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# GLOBAL STABILIZATION OF THE NAVIER-STOKES-VOIGHT AND THE DAMPED NONLINEAR WAVE EQUATIONS BY FINITE NUMBER OF FEEDBACK CONTROLLERS

VARGA K. KALANTAROV AND EDRISS S. TITI

**ABSTRACT.** In this paper we introduce a finite-parameters feedback control algorithm for stabilizing solutions of the Navier-Stokes-Voigt equations, the strongly damped nonlinear wave equations and the nonlinear wave equation with nonlinear damping term, the Benjamin-Bona-Mahony-Burgers equation and the KdV-Burgers equation. This algorithm capitalizes on the fact that such infinite-dimensional dissipative dynamical systems possess finite-dimensional long-time behavior which is represented by, for instance, the finitely many determining parameters of their long-time dynamics, such as determining Fourier modes, determining volume elements, determining nodes, etc.. The algorithm utilizes these finite parameters in the form of feedback control to stabilize the relevant solutions. For the sake of clarity, and in order to fix ideas, we focus in this work on the case of low Fourier modes feedback controller, however, our results and tools are equally valid for using other feedback controllers employing other spatial coarse mesh interpolants.

*This work is dedicated to the memory of Professor Igor Chueshov.*

## 1. INTRODUCTION

The stabilization problem of nonlinear parabolic equations, such as the Navier-Stokes equations and other related equations of hydrodynamics, the linear and nonlinear wave equations have been intensively investigated by various authors (see, e.g., [4], [5],[9], [10], [16], [19], [25], [29],[38], [42], [44], [50] and references therein). Some of these works have been specifically devoted to the problem of feedback stabilization by controllers depending only on finitely many parameters for nonlinear parabolic equations and related systems, such as the reaction-diffusion equations, the Navier-Stokes equations, the Oseen equations, the phase-field equations, the Kuramoto-Sivashynsky equations, etc... (see, e.g., [1],[3],[6]-[8], [10] [12],[33],[41], [43], [47] and references therein). This large body of work relies, whether explicitly or implicitly, on the fact that the asymptotic in time behavior of such infinite-dimensional dissipative dynamical systems is governed by finitely many degrees of freedom. This fact was first established for the two-dimensional Navier–Stokes equations in the pioneer works of C. Foias and G. Prodi [22], and of O.A. Ladyzhenskaya [36]. Specifically, the authors proved that the long-time behavior of solutions of 2D Navier-Stokes equations and the trajectories in the global attractor of are determined by the dynamics of finitely many, but large enough number, of Fourier modes. This seminal work triggered a subsequent investigation concerning

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the finite-dimensional asymptotic in time behavior of solutions of dissipative nonlinear PDE's (see, e.g., [2], [21]-[23], [27],[31], [37],[48] and references therein). Indeed, it was shown, for instance, that the long-time behavior of solutions of 2D Navier-Stokes equations, reaction-diffusion equations, complex Ginzburg-Landau equations, Kuramoto-Sivashinsky equation, 1D damped semilinear wave equations and a number of other dissipative equations can be determined by finitely many nodes and volume elements, etc... (see, e.g., [17], [18], [24], [21], [28], [31], [34] and references therein). The general concept of determining interpolant operators (determining functionals) was introduced in [17],[18] and which enabled the authors to present a unified approach for investigating these determining parameters. A further extension of the applicability of this unified approach for studying the long-time behavior of various nonlinear dissipative PDE's was developed in, e.g., [18] , [13] - [16] and references therein.

In this paper we study the problem of global stabilization of the solutions of the initial boundary value problems for the 3D Navier-Stokes-Voigt (NSV) equations

$$\partial_t v - \nu \Delta v - \alpha^2 \Delta \partial_t v + (v \cdot \nabla)v + \nabla p = h(x), \quad \nabla \cdot v = 0, \quad x \in \Omega, t \in \mathbb{R}^+, \quad (1.1)$$

and the following damped nonlinear dispersive equations:  
the strongly damped nonlinear wave equation

$$\partial_t^2 u - \Delta u - b \Delta \partial_t u - \lambda u + f(u) = h(x), \quad x \in \Omega, t > 0, \quad (1.2)$$

the nonlinear wave equation with a nonlinear damping term

$$\partial_t^2 u + g(\partial_t u) - \Delta u + f(u) = h(x), \quad x \in \Omega, t > 0, \quad (1.3)$$

the Benjamin-Bona-Mahoni-Burgers (BBMB) equation,

$$\partial_t v - \partial_x^2 \partial_t v + f(v) \partial_x v - \partial_x^2 v = h(x), \quad x \in (0, 1), t > 0 \quad (1.4)$$

under the Dirichlet boundary condition and the Korteweg-de Vries-Burgers (KdVB) equation

$$\partial_t v + \partial_x^3 v + v \partial_x v - \partial_x^2 v = h(x), \quad x \in (0, 1), t > 0, \quad (1.5)$$

under periodic boundary conditions. Here  $\Omega \subset \mathbb{R}^3$  is a bounded domain with sufficiently smooth boundary,  $\alpha$  and  $b$  are given positive parameters,  $h \in L^2$  is a given function,  $f(\cdot), g(\cdot)$  are given nonlinear terms satisfying natural growth conditions guaranteeing the global existence of corresponding initial boundary value problems.

Our main goal is to show that any arbitrary given solution of the initial boundary value problem for each of equations (1.1) -(1.5) can be stabilized by using a feedback controller depending only on finitely many large spatial-scale parameters, such as the low Fourier modes or other finite rank spatial interpolant operators that are based on coarse mesh spatial measurements. A common feature of these equations is that the semigroups generated by initial boundary value problems are asymptotically compact semigroups which have finite-dimensional global attractors in corresponding phase spaces (see e.g. [20], [27], [32]-[35], [46], [48], [49] and references therein). In [1], [33] and [41] the authors introduced a feedback control algorithm based on the above mentioned unified approach employing finitely many parameters for the global stabilization of solutions for a number nonlinear parabolic equations and to the damped nonlinear wave equations. In this work we develop further this algorithm and extend/demonstrate its applicability to a larger class of dissipative PDEs.

However, for the sake of clarity, and in order to fix ideas, we focus in this work on the case of low Fourier modes feedback controller. Nonetheless, it is worth stressing that our results and tools are equally valid when using other feedback controllers that are employing other spatial coarse mesh interpolants and determining functionals. This will be the subject of forthcoming publication.

Throughout this paper we will use the following notations

- $(\cdot, \cdot)$  and  $\|\cdot\|$  will denote the inner product and the norm for both  $L^2(\Omega)$  and  $(L^2(\Omega))^3$ .  
For  $u = (u_1, u_2, u_3), v = (v_1, v_2, v_3) \in (L^2(\Omega))^3$ ,  $(u, v) := \int_{\Omega} \sum_{j=1}^3 u_j v_j dx$ .
- $H$  will denote the closure of  $\mathcal{V} := \{u \in (C_0^\infty(\Omega))^3 : \nabla \cdot u = 0\}$  in  $(L^2(\Omega))^3$ , and  $V$  is the closure of  $\mathcal{V}$  in  $(H_0^1(\Omega))^3$ .
- $\dot{H}_{per}^k(0, 1) := \left\{ \phi \in H_{per}^k(0, 1) : \int_0^1 \phi(x) dx = 0, k = 1, 2, \dots \right\}$ .
- $w_1, w_2, \dots, w_n, \dots$  will denote the eigenfunctions of the Stokes operator and the Laplace operator  $-\Delta(-\frac{d^2}{dx^2})$  under the homogeneous Dirichlet boundary condition and of the operator  $-\frac{d^2}{dx^2}$  in  $\dot{H}_{per}^2(0, 1)$  corresponding to eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n, \dots$ .

In what follows we are using the following inequalities:

- *Young's inequality* For each  $a, b > 0$  and  $\epsilon > 0$

$$ab \leq \epsilon \frac{a^p}{p} + \frac{1}{\epsilon^{q/p}} \frac{b^q}{q}, \quad (1.6)$$

where  $p, q > 0$  and  $\frac{1}{p} + \frac{1}{q} = 1$ .

