\} = \text{pos}\{e_{2i+1}, \dots, e_{2d+2i-2}\}.$$

We now consider the geodesic segments $G(b, u^i)$ for $i = 1, 2, \dots, d$. Let $\tilde{u}^i$ denote the unique intersection point of these geodesic segments with the boundary of $\mathcal{O}_0$. Note that $\tilde{u}^1 = u^1$. The points $a, \tilde{u}^1, \dots, \tilde{u}^d$ lie in the orthant $\mathcal{O}_0 \simeq \mathbb{R}_{\geq 0}^{2d}$. By construction, they are also contained in the geodesic triangle $\text{gconv}(a, b, c)$. We shall prove that they are affinely independent.

The above point u^i can be written as $\sum_{k=2i-1}^{2d+2i-2} u_k^i e_k$ or as $\sum_{k=2i+1}^{2d+2i} u_k^i e_k$. Note that $u_k^i = 0$ for $k = 2i-1, 2i, 2d+2i-1, 2d+2i$. For $2i+1 \leq k \leq 2d+2i-2$, we claim:

$$u_{2j-1}^i = \frac{7(j-i)}{5i+1}, \quad u_{2j}^i = \frac{j-i}{5i+1}, \quad \text{for } i+1 \leq j \leq d; \quad (5.10)$$

$$u_{2d+2j-1}^i = u_{2d+2j}^i = \frac{5(i-j)}{5i+1}, \quad \text{for } 1 \leq j \leq i-1. \quad (5.11)$$

To prove the above claim, it suffices to verify that the proposed u^i satisfy

$$\sum_{i=0}^d d(u^i, u^{i+1}) = d(a, c), \quad \text{where } u^0 = a, u^{d+1} = c.$$

In fact, by (5.3) we have

$$d(a, c) = \sqrt{\sum_{i=1}^d (5i\sqrt{2} + \sqrt{2})^2} = \sqrt{2S_d} \quad \text{where } S_d = \sum_{i=1}^d (5i+1)^2.$$

For $0 \leq i \leq d-1$, we have

$$\begin{aligned}
 d(u^i, u^{i+1}) &= \sqrt{\sum_{k=2i+1}^{2d+2i} (u_k^i - u_k^{i+1})^2} \\
 &= \sqrt{\sum_{j=0}^{d-i-1} (7^2 + 1^2) \left(\frac{j+1}{5i+1} - \frac{j}{5i+6}\right)^2 + \sum_{j=0}^{i-1} 2 \left(\frac{5j}{5i+1} - \frac{5j+5}{5i+6}\right)^2} \\
 &= \frac{\sqrt{50}}{(5i+1)(5i+6)} \sqrt{\sum_{j=i+1}^d (5j+1)^2 + \sum_{j=1}^i (5j+1)^2} \\
 &= \frac{5}{(5i+1)(5i+6)} \sqrt{2S_d}.
 \end{aligned}$$

The sum of these d quantities simplifies to $(1 - \frac{1}{5d+1})\sqrt{2S_d}$. We next observe that

$$d(u^d, c) = \sqrt{\sum_{k=2d+1}^{4d} (u_k^d - c_k)^2} = \sqrt{2 \sum_{j=0}^{d-1} \left(\frac{5j}{5d+1} - 1\right)^2} = \frac{\sqrt{2S_d}}{5d+1}.$$

By adding this to the previous sum, we obtain $\sqrt{2S_d} = d(a, c)$. So, the claim is proved.

Next, we compute $\tilde{u}^i$ for $1 \leq i \leq d$. Recall $\tilde{u}^i$ is the unique intersection point of the geodesic segment $G(b, u^i)$ with $\text{pos}\{e_3, \dots, e_d\}$. We note two facts about $G(b, u^i)$:

(F1) The common coordinates of b and u^i are $e_{2i-1}, \dots, e_{2d}$.

(F2) By equation (5.11), the pair of ordered partitions determining $G(b, u^i)$ is

$$\begin{aligned}
 &((\{1\}, \{2\}, \dots, \{2i-2\}), \\
 &(\{2d+1\}, \{2d+2\}, \dots, \{2d+2i-2\})).
 \end{aligned}$$

Suppose $\tilde{u}^i = \sum_{k=1}^{2d} \tilde{u}_k^i e_k$. For $1 \leq k \leq 2i-2$, we compute $\tilde{u}_k^i$ by (5.5) and (5.10). For $2i-1 \leq k \leq 2d$, we compute $\tilde{u}_k^i$ by (5.6) and (5.11). Then we obtain $\tilde{u}_k^i$ as follows:

$$\tilde{u}_{2j}^i = \begin{cases} \frac{j+4i-5}{10i-4}, & \text{if } i+1 \leq j \leq d; \\ \frac{5j-5}{10i-4}, & \text{if } 1 \leq j \leq i \end{cases} \quad (5.12)$$

and

$$\tilde{u}_{2j-1}^i = \begin{cases} \frac{7j-2i-5}{10i-4}, & \text{if } i+1 \leq j \leq d; \\ \frac{5j-5}{10i-4}, & \text{if } 1 \leq j \leq i. \end{cases} \quad (5.13)$$

The $d+1$ points $\tilde{u}^0 = a, \tilde{u}^1, \dots, \tilde{u}^d$ are contained in $\mathcal{O}_0 \simeq \mathbb{R}^{2d}$ also in our geodesic triangle. To complete the proof of Theorem 5.14, we will now show that they are affinely

independent, so their convex hull is a d -simplex. Consider the $(d+1) \times (2d+1)$ matrix U , whose $(i+1)$ -th row is the vector of homogeneous coordinates of $\tilde{u}^i$. We must show that U has rank $d+1$. Let U' be the integer matrix obtained from U by multiplying each row by the denominator $10i-4$. Let U'' be the $(d+1) \times (d+1)$ submatrix of U' formed by the $2, 4, \dots, 2d$ -th and $(2d+1)$ -th columns of U' . We apply elementary column operators to U'' to obtain a triangular form. From this, we find that $|\det(U'')| = 4^{d-3}|10d-34| \neq 0$. This means that U'' has rank $d+1$, and hence so do the rectangular matrices U' and U . $\square$

Here is an example for the matrix U we have constructed in the above proof. Take $d=5$. The matrix representing $a, \tilde{u}^1, \tilde{u}^2, \tilde{u}^3, \tilde{u}^4, \tilde{u}^5$ in $\mathbb{R}^{10}$ is

$$U = \begin{bmatrix} 7 & 1 & 14 & 2 & 21 & 3 & 28 & 4 & 35 & 5 & 1 \\ 0 & 0 & \frac{7}{6} & \frac{1}{6} & \frac{14}{6} & \frac{2}{6} & \frac{21}{6} & \frac{3}{6} & \frac{28}{6} & \frac{4}{6} & 1 \\ 0 & 0 & \frac{5}{16} & \frac{5}{16} & \frac{12}{16} & \frac{6}{16} & \frac{19}{16} & \frac{7}{16} & \frac{26}{16} & \frac{8}{16} & 1 \\ 0 & 0 & \frac{5}{26} & \frac{5}{26} & \frac{10}{26} & \frac{10}{26} & \frac{17}{26} & \frac{11}{26} & \frac{24}{26} & \frac{12}{26} & 1 \\ 0 & 0 & \frac{5}{36} & \frac{5}{36} & \frac{10}{36} & \frac{10}{36} & \frac{15}{36} & \frac{15}{36} & \frac{22}{36} & \frac{16}{36} & 1 \\ 0 & 0 & \frac{5}{46} & \frac{5}{46} & \frac{10}{46} & \frac{10}{46} & \frac{15}{46} & \frac{15}{46} & \frac{20}{46} & \frac{20}{46} & 1 \end{bmatrix}_{6 \times 11}.$$

This matrix has rank 6, so its columns form a 5-simplex. Hence, the geodesic triangle spanned by the points a, b, c in the orthant space $\mathcal{F}_\Omega$ has dimension at least 5.

A nice feature of the construction above is that it extends to BHV tree space. The geodesic convex hull of three equidistant trees can have arbitrarily high dimension.

Corollary 5.15. *There exist three ultrametric trees with $2d+2$ leaves whose geodesic triangle in ultrametric BHV space $\mathcal{U}_{2d+2}$ has dimension at least d .*

Proof. Consider the following sequence of $4d$ clades on the set $[2d+2]$. Start with the $2d$ clades $\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \dots, \{1, 2, 3, \dots, 2d+1\}$. Then continue with the $2d$ clades $\{1, 3\}, \{1, 3, 4\}, \dots, \{1, 3, 4, 5, \dots, 2d+2\}$. For instance, for $d=5$, our sequence equals

$$\begin{aligned} &\{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}, \dots, \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}, \\ &\{1, 3\}, \{1, 3, 4\}, \{1, 3, 4, 5\}, \{1, 3, 4, 5, 6\}, \dots, \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}. \end{aligned}$$

In this sequence of $4d$ clades, every collection of $2d$ consecutive splits is compatible and forms a trivalent caterpillar tree. No other pair is compatible. Hence the induced subfan of the tree space $\mathcal{U}_{2d+2}$ is identical to the orthant space $\mathcal{F}_\Omega$ in the proof above. The two spaces are isometric. Hence our high-dimensional geodesic triangle exists also in tree space $\mathcal{U}_{2d+2}$. $\square$

5.4 Tropical convexity

In this section we shift gears, by turning to tropical convexity. For a textbook introduction to tropical geometry see [78]. Basics were summarized in Chapter 1. We here use the max-plus algebra, so our convention is opposite to that of Chapter 2. The connection between

phylogenetic trees and tropical lines identifies tree space with a tropical Grassmannian. This has been explained in many sources, including [78, §4.3] and [90, §3.5]. However, the restriction to ultrametrics [8, §4] offers a fresh perspective.

We first review some basic definitions from tropical geometry. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. The tropical arithmetic operations $\oplus$ and $\odot$ on the set $\overline{\mathbb{R}}$ as follows:

$$a \oplus b = \max(a, b), \quad a \odot b = a + b \quad \text{for } a, b \in \mathbb{R}. \quad (5.14)$$

In addition, we have $a \oplus -\infty = a$ and $a \odot -\infty = -\infty$. With these definitions, $(\overline{\mathbb{R}}, \oplus, \odot)$ has the structure of a semiring. It is called the *tropical semiring*.

Let L_m denote the cycle space of the complete graph on m nodes with $e := \binom{m}{2}$ edges. This is the $(m-1)$ -dimensional linear subspace of $\mathbb{R}^e$ defined by the equations $x_{ij} - x_{ik} + x_{jk} = 0$ for $1 \leq i < j < k \leq m$. The tropicalization of the linear space L_m is the set of points $D = (d_{ij})$ such that the following maximum is attained at least twice for all triples i, j, k :

$$d_{ij} \oplus d_{ik} \oplus d_{jk} = \max(d_{ij}, d_{ik}, d_{jk}).$$

Disregarding nonnegativity constraints and triangle inequalities, points in $\text{trop}(L_m)$ are precisely the ultrametrics on $[m]$; see [8, Theorem 3] and [78, Example 4.2.14].

As is customary in tropical geometry, we work in the quotient space $\mathbb{R}^e/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, 1, \dots, 1)$. The images of $\text{trop}(L_m)$ and $\mathcal{U}_m$ in that space are equal. Each point in that image has a unique representative whose coordinates have tropical sum $d_{12} \oplus d_{13} \oplus \dots \oplus d_{m-1,m} = \max(d_{12}, d_{13}, \dots, d_{m-1,m})$ equal to 1. Thus, in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$,

$$\mathcal{U}_m^{[1]} \subset \mathcal{U}_m = \text{trop}(L_m).$$

Given two elements D and D' in $\mathbb{R}^e$, their *tropical sum* $D \oplus D'$ is the coordinate-wise maximum. A subset of $\mathbb{R}^e$ is *tropically convex* if it is closed under the tropical sum operation. The same definitions apply to subsets of $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. The *tropical convex hull* of a subset $V \subseteq \mathbb{R}^e/\mathbb{R}\mathbf{1}$ is the smallest tropically convex set containing V .

Proposition 5.16. *The tree space $\mathcal{U}_m$ is a tropical linear space and is hence tropically convex. The compact tree space $\mathcal{U}_m^{[1]}$ is a tropically convex subset.*

Proof. We saw that $\mathcal{U}_m = \text{trop}(L_m)$ is a tropical linear space, so it is tropically convex by [78, Proposition 5.2.8]. We show that its subset $\mathcal{U}_m^{[1]}$ is closed under tropical sums. Suppose two real vectors $a = (a_1, \dots, a_e)$ and $b = (b_1, \dots, b_e)$ satisfy $|a_i - a_j| \leq 1$ and $|b_i - b_j| \leq 1$ for all i, j . Then the same holds for their tropical sum $a \oplus b$. Indeed, let $a_i \oplus b_i$ be the largest coordinate of $a \oplus b$ and let $a_j \oplus b_j$ be the smallest. There are four cases as to which attains the two maxima given by $\oplus$. In all four cases, one easily checks that $|(a_i \oplus b_i) - (a_j \oplus b_j)| \leq 1$. $\square$

We now recall some definitions and facts from tropical convexity [78, §5.2]. A *tropical segment* is the tropical convex hull $\text{tconv}(u, v)$ of two points $u = (u_1, u_2, \dots, u_e)$ and $v = (v_1, v_2, \dots, v_e)$ in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. It is the concatenation of at most $e - 1$ ordinary line segments,

with slopes in $\{0, 1\}^e$. Computing that tropical segment involves sorting the coordinates of $u - v$, and it can be done in time $O(e \cdot \log(e))$. This algorithm is described in the proof of [78, Proposition 5.2.5].

A *tropical polytope* $\mathcal{P} = \text{tconv}(\mathcal{S})$ is the tropical convex hull of a finite set $\mathcal{S}$ in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. This is a classical polyhedral complex of dimension at most $|\mathcal{S}| - 1$.

If $\mathcal{D}$ is a real $s \times e$ -matrix then the tropical convex hull of its s rows is a tropical polytope in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. The tropical convex hull of its e columns is a tropical polytope in $\mathbb{R}^s/\mathbb{R}\mathbf{1}$. It is a remarkable fact [78, Theorem 5.2.21] that these two tropical polytopes are identical. We write $\text{tconv}(\mathcal{D})$ for that common object. Example 5.17 illustrates this for a 3×10 -matrix $\mathcal{D}$. Here, $\text{tconv}(\mathcal{D})$ has dimension 2 and is shown in Fig. 5.4.

Example 5.17. We compute the tropical convex hull $\text{tconv}(\mathcal{D})$ of the 3×10 -matrix

$$\mathcal{D} = \begin{pmatrix} 77/100 & 1 & 21/25 & 1 & 1 & 21/25 & 1 & 1 & 13/25 & 1 \\ 1 & 1 & 1 & 1 & 8/25 & 3/4 & 3/4 & 3/4 & 3/4 & 23/50 \\ 1 & 1 & 1 & 49/50 & 16/25 & 16/25 & 1 & 3/100 & 1 & 1 \end{pmatrix}.$$

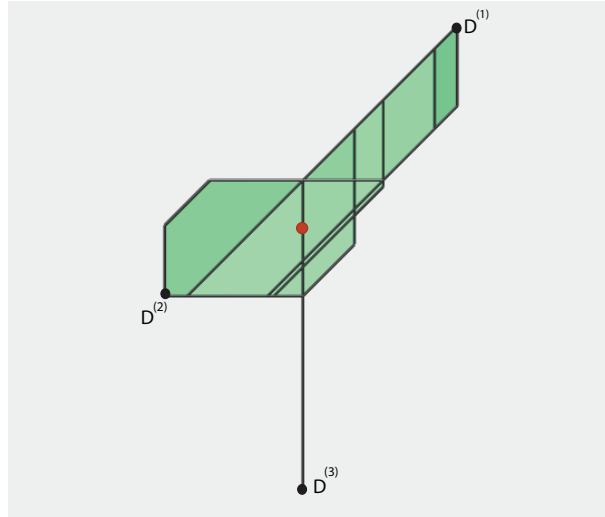


Figure 5.4: The tropical triangle formed by three equidistant trees on five taxa.

This is the tropical convex hull of the 10 points in the plane $\mathbb{R}^3/\mathbb{R}\mathbf{1}$ represented by the column vectors. Its type decomposition is a 2-dimensional polyhedral complex with 23 nodes, 35 edges and 13 two-dimensional cells. It is shown in Fig. 5.4. We can also regard $\text{tconv}(\mathcal{D})$ as a tropical triangle in $\mathbb{R}^{10}/\mathbb{R}\mathbf{1}$, namely as the tropical convex hull of the row vectors. The three rows of $\mathcal{D}$ are ultrametrics $(d_{12}, d_{13}, d_{14}, d_{15}, d_{23}, d_{24}, d_{25}, d_{34}, d_{35}, d_{45})$, i.e. points in $\mathcal{U}_5$. We denote the three row vectors of the matrix $\mathcal{D}$ by $D^{(1)}$, $D^{(2)}$ and $D^{(3)}$, respectively.

Each of the 23 nodes in our tropical triangle represents an equidistant tree. In Table 5.1 we list the 23 ultrametrics, along with their tree topologies. Those marked with a bullet $\bullet$ are

d_{12}	d_{13}	d_{14}	d_{15}	d_{23}	d_{24}	d_{25}	d_{34}	d_{35}	d_{45}	nested set
77/100	1	21/25	1	1	21/25	1	1	13/25	1	$\{12, 35, 124\} \bullet$
21/25	1	21/25	1	1	21/25	1	1	59/100	1	$\{35, 124\}$
1	1	1	1	1	21/25	1	1	3/4	1	$\{24, 35\}$
1	1	1	1	91/100	3/4	91/100	91/100	3/4	91/100	$\{24, 35, 2345\}$
1	1	1	1	3/4	3/4	3/4	3/4	3/4	3/4	$\{2345\}$
1	1	1	1	23/50	3/4	3/4	3/4	3/4	23/50	$\{23, 45, 2345\}$
1	1	1	1	8/25	3/4	3/4	3/4	3/4	23/50	$\{23, 45, 2345\} \bullet$
1	1	1	1	8/25	3/4	3/4	3/4	3/4	17/25	$\{23, 45, 2345\}$
1	1	1	1	39/100	3/4	3/4	3/4	3/4	3/4	$\{23, 2345\}$
1	1	1	1	16/25	3/4	1	3/4	1	1	$\{23, 234\}$
1	1	1	49/50	16/25	73/100	1	73/100	1	1	$\{15, 23, 234\}$
1	1	1	49/50	16/25	16/25	1	3/100	1	1	$\{15, 34, 234\} \bullet$
1	1	1	49/50	16/25	16/25	1	16/25	1	1	$\{15, 234\}$
1	1	1	49/50	4/5	16/25	1	4/5	1	1	$\{15, 24, 234\}$
1	1	1	49/50	89/100	73/100	1	89/100	1	1	$\{15, 24, 234\}$
1	1	1	49/50	49/50	41/50	1	49/50	1	1	$\{15, 24, 234\}$
1	1	1	1	1	21/25	1	1	1	1	$\{24\}$
21/25	1	21/25	1	1	21/25	1	1	21/25	1	$\{35, 124\}$
77/100	1	21/25	1	1	21/25	1	1	77/100	1	$\{12, 35, 124\}$
1	1	1	1	91/100	3/4	91/100	91/100	91/100	91/100	$\{24, 2345\}$
1	1	1	1	91/100	3/4	1	91/100	1	1	$\{24, 234\}$
1	1	1	1	3/4	3/4	1	3/4	1	1	$\{234\}$
1	1	1	49/50	73/100	73/100	1	73/100	1	1	$\{15, 234\}$

Table 5.1: The 23 ultrametries (with tree topologies) at the nodes in Fig. 5.4.

the rows of $\mathcal{D}$. The boundary of $\text{tconv}(\mathcal{D})$ is given by the first 19 rows, in counterclockwise order. Rows 1 to 6 form the tropical segment from $D^{(1)}$ to $D^{(2)}$, rows 7 to 12 form the tropical segment from $D^{(2)}$ to $D^{(3)}$, and rows 13 to 19 form the tropical segment from $D^{(3)}$ to $D^{(1)}$. The last four rows are the interior nodes, from top to bottom in Fig. 5.4. The tropical segment from $D^{(2)}$ to $D^{(3)}$ has depth 1, but the other two segments have depth 2. See Section 5.5 for the definition of “depth”. Note that some of the breakpoints, such as that given in row 6, lie in the interior of a maximal cone in the tree space $\mathcal{U}_5 = \text{trop}(L_5)$. The red circle in Fig. 5.4 is the tropical centroid, a concept to be introduced in Section 5.6.

The polyhedral geometry package **Polymake** [61] can compute the tropical convex hull of a finite set of points. It can also visualize such a tropical polytope, with its cell complex structure, provided its dimension is 2 or 3. Fig. 5.4 was drawn using **Polymake**.

Matroid theory furnishes the appropriate level of generality for tropical convexity. We refer to the text book reference [78, §4.2] for an introduction. Let M be any matroid of rank r on the ground set $[e] = \{1, 2, \dots, e\}$. The associated *tropical linear space* $\text{trop}(M)$ is a polyhedral fan of dimension $r - 1$ in the quotient space $\mathbb{R}^e / \mathbb{R}\mathbf{1}$. For historical reasons, the tropical linear space $\text{trop}(M)$ is also known as the *Bergman fan* of the matroid M ; see [6, 8]. The tree space $\mathcal{U}_m$ arises when M is the graphic matroid [78, Example 4.2.14] associated with the complete graph K_m . Here $[e]$ indexes the edges of K_m , so $e = \binom{m}{2}$ and the rank is $r = m - 1$. Fig. 5.1 (left) shows a rendition of the Petersen graph in [78, Figure 4.1.4].

The tropical linear space $\text{trop}(M)$ is tropically convex [78, Proposition 5.2.8]. Hence the

notions of tropical segments, triangles, etc. defined above extend immediately to $\text{trop}(M)$. This convexity structure on $\text{trop}(M)$ assigns to every subset a well-defined convex hull. This structure is extrinsic and global. It is induced from the ambient space $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. It does not rely on choosing a subdivision or local coordinates.

We can also regard $\text{trop}(M)$ as an orthant space. This requires the choice of a simplicial fan structure on $\text{trop}(M)$. Feichtner [53] developed a theory of such fan structures. Each is determined by a collection of n flats of M , known as a *building set*. From these one constructs a simplicial complex Ω on $[n]$, known as a *nested set complex*. This Ω may or may not be flag. The finest fan structure in this theory arises when all flats of M are in the collection. Here the nested set complex Ω is flag: it is the order complex of the geometric lattice of M . See Exercise 10 in [78, Chapter 4].

Example 5.18. *Let M be the uniform matroid of rank r on $[e]$. The tropical linear space $\text{trop}(M)$ is the set of all vectors u in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$ whose largest coordinate is attained at least $e - r + 1$ times. The proper flats of M are the non-empty subsets of $[e]$ having cardinality at most $r - 1$. Their number is $n = \sum_{i=1}^{r-1} \binom{e}{i}$. Ordered by inclusion, these form the geometric lattice of M . Its order complex is the first barycentric subdivision of the $(r - 1)$ -skeleton of the $(e - 1)$ -simplex. This is a flag simplicial complex Ω with n vertices. The corresponding orthant space $\mathcal{F}_\Omega$ in $\mathbb{R}^n$ defines the structure of an orthant space on $\text{trop}(M) \subset \mathbb{R}^e/\mathbb{R}\mathbf{1}$.*

While the order complex of the geometric lattice of any matroid M makes $\text{trop}(M)$ into an orthant space, many matroids have smaller building sets whose nested set complex is flag as well. Our primary example is ultrametric space $\mathcal{U}_m = \text{trop}(L_m)$. The flats of L_m correspond to proper set partitions of $[m]$. Their number is $n = B_m - 2$, where B_m is the *Bell number*. The resulting orthant space on $\mathcal{U}_m$ is the τ -space of Gavryushkin and Drummond [60]. The subdivision of $\mathcal{U}_m$ given by the nested sets of clades is much coarser. It has only $n = 2^m - m - 2$ rays, and its orthant space gives the BHV metric. This is different from the metric of the τ -space, by [60, Proposition 2]. Note that [60, Figure 4] is the same as our Example 5.6.

Each orthant space structure defines a Euclidean metric on $\text{trop}(M)$. These metrics differ dramatically from the tropical metric, to be defined next, in (5.15). Euclidean metrics on $\text{trop}(M)$ are *intrinsic* and do not extend to the ambient space $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. Distances are computed by identifying $\text{trop}(M)$ with the orthant space $\mathcal{F}_\Omega$ of a nested set complex Ω that is flag. On the other hand, the tropical metric is *extrinsic*. It lives on $\mathbb{R}^e/\mathbb{R}\mathbf{1}$ and is defined on $\text{trop}(M)$ by restriction. The tropical distance between two points is computed as follows:

$$d_{\text{tr}}(v, w) = \max\{|v_i - w_i - v_j + w_j| : 1 \leq i < j \leq e\}. \quad (5.15)$$

This is also known as the *generalized Hilbert projective metric* [2, §2.2], [41, §3.3]. Unlike in the Euclidean setting of Theorem 5.4, geodesics in that metric are not unique.

Proposition 5.19. *For any two distinct points $v, w \in \mathbb{R}^e/\mathbb{R}\mathbf{1}$, there are many geodesics between v and w in the tropical metric. One of them is the tropical line segment $\text{tconv}(v, w)$.*

Proof. After translating and permuting coordinates, we may assume that $v = (0, 0, 0, \dots, 0)$ and $w = (0, w_2, w_3, \dots, w_e)$ where $0 \leq w_2 \leq w_3 \leq \dots \leq w_e$. Consider any point $u = (0, u_2, u_3, \dots, u_e)$ in the classical convex polytope defined by $0 \leq u_2 \leq u_3 \leq \dots \leq u_e$ and $0 \leq w_2 - u_2 \leq w_3 - u_3 \leq \dots \leq w_e - u_e$. Note that if $d(v, w) > 0$, then such point u is not unique. Any such point u satisfies

$$d_{\text{tr}}(v, w) = w_e = u_e + (w_e - u_e) = d_{\text{tr}}(v, u) + d_{\text{tr}}(u, w). \quad (5.16)$$

The tropical segment from v to w is a geodesic, both for the max-plus arithmetic and for the min-plus arithmetic, because all points u on these segments satisfy the property (5.16). Now, let u be any point in the aforementioned polytope. We can then connect v and u by either the min-plus segment or the max-plus segment, and also connect u and w by either the min-plus segment or the max-plus segment. By iterating this construction, we obtain arbitrarily many geodesics from v to w . $\square$

One important link between the tropical metric and tropical convexity is the nearest-point map, to be described next. Let $\mathcal{P}$ be any tropically convex closed subset of $\mathbb{R}^e/\mathbb{R}\mathbf{1}$, and let u be any vector in $\mathbb{R}^e$. Let $\mathcal{P}_{\leq u}$ denote the subset of all points in $\mathcal{P}$ that have a representative $v \in \mathbb{R}^e$ with $v \leq u$ in the coordinate-wise order. In tropical arithmetic this is expressed as $v \oplus u = u$. If v and v' are elements of $\mathcal{P}_{\leq u}$ then so is their tropical sum $v \oplus v'$. It follows that $\mathcal{P}_{\leq u}$ contains a unique coordinate-wise maximal element, denoted $\max(\mathcal{P}_{\leq u})$.

Theorem 5.20. *Given any tropically convex closed subset $\mathcal{P}$ of $\mathbb{R}^e/\mathbb{R}\mathbf{1}$, consider the function*

$$\pi_{\mathcal{P}} : \mathbb{R}^e/\mathbb{R}\mathbf{1} \rightarrow \mathcal{P}, \quad u \mapsto \max(\mathcal{P}_{\leq u}). \quad (5.17)$$

Then $\pi_{\mathcal{P}}(u)$ is a point in $\mathcal{P}$ that minimizes the tropical distance to u .

This result was proved by Cohen, Gaubert and Quadrat in [41, Theorem 18]. See also [2]. In Section 5.5 we shall discuss the important case when $\mathcal{P} = \text{Trop}(M)$ is a tropical linear space. The subcase when $\mathcal{P}$ is a tropical hyperplane appears in [2, §7].

We close this section by considering a tropical polytope, $\text{tconv}(D^{(1)}, D^{(2)}, \dots, D^{(s)})$. The $D^{(i)}$ are points in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. For instance, they might be ultrametrics in $\mathcal{U}_m$. Then

$$\pi_{\mathcal{P}}(D) = \lambda_1 \odot D^{(1)} \oplus \lambda_2 \odot D^{(2)} \oplus \dots \oplus \lambda_s \odot D^{(s)}, \quad \text{where } \lambda_k = \min(D - D^{(k)}).$$

This formula appears for example in [78, (5.2.3)]. It allows us to project an ultrametric D (or any other point in $\mathbb{R}^e$) onto the tropical convex hull of s given ultrametrics.

5.5 Experiments with depth

It is natural to compare tropical convexity with geodesic convexity. One starts by comparing the line segments $\text{gconv}(v, w)$ and $\text{tconv}(v, w)$, where v, w are points in a tropical linear space $\text{trop}(M)$. Our first observation is that tropical segments generally do not obey the

combinatorial structure imposed by ordered partitions $(\mathcal{A}, \mathcal{B})$. For instance, if v and w represent equidistant trees then every clade used by a tree in the geodesic segment $\text{gconv}(v, w)$ must be a clade of v or w . This need not hold for trees in the tropical segment $\text{tconv}(v, w)$.

Example 5.21. *Let $v = D^{(1)}$ and $w = D^{(2)}$ in Example 5.17. Their tree topologies are given by the sets of clades $\{12, 35, 124\}$ and $\{23, 45, 2345\}$. Consider the breakpoints of the tropical segment $\text{tconv}(v, w)$ given in lines 3 and 4 of Table 5.1. Both trees have the new clade 24.*

This example suggests that tropical segments might be worse than geodesics in BHV space. However, as we shall now argue, the opposite is the case: for us, tropical segments are better. We propose the following quality measure for a path P in an orthant space $\mathcal{F}_\Omega$. Suppose that $\mathcal{F}_\Omega$ has dimension d . Each point of P lies in the relative interior of a unique orthant $\mathcal{O}_\sigma$, and we say that this point has *codimension* $d - \dim(\mathcal{O}_\sigma)$. We define the *depth* of a path P as the maximal codimension of any point in P . For instance, the depth of the Euclidean geodesic in Theorem 5.4 is the maximum of the numbers $\sum_{k=1}^i |A_k| - \sum_{k=1}^{i-1} |B_k|$, where i runs over $\{1, 2, \dots, q\}$. These are the codimensions of the breakpoints of $\text{gconv}(v, w)$.

Geodesics of small depth are desirable. A cone path has depth d . Optimal geodesics have depth 0. Such geodesics are line segments within a single orthant. These occur if and only if the starting point and target point are in the same orthant. If the two given points are not in the same orthant then the best-case scenario is depth 1, which means that each transition is through an orthant of codimension 1 in $\mathcal{F}_\Omega$. We conducted two experiments, to assess the depths of $\text{gconv}(v, w)$ and $\text{tconv}(v, w)$.

Experiment 5.22 (Euclidean Geodesics). *For each $m \in \{4, 5, \dots, 20\}$, we sampled 1000 pairs $\{v, w\}$ from the compact tree space $\mathcal{U}_m^{[1]}$, introduced at the end of Section 5.1, and for each pair we computed the depth of $\text{gconv}(v, w)$. The sampling scheme is described below. The depths are integers between 0 and $m - 2$.*

Algorithm 5.23 (Sampling normalized equidistant trees with m leaves). *Input: The number m of leaves, and the sample size s .*

Output: A sample of s random equidistant trees in the compact tree space $\mathcal{U}_m^{[1]}$.

1. Set $\mathcal{S} = \emptyset$.
2. For $i = 1, \dots, s$, do
 - a) Generate a tree D_i using the function `rcoal` from the `ape` package [91] in R.
 - b) Randomly permute the leaf labels on the metric tree D_i .
 - c) Change the clade nested structure of D_i by randomly applying the nearest neighbor interchange (NNI) operation m times.
 - d) Turn D_i into an equidistant tree using the `ape` function `compute.brtree`.
 - e) Normalize U_i so that the distance from the root to each leaf is $\frac{1}{2}$.

f) Add D_i to the output set $\mathcal{S}$.