- *Poincaré-Friedrichs inequality*

$$\|u\|^2 \leq \lambda_1^{-1} \|\nabla u\|^2, \quad \forall u \in H_0^1(\dot{H}_{per}^1, V) \quad (1.7)$$

and the inequality

$$\sum_{k=N+1}^{\infty} |(u, w_k)|^2 \leq \lambda_{N+1}^{-1} \|\nabla u\|^2, \quad \forall u \in H_0^1(\dot{H}_{per}^1, V), \quad (1.8)$$

where  $\lambda_1$  is the first and  $\lambda_{N+1}$  is the  $(N+1)$ th eigenvalue of the operator  $-\Delta$  (or  $-\frac{d^2}{dx^2}$ ) under the Dirichlet or periodic boundary conditions.

- *The 1D Agmon inequality*

$$\max_{x \in [0, 1]} |u(x)|^2 \leq c_0 \|u\| \|u'\|, \quad \forall u \in H_0^1(0, 1)(\dot{H}_{per}^1(0, 1)). \quad (1.9)$$

- *The 3D Ladyzhenskaya inequality*

$$\|u\|_{L^4(\Omega)}^2 \leq b_0 \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{3}{2}}, \quad \forall u \in V, \quad (1.10)$$

where  $\Omega \subseteq \mathbb{R}^3$ .

- *The 1D Gagliardo-Nirenberg inequality*

$$\|u^{(j)}\|_{L^p(0, 1)} \leq \beta \|u\|^{1-\alpha} \|u^{(m)}\|^\alpha, \quad \forall u \in H^2(0, 1) \cap H_0^m(0, 1), (\dot{H}_{per}^m(0, 1)). \quad (1.11)$$

where  $p > 2, m = 1, 2, \frac{j}{m} \leq \alpha \leq 1, \alpha = \left(\frac{1}{2} + j - \frac{1}{p}\right) m^{-1}$ .

2. STABILIZATION EMPLOYING FINITELY MANY FOURIER MODES  
FOR NAVIER-STOKES-VOIGHT EQUATIONS

**2.1. 3D Navier-Stokes-Voigt equations.** In this section we study the problem of global stabilization of 3D Navier-Stokes-Voigt equations by finitely many Fourier modes. Suppose that  $v$  is a given weak solution of the problem

$$\begin{cases} \partial_t v - \nu \Delta v - \alpha^2 \Delta \partial_t v + (v \cdot \nabla)v + \nabla p = h, & x \in \Omega, t \in \mathbb{R}^+, \\ \nabla \cdot v = 0, & x \in \Omega, t \in \mathbb{R}^+; v|_{\partial\Omega} = 0, & t \in \mathbb{R}^+, \\ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (2.1)$$

in a bounded domain  $\Omega \subset \mathbb{R}^3$  with sufficiently smooth boundary  $\partial\Omega$ ,  $v_0 \in V$  is a given initial data and  $h \in L^2(\Omega)$  is a given source term. The equation (2.1) was introduced by A.P.Oskolkov in [45] as a model of motion of linear viscoelastic non-Newtonian fluids. This equations were also proposed in [11] as a regularisation, for small values of the parameter  $\alpha$ , of the 3D Navier-Stokes equations. Here  $v = v(x, t)$  is the velocity vector field,  $p$  is the pressure,  $\nu > 0$  is the kinematic viscosity, and  $h$  is a given force field. The positive parameter  $\alpha$  is characterizing the elasticity of the fluid in the sense that  $\frac{\alpha^2}{\nu}$  is a characteristic relaxation time scale of the viscoelastic material. Our aim is to stabilize any strong solution  $v$ . That is for any initial data  $u_0$  we will show that the solution  $u$  of the feedback control system converges to function  $v$  as  $t \rightarrow \infty$ , provided  $N$  is large enough depending on physical parameters of the equation (2.1). That is we will show that each solution of the system, for  $N$  large enough,

$$\begin{cases} \partial_t u - \Delta(\nu u + \alpha^2 \partial_t u) + (u \cdot \nabla)u + \nabla \tilde{p} = -\mu \sum_{k=1}^N (u - v, w_k) w_k + h, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t \in \mathbb{R}^+; u|_{\partial\Omega} = 0, & t \in \mathbb{R}^+, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (2.2)$$

where  $u_0 \in V$  is given, will tend to solution of the problem (2.1) in  $V$  norm as  $t \rightarrow \infty$ . Existence and uniqueness of solution  $u \in C(\mathbb{R}^+, V)$  can be shown exactly the same way as it is done for the problem (2.1). It is well known that the problem (2.1) generates a continuous bounded dissipative semigroup  $S(t) : V \rightarrow V$  with an absorbing ball  $B_0(r_0) \subset V$  (see e.g. [32], [33].) In fact taking inner product of (2.1) in  $L^2(\Omega)$  with  $v$  we get

$$\frac{1}{2} \frac{d}{dt} [\|v(t)\|^2 + \alpha^2 \|\nabla v(t)\|^2] + \nu \|\nabla v(t)\|^2 = (h, v(t)). \quad (2.3)$$

Employing Young's inequality (1.6) and the Poincaré-Friedrichs inequality (1.7) we obtain

$$\frac{d}{dt} [\|v(t)\|^2 + \alpha^2 \|\nabla v(t)\|^2] + \nu \|\nabla v(t)\|^2 \leq \frac{1}{\nu \lambda_1} \|h\|^2,$$

which implies that

$$\frac{d}{dt} [\|v(t)\|^2 + \alpha^2 \|\nabla v(t)\|^2] + d_0 [\|v(t)\|^2 + \alpha \|\nabla v(t)\|^2] \leq \frac{1}{\nu \lambda_1} \|h\|^2,$$

where  $d_0 = \frac{\nu}{2} \min\{\lambda_1, \frac{1}{\alpha^2}\}$ . From this inequality we obtain that

$$\|v(t)\|^2 + \alpha^2 \|\nabla v(t)\|^2 \leq e^{-d_0 t} [\|v_0\|^2 + \alpha \|\nabla v_0\|^2] + \frac{1}{\nu \lambda_1 d_0} \|h\|^2.$$

That is each solution  $u \in C(\mathbb{R}^+; H_0^1(\Omega))$  of the problem the norm  $\|\nabla v(t)\|$  is uniformly bounded on  $\mathbb{R}^+$ , moreover there exists  $t_0 > 0$  such that

$$\|v(t)\|^2 + \alpha^2 \|\nabla v(t)\|^2 \leq \frac{2}{\nu \lambda_1 d_0} \|h\|^2, \quad t \geq t_0. \quad (2.4)$$

So the ball  $B_0(r_0) \subset V$  with the radius

$$r_0 = \frac{2}{\nu \lambda_1 d_0 \alpha^2} \|h\|^2 \quad (2.5)$$

is an absorbing ball for the problem (2.1).

By using standard Galerkin method it can be shown that system (2.2) has also a global strong solution. Moreover it has an absorbing ball in  $V$ . Really, multiplying (2.2) by  $u$  in  $L^2(\Omega)$  we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2] + \nu \|\nabla u(t)\|^2 = \\ - \mu \sum_{k=1}^N |(u(t), w_k)|^2 + \mu \sum_{k=1}^N (u(t), w_k)(v(t), w_k) + (h, u(t)) \\ \leq \frac{\mu}{4} \sum_{k=1}^N |(v(t), w_k)|^2 + \|h\| \|u(t)\| \leq \frac{\mu}{4} \|v(t)\|^2 + \frac{\nu}{2} \|\nabla u(t)\|^2 + \frac{1}{2\nu \lambda_1} \|h\|^2. \end{aligned}$$

Thanks to (2.4) we have

$$\frac{d}{dt} [\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2] + d_0 [\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2] \leq \frac{\mu}{2\lambda_1} r_0 + \frac{1}{\nu \lambda_1} \|h\|^2.$$

From this inequality we deduce that

$$\|u(t)\|^2 + \alpha^2 \|\nabla u(t)\|^2 \leq e^{-d_0 t} [\|u_0\|^2 + \alpha^2 \|\nabla u_0\|^2] + \frac{1}{d_0} \left[ \frac{\mu}{2\lambda_1} r_0 + \frac{1}{\nu \lambda_1} \|h\|^2 \right]$$

Hence the ball  $B_0(r_1) \subset V$  with radius  $r_1 = \frac{2}{d_0 \alpha^2} \left[ \frac{\mu}{2\lambda_1} r_0 + \frac{1}{\nu \lambda_1} \|h\|^2 \right]$  is an absorbing ball for problem (2.2).

**Theorem 2.1.** *Suppose that*

$$\mu \geq C_1(\nu) r_0^2 b_0^4, \quad \nu - 4\lambda_{N+1}^{-1} C_1(\nu) r_0^2 b_0^4 > 0, \quad (2.6)$$

where  $C_1(\nu) = \frac{1}{4} \left( \frac{3}{2\nu} \right)^3$ ,  $b_0$  is a constant in the Ladyzhenskaya inequality (1.10) and  $r_0$  is the radius of the absorbing ball of the problem (2.1).

Then there exists  $t_0 > 0$  such that the following inequality holds true

$$\|\nabla u(t) - \nabla v(t)\| \leq k_0 e^{-\kappa(t-t_0)} \|\nabla u(t_0) - \nabla v(t_0)\|, \quad t \geq t_0, \quad (2.7)$$

where  $k_0 = 1 + \frac{1}{\alpha^2 \lambda_1}$ ,  $\kappa = \frac{\nu}{4} \min\{\lambda_1, \alpha^{-2}\}$ .

*Proof.* If  $v$  is a solution of the problem (2.1) and  $u$  is a solution of system (2.2), then the function  $z = u - v$  is a solution of the system

$$\begin{cases} \partial_t z - \nu \Delta z - \alpha^2 \Delta \partial_t z + (u \cdot \nabla)z + (z \cdot \nabla)v + \nabla \pi = -\mu \sum_{k=1}^N (z, w_k) w_k, & x \in \Omega, t > 0, \\ \nabla \cdot z = 0, & x \in \Omega, t \in \mathbb{R}^+; \quad z|_{\partial\Omega} = 0, \quad t > 0, \\ z(x, 0) = z_0(x), & x \in \Omega, \end{cases} \quad (2.8)$$

where  $z_0 := u_0 - v_0, \pi := \tilde{p} - p$ .

Inner product of (2.8) in  $L^2(\Omega)$  with  $z$  gives

$$\frac{1}{2} \frac{d}{dt} [\|z(t)\|^2 + \alpha^2 \|\nabla z(t)\|^2] + \nu \|\nabla z(t)\|^2 + ((z(t) \cdot \nabla)v(t), z(t)) = -\mu \sum_{k=1}^N |(z(t), w_k)|^2. \quad (2.9)$$

Thanks to the Ladyzhenskaya inequality (1.10) and the Young inequality (1.6) we have

$$\begin{aligned} |((z(t) \cdot \nabla)v(t), z(t))| &\leq \|z(t)\|_{L^4(\Omega)}^2 \|\nabla v(t)\| \leq b_0 \|z(t)\|^{\frac{1}{2}} \|\nabla z(t)\|^{\frac{3}{2}} \|\nabla v(t)\| \\ &\leq \frac{\nu}{2} \|\nabla z(t)\|^2 + C_1(\nu) b_0^4 \|z(t)\|^2 \|\nabla v(t)\|^4, \end{aligned} \quad (2.10)$$

where  $C_1(\nu) = \frac{1}{4} \left(\frac{3}{2\nu}\right)^3$ .

Employing the fact that  $\|\nabla v(t)\|^4 \leq r_0^2$  for  $t \geq t_0$  and (2.10) we obtain from (2.9) the inequality

$$\frac{1}{2} \frac{d}{dt} [\|z(t)\|^2 + \alpha^2 \|\nabla z(t)\|^2] + \frac{\nu}{2} \|\nabla z(t)\|^2 = -\mu \sum_{k=1}^N |(z(t), w_k)|^2 + C_1(\nu) r_0^2 b_0^4 \|z(t)\|^2. \quad (2.11)$$

Since

$$\|z(t)\|^2 = \sum_{k=1}^N |(z(t), w_k)|^2 + \sum_{k=N+1}^{\infty} |(z(t), w_k)|^2 \leq \sum_{k=1}^N |(z(t), w_k)|^2 + \lambda_{N+1}^{-1} \|\nabla z(t)\|^2$$

we obtain from (2.11) the inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|z(t)\|^2 + \alpha^2 \|\nabla z(t)\|^2] + \left(\frac{\nu}{2} - \lambda_{N+1}^{-1} C_1(\nu) r_0^2 b_0^4\right) \|\nabla z(t)\|^2 \\ = (-\mu + C_1(\nu) r_0^2 b_0^4) \sum_{k=1}^N |(z(t), w_k)|^2. \end{aligned} \quad (2.12)$$

Taking into account conditions (2.6), (2.12) and the Poincaré-Friedrichs inequality (1.7) we obtain from (2.12) the inequality

$$\frac{d}{dt} [\|z(t)\|^2 + \alpha^2 \|\nabla z(t)\|^2] + \kappa [\|z(t)\|^2 + \alpha^2 \|\nabla z(t)\|^2] \leq 0, \quad \forall t \geq t_0,$$

where  $\kappa = \frac{\nu}{4} \min\{\lambda_1, \alpha^{-2}\}$ . Integrating the last inequality over the interval  $(t_0, t)$  we get

$$\|\nabla u(t) - \nabla v(t)\| \leq k_0 \|\nabla u(t_0) - \nabla v(t_0)\| e^{-\kappa(t-t_0)}, \quad \forall t \geq t_0.$$

Hence the inequality (2.7) holds true.  $\square$

3. STABILIZATION EMPLOYING FINITELY MANY FOURIER MODES FOR DAMPED NONLINEAR WAVE EQUATIONS.

3.1. **Benjamin-Bona-Mahony-Burgers equation.** Let  $v \in C(\mathbb{R}^+; H_0^1(0, 1))$  be a strong solution of the generalized Benjamin-Bona-Mahony(BBMB) equation

$$\partial_t v - \partial_x^2 \partial_t v + f(v) \partial_x v - \partial_x^2 v = h(x), \quad x \in (0, 1), t > 0, \quad (3.1)$$

satisfying the initial condition

$$v(x, 0) = v_0(x), \quad x \in (0, 1) \quad (3.2)$$

and the Dirichlet boundary conditions.

$$v(0, t) = v(1, t) = 0, \quad \forall t > 0. \quad (3.3)$$

Here  $h \in L^2(0, 1)$  is a given source term and  $f(\cdot) \in C^1(\mathbb{R})$  is a given function. First energy equality for this problem is the equality

$$\frac{1}{2} \frac{d}{dt} [\|v(t)\|^2 + \|\partial_x v(t)\|^2] + \|\partial_x v(t)\|^2 = (h, v(t)).$$

Applying the Poincaré-Friedrichs inequality (1.7) after some manipulations we get

$$\begin{aligned} \frac{d}{dt} [\|v(t)\|^2 + \|\partial_x v(t)\|^2] + \lambda_1 \|v(t)\|^2 + \|\partial_x v(t)\|^2 &\leq 2\|h\| \|v(t)\| \\ &\leq \frac{\lambda_1}{2} \|v(t)\|^2 + \frac{2}{\lambda_1} \|h\|^2. \end{aligned}$$

This inequality implies the inequality

$$\frac{d}{dt} [\|v(t)\|^2 + \|\partial_x v(t)\|^2] + \kappa_1 [\|v(t)\|^2 + \|\partial_x v(t)\|^2] \leq \frac{2}{\lambda_1} \|h\|^2, \quad (3.4)$$

where  $\kappa_1 = \min\{\frac{\lambda_1}{2}, 1\}$ .

Therefore there exists  $t_0 > 0$  such that

$$\|v(t)\|^2 + \|\partial_x v(t)\|^2 \leq \frac{4}{\lambda_1 \kappa_1} := R_1, \quad t \geq t_0. \quad (3.5)$$

We propose the following feedback system, to stabilize the solution  $v(x, t)$  of the problem (3.1)-(3.3):

$$\begin{cases} \partial_t u - \partial_x^2 \partial_t u + u \partial_x u - \partial_x^2 u = -\mu \sum_{k=1}^N (u - v, w_k) w_k + h(x), & x \in (0, 1), t > 0, \\ u(x, 0) = u_0(x), & x \in (0, 1), \\ u(0, t) = u(1, t) = 0, & t > 0. \end{cases} \quad (3.6)$$

Our aim is to show that for a given  $u_0 \in H_0^1(0, 1)$  and properly chosen  $\mu$  and  $N$  the function  $\|\partial_x u(t) - \partial_x v(t)\|$  tends to zero as  $t \rightarrow \infty$  with an exponential rate.