3. Return $\mathcal{S}$.

Table 5.2 shows the distribution of the depths. For instance, the first row concerns 1000 random geodesics on the 2-dimensional polyhedral fan $\mathcal{U}_4$ depicted in Figure 5.1. Of these geodesics, 8.4% were in a single orthant, 58.4% had depth 1, and 33.2% were cone paths. For $m = 20$, the fraction of cone paths was 6.2%. The data in Table 5.2 are based on the sampling scheme in Algorithm 5.23.

$m \backslash \text{depth}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	8.4	58.4	33.2																
5	1.6	26.4	47.4	24.6															
6	0.2	13.2	36.7	31.5	18.4														
7	0	4	25.9	29.9	22.2	18													
8	0	1.1	15	28.9	25	17.1	12.9												
9	0	0.8	8	22.1	25.9	18.3	14.5	10.4											
10	0	0.4	3.3	17.2	22.3	20.6	14.1	13.2	8.9										
11	0	0.2	1.5	10.4	17.6	20.3	16.8	12.8	11.1	9.3									
12	0	0.2	0.1	6	14.1	20.4	13.9	14.6	12.7	10.5	7.5								
13	0	0.2	0.4	4.2	10.1	17.2	15.9	12.5	11	9.8	9.1	9.6							
14	0	0.2	0	2.7	9.3	14.9	15.5	12.2	11.3	10.4	8.7	8	6.8						
15	0	0.1	0	1.4	5.9	12.7	13	13.1	11.3	9.2	8.9	8.5	7.5	8.4					
16	0	0	0	1	5	11.2	11.4	11.3	11.2	9.9	8.1	9.1	7.3	6.7	7.8				
17	0	0	0	0.2	3.4	5.9	10.7	11	11.2	11.5	8.4	7.9	7.9	6.2	8.5	7.2			
18	0	0	0.1	0.4	1.5	6.5	8.7	10.5	10.9	9.7	7.9	7.5	7.1	8.7	7.7	6.5	6.3		
19	0	0	0	0.2	1.6	5	7.2	9.3	9.6	8.5	7.5	8.3	7.4	6.1	9.2	7.4	6.8	5.9	
20	0	0	0	0	0.5	3	6.7	7.6	11.2	9.8	9.4	8.2	5.9	7.5	6.9	6.9	4.5	5.7	6.2

Table 5.2: The rows are labeled by the number m of taxa and the columns are labeled by the possible depths of a geodesic in tree space. The entries in each row sum to 100%. They are the frequencies of the depths among 1000 geodesics, randomly sampled using Algorithm 5.23.

Next we perform our experiment with the tropical line segments $\text{tconv}(v, w)$.

Experiment 5.24 (Tropical Segments). *For each $m \in \{4, 5, \dots, 20\}$, we revisited the same 1000 pairs $\{v, w\}$ from Experiment 5.22, and we computed their tropical segments $\text{tconv}(v, w)$. Table 5.3 shows the distribution of their depths. There is a dramatic difference between Tables 5.2 and 5.3. The depths of the Euclidean geodesics are much larger than those of the tropical segments. Since small depth is desirable, this suggests that the tropical convexity structure may have good statistical properties.*

Triangles show even more striking differences: While tropical triangles $\text{tconv}(a, b, c)$ are 2-dimensional, geodesic triangles $\text{gconv}(a, b, c)$ can have arbitrarily high dimension, by Theorem 5.14. In spite of these dimensional differences, $\text{tconv}(a, b, c)$ is usually not contained in $\text{gconv}(a, b, c)$. In particular, this is the case in the following example.

$m \backslash \text{depth}$	0	1	2	3	4	5	6	7	8	9	10
4	8.1	88.7	3.2								
5	1.5	84.7	13.8	0							
6	0.3	69.9	29.8	0	0						
7	0	55.7	44.1	0.2	0	0					
8	0	42.8	56.9	0.2	0.1	0	0				
9	0	28.0	71.4	0.6	0	0	0	0			
10	0	20.7	78.2	1.1	0	0	0	0	0		
11	0	10.8	88.0	1.1	0.1	0	0	0	0	0	
12	0	7.8	89.5	2.4	0.2	0.1	0	0	0	0	0
13	0	5.3	90.8	3.5	0.3	0.1	0	0	0	0	0
14	0	2.5	92.1	4.9	0.5	0	0	0	0	0	0
15	0	1.9	90.4	6.8	0.9	0	0	0	0	0	0
16	0	0.4	88.7	9.4	1.1	0.4	0	0	0	0	0
17	0	0.8	87.5	9.4	2.3	0	0	0	0	0	0
18	0	0.8	86.1	9.7	3.1	0.2	0	0.1	0	0	0
19	0	0.3	84.1	11.3	3.4	0.8	0.1	0	0	0	0
20	0	0.1	78.4	16.5	3.9	1.0	0	0.1	0	0	0

Table 5.3: Frequencies of the depths among 1000 tropical segments. The same input data as in Table 5.2 was used.

Example 5.25. Let M be the matroid on $e = 4$ elements having rank $r = 3$ whose bases are 124, 134 and 234. The corresponding tropical plane $\text{trop}(M)$ is defined in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ by the tropical linear form $x_1 \oplus x_2 \oplus x_3$, where the coordinates of a point are (x_1, x_2, x_3, x_4) . Geometrically, this tropical plane is the open book with three pages in Example 5.10. The following points lie in $\text{trop}(M)$:

$$a = (0, 1, 1, 2), \quad b = (1, 0, 1, 4), \quad c = (1, 1, 0, 6).$$

Fig. 5.5 shows the two triangles $\text{gconv}(a, b, c)$ and $\text{tconv}(a, b, c)$ inside that open book. See Example 5.10 and Fig. 5.3 for the derivation of the geodesic triangle.

5.6 Towards tropical statistics

Geometric statistics is concerned with the analysis of data sampled from highly non-Euclidean spaces [67, 83]. The section title above is meant to suggest the possibility that objects and methods from tropical geometry can play a role in this development. As an illustration consider the widely used technique of Principal Component Analysis (PCA). This serves to reduce the dimension of high-dimensional data sets, by projecting these onto lower-dimensional subspaces in the data space. The geometry of dimension reduction is essential in phylogenomics, where it can provide insight into relationships and evolutionary patterns of a diversity of organisms, from humans, plants and animals, to microbes and viruses.

To see how tropical convexity might come in, consider the work of Nye [86] (superseded by [54, 87]) in statistical phylogenetics. Nye developed PCA for the original tree space of Billera-Holmes-Vogtman. We here use the BHV metric on the space $\mathcal{U}_m$.

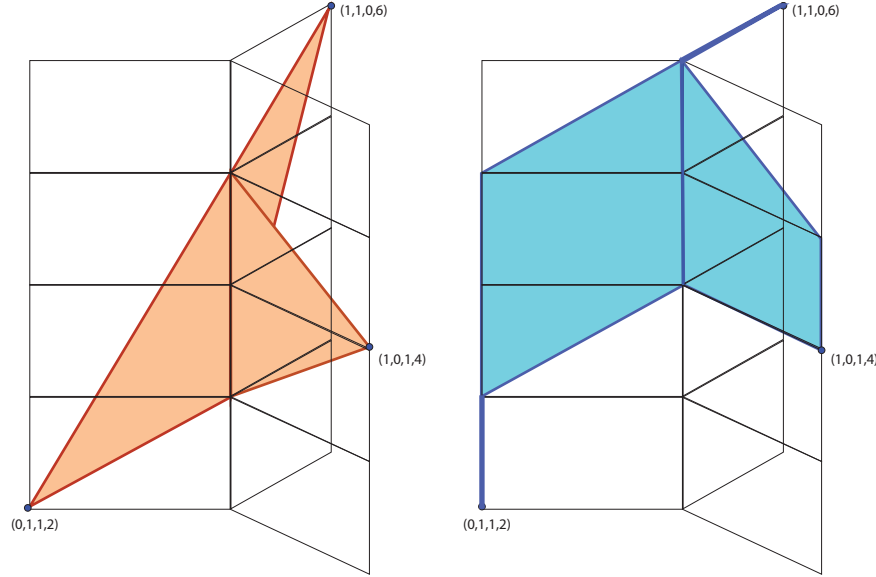


Figure 5.5: Two convex hulls of three points a, b, c inside the open book with three pages. The geodesic triangle is shown on the left, while the tropical triangle is on the right.

We sketch the basic ideas in [87]. Nye defined a line L in $\mathcal{U}_m$ to be an unbounded path such that every bounded subpath is a BHV geodesic. Suppose that L is such a line, and $u \in \mathcal{U}_m \setminus L$. Proposition 2.1 in [86] shows that L contains a unique point x that is closest to u in the BHV metric. We call x the *projection* of u onto the line L . Given u and L , we can compute x as follows. Fixing a base point $L(0)$ on the line, one chooses a geodesic parametrization $L(t)$ of the line. This means that t is the distance $d(L(0), L(t))$. Also let k be the distance from u to $L(0)$. By the triangle inequality and the CAT(0)-property, the desired point is $x = L(t^*)$ for some $t^* \in [-k, k]$. The distance $d(x, L(t))$ is a continuous function of t . Our task is to find the value of t^* which minimizes that function on the closed interval $[-k, k]$. This is done using numerical methods. The uniqueness of t^* follows from the CAT(0)-property.

Suppose we are given a collection $\{v^1, v^2, \dots, v^s\}$ of tree metrics on m taxa. This is our data set for phylogenetic analysis. Nye's method computes a first principal line (regression line) for these data inside BHV space. This is done as follows. One first computes the *centroid* x^0 of the s given trees (Nye used the majority-rule consensus tree instead). This can be done using the iterative method in [21, Theorem 4.1]. Now, the desired regression line L is one of the geodesics through x^0 . For any such line L , we can compute the projections $x^1, \dots, x^s$ of the data points $v^1, \dots, v^s$. The goal is to find the line L that minimizes (or, maximizes) a certain objective function. Nye proposes two such functions:

$$f_{\perp}(L) := \sum_{i=1}^s d(v^i, x^i)^2 \quad \text{or} \quad f_{\parallel}(L) := \sum_{i=1}^s d(x^0, x^i)^2.$$

The first function of L above can be minimized and the second function above can be maximized using an iterative numerical procedure.

While the paper [86] represents a milestone concerning statistical inference in BHV tree space, it left the open problem of computing higher-dimensional principal components. First, what are geodesic planes? Which of them is the regression plane for $v^1, v^2, \dots, v^s$? Ideally, a plane in tree space would be a 2-dimensional complex that contains the geodesic triangle formed by any three of its points. Outside a single orthant, do such planes even exist? Such questions were raised in [86, §6]. Our Theorem 5.14 suggests that the answer is negative.

On the other hand, tropical convexity and tropical linear algebra in $\mathcal{U}_m$ behave better. Indeed, each triangle $\text{tconv}(v^1, v^2, v^3)$ in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$ is 2-dimensional and spans a tropical plane; each tropical tetrahedron $\text{tconv}(v^1, v^2, v^3, v^4)$ is 3-dimensional and spans a tropical 3-space. These tropical linear spaces are best represented by their Plücker coordinates; cf. [78, §4.4].

In what follows we take first steps towards the introduction of tropical methods into geometric statistics. We discuss *tropical centroids* and *projections* onto tropical linear spaces.

In any metric space $\mathcal{M}$, one can study two types of “centroids”: one is the Fréchet mean and the other is the Fermat-Weber point. Given a finite sample $\{v^1, v^2, \dots, v^s\}$ of points in $\mathcal{M}$, a *Fréchet mean* minimizes the sum of squared distances to the points. A *Fermat-Weber point* y minimizes the sum of distances to the given points.

$$y := \arg \min_{z \in \mathcal{M}} \sum_{i=1}^s d(z, v^i). \quad (5.18)$$

Here we do not take the square. We note that y is generally not a unique point but refers to the set of all minimizers. Miller et.al. [84] took the Fréchet mean for the centroid in orthant spaces. In the non-Euclidean context of tropical geometry, we prefer to work with the Fermat-Weber points. If $v^1, \dots, v^s \in \mathcal{M} \subset \mathbb{R}^e/\mathbb{R}\mathbf{1}$ then a *tropical centroid* is any solution y to (5.18) with $d = d_{\text{tr}}$. In unconstrained cases, we can use the following linear program to compute tropical centroids:

Proposition 5.26. *Suppose $\mathcal{M} = \mathbb{R}^e/\mathbb{R}\mathbf{1}$. Then the set of tropical centroids is a convex polytope. It consists of all optimal solutions $y = (y_1, \dots, y_e)$ to the following linear program:*

$$\begin{aligned} & \text{minimize} && d_1 + d_2 + \dots + d_s && \text{subject to} \\ & y_j - y_k - v_j^i + v_k^i \geq -d_i && \text{for all } i = 1, \dots, s \text{ and } 1 \leq j < k \leq e, \\ & y_j - y_k - v_j^i + v_k^i \leq d_i && \text{for all } i = 1, \dots, s \text{ and } 1 \leq j < k \leq e. \end{aligned} \quad (5.19)$$

See Chapter 6 for details, proofs, and further results on the topic discussed here. If $\mathcal{M}$ is a proper subset of $\mathbb{R}^e/\mathbb{R}\mathbf{1}$ then the computation of tropical centroids is highly dependent on the representation of $\mathcal{M}$. For tropical linear spaces, $\mathcal{M} = \text{Trop}(M)$, we must solve a linear program on each maximal cone. The question remains how to do this efficiently.

Example 5.27. *Consider the rows of the 3×10 -matrix in Example 5.17. We compute the tropical centroid of these three points in $\mathcal{M} = \mathcal{U}_5 \subset \mathbb{R}^{10}/\mathbb{R}\mathbf{1}$. To do this, we first compute the*

set of all tropical centroids in $\mathbb{R}^{10}/\mathbb{R}\mathbf{1}$. This is a 6-dimensional classical polytope, consisting of all optimal solutions to (5.19). The intersection of that polytope with tree space $\mathcal{U}_5$ equals the parallelogram

$$\left\{ \left(1, 1, 1, 1, \frac{61}{100} + y, \frac{61}{100} + x + y, \frac{3}{4} + y, \frac{61}{100} + y, \frac{3}{4} + y, \frac{3}{4} + y \right) : 0 \leq x \leq \frac{43}{100}, 0 \leq y \leq \frac{7}{50} \right\}.$$

This is mapped to a single (red) point inside the tropical triangle in Fig. 5.4.

Example 5.27 shows that tropical centroids of a finite set of points generally do not lie in the tropical convex hull of those points. For instance, the tropical centroid of $\{D^{(1)}, D^{(2)}, D^{(3)}\}$ that is obtained by setting $x = y = 0$ in the parallelogram above does not lie in the tropical convex hull $\text{tconv}(D^{(1)}, D^{(2)}, D^{(3)})$.

We now come to our second and last topic in this section, namely projecting onto subspaces. Let $L_{\mathbf{w}}$ be a *tropical linear space* of dimension $r - 1$ in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. The notation is taken from [78, Definition 4.4.3]. Hence $\mathbf{w} = (w_{\sigma})$ is a vector in $\mathbb{R}^{\binom{[e]}{r}}$ that lies in the *Dressian* $\text{Dr}(r, e)$, which is the parameter space for all tropical linear spaces, as introduced in [78, Definition 4.4.1]. The Plücker coordinates w_{σ} are indexed by subsets $\sigma \in \binom{[e]}{r}$. Among the $L_{\mathbf{w}}$ are the tropicalized linear spaces [78, Theorem 4.3.17]. Even more special are linear spaces spanned by r points; cf. [55]. If $L_{\mathbf{w}}$ is spanned by $x^1, \dots, x^r$ in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$ then its Plücker coordinate w_{σ} is the *tropical determinant* of the $r \times r$ -submatrix indexed by σ of the $r \times e$ -matrix $X = (x^1, \dots, x^r)$. Note that all tropical linear spaces $L_{\mathbf{w}}$ are tropically convex.

We are interested in the nearest point map $\pi_{L_{\mathbf{w}}}$ that takes a point u to the largest point in $L_{\mathbf{w}}$ dominated by u , as seen in (5.17). From [69, Theorem 15] we have:

Theorem 5.28 (The Blue Rule). *The i -th coordinate of the point in $L_{\mathbf{w}}$ nearest to u is equal to*

$$\pi_{L_{\mathbf{w}}}(u)_i = \max_{\tau} \min_{j \notin \tau} (u_j + w_{\tau \cup i} - w_{\tau \cup j}) \quad \text{for } i = 1, 2, \dots, e. \quad (5.20)$$

Here τ runs over all $(r - 1)$ -subsets of $[e]$ that do not contain i .

The special case of this theorem when $L_{\mathbf{w}}$ has the form $\text{Trop}(M)$, for some rank r matroid M on $[e]$, was proved by Ardila in [6, Theorem 1]. Matroids correspond to the case when each tropical Plücker coordinate w_{σ} is either 0 or $-\infty$. The application that motivated Ardila's study was the ultrametric tree space $\mathcal{U}_m$. Here the nearest-point map computes the largest ultrametric dominated by a given dissimilarity map, a problem of importance in phylogenetics. An efficient algorithm for this problem was given by Chepoi and Fichet [38]. This was recently revisited by Apostolico *et al.* in [5].

Example 5.29. *Following Example 5.18, let M be the uniform matroid of rank r on $[e]$. Then $\text{Trop}(M) = L_{\mathbf{w}}$ where $\mathbf{w}$ is the all-zero vector, i.e. $w_{\sigma} = 0$ for $\sigma \in \binom{[e]}{r}$. By (5.20), the i -th coordinate of the nearest point $\pi_{L_{\mathbf{w}}}(u)$ equals $\max_{\tau} \min_{j \notin \tau} u_j$. That nearest point is obtained from u by replacing the $e - r$ largest coordinates in u by the r -th smallest coordinate.*

Returning to ideas for geometric statistics, the Blue Rule may serve as a subroutine for the numerical computation of regression planes. Let $u^1, \dots, u^s$ be data points in $\mathbb{R}^e/\mathbb{R}\mathbf{1}$, lying in a tropically convex subset $\mathcal{P}$ of interest, such as $\mathcal{P} = \mathcal{U}_m$. The tropical regression plane of dimension $r - 1$ is a solution to the optimization problem

$$\arg \min_{L_{\mathbf{w}}} \sum_{i=1}^s d_{\text{tr}}(u^i, L_{\mathbf{w}}). \quad (5.21)$$

Here $\mathbf{w}$ runs over all points in the Dressian $\text{Dr}(r, e)$, or in the tropical Grassmannian $\text{Gr}(r, e)$. One might restrict to *Stiefel tropical linear spaces* [55], i.e. those that are spanned by points. Even the smallest case $r = 2$ is of considerable interest, as seen in the study of Nye [86]. In his approach, we would first compute the tropical centroid inside $\mathcal{P}$ of the sample $\{u^1, u^2, \dots, u^s\}$. Fix x^1 to be that centroid. Now $x^2 \in \mathcal{P}$ is the remaining decision variable, and we optimize over all tropical lines spanned by x^1 and x^2 inside $\mathbb{R}^e/\mathbb{R}\mathbf{1}$. Such a line is a tree with e unbounded rays. If the ambient tropically convex set $\mathcal{P}$ is a tropical linear space, such as our tree space $\mathcal{U}_m$, then the regression tree $L_{\mathbf{w}}$ will always be contained inside $\mathcal{P}$.

Chapter 6

Tropical Fermat-Weber points

This chapter is a joint work with Ruriko Yoshida. It is on the [arXiv \[76\]](#) with the same title. It is a continuation of Chapter 5.

In this chapter we focus on a metric space motivated from tropical geometry. We investigate the computation of Fermat-Weber points under the tropical metric 5.15, motivated by its application to the space of equidistant phylogenetic trees with N leaves realized as the tropical linear space of all ultrametrics. While the Fréchet mean with the CAT(0)-metric of Billera-Holmes-Vogtman has been studied by many authors (for example [15, 84]), the Fermat-Weber point under tropical metric in tree spaces is not well understood. In this chapter we investigate the Fermat-Weber points under the tropical metric and we show that the set of tropical Fermat-Weber points is a classical convex polytope. We identify conditions under which this set is a singleton.

6.1 Introduction

As costs of genome sequencing have dropped precipitously, development of efficient bioinformatic methods to analyze genome structure and evolution have become ever more urgent. There is a major need to explore new approaches to undertake phylogenomic (phylogenetics applied on a genomic scale) studies for elucidating gene and genome evolution, relationships among species and populations, and processes of speciation and molecular evolution much more rapidly and robustly than existing tools allow (e.g. [20, 30, 66, 105, 111]). In order to do so, one needs a statistical tool to analyze a set of phylogenetic trees over the tree spaces. In this chapter we focus on one of the most classical statistics computed from the observed sample, namely, "median" on tree spaces.

The Fréchet mean and Fermat-Weber point of a sample are statistics to measure the central tendency of the sample [40, 85]. For any metric space with a distance metric $d(\cdot, \cdot)$ between any two points, the Fréchet population mean of a distribution ν is defined as follows:

$$\mu = \arg \min_{\mathbf{y}} \int d(\mathbf{y}, \mathbf{x})^2 d\nu(\mathbf{x}),$$

and thus, the Fréchet sample mean of a sample $\mathbf{x}_1, \dots, \mathbf{x}_m$ is defined as

$$\hat{\mu} = \arg \min_{\mathbf{y}} \sum_{i=1}^m d(\mathbf{y}, \mathbf{x}_i)^2$$

The Fermat-Weber point of a distribution ν is defined as follows:

$$\mu = \arg \min_{\mathbf{y}} \int d(\mathbf{y}, \mathbf{x}) d\nu(\mathbf{x}),$$

and thus, the Fermat-Weber point of a sample is defined as follows:

$$\hat{\mu} = \arg \min_{\mathbf{y}} \sum_{i=1}^m d(\mathbf{y}, \mathbf{x}_i).$$

The Fréchet sample mean with the Billera-Holmes-Vogtman (BHV) metric on tree spaces [21] of rooted phylogenetic trees with N leaves has been studied by many authors (for example, [15, 84] and references within) where $N \in \mathbb{N}$. A Fermat-Weber point of m points in the Euclidean n -dimensional space $\mathbb{R}^n$ has been well studied in the literature (for example, [16, 39, 49]) and the set of Fermat-Weber points of m points in the n -dimensional Minkowski spaces has been also studied by many authors (for example, [33, 50, 51, 71, 79]). However, to our best knowledge, the Fermat-Weber point of a sample under the *tropical metric* on tree spaces of rooted phylogenetic trees with N leaves has not been studied. Thus, in this chapter, we investigate the properties of Fermat-Weber points of a sample under the tropical metric on tree spaces of rooted phylogenetic trees with N leaves.

This tropical metric is also called *generalized Hilbert projective metric* [2, §2.2], [41, §3.3]. It is also known that the geodesic under this metric is not unique. For more details on this metric and tropical geometry see [78].

While a *geodesic metric* on BHV treespace is *intrinsic*, the tropical metric on the space is *extrinsic*. Therefore we first investigate the set of all the tropical Fermat-Weber points of a sample on $\mathbb{R}^{\binom{N}{2}}/\mathbb{R}\mathbf{1}$ (in this chapter $\mathbf{1}$ denoted the all-one vector in a space $\mathbb{R}^n$) and then we investigate the intersection of the set of all tropical Fermat-Weber points of a sample and the space of ultrametrics $\mathcal{U}_N$, the space of equidistant trees with N leaves.

In this chapter, we consider the quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ for a fixed positive integer n . Thus, in the case of a treespace, we consider $n = \binom{N}{2}$. In Section 6.2, we show an important property of a tropical Fermat-Weber point of m points $\mathbf{v}_1, \dots, \mathbf{v}_m$ over $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ in Theorem 6.3. Namely, the set of all tropical Fermat-Weber points of m points $\mathbf{v}_1, \dots, \mathbf{v}_m$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ forms a classical convex polytope. This is Proposition 6.7. Therefore, there are many cases when a set of m points has infinitely many tropical Fermat-Weber points. Then in Section 6.3, we investigate the condition when a random sample of m points has a unique tropical Fermat-Weber point. If we consider the space of families of m arbitrary points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ (which corresponds to $\mathbb{R}^{m(n-1)}$), the points forming an *essential set* (Definition 6.10) with a unique tropical Fermat-Weber point are contained in a finite union of proper linear subspaces

in $\mathbb{R}^{m(n-1)}$ (Theorem 6.12). This theorem implies that if we pick a random sample of m points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in $\mathbb{R}^{m(n-1)}$, then with probability 1 we get either a set of points such that one of them is already a unique tropical Fermat-Weber point of the others, or a set of points that has infinitely many tropical Fermat-Weber points. In Section 6.4 we investigate the intersection between the set of tropical Fermat-Weber points of the sample in $\mathbb{R}^{\binom{N}{2}}/\mathbb{R}\mathbf{1}$ and $\mathcal{U}_N$ (defined in Section 5.1) for small $N \in \mathbb{N}$. We show by experiments that it is very rare to obtain a unique tropical Fermat-Weber point of an essential random sample over tree spaces. From our experimental study, we conjecture that if an essential random sample has a unique tropical Fermat-Weber point then the unique tropical Fermat-Weber point is the vector with all ones in its coordinates. In Section 6.5 we generalize the locus of the tropical Fermat-Weber points of a sample of size k to the k -ellipses under tropical metric. We end this chapter with an open problem regarding the computation of the set of tropical Fermat-Weber points of a sample.

6.2 The tropical Fermat-Weber point

In this section, we define the tropical metric and derive some basic properties of the tropical Fermat-Weber point.

For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ we define their distance as in (5.15) to be

$$d_{tr}(\mathbf{u}, \mathbf{v}) = \max_{1 \leq i < j \leq n} \{|u_i - u_j - v_i + v_j|\}. \quad (6.1)$$

In other words, let $\mathcal{D}_{\mathbf{u}, \mathbf{v}} = \{u_i - v_i \mid 1 \leq i \leq n\}$, then

$$d_{tr}(\mathbf{u}, \mathbf{v}) = \max_{x, y \in \mathcal{D}_{\mathbf{u}, \mathbf{v}}} (|x - y|) = \max(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) - \min(\mathcal{D}_{\mathbf{u}, \mathbf{v}}). \quad (6.2)$$

By definition d_{tr} is symmetric. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have that

$$\begin{aligned} d_{tr}(\mathbf{u}, \mathbf{w}) &= \max_{1 \leq i < j \leq n} \{|u_i - u_j - w_i + w_j|\} \\ &= \max_{1 \leq i < j \leq n} \{|(u_i - u_j - v_i + v_j) + (v_i - v_j - w_i + w_j)|\} \\ &\leq \max_{1 \leq i < j \leq n} \{(|u_i - u_j - v_i + v_j| + |v_i - v_j - w_i + w_j|)\} \\ &\leq \max_{1 \leq i < j \leq n} \{|u_i - u_j - v_i + v_j|\} + \max_{1 \leq i < j \leq n} \{|v_i - v_j - w_i + w_j|\} \\ &= d_{tr}(\mathbf{u}, \mathbf{v}) + d_{tr}(\mathbf{v}, \mathbf{w}). \end{aligned}$$

Thus, d_{tr} satisfies the triangle inequality. Note that $d_{tr}(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} - \mathbf{v}$ is a scalar multiple of $\mathbf{1}$, and for any scalar multiple $c\mathbf{1}$ with a constant $c \in \mathbb{R}$, $d_{tr}(\mathbf{u} + c\mathbf{1}, \mathbf{v}) = d_{tr}(\mathbf{u}, \mathbf{v})$. So $d_{tr}(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$ in the quotient space $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Then d_{tr} is a *metric* on $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. It is called the *tropical metric* [2].

Remark 6.1. *The metric d_{tr} is invariant under vector addition in Euclidean space: for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$, by (6.2), we have $d_{tr}(\mathbf{u} + \mathbf{w}, \mathbf{v} + \mathbf{w}) = d_{tr}(\mathbf{u}, \mathbf{v})$.*

Given vectors $\mathbf{v}_1, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$, the set of their *tropical Fermat-Weber points* (if the context is clear, we simply use *Fermat-Weber points*) is

$$\arg \min_{\mathbf{u} \in \mathbb{R}^n / \mathbb{R}\mathbf{1}} \sum_{i=1}^m d_{tr}(\mathbf{u}, \mathbf{v}_i). \quad (6.3)$$

Definition 6.2. For points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$, we define the minimal sum of distances from them as

$$d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \min_{\mathbf{u} \in \mathbb{R}^n / \mathbb{R}\mathbf{1}} \sum_{i=1}^m d_{tr}(\mathbf{u}, \mathbf{v}_i). \quad (6.4)$$

The quantity $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ should be determined by the entries of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. However, at this point we do not even know whether $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ is well-defined, let alone any explicit formula of it. In addition, in order to find the set of Fermat-Weber points of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, we need to know the value of $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$. The following theorem gives a direct formula of $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$.

Theorem 6.3. Let M be an $m \times n$ matrix with real entries such that the row vectors are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Then

$$d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \max_{\sigma, \tau} \left| \sum_{i=1}^m M_{i, \sigma(i)} - \sum_{i=1}^m M_{i, \tau(i)} \right|, \quad (6.5)$$

where functions $\sigma, \tau : [m] \rightarrow [n]$ satisfy $\sigma([m]) = \tau([m])$ as multisets.

To prove this theorem, we need the following lemmas.

Lemma 6.4. The right hand side (RHS) of (6.5) is bounded above by the left hand side (LHS) of (6.5).

Proof. Let $\mathbf{u}$ be a Fermat-Weber point of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. Suppose σ, τ are functions with the same multiset of values. Since σ and τ are symmetric, we may assume that

$$\sum_{i=1}^m M_{i, \sigma(i)} \geq \sum_{i=1}^m M_{i, \tau(i)}.$$

Now for $1 \leq i \leq m$ we have

$$d_{tr}(\mathbf{u}, \mathbf{v}_i) \geq |(\mathbf{v}_i)_{\sigma(i)} - (\mathbf{v}_i)_{\tau(i)} - u_{\sigma(i)} + u_{\tau(i)}| \geq M_{i, \sigma(i)} - M_{i, \tau(i)} - u_{\sigma(i)} + u_{\tau(i)}.$$

Summing up over $1 \leq i \leq m$, the LHS of the sum is $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$. One part of the RHS of the sum is

$$\sum_{i=1}^m M_{i, \sigma(i)} - \sum_{i=1}^m M_{i, \tau(i)},$$

and the other part vanishes because $\sigma([m]) = \tau([m])$. Hence $d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \geq \sum_{i=1}^m M_{i, \sigma(i)} - \sum_{i=1}^m M_{i, \tau(i)}$. $\square$

Lemma 6.5. *If A and B are two $m \times n$ matrices that have the same multiset of entries, then there exist $m \times n$ matrices A' and B' such that:*

- (i) *for $1 \leq i \leq m$, the entries of the i -th row of A' and the entries of the i -th row of B' form the same multiset; and*
- (ii) *for $1 \leq j \leq n$, the entries of the j -th column of A' and the entries of the j -th column of A form the same multiset, and the entries of the j -th column of B' and the entries of the j -th column of B form the same multiset.*

Example 6.6. *If*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \\ 4 & 6 & 10 \\ 8 & 11 & 12 \end{bmatrix},$$

then we let

$$A' = \begin{bmatrix} 1 & 5 & 12 \\ 10 & 2 & 6 \\ 7 & 8 & 3 \\ 4 & 11 & 9 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & 5 & 12 \\ 2 & 6 & 10 \\ 8 & 3 & 7 \\ 4 & 11 & 9 \end{bmatrix}.$$

Proof of Lemma 6.5. If the entries of A are not all distinct, then we can label the equal entries to distinguish them. So we may assume that both A and B have mn distinct entries. Then we can replace *multiset* in the statement by *set*.

We use induction on m . If $m = 1$, we can take $A' = A$ and $B' = B$. Suppose Lemma 6.5 is true when $m \leq k$, we consider the case when $m = k + 1$. If there exist $m \times n$ matrices A'' and B'' such that (ii) is true and (i) is true for $i = 1$, let r_A and r_B be the vector of first row in A'' and B'' respectively, and we denote

$$A'' = \begin{bmatrix} r_A \\ A_2 \end{bmatrix}, \quad B'' = \begin{bmatrix} r_B \\ B_2 \end{bmatrix}.$$

We may apply the induction hypothesis of $m = k$ to the matrices A_2 and B_2 . Suppose we get new matrices A'_2 and B'_2 respectively. We let

$$A' = \begin{bmatrix} r_A \\ A'_2 \end{bmatrix}, \quad B' = \begin{bmatrix} r_B \\ B'_2 \end{bmatrix}.$$

So A' and B' satisfy both (i) and (ii). Now it suffices to show that we can find such a pair of matrices A'' and B'' . We denote by s the set of entries in r_A (and also the set of entries in r_B). The above claim is equivalent to the following statement: there exists a set s with $|s| = n$ and s has exactly one element in each column of A and B .