Multiplication of (3.23) by  $u$  in  $L^2$  gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|u\|^2 + \|\partial_x u\|^2] + \|\partial_x u\|^2 &= -\mu \sum_{k=1}^N |(u, w_k)|^2 + \mu \sum_{k=1}^N (u, w_k)(v, w_k) + (h, u) \\ &\leq \frac{\mu}{2} \sum_{k=1}^N |(v, w_k)|^2 + \frac{\lambda_1}{2} \|u\|^2 + \frac{2}{\lambda_1} \|h\|^2. \end{aligned}$$

From this inequality taking into account (3.5), similar to (3.4), we obtain for  $\Lambda(t) := \|u(t)\|^2 + \|\partial_x u(t)\|^2$  the following inequality

$$\frac{d}{dt} \Lambda(t) + a_1 \Lambda(t) \leq \frac{\mu R_1}{2} + \frac{2}{\lambda_1} \|h\|^2, \quad \forall t \geq t_0.$$

The last inequality implies that there exists some  $t_1 \geq t_0$  such that

$$\|u(t)\|^2 + \|\partial_x u(t)\|^2 \leq R_2, \quad t \geq t_1, \quad (3.7)$$

where  $R_2 = \mu R_1 + \frac{2}{\lambda_1} \|h\|^2$ .

It is clear that the function  $z = u - v$  is a solution of the problem

$$\begin{cases} \partial_t z + \partial_x^2 \partial_t z + f(u) \partial_x z + (f(u) - f(v)) \partial_x v - \partial_x^2 z = -\mu \sum_{k=1}^N (z, w_k) w_k, & x \in (0, 1), t > 0, \\ z(x, 0) = u_0(x) - v_0(x), & x \in (0, 1), \\ z(0, t) = z(1, t) = 0, & t > 0. \end{cases} \quad (3.8)$$

Multiplying (3.8) in  $L^2$  by  $z$  we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} [\|z\|^2 + \|\partial_x z\|^2] + \|\partial_x z\|^2 \\ + (f(u) \partial_x z, z) + f'(\theta u + (1 - \theta)v) z^2, \partial_x v = -\mu \sum_{k=1}^N |(z, w_k)|^2, \quad \theta \in (0, 1). \end{aligned} \quad (3.9)$$

Thanks to the the Sobolev inequality

$$\|z\|_{L^\infty(0,1)} \leq \|\partial_x z\|_{L^2(0,1)} \quad (3.10)$$

we have

$$\begin{aligned} \left| (f(u) \partial_x z, z) + f'(\theta u + (1 - \theta)v) z^2, \partial_x v \right| \\ \leq D_1 \|z\| \|\partial_x z\| + D_2 \|\partial_x v\| \|z\|^2 \leq \frac{1}{2} \|\partial_x z\|^2 + \left( \frac{1}{2} D_1^2 + D_2 \|\partial_x v\| \right) \|z\|^2, \end{aligned} \quad (3.11)$$

where

$$D_1 = \max_{|s| \leq R_2} |f(s)|, \quad D_2 = \max_{|s| \leq R_1 + R_2} |f'(s)|.$$

Thanks to (3.11), (3.5) and (3.7) we obtain from (3.9) the inequality

$$\frac{1}{2} \frac{d}{dt} [\|z\|^2 + \|\partial_x z\|^2] + \frac{1}{2} \|\partial_x z\|^2 = -\mu \sum_{k=1}^N |(z, w_k)|^2 + \left( \frac{1}{2} D_1^2 + D_2 \sqrt{R_1} \right) \|z\|^2, \quad \forall t \geq t_2. \quad (3.12)$$

Suppose that  $N$  and  $\mu$  are large enough such that

$$\lambda_{N+1}^{-1} \left( D_1^2 + 2D_2 \sqrt{R_1} \right) \leq \frac{1}{2}, \quad \text{and} \quad \mu \geq \frac{1}{2} D_1^2 + D_2 \sqrt{R_1}. \quad (3.13)$$

Then (3.12) yields

$$\frac{d}{dt} [\|z(t)\|^2 + \|\partial_x z(t)\|^2] + \frac{1}{2} \|\partial_x z(t)\|^2 \leq 0.$$

Applying Poincaré-Friedrichs inequality (1.7) we get

$$\frac{d}{dt} [\|z(t)\|^2 + \|\partial_x z(t)\|^2] + \frac{1}{4} \min\{1, \lambda_1\} [\|z(t)\|^2 + \|\partial_x z(t)\|^2] \leq 0, \quad t \geq t_1.$$

Hence

$$\|z(t)\|^2 + \|\partial_x z(t)\|^2 \leq e^{-a_0(t-t_1)} (\|z(t_1)\|^2 + \|\partial_x z(t_1)\|^2), \quad \forall t \geq t_1, \quad (3.14)$$

where  $a_0 = \frac{1}{4} \min\{1, \lambda_1\}$ . So we proved the following theorem

**Theorem 3.1.** *Suppose (3.13) are satisfied, then there exists a number  $t_1 > 0$  such that*

$$\|\partial_x u(t) - \partial_x v(t)\|^2 \leq e^{-a_0(t-t_1)} (\|\partial_x u(t_1) - \partial_x v(t_1)\|^2), \quad \forall t \geq t_1. \quad (3.15)$$

**Remark 3.2.** Statement of the Theorem 3.1 holds also for solutions of BBMB equation under the periodic boundary conditions

$$u(x, t) = u(x + 1, t), \quad \forall x \in \mathbb{R}, t > 0, \quad \int_0^1 u(x, t) dx = 0.$$

**3.2. Korteweg-de Vries-Burgers equation.** In this section we consider stabilization of Korteweg-de Vries-Burgers equation. Let  $v \in C(\mathbb{R}^+; \dot{H}_{per}^1(0, 1))$  be a strong solution of the equation

$$\partial_t v + \partial_x^3 v + v \partial_x v - \partial_x^2 v = h(x), \quad x \in \mathbb{R}, t > 0, \quad (3.16)$$

satisfying the initial condition

$$v(x, 0) = v_0(x), \quad x \in \mathbb{R} \quad (3.17)$$

and the periodic boundary conditions.

$$v(x, t) = v(x + 1, t), \quad \forall x \in \mathbb{R}, t > 0. \quad (3.18)$$

Here  $v_0 \in \dot{H}_{per}^1(0, 1)$  is a given initial function,  $h \in L_{per}^2(0, 1)$  is a given source term with  $\int_0^1 h(x) dx = 0$ .

There have been many studies on the long-time behavior of solutions of the KdVB equation (see e.g. [30], [40], [51]). Our goal here is to show that any solution of the initial boundary value problem (3.16)-(3.18) can be stabilised by using feedback controller employing finitely many Fourier modes as observables and controllers.

To this end we use the first energy equality

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|^2 + \|\partial_x v(t)\|^2 = (h, v(t))$$

to infer that there exists  $t_0 > 0$  such that

$$\|v(t)\|^2 \leq \frac{2}{\lambda_1^2} \|h\|^2 := \rho_1, \quad \forall t \geq t_0. \quad (3.19)$$

Next we show that the problem (3.16)-(3.18) generates a continuous bounded dissipative semigroup also in  $\dot{H}_{per}^1(0, 1)$ . Multiplication of (3.16) in  $L^2$  by  $v^2$  gives:

$$\frac{1}{3} \frac{d}{dt} (v^3, 1) - 2(v \partial_x v, \partial_x^2 v) - (v^2, \partial_x^2 v) = (h, v^2)$$

Using equation (3.16)

$$-v \partial_x v = \partial_t v + \partial_x^3 v - \partial_x^2 v - h$$

we get

$$\frac{d}{dt} \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] + 2 \|\partial_x^2 v\|^2 = -(v^2, \partial_x^2 v) - (h, v^2) - 2(h, \partial_x^2 v).$$

This equality implies that

$$\frac{d}{dt} \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] + \frac{3}{2} \|\partial_x^2 v\|^2 \leq 2 \|v\|_{L^4(0,1)}^2 + 5 \|h\|^2.$$