We construct a bipartite graph $G = (V, E)$, where the two parts of V correspond to the columns of A and the columns of B respectively:

$$V = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n\}.$$

For each entry x in the set of entries of A, B , if x is in the i -th column of A and in the j -th column of B , then we connect an edge between a_i and b_j . So $|E| = mn$. Since each column has m entries, the graph G is m -regular. By Hall's Theorem [65], G admits a perfect matching. Then we let s be the set of n elements corresponding to the edges in this perfect matching. The induction step is done. $\square$

For convenience, if $\sigma : [m] \rightarrow [n]$ is a function, then we view it as a vector in $[n]^m$ and we define a vector $\mathbf{w}_\sigma \in \mathbb{N}^n$ as follows: the i -th entry of $\mathbf{w}_\sigma$ is $|\sigma^{-1}(i)|$. So the sum of entries in $\mathbf{w}_\sigma$ is always m . For example, if $m = 3, n = 5$ and $\sigma(1) = 4, \sigma(2) = 3, \sigma(3) = 3$, then $\mathbf{w}_\sigma = (0, 0, 2, 1, 0)$.

Proof of Theorem 6.3. Let $\mathbf{u} = (u_1, \dots, u_n)$ be a point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. We denote $c_i = d_{tr}(\mathbf{u}, \mathbf{v}_i)$. Then c_i satisfies the inequalities

$$c_i \geq u_j - u_k - M_{i,j} + M_{i,k} \quad \forall 1 \leq j, k \leq n. \quad (6.6)$$

Let $s = \sum_{i=1}^m c_i$. By definition

$$d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) = \min_{\mathbf{u} \in \mathbb{R}^n/\mathbb{R}\mathbf{1}} s.$$

Then we can eliminate the parameters c_i and we get the following family of inequalities:

$$s \geq \sum_{i=1}^m M_{i,\sigma(i)} - \sum_{i=1}^m M_{i,\tau(i)} - \mathbf{u} \cdot \mathbf{w}_\tau + \mathbf{u} \cdot \mathbf{w}_\sigma, \forall \sigma, \tau \in [n]^m. \quad (6.7)$$

Let M be the value of the RHS in (6.5). Lemma 6.4 tells us that if (6.7) is feasible then $s \geq M$. It suffices to prove that (6.7) is feasible when $s = M$. Note that Fourier-Motzkin Elimination [112] could get rid of the variables u_i one at a time. After finite steps, the only variable remaining in the inequalities is s . It suffices to show that any lower bound for s we obtain from Fourier-Motzkin Elimination is at most M .

Note that, in each step of Fourier-Motzkin Elimination, we obtain a new inequality as a $\mathbb{Q}_+$ -linear combination of existing inequalities. Therefore if l is a lower bound, then $s \geq l$ is a $\mathbb{Q}_+$ -linear combination of the inequalities in (6.7). Multiplying by a positive integer we may assume that it's a $\mathbb{Z}_+$ -linear combination, therefore we have $r \in \mathbb{Z}_+$ and functions $\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_r : [m] \rightarrow [n]$ such that $s \geq l$ is equivalent to

$$rs \geq \sum_{j=1}^r \left(\sum_{i=1}^m M_{i,\sigma_j(i)} - \sum_{i=1}^m M_{i,\tau_j(i)} - \mathbf{u} \cdot \mathbf{w}_{\tau_j} + \mathbf{u} \cdot \mathbf{w}_{\sigma_j} \right). \quad (6.8)$$

So the RHS of (6.8) is the constant rl . Therefore

$$rl = \sum_{j=1}^r \left(\sum_{i=1}^m M_{i,\sigma_j(i)} - \sum_{i=1}^m M_{i,\tau_j(i)} \right), \quad (6.9)$$

and

$$\sum_{j=1}^r \mathbf{w}_{\sigma_j} = \sum_{j=1}^r \mathbf{w}_{\tau_j}. \quad (6.10)$$

Now let A and B be two matrices with the same size $r \times m$, with $A_{j,i} = \sigma_j(i)$ and $B_{j,i} = \tau_j(i)$ for $1 \leq j \leq r, 1 \leq i \leq m$. Then the multiset of their entries are equal because of (6.10). By Lemma 6.5 we can obtain matrices A' and B' satisfying the conditions in (i) and (ii). For $1 \leq j \leq r$, we let σ'_j and τ'_j be functions mapping from $[m]$ to $[n]$ such that for $1 \leq i \leq m$, $\sigma'_j(i) = A'_{j,i}$ and $\tau'_j(i) = B'_{j,i}$. Then the condition (i) implies that

$$\mathbf{w}_{\sigma'_j} = \mathbf{w}_{\tau'_j} \quad \forall 1 \leq j \leq r.$$

The condition (ii) implies that for each $1 \leq i \leq m$, the multisets $\{\sigma_j(i) | 1 \leq j \leq r\}$ and $\{\sigma'_j(i) | 1 \leq j \leq r\}$ are equal and the multisets $\{\tau_j(i) | 1 \leq j \leq r\}$ and $\{\tau'_j(i) | 1 \leq j \leq r\}$ are equal. Then

$$\sum_{j=1}^r \sum_{i=1}^m M_{i,\sigma_j(i)} = \sum_{j=1}^r \sum_{i=1}^m M_{i,\sigma'_j(i)} \quad (6.11)$$

because each entry of M is added by the same number of times in both sides of (6.11). Similarly

$$\sum_{j=1}^r \sum_{i=1}^m M_{i,\tau_j(i)} = \sum_{j=1}^r \sum_{i=1}^m M_{i,\tau'_j(i)}. \quad (6.12)$$

The inequality (6.8) is equivalent to

$$rs \geq \sum_{j=1}^r \left(\sum_{i=1}^m M_{i,\sigma'_j(i)} - \sum_{i=1}^m M_{i,\tau'_j(i)} - \mathbf{u} \cdot \mathbf{v}_{\tau'_j} + \mathbf{u} \cdot \mathbf{v}_{\sigma'_j} \right). \quad (6.13)$$

Now for each $1 \leq j \leq r$,

$$\sum_{i=1}^m M_{i,\sigma'_j(i)} - \sum_{i=1}^m M_{i,\tau'_j(i)} - \mathbf{u} \cdot \mathbf{v}_{\tau'_j} + \mathbf{u} \cdot \mathbf{v}_{\sigma'_j} = \sum_{i=1}^m M_{i,\sigma'_j(i)} - \sum_{i=1}^m M_{i,\tau'_j(i)}.$$

In addition σ'_j and τ'_j have the same multiset of values. Then by the definition of M ,

$$M \geq \sum_{i=1}^m M_{i,\sigma'_j(i)} - \sum_{i=1}^m M_{i,\tau'_j(i)}.$$

We sum over $1 \leq j \leq r$. This yields

$$rl = \sum_{j=1}^r \left(\sum_{i=1}^m M_{i,\sigma'_j(i)} - \sum_{i=1}^m M_{i,\tau'_j(i)} \right) \leq \sum_{j=1}^r M = rM,$$

hence $l \leq M$. So M is the greatest possible lower bound for s , which means $s = M$ would make the system of linear inequalities feasible. So $d(v_1, v_2, \dots, v_m) = M$. $\square$

We restate the following result in Chapter 5.

Proposition 6.7. [74, Proposition 6.1] *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The set of their Fermat-Weber points is a classical convex polytope in $\mathbb{R}^{n-1} \simeq \mathbb{R}^n/\mathbb{R}\mathbf{1}$.*

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Then $\mathbf{x}$ is a Fermat-Weber point of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ if and only if for all choices of indices $j_i, k_i \in [n], 1 \leq i \leq m$,

$$\sum_{i=1}^m (x_{j_i} - x_{k_i} + v_{i,k_i} - v_{i,j_i}) \leq d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m). \quad (6.14)$$

Then the set is a polyhedron in $\mathbb{R}^n$. Finally in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ we may assume $x_1 = 0$, and thus x_i is bounded for $2 \leq i \leq n$. $\square$

Example 6.8. *The polytope of the following three points in $\mathbb{R}^3/\mathbb{R}\mathbf{1} \simeq \mathbb{R}^2$*

$$(0, 0, 0), (0, 3, 1), (0, 2, 5)$$

is the triangle with vertices

$$(0, 1, 1), (0, 2, 1), (0, 2, 2).$$

In Figure 6.1, we draw the coordinates x_2 and x_3 since the first coordinate is $x_1 = 0$.

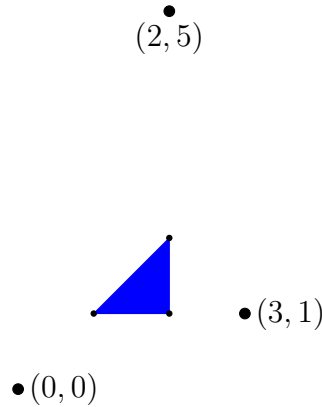


Figure 6.1: The Fermat-Weber points of three points in Example 6.8 is a closed triangle (blue).

6.3 Uniqueness of the Fermat-Weber point

In the previous section we have shown that, in some cases, there are infinitely many Fermat-Weber points of a given set of m points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. But how often does this case happen? In this section we investigate conditions on sets of points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ that have a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, i.e., we study when a random sample of m points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ has a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

Lemma 6.9. *Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ be points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ and $\mathbf{v}_0$ be a Fermat-Weber point of them. Then $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_m$ have a unique Fermat-Weber point, which is $\mathbf{v}_0$.*

Proof. For any point $\mathbf{x} \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$, suppose $\mathbf{x}$ and $\mathbf{v}_0$ are not the same point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Then we have

$$d_{tr}(\mathbf{x}, \mathbf{v}_0) > 0 = d_{tr}(\mathbf{v}_0, \mathbf{v}_0). \quad (6.15)$$

Since $\mathbf{v}_0$ is a Fermat-Weber point of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$, we have

$$\sum_{i=1}^m d_{tr}(\mathbf{x}, \mathbf{v}_i) \geq \sum_{i=1}^m d_{tr}(\mathbf{v}_0, \mathbf{v}_i). \quad (6.16)$$

So

$$\sum_{i=0}^m d_{tr}(\mathbf{x}, \mathbf{v}_i) > \sum_{i=0}^m d_{tr}(\mathbf{v}_0, \mathbf{v}_i). \quad (6.17)$$

Hence, by definition, $\mathbf{v}_0$ is the unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. $\square$

The situation in Lemma 6.9 is not desirable, because we don't know whether $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ have a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. So we introduce the following definition.

Definition 6.10. *Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ be a set of points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The set S is essential if for $1 \leq i \leq m$, the point $\mathbf{v}_i$ is not a Fermat-Weber point of the points in $S - \{\mathbf{v}_i\}$.*

Now we consider the following question: in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, what is the smallest integer $u(n)$ such that there exist an essential set of $u(n)$ points with a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$?

Proposition 6.11. *For $n \geq 3$, $u(n) \leq n$.*

Proof. First we suppose $n \geq 4$. We claim that the row vectors in the following $n \times n$ matrix M represent n points $\mathbf{v}_1, \dots, \mathbf{v}_n$ that form an essential set and have a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$.

$$M_{i,j} = \begin{cases} 1, & \text{if } j - i \equiv 0, 1 \pmod{n}; \\ -1, & \text{if } j - i \equiv 2, 3 \pmod{n}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that for $1 \leq i \leq n$ we have $d_{tr}(\mathbf{v}_i, \mathbf{0}) = 1 - (-1) = 2$. Thus,

$$\sum_{i=1}^n d_{tr}(\mathbf{v}_i, \mathbf{0}) = 2n. \quad (6.18)$$

Now suppose $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$ is a Fermat-Weber point of $\mathbf{v}_1, \dots, \mathbf{v}_n$. For convenience we denote that $a_{i+n} = a_i$ for all i . By (6.2), for $1 \leq i \leq n$ we have

$$d_{tr}(\mathbf{a}, \mathbf{v}_i) = \max_{1 \leq j \leq n} \{a_j - M_{i,j}\} - \min_{1 \leq j \leq n} \{a_j - M_{i,j}\}. \quad (6.19)$$

For $1 \leq i \leq n$, note that the i -th and $(i+1)$ -th coordinates of $\mathbf{v}_i$ are 1. Thus,

$$\max_{1 \leq j \leq n} \{a_j - M_{i,j}\} \geq 1 - \min\{a_i, a_{i+1}\}.$$

Similarly, since the $(i+2)$ -nd and $(i+3)$ -rd coordinates of $\mathbf{v}_i$ are -1 ,

$$\min_{1 \leq j \leq n} \{a_j - M_{i,j}\} \leq -1 - \max\{a_{i+2}, a_{i+3}\}.$$

We have

$$\begin{aligned} d_{tr}(\mathbf{v}_i, \mathbf{a}) &\geq (1 - \min\{a_i, a_{i+1}\}) - (-1 - \max\{a_{i+2}, a_{i+3}\}) \\ &= 2 + \max\{a_{i+2}, a_{i+3}\} - \min\{a_i, a_{i+1}\}. \end{aligned}$$

Summing over i , we get

$$\sum_{i=1}^n d_{tr}(\mathbf{v}_i, \mathbf{a}) \geq 2n + \sum_{i=1}^n [\max\{a_i, a_{i+1}\} - \min\{a_i, a_{i+1}\}] \geq 2n. \quad (6.20)$$

By (6.18) and (6.20), we know that $\mathbf{0}$ is a Fermat-Weber point. Since $\mathbf{a}$ is also a Fermat-Weber point, all equalities in (6.20) hold. Hence $\max\{a_i, a_{i+1}\} = \min\{a_i, a_{i+1}\}$ for all i , which means $a_i = a_{i+1}$ for all i . So $\mathbf{a} = \mathbf{0}$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Then $\mathbf{v}_1, \dots, \mathbf{v}_n$ have a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Finally since $\mathbf{v}_i \neq \mathbf{0}$ for each $i = 1, \dots, n$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, the set of points $\mathbf{v}_1, \dots, \mathbf{v}_n$ forms an essential set.

As for the case when $n = 3$, we have the following example of three points in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$:

$$(-1, 1, 1), (1, -1, 1), (1, 1, -1).$$

A simple computation shows that they have the unique Fermat-Weber point $(0, 0, 0)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. In particular, they form an essential set. $\square$

Proposition 6.11 shows the existence of essential sets of points with a unique Fermat-Weber point. However, the following theorem tells us that this case is very rare.

Theorem 6.12. *Fix positive integers m and n . Consider the space $\mathbb{R}^{m(n-1)}$ of m points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. The configurations representing an essential set of points with a unique Fermat-Weber point are contained in a finite union of proper linear subspaces in $\mathbb{R}^{m(n-1)}$.*

Definition 6.13. *Let $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n)$ be two points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ and $d = d_{tr}(\mathbf{u}, \mathbf{v}) > 0$. The peaks and valleys of $\mathbf{u}, \mathbf{v}$ are the following subsets of $[n]$:*

$$\text{peak}(\mathbf{u}, \mathbf{v}) = \arg \max_{1 \leq i \leq n} \{u_i - v_i\}, \quad \text{valley}(\mathbf{u}, \mathbf{v}) = \arg \min_{1 \leq i \leq n} \{u_i - v_i\}.$$

We prove a few lemmas before we prove Theorem 6.12.

Lemma 6.14. *Let $\mathbf{u} = (u_1, \dots, u_n), \mathbf{v} = (v_1, \dots, v_n)$ be two points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ and $d = d_{tr}(\mathbf{u}, \mathbf{v}) > 0$. Let ϵ be a positive real number less than the minimum of the set*

$$\{|(u_i - v_i) - (u_j - v_j)| : 1 \leq i < j \leq n\} - \{0\}.$$

(Since $d > 0$, the above set is nonempty.) For $1 \leq i \leq n$, we denote ϵ_i as the vector in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ whose i -th entry is ϵ and other entries are zero. Then we have

$$d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) = \begin{cases} d, & \text{if } i \notin \text{peak}(\mathbf{u}, \mathbf{v}) \cup \text{valley}(\mathbf{u}, \mathbf{v}); \\ d + \epsilon, & \text{if } i \in \text{peak}(\mathbf{u}, \mathbf{v}); \\ d - \epsilon, & \text{if } i \in \text{valley}(\mathbf{u}, \mathbf{v}) \text{ and } |\text{valley}(\mathbf{u}, \mathbf{v})| = 1; \\ d & \text{if } i \in \text{valley}(\mathbf{u}, \mathbf{v}) \text{ and } |\text{valley}(\mathbf{u}, \mathbf{v})| \geq 2. \end{cases} \quad (6.21)$$

Similarly,

$$d_{tr}(\mathbf{u} - \epsilon_i, \mathbf{v}) = \begin{cases} d, & \text{if } i \notin \text{peak}(\mathbf{u}, \mathbf{v}) \cup \text{valley}(\mathbf{u}, \mathbf{v}); \\ d + \epsilon, & \text{if } i \in \text{valley}(\mathbf{u}, \mathbf{v}); \\ d - \epsilon, & \text{if } i \in \text{peak}(\mathbf{u}, \mathbf{v}) \text{ and } |\text{peak}(\mathbf{u}, \mathbf{v})| = 1; \\ d & \text{if } i \in \text{peak}(\mathbf{u}, \mathbf{v}) \text{ and } |\text{peak}(\mathbf{u}, \mathbf{v})| \geq 2. \end{cases} \quad (6.22)$$

Proof. We use formula (6.2). Let $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$ be the set $\{u_i - v_i | 1 \leq i \leq n\}$ for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$.

We consider $d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v})$ first. If $i \notin \text{peak}(\mathbf{u}, \mathbf{v}) \cup \text{valley}(\mathbf{u}, \mathbf{v})$, then $u_i - v_i$ is between the maximum and minimum of $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$. So $\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}$ has the same maximum and minimum as $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$, then $d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) = d_{tr}(\mathbf{u}, \mathbf{v})$. If $i \in \text{peak}(\mathbf{u}, \mathbf{v})$, then $\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}$ has the same minimum as $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$, but $\max(\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}) = \max(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) + \epsilon$. So $d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) = d_{tr}(\mathbf{u}, \mathbf{v}) + \epsilon$.

If $i \in \text{valley}(\mathbf{u}, \mathbf{v})$, then $u_i - v_i$ is the minimum of $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$. So $\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}$ has the same maximum as $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$. As for the minimum, if $|\text{valley}(\mathbf{u}, \mathbf{v})| \geq 2$, then there exists $k \neq i$ with $u_k - v_k = u_i - v_i$. Then $u_k - v_k \in \mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}$ and $\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}$ has the same minimum as $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$. As a result, $d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) = d_{tr}(\mathbf{u}, \mathbf{v})$. If $|\text{valley}(\mathbf{u}, \mathbf{v})| = 1$, then all other elements in $\mathcal{D}_{\mathbf{u}, \mathbf{v}}$ are strictly greater than $u_i - v_i$, thus $\min(\mathcal{D}_{\mathbf{u} + \epsilon_i, \mathbf{v}}) = \min(\mathcal{D}_{\mathbf{u}, \mathbf{v}}) + \epsilon$. So $d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) = d_{tr}(\mathbf{u}, \mathbf{v}) - \epsilon$.

The cases of $d_{tr}(\mathbf{u} - \epsilon_i, \mathbf{v})$ can be analyzed in the same way. $\square$

Next, for (6.21) and (6.22), if we sum over i , we get the following corollary.

Corollary 6.15. *Let $\mathbf{u}, \mathbf{v}$ be two points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$. Let d, e and ϵ_i be the same as in Lemma 6.14. Then*

$$\sum_{i=1}^n (d_{tr}(\mathbf{u} + \epsilon_i, \mathbf{v}) + d_{tr}(\mathbf{u} - \epsilon_i, \mathbf{v})) = 2n \cdot d + [f(|\text{peak}(\mathbf{u}, \mathbf{v})|) + f(|\text{valley}(\mathbf{u}, \mathbf{v})|)] \cdot e,$$

where f is the function defined on $\mathbb{Z}_+$ by

$$f(n) = \begin{cases} 0, & \text{if } n = 1; \\ n, & \text{if } n \geq 2. \end{cases}$$

Definition 6.16. Let m and n be positive integers. Two subsets $S, T \subset [m] \times [n]$ are called similar if for $1 \leq i \leq m$ we have

$$|\{k \mid (i, k) \in S\}| = |\{k \mid (i, k) \in T\}|$$

and for $1 \leq j \leq n$ we have

$$|\{k \mid (k, j) \in S\}| = |\{k \mid (k, j) \in T\}|.$$

In other words, S and T are similar if and only if, given any row or column of M , they have the same number of elements in it.

The following lemma explicitly tells us the defining equations of the finite union of proper linear subspaces.

Lemma 6.17. Let $X = (x_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ be an $m \times n$ matrix. For any set $S \subset [m] \times [n]$, let

$$x_S = \sum_{(i,j) \in S} x_{i,j}.$$

If the row vectors of $X \in \mathbb{R}^{m(n-1)}$ form an essential set with a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, then there exist disjoint $S, T \subset [m] \times [n]$ such that S and T are similar and $x_S = x_T$.

Proof. Suppose X is an $m \times n$ matrix with entries $x_{i,j}$ such that the row vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ of X form an essential set of points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ with a unique Fermat-Weber point $\mathbf{c} \in \mathbb{R}^n/\mathbb{R}\mathbf{1}$. Then the points $\mathbf{v}_1 - \mathbf{c}, \dots, \mathbf{v}_m - \mathbf{c}$ also form an essential set and they have a unique Fermat-Weber point $\mathbf{0}$. Let $X' = (x'_{i,j})$ be the corresponding matrix of these points. Then $x'_{i,j} = x_{i,j} - c_j$ for any $1 \leq i \leq m, 1 \leq j \leq n$. Note that for $S, T \subset [m] \times [n]$, if S and T are similar, then $x_S = x_T$ if and only if $x'_S = x'_T$. Then we may assume the unique Fermat-Weber point of $\mathbf{v}_1, \dots, \mathbf{v}_m$ is $\mathbf{0}$.

Now we construct an undirected graph $G = (V, E)$. Let $V = [n]$. For $1 \leq i \leq m$, let $P_i = \text{peak}(\mathbf{0}, \mathbf{v}_i)$ and $Q_i = \text{valley}(\mathbf{0}, \mathbf{v}_i)$. Then $P_i, Q_i \subset [n]$. For $1 \leq i \leq n$, we choose an arbitrary tree T_{P_i} whose set of vertices is P_i and include its edges into E and we choose an arbitrary tree T_{Q_i} whose set of vertices is Q_i and include its edges into E . Note that if $|P_i| = 1$ then T_{P_i} has no edge. Here we allow parallel edges in G , because P_i may equal to P_j for different i and j .

It suffices to show that G contains a cycle. Suppose one minimal cycle in G has r distinct vertices $j_1, j_2, \dots, j_r \in [n]$, where for each $1 \leq t \leq r$ there is an edge connecting j_t and

j_{t+1} (we denote $j_{r+1} = j_1$). By definition, there exists $i_t \in [m]$ such that $\{j_t, j_{t+1}\} \subset P_{i_t}$ or $\{j_t, j_{t+1}\} \subset Q_{i_t}$. In either case we have that

$$x_{i_t, j_t} = x_{i_t, j_{t+1}}. \quad (6.23)$$

We define the two subsets S, T of $[m] \times [n]$ as follows:

$$S = \{(i_t, j_t) | 1 \leq t \leq r\}, \quad T = \{(i_t, j_{t+1}) | 1 \leq t \leq r\}.$$

We have that $x_S = x_T$ follows from (6.23). In addition, for $j \in [n]$, if $j = j_t$ for some t , then both S and T have one element in the j -th column of M ; otherwise both S and T have no elements in the j -th column of M . For $i \in [m]$, both S and T have $|\{t | i_t = i\}|$ elements in the i -th row of M . Then S and T are similar. Next we show that $S \neq T$. Suppose $S = T$, then for each $1 \leq t \leq r$, the unique element of S in the j_t -th column is equal to the unique element of T in the j_t -th column, which means $(i_t, j_t) = (i_{t-1}, j_t)$. Then we have $i_t = i_{t-1}$. So $i_1 = i_2 = \dots = i_r$, which means all r vertices in the cycle are chosen from $P_i \cup Q_i$. Since P_i and Q_i are disjoint, either all vertices are chosen from P_i or all vertices are chosen from Q_i . Then in either case, the edges in the cycles are either all chosen from $T_{P,i}$ or all chosen from $T_{Q,i}$, which contradicts the fact that both $T_{P,i}$ and $T_{Q,i}$ are trees. Therefore $S \neq T$. Finally if S and T have common elements, then we can delete them to get another pair of similar subsets S', T' , and we still have $x_{S'} = x_{T'}$. So we can choose disjoint S and T .

Finally we show that G contains a cycle. We compute the following sum

$$\mathcal{K} = \sum_{i=1}^m \sum_{j=1}^n (d_{tr}(\epsilon_j, \mathbf{v}_i) + d_{tr}(-\epsilon_j, \mathbf{v}_i)).$$

On the one hand, since $\mathbf{v}_1, \dots, \mathbf{v}_m$ have a unique Fermat-Weber point $\mathbf{0}$, we have

$$\sum_{i=1}^m d_{tr}(\mathbf{w}, \mathbf{v}_i) > \sum_{i=1}^m d_{tr}(\mathbf{0}, \mathbf{v}_i) \quad (6.24)$$

for any nonzero vector $\mathbf{w} \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$. By (6.21) and (6.22), for $1 \leq i \leq m$

$$d_{tr}(\pm \epsilon_j, \mathbf{v}_i) - d_{tr}(\mathbf{0}, \mathbf{v}_i)$$

is $\pm \epsilon$ or zero. Then difference between the LHS and the RHS of (6.24) is an integer multiple of ϵ . Hence

$$\sum_{i=1}^m d_{tr}(\mathbf{w}, \mathbf{v}_i) - \sum_{i=1}^m d_{tr}(\mathbf{0}, \mathbf{v}_i) \geq \epsilon \quad (6.25)$$

for $\mathbf{w} = \pm \epsilon_j$. Summing over j , we have

$$\mathcal{K} \geq 2n \sum_{i=1}^m d_{tr}(\mathbf{0}, \mathbf{v}_i) + 2n \cdot \epsilon. \quad (6.26)$$

On the other hand, by Corollary 6.15 we have

$$\sum_{j=1}^n (d_{tr}(\epsilon_j, \mathbf{v}_i) + d_{tr}(-\epsilon_j, \mathbf{v}_i)) = 2n \cdot d_{tr}(\mathbf{0}, \mathbf{v}_i) + [f(|P_i|) + f(|Q_i|)] \cdot \epsilon.$$

Summing over i we have

$$\mathcal{K} = 2n \sum_{i=1}^m d_{tr}(\mathbf{0}, \mathbf{v}_i) + \left[\sum_{i=1}^m (f(|P_i|) + f(|Q_i|)) \right] \cdot \epsilon. \quad (6.27)$$

Comparing (6.26) and (6.27), we get

$$\sum_{i=1}^m (f(|P_i|) + f(|Q_i|)) \geq 2n. \quad (6.28)$$

Next, for $x \geq 2$ we have

$$x - 1 \geq \frac{x}{2} = \frac{1}{2}f(x)$$

and when $x = 1$ both $x - 1$ and $f(x)$ are zero. This implies

$$\sum_{i=1}^m (|P_i| - 1) + (|Q_i| - 1) \geq \frac{1}{2} \sum_{i=1}^m (f(|P_i|) + f(|Q_i|)) \geq n.$$

So the graph G has at least n edges and it contains a cycle. $\square$

Proof of Theorem 6.12. For $(i, j) \in [m] \times [n]$, let $X_{i,j}$ be variables. For $S \subset [m] \times [n]$, let

$$X_S = \sum_{(i,j) \in S} X_{i,j}.$$

We define the polynomial

$$F = \prod_{\substack{S, T \subset [m] \times [n] \\ S \neq T \\ S, T \text{ are similar}}} (X_S - X_T).$$

Then $F \in \mathbb{R}[X_{1,1}, X_{1,2}, \dots, X_{m,n}]$. By Lemma 6.17, if an $m \times n$ matrix $M = (m_{i,j})$ corresponds to an essential set of m points with a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$, then there exist distinct $S, T \subset [m] \times [n]$ such that S and T are similar and $M_S = M_T$. So $F((m_{i,j})) = 0$. As a result, the points of $\mathbb{R}^{m(n-1)}$ corresponding to an essential set of m points with a unique Fermat-Weber point in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$ are contained in the union of proper linear subspaces $V(F)$. $\square$

The immediate consequence of Theorem 6.12 is as follows:

Corollary 6.18. *If we choose a random sample in the moduli space $\mathbb{R}^{m(n-1)}$ with any distribution ν with $\nu(L) = 0$ for any $L \subset \mathbb{R}^{m(n-1)}$ with the dimension of L is strictly less than $m(n-1)$, then we have probability 1 to get either a random sample that is not essential, or a random sample that has more than one (thus infinitely many) Fermat-Weber points.*

Proof. Let C be the random sample in $\mathbb{R}^{m(n-1)}$ that corresponds to an essential set of m points with a unique Fermat-Weber point. Then it suffices to show that the measure of C is zero. By Theorem 6.12, C is contained in the finite union of hypersurfaces $V(X_S - X_T)$, where $S, T \subset [m] \times [n]$, S and T are distinct and similar. Then for each pair of such S and T , the hypersurface $V(X_S - X_T)$ is isomorphic to $\mathbb{R}^{m(n-1)-1}$. So it has measure zero. Thus, the measure of this finite union is still zero, so is C . $\square$

Definition 6.19 (Tropical Determinant). *Let $X = (x_{ij})$ be an n by n matrix with real entries. Then its tropical determinant is defined as follows:*

$$\text{trop det } X = \min_{\pi \in S_n} \sum_{i=1}^n x_{i\pi(i)}. \quad (6.29)$$

A matrix X is tropically singular if the minimum is attained at least twice in (6.29).

In the proof of Lemma 6.17, the subsets S and T are very similar to the terms in the tropical determinant of matrices. However the following example shows that the matrix does not need to have a minor whose tropical determinant contains two equal terms.

Example 6.20. *[No equal terms in the tropical determinant of all minors] The following five points in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$*

$$(1, -1, -1), (-1, 1, -1), (1, 1, -1), (0, -1, 1), (-1, 0, 1)$$

form an essential set and they have a unique Fermat-Weber point $(0, 0, 0)$ in $\mathbb{R}^3/\mathbb{R}\mathbf{1}$. However, let M be the corresponding 5×3 matrix. No minor of M is tropically singular. In addition, for every minor of M , its tropical determinant has no equal terms.

Remark 6.21. *The converse of Theorem 6.12 is not true in general. Consider following three points in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$*

$$(0, 0, 0, 5), (0, 0, 3, 1), (0, 4, 5, 7)$$

This corresponds to a point in the finite union of proper linear subspaces of $\mathbb{R}^9$ as in Lemma 6.17, because we can take $S = \{(1, 1), (2, 2)\}$ and $T = \{(1, 2), (2, 1)\}$. However, their polytope of Fermat-Weber points is a line segment in $\mathbb{R}^4/\mathbb{R}\mathbf{1}$ with endpoints

$$(0, 2, 3, 5), (0, 3, 3, 5).$$

So these three points form an essential set and they have more than one Fermat-Weber point.

6.4 Fermat-Weber points within tree spaces

In this section we focus on the space of phylogenetic trees. An *equidistant tree* is a weighted rooted phylogenetic tree whose distance from the root to each leaf is the same real number for all its leaves. Suppose $\mathcal{U}_N$ is the space of all equidistant trees with N leaves, i.e., the set of leaves is $\{1, 2, \dots, N\}$. For positive integer N , we denote by $[N]$ the set $\{1, 2, \dots, N\}$.