Thanks to the Poincaré-Friedrichs inequality (1.7) and Young inequality (1.6) we have

$$\begin{aligned} \frac{d}{dt} \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] + \lambda_1 \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] + \frac{1}{2} \|\partial_x^2 v\|^2 &\leq \frac{\lambda_1}{3} |(v^3, 1)| \\ &+ 2 \|v\|_{L^4}^2 + 5 \|h\|^2 \leq 3 \|v\|_{L^4}^4 + 5 \|h\|^2 + \left( \frac{\lambda_1}{3} \right)^4 + 1. \end{aligned} \quad (3.20)$$

Next we use the Gagliardo-Nirenberg inequality (1.11) with  $p = 4$

$$\|u\|_{L^4}^4 \leq \beta^4 \|\partial_x^2 u\|^{\frac{1}{2}} \|u\|^{\frac{7}{2}}.$$

and the Young inequality (1.6) we get

$$3 \|v\|_{L^4}^4 \leq 3 \beta^4 \|\partial_x^2 v\|^{\frac{1}{2}} \|v\|^{\frac{7}{2}} \leq \frac{1}{2} \|\partial_x^2 v\|^2 + \frac{1}{2^{1/3}} (3 \beta^4 \|v\|^{7/2})^{4/3}.$$

Employing last four estimates we deduce from (3.20) the inequality

$$\frac{d}{dt} \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] + \left[ \|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \right] \leq \rho_2, \quad t \geq t_0, \quad (3.21)$$

where

$$\rho_2 := \frac{1}{2^{1/3}} \left( 3 \beta^4 \rho_1^{7/4} \right)^{4/3} + 5 \|h\|^2 + \left( \frac{\lambda_1}{3} \right)^4 + 1.$$

From (3.21) we infer that there exists  $t_1 \geq t_0$  such that

$$\|\partial_x v\|^2 - \frac{1}{3} (v^3, 1) \leq 2 \rho_2 \quad \forall t \geq t_1 \geq t_0.$$

Since

$$\frac{1}{3} |(v^3, 1)| \leq \frac{\beta^3}{3} \|\partial_x v\|^{\frac{1}{2}} \|v\|^{\frac{5}{2}} \leq \frac{1}{4} \|\partial_x v\|^2 + \frac{4}{5} \left( \frac{1}{3} \beta^3 \|v\|^{\frac{5}{2}} \right)^{\frac{5}{4}}$$

and thanks to (3.19) we get

$$\|\partial_x v(t)\|^2 \leq \frac{8}{3}\rho_2 + \frac{16}{15} \left( \frac{1}{3}\beta^3 \rho_1^{\frac{5}{2}} \right)^{\frac{5}{4}} := M_0, \quad \forall t \geq t_1. \quad (3.22)$$

We propose the following feedback system, to stabilize the solution  $v(x, t)$  of the problem (3.16)-(3.18):

$$\begin{cases} \partial_t u + \partial_x^3 u + u \partial_x u - \partial_x^2 u = -\mu \sum_{k=1}^N (u - v, w_k) w_k + h(x), & x \in (0, 1), t > 0, \\ u(x, 0) = u_0(x), & x \in (0, 1), \\ u(x, t) = u(x + 1, t), & \forall x \in \mathbb{R}, t > 0. \end{cases} \quad (3.23)$$

Our aim is to show that for given  $u_0 \in \dot{H}_{per}^1(0, 1)$  and properly choosen  $\mu$  and  $N$  to be large enough the function  $\|u(t) - v(t)\|$  tends to zero as  $t \rightarrow \infty$  with an exponential rate.

First we obtain some uniform estimates for solutions of the problem (3.23). The uniform estimate of  $L^2$  norm of solution we obtain from the first energy equality

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\partial_x u\|^2 = -\mu \sum_{k=1}^N |(u, w_k)|^2 + \mu \sum_{k=1}^N (u, w_k)(v, w_k) + (h, u).$$

Thanks to the Cauchy-Schwarz inequality and Poincaré-Friedrichs inequality (1.7) we obtain the inequality

$$\frac{d}{dt} \|u\|^2 + \lambda_1 \|u\|^2 = \frac{\mu}{2} \sum_{k=1}^N |(v, w_k)|^2 + \lambda_1 \|h\|^2 \leq \frac{\mu}{2} \rho_1 + \frac{1}{\lambda_1} \|h\|^2.$$

which implies, thanks to Gronwall's inequality, that there exists  $t_2 \geq t_1$ , depending on  $u_0$ , such that

$$\|u(t)\|^2 \leq \lambda_1^{-1} (\mu \rho_1 + \lambda_1^{-1} \|h\|^2) =: M_1, \quad \forall t \geq t_2.$$

To get the uniform estimate for  $\|\partial_x u(t)\|$  we multiply (3.23) by  $-2\partial_x^2 u$  in  $L^2$  and obtain the equality

$$\frac{d}{dt} \|\partial_x u\|^2 + \int_0^1 (\partial_x u)^3 dx + 2\|\partial_x^2 u\|^2 = -2\mu \sum_{k=1}^N \lambda_k [(u, w_k)^2 - (u, w_k)(v, w_k)] - 2(h, \partial_x^2 u). \quad (3.24)$$

Employing the 1D Gagliardo-Niranberg inequality (1.11), and the Young inequality (1.6) we get the estimate

$$|((\partial_x u)^3, 1)| \leq \beta^3 \|u\|^{\frac{11}{4}} \|\partial_x^2 u\|^{\frac{1}{4}} \leq \frac{1}{2} \|\partial_x^2 u\|^2 + \beta_1 \|u\|^{22/7}, \quad (3.25)$$

where  $\beta_1 = \beta^{24/7}$  and

$$|(u, w_k)(v, w_k)| \leq (u, w_k)^2 + \frac{1}{4}(v, w_k)^2, \quad 2|(h, \partial_x^2 u)| \leq \frac{1}{2} \|\partial_x^2 u\|^2 + 2\|h\|^2. \quad (3.26)$$

By using (3.25) and (3.26) we obtain from (3.24)

$$\frac{d}{dt} \|\partial_x u\|^2 + \|\partial_x^2 u\|^2 \leq \frac{\mu}{2} \sum_{k=1}^N \lambda_k (v, w_k)^2 + \beta_1 \|u\|^{22/7} + 2\|h\|^2, \quad (3.27)$$

Finally due to the Poincaré-Friedrichs inequality (1.7), and since

$$\sum_{k=1}^N \lambda_k(v, w_k)^2 \leq \|\partial_x v(t)\|^2 \leq M_0, \quad \|u(t)\|^2 \leq M_1, \quad \forall t \geq t_2$$

we deduce from (3.27) that there exists  $t_3 \geq t_2$ , depending on  $u_0$ , such that

$$\|\partial_x u(t)\|^2 \leq M_2, \quad \forall t \geq t_3 \geq t_2, \quad (3.28)$$

where  $M_2 = \mu M_0 + \beta_1 M_1^{11/7} + 2\|h\|^2$ .

It is clear that the function  $z = u - v$  is a solution of the problem

$$\begin{cases} \partial_t z + \partial_x^3 z + u \partial_x z + z \partial_x v - \partial_x^2 z = -\mu \sum_{k=1}^N (z, w_k) w_k, & x \in (0, 1), t > 0, \\ z(x, 0) = u_0(x) - v_0(x), & x \in (0, 1), \\ z(x, t) = z(x+1, t), & \forall x \in \mathbb{R}, t > 0. \end{cases} \quad (3.29)$$

Multiplying (3.29) in  $L^2$  by  $z$  we obtain

$$\frac{1}{2} \frac{d}{dt} \|z(t)\|^2 + (u \partial_x z, z) + (z^2(t), \partial_x v) + \|\partial_x z(t)\|^2 = -\mu \sum_{k=1}^N |(z(t), w_k)|^2. \quad (3.30)$$

By using the Gagliardo-Nirenberg inequality (1.11) and the Cauchy - Schwarz inequality and the estimates (3.22) and (3.28) we get

$$\begin{aligned} |(u \partial_x z, z)| &= \frac{1}{2} |(z^2, \partial_x u)| \leq \frac{1}{2} \|\partial_x u\| \|z\|_{L^4}^2 \leq \frac{1}{2} \beta^2 \|\partial_x u\| \|z\|^{\frac{3}{2}} \|\partial_x z\|^{\frac{1}{2}} \\ &\leq \frac{1}{2} \|\partial_x z\|^2 + \left( \frac{1}{2} \beta^2 \|\partial_x u\| \right)^{4/3} \|z\|^2, \quad \forall t \geq t_3. \end{aligned} \quad (3.31)$$

Similarly we obtain

$$|(z^2, \partial_x v)| \leq \|v\|_{L^4}^2 \|\partial_x v\| \leq \beta^2 \|z\|^{1/2} \|\partial_x z\|^{3/2} \leq \frac{1}{2} \|\partial_x z\|^2 + \frac{1}{4} (\|\partial_x v\| \beta^2)^4 \|z\|^2, \quad \forall t \geq t_3. \quad (3.32)$$

Thanks to (3.31) and (3.32) we get from (3.30) the inequality

$$\frac{d}{dt} \|z(t)\|^2 + \|\partial_x z(t)\|^2 \leq -2\mu \sum_{k=1}^N (z(t), w_k)^2 + M_3 \|z\|^2, \quad \forall t \geq t_3,$$

where  $M_3 := 2\beta^4 \lambda_1^{-\frac{3}{4}} (M_0^2 + \frac{1}{4} M_2^2)$ . Using the inequality (1.8) and the fact that

$$\|z(t)\|^2 = \sum_{k=1}^N (z(t), w_k)^2 + \sum_{k=N+1}^{\infty} (z(t), w_k)^2, \quad \forall t \geq t_3,$$

we get

$$\frac{d}{dt} \|z(t)\|^2 + \|\partial_x z(t)\|^2 \leq (-2\mu + M_3) \sum_{k=1}^N (z(t), w_k)^2 + M_3 \lambda_{N+1}^{-1} \|\partial_x z(t)\|^2, \quad \forall t \geq t_3.$$

Hence, if  $N$  and  $\mu$  are large enough, such that  $M_3\lambda_{N+1}^{-1} \leq \frac{1}{2}$  and  $M_3 \leq 2\mu$ , we have

$$\frac{d}{dt}\|z(t)\|^2 + \frac{\lambda_1}{2}\|z(t)\|^2 \leq 0, \quad \forall t \geq t_3.$$

Thus

$$\|z(t)\| \leq z(t_3)e^{-(t-t_3)}, \quad \forall t \geq t_3.$$

So we proved the following theorem

**Theorem 3.3.** *Suppose that  $v \in \dot{H}_{per}^1(0, 1)$  is a given weak solution of problem (3.16)-(3.18) and the following conditions hold true:*

$$2\beta^4\lambda_1^{-\frac{3}{4}}\left(M_0^2 + \frac{1}{4}M_2^2\right) \leq 2\mu, \quad \text{and} \quad \lambda_{N+1}^{-1}\left(2\beta^4\lambda_1^{-\frac{3}{4}}\left(M_0^2 + \frac{1}{4}M_2^2\right)\right) \leq \frac{1}{2},$$

where  $M_0, M_2$  are defined in (3.22) and (3.28). Then

$$\|u(t) - v(t)\| \rightarrow 0 \quad \text{with an exponential rate, as } t \rightarrow \infty,$$

where  $u(t) \in \dot{H}_{per}^1(0, 1)$  is an arbitrary solution of the feedback control system.

**3.3. Strongly damped nonlinear wave equation.** In this section we consider the initial boundary value problem for 3D strongly damped nonlinear wave equation:

$$\begin{cases} \partial_t^2 v - \Delta v - b\Delta\partial_t v - \lambda v + f(v) = 0, & x \in \Omega, t > 0, \\ v = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), \quad \partial_t v(x, 0) = v_1(x), & x \in \Omega, t > 0, \end{cases} \quad (3.33)$$

where  $b > 0$  is a given number,  $\Omega \subset \mathbb{R}^3$  is a bounded domain with a smooth boundary  $\partial\Omega$ ,  $f : C^1(\mathbb{R} \rightarrow \mathbb{R})$  is a given function which satisfies the condition

$$-m_0 + a|s|^p \leq f'(s) \leq m_0(1 + |s|^p), \quad \forall s \in \mathbb{R}^1 \quad (3.34)$$

with some  $m_0 > 0, a > 0, p \geq 2$ . The existence of a unique weak (energy) solution of the problem (3.33), i.e. a function

$$v \in L^\infty([0, T], H_0^1(\Omega) \cap L^{p+1}(\Omega)) \cap W^{1,\infty}([0, T], L^2(\Omega)) \cap W^{1,2}([0, T], H^1(\Omega)), \quad \forall t > 0,$$

which satisfies the equation in the sense of distributions, is established in [35]. Here we consider the feedback control problem for stabilizing the solution  $v$  of the 3D strongly damped nonlinear wave equation, based on finitely many Fourier modes, i.e. , we consider the feedback system of the following form:

$$\begin{cases} \partial_t^2 u - \Delta u - b\Delta\partial_t u - \lambda u + f(u) = -\mu \sum_{k=1}^N (z + \partial_t z, w_k) w_k, & x \in \Omega, t > 0, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \Omega, t > 0, \end{cases} \quad (3.35)$$

where  $z = u - v$ ,  $\mu > 0$  is a given numbers,  $w_1, w_2, \dots, w_n, \dots$  is the set of orthonormal (in  $L^2(\Omega)$ ) eigenfunctions of the Laplace operator  $-\Delta$  under the homogeneous Dirichlet's boundary condition, corresponding to eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n \leq \dots$$

In what follows we will use the following lemma:

**Lemma 3.4.** (see for instance [50]) *If the function  $f$  satisfies condition (3.34) then*

$$[f(u_1) - f(u_2)] \cdot (u_1 - u_2) \geq -\lambda|v|^2 + d_0(|u_1|^p + |u_2|^p)|v|^2, \quad (3.36)$$

for some positive  $\lambda, d_0$  and  $p \geq 2$ .

Our main result in this section is the following theorem:

**Theorem 3.5.** *Suppose that  $\mu$  and  $N$  are large enough such that*

$$\nu \geq (2a + 3b^2/4)\lambda_{N+1}^{-1}, \quad \text{and} \quad \mu \geq a + 3b^2/4. \quad (3.37)$$

Then the following decay estimate holds true

$$\|\partial_t z(t)\|^2 + \|\nabla z(t)\|^2 \leq E_0 e^{-\frac{b}{2}t}, \quad (3.38)$$

where

$$\begin{aligned} E_0 &:= \frac{1}{2}\|u_1\|^2 + \frac{\nu}{2}\|\nabla u_0\|^2 \\ &+ \left(\frac{b^2}{4} - \frac{a}{2}\right)\|u_0\|^2 + \frac{1}{p} \int_{\Omega} |u_0(x)|^p dx + \frac{\mu}{2} \sum_{k=1}^N (u_0, w_k)^2 + \frac{b}{2}(u_0, u_1). \end{aligned}$$

*Proof.* It is clear that the function  $z = u - v$  satisfies

$$\begin{cases} \partial_t^2 z - \Delta z - b\Delta \partial_t z + f(u) - f(v) = -\mu \sum_{k=1}^N (z + \partial_t z, w_k) w_k, & x \in \Omega, t > 0, \\ z = 0, \text{ for } x \in \partial\Omega, t > 0, \\ z(x, 0) = z_0(x), \partial_t z(x, 0) = z_1(x), & x \in \Omega, t > 0, \end{cases} \quad (3.39)$$

where  $v_0 = u_0 - v_0, z_1 = u_1 - v_1$ .