Definition 6.22. The distance $d(i, j)(T)$, between two leaves i and j in $T \in \mathcal{U}_N$, is the length of a unique path between leaves i and j . The distance matrix of $T \in \mathcal{U}_N$ is a $N \times N$ matrix $D(T) = (d_{ij})_{1 \leq i, j \leq N}$ such that

$$d_{ij} = d(i, j)(T) \quad \forall i, j (1 \leq i, j \leq N),$$

where N is the number of leaves in the tree T . The metric of $T \in \mathcal{U}_N$, denoted by $D = (d_{ij})_{1 \leq i < j \leq N}$, is a vector with $\binom{n}{2}$ entries.

Distance matrices of equidistant trees in $\mathcal{U}_N$ satisfy the following strengthening of the triangle inequalities:

$$d_{ik} \leq \max(d_{ij}, d_{jk}) \quad \text{for all } i, j, k \in [N]. \quad (6.30)$$

If (6.30) holds, then the metric D is called an *ultrametric*. The set of all ultrametries contains the ray $\mathbb{R}_{\geq 0}\mathbf{1}$ spanned by the all-one metric $\mathbf{1}$, defined by $d_{ij} = 1$ for $1 \leq i < j \leq N$. The image of the set of ultrametries in the quotient space $\mathbb{R}^{\binom{N}{2}}/\mathbb{R}\mathbf{1}$ is called the *space of ultrametries*.

Suppose we have a set of equidistant phylogenetic trees with N leaves. They are represented by their metrics D in $\mathbb{R}^{\binom{N}{2}}$ so that the space of equidistant phylogenetic trees $\mathcal{U}_N$ with fixed number of leaves N can be represented by a union of polyhedral cones in $\mathbb{R}^{\binom{N}{2}}$. In the previous sections, we have shown that there might be infinitely many Fermat-Weber points of them. However, many of those points may not correspond to any phylogenetic tree. In this section, for a sample of points in $\mathcal{U}_N$, we consider the set of their Fermat-Weber points within $\mathcal{U}_N$.

The spaces of equidistant phylogenetic trees $\mathcal{U}_N$ with N leaves have $(2N - 3)!!$ maximal cones with dimension $N - 2$ [21, 98]. Note that every maximal cone is a polyhedron, so the intersection of each maximal cone and the polytope of Fermat-Weber points is either empty or a polytope. Here we investigate the set of equidistant phylogenetic trees such that they form an essential set and there exists a unique equidistant phylogenetic tree that is a Fermat-Weber point of them.

We conducted simulations on Fermat-Weber points of a sample in $\mathcal{U}_N$ for $N = 4$. We generated 60 samples of equidistant phylogenetic trees with $N = 4$ leaves using R package `ape` [91]. Due to the computational time, we set 60 samples. Among these 60 samples, we chose subsamples of sizes 4, 5 and 6. For each subsample, we computed its Fermat-Weber

points within tree spaces by using **Maple** 2015 [19]. We counted the maximal dimension of the set of Fermat-Weber points, which is a finite union of classical convex polytopes in $\mathbb{R}^5$ by Proposition 6.7. The result is shown in Table 6.1.

Sample size \ Max Dim.	0	1	2
4	2	7	51
5	6	15	39
6	10	21	29

Table 6.1: The maximal dimension of the set of Fermat-Weber points within the treespace: 60 samples of 4 or 5 or 6 phylogenetic trees with 4 leaves.

Example 6.23. *The polytope of Fermat-Weber points of the following four trees with 4 leaves*

$$(32/109, 1, 124/673, 1, 32/109, 1), (1, 6/85, 1, 1, 203/445, 1), \\ (1, 1, 1, 310/783, 310/783, 1/265), (47/510, 1, 1, 1, 1, 125/151).$$

is 2-dimensional, while there is a unique Fermat-Weber point that corresponds a phylogenetic tree, which is $(1, 1, 1, 1, 1, 1)$.

We have the following conjecture based on our simulations.

Conjecture 6.24. *A sample in $\mathcal{U}_N$ like in Example 6.23 is the only case of a unique tree as Fermat-Weber point. In other words, if a sample in $\mathcal{U}_N$ has a unique Fermat-Weber point, then its unique Fermat-Weber point is the all-one vector $\mathbf{1}$.*

Remark 6.25. *We have tried to conduct a similar experiment for $N \geq 5$ but the computational time was not feasible. The computational bottleneck for our simulation study does not come from the number of cones in the treespace, but comes from the difficulty of computing the polytope of Fermat-Weber points. See Section 6.6 for details.*

6.5 The m -ellipses under the tropical metric

Let m be a positive integer. Given a sample of m points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$, the locus of points $\mathbf{u} \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$ such that

$$\sum_{i=1}^m d_{tr}(\mathbf{u}, \mathbf{v}_i) = d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m) \quad (6.31)$$

is the polytope of Fermat-Weber points of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$. In this section we generalize this locus and discuss the k -ellipses under the tropical metric.

Definition 6.26. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$ and $a \geq d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$. Then the m -ellipse with foci $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ and mean radius $\frac{a}{m}$ is the follow set of points in $\mathbb{R}^n / \mathbb{R}\mathbf{1}$:

$$\{\mathbf{u} \in \mathbb{R}^n / \mathbb{R}\mathbf{1} \mid \sum_{i=1}^m d_{tr}(\mathbf{u}, \mathbf{v}_i) = a\}. \quad (6.32)$$

Proposition 6.27. Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$ and $a \geq d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$. Then the m -ellipse with foci $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ and mean radius $\frac{a}{m}$ is a classical convex polytope in $\mathbb{R}^{n-1} \simeq \mathbb{R}^n / \mathbb{R}\mathbf{1}$.

Proof. The proof is very similar to that of Proposition 6.7. We can still eliminate the parameters c_i , and now the inequalities in (6.14) become

$$\sum_{i=1}^m (x_{j_i} - x_{k_i} + v_{i,k_i} - v_{i,j_i}) \leq a. \quad (6.33)$$

So this m -ellipse is also a polyhedron in $\mathbb{R}^{n-1}$ and for the same reason it is bounded. $\square$

Example 6.28. We consider Example 6.8 again. Let $\mathbf{v}_1 = (0, 0, 0)$, $\mathbf{v}_2 = (0, 3, 1)$, $\mathbf{v}_3 = (0, 2, 5)$. Then by Theorem 6.3, we have $d(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = (0 + 3 + 5) - (0 + 1 + 0) = 7$. We consider $a = 8, 10, 50, 100$. Figure 6.2 - 6.5 show the 3-ellipses with foci $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and mean radius $\frac{a}{3}$.

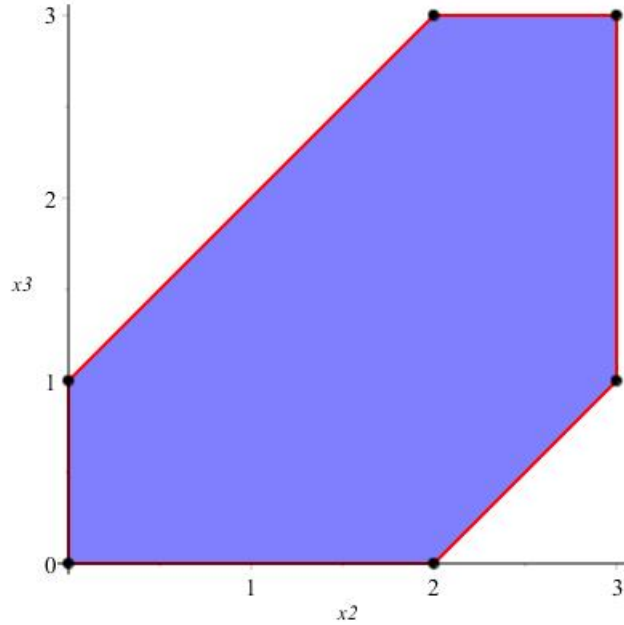


Figure 6.2: The 3-ellipse with $a = 8$. It is a hexagon.

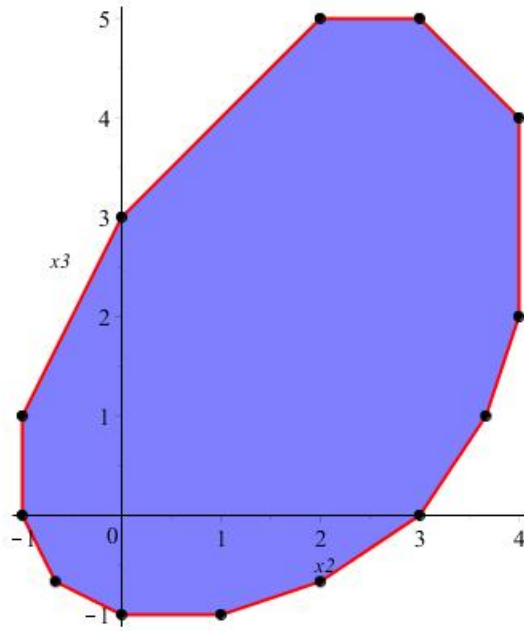


Figure 6.3: The 3-ellipse with $a = 10$. It is a 13-gon.

6.6 Computing the Fermat-Weber points

In this section we explain our method of computing the set of all Fermat-Weber points of a sample and discuss some computational issues. The Maple 2015 codes are available at

<https://sites.google.com/site/bolinmath/research/codes/fermat>.

Suppose the points in a sample are $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n / \mathbb{R}\mathbf{1}$. Then our method consists of two steps:

- (a) compute $d = d(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$;
- (b) given d , compute the set of Fermat-Weber points of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$.

One way to perform step (a) is to use Theorem 6.3. But then we have to compute all possible functions $\sigma, \tau : [m] \rightarrow [n]$ such that $\sigma([m]) = \tau([m])$ as multisets. The number of such functions is n^m , which means the time complexity of this step would be exponential in m . In practice we use a method of linear programming, which minimizes d such that all inequalities in (6.6) and $d \geq \sum_{i=1}^m c_i$ are feasible simultaneously.

However, for step (b), even if we have d , there are still many inequalities that define the polytope of Fermat-Weber points. In the proof of Proposition 6.7, if we eliminated the parameters c_i , then there are $\binom{n}{2}^m$ inequalities in (6.14); otherwise we may keep the parameters c_i and get another polytope in the ambient space $\mathbb{R}^{n+m-1}$ and then project it to $\mathbb{R}^{n-1}$, but for each c_i there are still $2\binom{n}{2} = n(n-1)$ inequalities, so we need $mn(n-1) + 1$

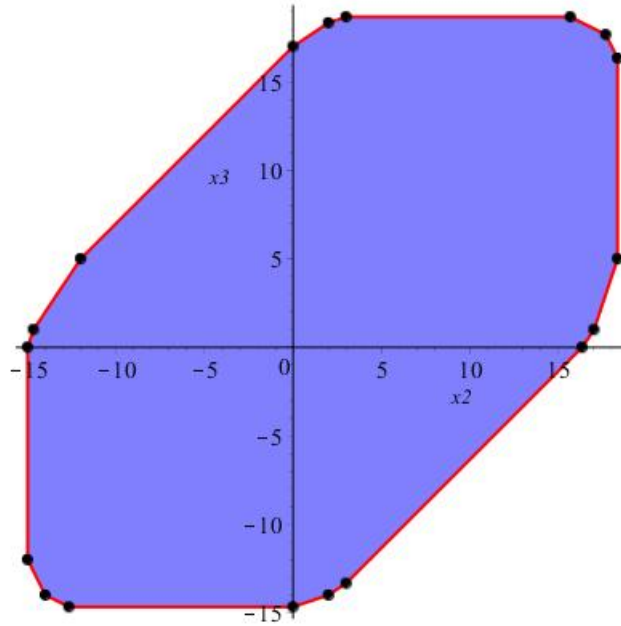


Figure 6.4: The 3-ellipse with $a = 50$. It is a 18-gon.

inequalities to define this polytope. From our computations with `polymake` [61], there seem to be some redundant inequalities but we do not know an efficient method for step (b).

Question 6.29. *What is the minimal set of inequalities needed to define the set of all tropical Fermat-Weber points of a given sample? What is the time complexity to compute the set of tropical Fermat-Weber points of a sample of m points in $\mathbb{R}^n/\mathbb{R}\mathbf{1}$?*

As costs of genome sequencing have dropped precipitously, development of bioinformatic methods to analyze genome structure and evolution have become ever more urgent.

There are several statistical methods to summarize genome data, such as the majority-rules consensus tree as well as the Fréchet mean over tree spaces. A Fermat-Weber point on tree spaces is another descriptive statistics of the given set of gene trees.

As we have discussed above, it is very hard to compute the set of all tropical Fermat-Weber points over tree spaces because of its computational time. At this moment, we can compute the Fermat-Weber points on tree spaces at most 4 leaves. As future research projects, it will be interesting to compare the set of all tropical Fermat-Weber points with summary/consensus trees, such as the majority-rules consensus tree as well as the Fréchet mean over tree spaces.

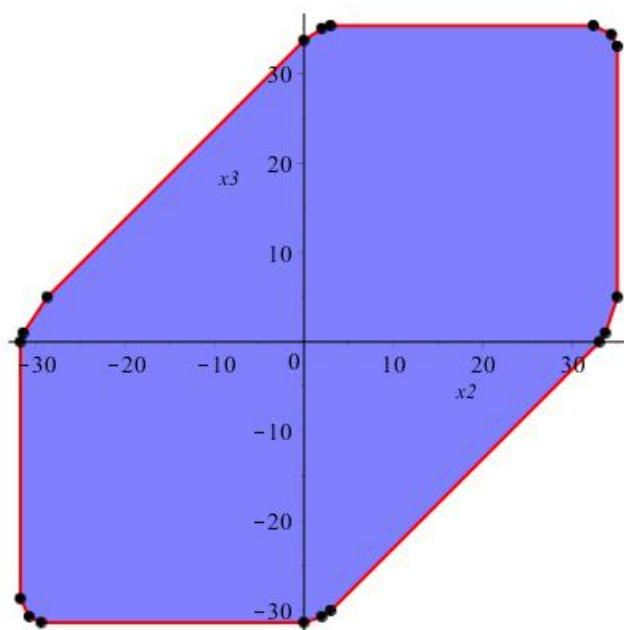


Figure 6.5: The 3-ellipse with $a = 100$. It is a 18-gon.

Bibliography

- [1] Dan Abramovich, Lucia Caporaso, and Sam Payne. The tropicalization of the moduli space of curves. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):765–809, 2015.
- [2] Marianne Akian, Stéphane Gaubert, Viorel Nițică, and Ivan Singer. Best approximation in max-plus semimodules. *Linear Algebra Appl.*, 435(12):3261–3296, 2011.
- [3] Omid Amini and Matthew Baker. Linear series on metrized complexes of algebraic curves. *Math. Ann.*, 362(1-2):55–106, 2015.
- [4] Omid Amini and Lucia Caporaso. Riemann-Roch theory for weighted graphs and tropical curves. *Adv. Math.*, 240:1–23, 2013.
- [5] Alberto Apostolico, Matteo Comin, Andres Dress, and Laxmi Parida. Ultrametric networks: a new tool for phylogenetic analysis. *Algorithms for molecular biology*, 8(1):7, 2013.
- [6] Federico Ardila. Subdominant matroid ultrametrics. *Ann. Comb.*, 8(4):379–389, 2004.
- [7] Federico Ardila, Tia Baker, and Rika Yatchak. Moving robots efficiently using the combinatorics of CAT(0) cubical complexes. *SIAM J. Discrete Math.*, 28(2):986–1007, 2014.
- [8] Federico Ardila and Caroline J. Klivans. The Bergman complex of a matroid and phylogenetic trees. *J. Combin. Theory Ser. B*, 96(1):38–49, 2006.
- [9] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Moeller. Compactification of strata of abelian differentials. *arXiv:1604.08834*, 2016.
- [10] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008. With an appendix by Brian Conrad.
- [11] Matthew Baker. Specialization of linear systems from curves to graphs. *Algebra Number Theory*, 2(6):613–653, 2008. With an appendix by Brian Conrad.
- [12] Matthew Baker and Xander Faber. Metrized graphs, Laplacian operators, and electrical networks. In *Quantum graphs and their applications*, volume 415 of *Contemp. Math.*, pages 15–33. Amer. Math. Soc., Providence, RI, 2006.