Let  $P = (-\Delta)^{-1}$  be the inverse of the Laplace operator under the homogeneous Dirichlet boundary condition. First we multiply the equation (3.39) by  $P\partial_t z$  and integrate over  $x \in \Omega$

$$\begin{aligned} \frac{d}{dt} \left[ \|P^{\frac{1}{2}}\partial_t z\|^2 + \|z\|^2 + \mu \sum_{k=1}^N \lambda_k^{-1} (z, w_k)^2 \right] &+ 2b\|\partial_t z\|^2 + 2(f(u) - f(v), P\partial_t z) \\ &+ 2\mu \sum_{k=1}^N \lambda_k^{-1} (\partial_t z, w_k)^2 = 0. \end{aligned} \quad (3.40)$$

Multiplying (3.39) by  $z$  and integrating over  $\Omega$  we get

$$\begin{aligned} \frac{d}{dt} \left[ \frac{b}{2}\|\nabla z\|^2 + (z, \partial_t z) + \frac{\mu}{2} \sum_{k=1}^N (z, w_k)^2 \right] &- \|\partial_t z\|^2 + \|\nabla z\|^2 \\ &= -(f(u) - f(v), z) - \mu \sum_{k=1}^N (z, w_k)^2. \end{aligned} \quad (3.41)$$

Now we multiply (3.41) by a positive parameter  $\varepsilon > 0$  (to be chosen below) and add to (3.40) to obtain:

$$\begin{aligned}
 \frac{d}{dt}E_\varepsilon(t) + (2b - \varepsilon)\|\partial_t z\|^2 + 2(f(u) - f(v), P\partial_t z) + 2\mu \sum_{k=1}^N \lambda_k^{-1}(\partial_t z, w_k)^2 \\
 + \varepsilon(f(u) - f(v), z) + \varepsilon\|\nabla z\|^2 + \varepsilon\mu \sum_{k=1}^N (z, w_k)^2 = 0,
 \end{aligned} \tag{3.42}$$

where

$$\begin{aligned}
 E_\varepsilon(t) := \|P^{\frac{1}{2}}\partial_t z(t)\|^2 + \|z(t)\|^2 + \mu \sum_{k=1}^N (\lambda_k^{-1} + \frac{\varepsilon}{2})(z(t), w_k)^2 + \frac{\varepsilon b}{2}\|\nabla z(t)\|^2 \\
 + \varepsilon(z(t), \partial_t z(t)).
 \end{aligned}$$

It is easy to see that if  $0 < \varepsilon \leq \frac{b}{2}$  then

$$E_\varepsilon(t) \geq \frac{1}{2}\|P^{\frac{1}{2}}\partial_t z\|^2 + \|z\|^2 + \mu \sum_{k=1}^N (\lambda_k^{-1} + \frac{\varepsilon}{2})(z, w_k)^2 + \frac{\varepsilon b}{4}\|\nabla z\|^2. \tag{3.43}$$

Employing the interpolation inequality

$$\|P\partial_t z\|^2 \leq c_0\|\partial_t z\|\|P^{\frac{1}{2}}\partial_t z\|$$

and the condition (3.34) we can estimate the term  $2(f(u) - f(v), P\partial_t z)$  as follows

$$\begin{aligned}
 |2(f(u) - f(v), P\partial_t z)| \leq \varepsilon_1(|u| + |v|)^p, |z|^2 + \varepsilon_1\|z\|^2 + \varepsilon_1\|\partial_t z\|^2 \\
 + C(\varepsilon_1)\|P^{\frac{1}{2}}\partial_t z\|^2.
 \end{aligned} \tag{3.44}$$

Since

$$( (|u| + |v|)^p, |z|^2 ) \leq C \left( \int_{\Omega} (|u|^p + |v|^p) dx \right) ( (|u| + |v|)^p, |z|^2 )$$

and the integral  $\int_{\Omega} (|u(x, t)|^p + |v(x, t)|^p) dx$  is uniformly bounded we obtain from (3.44) that

$$\begin{aligned}
 |2(f(u) - f(v), P\partial_t z)| \leq \varepsilon_1 C (|u|^p + |v|^p), |z|^2 + \varepsilon_1\|z\|^2 + \varepsilon_1\|\partial_t z\|^2 \\
 + C(\varepsilon_1)\|P^{\frac{1}{2}}\partial_t z\|^2,
 \end{aligned} \tag{3.45}$$

where  $C(\varepsilon_1) = \frac{C}{\varepsilon_1^3}$ .

By using the inequalities (3.45) and (3.36) we deduce from (3.42) the inequality:

$$\begin{aligned}
 \frac{d}{dt}E_\varepsilon(t) + (2b - \varepsilon - \varepsilon_1)\|\partial_t z\|^2 + (d_0\varepsilon - \varepsilon_1 C)(|u|^p + |v|^p), |z|^2 \\
 + \varepsilon\|\nabla z\|^2 - (\varepsilon_1 + \varepsilon\lambda)\|z\|^2 \\
 + \mu \sum_{k=1}^N [\varepsilon(z, w_k)^2 + 2\lambda_k^{-1}(\partial_t z, w_k)^2] - C(\varepsilon_1)\|P^{\frac{1}{2}}\partial_t z\|^2 \leq 0.
 \end{aligned} \tag{3.46}$$

By choosing  $\varepsilon_1 = \frac{d_0\varepsilon}{C}$  and  $0 < \varepsilon \leq \min\{\frac{b}{2}, \frac{bC}{C+d_0}\}$  in (3.45) we get

$$\begin{aligned} \frac{d}{dt}E_\varepsilon(t) + b\|\partial_t z\|^2 + \varepsilon\|\nabla z\|^2 - (\varepsilon_1 + \varepsilon\lambda)\|z\|^2 \\ + \mu \sum_{k=1}^N [\varepsilon(z, w_k)^2 + 2\lambda_k^{-1}(\partial_t z, w_k)^2] - C(\varepsilon_1)\|P^{\frac{1}{2}}\partial_t z\|^2 \leq 0. \end{aligned}$$

Let us rewrite the last inequality as follows

$$\begin{aligned} \frac{d}{dt}E_\varepsilon(t) + b\|\partial_t z\|^2 + \varepsilon\|\nabla z\|^2 + (\mu\varepsilon - \varepsilon_1 - \varepsilon\lambda) \sum_{k=1}^N (z, w_k)^2 \\ - (\varepsilon_1 + \varepsilon\lambda) \sum_{k=N+1}^{\infty} (z, w_k)^2 + (2\mu - C(\varepsilon_1)) \sum_{k=1}^N (\partial_t z, w_k)^2 \lambda_k^{-1} \quad (3.47) \\ - C(\varepsilon_1) \sum_{k=N+1}^{\infty} (\partial_t z, w_k)^2 \lambda_k^{-1} \leq 0. \end{aligned}$$

Finally by choosing  $\mu$  and  $\lambda_{N+1}$  large enough we infer from (3.48) the following inequality

$$\frac{d}{dt}E_\varepsilon(t) + \frac{b}{2}\|\partial_t z\|^2 + \frac{\varepsilon}{2}\|\nabla z\|^2 + \frac{1}{2}\mu\varepsilon \sum_{k=1}^N (z, w_k)^2 \leq 0. \quad (3.48)$$

Employing the last inequality and (3.43) we can show that there exists some  $\delta > 0$  depending on  $\varepsilon$  such that

$$\frac{d}{dt}E_\varepsilon(t) + \delta E_\varepsilon(t) \leq 0.$$

The last inequality implies that

$$\|P^{\frac{1}{2}}\partial_t z(t)\|^2 + \|\nabla z(t)\|^2$$

tends to zero with an exponential rate, as  $t \rightarrow \infty$ .

#### 4. WAVE EQUATION WITH NONLINEAR DAMPING TERM

In this section we consider the initial boundary value problem for a semilinear wave equation with nonlinear damping:

$$\begin{cases} \partial_t^2 v + g(\partial_t u) - \Delta u + f(u) = 0, & x \in \Omega, t > 0, \\ v = 0, & x \in \partial\Omega, t > 0, \\ v(x, 0) = v_0(x), \quad \partial_t v(x, 0) = u_1(x), & x \in \Omega, t > 0, \end{cases} \quad (4.1)$$

where  $f \in C(\mathbb{R} \rightarrow \mathbb{R})$  is a given function which satisfies the condition (3.34),  $g \in C(\mathbb{R} \rightarrow \mathbb{R})$  is a given function which satisfies the conditions

$$g(0) = 0, \quad |g(u_1) - g(u_2)| \leq a_0(1 + |u_1|^m + |u_2|^m)|u_1 - u_2|, \quad \forall u_1, u_2 \in \mathbb{R} \quad (4.2)$$

and

$$[g(u_1) - g(u_2)] \cdot (u_1 - u_2) \geq a_1|u_1 - u_2|^2 + a_2|u_1 - u_2|^{m+2}, \quad \forall u_1, u_2 \in \mathbb{R}. \quad (4.3)$$

It is well known that, under the above conditions, the stationary problem

$$\begin{cases} -\Delta\phi + f(\phi) = 0, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \quad (4.4)$$

corresponding to (4.5), has finitely many solutions.

Next we will show that the system (4.5) can be globally stabilized to a given stationary state  $\phi$  also by using a feedback controller involving finitely many Fourier modes. More precisely we will show that all solutions of the following feedback control problem

$$\begin{cases} \partial_t^2 u + g(\partial_t u) - \Delta u + f(u) = -\mu \sum_{k=1}^N (u - \phi, w_k) w_k, & x \in \Omega, t > 0 \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), & x \in \Omega, t > 0, \end{cases} \quad (4.5)$$

tend in the energy norm to the stationary state  $\phi$ , as  $t \rightarrow \infty$ . For global existence and long time behavior of solutions of (4.5) see e.g. [29], [39],[42], [46].

It is clear that the function  $z = u - \phi$  satisfies

$$\begin{cases} \partial_t^2 z + g(\partial_t z) - \Delta z + f(u) - f(\phi) = -\mu \sum_{k=1}^N (z, w_k) w_k, & x \in \Omega, t > 0, \\ z = 0, & x \in \partial\Omega, t > 0, \\ z(x, 0) = z_0(x), \quad \partial_t z(x, 0) = z_1(x), & x \in \Omega, t > 0, \end{cases} \quad (4.6)$$

where  $z = u - \phi$ ,  $z_0 = u_0 - \phi$ ,  $z_1 = u_1$ ,  $m$  is a given number such that  $m > 0$  if  $n = 1, 2$  and  $m \leq \frac{4}{n-2}$  if  $n \geq 3$ .

Our aim is to show that under some restrictions on  $\mu, N$  the function  $z$  tends to zero as  $t \rightarrow \infty$ .

First we multiply (4.6) by  $\partial_t z$  and integrate over  $\Omega$ :

$$\frac{1}{2} \frac{d}{dt} \left[ \|\partial_t z\|^2 + \|\nabla z\|^2 + \mu \sum_{k=1}^N (z, w_k)^2 \right] + (f(u) - f(\phi), \partial_t z) + (g(\partial_t z), p_t z) = 0.$$

Since

$$(f(u) - f(\phi), \partial_t z) = \frac{d}{dt} [(F(u), 1) - (f(\phi), z)]$$

and

$$(F(u), 1) = \int_0^1 f(\phi + sz) + (F(\phi), 1)$$

we have

$$\frac{d}{dt} \left[ \frac{1}{2} \|\partial_t z\|^2 + \frac{1}{2} \|\nabla z\|^2 + \frac{\mu}{2} \sum_{k=1}^N (z, w_k)^2 + \mathcal{F}(z) \right] + (g(\partial_t z), p_t z) = 0, \quad (4.7)$$

where

$$\mathcal{F}(z) = \int_0^1 (f(\phi + sz) - f(\phi), z) ds.$$

Due to the condition (3.34)

$$\mathcal{F}(z) \geq -\frac{m_0}{2}\|z\|^2 + \frac{d_0}{p+2} \int_G |z|^{p+2} dx.$$

Therefore if

$$\mu \geq m_0 \quad \text{and} \quad \lambda_{N+1} \geq 2\mu m_0 \quad (4.8)$$

then

$$\begin{aligned} E(t) &:= \frac{1}{2}\|\partial_t z\|^2 + \frac{1}{2}\|\nabla z\|^2 + \frac{\mu}{2} \sum_{k=1}^N (z, w_k)^2 + \mathcal{F}(z) \\ &\geq \frac{1}{2}\|\partial_t z\|^2 + \frac{1}{4}\|\nabla z\|^2 + \frac{d_0}{p+2} \int_G |z|^{p+2} dx. \end{aligned} \quad (4.9)$$

We will need also the following inequality which we obtain by integration of (4.7) over the interval  $(0, t)$  and using the condition (4.3)

$$E(t) + a_1 \int_0^t \|\partial_\tau z(\tau)\|^2 d\tau + a_2 \int_0^t \int_G |\partial_\tau z(x, \tau)|^{m+2} dx d\tau \leq E(0). \quad (4.10)$$

Next we multiply (3.34) by  $z$ :

$$\frac{d}{dt}(z, \partial_t z) = \|\partial_t z\|^2 - \|\nabla v\|^2 - (f(z + \phi) - f(\phi), z) - (g(\partial_t z), z) - \mu \sum_{k=1}^N (z, w_k)^2. \quad (4.11)$$

Employing the equality

$$-\frac{1}{2}\|\nabla z\|^2 = \frac{3}{2}\|\partial_t z\|^2 + \frac{\mu}{2} \sum_{k=1}^N (z, w_k)^2 + \mathcal{F}(z) - E(t)$$

we get from (4.11):

$$\begin{aligned} \frac{d}{dt}(z, \partial_t z) &= \frac{3}{2}\|\partial_t z\|^2 - \frac{1}{2}\|\nabla z\|^2 + [\mathcal{F}(z) - (f(z + \phi) - f(\phi), v)] - (g(\partial_t z), z) \\ &\quad - \frac{\mu}{2} \sum_{k=1}^N (z, w_k)^2 - E(t). \end{aligned} \quad (4.12)$$

Thanks to condition (3.34) we have

$$\begin{aligned} \mathcal{F}(z) - (f(z + \phi) - f(\phi), z) &= \int_0^1 (f(\phi + sz) - f(\phi + z), z) ds \\ &= - \int_0^1 (f(\phi + z) - f(\phi + sz), z) ds \leq \frac{m_0}{2}\|v\|^2 - \frac{d_0}{p+2} \int_G |v|^{p+2} dx. \end{aligned}$$

Utilizing the last inequality and conditions (4.8) we obtain from (4.13) that

$$\frac{d}{dt}(z, \partial_t z) \leq \frac{3}{2}\|\partial_t z\|^2 - (g(\partial_t z), z) - E(t).$$

Integrating the last inequality and employing condition (4.2) we get

$$\begin{aligned} \int_0^t E(\tau) d\tau &\leq (z(0), \partial_t z(0)) - (v(t), \partial_t z(t)) + \frac{3}{2} \int_0^t \|\partial_\tau z(\tau)\|^2 d\tau \\ &\quad + a_0 \int_0^t \|\partial_\tau z(\tau)\| \|z(\tau)\| d\tau + a_0 \int_0^t \int_G |\partial_\tau z(x, \tau)|^{m+1} |z(x, \tau)| dx d\tau. \end{aligned} \quad (4.13)$$

Thanks to (4.9) and (4.10) we have

$$\left| (z(0), \partial_t z(0)) - (z(t), \partial_t z(t)) + \frac{3}{2} \int_0^t \|\partial_\tau z(\tau)\|^2 d\tau \right| \leq C_1. \quad (4.14)$$

On the other hand, employing the Cauchy - Schwarz inequality and (4.10) we get

$$\int_0^t \|\partial_\tau z(\tau)\| \|z(\tau)\| d\tau \leq C_2 t^{\frac{1}{2}}, \quad (4.15)$$

Thanks to (4.10), the Hölder inequality and the continuous embedding of  $H_0^1(G)$  into  $L^{m+2}(G)$  we estimate the last term on the right-hand side of (4.13):

$$\begin{aligned} &\int_0^t |\partial_\tau z(x, \tau)|^{m+1} |z(x, \tau)| dx d\tau \\ &\leq \left( \int_0^t \int_G |\partial_\tau z(x, \tau)|^{m+2} dx d\tau \right)^{\frac{m+1}{m+2}} \left( \int_0^t \int_G |z(x, \tau)|^{m+2} dx d\tau \right)^{\frac{1}{m+2}} \\ &\leq C \left( \int_0^t \int_G |\partial_\tau z(x, \tau)|^{m+2} dx d\tau \right)^{\frac{m+1}{m+2}} \left( \int_0^t \|\nabla z(\tau)\|^{m+2} d\tau \right)^{\frac{1}{m+2}} \leq C_3 t^{\frac{1}{m+2}}. \end{aligned} \quad (4.16)$$

Hence due to (4.14)-(4.16) we obtain from (4.13) the inequality

$$\int_0^t E(\tau) d\tau \leq C_1 + C_2 t^{\frac{1}{2}} + C_3 t^{\frac{1}{m+2}}. \quad (4.17)$$

Estimates (4.7) and (4.9) imply that  $E(t)$  is nondecreasing function and therefore

$$tE(t) \leq \int_0^t E(\tau) d\tau, \quad \forall t \geq 0.$$

From the last inequality and (4.17) we finally obtain the desired decay estimate

$$E(t) \leq Ct^{-\frac{1}{2}}, \quad \text{provided } t > 0 \text{ is large enough.}$$

□

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