- [13] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math.*, 215(2):766–788, 2007.
- [14] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.
- [15] Dennis Barden, Huiling Le, and Megan Owen. Limiting behaviour of Fréchet means in the space of phylogenetic trees. *Annals of the Institute of Statistical Mathematics*, pages 1–31, 2014.
- [16] Miroslav Bačák. Computing medians and means in Hadamard spaces. *SIAM J. Optim.*, 24(3):1542–1566, 2014.
- [17] Marcel Berger. *A panoramic view of Riemannian geometry*. Springer-Verlag, Berlin, 2003.
- [18] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.
- [19] L. Bernardin, P. Chin, P. DeMarco, K. O. Geddes, D. E. G. Hare, K. M. Heal, G. Labahn, J. P. May, J. McCarron, M. B. Monagan, D. Ohashi, and S. M. Vorkoetter. *Maple 2015 Programming Guide*. Maplesoft, Waterloo ON, Canada, 2015.
- [20] Ricardo Betancur-R, Chenhong Li, Thomas A Munroe, Jesus A Ballesteros, and Guillermo Ortí. Addressing gene tree discordance and non-stationarity to resolve a multi-locus phylogeny of the flatfishes (teleostei: Pleuronectiformes). *Systematic biology*, page syt039, 2013.
- [21] Louis J. Billera, Susan P. Holmes, and Karen Vogtmann. Geometry of the space of phylogenetic trees. *Adv. in Appl. Math.*, 27(4):733–767, 2001.
- [22] Anders Björner, László Lovász, and Peter W. Shor. Chip-firing games on graphs. *European J. Combin.*, 12(4):283–291, 1991.
- [23] Brian H. Bowditch. Some results on the geometry of convex hulls in manifolds of pinched negative curvature. *Comment. Math. Helv.*, 69(1):49–81, 1994.
- [24] Silvia Brannetti, Margarida Melo, and Filippo Viviani. On the tropical Torelli map. *Adv. Math.*, 226(3):2546–2586, 2011.
- [25] Sarah Brodsky, Michael Joswig, Ralph Morrison, and Bernd Sturmfels. Moduli of tropical plane curves. *Res. Math. Sci.*, 2:Art. 4, 31, 2015.
- [26] Laurent Busé, David Cox, and Carlos D’Andrea. Implicitization of surfaces in $\mathbb{P}^3$ in the presence of base points. *J. Algebra Appl.*, 2(2):189–214, 2003.

- [27] Lucia Caporaso. Algebraic and tropical curves: comparing their moduli spaces. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 119–160. Int. Press, Somerville, MA, 2013.
- [28] Lucia Caporaso. Tropical methods in the moduli theory of algebraic curves. *arXiv:1606.00323*, 2016.
- [29] Lucia Caporaso and Filippo Viviani. Torelli theorem for graphs and tropical curves. *Duke Math. J.*, 153(1):129–171, 2010.
- [30] Matt D. Carling and Robb T. Brumfield. Integrating phylogenetic and population genetic analyses of multiple loci to test species divergence hypotheses in passerina buntings. *Genetics*, 178(1):363–377, 2008.
- [31] Renzo Cavalieri, Simon Hampe, Hannah Markwig, and Dhruv Ranganathan. Moduli spaces of rational weighted stable curves and tropical geometry. *Forum Math. Sigma*, 4:e9, 35, 2016.
- [32] Renzo Cavalieri, Hannah Markwig, and Dhruv Ranganathan. Tropicalizing the space of admissible covers. *Math. Ann.*, 364(3-4):1275–1313, 2016.
- [33] Gulbank D. Chakerian and Mostafa Ghandehari. The Fermat problem in Minkowski spaces. *Geom. Dedicata*, 17(3):227–238, 1985.
- [34] Melody Chan. Combinatorics of the tropical Torelli map. *Algebra Number Theory*, 6(6):1133–1169, 2012.
- [35] Melody Chan. Topology of the tropical moduli spaces $M_{2,n}$. *arXiv:1507.03878*, 2015.
- [36] Melody Chan, Soren Galatius, and Sam Payne. The tropicalization of the moduli space of curves ii: Topology and applications. *arXiv:1604.03176*, 2016.
- [37] Melody Chan, Margarida Melo, and Filippo Viviani. Tropical Teichmüller and Siegel spaces. In *Algebraic and combinatorial aspects of tropical geometry*, volume 589 of *Contemp. Math.*, pages 45–85. Amer. Math. Soc., Providence, RI, 2013.
- [38] Victor Chepoi and Bernard Fichet. l_∞ -approximation via subdominants. *J. Math. Psych.*, 44(4):600–616, 2000.
- [39] Dietmar Cieslik. The Fermat-Steiner-Weber-problem in Minkowski spaces. *Optimization*, 19(4):485–489, 1988.
- [40] Dietmar Cieslik. *Shortest connectivity*, volume 17 of *Combinatorial Optimization*. Springer-Verlag, New York, 2005. An introduction with applications in phylogeny.

- [41] Guy Cohen, Stéphane Gaubert, and Jean-Pierre Quadrat. Duality and separation theorems in idempotent semimodules. *Linear Algebra Appl.*, 379:395–422, 2004. Tenth Conference of the International Linear Algebra Society.
- [42] David Cox. Equations of parametric curves and surfaces via syzygies. In *Symbolic computation: solving equations in algebra, geometry, and engineering (South Hadley, MA, 2000)*, volume 286 of *Contemp. Math.*, pages 1–20. Amer. Math. Soc., Providence, RI, 2001.
- [43] David Cox. What is a toric variety? *Contemporary Mathematics*, 334:203–224, 2003.
- [44] David Cox, John Little, and Donal O’Shea. *Using algebraic geometry*, volume 185 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2005.
- [45] David Cox, John Little, and Donal O’Shea. *Ideals, varieties, and algorithms*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007. An introduction to computational algebraic geometry and commutative algebra.
- [46] David Cox, John Little, and Henry Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [47] Maria Angelica Cueto. *Tropical implicitization*. PhD thesis, University of California, Berkeley, 2010.
- [48] Mike Develin and Bernd Sturmfels. Tropical convexity. *Doc. Math.*, 9:1–27, 2004.
- [49] Roland Durier. The Fermat-Weber problem and inner product spaces. *J. Approx. Theory*, 78(2):161–173, 1994.
- [50] Roland Durier and Christian Michelot. Geometrical properties of the Fermat-Weber problem. *European J. Oper. Res.*, 20(3):332–343, 1985.
- [51] Roland Durier and Christian Michelot. On the set of optimal points to the Weber problem: further results. *Transportation Sci.*, 28(2):141–149, 1994.
- [52] David Eisenbud and Joe Harris. Limit linear series: basic theory. *Invent. Math.*, 85(2):337–371, 1986.
- [53] Eva Maria Feichtner. Complexes of trees and nested set complexes. *Pacific J. Math.*, 227(2):271–286, 2006.
- [54] Aasa Feragen, Megan Owen, Jens Petersen, Mathilde M. W. Wille, Laura H. Thomsen, Asger Dirksen, and Marleen de Bruijne. *Tree-Space Statistics and Approximations for Large-Scale Analysis of Anatomical Trees*, pages 74–85. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

- [55] Alex Fink and Felipe Rincón. Stiefel tropical linear spaces. *J. Combin. Theory Ser. A*, 135:291–331, 2015.
- [56] Thomas Fletcher, John Moeller, Jeff Phillips, and Suresh Venkatasubramanian. Computing hulls, centerpoints, and vc-dimension in positive definite space. In *Algorithms and Data Structures Symposium, New York; original version at arXiv:0912.1580*, 2011.
- [57] Andreas Gathmann and Michael Kerber. A Riemann-Roch theorem in tropical geometry. *Math. Z.*, 259(1):217–230, 2008.
- [58] Andreas Gathmann, Michael Kerber, and Hannah Markwig. Tropical fans and the moduli spaces of tropical curves. *Compos. Math.*, 145(1):173–195, 2009.
- [59] Andreas Gathmann and Hannah Markwig. Kontsevich’s formula and the WDVV equations in tropical geometry. *Adv. Math.*, 217(2):537–560, 2008.
- [60] Alex Gavryushkin and Alexei J. Drummond. The space of ultrametric phylogenetic trees. *J. Theoret. Biol.*, 403:197–208, 2016.
- [61] Ewgenij Gawrilow and Michael Joswig. polymake: a framework for analyzing convex polytopes. In *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, volume 29 of *DMV Sem.*, pages 43–73. Birkhäuser, Basel, 2000.
- [62] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.
- [63] Mark Gross. *Tropical geometry and mirror symmetry*, volume 114 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2011.
- [64] Christian Haase, Gregg Musiker, and Josephine Yu. Linear systems on tropical curves. *Math. Z.*, 270(3-4):1111–1140, 2012.
- [65] Philip Hall. On representatives of subsets. *J. London Math. Soc.*, 10(1):26–30, 1935.
- [66] Joseph Heled and Alexei J. Drummond. Bayesian inference of species trees from multilocus data. *Molecular biology and evolution*, 27(3):570–580, 2010.
- [67] Thomas Hotz, Stephan Huckemann, Huiling Le, James Stephen Marron, Jonathan C. Mattingly, Ezra Miller, James Nolen, Megan Owen, Vic Patrangenaru, and Sean Skwerer. Sticky central limit theorems on open books. *Ann. Appl. Probab.*, 23(6):2238–2258, 2013.
- [68] Nathan Owen Ilten and Hendrik Süß. Polarized complexity-1 T -varieties. *Michigan Math. J.*, 60(3):561–578, 2011.

- [69] Michael Joswig, Bernd Sturmfels, and Josephine Yu. Affine buildings and tropical convexity. *Albanian J. Math.*, 1(4):187–211, 2007.
- [70] Thomas Jukes, Charles Cantor, et al. Evolution of protein molecules. *Mammalian protein metabolism*, 3(21):132, 1969.
- [71] Yaakov Shimeon Kupitz and Horst Martini. Geometric aspects of the generalized Fermat-Torricelli problem. In *Intuitive geometry (Budapest, 1995)*, volume 6 of *Bolyai Soc. Math. Stud.*, pages 55–127. János Bolyai Math. Soc., Budapest, 1997.
- [72] Bo Lin. Almost-toric hypersurfaces. *Beitr. Algebra Geom.*, 57(2):343–359, 2016.
- [73] Bo Lin. Computing linear systems on metric graphs. *arXiv:1603.00547*, 2016.
- [74] Bo Lin, Bernd Sturmfels, Xiaoxian Tang, and Ruriko Yoshida. Convexity in tree spaces, 2015.
- [75] Bo Lin and Martin Ulirsch. Towards a tropical hodge bundle. In *Combinatorial Algebraic Geometry, Selected papers from the 2016 Apprenticeship Program, Fields Institute*. Springer, 2017. To appear.
- [76] Bo Lin and Ruriko Yoshida. Tropical Fermat-Weber points. *arXiv:1604.04674*, 2016.
- [77] Anna Lubiw, Daniela Maftuleac, and Megan Owen. Shortest paths and convex hulls in 2d complexes with non-positive curvature. *arXiv:1603.00847*, 2016.
- [78] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.
- [79] Horst Martini, Konrad J. Swanepoel, and Gunter Weiss. The Fermat-Torricelli problem in normed planes and spaces. *J. Optim. Theory Appl.*, 115(2):283–314, 2002.
- [80] Grigory Mikhalkin. Tropical geometry and its applications. In *International Congress of Mathematicians. Vol. II*, pages 827–852. Eur. Math. Soc., Zürich, 2006.
- [81] Grigory Mikhalkin. Moduli spaces of rational tropical curves. In *Proceedings of Gökova Geometry-Topology Conference 2006*, pages 39–51. Gökova Geometry/Topology Conference (GGT), Gökova, 2007.
- [82] Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 203–230. Amer. Math. Soc., Providence, RI, 2008.
- [83] Ezra Miller. Fruit flies and moduli: interactions between biology and mathematics. *Notices Amer. Math. Soc.*, 62(10):1178–1184, 2015.

- [84] Ezra Miller, Megan Owen, and J. Scott Provan. Polyhedral computational geometry for averaging metric phylogenetic trees. *Adv. in Appl. Math.*, 68:51–91, 2015.
- [85] Frank Nielsen and Rajendra Bhatia, editors. *Matrix information geometry*. Springer, Heidelberg, 2013.
- [86] Tom M. W. Nye. Principal components analysis in the space of phylogenetic trees. *Ann. Statist.*, 39(5):2716–2739, 2011.
- [87] Tom M. W. Nye. An algorithm for constructing principal geodesics in phylogenetic treespace. *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, 11(2):304–315, 2014.
- [88] Brian Osserman. Limit linear series for curves not of compact type. *arXiv:1406.6699*, 2014. To appear in Journal für die reine und angewandte Mathematik (Crelle’s journal).
- [89] Megan Owen and J. Scott Provan. A fast algorithm for computing geodesic distances in tree space. *IEEE/ACM Transactions on Computational Biology and Bioinformatics (TCBB)*, 8(1):2–13, 2011.
- [90] Lior Pachter and Bernd Sturmfels, editors. *Algebraic statistics for computational biology*. Cambridge University Press, New York, 2005.
- [91] Emmanuel Paradis, Julien Claude, and Korbinian Strimmer. Ape: analyses of phylogenetics and evolution in R language. *Bioinformatics*, 20(2):289–290, 2004.
- [92] Patrice Philippon and Martín Sombra. A refinement of the Bernštein-Kušnirenko estimate. *Adv. Math.*, 218(5):1370–1418, 2008.
- [93] Jean-Eric Pin. Tropical semirings. In *Idempotency (Bristol, 1994)*, volume 11 of *Publ. Newton Inst.*, pages 50–69. Cambridge Univ. Press, Cambridge, 1998.
- [94] Dhruv Ranganathan. Skeletons of stable maps i: Rational curves in toric varieties. *arXiv:1506.03754*, 2015. To appear in Journal of the London Mathematical Society.
- [95] Sebastien Roch and Mike Steel. Likelihood-based tree reconstruction on a concatenation of aligned sequence data sets can be statistically inconsistent. *Theoretical population biology*, 100:56–62, 2015.
- [96] Charles Semple and Mike Steel. *Phylogenetics*, volume 24 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2003.
- [97] Imre Simon. Recognizable sets with multiplicities in the tropical semiring. In *Mathematical foundations of computer science, 1988 (Carlsbad, 1988)*, volume 324 of *Lecture Notes in Comput. Sci.*, pages 107–120. Springer, Berlin, 1988.

- [98] David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Adv. Geom.*, 4(3):389–411, 2004.
- [99] David Speyer and Bernd Sturmfels. Tropical mathematics. *Math. Mag.*, 82(3):163–173, 2009.
- [100] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1996.
- [101] Bernd Sturmfels. *Solving systems of polynomial equations*, volume 97 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2002.
- [102] Bernd Sturmfels and Jenia Tevelev. Elimination theory for tropical varieties. *Math. Res. Lett.*, 15(3):543–562, 2008.
- [103] Bernd Sturmfels, Jenia Tevelev, and Josephine Yu. The Newton polytope of the implicit equation. *Mosc. Math. J.*, 7(2):327–346, 2007.
- [104] Bernd Sturmfels and Josephine Yu. Tropical implicitization and mixed fiber polytopes. In *Software for algebraic geometry*, pages 111–131. Springer, 2008.
- [105] Katherine L. Thompson and Laura S. Kubatko. Using ancestral information to detect and localize quantitative trait loci in genome-wide association studies. *BMC bioinformatics*, 14(1):200, 2013.
- [106] Martin Ulirsch. Functorial tropicalization of logarithmic schemes: The case of constant coefficients. *arXiv:1310.6269*, 2013. To appear in Proceedings of the London Mathematical Society.
- [107] Martin Ulirsch. Tropical geometry of moduli spaces of weighted stable curves. *J. Lond. Math. Soc. (2)*, 92(2):427–450, 2015.
- [108] Ravi Vakil. The moduli space of curves and its tautological ring. *Notices Amer. Math. Soc.*, 50(6):647–658, 2003.
- [109] Filippo Viviani. Tropicalizing vs. compactifying the Torelli morphism. In *Tropical and non-Archimedean geometry*, volume 605 of *Contemp. Math.*, pages 181–210. Amer. Math. Soc., Providence, RI, 2013.
- [110] Tony Yue Yu. Tropicalization of the moduli space of stable maps. *Math. Z.*, 281(3-4):1035–1059, 2015.
- [111] Yun Yu, Tandy Warnow, and Luay Nakhleh. Algorithms for mdc-based multi-locus phylogeny inference: beyond rooted binary gene trees on single alleles. *Journal of Computational Biology*, 18(11):1543–1559, 2011.

- [112] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.